Mechanics – *i + j notation*

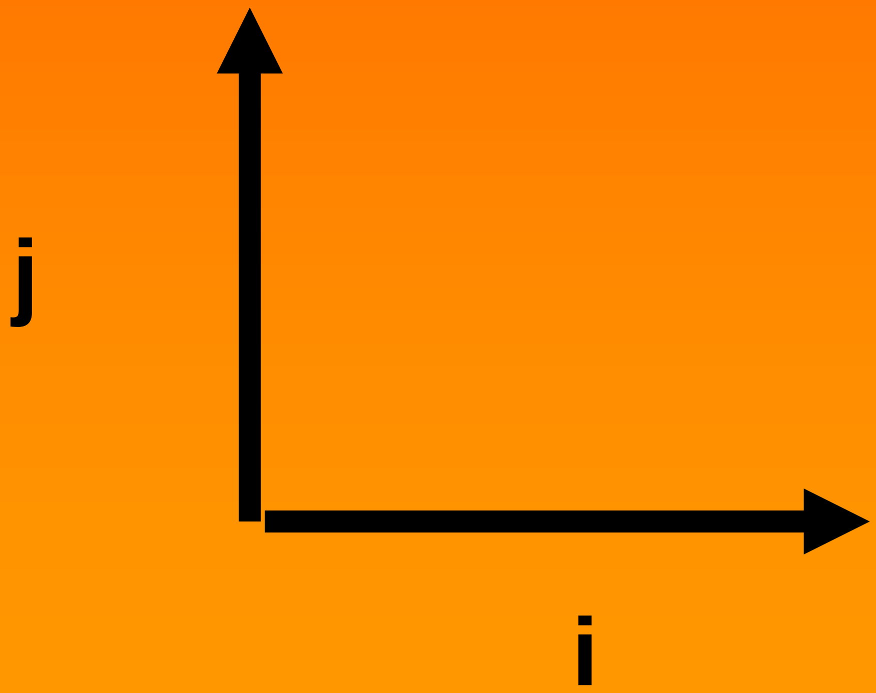
Guaranteeing university rejection since the year whenever

(this presentation will be focusing on the bits most likely causing university rejection)

By Stephen the Mechanical Master of i and j notation

The basics you need to know to answer the hard stuff

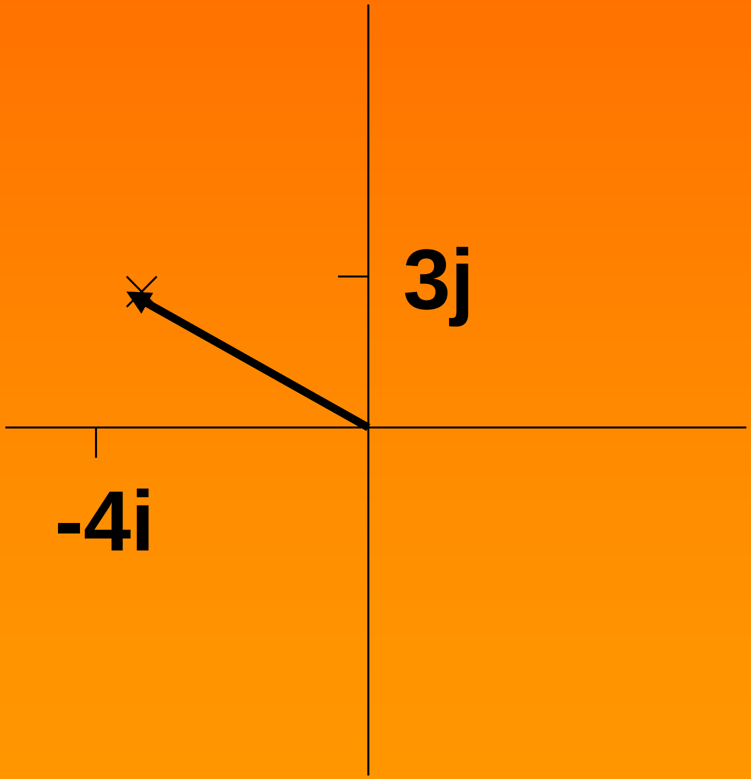
- ‘i’ is the horizontal component
- ‘j’ is the vertical component



For example…

- If a boat has position vector *A*:

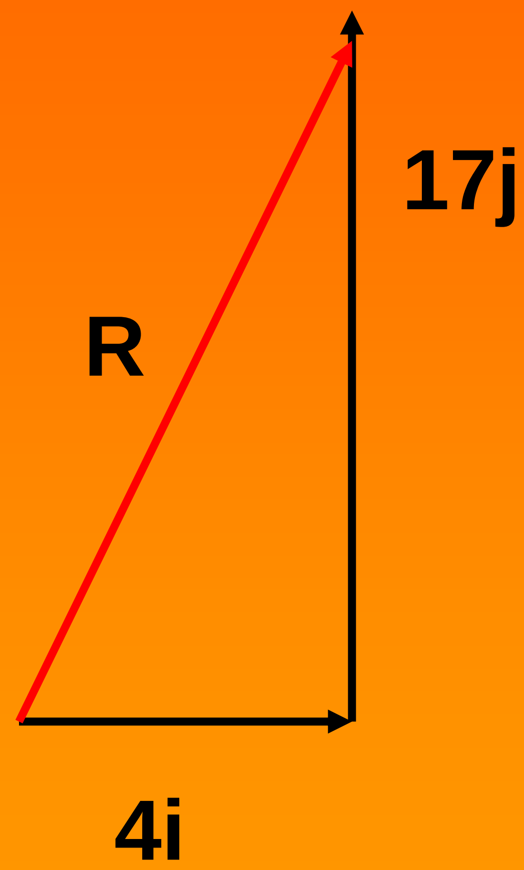
(-4i + 3j)km is:



And if its velocity is…

- $(4i + 17j)\text{kmh}^{-1}$

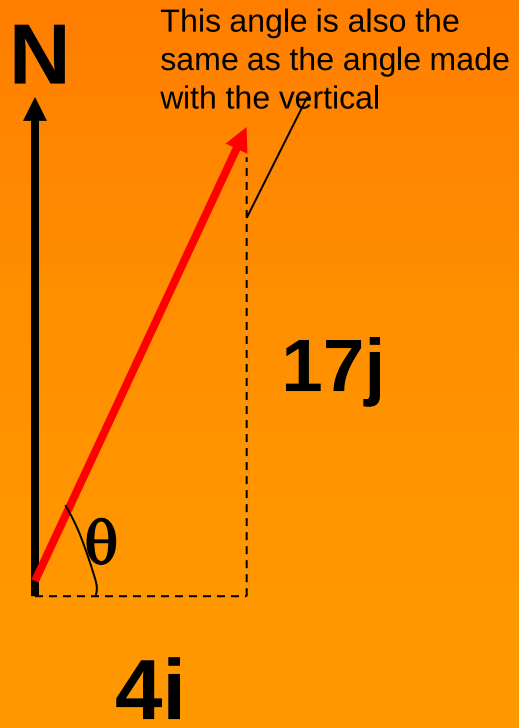
Using Pythagoras' Theorem we can calculate what the magnitude of the velocity |v| will be, finding the resultant:



- $R^2 = 42 + 172$
- $R^2 = 16 + 289$
- $R^2 = 305$
- $R = \sqrt{305} = 17.46\text{kmh}^{-1}$

But what will be the direction of the resultant velocity?

- **The bearing is <u>ALWAYS</u> measured from the north, and the angle is the angle made with the <u>vertical</u> axis. So using the previous example:**



Using trigonometry:

S =O/H C=A/H T=O/A

Where θ is the angle made with the horizontal axis, we can deduce θ by using tanθ = opposite / adjacent

So that $\theta = \tan^{-1}$(opposite/adjacent)

So $\theta = \tan^{-1}(17/4)$

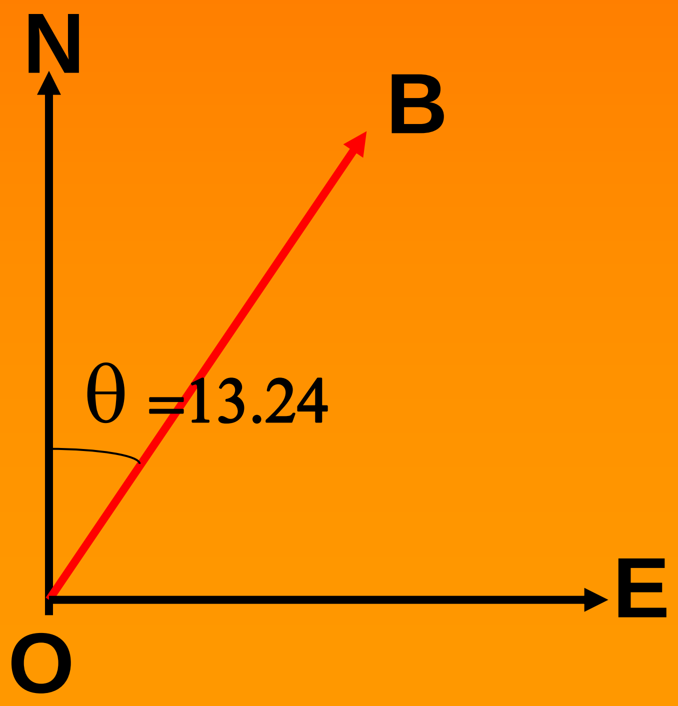
θ= 76.76°

But as we measure the angle from the north line, the angle made with N is

90 – 76.76 = 13.24°

So…

- **The angle made with the north line is 13.24˚, and we can therefore say that the boat is travelling N 13.24˚E:**



$\overrightarrow{OB}$ is in the direction

N 13.24 E

Velocity in the form (i + j)ms^{-1}

- **We know that:**

$$v = \frac{\Delta s}{\Delta t}$$

- **So: $\Delta s = v\Delta t$**
- **Therefore after a given time, a moving object will be:**

(its current position) + (its velocity x time)

Finding velocity at time ‘t’

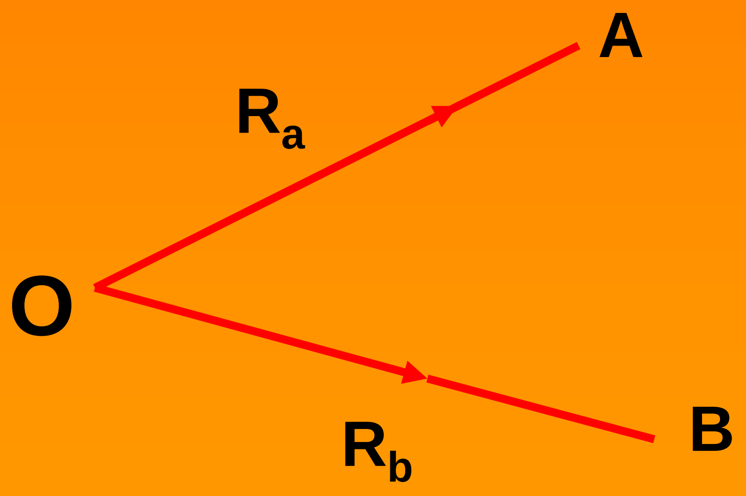
- **Therefore if an object, say a boat, is at position vector (-4i + 3j)km and is moving at a speed of (4i + 17j)kmh^{-1} then after time t, its new position will be:**
- **(initial position) + (displacement)**

  (Postion) + (vΔt)

  (-4i + 3j)km + (4i + 17j)t

Relative Position Vectors

- What if two moving objects have different positions? How can we find how far they are from each other?



Relative Position Vectors

The position vector of A relative to B is:

A – B

A *relative to* B essentially means A – B

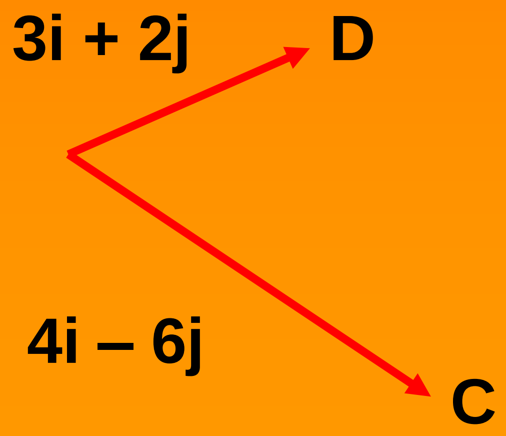
The position vector of B relative to A is:

B - A

B *relative to* A essentially means B - A

Example

- At a given time, particle C has position vector (4i – 6j)m relative to a fixed origin and particle D has position vector (3i + 2j)m relative to O. Find the position vector of D relative to C [D – C].



D – C

= (3i + 2j) – (4i – 6j)

= (-i + 8j)m

Colliding objects

- So if two objects are travelling at different speeds in different directions, in order for them to meet, **their positions must be the same.**

- **We've already said that after time 't' the <u>new position</u> of a moving object will be:**

(its initial position) + (its velocity x time).

So if we know the positions of two objects and their velocities, we can work out when they will collide, for example…

Example Question WRITE ME DOWN

- At noon, William Turner observes two ships, the black pearl – **ship A**, with position vector

$(-4i + 3j)$km and constant velocity of **$(4i + 17j)$kmh^{-1}** and the interceptor – **ship B**, with position vector

$(4i + 9j)$km, travelling at a constant velocity of

$(-12i + 5j)$kmh^{-1}.

- **Show that a) the ships will collide and b) find the time when the collision will occur and c) the position vector of the collision.**

Method

1) Find their positions after time t:

Ship A → position vector: (-4i + 3j)km

velocity: (4i + 17j)kmh^{-1}

So after time t the new position will be:

Initial Position + vΔt (displacement)

= (-4i + 3j)km + (4i + 17j)t

Ship B → position vector: (4i + 9j)km

velocity: (-12i + 5j)kmh^{-1}

So after time t, the new position will be:

Initial Position + vΔt (displacement)

(4i + 9j) + (-12i + 5j)t

2) If the boats will collide then their position vectors after time 't' are equal:

(-4i + 3j)km + (4i + 17j)t = (4i + 9j) + (-12i + 5j)t

Take out i and j <u>but</u> remember to times by t:

i(-4 + 4t) + j(3 +17t) = i(4 – 12t) + j(9 + 5t)

If we equate the i parts on each side so that:

-4 + 4t = 4 – 12t

Rearranging to get:

-8 = -16t

So t = -8/-16 = 0.5 or ½

And the same for j

If we equate the j parts from each side:

$3 + 17t = 9 + 5t$

We can rearrange to get:

$12t = 6$

So $t = 6/12 = 0.5$ or ½

So when $t = ½$ the two boats will collide.

As William turner observed the ships at noon 12:00, after 0.5 hours the ships will collide, so at 12:30 the ships will collide

But what is the position vector of the collision?

If we just look at ship A, when the ships collide, $t = 1/2$, so after time 't':

$(-4i + 3j) + (4i + 17j)\frac{1}{2}$

$= -4i + 2i + 3j + 8\frac{1}{2}j$

$= -2i + 11\frac{1}{2}j$

So the position vector of the point of collision is : $-2i + 11\frac{1}{2}j$

If we were to do the same with ship B, we get the same answer.

But wait there's more…

- In order to prevent a collision, at 12.15pm, Captain Jack Sparrow increases the black pearls speed (Ship A) to $(16i + 17j)kmh^{-1}$. But the interceptor, Ship B, continues at its original velocity. Find the distance between A and B at 12.30 now.

Approaching this question

1) The position vector of A will now be made up of 3 parts:

* the initial displacement (-4i + 3j)km

* the displacement from 12.00pm to 12.15pm

* the displacement from 12.15pm to 12.30pm

As we are measuring things in terms of kilometres and hours, 15 minutes = $\frac{1}{4}$ of an hour, so t = $\frac{1}{4}$ and 30 minutes = $\frac{1}{2}$ of an hour, so t = $\frac{1}{2}$.

The Displacement from 12 noon to 12.15pm will be (velocity x time):

$(4i + 17j)\frac{1}{4}$

The Displacement from 12.15pm to 12.30pm will be (new velocity x time):

$(16i + 17j)\frac{1}{4}$

So overall the position vector of ship A at 12.30pm will be:

(original displacement) + (displacement from 12.00 – 12.15) + (displacement from 12.15 – 12.30)

$= (-4i + 3j)\text{km} + (i + 4.25j)\text{ km} + (4i + 4.25j)\text{ km}$

$= (i + 11.5j)\text{km}$

But what is the distance between A and B?

The position vector of A relative to B =

the position vector of ship A – the position vector of ship B

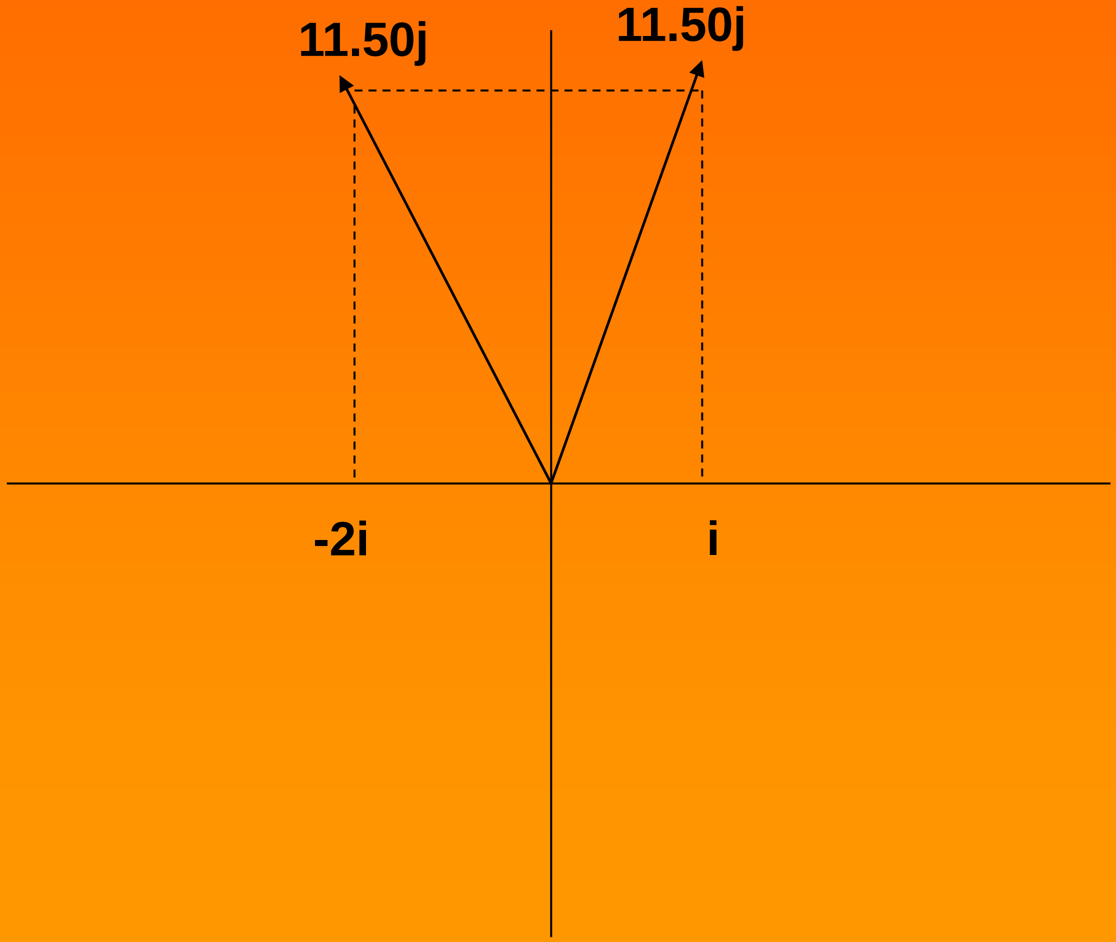
*The position vector of ship B will be the same as it does not change velocity. So it will have the position vector: 2i + 11½j

*The position vector of Ship A is:

- (i + 11½j)km
- A – B = (i + 11½ j) – (-2i + 11½j)

=3i

Here you can see that the distance between each boat is 3i



Another mind-numbing example…

If this bit of blood is travelling at $(4i + 23j)ms^{-1}$, and has position vector $(4j)$ after time t it will have position vector: $(4j) +$

$(4i + 23j)t$

A cruiser *C* is sailing due east at a constant speed of $20kmh^{-1}$ and a destroyer *D* is sailing due north at a constant speed of $10kmh^{-1}$. At noon *C* and *D* are at position vectors (-5i)km and (-20j)km, respectively, relative to a fixed origin O.

a) **Show that at time t hours after noon the position vector of C relative to D is given by: [20t – 5)i + (20 – 10tj]km**

- **Ship C: Position Vector → (-5i)km**

  Velocity → $20kmh^{-1}$

- **Ship D: Position Vector → (-20j)km**

  Velocity → $10kmh^{-1}$

After time t, their position vectors will be:

(initial position) + (velocity x time)

Ship C → (-5i) + 20t

Ship D → (-20j) + 10t

Therefore C relative to D = C – D which is:

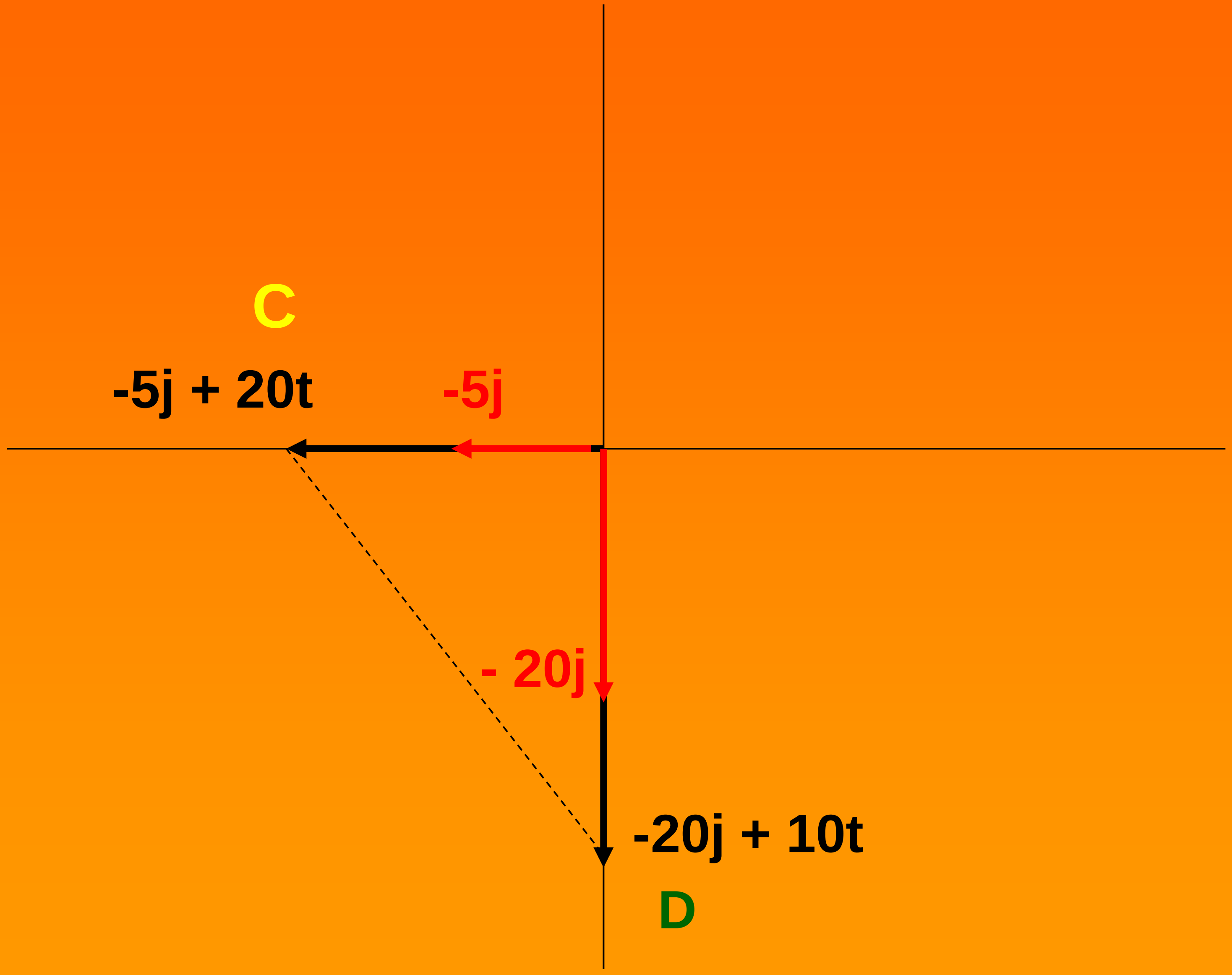
= (-5i + 20t) – (-20j + 10t)

= (-5i + 20t) + (20j – 10t)

Rearrange, taking out i and j

= i(5 + 20t) + j(20 - 10t)

This would look something like this:



b) Show that the distance *dkm* between the vessels at this time, is given by:

$d^2 = 25[20t^2 – 24t + 17]$

The distance between them is C relative to D, which we know from before is:

$d = (-5i + 20t) + (20j – 10t)$

So $d^2 =$

$(20t – 5)^2 + (20-10t)^2$

NOW EXPAND AND FACTORISE

Which gives…

$(20t - 5)^2$

$= (20t - 5)\ (20 - 5)$

$= 400t^2 - 200t + 25$

$(20 - 10t)^2$

$= (20 - 10t)\ (20 - 10t)$

$= 400 - 400t + 100t^2$

$(20t - 5)^2 + (20\text{-}10t)^2 =$

$(400t^2 - 200t + 25) + (400t^2 - 200t + 25)$

$= 500t^2 - 600t + 425$

FACTORISE IT

$= 500t^2 – 600t + 425$

25 is common to all terms so divide by 25

$d^2 = 25[\ 20t^2 – 24t + 17]$

Now as we all know, the ability to complete the square is an innate ability, the womb is lined with formulae and mathematical methods so a crappy vague description in the textbook is completely understandable. We cannot find ‘t’ fully by using the quadratic formulae or completing the square as we end up with the square of a negative number, and this would require the use of imaginary numbers, but mechanics is as real as the tears it makes you cry, so we have to complete the square…but how?

Example:

$2x^2 – 8x + 7$

Question

Hence show that C and D are closest together at 12.36pm and find the distance between them at this time.

Since time is measured in hours, 36 minutes: 1/ 60 = 1 minute

36 minutes = 36/60 = 6/10 = 3/5

So we want to aim for t giving a value of 3/5

Completing the Square

If we just focus on the quadratic equation: $20t^2 – 24t + 17$

$20(t^2 – 24/20 + 17/20) = 0$

$t^2 – 24/20 = - 17/20$

$t^2 – 6/5 = - 17/20$

$(t – 6/10)^2 – (6/10)^2 = - 17/20$

$(t – 3/5)^2 – (36/100) = - 17/20$

$(t – 3/5)^2 – (36/100) = - 85/100$

$(t – 3/5)^2 – 36/100 + 85/100 = 0$

$(t – 3/5)^2 +49/100 = d^2$

- **So $(t - 3/5)^2 \geq 0$** (as we cannot have negative time, only doctor who has that, and yet he still let 911 happen and where was he when I kicked a poo that one time? Hmm?) and t only = 0, when t = 3/5, because if we subbed it in, we'd get 0^2.
- **So the minimum value of the quadratic: $(t - 3/5)^2 + 49/100$ occurs when t = 3/5, putting this in leaves us with the + 49/100.**

So after 3/5 hours, which equals: 60 x 3/5

= 36 minutes after noon, the ships will be closest together, at 12.36pm.

So the minimum value of the quadratic:
$(t – 3/5)^2 + 49/100$ occurs when $t = 3/5$, putting this in leaves us with the + 49/100.
However, **$d^2 = 25[\ 20t^2 – 24t + 17]$**
So subbing in the minimum value of t: **$25 \times 20 \times 49/100 = 245$**

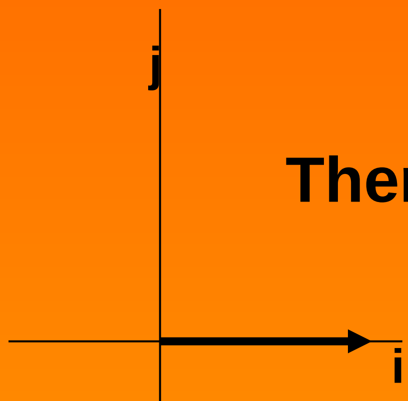
- **But this is d^{2}, so d = the square root of this value, which gives: $7\sqrt{5}$ = 15.65km**
- **So the minimum distance between the ships is 15.65km at 12.36pm.**

- **If the cruisers radar can detect other ships that are at a maximum distance of 15km, will the destroyer be detected?**

… No. The destroyer will just be out of the cruisers detecting range. But if either of the captains use this method to calculate this, they're obviously gay and should meet up.

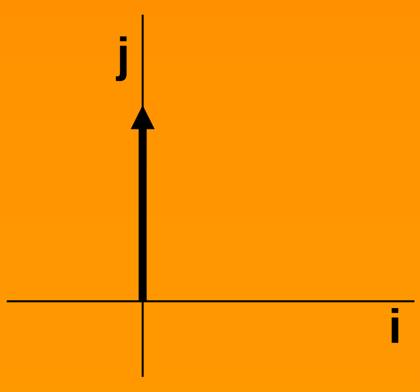
Other stuff that we should know:

- If a particle is moving parallel to i:



Then j = 0

- If a particle is moving parallel to j:



Then i = 0

Thank You for Watching

Now you can use is and js proper good, and if not, I can because I did 37 slides on it so HA

Mechanics

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