Mechanics Lecture Notes

1 Lecture 1: Statics — equilibrium of a particle

1.1 Introduction

This lecture deals with forces acting on a particle which does not move, i.e. is in *equilibrium*. The important concept is the resolution of forces to obtain the equations determining equilibrium. It is essential when solving such problems to start with a good diagram showing all the forces.

The example introduces the idea of friction which, although simple at first sight, turns out to be quite subtle. The idea of limiting friction is introduced: this occurs when something is just on the point of slipping.

1.2 Key concepts

- Reduction of a number of forces to one *resultant* force by vector addition.
- Condition for equilibrium: the resultant force is zero.
- Resolution of forces in orthogonal directions to determine an unkown force.
- Frictional force; *limiting friction* and relation via the *coefficient of friction* to the normal reaction.

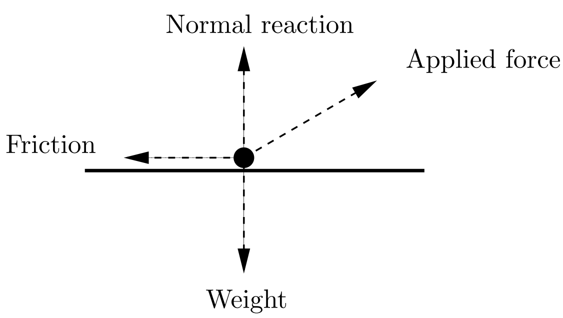
1.3 Forces

We consider here the the situation of a stationary particle acted on by a number of forces. It is not very useful to attempt to define exactly what we mean by a *force*: examples of forces will suffice.[1] But we can think of a force as something that tends to produce motion. A force is therefore obviously a vector quantity.

There are (as far as is known at present) four fundamental forces: gravity, electromagnetism, weak nuclear force and strong nuclear force. Each force is accompanied by a theory and a set of equations governing the behaviour of the force and objects affected by the force.[2]

All other forces are derived from these forces. Examples in no particular order are friction, tension in strings, normal reaction forces, air resistance, viscosity, magnetism, gravity, van der Waals forces between molecules, etc. I'm sure that you can think of many more examples.

For example, for a particle on a rough horizontal table being pulled by a string (though not hard enough to make the particle move), the forces are as shown in the diagram. There are two external forces, namely the applied (pulling) force acting along the string, and the weight acting downwards. The table exerts two forces on the particle: one is the force of friction which tends to oppose motion; the other is the reaction of the table on the particle that stops the particle falling through the table. This latter force is normal to the surface of the table and is called the *normal reaction.*



[1] It is tempting to use Newton's second law to define force, but there is a danger of a circular argument.

[2] Currently accepted theories are General Relativity, Quantum electrodynamics, Electroweak, and Quantum Chromodynamics, respectively. Other theories, such as string theory, attempt to combine these forces.

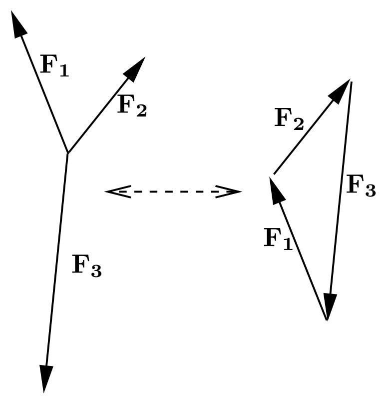
1.4 Equilibrium

A particle or body is said to be *in equilibrium* when all the forces acting on it balance and it is not in motion. Algebraically, this just means that the vector sum of the forces is zero:

$$\sum_i \mathbf{F}_i = \mathbf{0}$$

or, equivalently, the components of the vectors in three directions (which must be linearly independent, of course, but not necessarily orthogonal) sum to zero.

Geometrically, this means that the vectors representing the forces (in both direction and magnitude) can be joined to form a closed polygon.



In order to determine whether a particle is in equilibrium (or, given that it is in equilibrium, to determine an unkown force) we have to check that the vector sum of the forces, i.e. the *resultant force*, is zero. That means that the resultant force should have no non-zero component in any direction. Normally, the way to check this is to check the components of the resultant force in three independent directions, which need not be orthogonal but are usually, for convenience, orthogonal. This process is called *resolving forces*. It can best be understood in a concrete example.

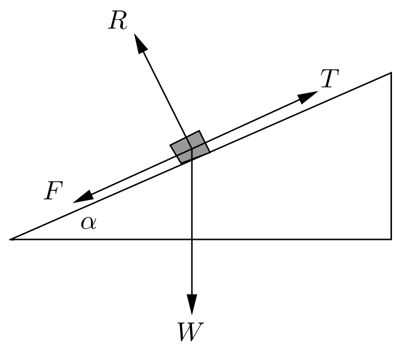
Example

A particle of weight[3] *W lies on a fixed rough plane inclined at angle α to the horizontal. It is held in position by a force of magnitude T acting up the line of greatest slope of the plane. Find the frictional force, F, in terms of W, α and T.*

Before anything else, we must draw a good diagram showing all the forces. The importance of a diagram is seen immediately: as soon as we try to draw in the frictional force F we realise that we don't know which way it acts — up or down the plane.

As was stated earlier, the frictional force opposes the motion, so if, in the absence of friction, the force T is large enough to pull the particle up the plane, friction acts down the plane. If, in the absence if friction, the weight is enough to pull the particle down the plane, then friction acts up the plane. We assume first the former: friction acts down the plane.

[3]The weight of the particle is the magnitude of the force it experiences due to gravity; for a particle of mass m, $W = mg$, where g is the (constant) acceleration due to gravity.



The strategy for all similar problems is to determine the equations of equilibrium by resolving (i.e. taking components of the vectors) forces in two directions and equating to zero. It helps to choose the directions carefully in order to reduce the number of terms in each equation.

Clearly, for our problem, it is a good plan to resolve parallel and perpendicular to the plane. We have, respectively:

$$T = F + W \sin\alpha \tag{1}$$
$$R = W \cos\alpha \tag{2}$$

Thus $F = T - W \sin\alpha$, using only the first equation.

Normally, we are interesting in finding the value of T that will support the particle on the plane. To accomplish this, we have to know something about the frictional force. The experimental result relating the frictional force to the normal reaction

$$\boxed{F \le \mu R} \tag{3}$$

is generally used. Here μ is the *coefficient of friction*, the value of which depends on the surfaces involved[4]. When the equality holds, the friction is said to be *limiting*.

In our example, combining equations (1) and (2) with the experimental law (3) gives

$$T \le W(\sin\alpha + \mu\cos\alpha).$$

Note that in the case of limiting friction, T is determined by this equation.

If, instead of assuming that the particle is tending to slip up the plane, we assume that it is tending to slip down the plane, then the frictional force would act up the plane. In this case (check this!) we find

$$T \ge W(\sin\alpha - \mu\cos\alpha)$$

and combining the two results gives the range of values of T for which equilibrium is possible, for a given value of μ:

$$W(\sin\alpha - \mu\cos\alpha) \le T \le W(\sin\alpha + \mu\cos\alpha).$$

Not surprisingly, in order for the particle to remain in equilibrium — i.e. not move — T cannot be too big or too small.

Note that if T were given (for example, if it were the tension in a string that passes over a pulley and has a weight dangling on the other end) the above equation would give bounds on the values of α allowed for equilibrium.

[4]For most common materials, the coefficient of friction lies between 0.3 and 0.6, though it can be considerably higher: for silicone rubber on tarmac it is over 1 (which is a good thing).

Mechanics Lecture Notes

1 Notes for lectures 2 and 3: Equilibrium of a solid body

1.1 Introduction

This lecture deals with forces acting on a body at rest. The difference between the particle of the last lecture and the body in this lecture is that all the forces on the particle act through the same point, which is not the case for forces on an extended body. The important concept, again, is the resolution of forces to obtain the equations determining equilibrium.

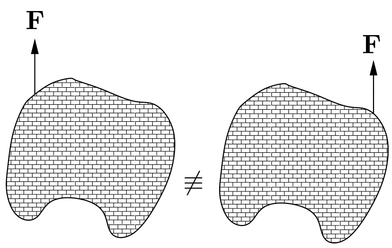
The simplest examples involve essentially one-dimensional bodies such as ladders. Again, it is essential start with a good diagram showing all the forces.

1.2 Key concepts

- Resolution of forces into a single *resultant force* or a *couple.*
- The *moment* of a force about a fixed point.
- Condition for equilibrium: zero resultant force and zero total couple.

1.3 Resolving forces

The difference between forces acting on a particle and forces acting on an extended body is immediately obvious from the intuitive inequivalence of the two situations below: for an extended body, it matters through which points the forces act — i.e. on the position of the *line of action* of the force.



In general, each force acting on a body can be thought of as having two effects: a tendency to translate the body in the direction parallel to the line of action of the force; and a tendency to rotate the body.[1] Clearly, for the body to be in equilibrium these effects must separately balance.

For the translational effects to balance, we need (as in the case of a particle) the vector sum of the forces to be zero:

$$\boxed{\sum_i \mathbf{F}_i = 0.} \tag{1}$$

For the rotational effects about a point P to balance, we need the sum of the effects to be zero, but what does this mean? Intuitively, we expect that a force whose line of action is a long way from P to have more rotating effect than a force of the same magnitude that is nearer and it turns out (see below) that the effect is linear in distance. The rotation effect of a force is called the *moment* of the force.

1.4 The moment of a force

In two dimensions, or in three dimensions in the case of a planar body and forces acting in the same plane as the body, any force tends to rotate the body within the plane or, in other words, about an axis perpendicular to the plane. In this case, we define:

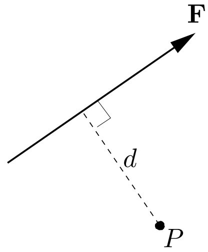
Moment of a force about a point P
$=$
magnitude of the force
$\times$
the shortest distance between the line of action of the force and P.

[1]Imagine that one point, not on the line of action of the force, is fixed.

with account taken of the direction of the effect: either clockwise or anticlockwise.[2]

In general (in three dimensions when the body and forces are not coplanar), the effect of different forces will be to tend to rotate the body about different axes. In this case, the 'force times distance from line of action to the point' definition of the moment of a force is not adequate. We have to represent the moment as a vector. The important thing to understand is that the direction of vector representing the moment of the force **not** in the direction in which the body might move; it is along an axis about which the body might rotate.

We can obtain the required vector expression for the moment of a force from a diagram. In the diagram below, the magnitude of the moment of the force $\mathbf{F}$ about the point P is $|\mathbf{F}| \times d$.



As mentioned above, the moment of a force is a vector quantity, the direction of the vector being parallel to the axis through P about which the body would rotate under the action of the force. This can be very conveniently expressed using the vector cross product:

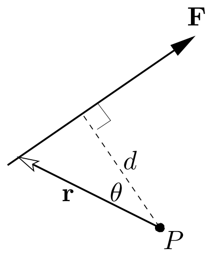
$$\boxed{\text{moment of } \mathbf{F} \text{ about } P = \mathbf{r} \times \mathbf{F}} \tag{2}$$

where $\mathbf{r}$ is the position vector from P to any point on the line of action of $\mathbf{F}$.

Why is this cross product the right expression for the moment? Again, we can see from a diagram. The vector $\mathbf{r}$ in the diagram below goes from the point P to an arbitrary point on the line of action of the force. Clearly, $\mathbf{r} \times \mathbf{F}$ is in the correction direction (into the paper). And we have

$$|\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\ |\mathbf{F}|\ \sin\theta = |\mathbf{F}|\ d$$

which agrees with the 2-dimensional case.



To summarise: the magnitude of the moment of a force about a given point is given by the rule 'magnitude of moment equals magnitude of force times shortest distance between line of action of force and the point'. The vector moment has direction normal to the plane containing the point and the line of action of the force.

For the body to be in equilibrium, we require that there is no tendency to turn about any axis. The condition for equilibrium, in addition to (1), is therefore, (in the obvious notation),

$$\boxed{\sum(\text{vector moments of the forces about any point}) \equiv \sum_i \mathbf{r}_i \times \mathbf{F}_i = 0} \tag{3}$$

There will still perhaps be a question mark in your mind about this result: why doesn't it matter what point we choose to take moments about? This is easily addressed. Suppose we change the point from P to P', where the position vector of P' with respect to P is a fixed vector $\mathbf{a}$. The position vectors in the condition (3) change from $\mathbf{r}_i$ to $\mathbf{r}'_i$, where $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{a}$. If we consider moments about P' instead of moments about P, we have

$$\sum_i \mathbf{r}'_i \times \mathbf{F}_i = \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i - \sum_i \mathbf{a} \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i - \mathbf{a} \times \sum_i \mathbf{F}_i$$

the vanishing of which is equivalent to the condition (3) provided the equilibrium condition (1) holds.

[2] In section 1.6, I will explain why the moment of the force, defined like this, is the correct measure of the rotational effect of the force; for the time being (and for ever if you perfectly sensibly don't want to wade through section 1.6) you should just accept that it is what we need.

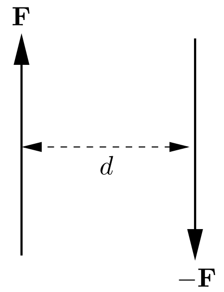
1.5 Couple

A *couple* is a pair of equal and opposite forces.[3] We define the moment of a couple about any point in the obvious way, as the sum of the moments of the two forces about that point. The sum of the moments of two forces will in general depend on the point about which the moment of the individual forces is taken; but this is not the case for a couple. Let the two forces be $\mathbf{F}$ and $-\mathbf{F}$, and let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of any fixed points on their respective lines of action, with respect to a point P. Then

$$\text{moment about } P = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F}) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}$$

and this does not depend on the choice of P.

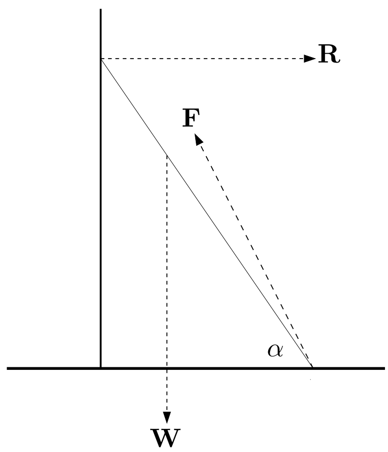
Choosing P on the line of action of one of the forces shows that the magnitude of the couple is just $|\mathbf{F}| \times d$, where d is the distance between the lines of action of the forces.



Example

A light rod[4] of length a stands on rough ground leaning against a smooth wall, and inclined at an angle α to the horizontal. A particle of weight W is placed a distance $\frac{2}{3}a$ up the rod. What is the magnitude of the normal reaction of the wall on the end of the rod?

First, as always, a good picture showing all the forces:



The normal reaction of the ground on the foot of the ladder and the frictional force acting on the foot of the ladder are combined into a single (unknown) force $\mathbf{F}$, acting in an unknown direction. There is no friction at the upper end of the ladder because the wall is smooth.

Before we go any further, we need to sit back and think. We have only three weapons in our armoury: resolving forces in each of two directions of our choosing; and taking moments about a point of our choosing. If we choose the directions and the point well, we can simplify our task enormously. In this rather straightforward case, we can eliminate the force that we are not interested in (the force on the foot of the ladder) in one go by taking moments about the foot of the ladder.

Taking moments ('force times shortest distance from the line of action of the forces to the point') clockwise gives

$$W \times \frac{2}{3}a\cos\alpha - R \times a\sin\alpha = 0$$

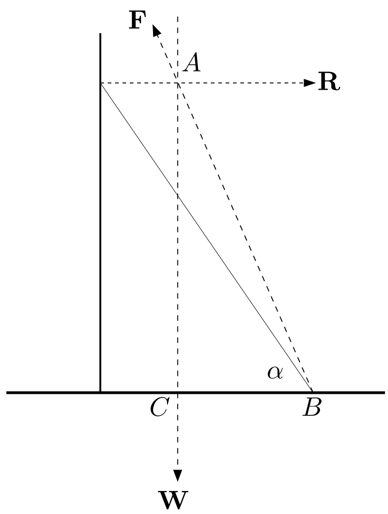
[3]Think of turning on an old-fashioned tap.

[4]In the idealisation of the elementary mechanics, rods are straight, rigid and one-dimensional; this one is massless ('light').

so $R = \frac{3}{2}\cot\alpha$.

We could now, by resolving forces horizontally and vertically, find the horizontal and vertical components of the force on the foot of the ladder, and then the whole problem would be solved.

Sometimes it is possible to solve such problems elegantly by geometry (but not necessarily more easily; you have to be good at geometry). Let us draw the diagram again, this time paying attention to the point at which the lines of actions of the forces intersect.



Note first that the three lines of action must intersect. Otherwise, we could take moments about the point of intersection of any pair (if not parallel); the two corresponding forces would have no moment about this point, since their lines of action passes through the point, leaving a non-zero moment from the third force. The total moment would thus be non-zero and the rod could not be in equilibrium.

The triangle ABC can be thought of completely geometrically, in which case it is soluble since we know two sides (horizontal and vertical) and the included angle (a right-angle).

But the the three forces $\mathbf{R}$, $\mathbf{F}$ and $\mathbf{W}$ are parallel to the sides of the triangle and, since they sum to zero (equilibrium condition) they can be represented as the sides of a triangle. This triangle must be similar to ABC, so we can find the relationships between the forces. For example, $AC/CB = |\mathbf{W}|/|\mathbf{R}|$ and $CB = \frac{2}{3}a\cos\alpha$ and $AC = a\sin\alpha$ so we obtain W in terms of R as before.

1.6 Moment of a force: justification of definition

NB this section is an optional extra: a gigantic footnote[5]

To emphasise the point in the heading: you do not need to know the material in this section; in fact, hardly anyone knows it.[6]

However, if you ever ask yourself why the moment of a force is defined as above (i.e. why is this the appropriate tool for investigating equilibrium), you will find this interesting.

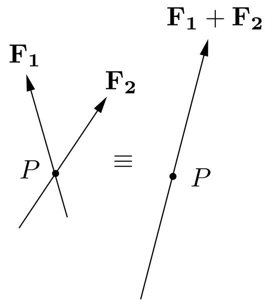
We will investigate the *resultant* of a number of forces acting on a body, which means, as in the case of a single particle, a reduced system of forces that has exactly the same effect on the body as the original system of forces. In the case of a single particle the reduced system is just one force; in the case of a system of forces acting on a body, the reduced system turns out to be a single force or, in very special cases, a couple.

We consider the case of just two forces, $\mathbf{F_1}$ and $\mathbf{F_2}$, in two dimensions; the generalisation to more forces in two dimensions is obvious (you just reduce the forces in pairs) and the three-dimensional case can be reduced to three two-dimensional cases by looking at the components of the forces in, for example, the x-y plane.

There are three cases to consider.

Case (i) $\mathbf{F_1}$ and $\mathbf{F_2}$ are not parallel

In this case, the lines of actions of $\mathbf{F_1}$ and $\mathbf{F_2}$ intersect, at P, say. The resultant, $\mathbf{F}$, of the two forces is just $\mathbf{F_1} + \mathbf{F_2}$ and it acts through P.



It is a simple exercise to check that the moment of $\mathbf{F}$ about any point Q is the same as the sum of the moments of $\mathbf{F}_1$ and $\mathbf{F}_2$ about Q.

This means that $\mathbf{F}$ has the same translational and rotational effect as $\mathbf{F}_1$ and $\mathbf{F}_2$ combined, provided we use the definition of the moment of a force given in section (1.4).

Case (ii) $\mathbf{F_1}$ and $\mathbf{F_2}$ are parallel, but $\mathbf{F_1} + \mathbf{F_2} \neq \mathbf{0}$.

In this case, the lines of action of the forces do not act through a common point, so we cannot immediately use the method above. Instead, the method can be used indirectly through a lovely construction. All we do is to add a pair of equal and opposite forces[7] as shown in the diagrams, to give two new forces that are no longer parallel. The diagrams on the next page illustrate the construction.

The diagrams show:

(i) Two parallel forces, $\mathbf{F_1}$ and $\mathbf{F_2}$, with $\mathbf{F_1} + \mathbf{F_2} \neq \mathbf{0}$ (i.e. exactly the situation we are considering).

(ii) In this diagram, two equal and opposite forces, $\mathbf{F}$ and $-\mathbf{F}$, have been added to the previous diagram. Adding these forces clearly has no effect: they just cancel each other out.

(iii) But if instead of cancelling them out we add them to $\mathbf{F_1}$ and $\mathbf{F_2}$, respectively, we obtain two resultant forces (by Case (i) above) which are the diagonal forces in the diagram.

(iv) The forces in **(i)** are therefore equivalent to the two forces in this diagram.

(v) Since the forces in **(iv)** are not parallel, they can be resolved (by Case (i) above) into a single force of magnitude (unsurprisingly) $\mathbf{F_1} + \mathbf{F_2}$ as shown in this diagram. But where does the line of action of this resultant lie?

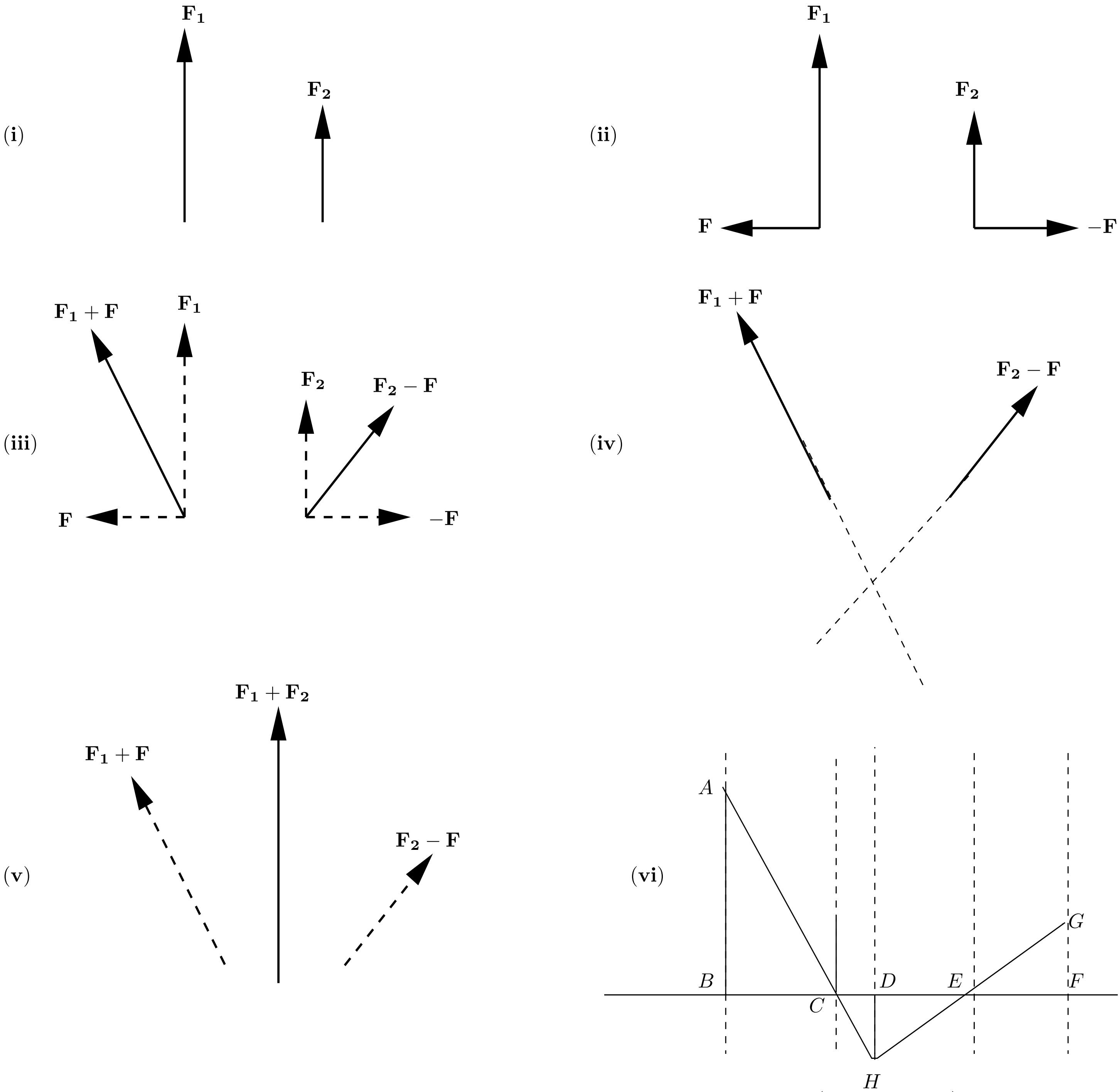
(vi) A bit of geometry does the trick. In this diagram, the AB and FG represent in direction and magnitude the original forces $\mathbf{F_1}$ and $\mathbf{F_2}$. CB and EF represent the equal and opposite forces we added in, and the two resultants are represented by CA and EG. Now using similar triangles gives

[5] which I put in because I found it interesting.

[6] It is not covered in any recent A-level text book that I could find, though it was covered in the book I had at school, by Humphrey and Topping.

[7] So simple!

the magic result $DC \times |\mathbf{F_1}| = DE \times |\mathbf{F_2}|$ — i.e. the resultant acts through the point where the moments of the original forces balance.



We thus find that the resultant of the two forces is a single force $\mathbf{F}_1 + \mathbf{F_2}$ (unsurprisingly), the line of action of which — and this is the important result — lies at distances d_1 and d_2 from the lines of action of $\mathbf{F_1}$ and $\mathbf{F_2}$, respectively, such that $d_1|\mathbf{F_1}| = d_2|\mathbf{F_2}|$; i.e. so that the *moments of the two forces* as defined above are equal.

Case (iii) $\mathbf{F_1}$ and $\mathbf{F_2}$ are parallel, and $\mathbf{F_1} + \mathbf{F_2} = \mathbf{0}$.

This is the exceptional case, when the system of forces cannot be reduced to a single force. No further reduction is possible and we are left with a couple. It is easy to see that the construction of Case (ii) breaks down: the construction gives another pair of equal and opposite forces.

Mechanics Lecture Notes

1 Lecture 4: Centre of mass

1.1 Introduction

This lecture deals with

1.2 Key concepts

- Definition of *centre of mass* as the point through which the resultant of the gravitational forces may be considered to act; the point about which the total moment of the gravitational forces is zero.

1.3 Centre of mass

The centre of mass (or centre of gravity) of a body is the point through which gravity may be considered to act. In other words, for the purposes of any calculation involving gravity, we can replace the solid body with a single massive particle at the centre of gravity; it is as if the mass of the body were concentrated at the centre of mass. [1]

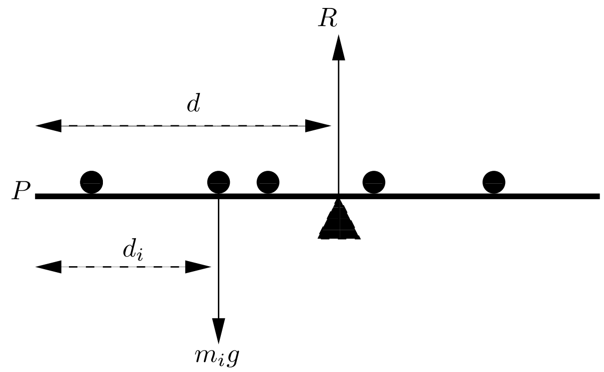
For a one dimensional body (a stick or ladder), the centre of mass is just the point at which the body balances, and the same is true of a two-dimensional body (think of a lamina balancing on a pencil point). It is harder to imagine this for a three dimensional body, but you could think of the centre of mass as being the point such that, for any axis passing through it, there is no tendency for gravity to rotate the body about that axis.

The following example illustrates the process.

Example: the centre of mass of a set of particles

Let n particles be attached to a light straight rod which rests on a smooth pivot, as shown. The ith particle has mass m_i and is distance d_i from P, the left end of the rod. The pivot is distance d from one end of the rod. The reaction of the pivot on the rod is R. The system is in equilibrium with the rod horizontal.

As always, the first move is to draw a good diagram.



The blobs in the diagram are the (point!) particles and the ith blob from P, the left end, of the rod has mass m_i and is a distance d_i from P.

For equilibrium we need the both the forces and their moments (about any point) to balance. Summing the forces gives

$$\sum_{i=1}^{n} m_i g - R = 0$$

so the reaction of the pivot on the rod is equal to the total weight of the particles — not entirely unexpected!

[1] This approach works because the forces of gravity acting on the individual particles of the body are all parallel. A formal proof will be given in Relativity and Dynamics course.

Summing the moments of the forces about P gives

$$Rd = \sum_{i=1}^{n} d_i m_i g.$$

So, provided we choose the position of the pivot according to

$$d = \frac{\sum_{i=1}^{n} d_i m_i}{\sum_{i=1}^{n} m_i},$$

the rod and the masses will balance on the pivot just as if it were a single particle of mass equal to the total mass situated on the pivot.

We could of course have taken moments about the pivot to obtain the same result:

$$0 = \sum_{i=1}^{n} (d - d_i) m_i g,$$

and had we started from this equation for d we would not have had to involve R at all.

1.4 Centre of mass: general definition

Extending the above idea to a general system of particles, we define the centre of mass to be at the point with position vector

$$\boxed{\frac{\sum m_i \mathbf{r}_i}{\sum m_i}.}$$

There is an analogous formula for a continuous mass distribution along a one dimensional body (a thin rod, say), replacing the sum by an integral and the mass at the point with coordinate x by the $\rho(x)dx$, where $\rho(x)$ is the mass density per unit length at that point. Thus the distance of the centre of mass from the origin (somewhere on the rod, not necessarily at the end) $\bar{x}$, is given by

$$\bar{x} = \frac{\int x\rho dx}{\int \rho dx}.$$

The numerator is the moment of the element of mass at x and the denominator is the total mass.

We can extend this to three dimensions. The position vector of the centre of mass is

$$\frac{\int \mathbf{r}\rho(\mathbf{r})dV}{\int \rho(\mathbf{r})dV}$$

though you may not properly understand these volume integrals until the Vector Calculus course in the Lent term.

Mechanics Lecture Notes

1 Lecture 5: Kinematics of a particle

1.1 Introduction

Kinematics[1] is the study of particle motion without reference to mass or force. In some ways, studying kinematics is rather artificial: in almost all realistic situations, the motion would have been produced by forces and the problem can only be solved by investigating the equations of motion appropriate to the forces acting. The study of motion produced by forces is called *Dynamics*[2]. Note that we deal with particles, which, by definition, are point-like; they can have mass (though that is not needed in kinematics) but they have no internal structure, so they cannot, for example, spin.[3]

The example of projectile flight is important, historically and in terms of applications. From our point of view, it is a first stab at tackling equations of motion, which is fundamental to all theoretical physics courses.

1.2 Key concepts

- Differentiation of a vector is the same as differentiation of its Cartesian components.
- Definitions of speed, velocity and acceleration in one and in three dimensions.
- Formulae for particles moving with constant acceleration in one dimension.
- Motion of projectiles.

1.3 Notation

Motion on a line

In one (spatial) dimension, the variables are time, position or distance or displacement from a fixed point, speed or velocity, and acceleration. We make a distinction between speed and velocity even in one dimension: velocity may be positive or negative, corresponding to the particle moving (say) to the right or left; speed is the magnitude of velocity and is therefore always positive or zero. Acceleration can also be either positive or negative.[4] We denote time by t, position by x, velocity by u or v and acceleration by a. Sometimes, displacement from the original position of the particle is denoted by s. We might write, for example, $x(t)$ to emphasise that x is a function of time.

Velocity, by definition, is rate of change of position, so

$$v = \frac{dx}{dt} \equiv \dot{x}\,.$$

The overdot always denotes differentiation with respect to time.

Acceleration, by definition, is rate of chance of velocity, so

$$a = \frac{dv}{dt} \equiv \dot{v} = \frac{d^2x}{dt^2} \equiv \ddot{x}\,.$$

Motion in space

In two or three dimensions, time is still t, and the other variables are vector quantities. We denote position by $\mathbf{r}$, or sometimes[5] $\mathbf{x}$; we might write $\mathbf{r}(t)$ to emphasise that the position is a function of time. Velocity is denoted by $\mathbf{u}$ or $\mathbf{v}$ and acceleration by $\mathbf{a}$, both vector quantities having magnitude and direction (of course).

[1]From the Greek $\kappa\iota\nu\eta\mu\alpha$, meaning motion; think of cinema. (I put this in because I thought you might like to see some Greek letters in words instead of in equations.)

[2]From the Greek $\delta\upsilon\nu\alpha\mu\iota\kappa o\varsigma$, meaning powerful.

[3]At least, not in classical mechanics; in quantum mechanics, particles can have spin — but all sorts of other strange things happen in quantum mechanics.

[4]Even in one dimension, speed and acceleration are vector quantities: they have magnitude and direction, the direction being either to the left or to the right (say).

[5]I'm going to use $\mathbf{r}$, because students say that they can't tell the difference between my handwritten x and my multiplication or vector product sign $\times$.

With respect to an origin and in standard Cartesian axes, we write

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or, to save space, just (x, y, z).

Velocity, by definition, is rate of change of position, so

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} \equiv \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

This last equality (equivalence), obvious though it seems, actually needs proving. Is differentiating a vector the same as differentiating its components? The answer is yes, provided that the axes are fixed (for example, provided you are not in a train accelerating or going round a bend). You can just take my word for this, or read the footnote.[6]

Speed is the magnitude of the velocity vector $|\mathbf{x}|$, which is non-negative.

As in one dimension, acceleration is rate of change of velocity, so

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}.$$

1.4 Constant acceleration on a line

We can obtain standard results for constant acceleration by (as is often the case) writing down the definitions and integrating the resulting differential equations. We have

$$\ddot{x} = a,$$

where a is constant, so integrating once gives

$$\boxed{\dot{x} \equiv v = at + u} \tag{1}$$

where u is a constant of integration corresponding to the velocity at $t = 0$. Integrating again gives

$$x = \tfrac{1}{2}at^2 + ut + x_0 \tag{2}$$

where x_0 is a constant of integration corresponding to the position at $t = 0$. Sometimes, this is written as

$$\boxed{s = \tfrac{1}{2}at^2 + ut}$$

where s is displacement from the initial position. It is worth checking dimensions at each stage as a quality control check. The dimension of s is length (L), the dimension of u is L/T and the dimension of a is L/T^2; substituting these into this last equation reveals it is dimensionally consistent.

Equation (2) gives distance as a function of time. We can find distance as a function of velocity by using (1) to eliminate time from (2):

$$t = \frac{v-u}{a} \Longrightarrow s = \tfrac{1}{2}a\left(\frac{v-u}{a}\right)^2 + u\left(\frac{v-u}{a}\right)$$

which simplifies to

$$\boxed{2as = v^2 - u^2} \tag{3}$$

[6] We can't make progress with this without knowing what differentiation of a vector means. Taking the usual definition in terms of limits, we have

$$\frac{d\mathbf{r}}{dt} \equiv \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \lim_{h\to 0}\frac{\begin{pmatrix} x(t+h) \\ y(t+h) \\ z(t+h) \end{pmatrix} - \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}}{h} = \lim_{h\to 0}\frac{\begin{pmatrix} x(t+h)-x(t) \\ y(t+h)-y(t) \\ z(t+h)-z(t) \end{pmatrix}}{h} = \lim_{h\to 0}\begin{pmatrix} \dfrac{x(t+h)-x(t)}{h} \\ \dfrac{y(t+h)-y(t)}{h} \\ \dfrac{z(t+h)-z(t)}{h} \end{pmatrix} \equiv \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

We will see later that this formula relates the change in kinetic energy to the work done by the accelerating force.

This formula could have been obtained directly from the equations of motion by means of the following very important idea. Essentially, it is a method of changing variable in the differential equation itself using the chain rule rather than changing variable in the solution.[7]

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dt}$$

so

$$\frac{dv}{dt} = a \Longrightarrow v\frac{dv}{dx} = a \Longrightarrow \tfrac{1}{2}(v^2 - u^2) = a(x - x_0) \equiv as$$

The three formulae in boxes provide everything you need for constant acceleration problems.[8]

1.5 Constant acceleration in three dimensions

We can integrate the vector equations for constant acceleration more or less as we did in the one dimensional case. We have

$$\ddot{\mathbf{x}} = \mathbf{a}$$

so

$$\dot{\mathbf{x}} = \mathbf{a}t + \mathbf{u}$$

where $\mathbf{u}$ is the velocity at $t = 0$. Integrating again gives

$$\mathbf{x} = \tfrac{1}{2}\mathbf{a}t^2 + \mathbf{u}t + \mathbf{x}_0 \tag{4}$$

where $\mathbf{x}_0$ is the position at $t = 0$. There is no easy formula corresponding to (3).

1.6 Projectiles

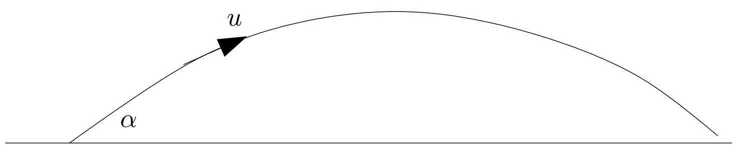
Projectiles are normally particles fired in the Earth's gravitational field. Properly, this should be treated as a problem in dynamics, since it involves forces but, since the gravitational field may be treated as uniform, the problem reduces to one of constant acceleration and the mass of the particle does not matter.[9]

Example

A particle is projected from a point on a horizontal plane at speed u *and at angle of projection*[10] α*. Find the equation of the trajectory.*

First step: draw a diagram

In most mechanics problems, the first step is to draw a good diagram with all the information in it. This may just be a good way of keeping the data in front of you, but it may also give some important insight into the problem.



[7] A similar idea occurs when dealing with orbits in the Dynamics and Relativity course: instead of solving for the position in terms of time, one changes variable in the differential equations from time to polar angle, giving a much simpler equation, the solution of which is the geometrical path in polar coordinates.

[8] Note that they work when the acceleration is *constant*. Many candidates for STEP learn these formulae and churn them out without asking themselves whether they apply.

[9] This fact will be discussed in the Dynamics and Relativity course: it is of crucial importance both in Newtonian Dynamics and General Relativity.

[10] The angle that the initial trajectory makes with the horizontal.

Equations of motion

The next thing to do is to write down the equations of motion, which are in this case just the mathematical expression of the fact that the acceleration the particle is given:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$$

where here $\mathbf{a}$ is the gravitational acceleration, which is constant in magnitude and direction.[11]

Choice of axes

The next thing to do, as in most similar problems, is to choose suitable axes and coordinates. We choose Cartesian axes with origin at the point of projection. We are free to choose the orientation of the axes, so let the z axis be vertically upwards and let the x-axis be aligned with the initial velocity.

With this choice, the initial velocity is $(u\cos\alpha, 0, u\sin\alpha)$ and the acceleration $\mathbf{a}$ (due to gravity) is $(0, 0, -g)$, where g is a positive constant.[12]

Set out the problem in the chosen axes

We now write down the equations of of motion in our chosen axes:

$$\mathbf{a} \equiv (\ddot{x}, \ddot{y}, \ddot{z}) = (0, 0, -g).$$

This gives us three differential equations:

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = -g.$$

Solve the equations with the given initial conditions

Integrating each one twice and using the initial conditions $x = y = z = 0$ and $\dot{x} = u\cos\alpha$, $\dot{y} = 0$, $\dot{z} = u\cos\alpha$ at $t = 0$ gives

$$x = ut\cos\alpha, \quad y = 0, \quad z = -\tfrac{1}{2}gt^2 + ut\sin\alpha. \tag{5}$$

It can be seen, by eliminating t, that this is a parabola.

$$t = \frac{x}{u\cos\alpha} \Longrightarrow z = -\frac{g}{2u^2\cos^2\alpha}x^2 + x\tan\alpha. \tag{6}$$

Equation (5) is the dynamical (or kinematic) equation of the trajectory, a parabola parametrised by time t. Equation (6) is the geometric equation: it describes a geometric object with no sense of motion.

We could, of course, have just substituted the initial conditions $\mathbf{u} = (u\cos\alpha, 0, u\sin\alpha)$ and $\mathbf{x}_0 = 0$ into the formula (4). It is probably better, though, to start from the equations of motion rather than to quote elaborate formulae.

[11] There is of course a modelling assumption here: the Earth's gravitational field is not constant in either magnitude (it falls off as inverse distance squared) or direction (it is radial), but for the sort of projectile we normally mean, these effects are negligible.

[12] g is the acceleration due to gravity at the surface of the Earth, and is about 9.8 ms^{-2}.

Mechanics Lecture Notes

1 Lecture 6: Newton's Laws: dynamics of particles

1.1 Introduction

This lecture deals with the application of Newton's second law to single particles and to simple systems (e.g. particles in contact or joined by strings). For particles or bodies in contact, Newton's third law plays a big part.

The worked examples are one-dimensional, and need to be understood before the vector versions are tackled in the Dynamics and Relativity course.

It hardly needs to be said that the application of Newton's laws is fundamental to all of theoretical physics, either directly (as for example in basic fluid dynamics) or indirectly as a model for more sophisticated theories (such as General Relativity).

1.2 Key concepts

- Newton's second law, in the form 'force equals mass times acceleration'.
- Newton's third law: 'to every action there is an equal and opposite reaction'.

1.3 Newton's Laws

Newton's Laws of motion were formulated in the mid-seventeenth century, in Latin and before some of the terms we are familiar with were current. There is therefore some ambiguity in their interpretation, which need not concern us. They are (in colloquial English)

(N1) A body persists in a state of rest or of uniform motion unless acted upon by an external force.

(N2) Force equals mass times acceleration ($F = ma$).

(N3) To every action there is an equal and opposite reaction.

For our purposes, N1 can be ignored; or, rather, regarded as a special case of N2.[1]

N2 is given here in its simplest form, applicable when the mass of the body is fixed; if the mass is variable (for example, a rocket or a particle in the theory of Special Relativity) the right hand side must be replaced by rate of change of momentum.[2]

It is not stated, though it is fundamental, that the law applies in inertial frames[3]: not, for example, in rotating or accelerating frames.

N3 seems natural and obvious: if you and I stand firm and I push you, then I feel you pushing me equally hard. It is less obvious when dealing with, say, magnets that act at a distance. But it seems that N3 holds in all situations and for all known forces.

1.4 Application of N2 to a single particle

Applied to a single particle of mass m with position vector $\mathbf{r}$ (a function of time t), we have

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} \tag{1}$$

where the force $\mathbf{F}$ could depend on t, on position $\mathbf{r}$ (a particle on a spring, for example), on velocity $\dot{\mathbf{r}}$ (air resistance, for example), on the mass m, or on any combination of these[4].

[1] There is a different interpretation, which does not assume that Newton was so dim as to formulate a law which was just a special case of another law — and, further, to formulate it many years *after* he formulated the second law! However, one could just assume that he wished to emphasise this special case because it is so surprising and counter-intuitive, given that in every familiar case moving bodies do in fact slow down.

[2] A rocket, together with its fuel, can be regarded as a system of particles, the mass of each particle being fixed, and so can be dealt with using $\mathbf{F} = m\mathbf{a}$; for a particle in Special Relativity there is no way of avoiding the 'force equals rate of change of momentum' formulation.

[3] The question of how to define an inertial frame then arises. One possibility would be to say that it is the set of frames (i.e. coordinate axes) in which N1 holds; or one could define an inertial frame to be one that is fixed (or moves with constant velocity) with respect to distant stars/galaxies/etc. Let's not worry too much about this now.

[4] Or on a multitude of other quantities such as electric field strength.

If $\mathbf{F}$ is constant, or a function just of t, then equation (1) can be integrated twice (at least in principle), taking each component separately. Physical problems normally come with initial conditions (the initial position and the initial velocity, normally), or boundary conditions, which can be applied at each integration. Altogether 6 conditions are required for what is essentially three second-order differential equations.[5]

In one dimension, if F is a function of x only (not t) we can integrate (1) by means of the chain rule using the change of variable discussed in lecture 5:

$$F = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dt} = \tfrac{1}{2}m\frac{d(v^2)}{dx}$$

so

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = \int_0^x F(s)ds$$

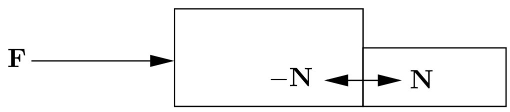
where u is the speed of the particle at $x = 0$.[6] Having done the integral on the right hand side, for a given function F, we can integrate again to find $x(t)$ by setting $v = dx/dt$.

1.5 Worked examples

The general plan for solving problems always starts with writing down the equations of motion[7], here N2. For a system of particles or body, we can apply N2 to the whole, or to each particle individually remembering to include the forces between particles.

(i) Two bodies of masses m and M are joined together and move in contact, subject to a single force $\mathbf{F}$. Find the acceleration of the bodies.

A small variation of this problem would be a model of a car towing a caravan.



First we apply N2 to the whole system

$$F = (M + m)a$$

where F is the magnitude of $\mathbf{F}$ and a is the acceleration of the two bodies. Thus $a = F/(M + m)$.

We can obtain the same result by applying N2 to the bodies individually, using N3. We have to include all the forces, including internal forces, on the left hand side of N2: each body experiences a force of reaction from the other body, which is internal to the system, and N3 says that these force are equal and opposite. Thus

$$(F - N) = Ma \quad \text{and} \quad N = ma$$

Adding these gives the previous result for a.

(ii) Two particles of masses m and M are attached to a light inextensible string that passes over a light frictionless fixed pulley. Find the accelerations of each particle and the force on the axle of the pulley.

As always, we start with a good picture and then write down the equations of motion. The equation of motion in this case is N2, again taking into account all the forces on each particle.

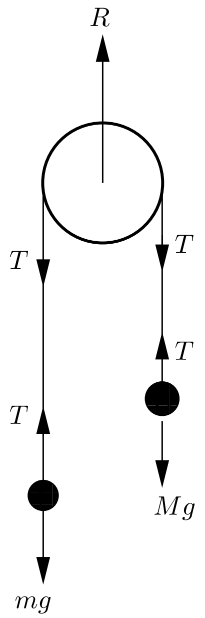
[5] Rather surprisingly, until you learn about Green's functions in Part IB Methods, the solution corresponding (for example) to a particle initially at rest at the origin can be written as a *single* integral:

$$x(t) = \int_0^t (t - \tau)F_x(\tau)d\tau$$

where F_x is the x-component of the force, and similar formulae hold for y and z. You can fairly easily verify that this satisfies the differential equation, and the initial conditions.

[6] We will see in the next lecture that this equation relates the change of kinetic energy of a particle to the work done by the force on the particle.

[7] Or first integrals such as energy and momentum conservation — see lectures 6 and 7.



In the diagram, T is the tension in the string, which is the same on both sides of the pulley, because the string is light and the pulley is light and moves without friction.[8] R is the reaction of the axle on the pulley (i.e. the force that stops the pulley falling).

Note that the accelerations of the two particles are equal and opposite, because the string is inextensible.

The equations of motion are as follows.

$$\begin{aligned} \text{For the mass } m \quad & T - mg = ma; && (2) \\ \text{For the mass } M \quad & T - Mg = -Ma; && (3) \\ \text{For the pulley} \quad & 2T - R = 0. && (4) \end{aligned}$$

Eliminating T, we find[9]

$$a = \frac{M-m}{m+M} g.$$

Eliminating a gives

$$T = \frac{2mM}{m+M} g.$$

Thus

$$R = 2T = \frac{4mM}{m+M} g.$$

[8] If, for example, the pulley had friction, then there would be an extra equation of motion relating the difference between the tensions on the two sides to the frictional force.

[9] This result makes sense. If we write it in the form $(M-m)g = (M+m)a$ and compare with N2, we see that the force is provided by gravity acting on what could be thought of as the gravitational mass of the system, namely $M-m$, whereas $M+m$ is the inertial mass of the system (it all needs to be accelerated).

Mechanics Lecture Notes

1 Lectures 7 and 8: Energy

1.1 Introduction

This lecture covers several absolutely fundamental concepts: kinetic energy, potential energy, conservation of energy, use of the equation of conservation of energy to replace one of the equations of motion. The definition of kinetic energy is generally unproblematic since it relates to the motion of the body. Potential energy is more difficult and we discuss it here only in the context of a uniform gravitational field. Similarly, the discussion of conservation of energy, which involves potential energy, is restricted to this simple case. It is important to recognise conservation of energy both as a fundamental principle and as a practical tool for investigating the motion of a body: the equation of conservation of energy is a first order differential equation and can be used to replace one of the second order differential equations of motion.

1.2 Key concepts

- The work done against a force.
- Power as the instantaneous rate of doing work.
- Kinetic energy.
- Potential energy due to gravity.
- Conservation of energy as first integral of equations of motion for gravity.

1.3 Work done by a force

The work done (WD) by a force is, loosely speaking, the product of the force and the distance moved by a body against the force. It represents the total effort expended by the force in pushing the body along. It is equal in magnitude to the effort expended by a machine pushing the body against the force.

For a constant force in one dimension, we have simply

$$\text{WD} = F(x_2 - x_1)$$

where $x_2 - x_1$ is the distance moved. If the force is a function of position, we have to integrate:

$$\text{WD} = \int_{x_1}^{x_2} F(x)dx.$$

In three dimensions, when the path of the particle is a straight line from $\mathbf{x_1}$ to $\mathbf{x_2}$ and the force is constant,[1]

$$\text{WD} = \mathbf{F}.(\mathbf{x}_2 - \mathbf{x}_1).$$

1.4 Power

Power, P, is the (instantaneous) rate of doing work. For a body moving in one dimension, we can express this in terms of the work done by the force in moving the body an infinitesimal distance and the time taken to do so:

$$P = \frac{\text{WD in time } dt}{dt} = \frac{F\,dx}{dt} = Fv \tag{1}$$

and the three-dimensional equivalent is $\mathbf{F}.\mathbf{v}$.

[1] We can extend the definition to varying forces on curved paths using the concept of a *path integral* which comes up in the Vector Calculus course:

$$\text{WD} = \int \mathbf{F}(\mathbf{x}).d\mathbf{x}\,.$$

Power is measured in watts (using SI units)[2] so, for example, the brightness of a light bulb is a measure of its power. The power of a car, normally given as the time take to accelerate from 0 to 60 miles an hour, could also be measured in watts as in the following example. The practical 0 to 60 definition has the advantage of including information about the smoothness of the bearings and the aerodynamicness of the shape.

1.5 Worked example

A car of mass m, has an engine that can work at a maximum power P. The total resistance force (including friction and air resistance) on the car at speed v is Rv.[3] Find the time, T, it takes to accelerate from 0 to u at maximum power.

As always, we write down the equation of motion:

$$m\frac{dv}{dt} = F - Rv = \frac{P}{v} - Rv$$

using the definition (1) of power, so

$$\int_0^u \frac{mv\,dv}{P - Rv^2} = T \Longrightarrow 2RT = m\log(1 - Ru^2/P)\,.$$

1.6 Energy

Energy comes in many different forms. We consider here just potential energy, related to the position of a particle in a force field[4] and kinetic energy, related to the motion of a particle.[5]

We define the kinetic energy (KE) of a particle of mass m moving with velocity $\mathbf{v}$ by

$$\text{KE} = \tfrac{1}{2}m\mathbf{v}.\mathbf{v}.$$

Kinetic energy is additive, so that the KE of two particles is just the sum of the KE of the individual particles.

The potential energy (PE) of a particle in a force field relates to the ability of the particle to do work by virtue of its position in the force field. For example, water at the top of a waterfall has PE because it could be used to generate hydro-electric power. As will be discussed in the course Dynamics and Relativity, not all force fields have potential energy associated with them[6]. To avoid any complications, we will consider only the force of gravity, and define the PE of a particle of mass m by

$$\text{PE} = mgh, \tag{2}$$

where h is the height of the particle above some fixed level. The PE is only defined up to an additive constant, because if the fixed level is chosen a distance h_0 lower the PE will increase by mgh_0.

Since the work done against gravity in raising a particle a distance d vertically upwards is mgd (force times distance moved by particle), we see that the change in PE of the particle is just the work done and this coincides with the idea of PE as representing the ability to do work.

[2] A more old-fashioned unit of power is the *horsepower*. There are various different definitions. The unit was invented by James Watt in order to compare the output of his steam engines with other steam engines and with that of a horse. One horsepower is about 745 watts, which was calculated by James Watt the power of a typical brewery horse. It seems surprising to me that a horse working at full power could light only 7 conventional light bulbs, but maybe that just shows how wasteful non-energy-saving bulbs are. A healthy human can produce 1.2 horsepower for a short burst and can sustain 0.1 horsepower indefinitely.

[3] Which is not very realistic: more likely, the resistive force would be modelled by a polynomial of degree 2, to take into account inertial and viscous drag and a constant friction term; but the resulting cubic denominator in the integral would not be terribly attractive for a simple example.

[4] A *force field* is a force that acts at each point in space, like gravity, but not like friction which only acts at a specific point moving with the particle. It is represented by a vector field, $\mathbf{F}(\mathbf{x})$ say, which is defined at each point $\mathbf{x}$ and defines at that point the force that would be experienced by a susceptible particle.

[5] Note, though, that what seems a completely disparate form of energy may be expressible in terms of these: for example, heat energy may be due entirely to motion of molecules which could then be regarded as kinetic energy.

[6] Forces for which the concept of potential energy makes sense are called *conservative forces*; for a conservative force, a particle at a point $\mathbf{x}$ in the field has a potential energy which is uniquely determined (up to an additive constant) by $\mathbf{x}$. The defining characteristic of a conservative force is that the work done to move a particle between two fixed points is independent of the path taken.

1.7 Conservation of energy

The principle of conservation of total energy — that energy in a closed system can be neither created not destroyed — is absolutely fundamental in classical mechanics.[7] We consider here only situations in which there are no *dissipative* forces, such as friction.[8]

It is very important to understand that in systems such as we are considering that governed by equations of motion, conservation of energy is not an additional equation: it must be consistent with — indeed, derived from — the equations of motion. There is therefore the option to replace one of the equations of motion with the equation of conservation of energy, and this is often a very sensible plan, for two reasons:

1. conservation of energy involves first derivatives of position (through the velocity $\frac{dx}{dt}$ in the KE) whereas the equations of motion are second order (through the acceleration $\frac{d^2x}{dt^2}$ in N2);
2. forces that do no work, such as normal reaction or tension in an inelastic string (essentially forces of constraint) appear in the equations of motion but not in the equation of conservation of energy).

In the case of a particle moving vertically in a gravitational field we can easily demonstrate conservation of energy. Conservation means that a quantity is unchanged in the motion, which here means unchanged in time. Thus to prove that total energy is conserved, we just write down the expression for total energy, differentiate it and apply the equations of motion:

$$\text{Total energy} = \text{KE} + \text{PE} = \tfrac{1}{2}mv^2 + mgz$$

where z is the height of the particle and $v = \dot{z}$. Thus

$$\frac{d}{dt}(\text{Total energy}) = mv\dot{v} + mgv.$$

But the equation of motion is $ma = -mg$, where $a = \dot{v}$, so the result follows immediately.

More generally, allowing the particle to move horizontally as well as verically, we have

$$\text{Total energy} = \text{KE} + \text{PE} = \tfrac{1}{2}m\mathbf{v}.\mathbf{v} + mgz.$$

Differentiating as before gives

$$\frac{d}{dt}(\text{Total energy}) = m\mathbf{v}.\dot{\mathbf{v}} + mg\dot{z}.$$

But now Newton's second law is $m\dot{\mathbf{v}} = (0, 0, -g)$ and, since $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$, the above expression sums to zero as before.

We have therefore just proved the fundamental result that for a single particle in a uniform gravitational field (subject to no other forces) the total energy is conserved. Conservation of energy for systems of particles (such as a solid body) and more general forces will be discussed in the Dynamics and Relativity course.

1.8 Example: the pulley from lecture 6

In this example, two particles of mass m and M are joined by a light string which passes over a smooth fixed pulley. We wish to find the acceleration of the particles.

Let x and $l - x$ be the distances of the two particles below the axle of the pulley.

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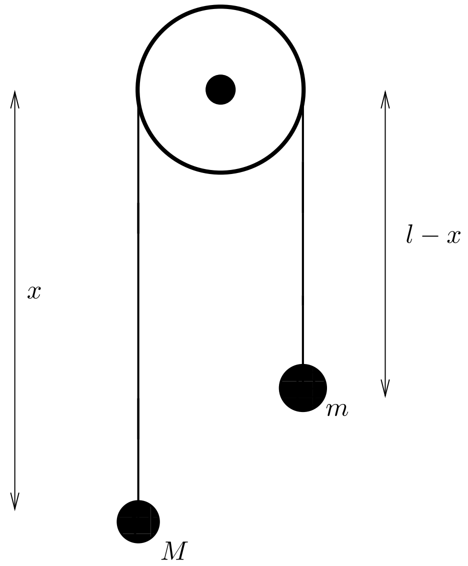
[7]In elementary quantum mechanics, the governing equation describing the behaviour of a single particle is essentially that of conservation of energy; in thermodynamics, the first law is precisely an expression of conservation of energy, including heat energy; in Special Relativity, conservation of energy holds but mass has to regarded as a form of energy via the famous equation $E = mc^2$; but in General Relativity, the concept of total energy is problematic, since there may not be any conserved quantity.

[8]The principle still holds even when there are dissipative forces, but then you have to take into account the mechanical energy that is converted into heat energy. Heat energy can be regarded as the kinetic energy of moving molecules but is more conveniently treated through the theory of thermodynamics .



We use the conservation of total energy of the two particles. The velocities of the particles (differentiating distance below the axle) are $\dot{x}$ and $-\dot{x}$ so the total kinetic energy is

$$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2.$$

The potential energies of the particles are $-Mgx$ and $-mg(l-x)$ (the minus signs because the distances are measured downwards). Thus the total energy, which is constant, is

$$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2 + gx(m-M) - mgl.$$

We could integrate this equation to find x as a function of t. Instead, since we are seeking the acceleration, we differentiate:

$$M\dot{x}\ddot{x} + m\dot{x}\ddot{x} + g\dot{x}(m-M) = 0\,.$$

Cancelling the factor of $\dot{x}$ gives the result obtained in lecture 6 by equating forces:

$$\ddot{x} = \frac{(M-m)g}{m+M}.$$

Notice that we did not have to worry about the tension in the string at all by this method.

Mechanics Lecture Notes

1 Notes for lecture 9 and 10: Momentum and Impulse

1.1 Introduction

This lecture covers another fundamental concept: momentum. The momentum of a body is the quantity that changes directly as a result of applying a force: the greater the force, the faster the momentum changes. This is a consequence — in fact, a statement of — Newton's second law of motion. The reason that momentum is so important is that in closed systems[1] not subject to external forces, the total momentum is conserved. This result follows from Newton's equations and will be proved in the Dynamics and Relativity course. Momentum is conserved in quite surprising situations, even when kinetic energy is not conserved, such as during explosions.

As with conservation of energy, the equations of conservation of momentum can be used very conveniently as a substitute for one or more of the equations of motion.

You might ask whether we have an intuitive feel for the quantity momentum. The answer is yes. When a large truck approaches you, it it not the kinetic energy that you worry about: it could be huge (compared with you), but travelling very slowly indeed so that you would hardly notice if it hit you. What you are worried about is its momentum.

When one particle collides with another, they each experience a force of large magnitude for a very short duration. This is normally idealised as an infinite force for an infinitesimal time[2]. There are various ways of describing this idealisation mathematically[3] but for our purposes it is easier to deal not with the (infinite) force with a quantity called *impulse* which is, roughly speaking, the product of the very large force and the very small time during which it acts.

1.2 Key concepts

- Momentum.
- Conservation of momentum in closed systems with no external forces.
- Impulse.
- Relation between impulse and change of momentum during particle collisions.
- Newton's experimental law and coefficient of restitution.

1.3 Momentum

The *momentum* of a particle of mass m moving with velocity $\mathbf{v}$ is defined to be $m\mathbf{v}$. It is, of course, a vector quantity. The total momentum of a system of particles is just the sum of the individual momenta.

For a particle with momentum $\mathbf{p}$, Newton's second law gives

$$\frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

so if there is no force on the particle, $\mathbf{p}$ is constant: momentum is conserved. Similarly, as will be proved in the Dynamics and Relativity course, the momentum of a system of particles on which no external forces act is conserved.[4]

Thus for two colliding particles, in obvious notation (with initial velocities denoted by $\mathbf{u}$ and final velocities denoted by $\mathbf{v}$), conservation of momentum requires

$$m_1\mathbf{u}_1 + m_2\mathbf{u_2} = m_1\mathbf{v}_1 + m_2\mathbf{v_2}\,. \tag{1}$$

[1] No particles join or leave the system.

[2] In a less idealised situation, if (say) two snooker balls collide, there would be a small compression of each ball at the contact point, during which the balls instantaneously move together; then the elastic forces in the ball cause the compression to spring back to the original shape and the balls rebound. This process would take a short time and could be analysed using Newton's laws if the internal structure of the billiard balls were properly understood.

[3] For example, by means of the Dirac delta function, which you will come across in the Differential Equations course and later in the Part IB Methods course; and more rigorously later still if you study the theory of distributions.

[4] This is not quite 'similarly': it requires the use of Newton's 3rd law to show that the effects of the *internal* forces cancel out.

1.4 Newton's experimental law: coefficient of restitution

Assuming that the initial velocities are given, equation (1) gives three equations (because it is a vector equation) for six unknowns (the three components of each of the two final velocities): not enough equations!

In general the kinetic energy of the colliding bodies (for example, snooker balls) will not be conserved in a realistic collision: the fact that a collision can be *heard* tells us this immediately (some kinetic energy is converted into sound energy); and generally the internal effects will heat up the balls slightly (some kinetic energy is converted into heat energy). Therefore, conservation of kinetic energy cannot be used as an additional equation to determine the outcome of a collision.

However, we have one more tool in the bag. *Newton's experimental law* furnishes us with one further equation, which is sufficient in the case of head-on collisions in one dimension or in the case of a particle bouncing on a plane, which are the only two situations we will cover here. By experiment, Newton discovered that

$$\text{relative velocity after collision} = -e \times \text{relative velocity before collision} \tag{2}$$

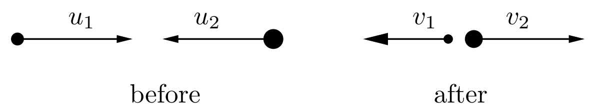
where e is a constant, called the *coefficient of restitution*, that depends on the two colliding particles (or bodies) and not, for example, on their velocities. The term *relative velocity* just means the velocity of one particle with respect to the other; i.e. in moving axes chosen so that the velocity of the second particle is zero.

If the two bodies coalesce, like putty, the relative velocity after the collision is zero, which corresponds to $e = 0$; this is called an *inelastic* collision. If the relative speed before is equal in magnitude to the relative speed after, which corresponds to $e = 1$, the collision is called *perfectly elastic*. In this case, no energy is lost in the collision: the total kinetic energy is the same as before (and you can't hear the collision!).

One has to be quite careful with the signs when applying (2), but normally common sense prevails.

1.5 Worked examples

(i) *Two particles of masses m_1 and m_2 collide head-on. Their initial velocities are u_1 and $-u_2$ and their final velocities are $-v_1$ and v_2. The coefficient of restitution between the two particles is e. Find the final speeds in terms of the initial speeds, the masses and e.*



In the picture, we are assuming that that u_1 and u_2 are positive — but it doesn't matter provided the velocities are such that the particles do in fact collide. We are also assuming, in the picture, that v_1 and v_2 are positive though they may not be: if m_2 is much larger than m_1, one could easily imagine both particles moving to the left after the collision.

Conservation of momentum in the positive x direction (to the right in the picture) gives

$$m_1u_1 - m_2u_2 = -m_1v_1 + m_2v_2.$$

Newton's experimental law, taking the velocity of the second particle relative to the first and again taking positive velocity to mean motion to the right, gives

$$v_2 - (-v_1) = -e[-u_2 - u_1]$$

Solving these two equations simultaneously gives

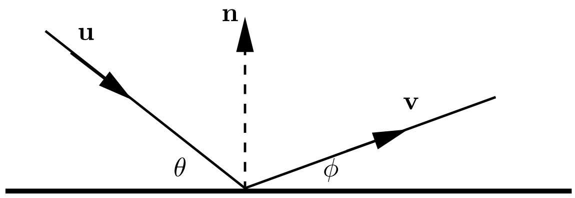
$$v_1 = \frac{(em_2 - m_1)u_1}{m_1 + m_2} + \frac{(e+1)m_2u_2}{m_1 + m_2} \qquad \text{and} \qquad v_2 = \text{Something similar - it really doesn't matter.}$$

Note that in the case $e = 1$ a direct calculation would show[5] that $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$; i.e. kinetic energy is conserved, as claimed above.

[5] I didn't do it, but I know it must: please don't spend your time on this unrewarding calculation.

(ii) *A particle is projected at velocity* $\mathbf{u}$ *on a horizontal smooth table. It hits a smooth vertical barrier, its trajectory making an angle of* θ *with the barrier. It rebounds with velocity* $\mathbf{v}$ *at an angle* ϕ *to the barrier. The coefficient of restitution between the particle and the barrier is* e. *Find* $\mathbf{v}$ *in terms of* $\mathbf{u}$ *and* e.

The first step, as so often in dynamics, is to choose sensible axes.[6] Let us take $\mathbf{u} = (u\cos\theta, u\sin\theta)$ and $\mathbf{v} = (v\cos\phi, v\sin\phi)$ which corresponds to taking the unit normal to the barrier to be $(0, 1)$, as shown in the diagram.



Now we consider components of velocity and momentum parallel and perpendicular to the barrier. The important thing to realise is that parallel to the barrier *nothing happens*: it is as if the barrier were not there. The barrier exerts no force parallel to itself, so momentum is conserved.

$$u\cos\theta = v\cos\phi.$$

The perpendicular components of velocity satisfy Newton's experimental law:

$$v\sin\phi = eu\sin\theta$$

and these two equations are sufficient to determine ϕ and v in terms of θ and v.

1.6 Impulse

In the example above of a bouncing particle, the momentum of the particle is not conserved: indeed, the normal component changes direction as well as magnitude. Why not?

The answer can only be that external forces act on the system and a moment's thought reveals that this is the case: there must be a force holding the barrier in place on the table. The force acts for a very short time and must be very large to turn the particle round.

In this situation, and other similar situations which involve very large forces acting for very short time, it is convenient work with the time-integrated force, which is called the *impulse*:

$$\int \mathbf{F}dt \equiv \mathbf{I}.$$

The impulse can be thought of as a measure of the amount by which momentum fails to be conserved. Using Newton's second law, we have:

$$\mathbf{I} = \int m\frac{d\mathbf{v}}{dt}dt = \Delta(m\mathbf{v})$$

i.e. impulse equals change of momentum. Thus in the above example, the barrier feels an impulse equal to $m(v\sin\phi + u\sin\theta)$, where m is the mass of the particle, in the direction normal to the barrier, and the particle feels an impulse equal in magnitude but in the opposite direction. There must be some external force of magnitude $m(v\sin\phi + u\sin\theta)$ acting on the system to prevent the barrier from moving.

[6] We could, if we were feeling masochistic, stick with vectors (no components), giving the normal to the barrier the direction $\mathbf{n}$, and $u\sin\theta = \mathbf{u}.\mathbf{n}$, etc. However, we can easily go back into vectors after we have solved the problem.

Mechanics Lecture Notes

1 Lecture 11: Simple harmonic motion

1.1 Introduction

This lecture covers simple harmonic motion (SHM). SHM occurs when a particle or body is displaced from an equilibrium position and experiences a restoring force (i.e. a force tending to pull it towards the equilibrium point) that is proportional to the displacement from the equilibrium point. It may seem that this would only occur in rather special conditions, but in fact it is ubiquitous: at the equilibrium point, the force acting on the particle is zero (by definition), so for a *small* displacement, the force is approximately linear (assuming that the force can be expanded in a Maclaurin series about the equilibrium point).

The second order differential equation that arises from the linear force law is easily solved, giving sin/cos or complex exponential solutions, and these describe the oscillatory motion with period independent of amplitude characteristic of SHM.

Two important examples are discussed: a particle hanging on a spring, which is the model for any one-dimensional SHM situation; and the simple pendulum. which will arise in the Dynamics and Relativity course, in the rotating frame of the Earth, in the form of the Foucault pendulum.

1.2 Key concepts

- Simple harmonic motion as a consequence of a linear restoring force: period and frequency.
- Hooke's law, which implies a linear restoring force when elastic materials are deformed.
- Derivation of equations of motion for the spring and simple pendulum.

1.3 Simple harmonic motion

Simple harmonic motion occurs when a particle experiences a force that is proportional to its displacement from a fixed point, and the constant of proportionality is negative. Choosing a sensible coordinate x to be the distance from the fixed point,[1] the equation of motion, using Newton's second law, is

$$m\frac{d^2x}{dt^2} = -kx \tag{1}$$

where $k > 0$. This is one of the few equations for which you need the solution at your finger tips. It is

$$x = A\cos\omega t + B\sin\omega t \equiv a\cos(\omega t + \epsilon) \equiv Pe^{i\omega t} + Qe^{-i\omega t}, \tag{2}$$

where $\omega = \sqrt{k/m}$. These three forms are completely equivalent, though it may be convenient in any given situation to prefer one of them. The solution is periodic and the quantity ω is called the *angular frequency*[2] of the motion. The *period* is the time taken for the cycle to repeat, namely $2\pi/\omega$; the *frequency* is $(\text{period})^{-1}$. A graph of x against t would show a wave.

In the $a\cos(\omega t + \epsilon)$ form of the solution, a is called the *amplitude*, $\omega t + \epsilon$ is called the *phase*, and ϵ may be referred to as the *phase shift*.

The SHM equation (1) has a first integral which can be obtained by writing

$$\frac{d^2x}{dt^2} \equiv \frac{dv}{dt} = \frac{dx}{dt}\,\frac{dv}{dx}$$

or, equivalently, by multiplying the equation by $\dot{x}$:

$$m\dot{x}\ddot{x} = -k\dot{x}x$$

and integrating:

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E \tag{3}$$

where E is a suggestively named constant of integration. The first term in (3) has the obvious interpretation of the kinetic energy of the particle. It would not be surprising, therefore, if the

[1] Which might require a transformation of the form $x \to x - x_0$ to shift the origin given in the problem to the equilibrium point.

[2] measured in radians per second if t is in seconds

second term could be interpreted as its potential energy, and this is easily demonstrated. Recall that potential energy is related to the ability of the particle to do work. The work done on the particle to move it from the equilibrium point to displacement x is

$$\int (-F)\, dx = \int (kx)\, dx = \tfrac{1}{2}kx^2$$

as expected.

We can check that the solution (2) satisfies (3) (it must, of course):

$$x = a\cos(\omega t + \phi) \Longrightarrow \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}ma^2\omega^2\sin^2(\omega t + \phi) + \tfrac{1}{2}ka^2\cos^2(\omega t + \phi) = \tfrac{1}{2}ka^2$$

using $k = m\omega^2$. Thus $E = \frac{1}{2}ka^2$ which is constant, as required.

1.4 Hooke's Law

Hooke wrote down his famous law, somewhat cryptically, in 1676 in the form *ceiiinosssttuv*. This turned out to be an anagram of the Latin 'ut tensio sic vis' which translates to 'as the extension, so the force'.[3] In more helpful language, this would express the idea that strain (i.e. deformation, stretching, compression, etc, divided by natural length) is proportional stress (i.e. applied force per unit area).

This experimental law holds for a wide variety of materials under a wide range of conditions; for example, it holds for steel wire until what it called the elastic limit is reached and the wire starts to stretch like plasticine. It does not hold for rubber[4].

1.5 A note on springs and strings

Springs and elastice strings are fundamentally different: the restoring force in a string is tension due to longitudinal stretching; the restoring force in a spring is due to bending and twisting of the material from which the spring is made. Furthermore, a string can go slack, which is a godsend to setters of STEP problems, whereas if a spring is compressed so that it is shorter than its natural length it exerts a restoring force towards the equilibrium. For both string (when it is not slack) and spring, to a good approximation, the restoring force is proportional to the extension.

In the case of a string, the restoring force comes from Hooke's law:

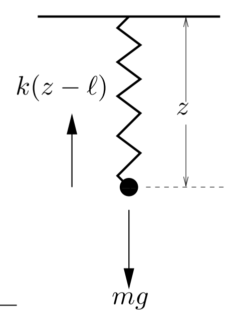
$$\text{tension} = \text{modulus of elasticity} \times \text{extension/natural length.}$$

The natural length of the string occurs in the denominator because the definition of 'strain' in Hooke's law is extension/natural length. The modulus of elasticity of the string takes into account its cross-sectional area, which is assumed to be constant. This explains why the left hand side of the equation is a force (tension), not a force per unit area (stress) as in Hooke's law.

For a spring, the restoring force is just $k \times$ extension, where k is the spring constant.

1.6 Example: particle hanging on a spring

A particle of mass m is attached to the lower end of a spring the upper end of which is fixed. The spring has unstretched length l and spring constant (i.e. the constant of proportionality between restoring force and extension of the spring) k. Initially, the spring hangs vertically at rest. The mass is then displaced a distance a downwards and released.



[3] Newton was also not averse to publishing his discoveries in the form of anagrams: the advantage was that he could claim priority in case anyone else discovered the same thing later, but for the moment it was kept secret to prevent others developing the idea.

[4] When you blow up a balloon it is hard to start then gets easier.

Let z be the distance of the mass below the top of the spring, to that the extension of the spring is $z - l$. Then the equation of motion of the particle is

$$m\ddot{z} = -k(z - l) + mg.$$

In equilibrium, $\ddot{z} = 0$, the equilibrium position is given by $z = l + mg/k$. If we choose a new coordinate x to be the downwards displacement from equilibrium, i.e. $x = z - (l + mg/k)$, we have[5]

$$m\ddot{x} = -kx$$

which is SHM. The solution is

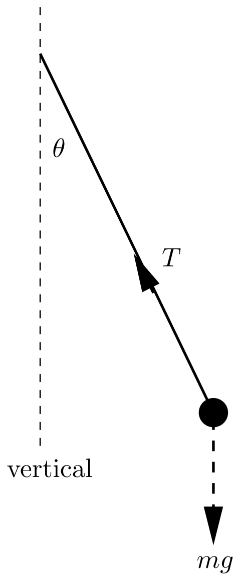
$$x = A\cos\omega t + B\sin\omega t.$$

Initially, $x = a$ and $\dot{x} = 0$, so

$$x = a\cos\omega t \quad \text{and} \quad z = a\cos\omega t + l + mg/k.$$

1.7 The simple pendulum

A particle of mass m is attached to one end of a light inextensible rod of length ℓ which is smoothly pivoted at the other end so that it can swing freely in a fixed vertical plane. Find the motion.



Let θ be the angle that the rod makes with the vertical. The easiest way of getting finding the motion is to use the energy integral: the reason that it is easier than writing down the (second order) equations of motion is that the tension T never needs to be considered. The speed of the mass is $\ell\dot{\theta}$ and its potential energy, relative to its lowest point, is $mg\ell(1 - \cos\theta)$. Thus

$$\tfrac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1 - \cos\theta) = E.$$

Here E is a constant that would be determined by the initial conditions. One way forward would be to solve this to obtain $\dot{\theta}$ then integrate[6] but the solution cannot be expressed in terms of elementary functions. However, if the oscillations are small ($|\theta| << 1$), the motion turns out to be SHM. Setting $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ gives

$$\tfrac{1}{2}m(\ell\dot{\theta})^2 + \tfrac{1}{2}mg\ell\theta^2 = E.$$

We could solve this directly, but differentiating with respect to time and cancelling an overall factor of $m\ell\dot{\theta}$ gives immediately

$$\ell\ddot{\theta} + g\theta = 0$$

which is SHM with period $2\pi\sqrt{\ell/g}$.

[5]We don't need to do this: we could solve the differential equation directly for z, using a (constant) particular integral and a complementary function.

[6]We can obtain θ as a function of t by doing the following integral and inverting the result:

$$t = \sqrt{\frac{m\ell^2}{2}} \int \frac{d\theta}{\sqrt{E - mg\ell(1 - \cos\theta)}}$$

This is an *elliptic* integral and can be evaluated in terms of rather unfriendly functions called *elliptic* functions — elliptic because one way such integrals arise is in calculating the length of an arc of an ellipse.

Mechanics Lecture Notes

1 Lectures 11 and 12: Simple harmonic motion

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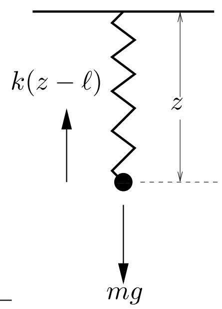

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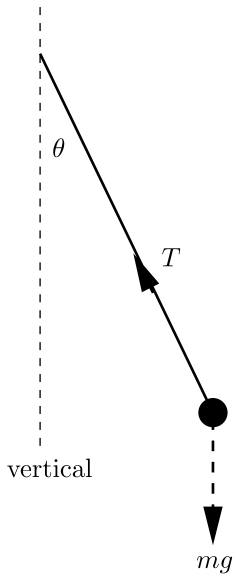

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Here E is a constant that would be determined by the initial conditions. One way forward would be to solve this to obtain $\dot{\theta}$ then integrate[6] but the solution cannot be expressed in terms of elementary functions. However, if the oscillations are small ($|\theta| << 1$), the motion turns out to be SHM. Setting $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ gives

$$\tfrac{1}{2}m(\ell\dot{\theta})^2 + \tfrac{1}{2}mg\ell\theta^2 = E.$$

We could solve this directly, but differentiating with respect to time and cancelling an overall factor of $m\ell\dot{\theta}$ gives immediately

$$\ell\ddot{\theta} + g\theta = 0$$

which is SHM with period $2\pi\sqrt{\ell/g}$.

[5]We don't need to do this: we could solve the differential equation directly for z, using a (constant) particular integral and a complementary function.

[6]We can obtain θ as a function of t by doing the following integral and inverting the result:

$$t = \sqrt{\frac{m\ell^2}{2}} \int \frac{d\theta}{\sqrt{E - mg\ell(1 - \cos\theta)}}$$

This is an *elliptic* integral and can be evaluated in terms of rather unfriendly functions called *elliptic* functions — elliptic because one way such integrals arise is in calculating the length of an arc of an ellipse.

Mechanics Lecture Notes

1 Notes for lectures 12 and 13: Motion in a circle

1.1 Introduction

The important result in this lecture concerns the force required to keep a particle moving on a circular path: if the radius of the circle is a, the speed of the particle is v and the mass of the particle is m, this force is mv^2/a, directed *towards* the centre of the circle.

This force is sometimes misleadingly described as 'centrifugal' or 'centripetal' and there is much confusion about whether it is directed towards the centre or away from it. But clearly, a force is needed to prevent the particle moving in a straight line (according to Newton's first law), and this force must be in the direction of deviation from a straight line path; i.e. towards the centre. The force may be provided by the tension in a string, or by the normal reaction if the particle is constrained to move on a circular hoop, or by gravity in the case of a planet orbiting the Sun, or by magnetic fields in the case of the Large Hadron Collider.

The confusion in the direction of the force arises because if you imagine yourself moving freely *in the rotating frame*[1] (in a car, say) and you would feel yourself being pushed *outwards* relative to the rotating frame; but this is just because you want to move in a straight line, and the car isn't moving in a straight line. Thus the 'force' you feel in a car as it goes round a bend is merely due to the tendency you have to move in a straight line: it is a *fictitious force*. It is this fictitious force, measured in the rotating frame, that is the centrifugal ('fleeing from the centre') force.

1.2 Key concepts

- Use of Cartesian coordinates and angular coordinates to describe motion in a circle.
- The force towards the centre required for circular motion.

1.3 Motion in a circle

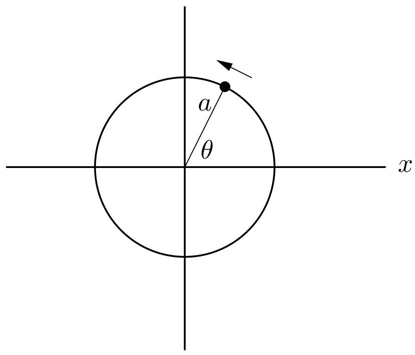
For a particle moving on the surface of a sphere centred on the origin and of radius a, we have

$$\mathbf{r}.\mathbf{r} = a$$

and differentiating[2] gives

$$2\mathbf{r}.\dot{\mathbf{r}} = 0,$$

which shows, not very surprisingly, that the velocity is perpendicular to the radius — i.e., it is tangent to the sphere. Similarly, if the speed is constant, we find that the acceleration is perpendicular to the velocity, which is an important (but not again not surprising, if you think about it) result. In particular, for a particle moving at constant speed in a circle, the acceleration is radial. That doesn't of course mean that the particle is moving towards the centre: only that the change in the velocity vector is radial.



[1]The rotating frame is accelerating, so Newton's laws do not apply. This point is discussed at length in the Dynamics and Relativity course.

[2]Using the Leibniz rule for vectors:

$$\frac{d}{dt}(\mathbf{a}.\mathbf{b}) = \mathbf{a}.\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}.\mathbf{b}$$

which can easily be demonstrated using the definition of the dot product in components.

The diagram shows a particle moving on a smooth horizontal ring of radius a. Although the motion is most conveniently described by the angular coordinate θ, it is often easier to work in Cartesian coordinates. Choosing the obvious axes, we have for the coordinates of the particle

$$x = a\cos\theta, \qquad y = a\sin\theta\,.$$

We can find the velocity $(\dot{x}, \dot{y})$ by differentiation:

$$\mathbf{v} = (-a\dot{\theta}\sin\theta, a\dot{\theta}\cos\theta)$$

which satisfies $\mathbf{r}.\mathbf{v} = 0$ as expected. The speed of the particle is given by

$$v = |\mathbf{v}| = \sqrt{(-a\dot{\theta}\sin\theta)^2 + (a\dot{\theta}\cos\theta))^2} = a|\dot{\theta}|.$$

Differentiating again gives the acceleration:

$$\mathbf{a} = (-a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\,,\, a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta)$$

which we can write in the form

$$a\ddot{\theta}\,\underbrace{(-\sin\theta,\cos\theta)}\,-a\dot{\theta}^2\,\underbrace{(\cos\theta,\sin\theta)}\,.$$

The two vectors with underbraces are unit vectors pointing tangentially to the circle and radially outwards, respectively. The magnitude of the acceleration is

$$|\mathbf{a}| = \sqrt{a^2\dot{\theta}^2 + a^2\ddot{\theta}^2}\,.$$

Thus, in order to move in a circle, the particle must experience a force directed towards the centre of the circle, of magnitude

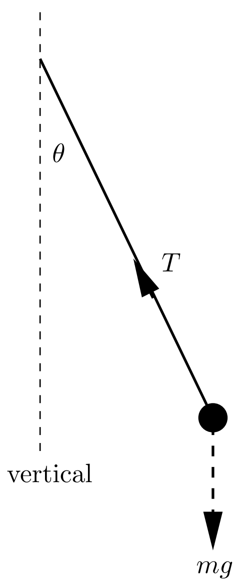
$$ma = ma\dot{\theta}^2 \equiv \frac{mv^2}{a}\,.$$

If $\ddot{\theta} = 0$, the particle moves with constant speed and no net force other than the central force acts on the particle.

1.4 Examples

(i) The simple pendulum

A particle of mass m is attached to one end of a light inextensible rod of length ℓ which is smoothly pivoted at the other end so that it can swing freely in a fixed vertical plane. Find the motion.



Let θ be the angle that the rod makes with the vertical. The easiest way of getting finding the motion is to use the energy integral: the reason that it is easier than writing down the (second order) equations of motion is that the tension T never needs to be considered. The speed of the mass is $\ell\dot\theta$ and its potential energy, relative to its lowest point, is $mg\ell(1-\cos\theta)$. Thus

$$\tfrac{1}{2}m(\ell\dot\theta)^2 + mg\ell(1-\cos\theta) = E.$$

Here E is a constant that would be determined by the initial conditions. One way forward would be to solve this to obtain $\dot\theta$ then integrate[3] but the solution cannot be expressed in terms of elementary functions. However, if the oscillations are small ($|\theta| << 1$), the motion turns out to be SHM. Setting $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ gives

$$\tfrac{1}{2}m(\ell\dot\theta)^2 + \tfrac{1}{2}mg\ell\theta^2 = E.$$

We could solve this directly, but differentiating with respect to time and cancelling an overall factor of $m\ell\dot\theta$ gives immediately

$$\ell\ddot\theta + g\theta = 0$$

which is SHM with period $2\pi\sqrt{\ell/g}$

[This example is not complete: it needs a calculation the tension in the rod using the central acceleration — coming shortly.]

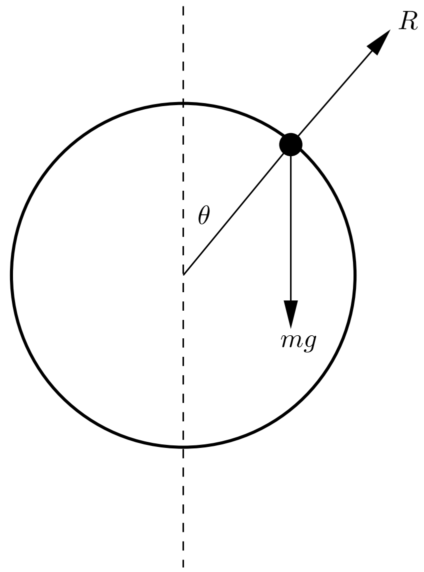
[3]We can obtain θ as a function of t by doing the following integral and inverting the result:

$$t = \sqrt{\frac{m\ell^2}{2}}\int \frac{d\theta}{\sqrt{E - mg\ell(1-\cos\theta)}}$$

This is an *elliptic* integral and can be evaluated in terms of rather unfriendly functions called *elliptic* functions — elliptic because one way such integrals arise is in calculating the length of an arc of an ellipse.

(ii) *A particle of mass m slides on the surface of a smooth cylinder under the action of gravity. The axis of the cylinder is horizontal and the motion is in a vertical plane. The particle was released from rest from a point very close to the top of the cylinder. Find the position of the particle when it leaves the surface of cylinder.*

How can we determine the point at which the particle leaves the cylinder? While in contact with the cylinder, the particle experiences a normal reaction from the surface and the cylinder experiences an equal and opposite normal reaction, by Newton's third law. When this force is zero, contact is broken and the particle then falls freely under gravity in a parabola.[4]



We can use conservation of energy to obtain the speed of the particle directly in terms of its angular position (comparing KE+PE with initial KE+PE):

$$\tfrac{1}{2}mv^2 - mga(1-\cos\theta) = 0 + 0. \quad \text{i.e.} \qquad v^2 = 2ga(1-\cos\theta).$$

To find the normal reaction of the cylinder on the particle, we use the radial component of the equation of motion:

$$mg\cos\theta - R = mv^2/a$$

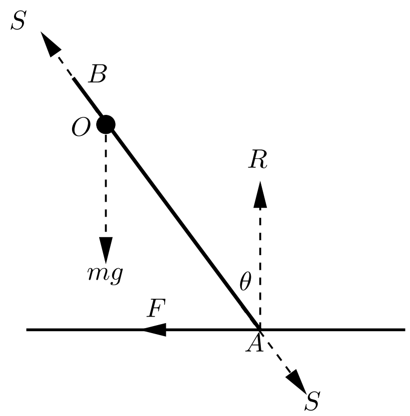
so

$$R/m = g\cos\theta - v^2/a = g(3\cos\theta - 2).$$

This is positive until $\cos\theta = 2/3$ at which point the particle leaves the surface of the cylinder.

[4]We can think of it another way. If the cylinder were taken away, the particle would fall in a parabola under gravity. If the radius of curvature of this parabola were less than a (i.e. of the path were more curved than the surface of the cylinder) , the trajectory would bend into the region previously occupied by the cylinder. In this case, if the cylinder were replaced, the particle would have to move on its surface. The condition that the particle loses contact with the cylinder is therefore that the radius of curvature of the parabolic trajectory exceeds a. For more details about radius of curvature, wait for the Vector Calculus course.

(iii) *A particle of mass* m *is riding a massless bicycle at constant speed* v *round a rough circular path of radius* a *at speed* v*. At what angle to the vertical must it lean?*



The diagram shows a bicycle AB, the point A representing the point of contact between a tyre and the path. The particle is sitting at O. To keep it simple, we assume that the particle is going round a circle of radius a (rather than the more natural assumption that the point A is going round a circle of given radius). The point O is a distance d from A.

The forces on the bicycle are: the normal reaction R of the path on the tyre; gravity, acting through O; and the friction between the tyre and the path which prevents slipping. Since there must be a horizontal force on the bicycle making it go round the corner, and the only horizontal force is friction, it is clear that the direction of the frictional force must be as shown in the diagram (inwards).

There is also an internal force of stress, S, in the frame of the bicycle. If we take this into account, we can use Newton's second law for any point of the bicycle. For the particle at O, we have (taking vertical and horizontal components)

$$0 = mg - S\cos\theta \quad \text{and} \quad \frac{mv^2}{a} = S\sin\theta.$$

Dividing gives

$$\tan\theta = \frac{v^2}{ag}.$$

For example, if $v = 5$ metres/sec (which is about 12mph) and $a = 10$ metres, and taking $g = 10$ metres per second per second results in $\tan\theta = 1/4$, i.e. $\theta \approx 15°$, which is not absurd despite the simplicity of the model.

Note that we could find R and F in terms of S and θ, and hence in terms of mg and v, by using Newton's second law on the point A instantaneously in contact with the path. We don't have to worry about the acceleration of this point, since it is massless by assumption:

$$0 = S - R\cos\theta - F\sin\theta \quad \text{and} \quad 0 = R\sin\theta - F\cos\theta.$$

Solving these equations and using $F = \mu R$ at limiting friction, we would find the maximum speed that the bicycle could go round the corner without slipping. (It is easy to see that $\mu = \tan\theta$; setting $\mu \approx 0.6$ for dry rubber on asphalt corresponds to $31°$, which is again not completely ridiculous).

The above equations will yield $R = mg$, which we could also have obtained directly by considering the vertical forces on the whole bicycle. The reason I didn't do this is because we have not discussed the motion of a rigid body (even though we did discuss the equilibrium of a rigid body). In this case, since no part of the bicycle is accelerating in the vertical direction, we can simply apply the equilibrium condition that the net force in the vertical direction (i.e. $R - mg$ is zero).