Elementary Students' Understanding of Basic Properties of Operations in US and China

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Abstract

The commutative, associative, and distributive properties are at the heart of mathematics. This study examines 97 US and 167 Chinese third and fourth graders' understanding of these basic properties through paper-and-pencil assessments. Even though students' understanding in both countries are not ideal, Chinese students demonstrate much better understanding than their US counterparts. Among these properties, the associative and distributive properties are most challenging, especially for the US students. By the end of grade 4, many Chinese students demonstrate full understanding of the associative and distributive properties across tasks however, almost none of the US students have achieved a comparable level of understanding. Student understanding in different contexts also reveals cross-cultural differences. For instance, Chinese students tend to reason upon concrete contexts for sense-making, which is rare with US students. Finally, there is a clear growth of student understanding with Chinese but not US students. In fact, the understanding gap dramatically increases across grades. Implications are discussed.

The commutative, associative, and distributive properties (CP, AP, and DP, respectively) are at the heart of mathematics (Carpenter et al., 2003) because these basic properties allow tremendous freedom in doing arithmetic (National Research Council [NRC], 2001), serve as fundamentals when working with equations (Bruner, 1977; Wu, 2009), and provide a foundation for generalizations and proofs (e.g., Schifter, Monk, Russell, & Bastable, 2008). Thus, an extensive use of these properties can serve as a good introduction to algebra (Wu, 2009). Unfortunately, many US students, including college students, conflate CP and AP (Ding, Li, & Capraro, 2013) and have difficulties with using DP to solve equations (Koedinger, Alibali, & Nathan, 2008). Given that the learning and understanding of basic properties should take place in elementary school (Common Core State Standards Initiative, 2010), it is necessary to explore the current status of elementary students' understanding of these basic properties. To date, few studies have explored how elementary students understand these basic properties. The purpose of this study is to systematically examine elementary students' understanding of the basic properties of operations from a cross-cultural perspective. Specifically, this study examines how US elementary students perform on tasks involving these basic properties in comparison with their Chinese counterparts. Focusing on third and fourth graders' understanding of the commutative, associative, and distributive properties (CP, AP, and DP), we ask three questions: (1) How do US and Chinese elementary students understand the CP, AP, and DP, respectively? (2) How do US and Chinese elementary students demonstrate understanding of the basic properties in different contexts? And (3) How do US and Chinese elementary students develop understanding of the basic properties over time?

Literature Review

The commutative property, associative property, and distributive property undergird the arithmetic operations of addition and multiplication (NRC, 2001). The commutative property (CP) refers to the invariance of the addition and multiplication operations to reordering of the operands. Algebraically, this property can be denoted as "$a + b = b + a$" (CP of addition, or "CP+" hereafter) and "$a \times b = b \times a$" (CP of multiplication, or "CP×" hereafter), where *a* and *b* standard for "arbitrary numbers in a given number system" (CCSSI, 2010, p.90). The associative property (AP) refers to the invariance of expressions in which three numbers are added or multiplied together to the order in which the operations are carried out. Elementary students are expected to know that operating on either the first two numbers or the latter two numbers will not lead to different results (CCSSI, 2010). Still using *a*, *b*, and *c* to stand for any arbitrary numbers, AP can be denoted as either "$(a + b) + c = a + (b + c)$" or "$(a \times b) \times c = a \times (b \times c)$" ("AP+" and "AP×" hereafter). The distributive property (DP) is unique in that it involves the *interaction* between two different operations. This property states that multiplying a given number by a sum of two numbers is equivalent to multiplying each summand individually and then adding the two products. One may denote this property as "$a \times (b + c) = a \times b + a \times c$." Accordingly, one may say that the factor "*a*" distributes to each summand "*b*" and "*c*," or that multiplication distributes over addition.

As noted, these properties of operations are of fundamental importance as computational tools for arithmetic, rules for algebraic manipulation, and as foundations for reasoning, generalization and proof. Moreover, properties of arithmetic are also crucial for developing a structural notion of operations – an "operation sense" (Slavit,

1998) – in which arithmetic operations are conceptualized as mental objects. For instance, to compute 8×6, students in early grades may use the known facts $5 \times 6 = 30$ and $3 \times 6 = 18$ to compute the answer. With a structural guidance on more arithmetic examples such as $8 \times 6 = (5 + 3) \times 6 = 5 \times 6 + 3 \times 6$, students may be prompted to see the distribute property, $a \times (b + c) = a \times b + a \times c$. The transition towards this "structural" thinking is of great importance to the development of mathematical understanding (Sfard, 1991), and to the development of students' success in algebra (Slavit, 1998). Explicit understanding of the properties of arithmetic is necessary for students to grasp algebra as a generalization of arithmetic (Tent, 2006).

In addition to their aforementioned importance for the learning of computational skills and development of algebraic ability, these three properties of arithmetic operations – commutative, associative, and distributive – serve an important foundational basis of what arithmetic operations *are* (in that they are important parts of the standard axiomatization of the real field). Thus, they are equally crucial for developing fundamental understanding of the *meaning* of abstract arithmetic operations, which is an important step in the development of mathematical understanding (Pirie & Kieren, 1994).

Student Poor Understanding of the basic properties

Among the three properties, CP (especially of addition) is straightforward and used intuitively and extensively by children from early grades (Baroody, Ginsburg, & Waxman, 1983; Slavit, 1998). However, with the learning of more properties, many students struggle to differentiate between CP and AP (Fletcher, 1972; Tent, 2006). In fact, difficulty disambiguating these two properties persists even to the undergraduate level (Ding et al., 2013; Larsen, 2010). This may be partly due to the fact that the two often co-occur in the same problem (Tent, 2006). For instance ubiquitous facts such as *"a + b + c = c + b + a"* involve both AP and CP. Additionally, CP and AP both relate to the notion of reordering — while CP refers to the reordering of *operands*, AP refers to the reordering of *operations* — and student difficulty in understanding this distinction may be the cause of much of their difficulty in distinguishing the two properties (Larsen, 2010).

In addition, many students also have difficulty learning the DP. Koedinger, Alibali, and Nathan (2008) reported that 71% of the U.S. undergraduates in their study could not solve an equation x-0.15x = 38.24. If these undergraduates had a deep understanding of the DP, they might have been able to apply this property to simplify "x-0.15x" as "(1-0.15)x," which most likely would have led them to successfully solve the equation. Students' learning difficulties with the basic properties indicate their lack of explicit understanding of these mathematical principles. This may be traced back to students' initial learning in elementary school, where formal learning of the properties occurs (CCSSI, 2010).

Unfortunately, prior research indicates a common instructional limitation in elementary school – focusing on strategies rather than the underlying properties – which may be a source of students' learning difficulties and poor understanding. For example, Schifter et al. (2008) reported classroom scenarios involving US third and fourth graders' creative problem solving strategies related to DP and AP. The teachers in those classrooms did not promote and utilize students' informal understanding to reveal the underlying properties. In fact, many existing US textbooks also do not provide effective support for students' learning of the basic properties. For instance, in regards to the distributive property, Ding and Li (2010) reported that existing US textbooks presented many computation strategies such as using a known fact (e.g., $3\times8=2\times8+8$), double strategy, (e.g., $8\times5=4\times5+4\times5$), and breaking apart a number to multiply (e.g., $18\times12=18\times10+18\times2$). Regardless of the variety of strategy, the underlying DP was rarely made explicit (Ding & Li, 2010). Focusing on the teaching of strategies rather than the underlying property not only increases student cognitive load, but also hinders students' learning and transfer of the basic properties to new contexts. In this study, we are interested in exploring to what extent our student participants may possess explicit understanding of each of the basic properties.

Understanding Basic Properties through Varied Contexts

Research indicates that elementary students are mainly exposed to the basic properties in numerical contexts where CP, AP, and DP are used implicitly or explicitly to solve computational tasks (Baroody et al, 1983; Schifter et al., 2008). To compute 8×6, a third grader in the US may be expected to add two known facts ($5 \times 6 = 30$ and $3 \times 6 = 18$) to find the answer, which involves the DP (CCSSI, 2010; Ding & Li, 2010). While these non-contextual tasks are important learning opportunities, these basic properties are abstract for elementary students who most likely need concrete support for sense-making during initial learning. Otherwise, the learned properties may be become "inert" knowledge that cannot be flexibly and actively retrieved to solve new problems (Koedinger et al., 2008). As such, elementary children's understanding of the basic properties through concrete contexts (e.g., real objects and story situations) should be acknowledged and valued. This is because the contextual support may enable students to justify the to-be-learned property, serving as a path for retrieval of the basic properties as needed.

NRC (2001) provided explanations for how CP, AP, and DP can be learned through experiences with and observations of problems set in context. For instance, using cube trains with different colors, CP+ and AP+ can be illustrated. In addition, CP× or AP× can be modeled by using an array or volume model. NRC also suggests that contextual tasks such as solving for the perimeter of a rectangle in two different ways, 2L+ 2W and 2(L+W), presents a great opportunity for helping students make sense of the DP. In a similar vein, Ding and Li (2010) found that Chinese textbooks situate the initial learning of the DP in story situations. By solving a story problem in two ways, students can compare the two solutions and make sense of the DP. In fact, this approach was observed with all basic properties in Chinese textbooks. In contrast, existing US curriculum and instruction on these properties seems to mainly be limited to number manipulations without sense-making. For example, even though some US textbooks introduced the DP and AP by using word problem contexts similar to the Chinese textbooks, the US word problems were not unpacked to illustrate the properties and were used primarily as a pretext for computation (Ding & Li, 2010; Ding, 2016). This type of textbook presentation creates missed opportunities for helping students make sense of the basic properties. In this study, we are interested in exploring whether students can recognize and explain the basic properties through contextual tasks.

Based on the literature about how students learn the basic properties, this study will explore students' understanding through different contexts. It is widely reported that students' understanding depends on the contexts to which they are exposed (Bisanz &

LeFevre, 1992; Bisanz, Watchorn, Piatt, & Sherman, 2009; Greeno & Riley, 1987). As such, it is necessary to assess students' understanding using both contextual and non-contextual tasks (Bisanz & LeFevre, 1992; Bisanz et al., 2009). Through these contexts, students may be asked to evaluate (whether a strategy works), apply (the properties to do computation), and explain (why a strategy or a computation procedure work), which together may provide valid assessment of students' procedural and conceptual understanding. Additionally, those that are able to explain the use of these properties, still provide explanations that differ in quality (Ding & Auxter, 2017). For instance, providing a numerical example of the CP+ (e.g., 3+7=7+3) may indicate only an implicit understanding of a single case, whereas providing an algebraic formula (e.g., $a + b = b + a$) may reveal a students' more general understanding of the CP+.

Learning the Basic Properties across Grades

The understanding of the basic properties is not an all-or-nothing phenomenon. Rather, through different contexts, understanding develops over time. Greeno and Riley (1987) pointed out that the development of understanding is progressive, often moving from implicit to explicit. To "apply" a property for computation/operation intuitively and subconsciously might indicate implicit understanding. On the other hand, to verbalize and explain the property in a more general/structural sense may show explicit understanding, which is at a higher cognitive level. The transition from implicit to explicit understanding is in some sense similar to the notion of shifting from operational (process-oriented) to structural (object-oriented) conception, which is developed through reification of the processes (Sfard & Linchevski, 1994). In theory, as grade level increases, students should progressively shift implicit understanding to explicit understanding of the basic properties. However, previous empirical studies do not necessarily support this prediction. For instance, Canobi (2005) found that as grade level increased, students' computation accuracy improved; yet, their explanations did not necessarily improve. In this study, we are interested in exploring whether there is a developmental trend in elementary students' understanding of the CP, AP, and DP as indicated by their paper-and-pencil responses.

Method

To examine US and Chinese elementary students' understanding of the CP, AP, and DP, we use both quantitative and qualitative methods. The quantitative analysis provides relatively accurate pictures about the levels of understanding across grades and countries. The qualitative analysis suggests interesting cross-cultural patterns that may further inform both curriculum and instruction.

Participants

This study is part of a NSF funded project on early algebra. A total of 97 US and 167 Chinese third and fourth grade students who took both a pre- and post-test involving the properties were included in the current study. In China, students formally learn the basic properties in fourth grade, over the span of two focused chapters (an entire chapter devoted to the DP is taught in the second semester; the rest of the properties are all included in one chapter taught during the first semester). Prior to fourth grade, Chinese students only learn the properties in implicit and informal ways. In contrast, US students are formally introduced to the basic properties much earlier (e.g., learning CP in the first grade). They also have recurring opportunities to constantly learn these properties across several grades. Regardless of the differences, both the US and Chinese students will have formally learned the CP, AP, and DP by the end of the fourth grade. As such, we are interested in exploring the current status of third and fourth graders' understanding of these properties. The US students in this study were from an urban school district in the Northeast, and the Chinese students were also from an urban district in Southeast China. Students were also selected from multiple classrooms within each grade in each country. Given that the school districts were large, sampled students were also from various schools. All of the sampled US and Chinese students were taught by expert teachers who had more than 10-years teaching experiences with good teaching reputations (e.g., earned teaching awards or recommended by school district or their principals). Table 1 shows detailed information about the student sample sizes for this study.

Table 1. *The Sample Size for both Pre- and Post-tests*

US (*N* = 97)			China (*N* = 167)		
Grade 3 (*N* = 32)	Class 1	*N* = 10	Grade 3 (*N* = 84)	Class 1	*N* = 48
	Class 2	*N* = 22		Class 2	*N* = 36
Grade 4 (*N* = 65)	Class 1	*N* = 17	Grade 4 (*N* = 83)	Class 1	*N* = 44
	Class 2	*N* = 27		Class 2	*N* = 39
	Class 3	*N* = 21			

Instrument

The student assessment instrument was developed for this NSF project in order to measure students' understanding of the CP, AP, and DP in grades 1-4. The instrument design was guided by Bisanz and colleagues (Bisanz & LeFevre, 1992; Bisanz et al., 2009). Items used to measure each property included both contextual and non-contextual tasks that demand evaluation, application and explanation of these properties. Students who have a good understanding of the basic properties should be able to identify the basic properties presented across various contexts. Figure 1 illustrates a sample of three of the tasks. The first two items are non-contextual tasks that require students' evaluation/application and explanation based on the CP or the AP. The last item is a contextual task that require students' recognition of the DP.

Q1: If you know 7 + 5= 12 does that help you solve 5 + 7? Why?	*Non-contextual evaluation/explanation task: CP+*
Q7a: Please use efficient strategies to solve. Show your strategy and explain why it works. (3 × 25) × 4.	*Non-contextual application/explanation task: AP×*
Q10: The length of a rectangular playground is 118 m and the width is 82 m. What is the perimeter? *John solved it with*: 2 × 118 + 2 × 82 *Mary solved it with:* 2 × (118 + 82) Both are correct. Compare the two strategies, what do you find?	*Contextual recognition task: DP*

Figure 1. Sample tasks used in the instrument.

Overall, there were 10 items (16 sub-tasks) gleaned from literature (e.g., Baroody, 1999) and elementary textbooks. Table 2 shows the structure of the instrument (the full instrument is presented in Appendix 1). This instrument has been reviewed by two experts (one mathematician and one mathematics educator) in the field and then validated through pilot testing in a 2^{nd} and a 4^{th} grade classroom. Note that for item 2, the second teacher of the piloting class suggested removing the words "instead of counting from 3" due to concerns of children's reading ability. Unfortunately, the revised task seemed to shift students' attention to the counting process rather than CP+. As such, we excluded item 2 and used the remaining 15 subtasks (5 for each CP, AP and DP) to assess students' understanding of the properties.

Table 2. *The structure of the instrument*

Property		Item	Task	Context	Nature	Points
CP	(+)	1	If you know 7 + 5= 12 does that help you solve 5 + 7? Why?	Non-contextual	Evaluation, Explanation	2
		3(a)	2 + 7 + 8 (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		4	Story problem about 8 boys and 5 girls. Solved by: 8 + 5 and 5+8	Contextual	Recognition	2
	(×)	6	To solve 3 × 28, Mary wrote: $\begin{array}{r} 2\ 8 \\ \times\ \ \ 3 \\ \hline \end{array}$	Non-contextual	Evaluation, Explanation	2
		8(c)	8 × 6 is solved by: Since 6 × 8 = 48, 8 × 6 = 48.	Non-contextual	Explanation	2
AP	(+)	3(b)	(7 + 19) + 1 (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		3(c)	2 + (98 + 17) (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		5	Story problem about three book shelfs with 7, 8, and 5 books on each. Solved by: (7 + 8) + 5 and 7 + (8 + 5)	Contextual	Recognition	2
	(×)	7(a)	(3 × 25) × 4 (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		9	Story problem about 3 tables of 2 plates of 5 mangos. Solved by: (3 × 2) × 5 and 3 × (2 × 5)	Contextual	Recognition	2
DP		7(b)	102 × 7 (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		7(c)	98 × 7 + 2 × 7 (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		8(a)	8 × 6 is solved by: 3 × 6 = 18, 5 × 6 = 30, and 18 + 30 = 48	Non-contextual	Explanation	2
		8(b)	8 × 6 is solved by: 10 × 6 = 60, 2 × 6 =12, and 60 –12 = 48	Non-contextual	Explanation	2
		10	Story problem about a playground perimeter with the length of118m and width of 82m. Solved by: 2 × 118 + 2 × 82 and 2 × (118 + 82)	Contextual	Recognition	2

Note. Full instrument is available in Appendix 1.

Data Coding and Procedures

We first developed a coding rubric based on two authors' collaborative coding of student responses from one US and one Chinese class. It was decided that subtasks would be assigned 2 points, resulting in a total of 10 points for each of the CP, AP, and DP. For an item that contains both evaluation/application and explanation [items 1, 3(a), 3(b), 3(c), 6, 7(a), 7(b), and 7(c)], we assigned 1 point to evaluation/application and the other 1 point to the explanation. The rest of the explanation/recognition tasks [4, 5, 8(a), 8(b), 8(c), 9, and 10] were each assigned 2 points. Regardless of the assigned points, students' explanation/recognition were classified into three levels: explicit, implicit and no/wrong, with full, half, or no score applied. For each item, we have supplemented the coding rubric with typical Chinese and US responses. Figure 2 illustrates the coding rubric for items 1 and 4. Note that we considered US students' "turn around facts" as full-understanding of the CP because some US textbooks call the CP the "turn-around property." After the rubric was defined, we trained the other two coders who coded part of the US and Chinese data for reliability checking. Our reliability (the # of common codes/the # of total codes) for coding the US data was 94% and for coding the Chinese data was 97%.

	Item 1: Evaluation/Explanation Task of CP+ (Non-contextual) If you know 7+5=12 does that help you solve 5+7? Why?	Item 4: Recognition Task of CP+ (contextual) There are 8 boys and 5 girls in a swimming pool. How many children are there altogether? John solved it with: 8+5; Mary solved it with: 5+8 Both are correct, comparing the two strategies, what do you find?
Full understanding	**Correct evaluation (1)** **Explicit explanation (1)** Chinese example: 有，因为学过加法交换律的同学知道"a+b=b+a"，变成现在的数字就是"5+7=7+5"。 *Yes. If one has learned the commutative property of addition, he knows a+b=b+a. In this case, It is 5+7=7+5.* US example: Yes because it is Communative property you will get the same anwser on both of them	**Explicit recognition (2)** Chinese example: 我发现"8+5=5+8"，还发现了我们学过的加法交换律。 *I found that 8+5=5+8 and I discovered the commutative property of addition that we have learned.* US example: Both are correct. Compare the two strategies, what do you find? I realized that the strategies are turn around problems. John added 8+5 and mary added 5+8.
Partial understanding	**Correct evaluation (1)** **Implicit explanation (0.5)** Chinese example: 答：有帮助因为5+7=7+5。 *Answer: Yes, helpful, because 5+7=7+5* US example: Knowing 7+5=12 does help me solve 5+7 because it is the Same problem but it is backwards.	**Implicit recognition (1)** Chinese example: 答：我发现这两道算式都一样，只是调了个头。 *Answer: I found the two number sentences are the same, only the numbers are flipped.* Chinese example: 答：比较这两种方法，我发现小明用的是先算男孩在算女孩，小芳的方法是先算女孩在算男孩。 *Answer: Comparing these two methods, I found Xiaoming's computation first considered boys and then girls while Xiaofang first considered girls and then boys.* US example: I found out both Strategies are the Same there Just backwards.
	Correct evaluation (1) **No/wrong explanation (0)** Chinese example: 答：是对的。 *Answer: Yes, it is correct.*	
No Understanding	**No/Wrong evaluation (0)** **No/wrong explanation (0)** Chinese example: 答：没有帮助因为这两道算式一样。 Answer: Not helpful, because the two number sentences are the same. US example: No response at all.	**No/wrong recognition (0)** Chinese example: 答：它们的答案一样。 *Answer: They have the same answer.*

Figure 2. Example coding rubric for items 1 and 4.

Data Analysis

All coded data was merged for analysis. To answer the first research question about student understanding of the CP, AP, and DP, we considered the third and fourth graders together and computed the overall pre- and post-test scores for each property within each country. Independent *t*-tests were conducted to determine the significance of differences between different groups. For all tests, type 1 error was controlled with Bonferoni post-hoc control. To obtain a clearer picture of students' understanding by the end of grade 4 (by then, the Chinese students have formally learned all properties), we conducted a further inspection on grade 4 students' post-tests in terms of the percentage of full, partial, and no understanding for each property. For each item, 2 points (full score) indicates a full understanding and "0 point" indicates no understanding. The other assigned scores (1.5 points and 1 point) indicate a partial understanding (see Figure 2 for an example). In these "partial" situations, students may have correctly evaluated a situation or applied a right property for the computation, but their explanations were implicit (score of 0.5) or even no/wrong (score of 0). Initially, we disagreed whether a correct evaluation/application with no/wrong explanation could be considered as "partial understanding." However, after discussion we agreed that since evaluation, application, and explanations were all indicators of understanding, it was reasonable to consider such responses as partial understanding even if the explanations were missing or were incorrect.

To answer the second research question about students' understanding across contexts, student performance on contextual and non-contextual tasks involving the three basic properties were analyzed. Cross-cultural differences were identified. Likewise, we further compared the fourth graders' post-tests in terms of full, partial, and no understanding under different contexts (tasks). Typical student examples were identified and are reported to substantiate the findings.

Finally, to answer the third research question about students' understanding of the basic properties over time, we examined US and Chinese third and fourth graders' responses to the pre- and post-tests from different angles by using matched pairs *t*-tests and independent *t*-tests. In addition, we examined the trends over several time spots in order to obtain a general sense of student understanding: the beginning and end of G3, and the beginning and end of G4. We are cautions of the limitation that the two student groups (grade 3 and grade 4) were different. Nevertheless, given that these students were selected from the same school district in each country, we believe these comparative results are at least to some degree insightful and informative.

Results

In this section, we report US and Chinese students' performance in alignment with three research questions: (1) students' understanding of the CP, AP, and DP in general, (2) students understanding of the basic properties in different contexts, and (3) students' understanding of the basic properties over time.

Student Understanding of the CP, AP, and DP

Figure 3 indicates the overall mean difference for the CP, AP, and DP, treating all students across grades in each country as a single group. The mean score for each property was obtained by averaging the scores of the corresponding five items (see Table 2). Even though students in both countries show improvement from the pre- to post-tests, independent *t*-tests reveal that Chinese students performed better than US students for each property in both the pre- and the post-tests: $t_{CP\text{-}pre}(200.78) = 8.8$, $p < 0.01$; $t_{CP\text{-}pst}(247.89) = 5.2$, $p < 0.01$; $t_{AP\text{-}pre}(254.7) = 12.1$, $p < 0.01$; $t_{AP\text{-}pst}(258.83) = 9.6$, $p < 0.01$; $t_{DP\text{-}pre}(235.6) = 10.97$, $p < 0.01$; $t_{DP\text{-}pst}(248.71) = 10.1$, $p < 0.01$.

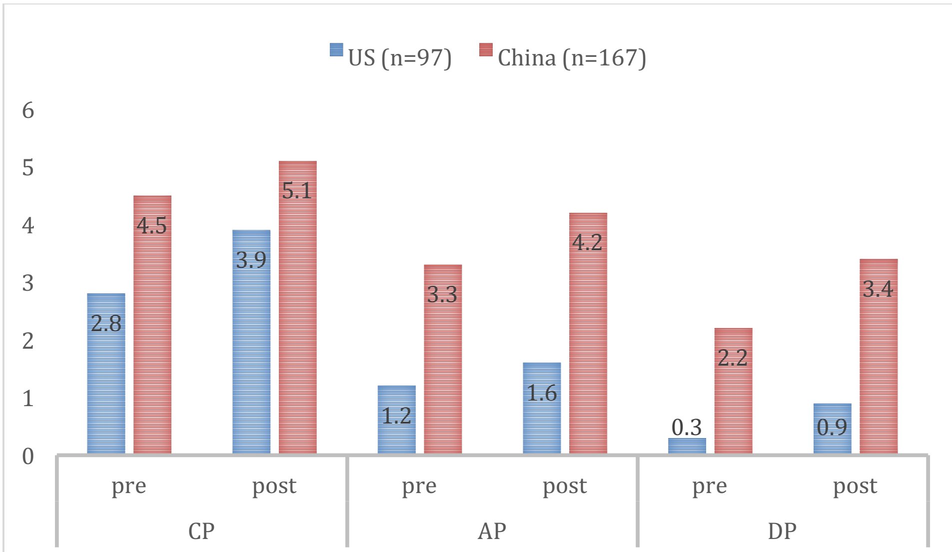


Figure 3. US and Chinese students' overall performance on CP, AP, and DP.

As indicated by Figure 3, among the three properties, students in both countries performed better in the CP than they did with the AP and DP. In fact, the cross-cultural gap in student understanding was most evident in the AP and the DP. Many US students conflated the AP and the CP. For instance, students in Q5 were expected to identify the AP+ from the story situation. A typical US response was, "The strategy I found was both Mary and John used the commutative property. It is just like the turn-around fact." Similar responses were found with Q9, a contextual task for the AP×, "… all they did was changed the numbers order. Just like the commutative property." Such a conflation between the CP and the AP was rare with Chinese students. For the DP, many US students referred to it as the "breaking down" strategy or simply mentioned the level of easiness of a strategy. In Q8, when asked to explain Mary's strategy (3 × 6 =18, 5 × 6 = 30, 18 + 30 = 48) to solve 8 × 6, a student responded, "Mary's strategy works because she is breaking down the problem to make it easier." Similarly, a typical response to Q10 was, "One is making it harder to do and the other one is doing it the simple."

One might notice that even though the Chinese students performed better in each property, the mean scores were relatively low (e.g., $M_{CP\text{-}pst} = 5.1$, $M_{AP\text{-}pst} = 4.2$, and $M_{DP\text{-}pst} = 3.4$; out of 10). This may be due to the fact that Chinese students do not formally learn these properties until fourth grade. As such, we further examined the US and Chinese fourth graders' post-tests to obtain a truer picture about students' understanding by the end of grade 4. Table 3 summarizes the fourth graders' mean scores for the CP, AP, and DP in the post-tests of both countries.

Table 3. *US and Chinese students' mean score of CP, AP, and DP at the end of grade 4.*

	CP		AP		DP	
	Mean	SD	Mean	SD	Mean	SD
US_G4Pst	3.7	1.3	1.5	1.5	0.9	1.3
China_G4Pst	6.0	2.0	6.0	2.2	5.1	2.5

Table 3 shows that by the end of grade 4, Chinese students have improved their understanding of the AP and the DP to almost the same level as their understanding of the CP. On average, Chinese fourth graders can answer about 60% (6 out of 10) AP tasks and 51% (5 out of 10) DP tasks. In fact, there were 7 Chinese fourth graders who correctly answered 90-97% of all tasks. On the other hand, on average US fourth grade students correctly answered 15% of the AP and 9% of the DP tasks. Table 4 provides further information about the extent to which students demonstrated full, partial, and no understanding of each property through each relevant task.

Table 4. *Students' Levels of Understanding of CP, AP, and DP by the end of grade 4.*

		CP					AP					DP				
		+	+	+	×	×	+	+	+	×	×					
		Q1	Q3a	Q4	Q6	Q8c	Q3b	Q3c	Q5	Q7a	Q9	Q7b	Q7c	Q8a	Q8b	Q
	Under-standing	Eva, Exp	App, Exp	Rec	Eva, Exp	Exp	App, Exp	App, Exp	Rec	App, Exp	Rec	App, Exp	App, Exp	Exp	Exp	R
US (n=65)	**Full**	22%	0%	15%	2%	3%	0%	0%	0%	0%	0%	0%	0%	0%	0%	2
	Partial	74%	25%	46%	75%	26%	28%	38%	42%	15%	29%	23%	6%	23%	20%	9
	No	5%	75%	38%	23%	71%	72%	62%	58%	85%	71%	77%	94%	77%	80%	8
China (n=83)	**Full**	46%	16%	59%	16%	30%	22%	25%	49%	24%	51%	20%	43%	16%	16%	6
	Partial	46%	81%	17%	82%	8%	75%	71%	22%	73%	16%	70%	51%	31%	33%	8
	No	8%	4%	24%	2%	61%	4%	4%	29%	2%	34%	10%	6%	53%	52%	2

Note. Due to rounding totals of percentages may not equal 100%.

As explained in Methods, students' full understanding of a property refers to using correct terminology, a formula (e.g., $a + b = b + a$), or verbally stating a property at a general level (see Figure 2 for examples). In Table 4, there were about 22% of US students in Q1 and 15% in Q4 who demonstrated full understanding of CP+. However, students' full understanding of the CP+ was inconsistent across tasks (e.g., no students in Q3 demonstrated understanding). Overall, much fewer US students (2%-3%) demonstrated full understanding of the CP×. With regards to the AP and the DP no US students demonstrated full understanding of these two properties across items (note: one student who correctly mentioned the DP in one task: Q10). In contrast, many more Chinese students demonstrated full understanding of all of these properties across all tasks (CP: 16%-59%; AP: 24%-51%; and DP: 16%-63%). In particular, while there were between 58%-85% of US students who had no understanding of the AP across tasks, only between 2%-34% Chinese students completely lacked this understanding.

Student Understanding of the Basic Properties in Different Contexts

Combining tasks for all three properties, Figure 4 shows the average percentages of the total points (22 points) that US and Chinese students earned. A first glance, these data appear to reveal a similar pattern of understanding between contextual and non-contextual tasks (see Figure 4) across grades and across countries. Independent *t*-tests, however, reveal a difference between US and Chinese students in all pre- and post-tests, except for the G3-post-test non-contextual tasks (see Figure 4). The standardized t-scores for non-contextual are: $t_{G3\text{-}pre}(79.1) = -6.6$, $p < 0.01$; $t_{G3\text{-}pst}(62) = -0.45$, $p = 0.65$; $t_{G4\text{-}pre}(129.2) = -10.9$, $p < 0.01$; $t_{G4\text{-}pst}(131.54) = -13.135$, $p < 0.01$. The standardized t-scores for contextual tasks are: $t_{G3\text{-}pre}(92.8) = -5.1$, $p < 0.01$; $t_{G3\text{-}pst}(94.5) = -3.13$, $p < 0.01$; $t_{G4\text{-}pre}(145.4) = -9.1$, $p < 0.01$; $t_{G4\text{-}pst}(124.3) = -9.4$, $p < 0.01$.

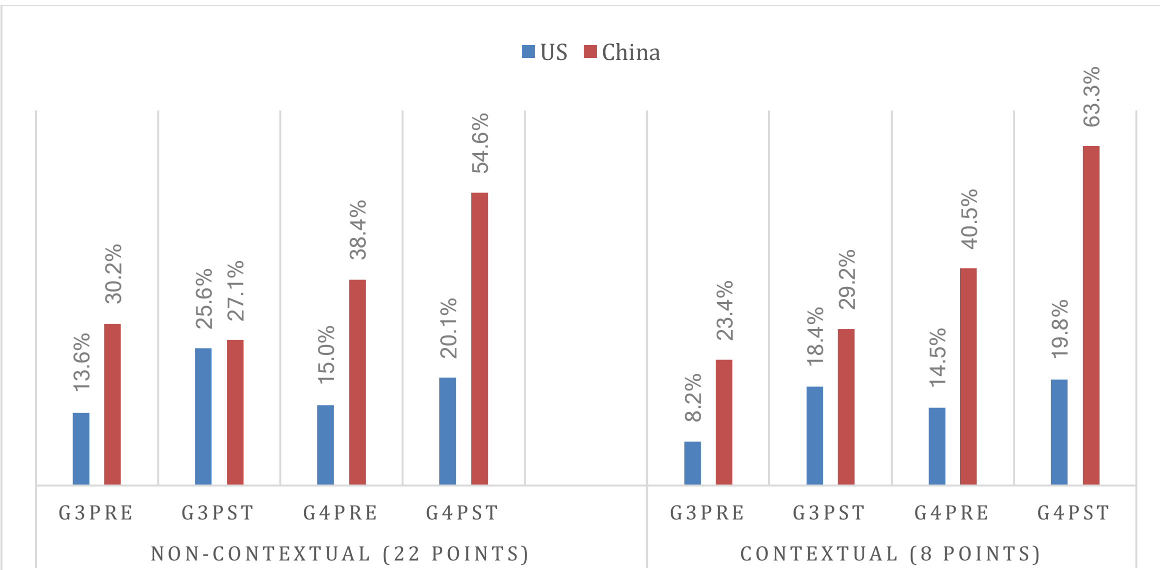


Figure 4. Percentage of points students earned in different contexts.

A closer inspection of the Chinese G3 non-contextual tasks revealed that some Chinese third graders in the post-tests computed tasks Q3 and Q7 [e.g., solving $(3 \times 25) \times 4$] based on the order of operations. These students explained that one should first perform the computations inside the parenthesis. It seems that the third graders' learning of the order of operations has perhaps overshadowed their intuitive/implicit knowledge of the basic properties. It is therefore encouraging to know that Chinese students formally learn the basic properties in fourth grade. As indicated by Figure 4, once Chinese students have learned these properties, many of the Chinese fourth graders can flexibly use these properties to do computations (see Figure 4, G4post). Table 5 below provides further information about the US and Chinese students' full, partial, and no understanding across contexts by the end of grade 4.

Table 5. *Fourth Graders' Understanding of the Basic Property across Contexts*

		Non-contextual (22 points)											Contextual (8points)			
		Eva/Exp		App/Exp						Exp			Rec			
		Q1 CP+	Q6 CP×	Q3a CP+	Q3b AP+	Q3c AP+	Q7a AP×	Q7b DP	Q7c DP	Q8a DP	Q8b DP	Q8c CP×	Q4 CP+	Q5 AP+	Q9 AP×	Q10 DP
US (n=65)	Full	22%	2%	0%	0%	0%	0%	0%	0%	0%	0%	3%	15%	0%	0%	2%
	Partial	74%	75%	25%	28%	38%	15%	23%	6%	23%	20%	26%	46%	42%	28%	9%
	No	5%	23%	75%	72%	62%	85%	77%	94%	77%	80%	71%	38%	58%	72%	89%
China (n=83)	Full	46%	16%	16%	22%	25%	24%	20%	43%	16%	16%	30%	59%	49%	51%	63%
	Partial	46%	82%	81%	75%	71%	73%	70%	51%	31%	33%	8%	17%	22%	16%	8%
	No	8%	2%	4%	4%	4%	2%	10%	6%	53%	52%	61%	24%	29%	34%	29%

An inspection of Table 5 indicates an overall consistence in student understanding within each type of task. The cross-cultural differences were mostly evident with the non-contextual application (explanation) tasks and the contextual recognition tasks, while the difference in evaluation tasks appeared to be minimal.

Non-contextual –evaluation/explanation. When students were asked whether knowing 7 + 5= 12 helped them solve 5 + 7 (Q1), about 22% of the US students demonstrated full understanding of the CP+. In contrast, when asked whether Mary can write $\begin{array}{r} 2\ 8 \\ \times\quad 3 \\ \hline \end{array}$ to solve 3 × 28 (Q6), only 2% of US students (*n*=1) fully understood this to be the CP×. In fact, to describe these two tasks, most of the US students used the term "turn-around facts." The Chinese students on the other hand performed relatively better on these two tasks with 46% (Q1) and 16% (Q6) demonstrating full understanding. Also supporting the notion that Chinese students demonstrated better understanding of the CP×, is the finding that 23% of US students but only 2% of Chinese students showed no understanding of Q6. Students' partial understanding indicated by their responses, supports the finding that students have more sensitivity to CP+ but not to CP×. For instance, in addition to correct evaluation, there were 29 (out of 38) Chinese students and 44 (out of 48) US students who provided partial explanations in Q1 (CP+); yet, only 6 (out of 68) Chinese and 3 (out of 49) US students could do so in Q6 (CP×). The rest of the students either provided wrong or procedural explanations, "because the number with more digits should be placed above" (a Chinese example) and "because you always do big number first" (a US example).

Non-contextual-application/explanation. Q3 and Q7 asked students to apply the basic properties to compute 2 + 7 + 8, (7 + 19) + 1, 2 + (98 + 17), (3 × 25) × 4, 102 × 7, and 98 × 7 + 2 × 7. We expected students to apply and explain the undergirding CP+, AP+, AP×, and the DP respectively (see Table 5). Regardless of the properties, none of the US students demonstrated full understanding on these tasks. In contrast, between 16% and 43% of the Chinese students explicitly mentioned the properties involved in their computations for these tasks (see Figure 5 for examples). Consistent with this observation, between 62% and 94% of US students, but only between 2% and10% of Chinese students demonstrated no understanding of the properties on these tasks (i.e., they simply followed the order of operations, see Figure 5). In other words, most of Chinese students were able to apply the basic properties and use efficient strategies to do computations, while most of the US students conveyed no knowledge of how to compute these problems efficiently.

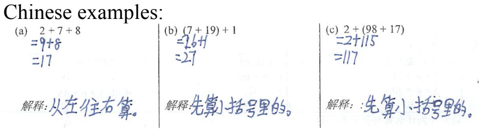
Note that there were 6-38% US and 51-81% Chinese fourth graders who demonstrated partial understanding on these non-contextual application/explanation tasks. In these cases, all US students (except for one student in Q3) and most Chinese students correctly applied the properties to do computations, but their explanations only described procedures without explanation of the undergirding properties. For instance, students from both countries described that they first attempted to make a 10 or a 100 out of the numbers. And, some US students described how they "break down" a number prior to multiplying (see Figure 5 for examples). Uniquely, there were a few Chinese students who provided partial explanations which shows better understanding than the procedural descriptions reported above (e.g., 10 students in Q3a, 4 in Q7c, and a few in other tasks). These partial explanations either referred to the surface features of the targeted property or referred to the meaning of operations within a task (see Figure 5). For instance, to explain their strategy for solving $98 \times 7 + 2 \times 7= (98 + 2) \times 7 = 100 \times 7 = 700$, some Chinese students pointed out "combining 98 groups of 7 and 2 groups of 7." These partial explanations were not observed with any of the US students. Note that US students particularly struggled with Q7c where the DP needs to be applied in an opposite direction from how this property is traditionally introduced to students in US (Ding & Li, 2010). Specifically, only 6% of US students could solve this problem, which is in sharp contrast to 94% of their Chinese counterparts.

Common Responses between US and China

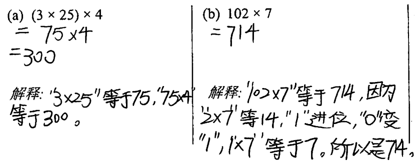
No understanding

Computation does not apply the properties. Explanation confirms the procedures. (A total of 0 point)

Chinese examples:

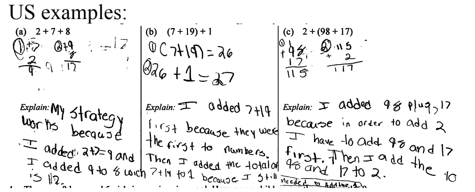


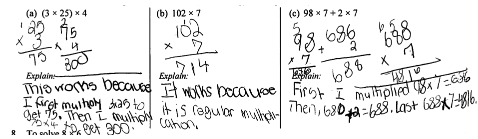
Translation: (a) from left to right. (b) and (c) first compute what is inside of the parenthesis.



Translation: (a) 3 × 25 equals to 75, 75 × 4 equals to 300. (b) 102 × 7 = 714, because 2 × 7 = 14, carrying 1, then 0 is changed to 1, 1 × 7 = 7. Thus, it is 714.

US examples:

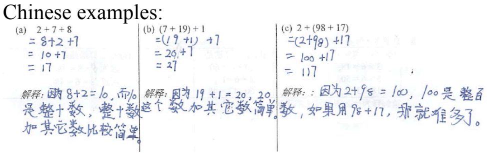




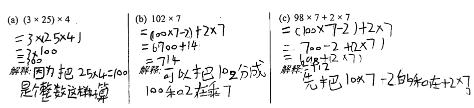
Partial understanding

Correctly applies the properties to compute. Explanation repeats the computation procedures such as making a 10 or 100 first. (A total of 1 point)

Chinese examples:

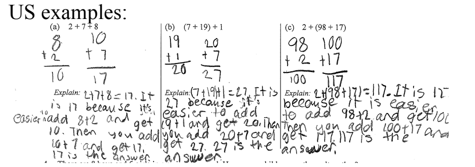


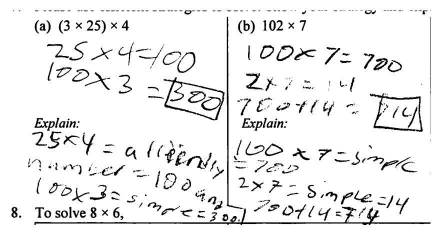
Translation: (a) Because 8+2=10, 10 is tens. Tens is easier to compute than the other number. (b) Because 19+1=20, Adding 20 is easier to compute than the other number. (c) Because 2+98=100, 100 is hundreds. If you add 98+17, that is be harder.



Translation: (a) because 25 × 4 = 100 which is a whole number. (b) You can separate 102 as 100 and 2, and then multiplying by 7. (c) First compute 100 × 7 and then add 2 × 7.

US examples:

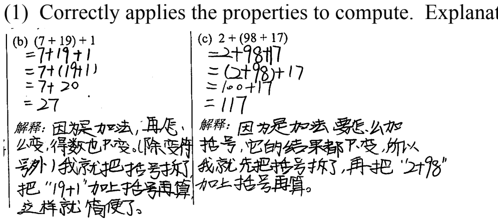




Unique Responses of China

Partial Understanding

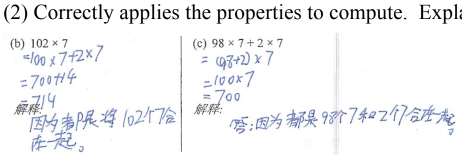
(1) Correctly applies the properties to compute. Explanation refers to surface features of a property (1.5 points)



Translation

(b) Because for addition, however you change, the answer will not be changed (expect for changing the sign). I removed the parenthesis and then added it to 19+1. This makes computation easy. (c) Because for addition, however you add the parenthesis, the answer will not be changed. As such, I first removed the parenthesis and then added it to 2+98.

(2) Correctly applies the properties to compute. Explanation referring to meaning of an operation (1.5 points).

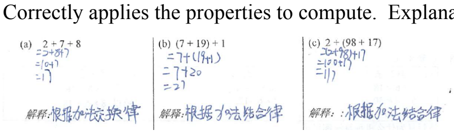


Translation

(b) because combines 102 groups of 7.
(c) because it combines 98 groups of 7 and 2 groups of 7.

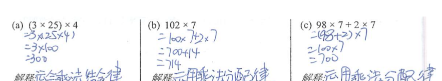
Full understanding

Correctly applies the properties to compute. Explanation explicitly pointed out the properties involved (2 points).



Translation

(a) according to CP of addition, (b) and (c) according to AP of addition.



Translation: *(a) using AP of multiplication, (b) and (c) using DP.*

Figure 5. Examples of students work with non-contextual application/explanation tasks.

Non-contextual - explanation. Q8 asked students to explain how the properties were used in solving 8 × 6, a modified task based on US curricula. In nature, this item is similar to Q3 and Q7 except for the application that is given. This task turned out to be challenging for Chinese students. For instance, only 16% demonstrated full understanding of the DP (8a, 8b) and only 30% fully explained the CP (8c, see Figure 6). Interestingly, even though this item was modified from US textbooks, no US students (0%) could explicitly point out the undergirding DP and CP. Some US students did refer to the DP as the "breaking down strategy" and thus received partial credit. In fact, a much higher proportion of US students demonstrated no understanding (77%, 80%, and 71% for Q8a, 8b, 8c, respectively) than did the Chinese students (53%, 52%, and 61%).

8. To solve 8 × 6,

(a) Mary thought:	(b) John thought:	(c) Kate thought:
3 × 6 = 18, 5 × 6 = 30, 18 + 30 = 48	10 × 6 = 60, 2 × 6 =12, 60 –12 = 48	Since 6 × 8 = 48, 8 × 6 = 48

Please explain why each strategy works.

8. 计算 8 × 6：

(a) 小红算法是：	(b) 小华算法是：	(c) 小丽算法是：
3 × 6 = 18, 5 × 6 = 30, 18 + 30 = 48	10 × 6 = 60, 2 × 6 =12, 60 –12 = 48	因为 6 × 8 = 48 所以 8 × 6 = 48

请解释为什么每种算法都正确。

(Note: The English names Mary, John, and Kate were replaced with popular Chinese names, Xiaohong, Xiaohua, and XiaoLi)

Ex1

答：小红把8拆成3和5，运用了乘法分配律。小华把8算成10−2，再运用乘法分配律。小丽运用了乘法交换律，6×8=8×6（a×b=b×a）。

Translation: Xiaohong (Mary) broke down 8 into 3 and 5 and used the DP of multiplication. XiaoHua (John) viewed 8 as 10-2 and then used DP of multiplication. XiaoLi (Kate) used the commutative property of multiplication.

Ex2

答 红：用了乘法分配律，8×6=(3+5)×6=3×6+5×6=18+30=48。
华：用了乘法分配律，8×6=(10−2)×6=10×6−2×6=60−12=48。
丽：用了乘法交换律，8×6=6×8=48。

Translation:
Hong (Mary): Used DP of multiplication. 8× 6=(3+5)×6=3×6 + 5×6 = 18 + 30= 48.
Hua (John): Used DP of multiplication. 8× 6=(10-2) × 6=10×6 – 2×6 = 60 – 12= 48.
Li (Kate): Used CP of multiplication. 8× 6=6 × 8=48.

Figure 6. Typical Chinese student responses that show understanding of the DP.

Contextual-recognition. In this study, students were expected to "recognize" the basic properties (CP, AP, DP) illustrated by the story problem solutions (Q4, Q5, Q9, and Q10). As indicated by Table 5, Chinese students performed the best in this type of task in a consistent manner (59%, 49%, 51%, 63%). This is in stark contrast to only 15% of the US students who recognized the CP and almost no students explicitly recognized the AP or the DP. In fact, there were about 37%, 58%, 71%, and 88% of US students who showed no understanding of CP+, AP+, AP×, and DP, respectively. An interesting observation lies in the differences between the US and Chinese students' partial understanding of these properties. At this understanding level, even though both Chinese and US students often partially described the pattern observed with the number sentences (e.g., "the parenthesis moved place," "the order of the parenthesis is changed"), across items, many Chinese students reasoned upon the two solutions based on the meaning of the story situations (see Figure 2, task 4). Note that Q9, the Mango problem, was a task taken from a US textbook (Ding, 2016). Chinese students' tendency of making sense of the numerical solutions based on contextual support was not observed with the US students including their responses to Q9. Below are common Chinese examples of this reasoning strategy.

> (*Q5, CP+*) I found that Xiaoming first figured out the number of books on the first and second bookshelves while Xiaofang first figured out the books on the second and third bookshelves. They both get the same answer.
>
> (*Q9, AP×*) I found that Xiaoming first figured out the total of six plates and then the total of 30 mangos; Xiaofaing first figured out that there were 10 mangos on each table and then a total of 30 mangos. They got the same answer.
>
> (*Q10, DP*) I found that Xiaoming first computed the (total) length of the playground and then the (total) width of the playground; Xiaofaing first compute

"length + width" of the playground, and then find two "length + width." Both answers are correct.

Student Understanding of the Basic Properties Over Time

To examine the progression of student learning, we analyzed student performance across grades. Being cognizant of the fact that the third and fourth grade students have inherently different mathematical ability levels, we analyzed the data from different angles. Figure 7 illustrates students' average scores for each property in the pre- and post-tests (10 points for each property; 30 points total). Matched pairs *t*-tests were conducted to examine the performance difference between pre- and post-tests in each grade and in each country. It was found that the Chinese third grade students did not have a difference in their pre- and post-tests of their total scores, $t(83) = -.82$, $p = .79$. This indicates that Chinese third graders did not have significant learning gains with the basic properties from the pre- to the post-tests. However, a matched pairs *t*-test did in fact show that Chinese fourth graders did perform differently on their pre- and post-tests, $t(82) = 7.6$, $p < 0.01$. This difference was found to be positive (post-test minus pre-test). For the US students, both the third and fourth graders scored significantly better on their post-tests than their pre-test, $t_{G3}(31) = 4.99$, $p_{G3} < 0.01$; $t_{G4}(64) = 4.54$, $p_{G4} < 0.01$. This indicates that the US third and fourth graders in this study both made progress over the course of corresponding grades.

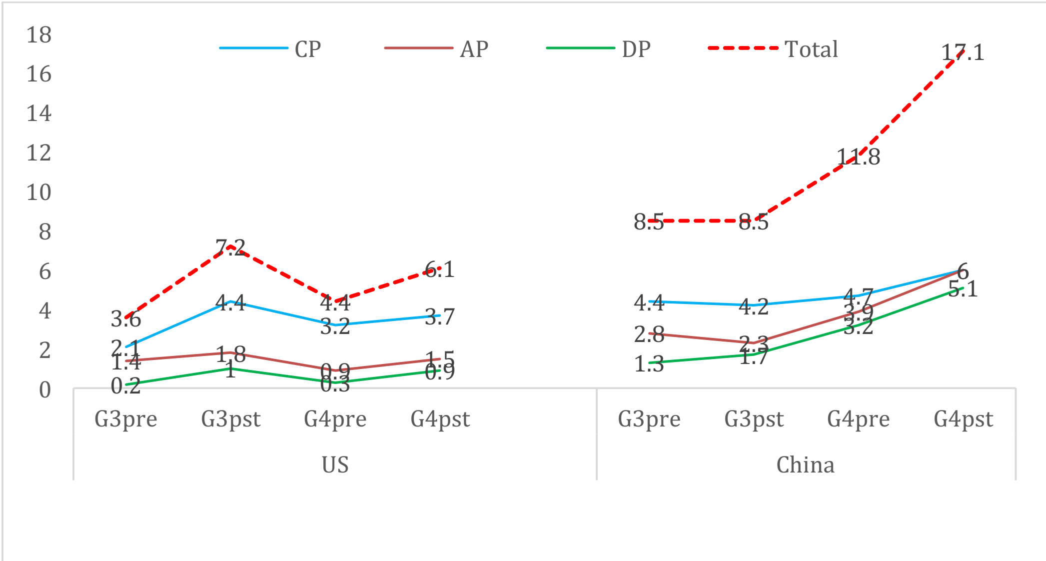


Figure 7. US and Chinese students' understanding of each property over time.

Independent *t*-tests per each property were conducted to compare Chinese and US students' performances at different grades. For the third graders, it was found that the Chinese students performed better than their US counterparts in each property on the pre-tests: $t_{CP\text{-}pre}(53.6) = 7.1$, $p < 0.01$; $t_{AP\text{-}pre}(71.5) = 4.7$, $p < 0.01$; ; $t_{DP\text{-}pre}(109.4) = 5.1$, $p < 0.01$. However, there was no difference between the US and Chinese students' understanding of the post-tests: $t_{CP\text{-}post}(51.7) = 0.62$, $p=0.54$; $t_{AP\text{-}post}(85.6) = -1.63$, $p = 0.11$; $t_{DP\text{-}post}(85.5) = -2.44$, $p = 0.02$. This indicates that by the end of third grade, the US students in this study had eliminated the cultural differences that existed between all properties. When it comes to the fourth grade, the US fourth graders' pre-test indicated a similar level of understanding as the third graders in their pre- but not post-tests. This may be due to sampling issue because the third and fourth graders were different students. This may also be due to fourth graders' forgetting about what they had learned in the third grade. Regardless of the interpretations, the fourth graders did increase understanding in their post-test. However, in all pre- and in all post-tests, Chinese fourth graders performed significantly better than the US counterparts in each property: $t_{CP\text{-}pre}(144.19) = 6.29$, $p < 0.01$; $t_{AP\text{-}pre}(142.87) = 12.5$, $p < 0.01$; $t_{DP\text{-}pre}(112.86) = 11.5$, $p < 0.01$. $t_{CP\text{-}pst}(140.02) = 8.3$, $p < 0.01$; $t_{AP\text{-}pst}(143.8) = 14.9$, $p < 0.01$; ; $t_{DP\text{-}pst}(127.1) = 13.6$, $p < 0.01$.

Finally, we inspected the progress that students made from grade 3 to grade 4 with regards to understanding these basic properties. Combining all three properties, the progress in the Chinese students was very evident after classroom instruction in the fourth grade (see Figure 7). In contrast, even though the US students formally learned all properties by grade 3 and relearned them during grade 4, they seemed to make little progress overall. In particular with the AP and the DP, the Chinese and US students' understanding gap dramatically increased from the beginning of grade 3 to the end of grade 4. The initial gaps for AP and DP were 1.4 and 1.1 points respectively (2.8-1.4=1.4 for AP; 1.3-0.2=1.1 for DP). However, by the end of the fourth grade, the gaps were increased to 4.5 and 4.2 points (6-1.5=4.5 for AP; 5.1-0.9 = 4.2 for DP). In other words, the US and Chinese participating students' understanding gap was magnified approximately four-fold once the Chinese students had formally learned all properties.

Discussion

Deep initial learning matters in students' development of structural understanding (Chi & VanLehn, 2012). As such, an emphasis on the understanding of basic properties of operations in elementary school is never overstated because these properties are foundations for future learning of more advanced topics such as algebra (Carpenter et al., 2003; CCSSI, 2010; Schifter et al., 2008; Wu, 2009). This study is one of the very first to systematically examine students' understanding of the basic properties from a cross-cultural perspective. Cross-cultural findings reveal that Chinese students demonstrated better understanding of all properties, even though their understanding does not reach an ideal level. These findings shed light on students' learning challenges in both countries, especially those of US students. Our findings also suggest possible ways to develop students' explicit understanding of these basic properties.

Student Difficulties in Understanding the Basic Properties

The cross-cultural difference between US and Chinese students' understanding of the basic properties does not appear to be as large as other algebraic topics such as the equal sign (Li, Ding, Capraro, & Capraro, 2008). On the one hand, this suggests the basic properties are a harder early algebraic topic to be grasped. On the other hand, this suggests there is room for students in both countries to improve their learning. In fact, in comparison with their Chinese counterparts, the US students performed worse in all properties except for the CP+. This finding confirms the literature that the CP+ is relatively easy and students in early grades use it intensively (Baroody et al., 1983, Slavit, 1998). However, when referring to students' responses to the AP tasks, we found that US students do not have a solid understanding of the CP because they tended to overgeneralize this property to the AP related tasks. In other words, many US students conflated the CP and the AP. The conflation is consistent with prior report about undergraduate students' confusion (Ding et al., 2013, Fletcher, 1972; Larsen, 2010; Tent, 2006). Our finding suggests that students' confusion between the CP and the AP may stem from their initial learning of these properties in elementary school.

In addition to the above difficulty, US students' understanding of the AP and the DP is disappointing. As reported, there were almost no US students who demonstrated full understanding of both properties. Many students viewed the AP as merely the use of parenthesis and the DP as breaking-down a factor. Even with partial understanding, while 94% of Chinese fourth graders could use the DP in the direction of $ab + ac = a(b+c)$ to perform a computation, only 6% of US students could do so. Note that such a use of DP is in a opposite direction from $a(b+c) = ab + ac$, and appears frequently in the Chinese but not the US textbooks (Ding & Li, 2010). Given that an understanding of DP in an opposite direction is critical for solving algebraic equations such as $x - 0.15x = 38.24$ (Koedinger et al., 2008), our finding about US students' lack of ability to use DP in an opposite direction calls for attention. In the current study, different from the US students, more than half of the Chinese students demonstrated full understanding once they had learned these properties in fourth grade. In fact, there were seven Chinese fourth graders who scored above 90% for all properties across all tasks. This indicates that all of the basic properties are learnable if taught appropriately. In contrast, even though US students are repeatedly taught these properties in both grades 3 and 4, almost none of them could obtain full understanding. This calls for examination of what happens inside US classrooms. Further, an investigation into how teachers may transform textbooks into classroom instruction seems warranted. In fact, our finding about the increased understanding gap between US and China by the end of the fourth grade confirms this pressing need.

Develop Student Understanding through varied Contexts

Students in this study demonstrated different understanding in different contexts. Our findings reveal that Chinese students did much better in contextual tasks than their US counterparts. In fact, many Chinese students (especially before their formal learning of the properties) could explain the meaning of each step of both solutions to make sense of them. This contextual-based implicit understanding will likely enable students to make sense of the abstract properties once they are formally introduced to students. Indeed, this is consistent with the Chinese textbook approach to instruction of these properties as reported by Ding and Li (2010). For instance, Chinese textbook lessons situate the initial teaching of the DP in a word problem context. Through solving a word problem in two different ways and making sense of both solutions, students are asked to compare the two number sentences and then pose additional similar examples. Based on these conversations, the curriculum then formally reveals the terminology and algebraic formula of the DP. This approach - from concrete to abstract and from specific to general – is consistent with the concreteness fading method (Goldstone & Son, 2005) that has been shown to effectively develop students' mathematical understanding (Fyfe, McNeil, & Borias, 2015; Fyfe, McNeil, Son, & Goldstone, 2014; McNeil & Fyfe, 2012). In our study, Chinese students' responses to the contextual items are in nature different from US students who tended to reason upon computation results rather than quantitative relationships. In fact, the use of concrete support to teach the basic properties has been emphasized by NRC (2001) with suggested methods for teaching each property. Possibly, a better path for developing US students' initial understanding of the basic properties is to better use "concrete contexts" for the purpose of sense-making.

Of course, students' understanding of the basic properties demand varied contexts including both contextual and non-contextual ones. In this study, students in both countries demonstrated inconsistent understanding across tasks. For instance, even though many Chinese students can use the terminologies of the basic properties in contextual tasks, they did not perform equally well with non-contextual tasks, especially the one from the US textbooks (explaining the strategies of 8×6). In fact, most Chinese students were able to apply the basic properties to solve problems with larger numbers (e.g., 102×7) and some even explicitly pointed out the underlying properties. Ironically, when facing an easier task with smaller numbers (8×6), more Chinese students failed to explain the involved properties. This may be due to the fact that Chinese students in second grade are expected to master the basic multiplication facts like 8×6 and they do not use "breaking up a factor" to find the answer in later grades. However, their lack of explanation of the given computation strategies may reflect a lack of completeness in their understanding about the DP. On the one hand, this confirms the importance of assessing students' understanding using a variety of tasks. On the other hand, to develop students' understanding, varied contexts and different types of tasks should be used. In fact, the US students performed poorly in almost all types of contexts, which indicates their lack of awareness and understanding of the basic properties. The US students were particularly weak in applying the properties to solve problems efficiently, which is in sharp contrast to the Chinese students. The cross-cultural differences indicate that students' difficulties with certain types of problems are not universal. Likely, this understanding gap stems from the different types of problems and contexts involved in classroom instruction in each country.

Develop Explicit Understanding through Spaced Learning

In this study, the US students did not perform progressively better in understanding the basic properties over grades, which is quite different from the observed Chinese pattern. Are these different learning patterns caused by different curriculum designs? For instance, the Chinese students learn the properties informally in grade 3 while the US students have repeatedly learned both properties (AP, DP) in grades 3 and 4. In fact, even though the US students have formally learned the properties in two different years, their explicit understanding of the AP and the DP by the end of grade 4 was quite low. Since explicit understanding is a necessary level of understanding that enables transfer of learning to new context (Greeno & Riley, 1987), US students' systematic lack of explicit understanding of these basic properties might be a fundamental reason of low performance in many international studies. In addition, we observed that many US third and fourth graders still called the CP a "turn-around fact." Even though we coded this as "full understanding," we question why these students do not know or use the actual terminology of the CP. It is understandable that when the CP is initially introduced to US first graders that the vivid metaphor of "turn-around" may be helpful however, why haven't students been introduced to the formal terminology of this property over time? Lacking the common terminology about this basic property, if not harmful, may not be helpful for students' future learning of more advanced mathematics. The above questions call for future exploration of what is happening inside US classrooms when teaching the basic properties.

Encouragingly, we notice that many US students possess implicit understanding of these properties. Even though this is demonstrated by often unclear and even inaccurate descriptions, such understanding can serve as a springboard for developing students' explicit and full understanding (Ding & Auxter, 2017). In fact, prior research argues for an implicit to an explicit learning progression (Fyfe et al., 2015; Greeno & Riley, 1987; Pirie & Kieren, 1994; Sfard & Linchevski, 1994). However, students in this study demonstrated difficulties with transforming implicit understanding of the properties into explicit understanding. Many US third and fourth graders only focused on superficial features such as “moving a parenthesis” or “breaking down a factor.” This seems to be consistent with Ding and Li’s (2010) textbook report where US textbooks presented many strategies but failed to help the teacher and students perceive and comprehend the underlying properties. In fact, US textbooks sometimes treat the properties as one of the computation strategies. Once such intended curriculum is delivered in classrooms, students might only learn procedures. This may explain why US students attended to strategies, but not the basic properties, even though they had repeatedly learned them across multiple grades.

In this study, the Chinese students’ learning pattern is quite different. Although there is room for improvement, many Chinese students have shifted their understanding of the basic properties from implicit to explicit by the end of fourth grade. This suggests an exploration of Chinese formal lessons on the basic properties. Of course, Chinese students’ underperformance on the easy task 8×6 calls for re-thinking of how to deepen understanding by linking back to familiar knowledge of basic multiplication facts. In China, students learn and master these basic facts like 8×6 in second grade and during initial learning students are guided to use the DP-based strategies (similar to Q8); yet, the undergirding properties are not expected to be made explicit at that time. Therefore, when students formally learn the DP in the fourth grade, in addition to asking students to apply the DP to solve problems with large numbers, bringing their attention back to familiar knowledge may also be helpful for deepening their understanding. An exploration of US and Chinese lessons on both implicit and explicit teaching of the basic properties would most likely be informative for detecting the causes of students' different learning patterns. More specifically, future studies may explore what are the important instructional factors that contribute to the shift of student thinking from implicit to explicit? Furthermore, what exactly does spacing learning over time mean if one expects students to shift thinking from implicit to explicit?

Conclusion, Limitation, and Implication. This study compares US and Chinese third and fourth graders' understanding of the basic properties. Our findings call for an increased attention to the learning and teaching of the AP and the DP in both countries. It seems that explicit classroom instruction involving varied contexts, especially the contextual ones, is critical. Of course, we are cognizant of the limitations in this study. First, as previously acknowledged, the third and fourth graders are different groups of students which may affect the observed learning patterns over time. Second, the instruments were self-designed. Even though the various tasks had been piloted, reviewed and revised, perhaps some tasks may be further improved. Third, we only assessed students' understanding through paper-and-pencil tests and did not conduct interviews to confirm our interpretations. Nevertheless, our cross-cultural findings reveal different learning patterns that may be otherwise neglected. These findings have practical implications for classroom practices, teacher education, and textbook design. For instance, textbooks and classroom instruction should treat the basic properties as mathematics principles that go beyond computation strategies, help students develop meaningful initial learning through contextual support, and use varied contexts to prompt students' shift of understanding from implicit to explicit over time. With joint efforts on improving current teaching and learning environments, students can be expected to develop better understanding of the basic properties.

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Appendix 1. The Full Instrument

Name ____________ Grade_______ School ___________ Teacher __________Date

1. If you know 7 + 5= 12 does that help you solve 5 + 7? Why?

2. When solving 3 + 8, Mary's strategy was to start with 8 and count 9, 10, 11. She ended up with her answer as 11. Is this strategy correct? Why?

3. Please use efficient strategies to solve. Show your strategy and explain why it works.

(a) 2 + 7 + 8	(b) (7 + 19) + 1	(c) 2 + (98 + 17)
Explain:	*Explain:*	*Explain:*

4. There are 8 boys and 5 girls in a swimming pool. How many children are there altogether?

John solved it with: 8 + 5	*Mary solved it with*: 5 + 8

Both are correct. Compare the two strategies, what do you find?

5. There is a bookcase with three shelves. The first shelf has 7 books, the second shelf has 8 books, and the third shelf has 5 books. How many books are there in all?

John solved it with: (7 + 8) + 5	*Mary solved it with:* 7 + (8 + 5)

Both are correct. Compare the two strategies, what do you find?

6. When solving 3 × 28, Mary wrote it as $\begin{array}{r} 28 \\ \times\ \ 3 \\ \hline \end{array}$. Is this order correct? Why?

7. Please use efficient strategies to solve. Show your strategy and explain why it works.

(a) (3 × 25) × 4	(b) 102 × 7	(c) 98 × 7 + 2 × 7
Explain:	*Explain:*	*Explain:*

8. To solve 8 × 6,

(a) Mary thought:	(b) John thought:	(c) Kate thought:
3 × 6 = 18, 5 × 6 = 30, 18 + 30 = 48	10 × 6 = 60, 2 × 6 =12, 60 –12 = 48	Since 6 × 8 = 48, 8 × 6 = 48

Please explain why each strategy works.

9. Mr. Levin's students are tasting foods grown in rainforests. He put 5 pieces of mango on each plate and put 2 plates on each table. There are 3 tables. How many pieces of mango are there?

John solved it with: (3 × 2) × 5	*Mary solved it with:* 3 × (2 × 5)

Both are correct. Compare the two strategies, what do you find?

10. The length of a rectangular playground is 118 m and the width is 82 m. What is the perimeter?

John solved it with : 2 × 118 + 2 × 82	*Mary solved it with:* 2 × (118 + 82)

Both are correct. Compare the two strategies, what do you find?