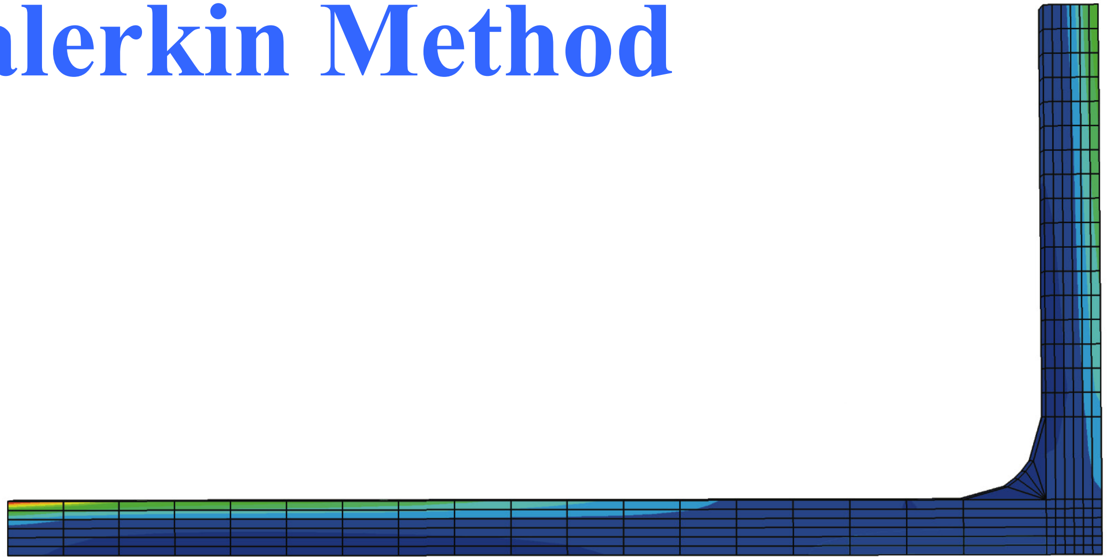
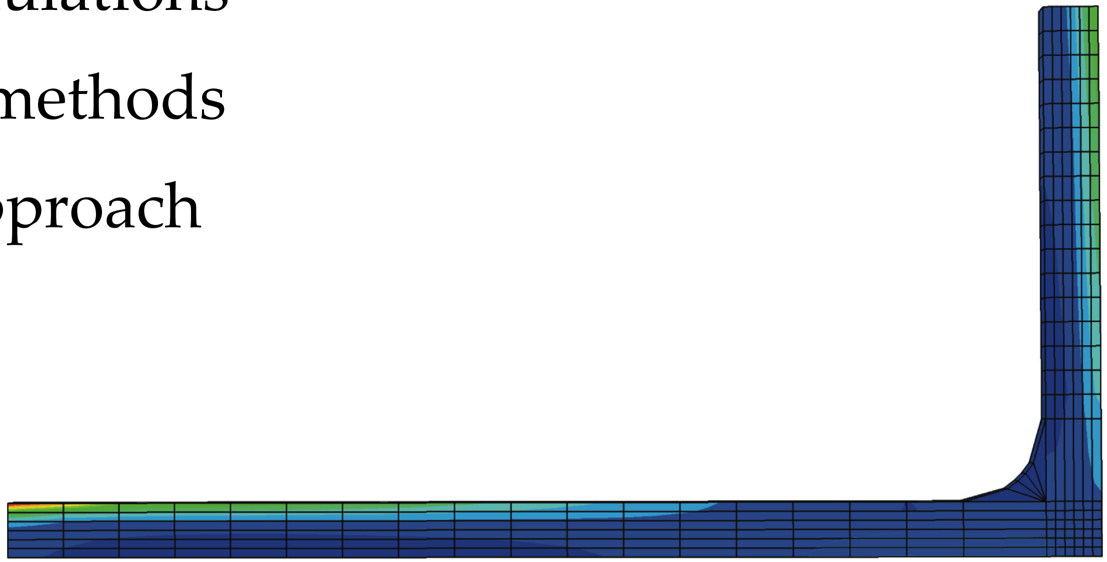
Chapter 3

Variational Formulation & the Galerkin Method



Today's Lecture Contents:

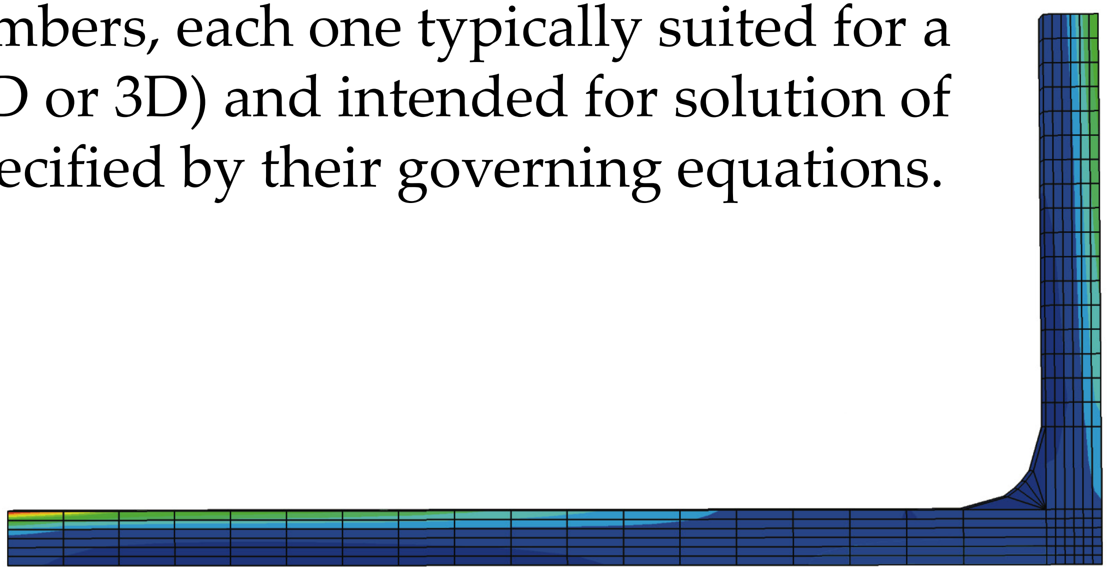
- Introduction
- Differential formulation
- Principle of Virtual Work
- Variational formulations
- Approximative methods
- The Galerkin Approach



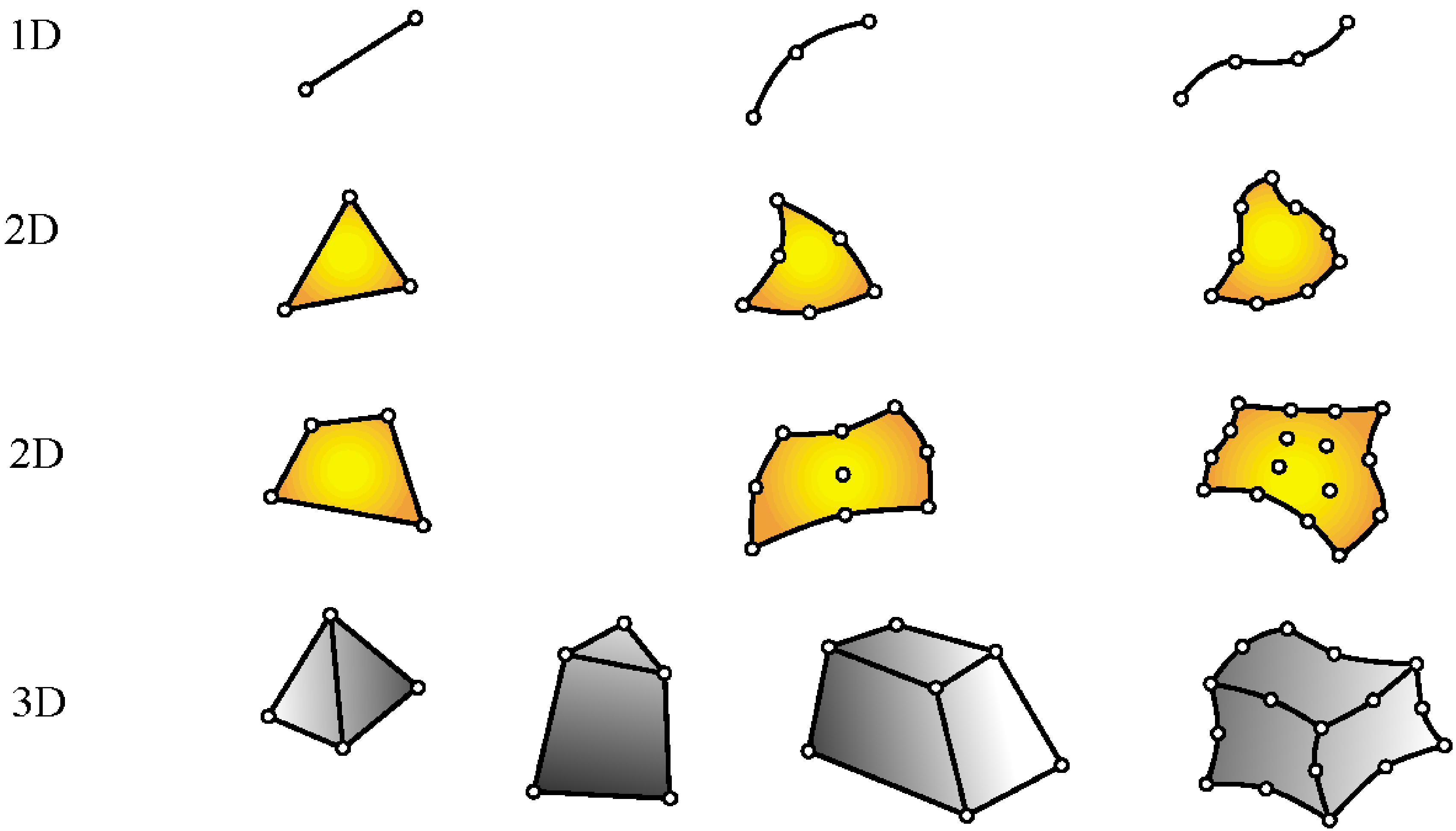
FE across different dimensions

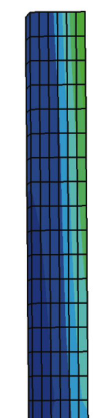
The lectures so far have deal with the Structural Approach to Finite Elements, namely the Direct Stiffness Method. These were elements serving specific loading conditions (truss→ axial loads / beam → bending)

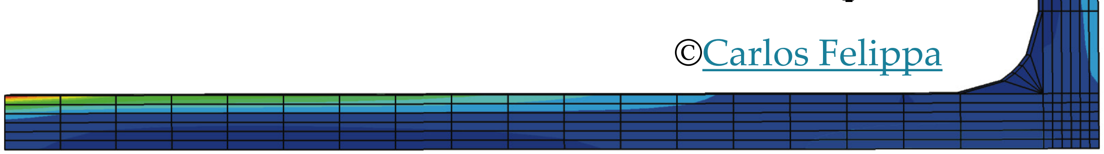
However, the Finite Elements family includes further and usually more generic members, each one typically suited for a particular domain (1D, 2D or 3D) and intended for solution of problems, as these are specified by their governing equations.



FE across different dimensions

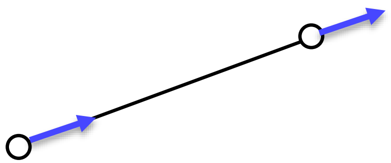




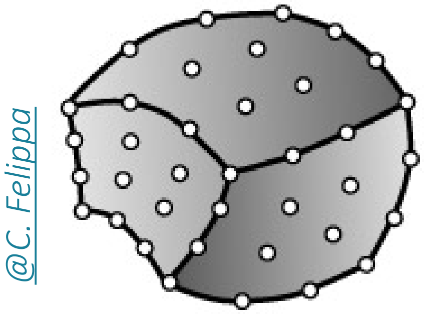


FE across different dimensions

Different Elements demonstrate a different degree of complexity

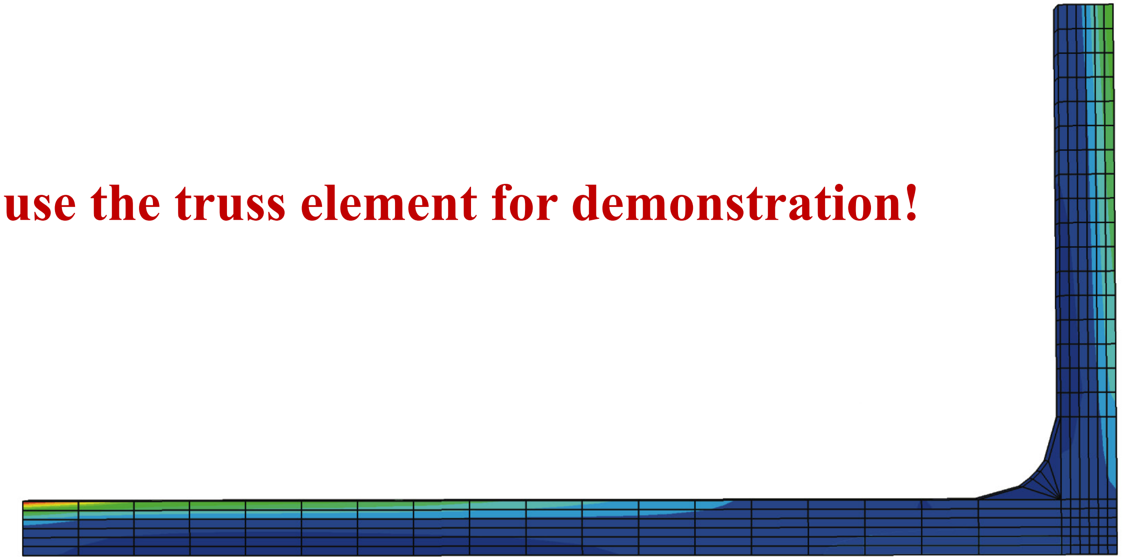


Truss element
2nodes, 2DOFs



Tri-cubic brick element
64nodes, 192DOFs

No wonder we will here use the truss element for demonstration!

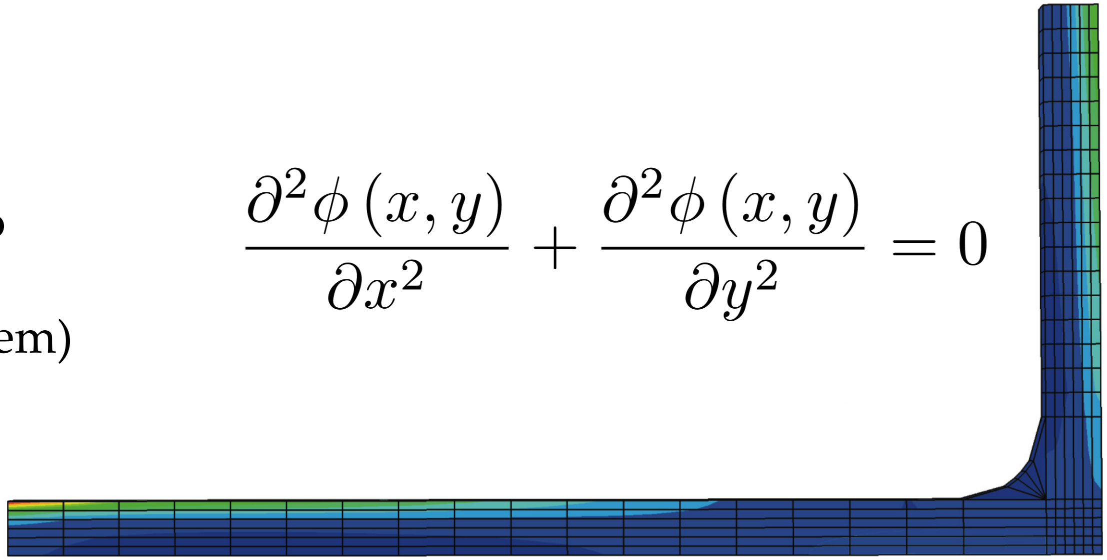


The differential form of physical processes

The governing laws of physical processes are usually expressed in a **differential form:**

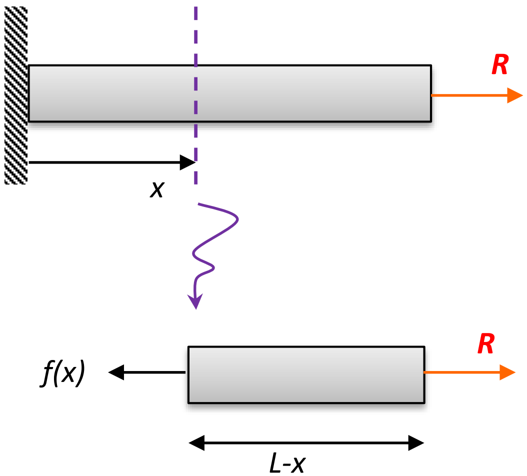
- The axially loaded bar equation: $$EA\frac{d^2u}{dx^2} = -F(x)$$
- The isotropic slab equation: $$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$
- The Laplace equation in two dimensions: $$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0$$

(e.g. the heat conduction problem)



The differential form of physical processes

1D: The axially loaded bar example.
Consider a bar loaded with constant end load *R*.



Given: Length L, Section Area A, Young's modulus E
Find: stresses and deformations.

Assumptions:
The cross-section of the bar does not change after loading.
The material is linear elastic, isotropic, and homogeneous.
The load is centric.
End-effects are not of interest to us.

Strength of Materials Approach

Equilibrium equation

$$f(x) = R \Rightarrow \sigma(x) = \frac{R}{A}$$

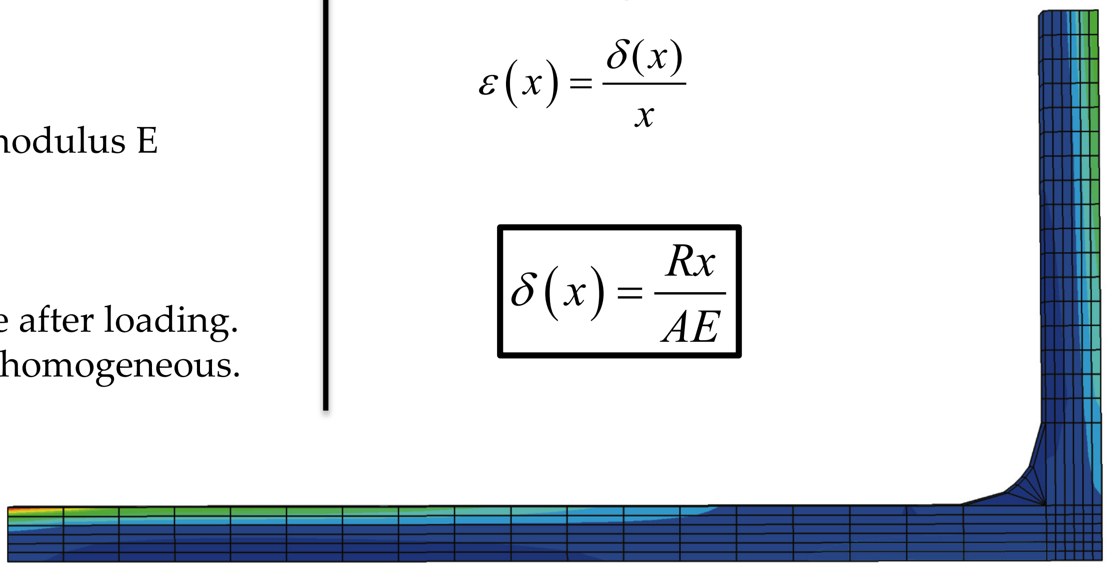
Constitutive equation (Hooke's Law)

$$\varepsilon(x) = \frac{\sigma(x)}{E} = \frac{R}{AE}$$

Kinematics equation

$$\varepsilon(x) = \frac{\delta(x)}{x}$$

$$\boxed{\delta(x) = \frac{Rx}{AE}}$$



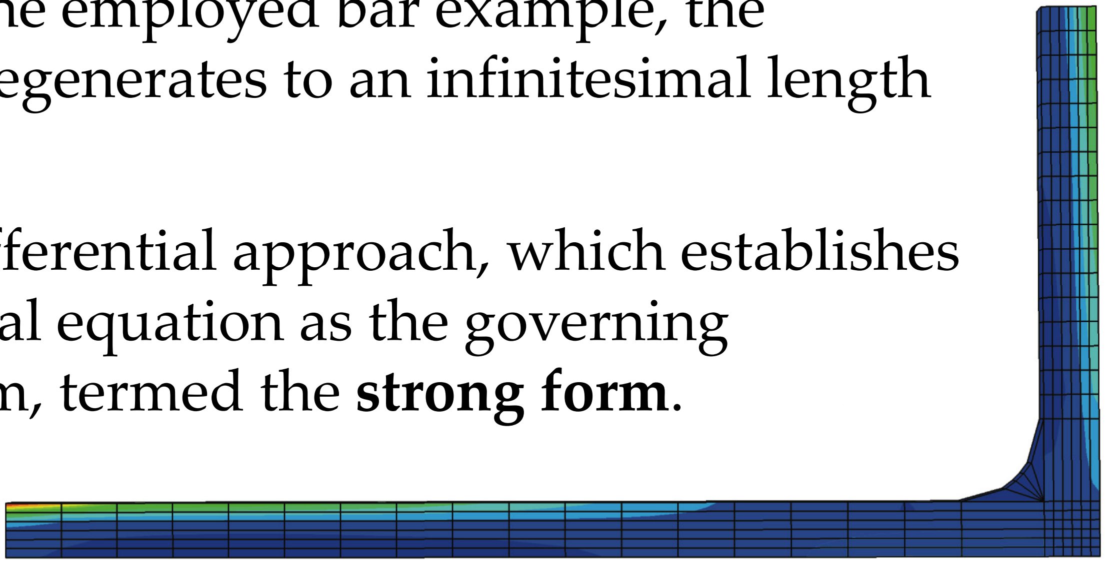
The differential form of physical processes

The Mechanics of Materials approach exemplified in the previous slide, is an approach that is not easily generalizable.

Instead, we would like to follow an approach, which initiates from a generic infinitesimal volume of our given structure.

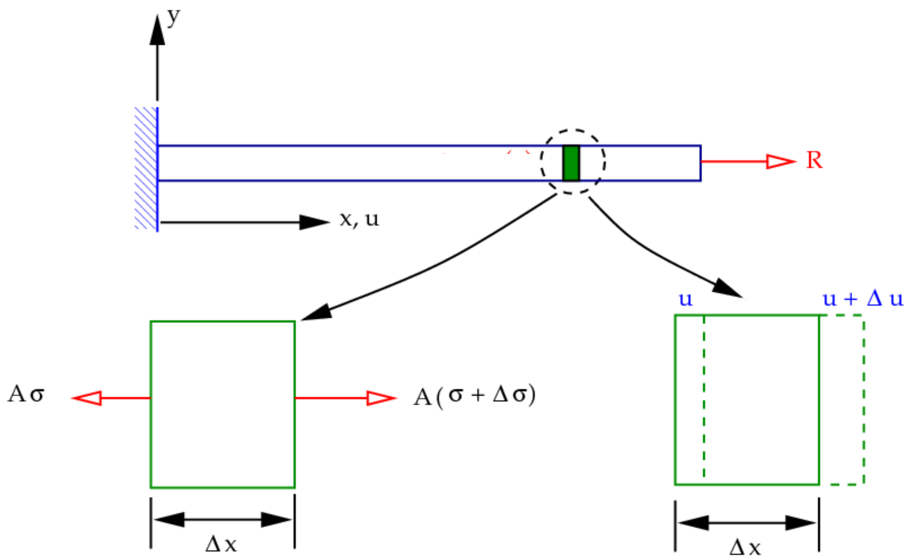
For the 1D case, as in the employed bar example, the infinitesimal volume degenerates to an infinitesimal length Δx (see next slide)

This is the so-called differential approach, which establishes a continuous differential equation as the governing equation of the problem, termed the **strong form**.



The differential form of physical processes

1D: The axially loaded bar example.
Consider and infinitesimal element of the bar:



Given: Length L, Section Area A, Young's modulus E
Find: stresses and deformations.

Assumptions:
The cross-section of the bar does not change after loading.
The material is linear elastic, isotropic, and homogeneous.
The load is centric.
End-effects are not of interest to us.

The Differential Approach

Equilibrium equation

$$A\sigma = A(\sigma + \Delta\sigma) \Rightarrow A \underbrace{\lim}_{\Delta x \to 0} \frac{\Delta\sigma}{\Delta x} = 0 \Rightarrow A\frac{d\sigma}{dx} = 0$$

Constitutive equation (Hooke's Law)

$$\sigma = E\varepsilon$$

Kinematics equation

$$\varepsilon = \frac{du}{dx}$$

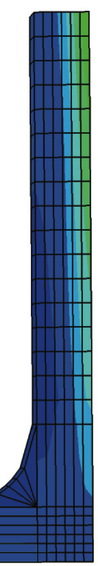
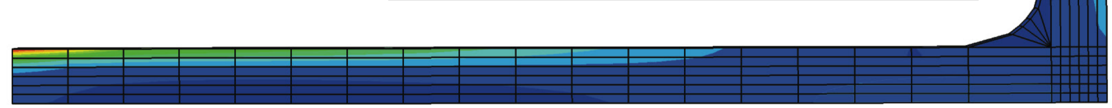
$$AE\frac{d^2u}{dx^2} = 0 \qquad \textit{Strong Form}$$

Boundary Conditions (BC)

$$u(0) = 0 \qquad \textit{Essential BC}$$

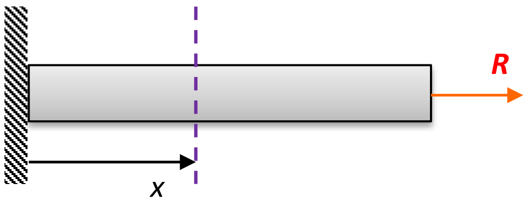
$$\sigma(L) = 0 \Rightarrow$$

$$AE\left.\frac{du}{dx}\right|_{x=L} = R \qquad \textit{Natural BC}$$

The differential form of physical processes

1D: The axially loaded bar example.



The Differential Approach

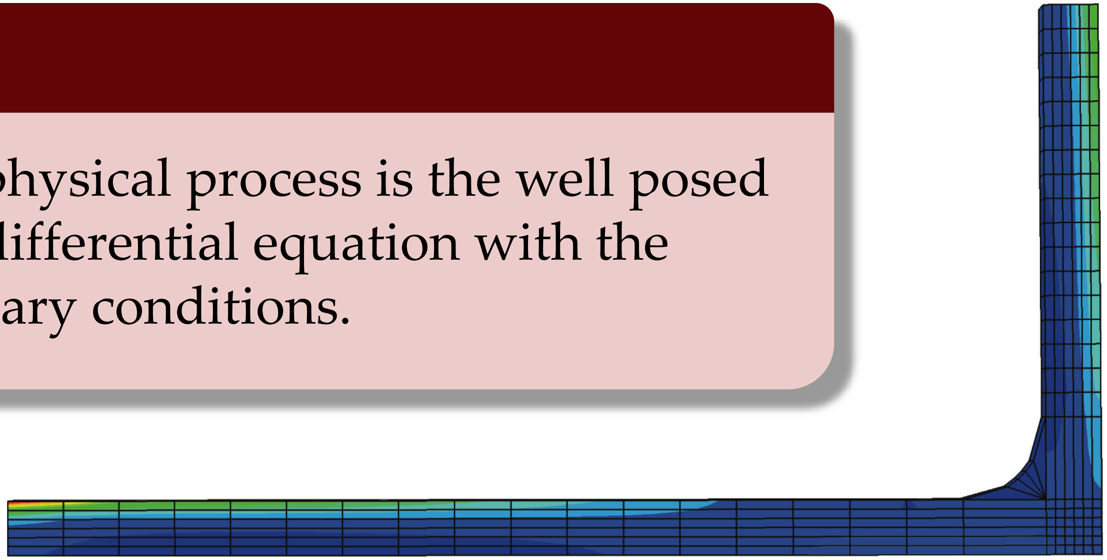
$$AE\frac{d^2u}{dx^2} = 0 \qquad \textit{Strong Form}$$

$$u(0) = 0 \qquad \textit{Essential BC}$$

$$AE\left.\frac{du}{dx}\right|_{x=L} = R \qquad \textit{Natural BC}$$

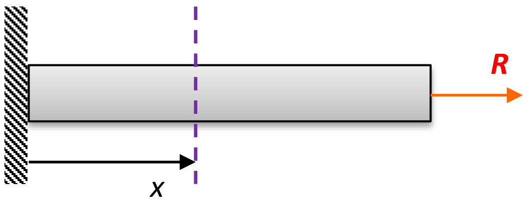
Definition

The **strong form** of a physical process is the well posed set of the underlying differential equation with the accompanying boundary conditions.



The differential form of physical processes

1D: The axially loaded bar example.



The Differential Approach

$$AE\frac{d^2u}{dx^2}=0 \qquad \textit{Strong Form}$$

$$u(0)=0 \qquad \textit{Essential BC}$$

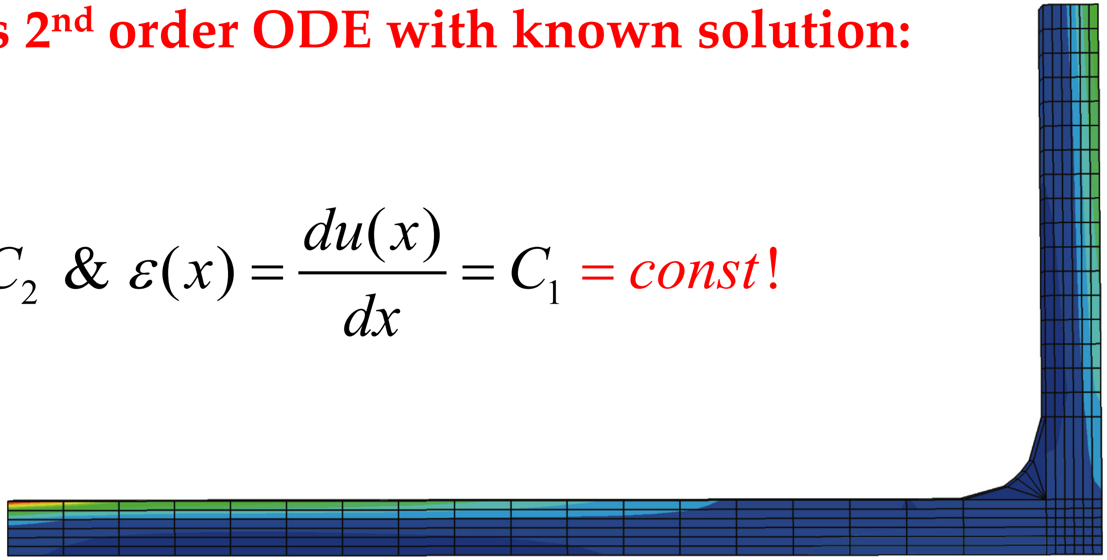
$$AE\left.\frac{du}{dx}\right|_{x=L}=R \qquad \textit{Natural BC}$$

Let's attempt to find the solution to this problem:

❖ **This is a homogeneous 2nd order ODE with known solution:**

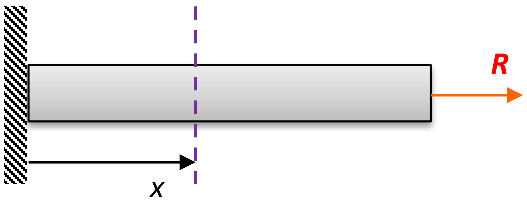
Analytical Solution:

$$u(x)=u_h=C_1x+C_2 \;\; \& \;\; \varepsilon(x)=\frac{du(x)}{dx}=C_1 = const!$$



The differential form of physical processes

1D: The axially loaded bar example.



The Differential Approach

$$AE\frac{d^2u}{dx^2}=0 \qquad \textit{Strong Form}$$

$$u(0)=0 \qquad \textit{Essential BC}$$

$$AE\left.\frac{du}{dx}\right|_{x=L}=R \qquad \textit{Natural BC}$$

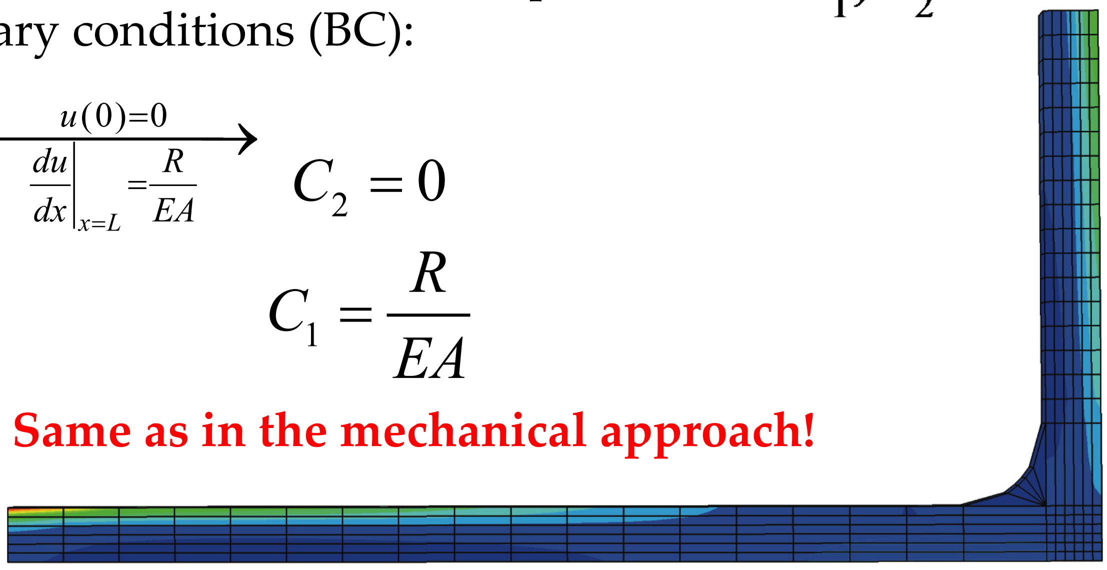
Analytical Solution:

To fully define the solution (i.e., to evaluate the values of parameters (C_1, C_2) we have to use the given boundary conditions (BC):

$$u(x)=u_h=C_1x+C_2 \xrightarrow[\left.\frac{du}{dx}\right|_{x=L}=\frac{R}{EA}]{u(0)=0} \quad C_2=0, \quad C_1=\frac{R}{EA}$$

$$\Rightarrow u(x)=\frac{Rx}{AE}$$

❖ Same as in the mechanical approach!

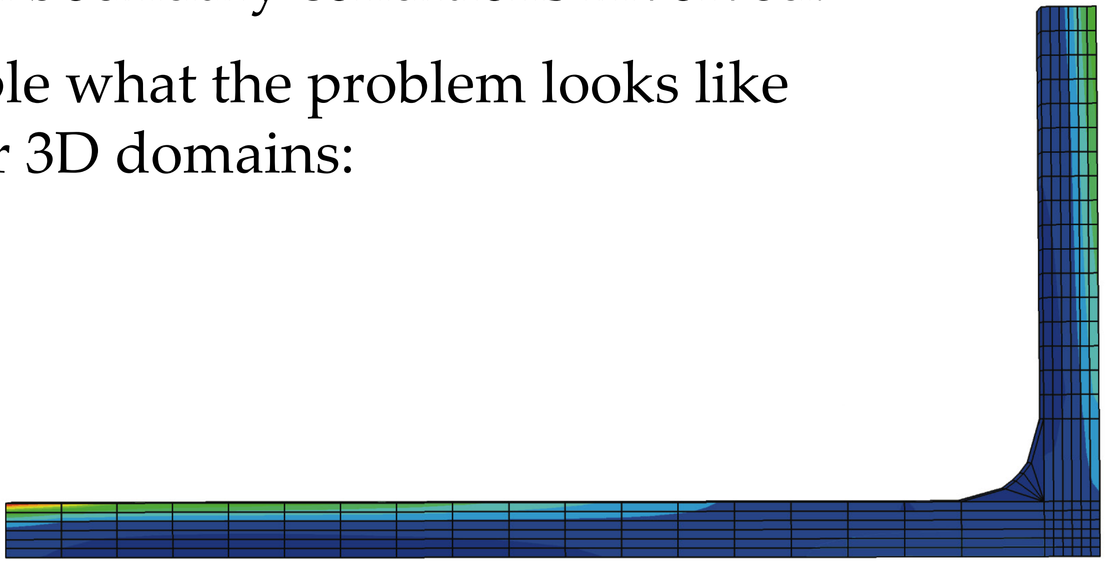


The differential form of physical processes

However, determination of the problem solution via an analytical approach is not always straightforward nor feasible.

Beyond the very simple example employed herein, actual problems are far more complex, with more members, and more complex loads and boundary conditions involved.

Let’s examine for example what the problem looks like when we go to the 2D or 3D domains:



The Strong form – 2D case

A generic expression of the **two-dimensional strong form** is:

$$A\left(x,y\right)\frac{\partial^2 u}{\partial x^2} + 2B\left(x,y\right)\frac{\partial^2 u}{\partial x \partial y} + C\left(x,y\right)\frac{\partial^2 u}{\partial y^2} = \phi\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial y}{\partial x}\right)$$

and a generic expression of the accompanying set of boundary conditions:

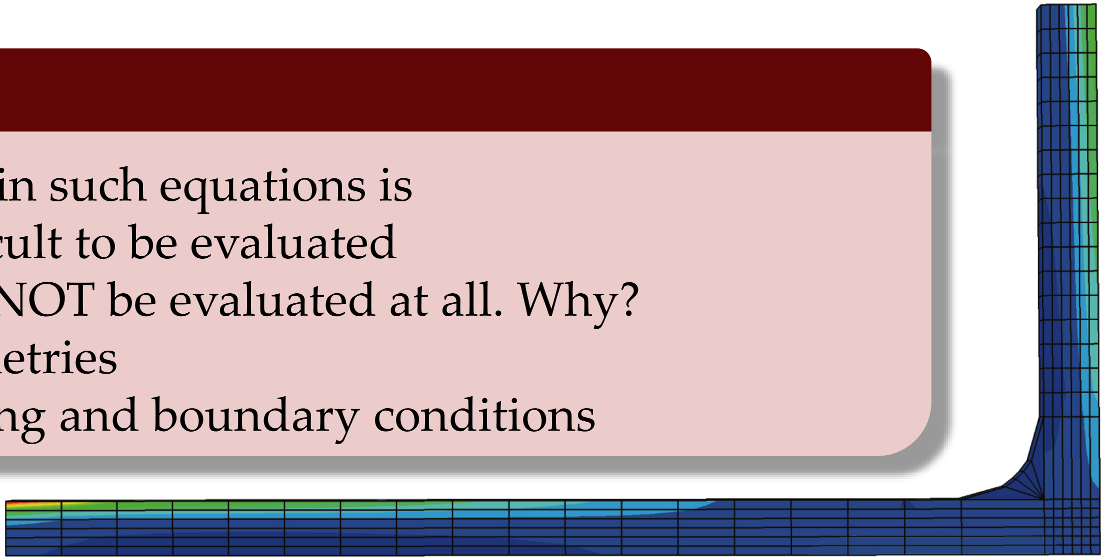
$$u\left(x_0,y_0\right) = 0$$ ***: Essential or Dirichlet BCs***

$$\frac{\partial u}{\partial x} = \dot{u}_0$$ ***: Natural or von Neumann BCs***

Disadvantages

The analytical solution in such equations is

i. In many cases difficult to be evaluated
ii. In most cases CANNOT be evaluated at all. Why?
- Complex geometries
- Complex loading and boundary conditions



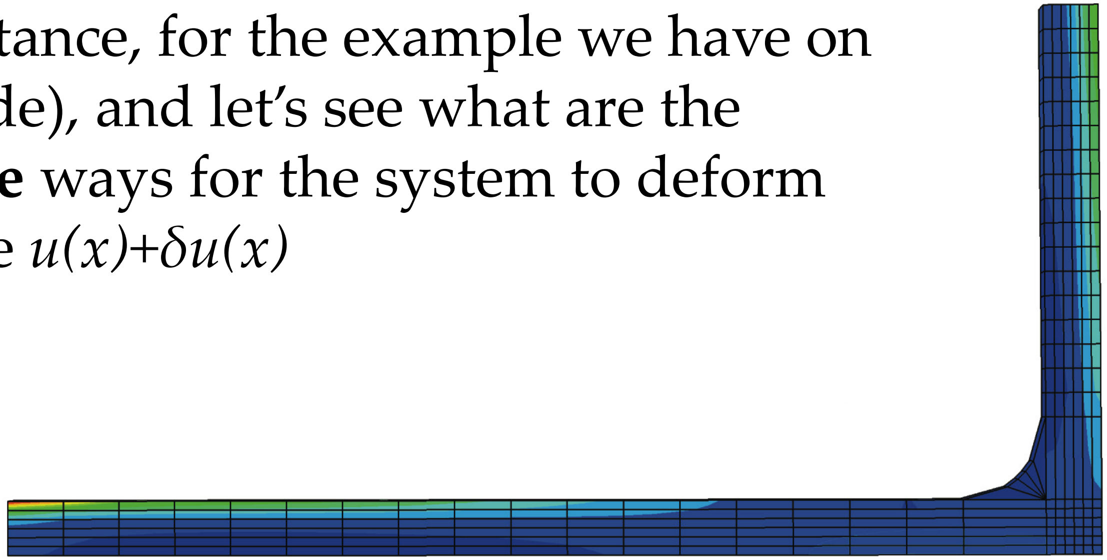
The differential form of physical processes

Solution by approximation

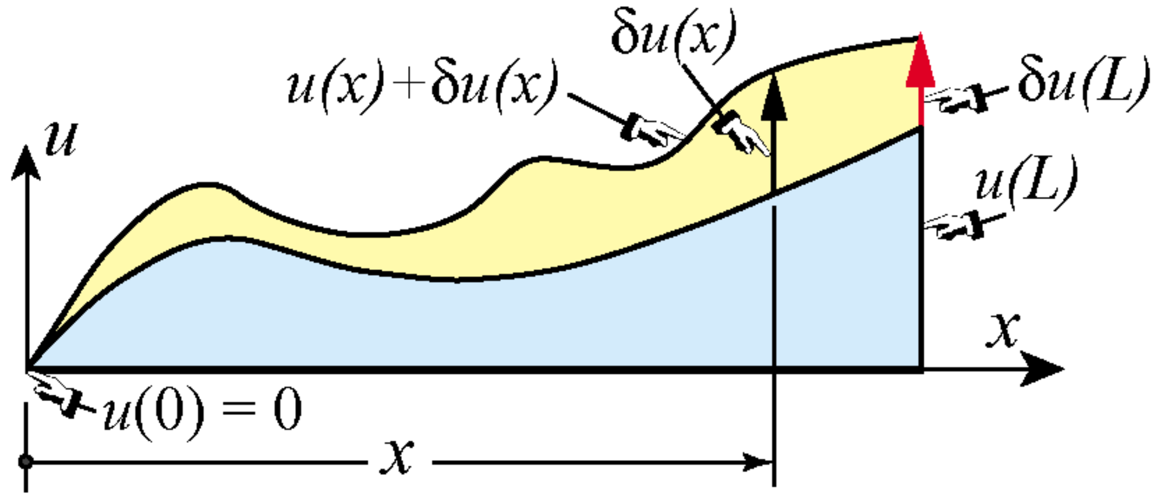
Instead of trying to solve the problem analytically, we would like to do it in an approximate, i.e., numerical way.

To this end, let us first consider what are the possible ways in which the system is allowed to deform.

Let's consider this for instance, for the example we have on the bar problem (next slide), and let's see what are the **kinematically admissible** ways for the system to deform from a state $u(x)$ to a state $u(x)+\delta u(x)$



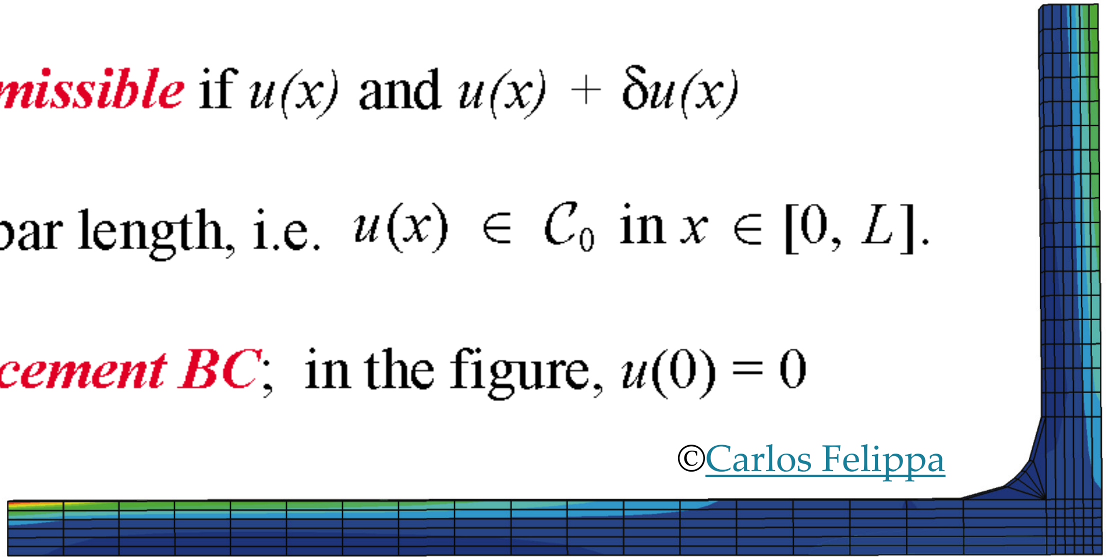
Admissible Displacements δu(x)



$\delta u(x)$ is ***kinematically admissible*** if $u(x)$ and $u(x) + \delta u(x)$

(i) are ***continuous*** over bar length, i.e. $u(x) \in C_0$ in $x \in [0, L]$.

(ii) ***satisfy exactly displacement BC***; in the figure, $u(0) = 0$



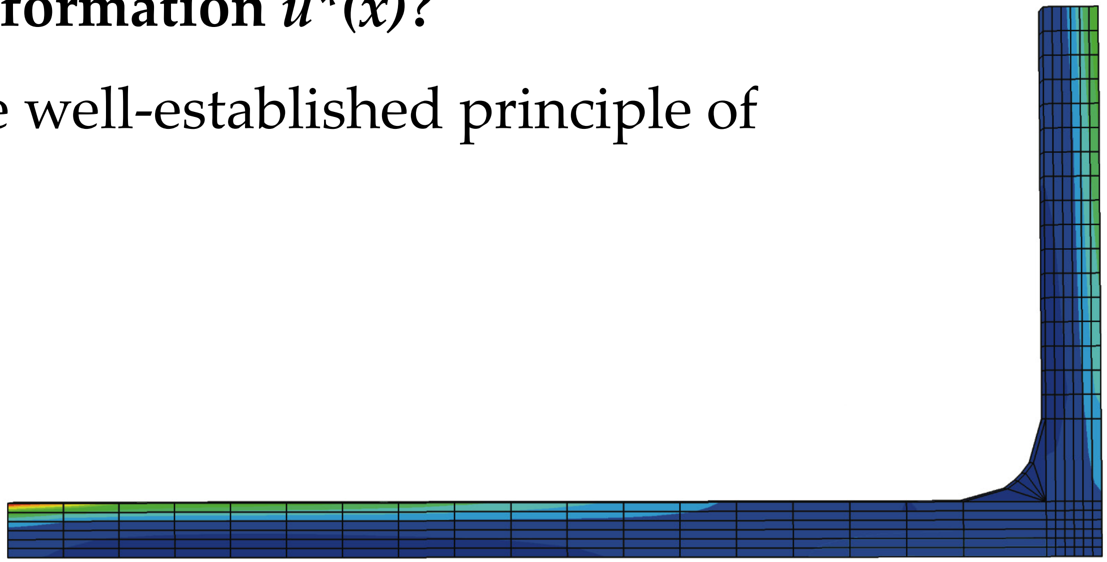
The differential form of physical processes

Solution by approximation

All **kinematically admissible** deformations are candidates for our system to deform. However, only one of these possible paths $u^*(x)$ will be the true one under a given set of loads.

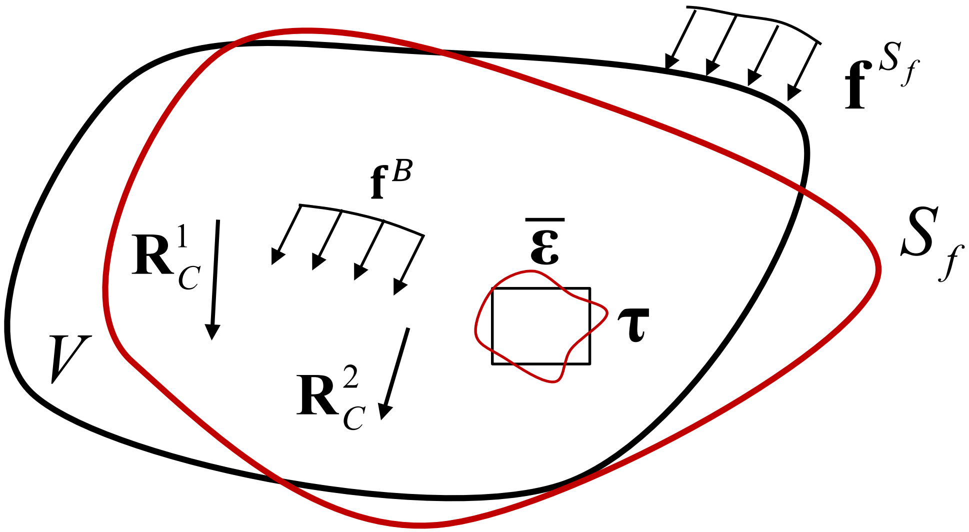
Which one is the true deformation $u^*(x)$?

It is the one satisfying the well-established principle of virtual work



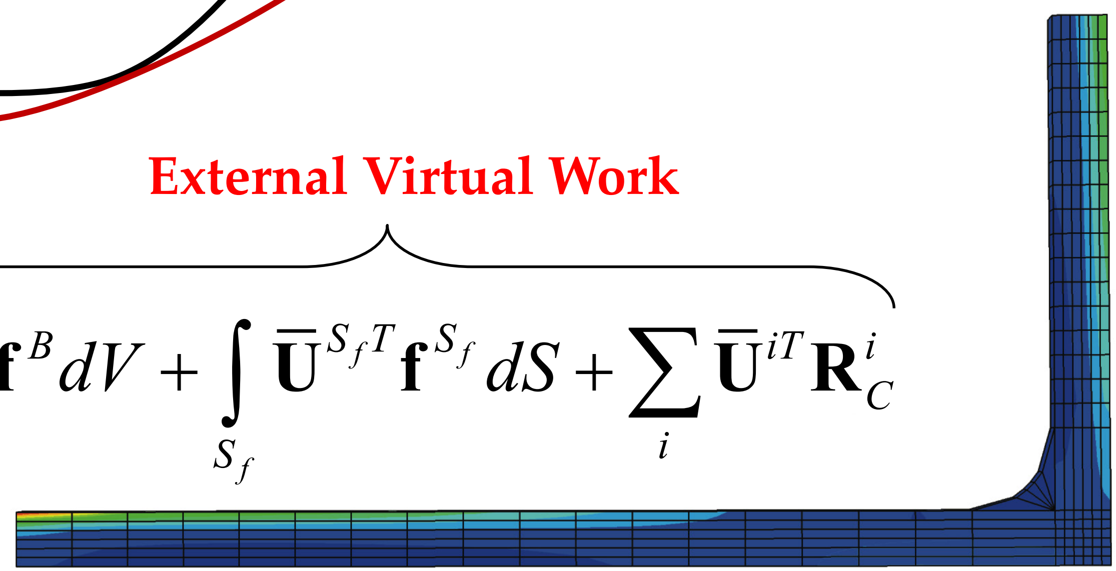
Principle of Virtual Work

The virtual work of a system of equilibrium forces vanishes when compatible virtual displacements *δu(x)* are imposed:



Internal Virtual Work — External Virtual Work

$$\underbrace{\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau}\, dV}_{\text{Internal Virtual Work}} = \underbrace{\int_V \bar{\mathbf{U}}^T \mathbf{f}^B\, dV + \int_{S_f} \bar{\mathbf{U}}^{S_f T} \mathbf{f}^{S_f}\, dS + \sum_i \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i}_{\text{External Virtual Work}}$$



Principle of Virtual Work

The virtual work of a system of equilibrium forces vanishes when compatible virtual displacements $\delta u(x)$ are imposed:

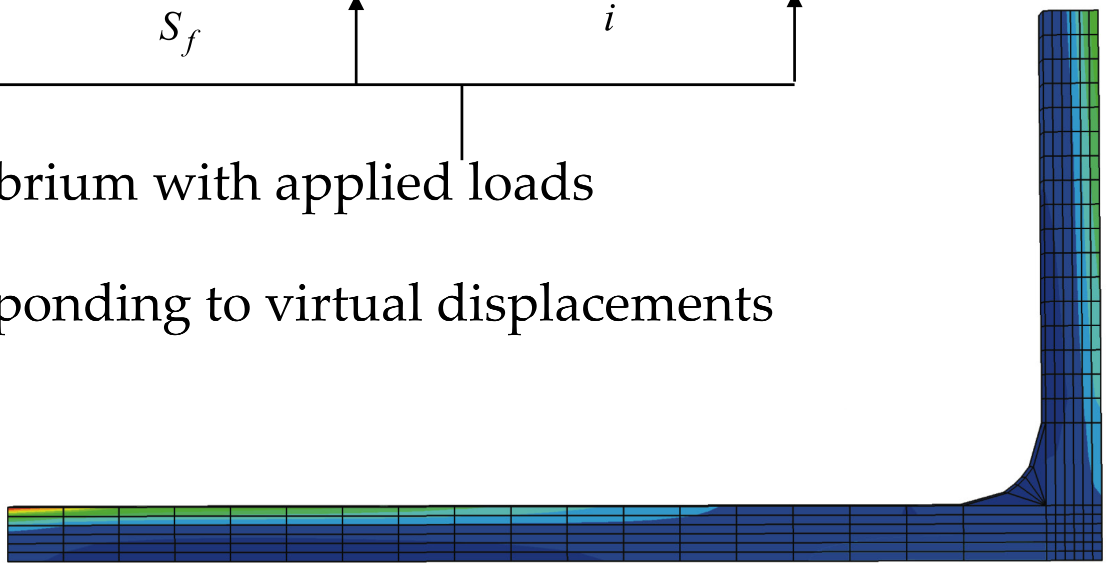
Internal Virtual Work

External Virtual Work

$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau}\, dV = \int_V \bar{\mathbf{U}}^T \mathbf{f}^B dV + \int_{S_f} \bar{\mathbf{U}}^{S_f T} \mathbf{f}^{S_f} dS + \sum_i \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i$$

Stresses in equilibrium with applied loads

Virtual strains corresponding to virtual displacements



The Minimum Potential Energy (MPE) Principle

For elastic systems subject to conservative forces (which is the case of systems we are dealing with), the principle of Virtual Work is equivalent to **principle of minimum total potential energy** (MPE).

The MPE principle states that the actual displacement solution $u^*(x)$, out of possible trial solutions, that satisfies the governing equations is the one which renders the Total Potential Energy functional Π stationary:

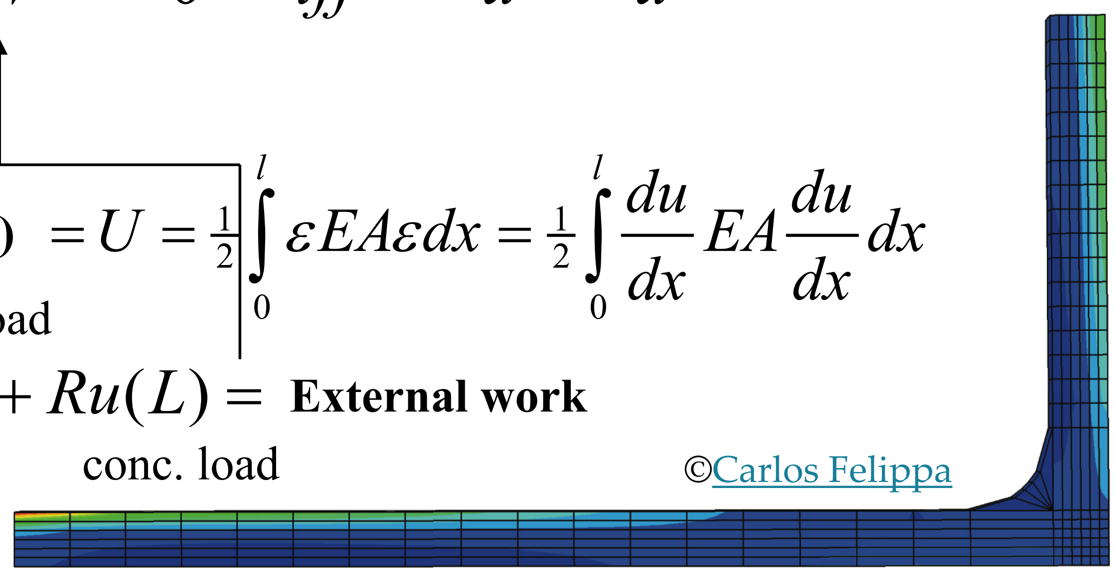
$$\delta\Pi = \delta U - \delta W = 0 \quad \textit{iff} \quad u = u^*$$

Internal energy (= strain energy) $= U = \frac{1}{2}\int_0^l \varepsilon EA \varepsilon \, dx = \frac{1}{2}\int_0^l \frac{du}{dx} EA \frac{du}{dx} \, dx$

distr. load

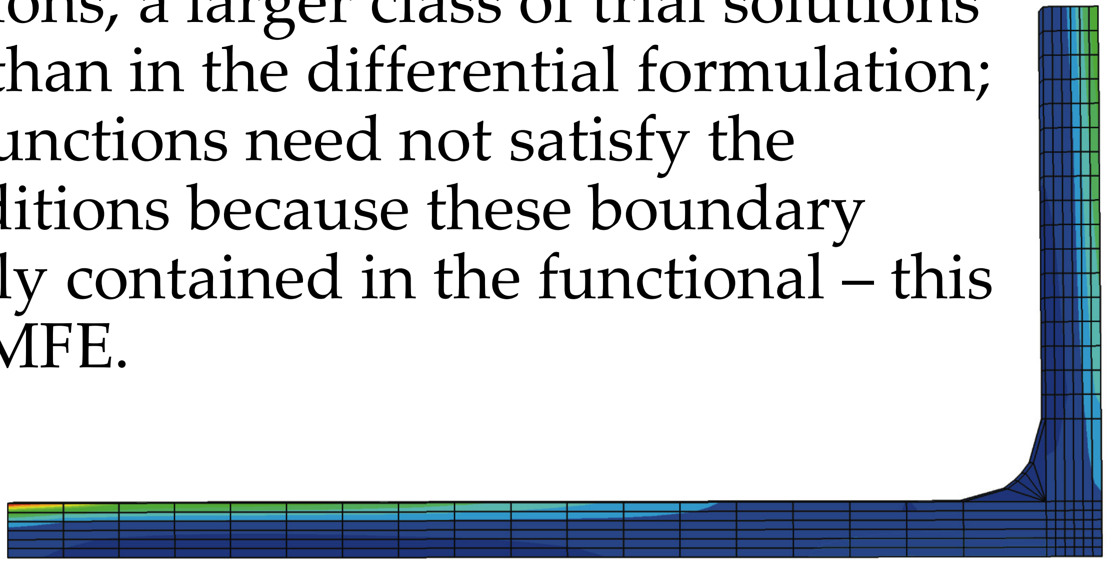
$$W = \int_0^l q u \, dx + R u(L) =$$ **External work**

conc. load



Variational Formulation

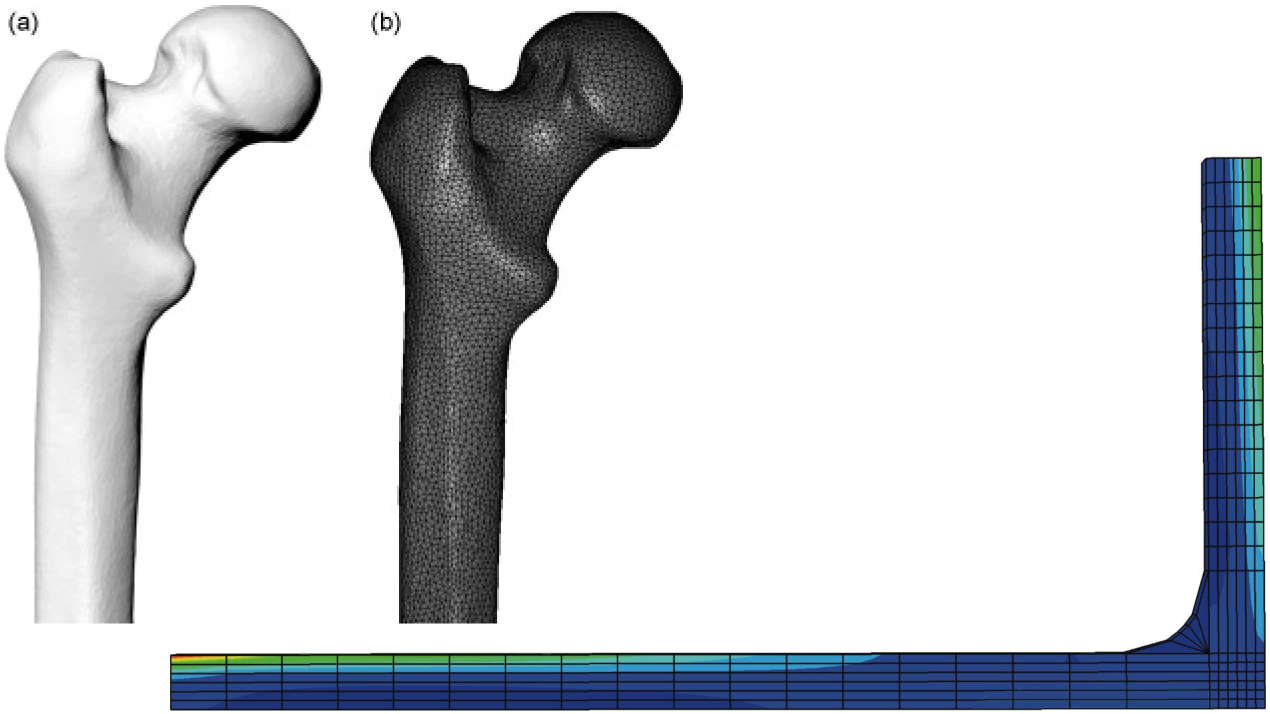
- By utilizing the previous variational formulation, it is possible to obtain a formulation of the problem, which is of lower complexity than the original differential form (strong form).
- This is also known as the **weak form**, which however can also be attained by following an alternate path (see Galerkin formulation).
- For approximate solutions, a larger class of trial solutions u(x) can be employed than in the differential formulation; for example, the trial functions need not satisfy the natural boundary conditions because these boundary conditions are implicitly contained in the functional – this is extensively used in MFE.



Approximative Methods

Instead of trying to find the **exact solution** of the continuous system, i.e., of the strong form, try to derive an **estimate of what the solution should be at specific points within the system.**

The procedure of **reducing** the physical process to its discrete counterpart is the **discretisation** process.



Approximative Methods

Variational Methods

approximation is based on the minimization of a functional, as those defined in the earlier slides.

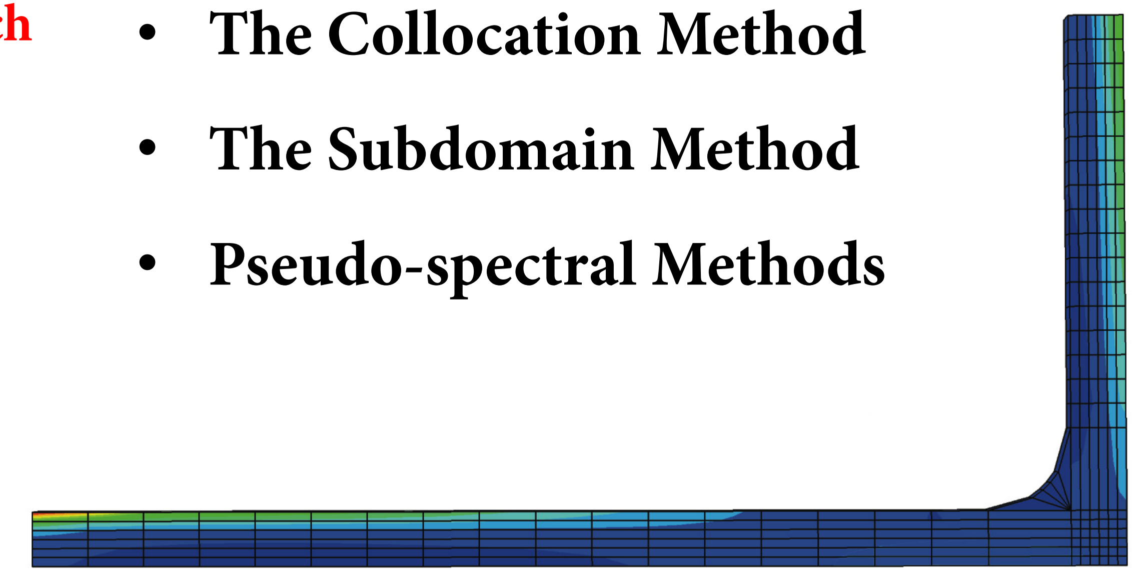
- **Rayleigh-Ritz Method**

Here, we focus on this approach

Weighted Residual Methods

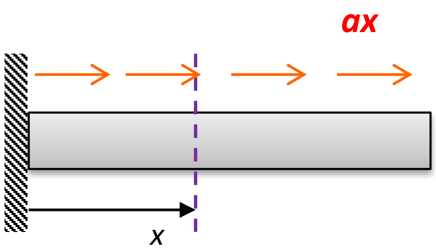
start with an estimate of the the solution and demand that its weighted average error is minimized

- **The Galerkin Method**
- **The Least Square Method**
- **The Collocation Method**
- **The Subdomain Method**
- **Pseudo-spectral Methods**



Bar Problem: Strong Form

Focus: Now assume the axially loaded bar **BUT** with **distributed load:**



$$AE\frac{d^2u}{dx^2} = -ax \qquad \textit{Strong Form}$$
$$u(0) = 0 \qquad \textit{Essential BC}$$
$$AE\left.\frac{du}{dx}\right|_{x=L} = 0 \qquad \textit{Natural BC}$$

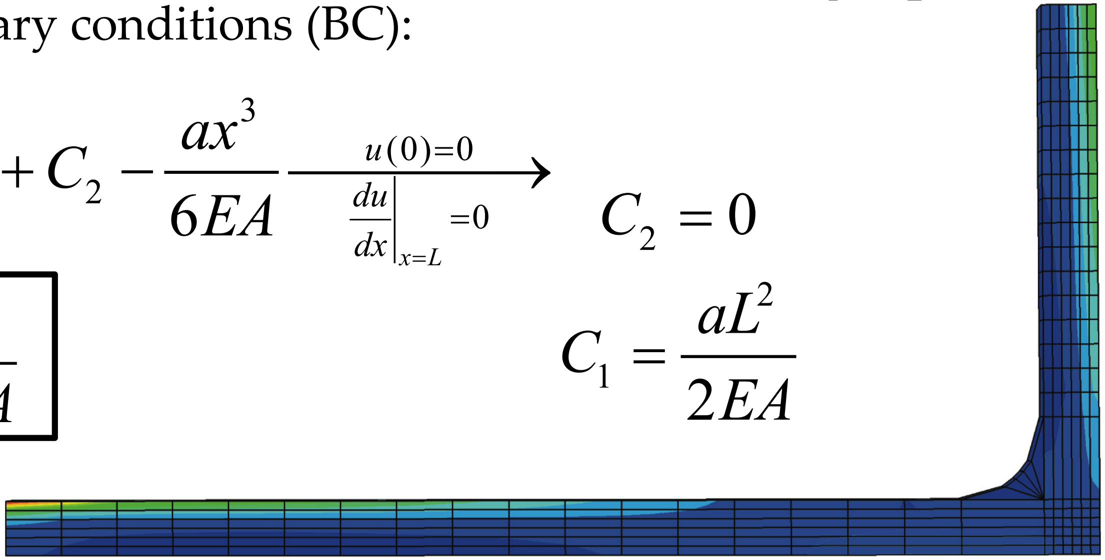
Analytical Solution:

To fully define the solution (i.e., to evaluate the values of parameters C_1, C_2) we have to use the given boundary conditions (BC):

$$u(x) = u_h + u_p = C_1 x + C_2 - \frac{ax^3}{6EA} \xrightarrow[\left.\frac{du}{dx}\right|_{x=L}=0]{u(0)=0} \quad C_2 = 0$$

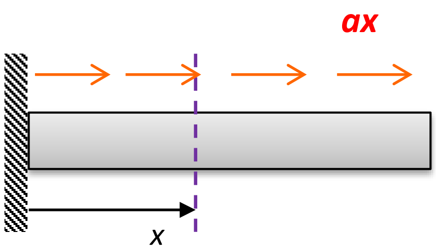
$$C_1 = \frac{aL^2}{2EA}$$

$$\Rightarrow u(x) = \frac{aL^2}{2EA}x - \frac{ax^3}{6EA}$$

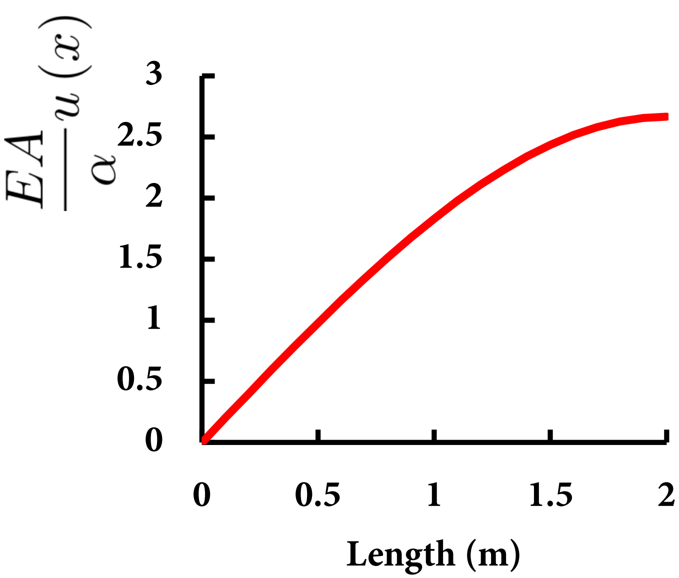


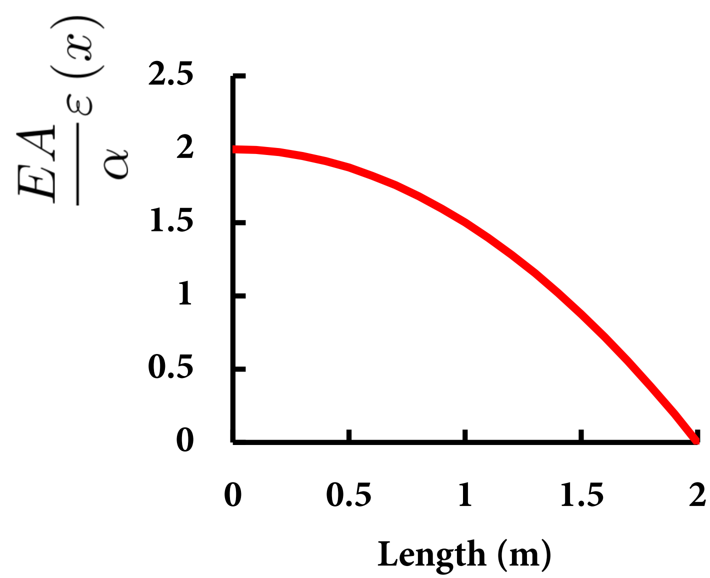
Bar Problem: Strong Form

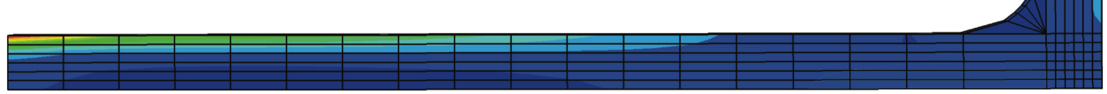
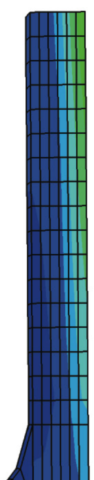
Focus: The axially loaded bar with distributed load deformation and strain plots:



$$u(x) = \frac{aL^2}{2EA}x - \frac{ax^3}{6EA}$$

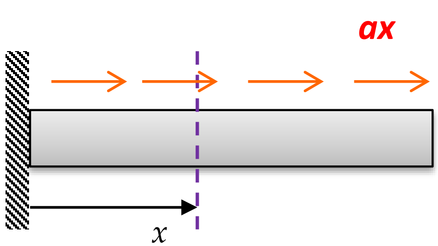




The Weak Form from the PVW

Let's now examine the same problem, and let us use the Principle of Virtual Work:



If we imagine there were forces (virtual forces) inside and outside of the bar, then the virtual work generated by these 'virtual forces' should conserve energy. For the bar, this principle can be stated as:

$$\delta W_{\text{int}} = \delta W_{ext} \quad \text{where}$$

$$\delta W_{\text{int}} = A\int_{l} \sigma \cdot \delta\varepsilon \; dx, \quad \delta W_{ext} = \int_{o}^{L} ax \cdot \delta u \; dx, \quad \delta u = \delta u(x)$$

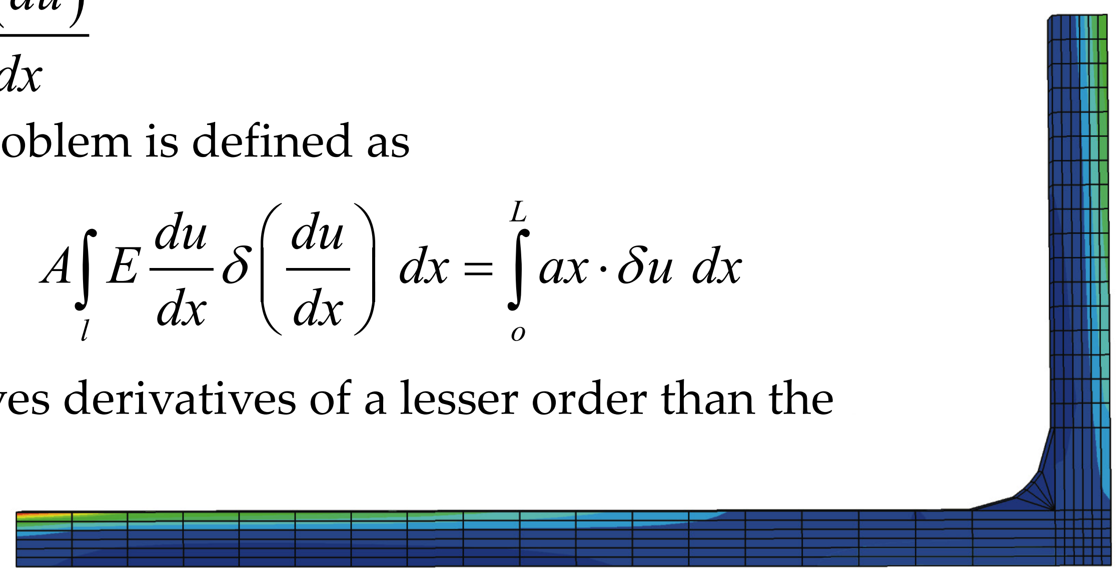
where $\sigma = E\dfrac{du}{dx}$, and $\delta\varepsilon = \dfrac{\delta(du)}{dx}$

Therefore, the ***weak form*** of the problem is defined as

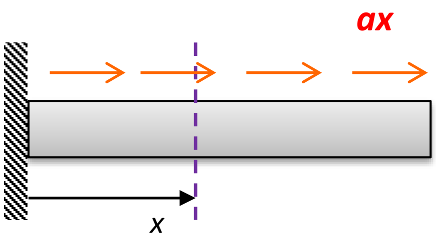
Find $u(x) \in S$ such that:

$$A\int_{l} E\frac{du}{dx}\delta\left(\frac{du}{dx}\right) dx = \int_{o}^{L} ax \cdot \delta u \; dx$$

Observe that the weak form involves derivatives of a lesser order than the original strong form.



Approximating the Strong Form
The Galerkin Method



Let us look at an alternative approximation:

$$AE\frac{d^2u}{dx^2} = -ax \qquad \textit{Strong Form}$$

Boundary Conditions (BC)

$$u(0) = 0 \qquad \textit{Essential BC}$$

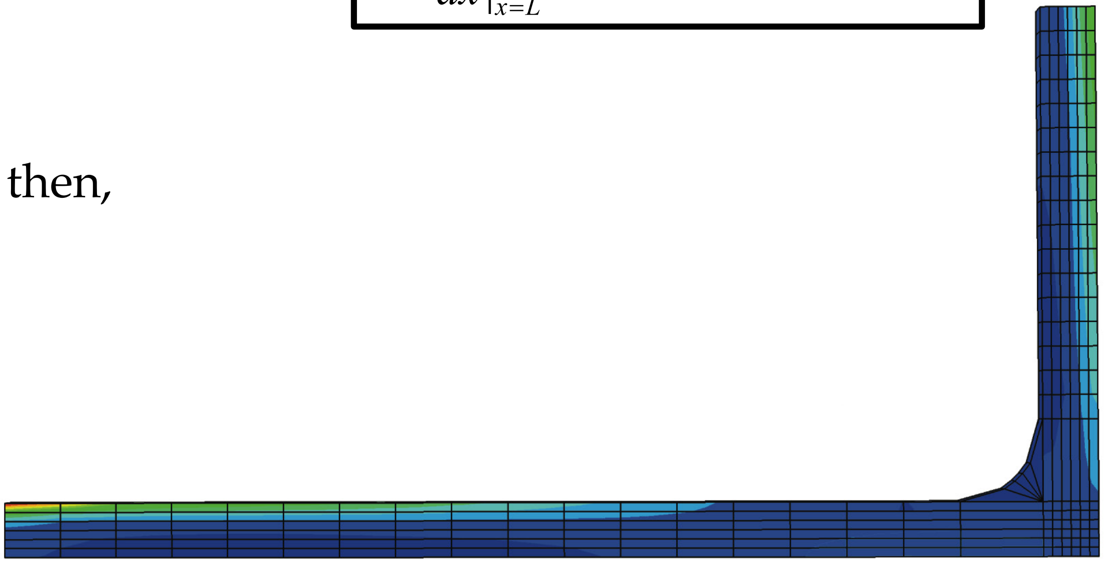
$$\sigma(L) = 0 \Rightarrow$$

$$AE\left.\frac{du}{dx}\right|_{x=L} = 0 \qquad \textit{Natural BC}$$

The method uses an arbitrary weighting function w that satisfies the essential conditions and additionally:

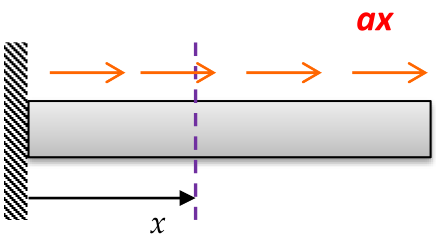
If $S = \{u|\, u \in C^0, u(0) = 0\}$ then,

$$S_w = \{w|\, w \in C^0, w(0) = 0\}$$



Weighted Residual Methods

Focus: The axially loaded bar with distributed load example.

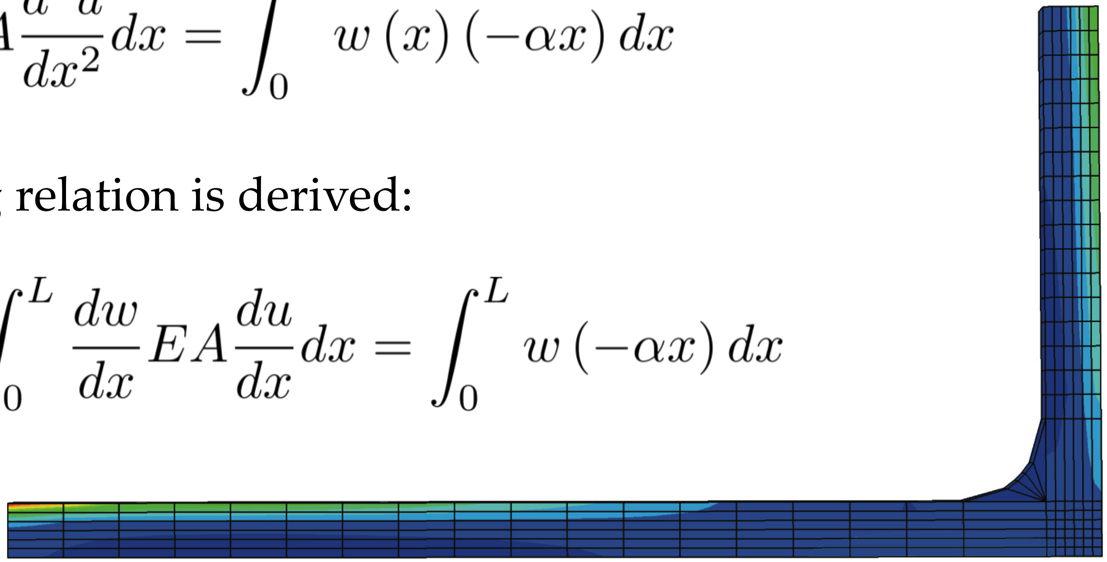


Multiplying the strong form by w and integrating over L:

$$\int_0^L w(x)\, EA \frac{d^2u}{dx^2} dx = \int_0^L w(x)\,(-\alpha x)\, dx$$

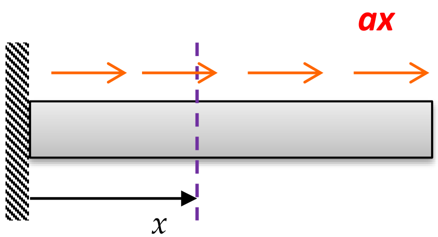
Integrating by parts, the following relation is derived:

$$EA\left[w\frac{du}{dx}\right]_0^L - \int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = \int_0^L w\,(-\alpha x)\, dx$$



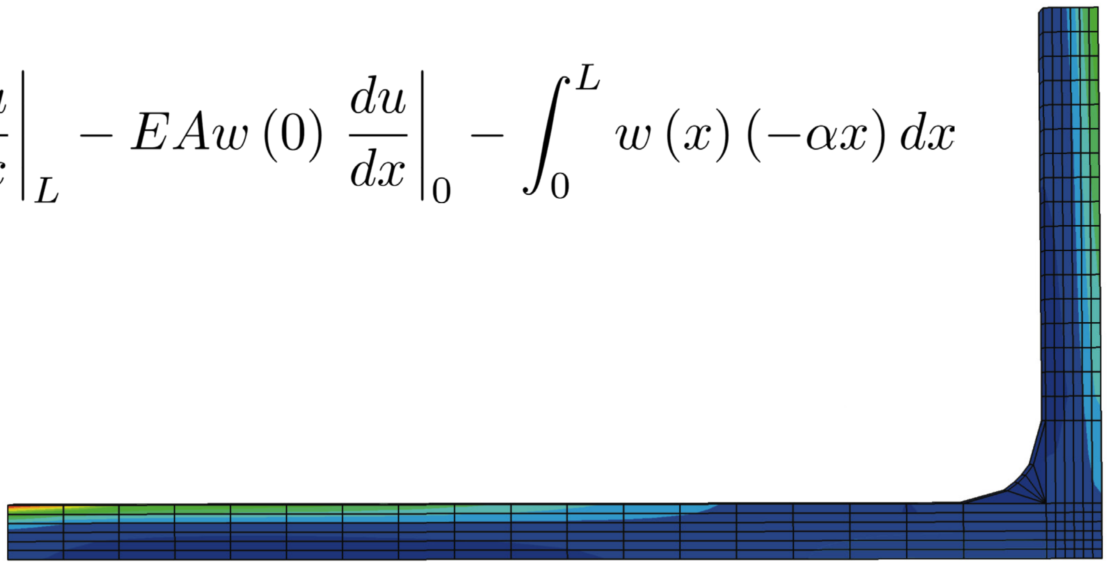
Weighted Residual Methods

Focus: The axially loaded bar with distributed load example.



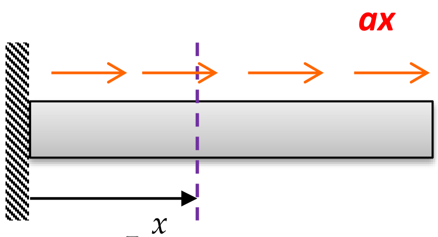
Elaborating a little bit more on the relation:

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = EAw(L) \left.\frac{du}{dx}\right|_L - EAw(0) \left.\frac{du}{dx}\right|_0 - \int_0^L w(x)(-\alpha x)\, dx$$



Weighted Residual Methods

Focus: The axially loaded bar with distributed load example.



Elaborating a little bit more on the relation:

why? why?

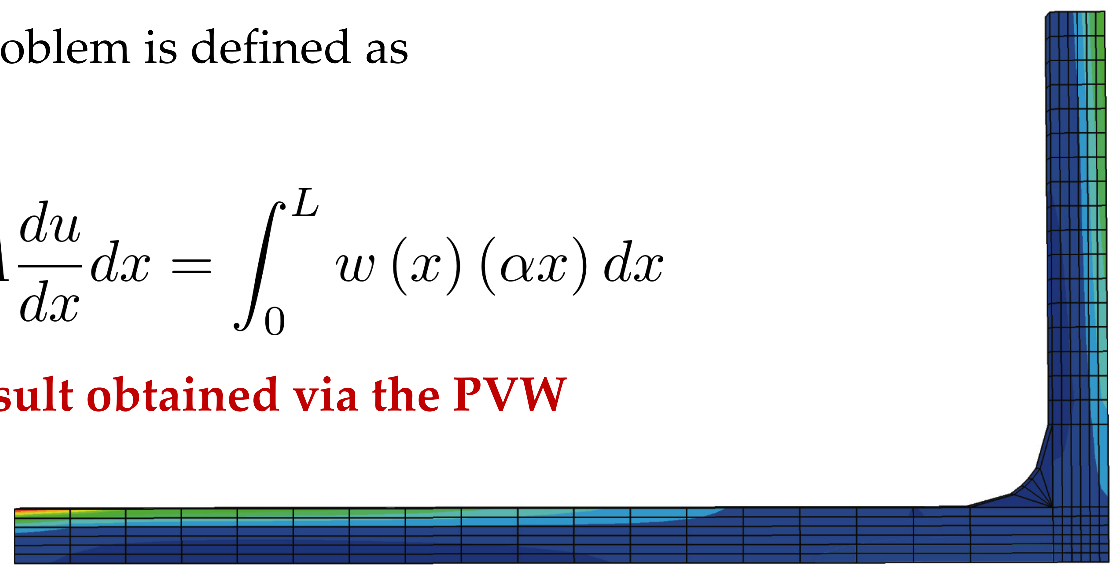
$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = EAw(L) \left.\frac{du}{dx}\right|_L - EAw(0) \left.\frac{du}{dx}\right|_0 - \int_0^L w(x)(-\alpha x)\, dx$$

Therefore, the ***weak form*** of the problem is defined as

Find $u(x) \in S$ such that:

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = \int_0^L w(x)(\alpha x)\, dx$$

This result is equivalent to the result obtained via the PVW



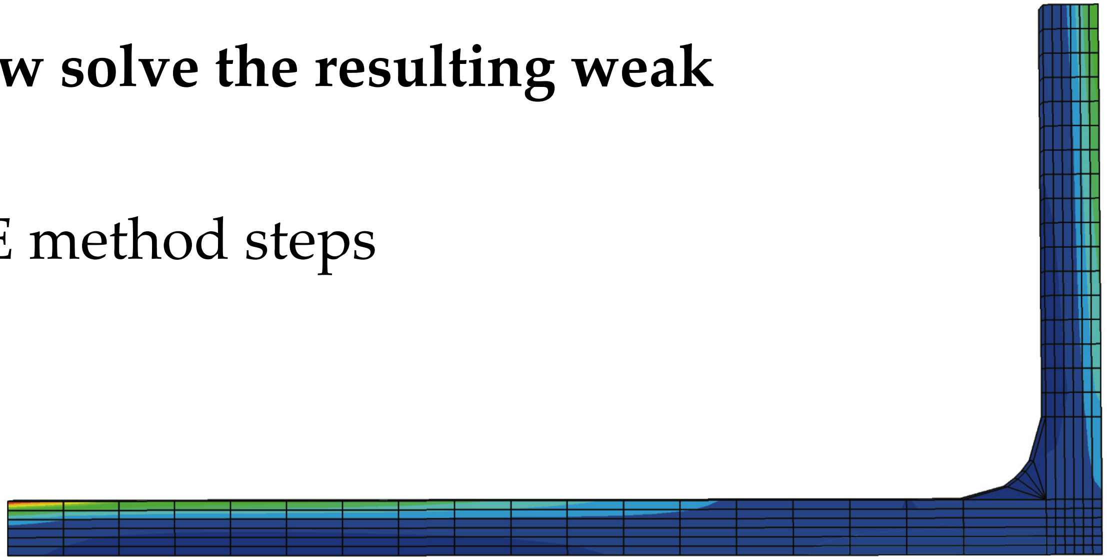
Weighted Residual Methods

The Galerkin method

The steps performed using the Galerking method, actually lead to the same kind of weak formulation we would have obtained via the principle of Virtual work or the MKE (remember slide). This is just an alternative.

However, how do we now solve the resulting weak formulation?

Well, we introduce the FE method steps



The Galerkin Method

Theory *– Consider the general case of a differential equation:* $Lu = f$

Try an **approximate solution** to the equation of the following form

$$\tilde{u} = \sum_{i=1}^{N} N_i u_i$$

where N_i are test functions (input) and u_i are unknown quantities that we need to evaluate. The solution must satisfy the boundary conditions.

Since $\tilde{u}$ is an approximation, substituting it in the initial equation will result in an error:

$$L\tilde{u} = f + \tilde{\epsilon} \Rightarrow L\tilde{u} - f = \tilde{\epsilon}$$

Example *– The axially loaded bar:*

$$EA\frac{d^2 u(x)}{dx^2} = -\alpha x$$

$$L = EA\frac{d^2 x}{dx^2}$$

$$f = -\alpha x$$

Choose the following approximation

$$\tilde{u} = u_1 + u_2 x + u_3 x^2 + u_4 x^3$$

Demand that the approximation satisfies the essential conditions:

$$\tilde{u}(0) = 0 \Rightarrow u_1 = 0$$

The approximation error in this case is:

$$EA\frac{d^2 \tilde{u}(x)}{dx^2} + \alpha x = \tilde{\epsilon}$$

Weighted Residual Methods

Comments

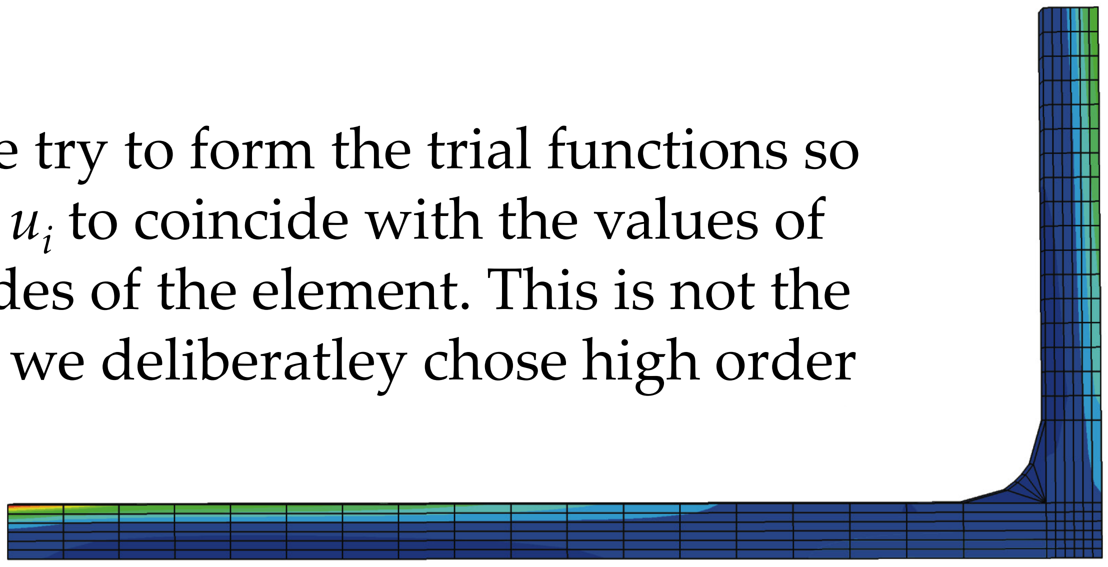
Notice how we chose an approximate solution which is a 3rd degree polynomial. Indeed, in the approximate solution chosen here, the **trial functions** are

$$N_1 = 1,\ N_2 = x,\ N_3 = x^2,\ N_4 = x^3$$

This implies that we made a good guess, since we already know that the true solution is a 3rd order polynomial!

(see slide)

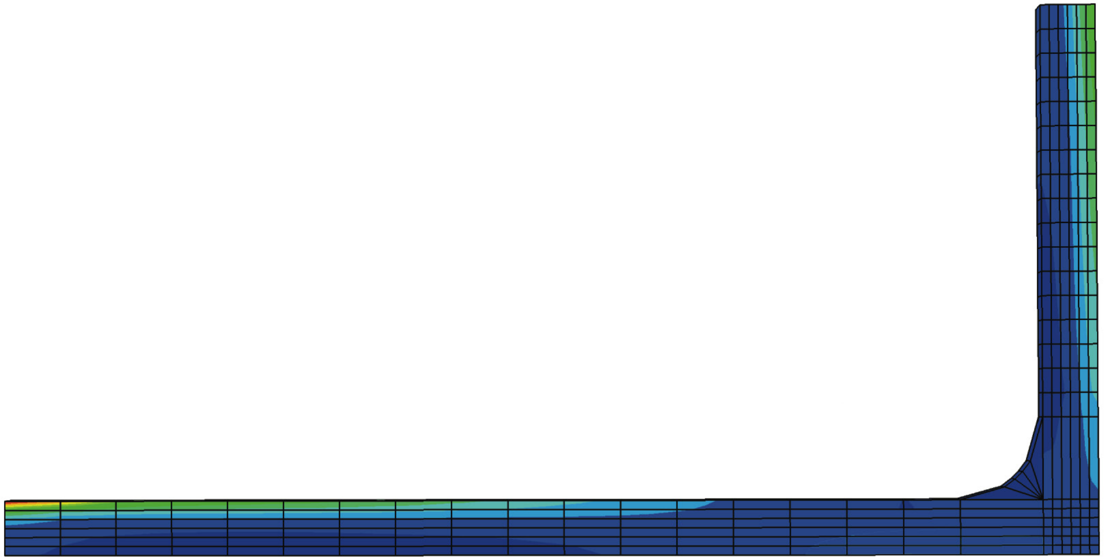
We will show later that we try to form the trial functions so as to force the coefficients u_i to coincide with the values of the deformation at the nodes of the element. This is not the case here however, where we deliberatley chose high order test functions.



The Galerkin Method

Assumption 1: The weighted average error of the approximation should be zero

$$\int_0^L \tilde{\epsilon} w dv = \int_0^L EA\left(\frac{d^2\tilde{u}(x)}{dx^2} + \alpha x\right) w dx = 0$$



The Galerkin Method

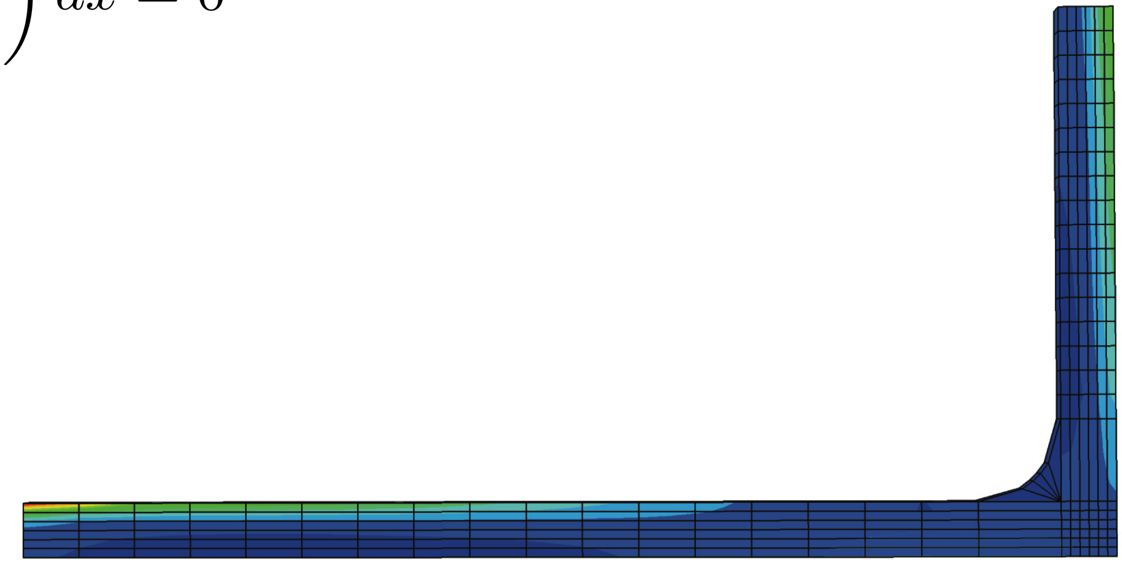
Assumption 1: The weighted average error of the approximation should be zero

$$\int_0^L \tilde{\epsilon} w dv = \int_0^L EA\left(\frac{d^2\tilde{u}(x)}{dx^2} + \alpha x\right) w dx = 0$$

But that's the Weak Form!!!!!

Therefore once again integration by parts leads to

$$\int_0^L \left(\frac{dw}{dx} EA \frac{d\tilde{u}}{dx} - w(x)(\alpha x)\right) dx = 0$$



The Galerkin Method

Assumption 1: The weighted average error of the approximation should be zero

$$\int_0^L \tilde{\epsilon} w dv = \int_0^L EA\left(\frac{d^2\tilde{u}(x)}{dx^2} + \alpha x\right) w dx = 0$$

But that's the Weak Form!

Therefore once again integration by parts leads to

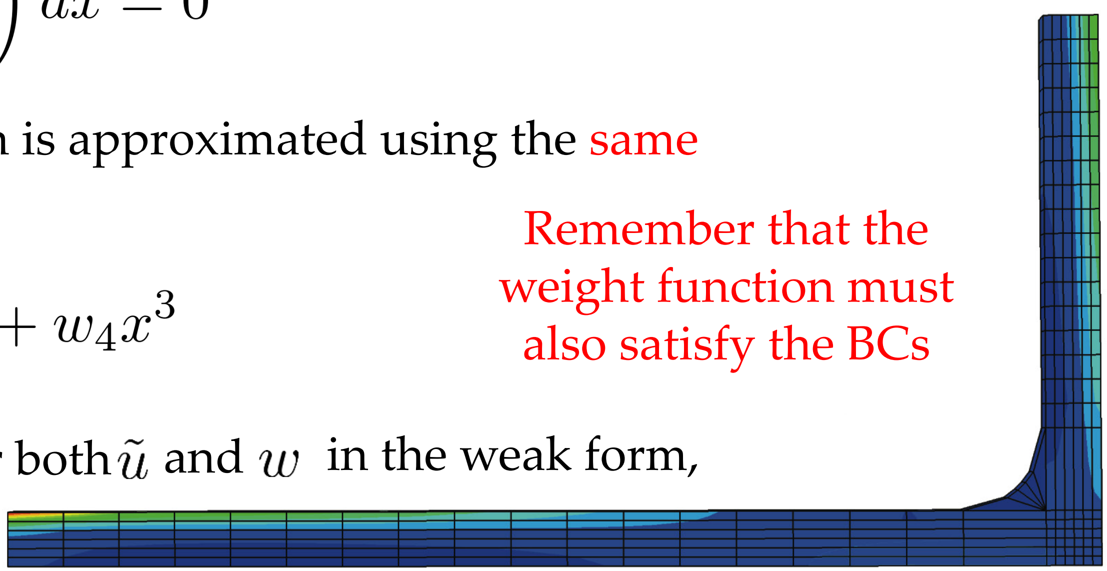
$$\int_0^L \left(\frac{dw}{dx} EA \frac{d\tilde{u}}{dx} - w(x)(\alpha x)\right) dx = 0$$

Assumption 2: The weight function is approximated using the same scheme as for the solution

$$w = \sum_{i=1}^{N} N_i w_i = w_2 x + w_3 x^2 + w_4 x^3$$

Remember that the weight function must also satisfy the BCs

Substituting the approximations for both $\tilde{u}$ and w in the weak form,



The Galerkin Method

The following relation is retrieved:

$$w_2 I_2 + w_3 I_3 + w_4 I_4 = 0$$

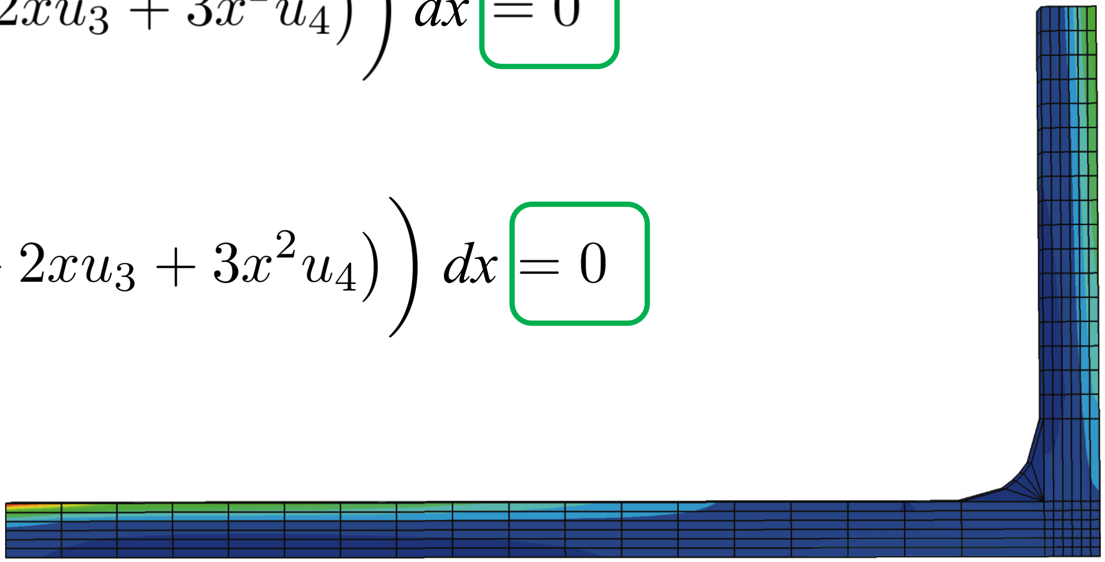
where:

Remember that the weight function is (almost) arbitrary! Therefore, the only way this holds for any w(x) is:

$$I_2 = \int_0^L \left(-\frac{x^2 \alpha}{EA} + u_2 + 2x u_3 + 3x^2 u_4 \right) dx = 0$$

$$I_3 = \int_0^L \left(-\frac{x^3 \alpha}{EA} + 2x \left(u_2 + 2x u_3 + 3x^2 u_4 \right) \right) dx = 0$$

$$I_4 = \int_0^L \left(-\frac{x^4 \alpha}{EA} + 3x^2 \left(u_2 + 2x u_3 + 3x^2 u_4 \right) \right) dx = 0$$



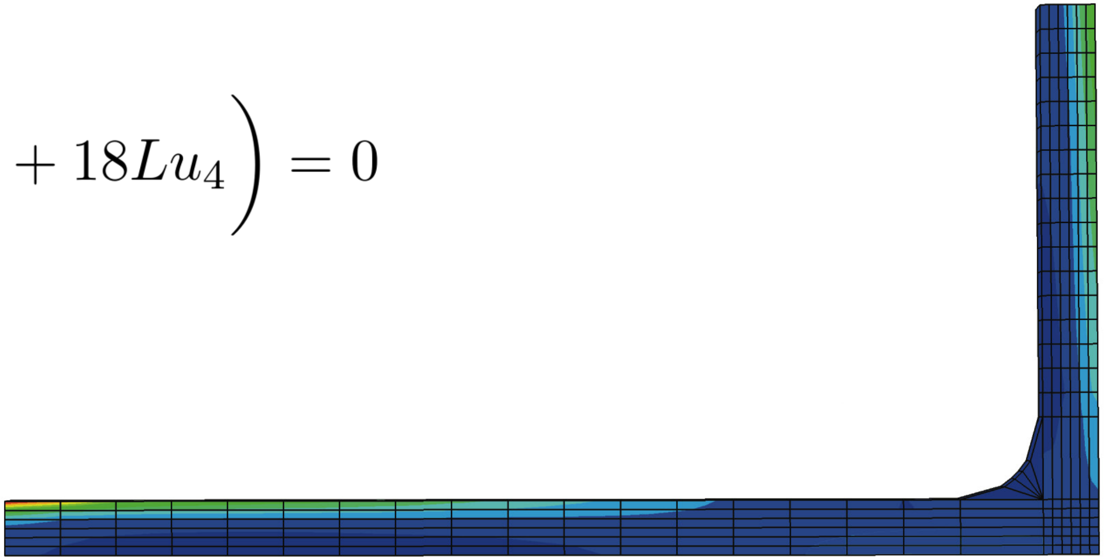
The Galerkin Method

Performing the integration, the following relations are established:

$$-\frac{L^3\alpha}{3AE} + Lu_2 + L^2u_3 + L^3u_4 = 0$$

$$L^2u_2 + \frac{1}{12}L^3\left(-\frac{3L\alpha}{EA} + 16u_3 + 18Lu_4\right) = 0$$

$$L^3u_2 + \frac{1}{10}L^4\left(-\frac{2L\alpha}{EA} + 15u_3 + 18Lu_4\right) = 0$$



The Galerkin Method

Or in matrix form:

$$\begin{bmatrix} L & L^2 & L^3 \\ L^2 & \frac{16}{12}L^3 & \frac{18}{12}L^4 \\ L^3 & \frac{15}{10}L^4 & \frac{18}{10}L^5 \end{bmatrix} \left\{ \begin{array}{c} u_2 \\ u_3 \\ u_4 \end{array} \right\} = \left\{ \begin{array}{c} \frac{\alpha L^3}{3EA} \\ \frac{\alpha L^4}{4EA} \\ \frac{\alpha L^5}{5EA} \end{array} \right\}$$

that's a linear system of equations:

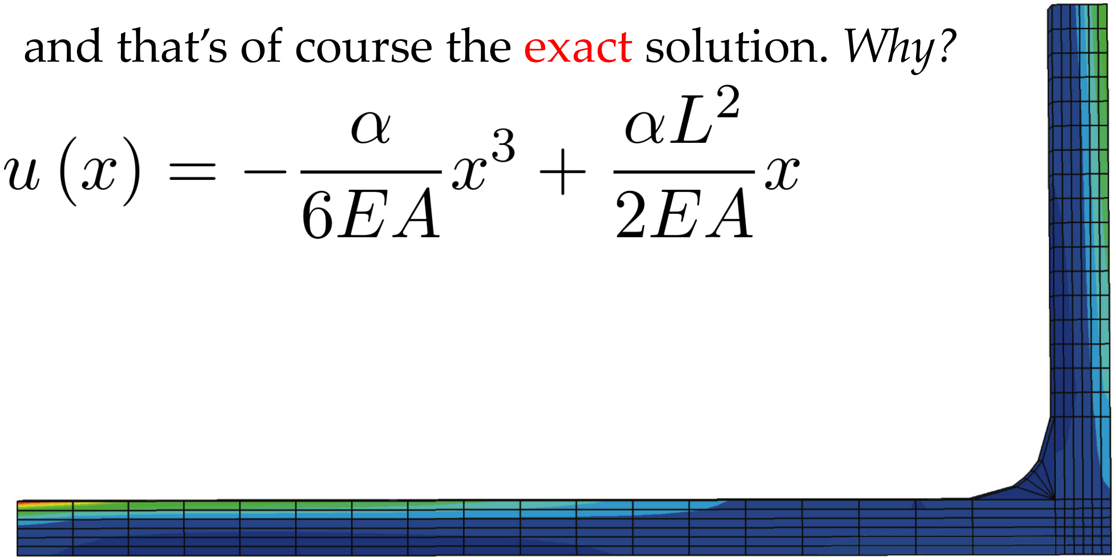
$$u_2 = \frac{L^2\alpha}{2EA}$$

$$u_3 = 0$$

$$u_4 = -\frac{\alpha}{6EA}$$

and that's of course the exact solution. *Why?*

$$u(x) = -\frac{\alpha}{6EA}x^3 + \frac{\alpha L^2}{2EA}x$$



The Galerkin Method

What is we do not make such a good guess, i.e., a 3rd order polynomial and instead try the following (suboptimal) approximation:

$$\tilde{u} = u_1 + u_2 x$$

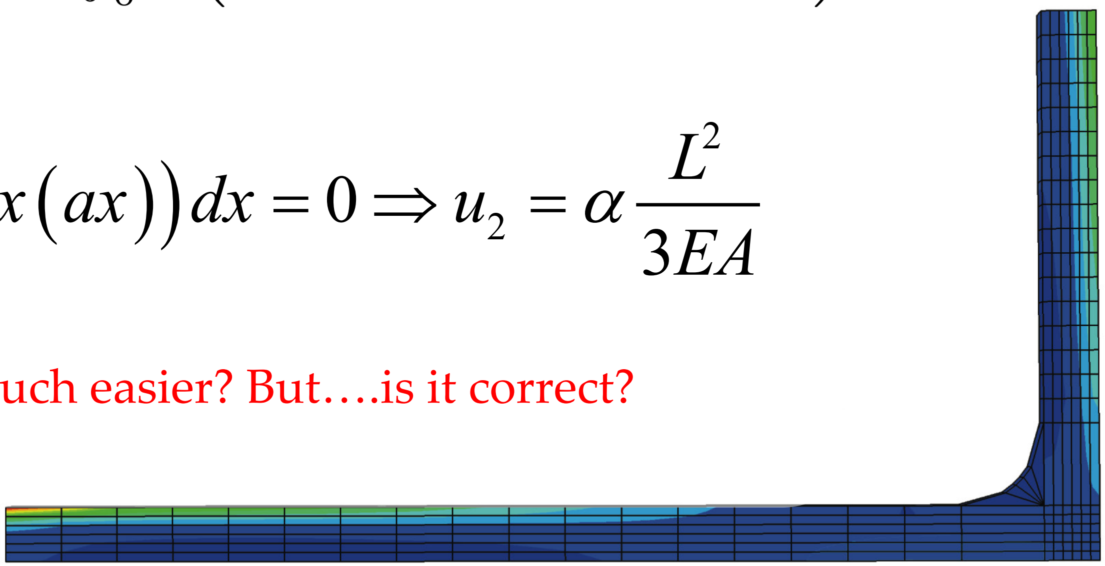
Which again needs to satisfy the essential BCs, therefore $\tilde{u}(0) = 0 \Rightarrow u_1 = 0$

The weight function assumes the same form: $w = w_1 + w_2 x, w(0) = 0 \Rightarrow w_1 = 0$

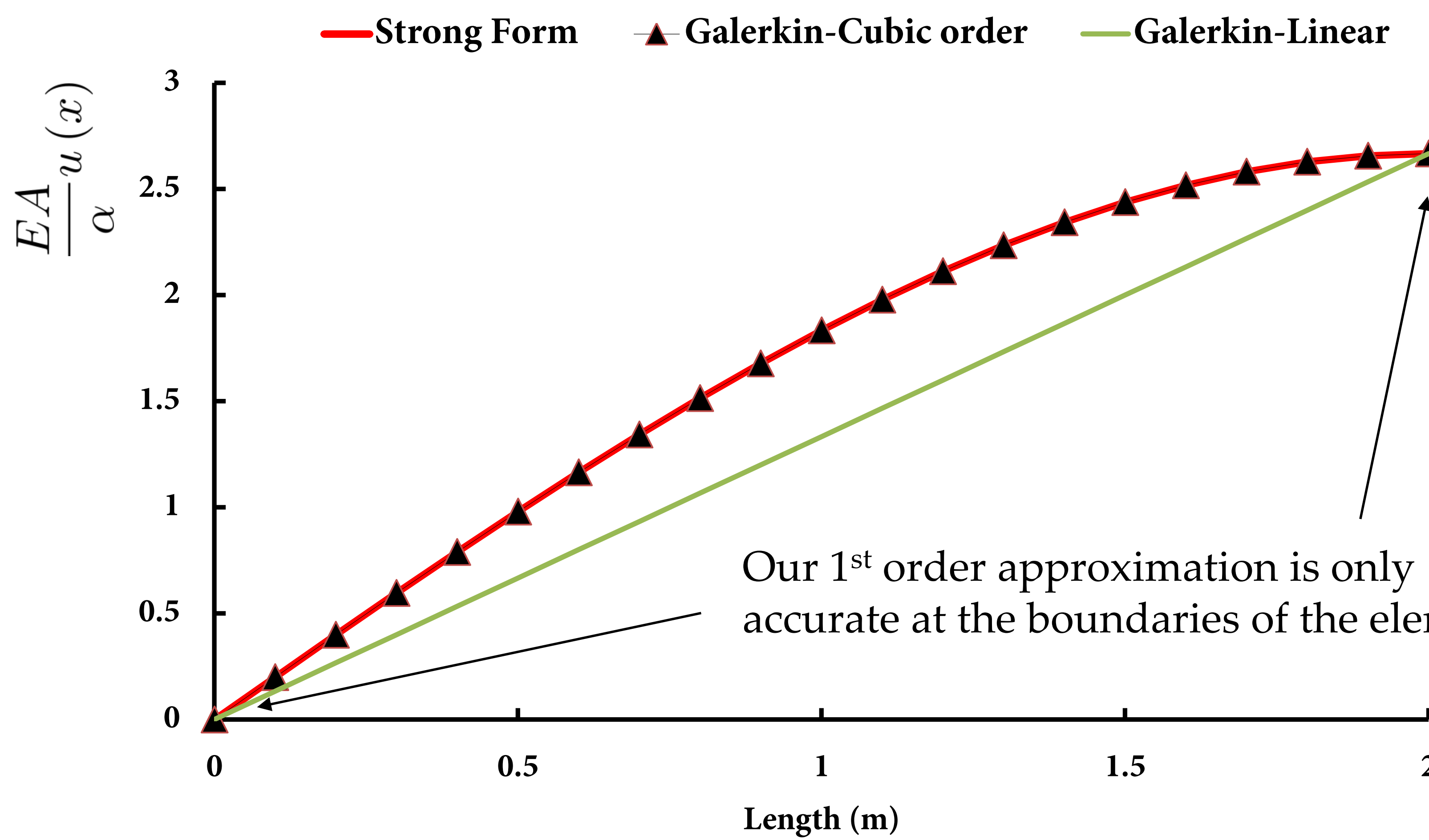
Substituting now into the weak form: $\int_0^L \left(\frac{dw}{dx} EA \frac{d\tilde{u}}{dx} - w(x)(\alpha x) \right) dx = 0$

$$w_2 \int_0^L \left(EAu_2 - x(ax) \right) dx = 0 \Rightarrow u_2 = \alpha \frac{L^2}{3EA}$$

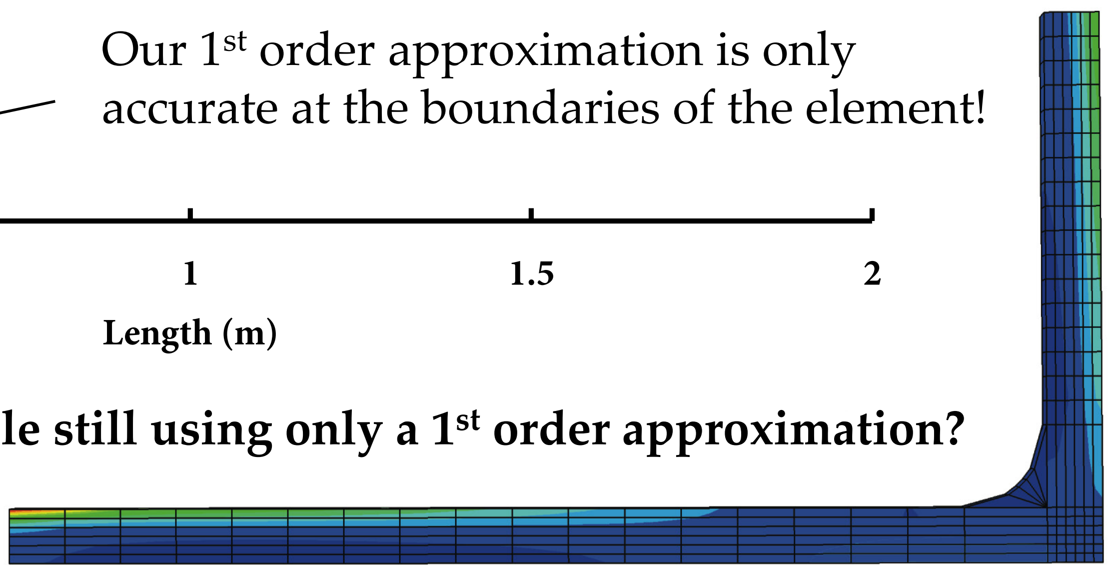
Wasn't that much easier? But….is it correct?



The Galerkin Method

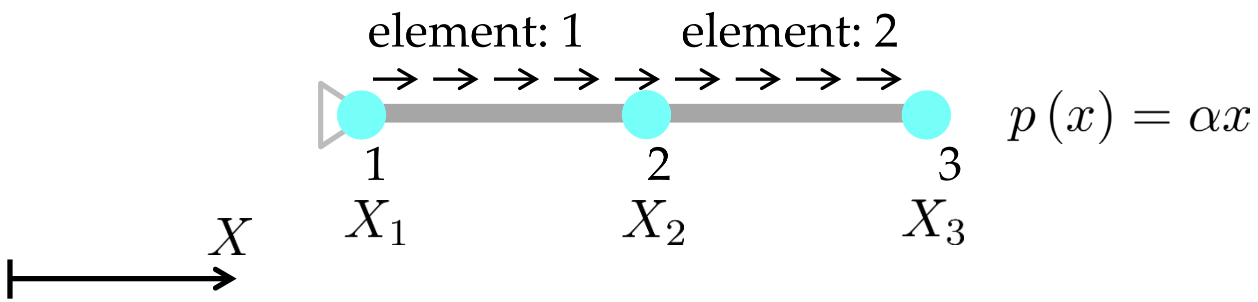


How to improve accuracy, while still using only a 1st order approximation?

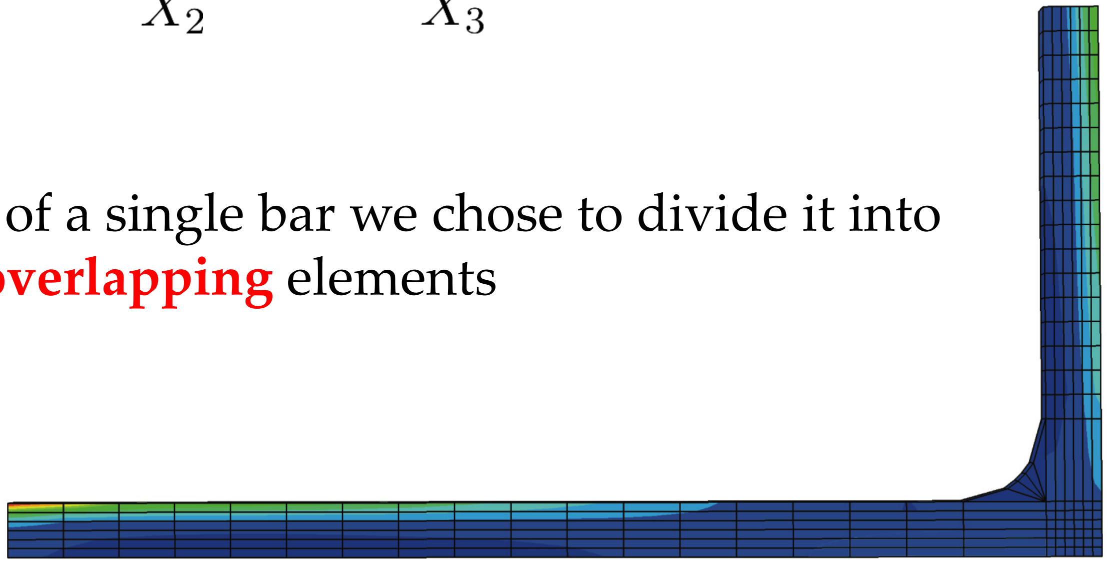


The Galerkin Method

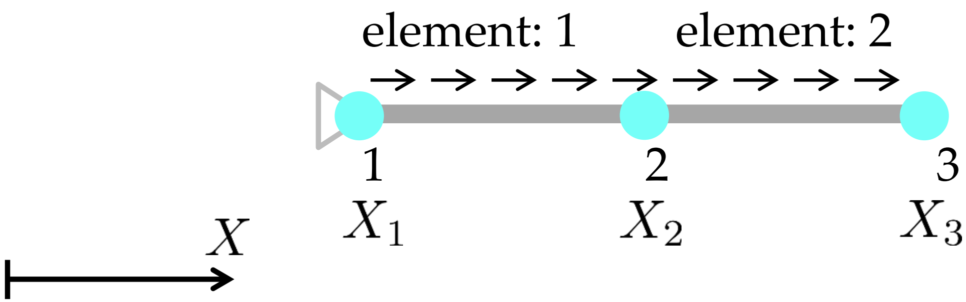
We saw in the previous example that the Galerkin method is based on the approximation of the strong form solution using a set of **basis functions**. These are **by definition absolutely accurate** at the boundaries of the problem. So, why not **increase the boundaries?**



Instead of seeking the solution of a single bar we chose to divide it into three **interconnected** and **not overlapping** elements



The Galerkin Method



$p(x) = \alpha x$

$$X_1 \leq X \leq X_2$$

$$\tilde{u} = u_1 + u_2 x$$
$$w = w_1 + w_2 x$$

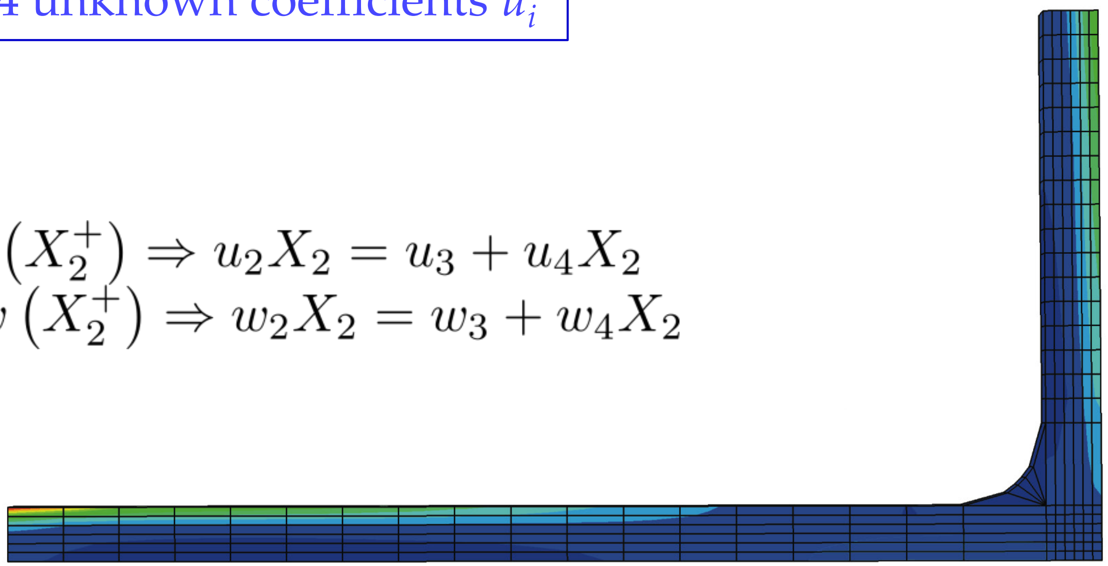
$$\tilde{u}(0) = 0 \Rightarrow \alpha_1 = 0$$
$$w(0) = 0 \Rightarrow w_1 = 0$$

We still have 1st order polynomials, but they are as many as the sub-elements, i.e., 4 unknown coefficients u_i

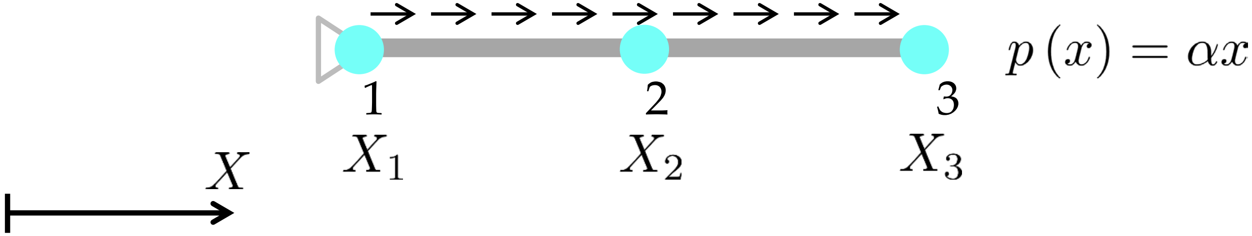
$$X_2 \leq X \leq X_3$$

$$\tilde{u} = u_3 + u_4 x$$
$$w = w_3 + w_4 x$$

$$\tilde{u}\left(X_2^-\right) = \tilde{u}\left(X_2^+\right) \Rightarrow u_2 X_2 = u_3 + u_4 X_2$$
$$w\left(X_2^-\right) = w\left(X_2^+\right) \Rightarrow w_2 X_2 = w_3 + w_4 X_2$$

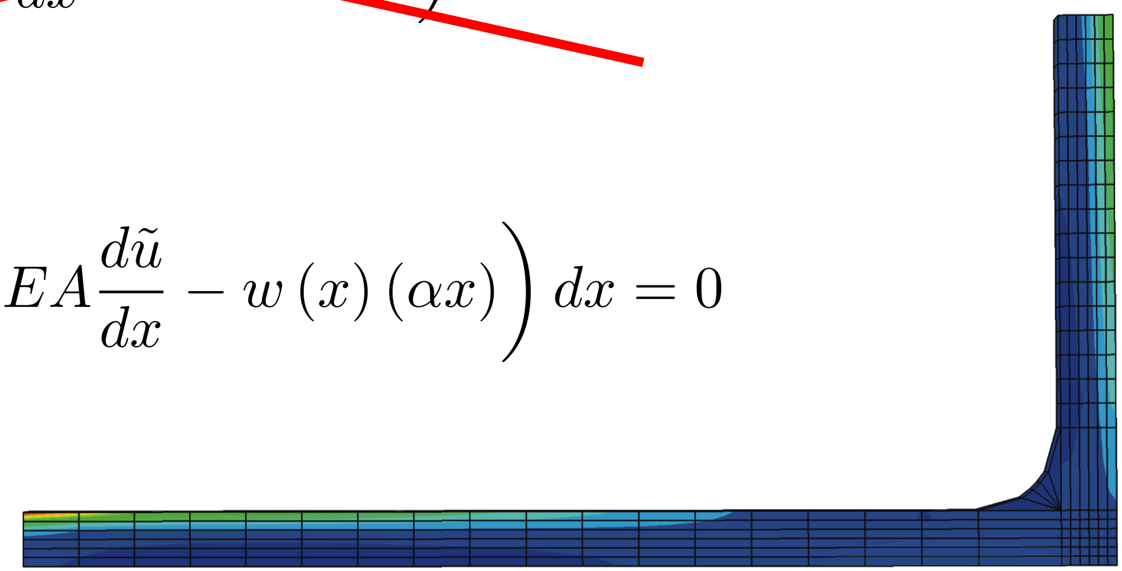


The Galerkin Method



$$\int_0^L \left(\frac{dw}{dx} EA \frac{d\tilde{u}}{dx} - w(x)(\alpha x) \right) dx = 0$$

$$\sum_{i=1}^{2} \int_{X_i}^{X_{i+1}} \left(\frac{dw}{dx} EA \frac{d\tilde{u}}{dx} - w(x)(\alpha x) \right) dx = 0$$



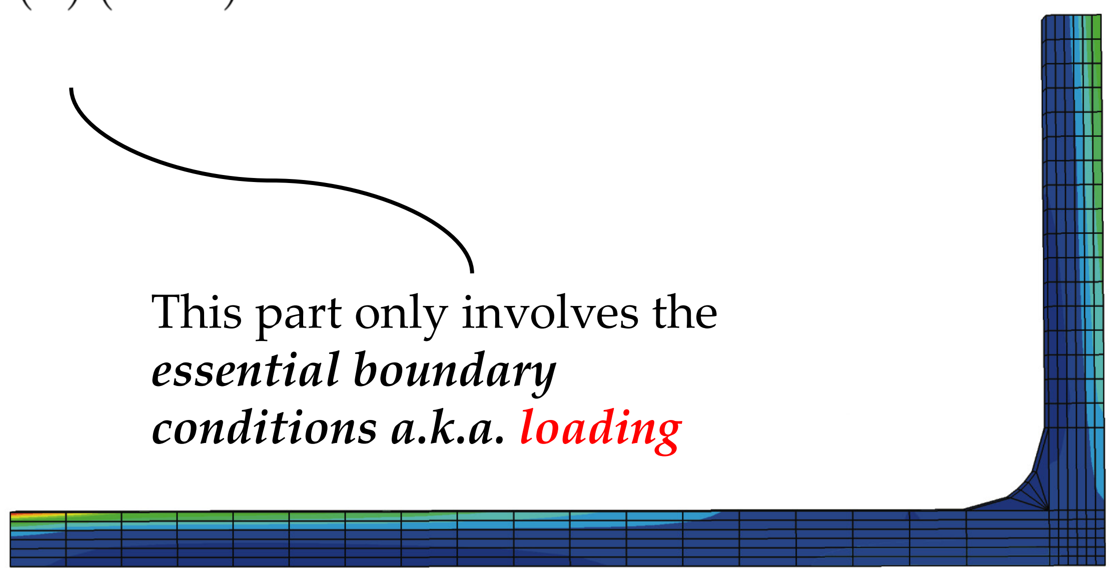
The Galerkin Method

Summarizing, the weak form of a continuous problem was derived in a systematic way:

This part involves only the solution ***approximation***

$$\boxed{\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx} = -\int_0^L w(x)\,(-\alpha x)\,dx$$

This part only involves the ***essential boundary conditions a.k.a. loading***



The Galerkin Method

And then an approximation was defined for the displacement field, for example

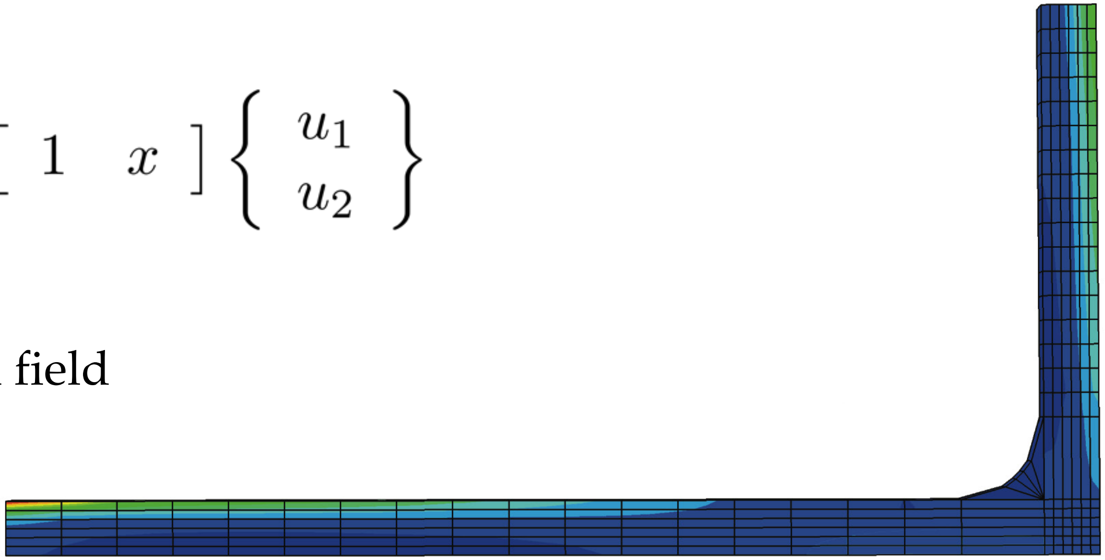
$$u = u_1 + u_2 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Displacement field

The weak form also involves the first derivative of the approximation

$$\frac{du}{dx} = \frac{d}{dx} \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Strain field



The Galerkin Method

where u_1,u_2 are so far random coefficients. Instead, we can choose to write the same relationship using a different basis **N**(x):

$$u = [N(x)]\{d\}$$

- u: Displacement field
- $[N(x)]$: Shape Function Matrix
- $\{d\}$: Vector of degrees of freedom

In this case we express u in term of the degrees of freedom, i.e., the displacements at the ends of the bar:

$$\{d\} = \begin{Bmatrix} u(x=0) \\ u(x=L) \end{Bmatrix} = \begin{Bmatrix} u_0 \\ u_L \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}^{-1} \begin{Bmatrix} u_0 \\ u_L \end{Bmatrix}$$

Substituting in our initial displacement approximation we obtain:

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}^{-1} \begin{Bmatrix} u_0 \\ u_L \end{Bmatrix} = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_L \end{Bmatrix} \Rightarrow N(x) = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$$

The Galerkin Method

Then:

$$u = [N(x)]\{d\}$$

Displacement field (u)

Vector of degrees of freedom ($\{d\}$)

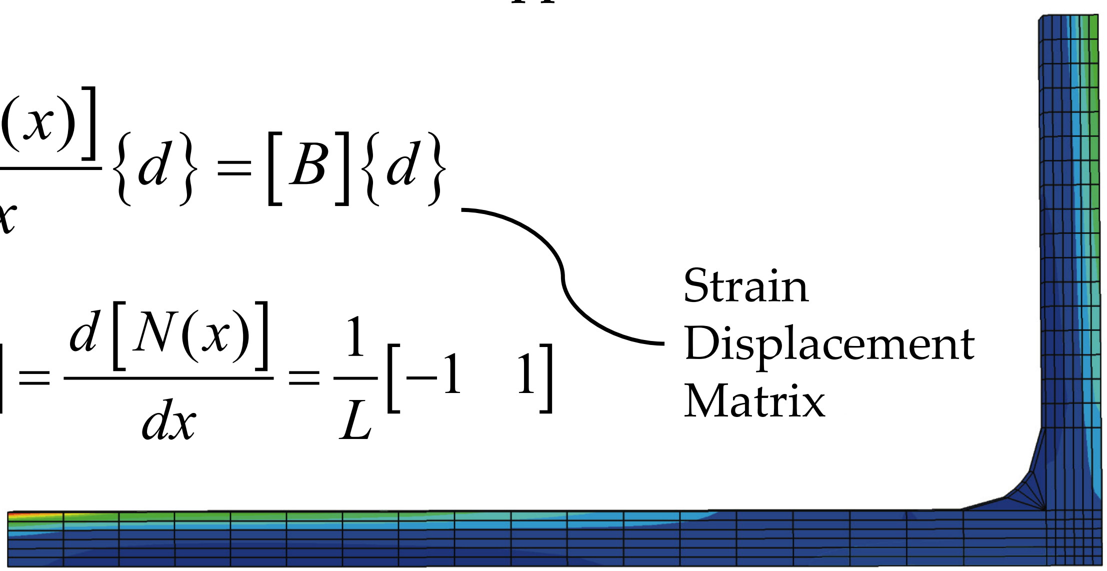
The weak form also involves the first derivative of the approximation

$$\varepsilon = \frac{d[N(x)]}{dx}\{d\} = [B]\{d\}$$

Strain field (ε)

Strain Displacement Matrix ($[B]$)

where $$[B] = \frac{d[N(x)]}{dx} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$



The Galerkin Method

Therefore if we return to the weak form :

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = -\int_0^L w(x)(-\alpha x)\, dx$$

and set:

$$u = [N]\{d\} \qquad \varepsilon = [B]\{d\} \qquad w = [N]\{\tilde{w}\}$$

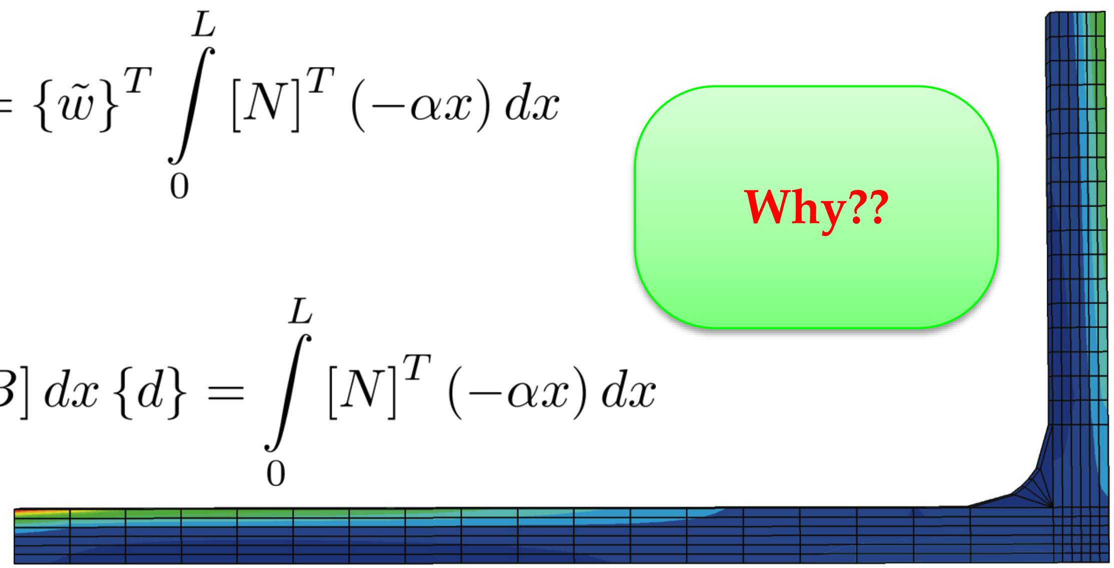
The following FUNDAMENTAL FEM expression is derived

$$\{\tilde{w}\}^T \int_0^L [B]^T EA [B]\, dx \{d\} = \{\tilde{w}\}^T \int_0^L [N]^T (-\alpha x)\, dx$$

Why??

or even better

$$\int_0^L [B]^T EA [B]\, dx \{d\} = \int_0^L [N]^T (-\alpha x)\, dx$$



The Galerkin Method

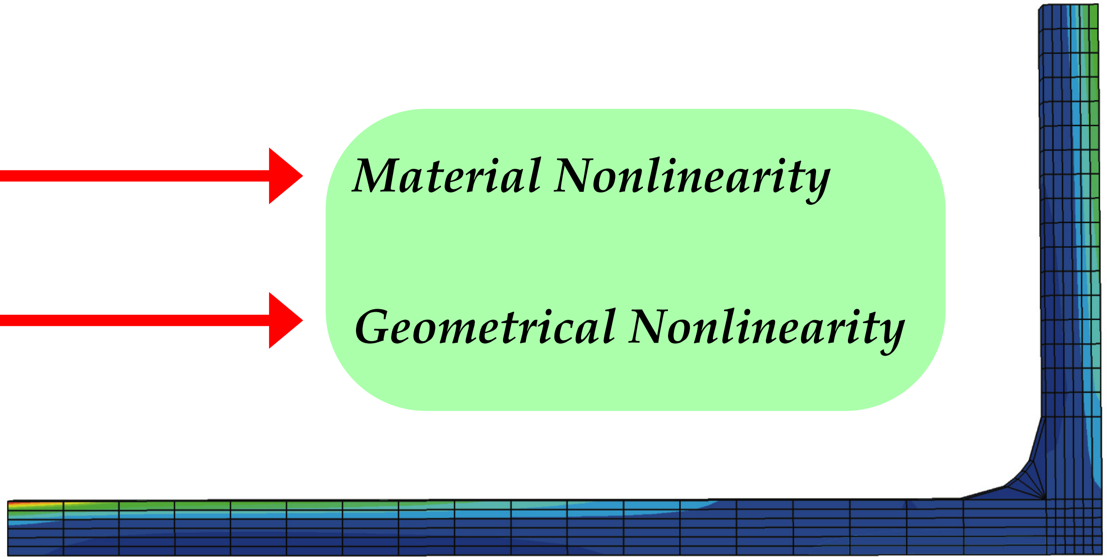
$$\int_0^L [B]^T EA\,[B]\,dx\,\{d\} = \int_0^L [N]^T(-\alpha x)\,dx$$

EA has to do only with material and cross-sectional properties

We call $$[k] = \int_0^L [B]^T EA\,[B]\,dx\,\{d\}$$ ***The Finite Element stiffness Matrix***

If E is a function of {d} ⟷ ***Material Nonlinearity***

If [B] is a function of {d} ⟷ ***Geometrical Nonlinearity***



The Galerkin Method

Indeed, if we use the proposed formulation for [N], [B]:

$$\left.\begin{aligned} k &= \int_0^L [B]^T EA[B]\,dx \\ [B] &= \begin{bmatrix} -1/L & 1/L \end{bmatrix} \end{aligned}\right\} \Rightarrow k = \int_0^L \frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix} EA \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix} dx \Rightarrow$$

$$= 1/L^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} EA \begin{bmatrix} -1 & 1 \end{bmatrix} L \Rightarrow$$

$$k = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

bar stiffness!