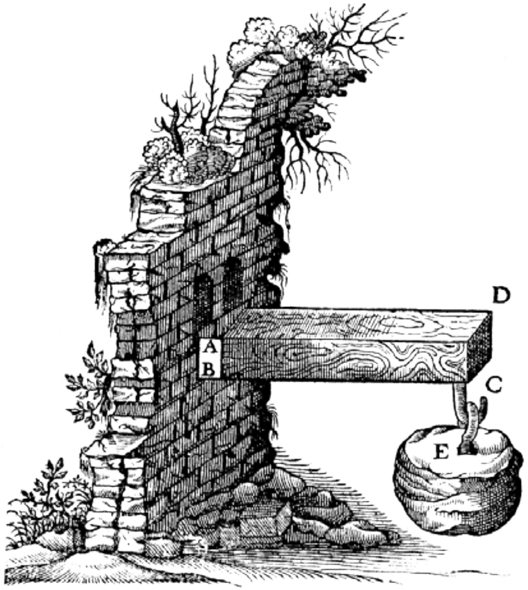
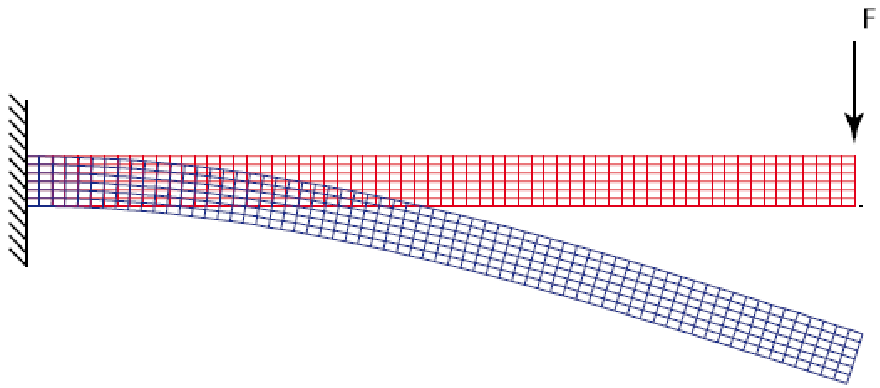
Method of Finite Elements I

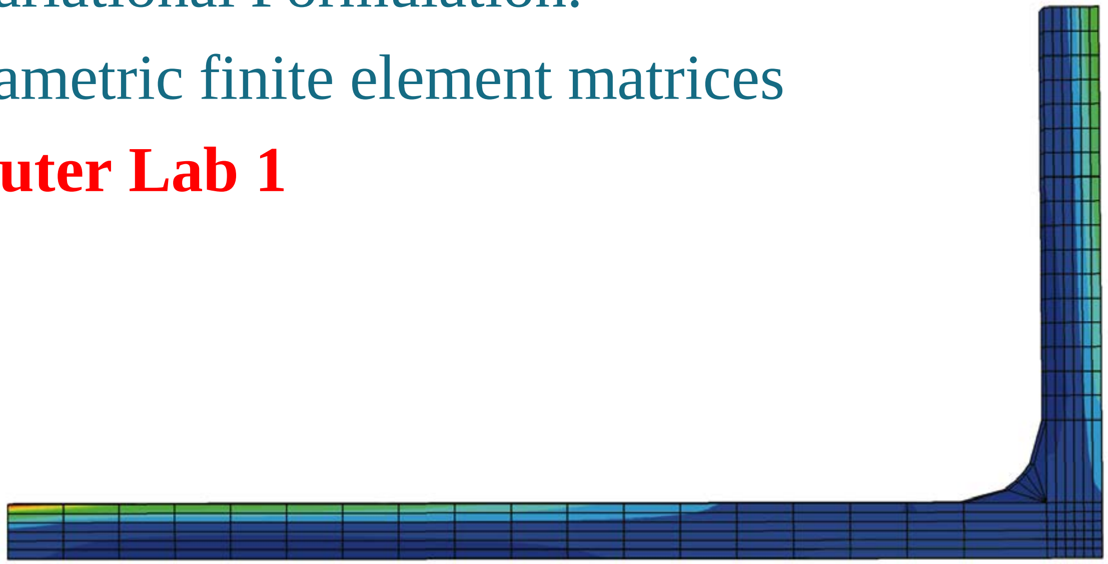




- Held by Prof. Dr. E. Chatzi, Dr. P. Steffen
- Assistant: Adrian Egger, HIL E 13.3
- Lectures homepage: http://www.ibk.ethz.ch/ibk/ibk/ch/education/femi/index_EN
- Course book: “Finite Element Procedures” by K.J. Bathe
- Performance assessment

Course Overview

- 22.02.2016 – Introductory Concepts
  Matrices and linear algebra - short review.
- 2.02.2016– The Direct Stiffness Method
- 07.03.2016 – Demos and exercises in MATLAB
- 14.03.2016 – The Variational Formulation.
- 16.03.2016 – Isoparametric finite element matrices
- **21.03.2016 – Computer Lab 1**

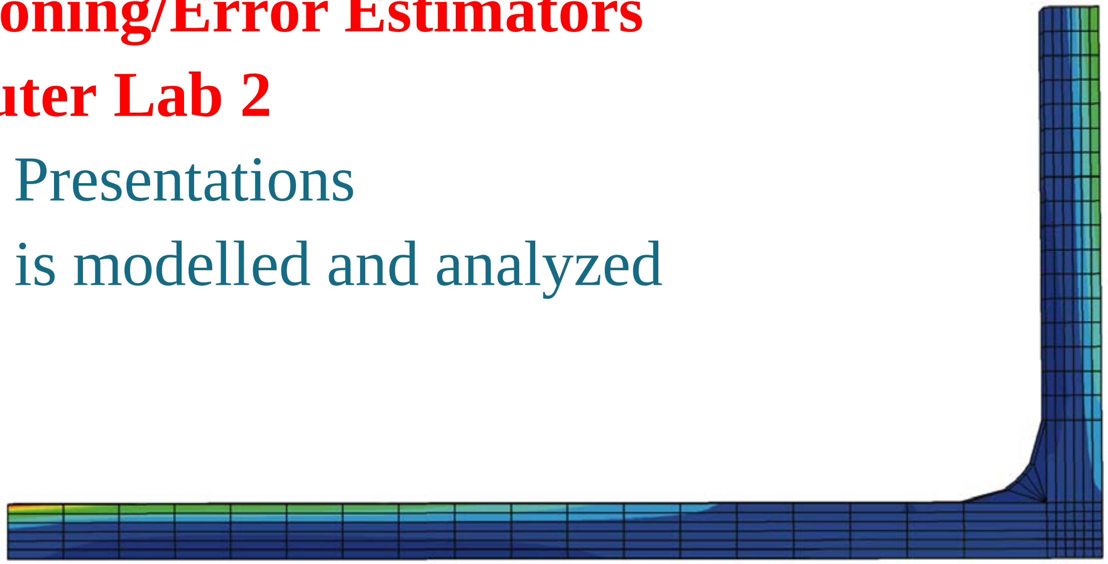


Course Overview

- 04.04.2016 – 1D Elements (truss/beam)
- 11.04.2016 – 2D Elements (plane stress/strain)
- 25.04.2016 – Practical application of the MFE

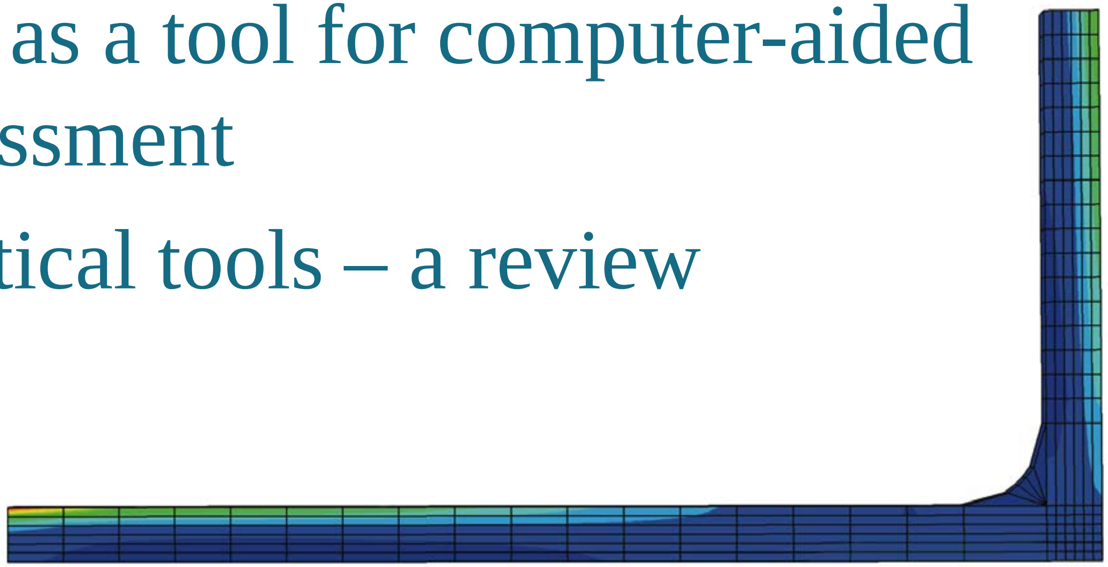
  Practical Considerations
- 02.05.2016 – Results Interpretation
- **09.05.2016 - Demo Session: Integration/Conditioning/Error Estimators**
- 23.05.2016 – **Computer Lab 2**
- 30.05.2016 – Project Presentations

  A Real Test Case is modelled and analyzed



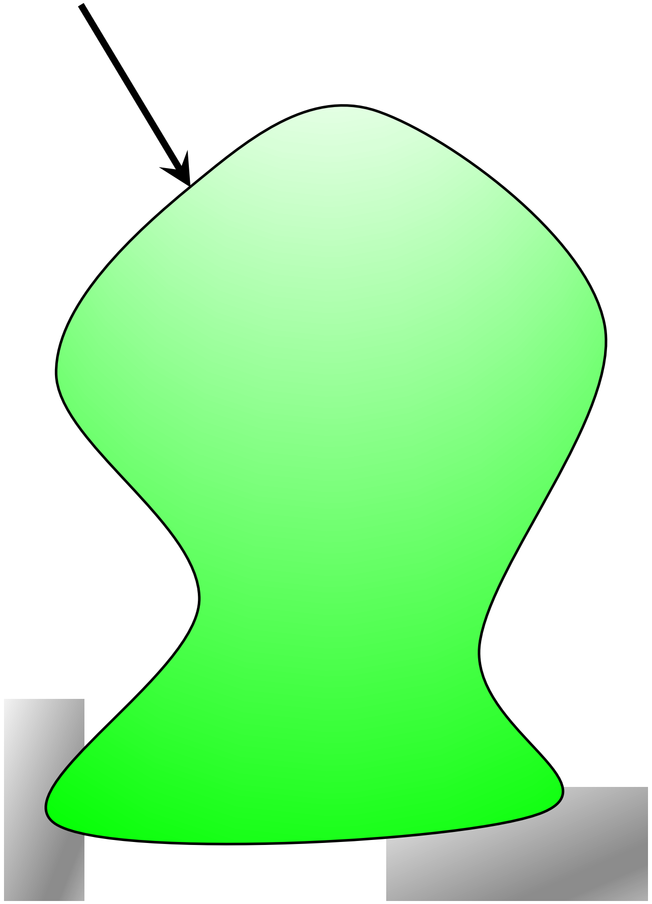
Today's Lecture

- An overview of the MFE I course
- MFE development
- Introduction to the use of Finite Elements
- Modelling the physical problem
- Finite elements as a tool for computer-aided design and assessment
- Basic mathematical tools – a review



FE Analysis in brief…

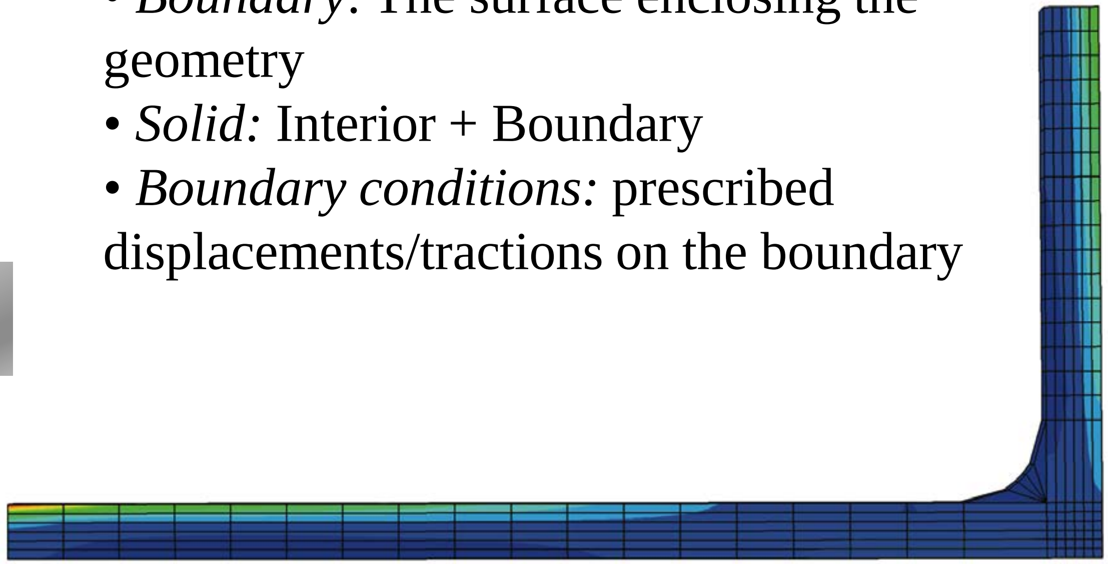
FEA was originally developed for solid mechanics applications.



Object: A Solid with known mechanical properties. (a skyscraper; a shaft; bio tissue …)

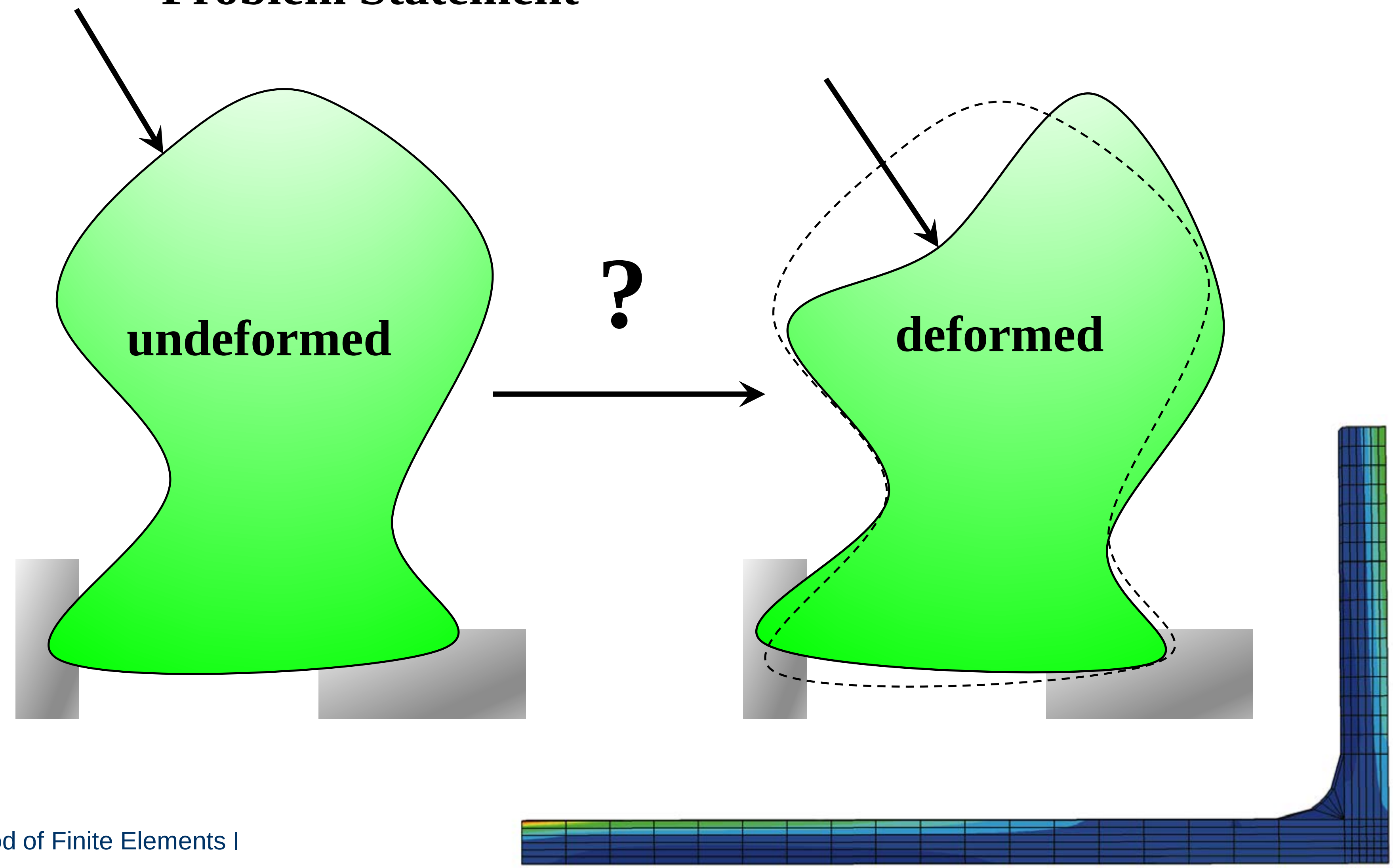
Main Features

- *Boundary*: The surface enclosing the geometry
- *Solid:* Interior + Boundary
- *Boundary conditions:* prescribed displacements/tractions on the boundary



FE Analysis in brief…

Problem Statement



How does it work?

Pre-processor

Physical Model

Describe the problem: Simplify a real engineering problem into a problem that can be solved by FEA

FE model

Discretize/mesh the solid, define material properties, apply boundary conditions

Solver

Theory

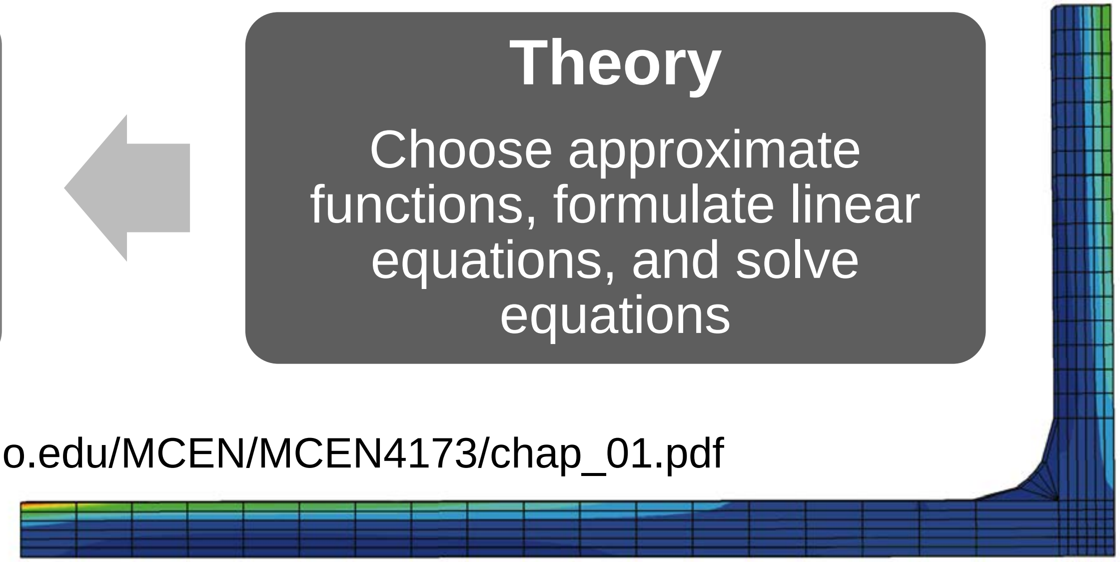
Choose approximate functions, formulate linear equations, and solve equations

Post-processor

Results

Obtain, visualize and explain the results

Source: http://www.colorado.edu/MCEN/MCEN4173/chap_01.pdf



MFE development

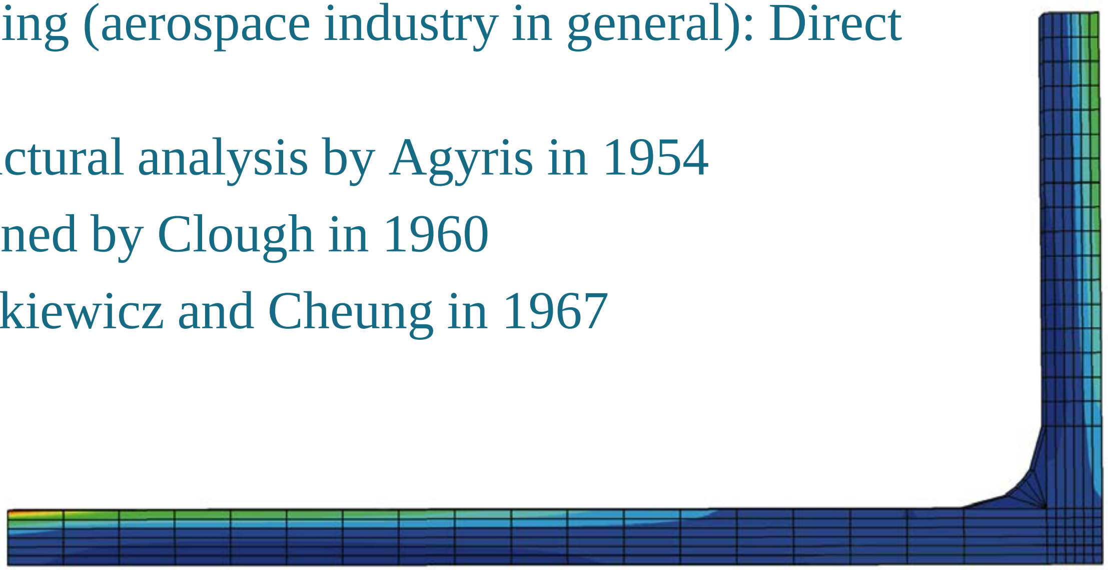
The MFE is the confluence of three ingredients: matrix structural analysis, variational approach and a computer

Theoretical Formulation

1. “Lösung von Variationsproblemen” by W. Ritz in 1908
2. “Weak formulation” by B. Galerkin in 1915
3. “Mathematical foundation” by R. Courant ca. 1943

Formulation & First Applications (1950s and 1960s)

1. 1950s, M.J. Turner at Boeing (aerospace industry in general): Direct Stiffness Method
2. Matrix formulation of structural analysis by Agyris in 1954
3. Term ‚Finite Element‘ coined by Clough in 1960
4. First book on EM by Zienkiewicz and Cheung in 1967



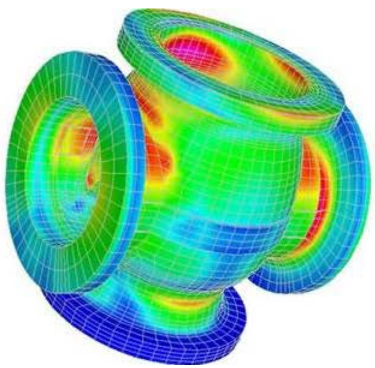
MFE development

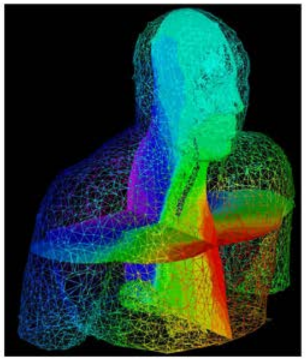
Commercial Software (since 1970s)

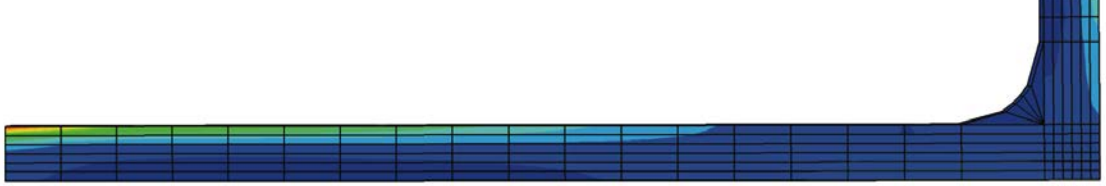
1. General purpose packages for main frames (Abaqus..) in 1970s
2. Special purpose software for PCs in 1980s

During this class, the following software packages will be used:

ABAQUS, ANSYS, CUBUS, SAP2000

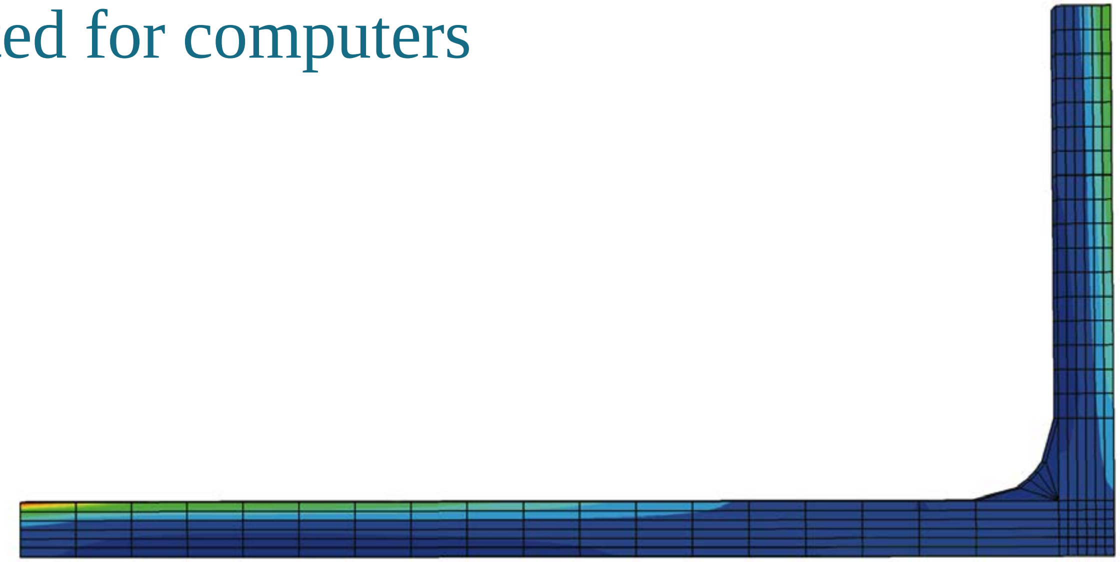






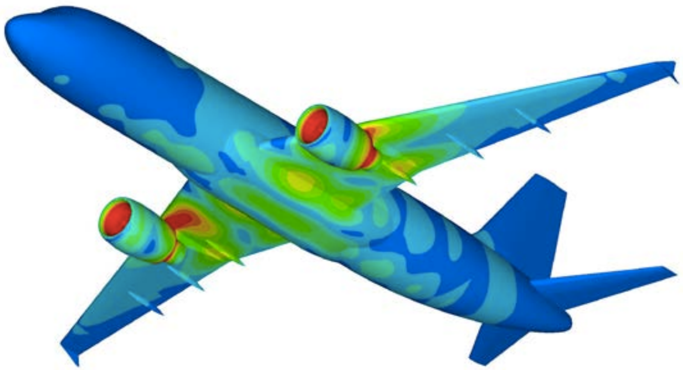
FEM is a big success story, because it…

1. can handle very complex geometry
2. can handle a wide variety of engineering problems
   - mechanics of solids & fluids
   - dynamics/heat/electrostatic problems…
3. can handle complex restraints & loading
4. is very well suited for computers

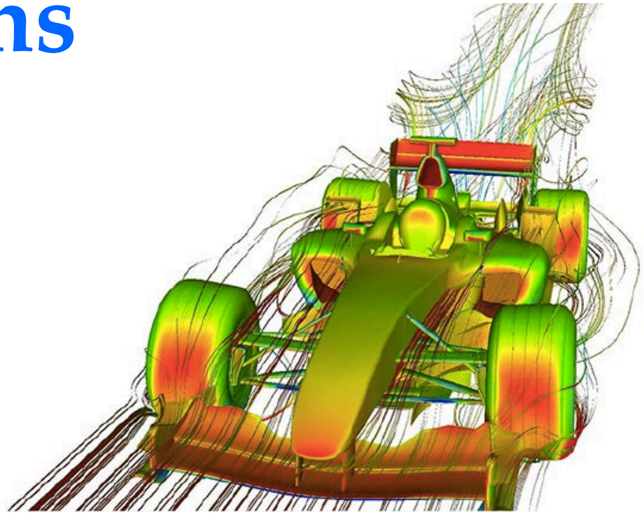


Applications

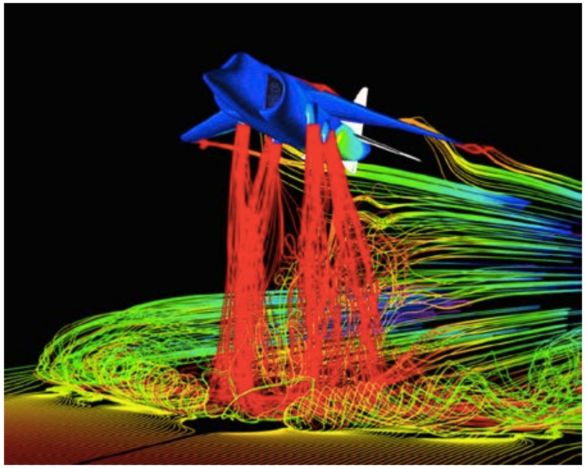
aerospace



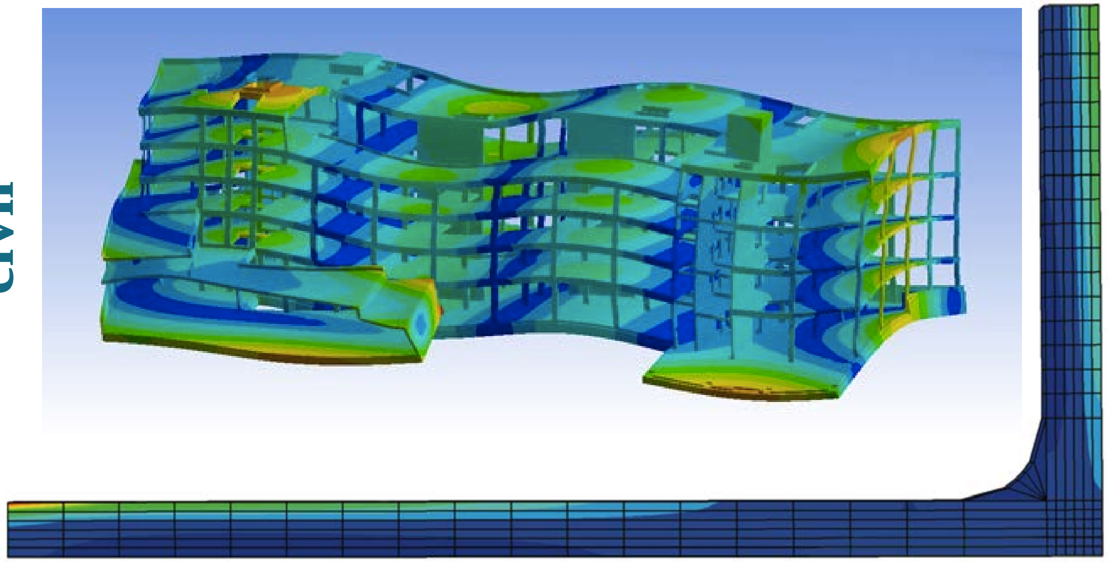
mechanical



fluid dynamics

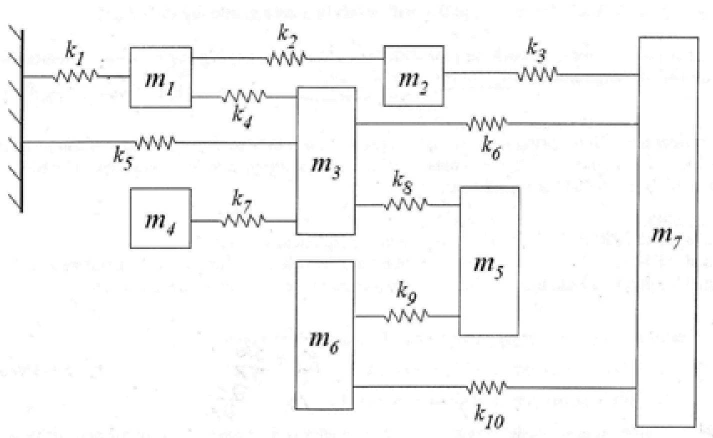


civil



Classification of Engineering Systems

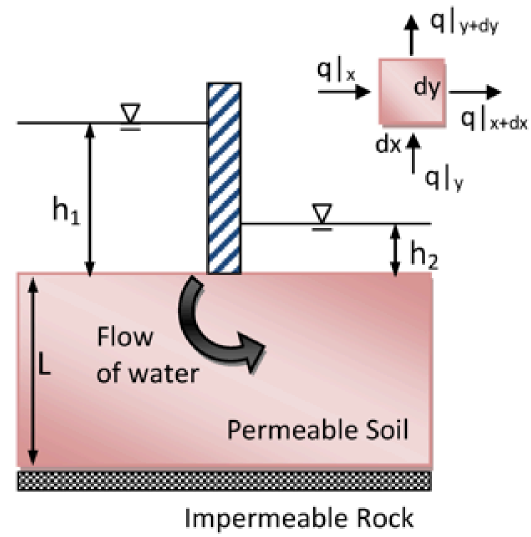
Discrete



$$F = KX$$

Direct Stiffness Method

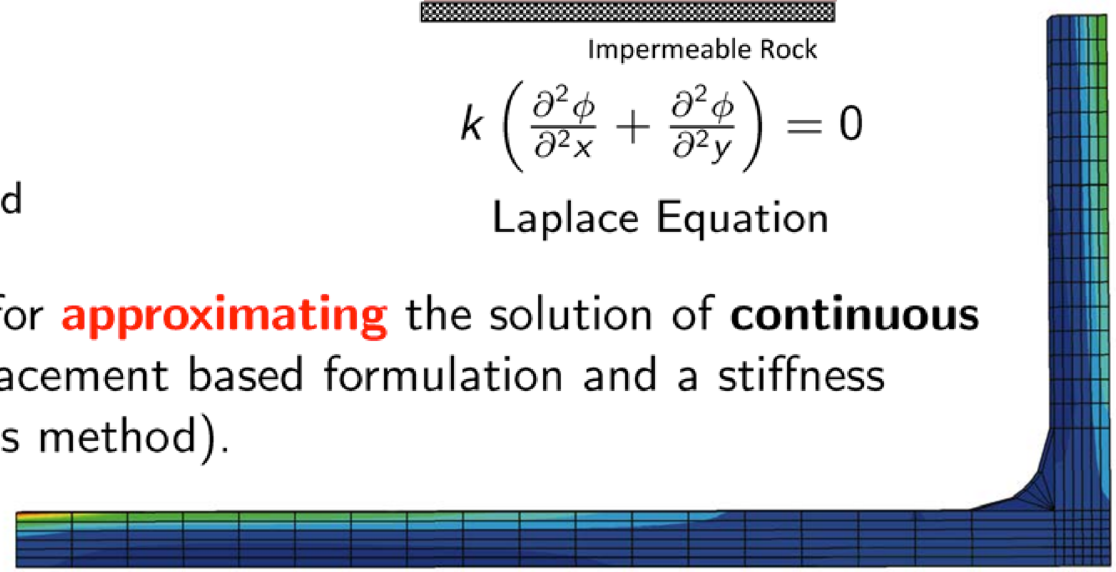
Continuous



$$k\left(\frac{\partial^2\phi}{\partial^2 x} + \frac{\partial^2\phi}{\partial^2 y}\right) = 0$$

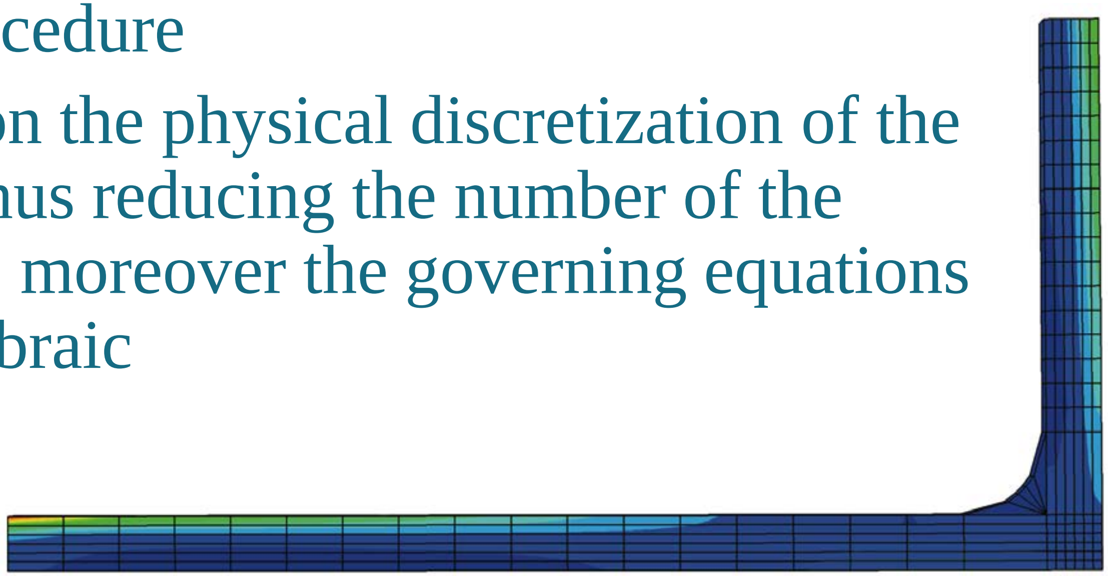
Laplace Equation

FEM: Numerical Technique for **approximating** the solution of **continuous systems**. We will use a displacement based formulation and a stiffness based solution (direct stiffness method).



Introduction to the Use of Finite Elements

- Within the framework of continuum mechanics dependencies between geometrical and physical quantities are formulated on a differentially small element and then extended to the whole continuum
- As a result we obtain differential, partial differential or integral equations for which, generally, an analytical solution is not available – they have to be solved using some numerical procedure
- The MFE is based on the physical discretization of the observed domain, thus reducing the number of the degrees of freedom; moreover the governing equations are, in general, algebraic

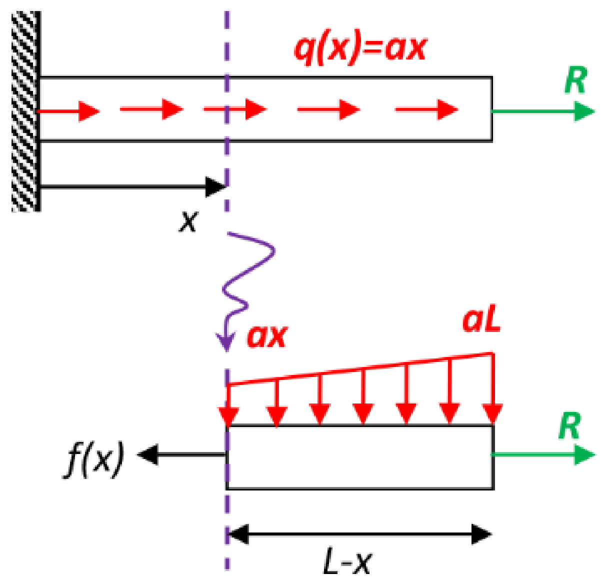


How is the Physical Problem formulated?

The formulation of the equations governing the response of a system under specific loads and constraints at its boundaries is usually provided in the form of a differential equation. The differential equation also known as the **strong form** of the problem is typically extracted using the following sets of equations:

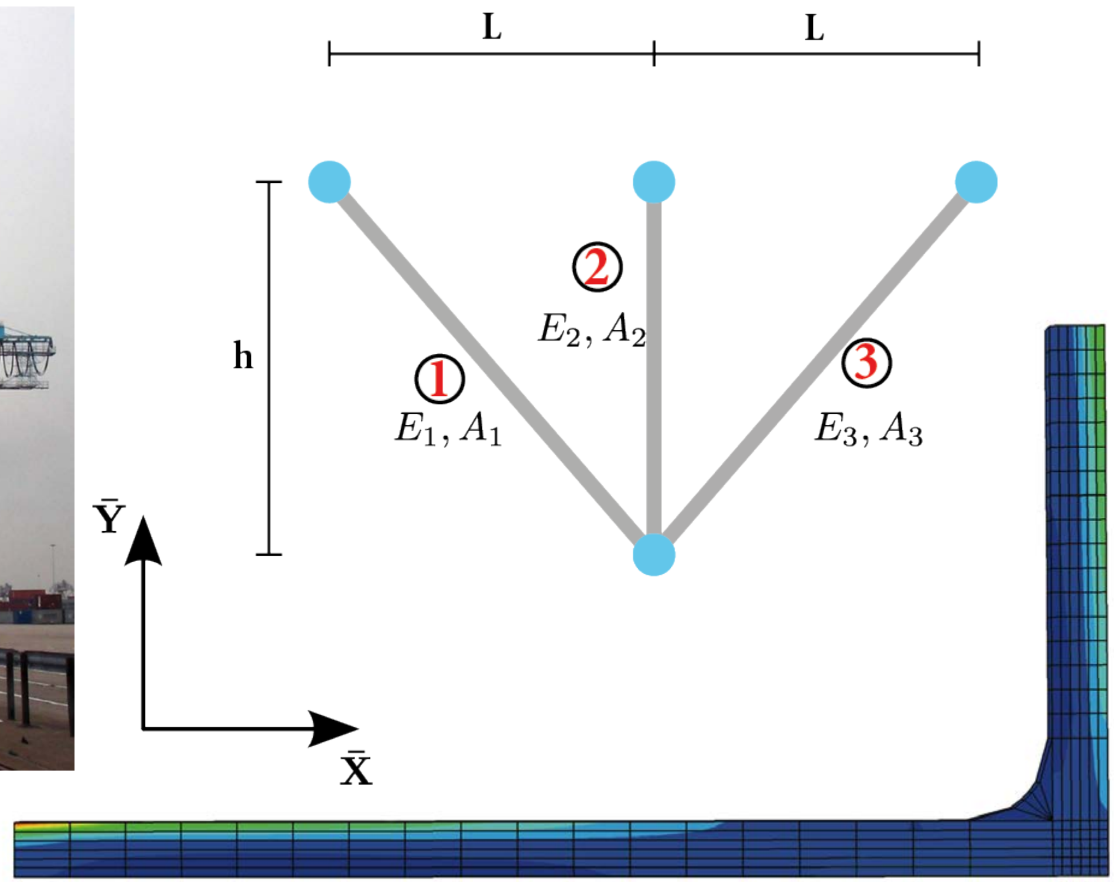
1. Equilibrium Equations
   ex. $f(x) = R + \frac{aL + ax}{2}(L - x)$
2. Constitutive Requirements Equations
   ex. $\sigma = E\epsilon$
3. Kinematics Relationships
   ex. $\epsilon = \frac{du}{dx}$

Axial bar Example



Steps in the MFE

- The continuum is discretized using a mesh of finite elements.



Steps in the MFE

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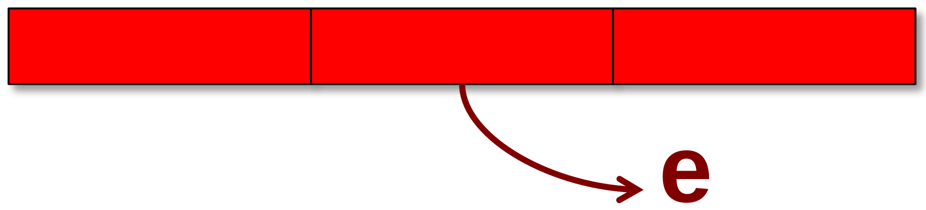
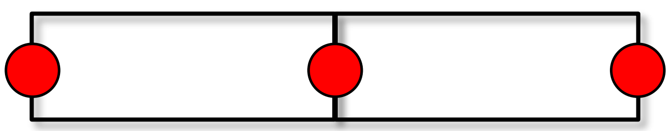
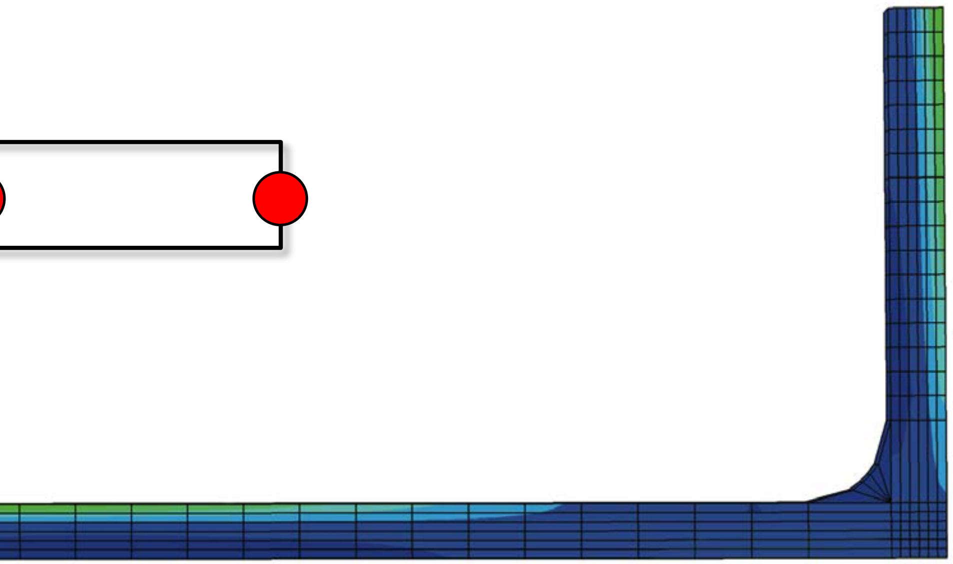


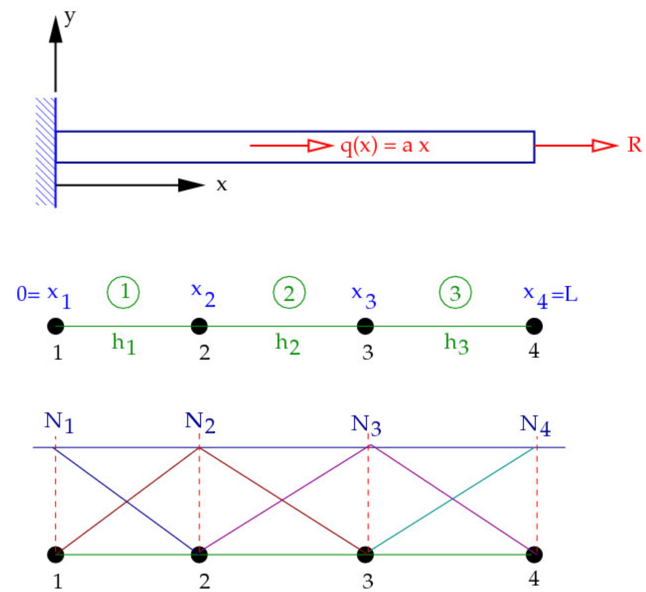
- These elements are connected at nodes located on the element boundaries.





Steps in the MFE

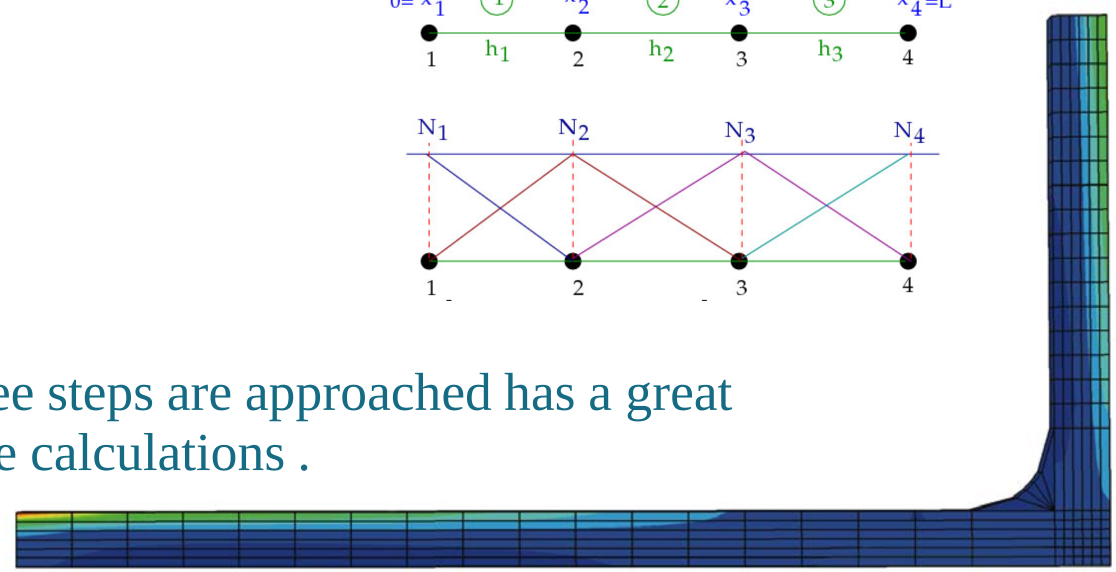
- State of deformation, stresses, etc. in each element is described by interpolation (shape) functions and corresponding values in the nodes; these nodal values are the basic unknowns of the MFE.



- Bounded and Continuous
- One for each node
- $N_i^e(x_j^e) = \delta_{ij}$, where

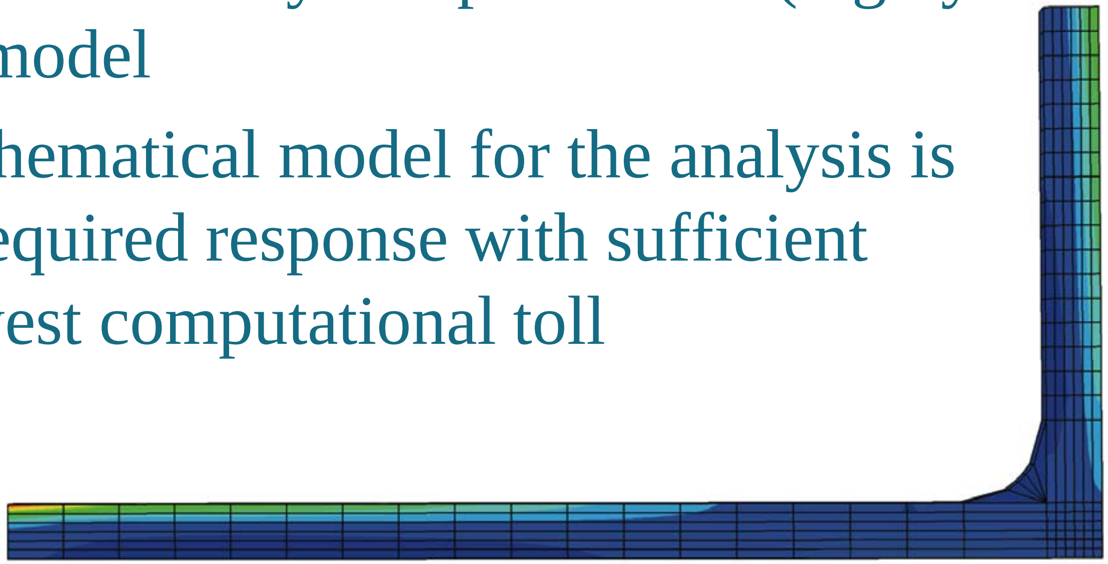
$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The way in which these three steps are approached has a great influence on the results of the calculations .

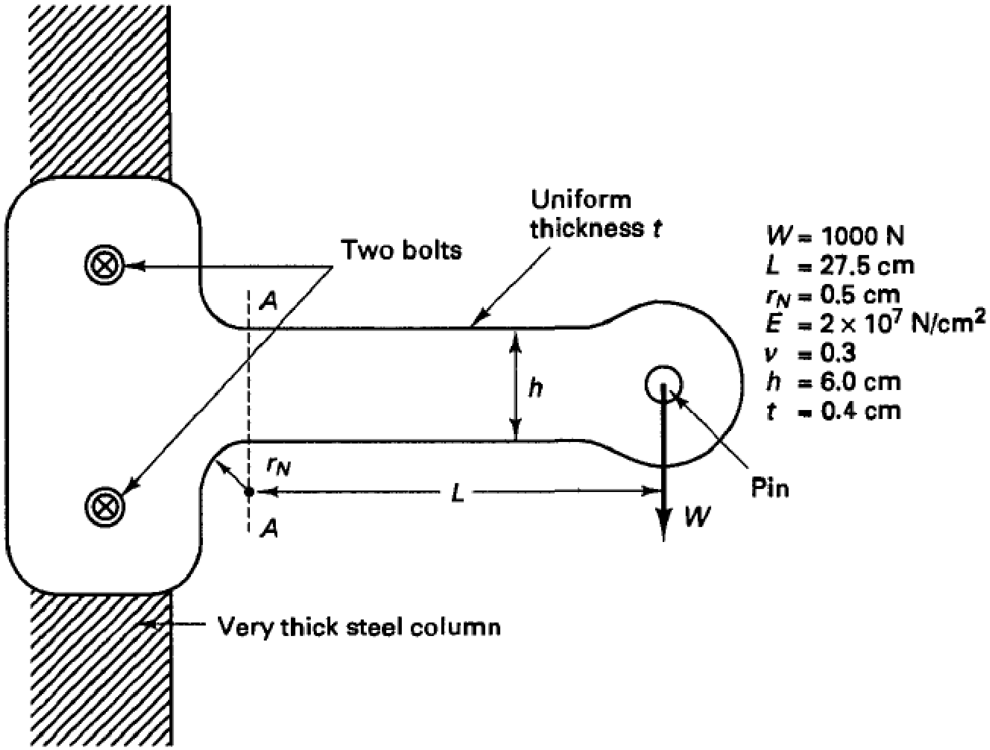


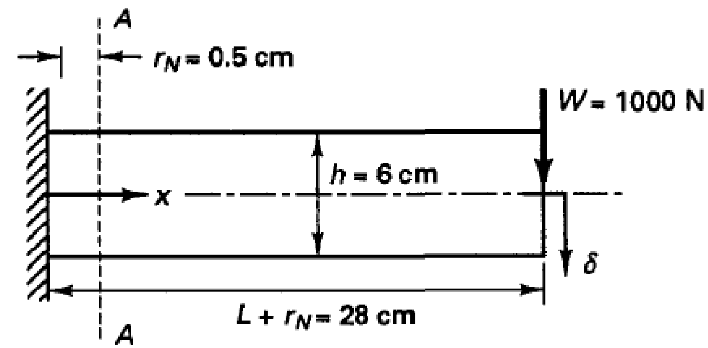
Modelling of the Physical Problem

- The MFE is only a way of solving the mathematical model
- The solution of the physical problem depends on the quality of the mathematical model – the choice of the mathematical model is crucial
- The chosen mathematical model is reliable if the required response can be predicted within a given level of accuracy compared to the response of a very comprehensive (highly refined) mathematical model
- The most effective mathematical model for the analysis is the one that gives the required response with sufficient accuracy and at the lowest computational toll



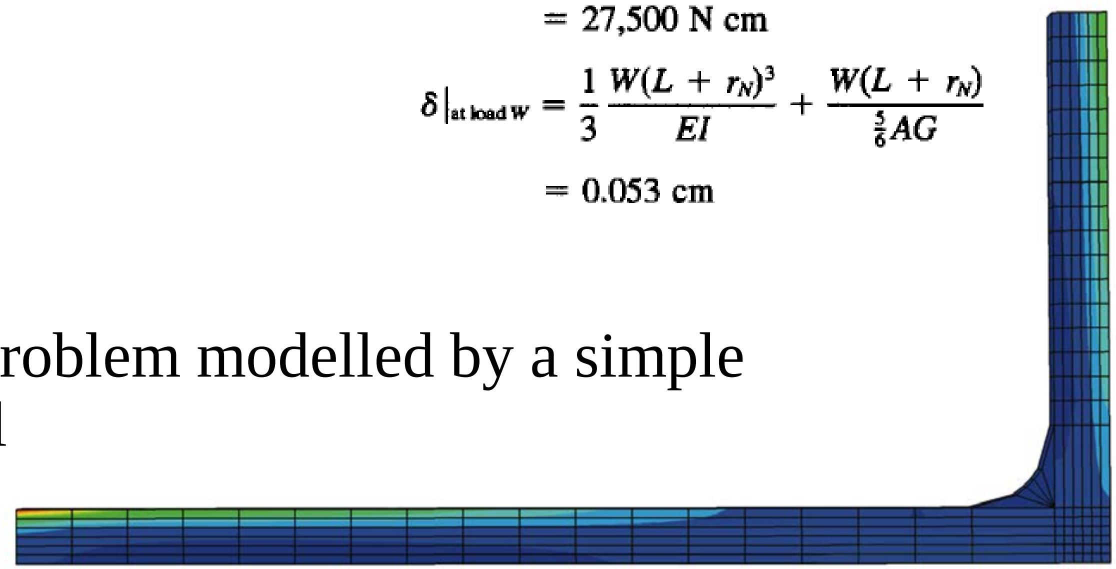
Simple Example



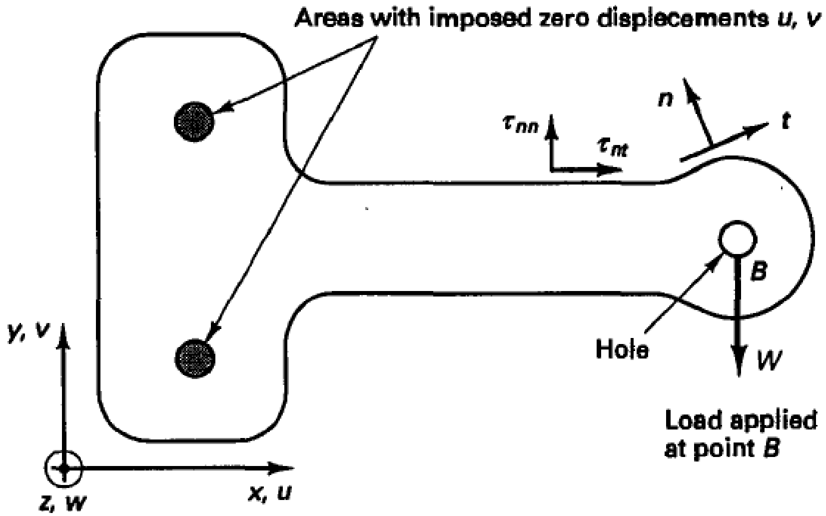


$$M = WL$$
$$= 27{,}500 \text{ N cm}$$
$$\delta\big|_{\text{at load } W} = \frac{1}{3}\frac{W(L + r_N)^3}{EI} + \frac{W(L + r_N)}{\frac{5}{6}AG}$$
$$= 0.053 \text{ cm}$$

Complex physical problem modelled by a simple mathematical model



Simple Example



$$\delta\big|_{\text{at load } W} = 0.064 \text{ cm}$$

$$M\big|_{x=0} = 27{,}500 \text{ N cm}$$

Equilibrium equations (see Example 4.2)

$$\left.\begin{aligned}\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0\end{aligned}\right\} \text{ in domain of bracket}$$

$\tau_{nn} = 0,\ \tau_{nt} = 0$ on surfaces except at point B and at imposed zero displacements

Stress-strain relation (see Table 4.3):

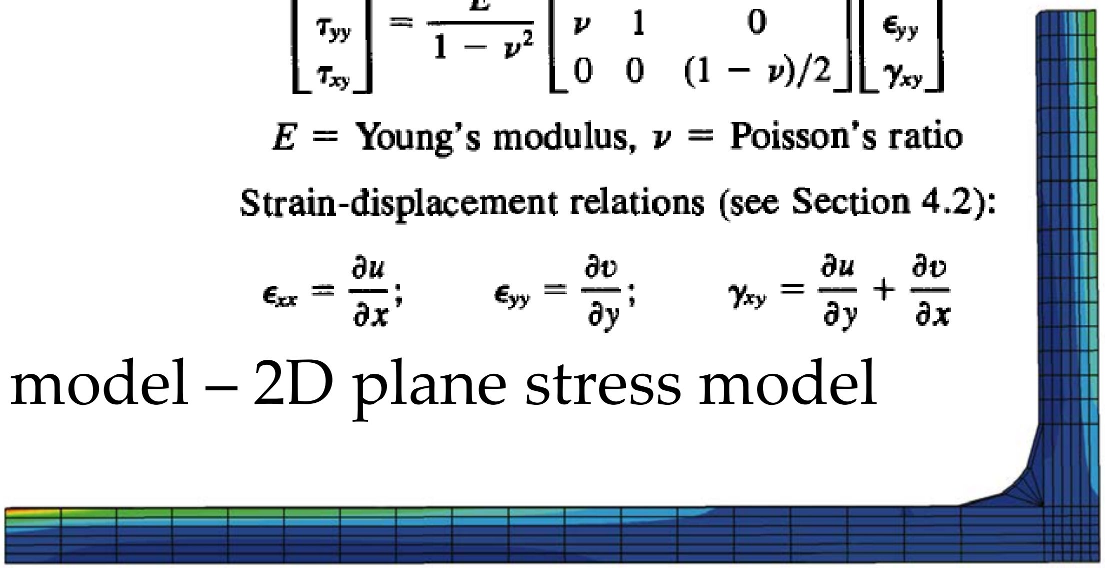
$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{bmatrix}$$

E = Young's modulus, ν = Poisson's ratio

Strain-displacement relations (see Section 4.2):

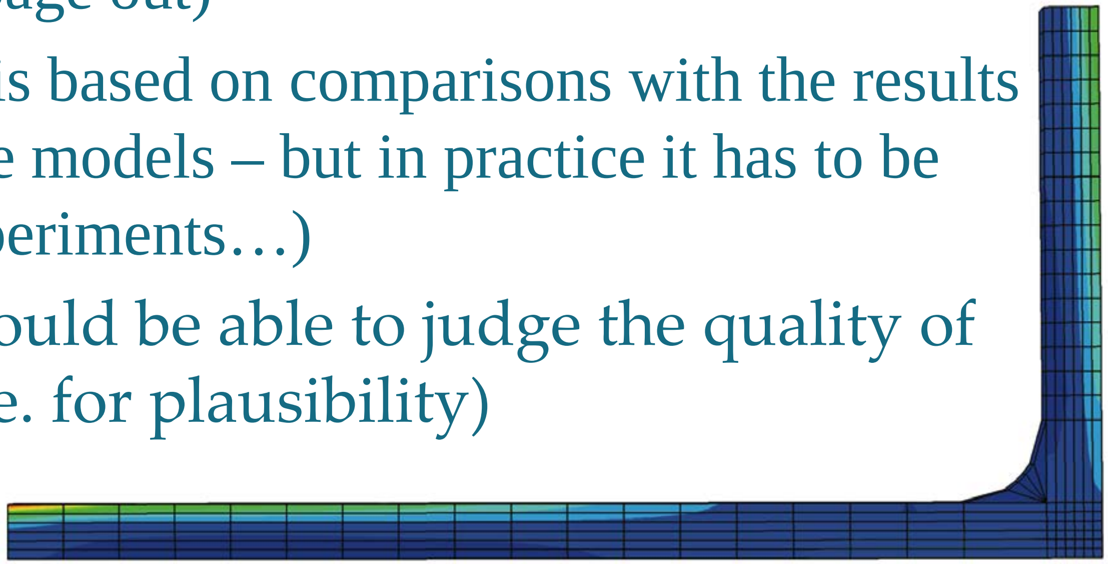
$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Detailed reference model – 2D plane stress model

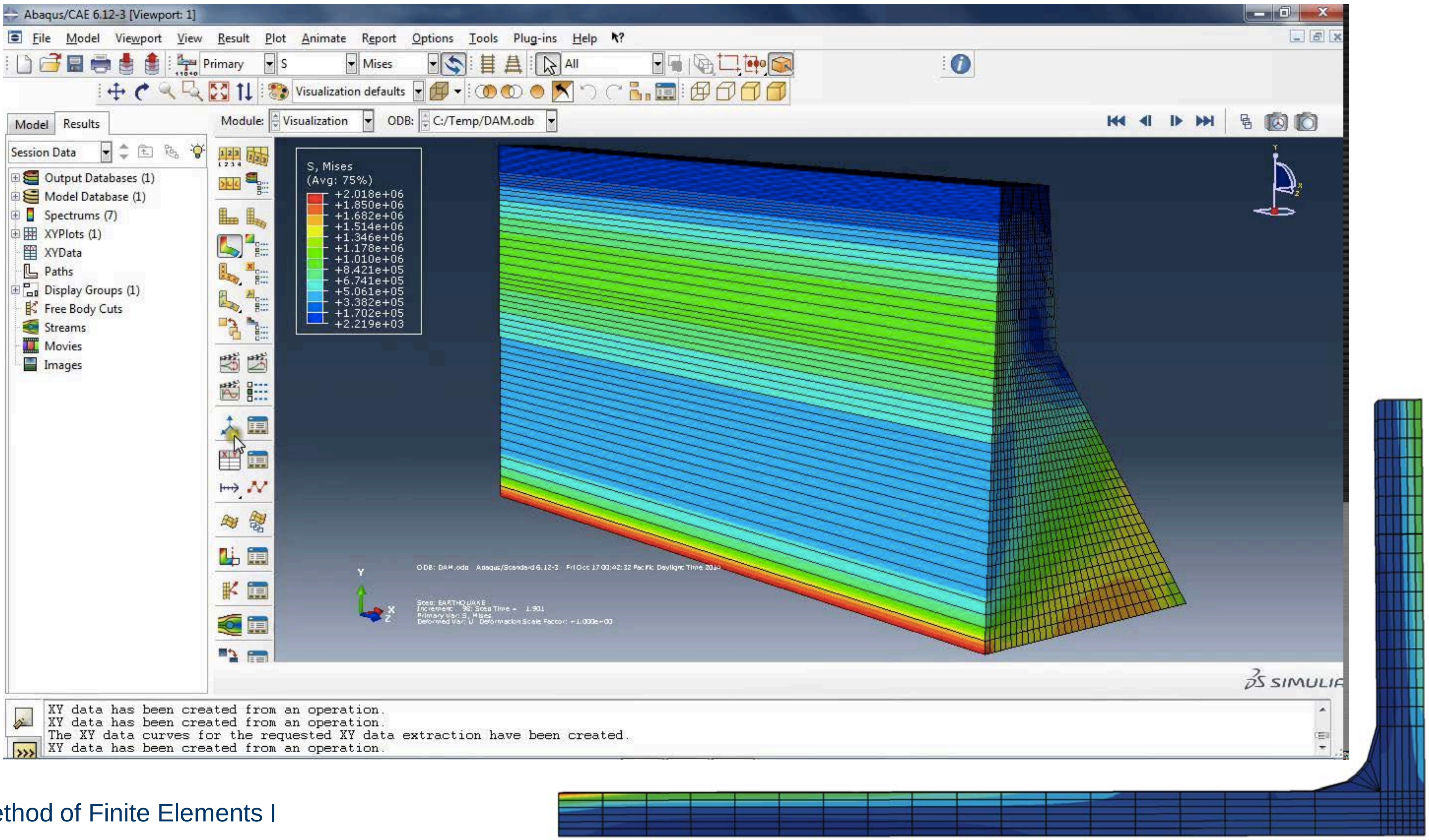


Considerations

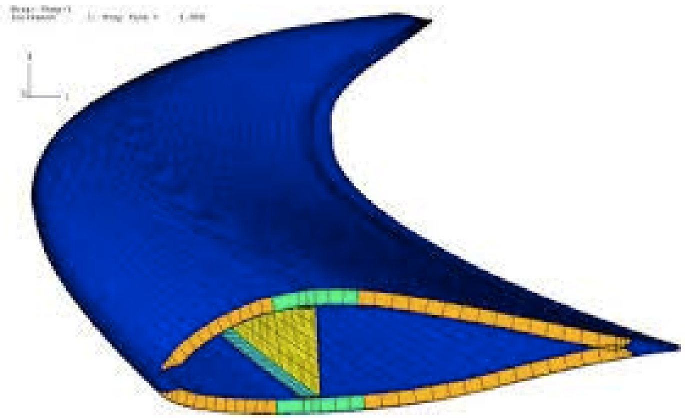
- Choice of mathematical model must correspond to desired response
- The most effective mathematical model delivers reliable answers with the least amount of effort
- Any solution (including MFE) of a mathematical model is limited to information contained in or fed into the model: bad input – bad output (garbage in – garbage out)
- Assessment of accuracy is based on comparisons with the results from very comprehensive models – but in practice it has to be based on experience (experiments…)
- The engineer (user) should be able to judge the quality of the obtained results (i.e. for plausibility)

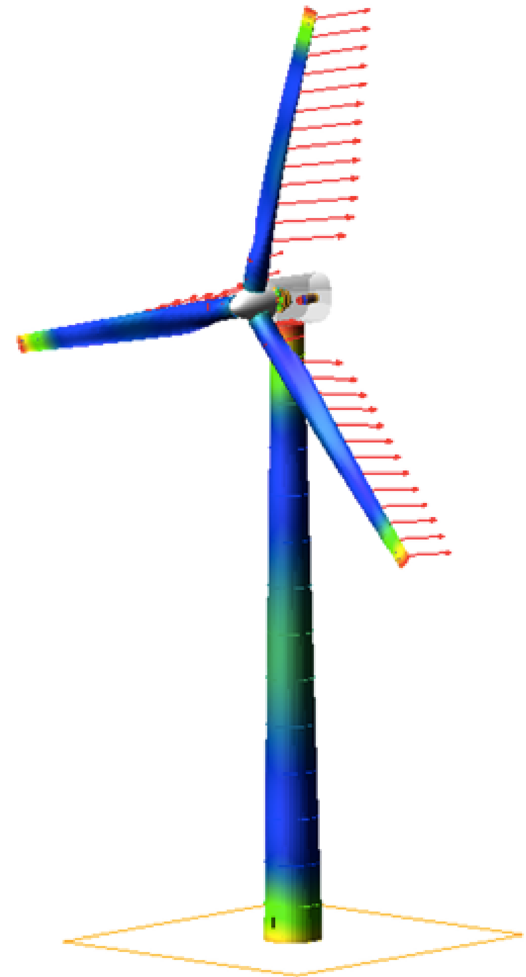


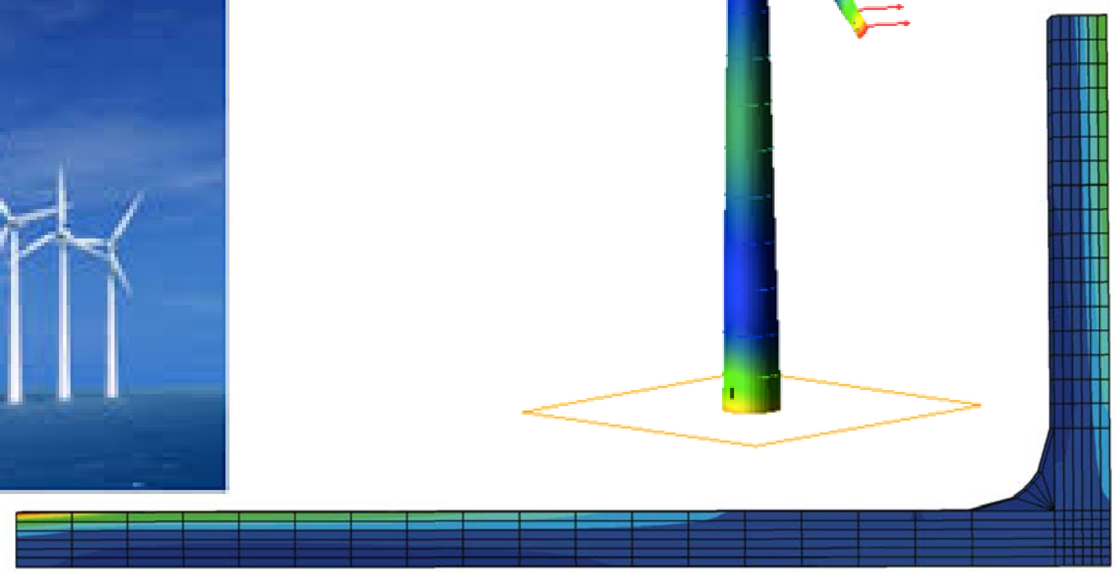
Seismic Analysis of a Concrete Gravity Dam in ABAQUS



Analysis of a Wind Turbine Structure in ANSYS

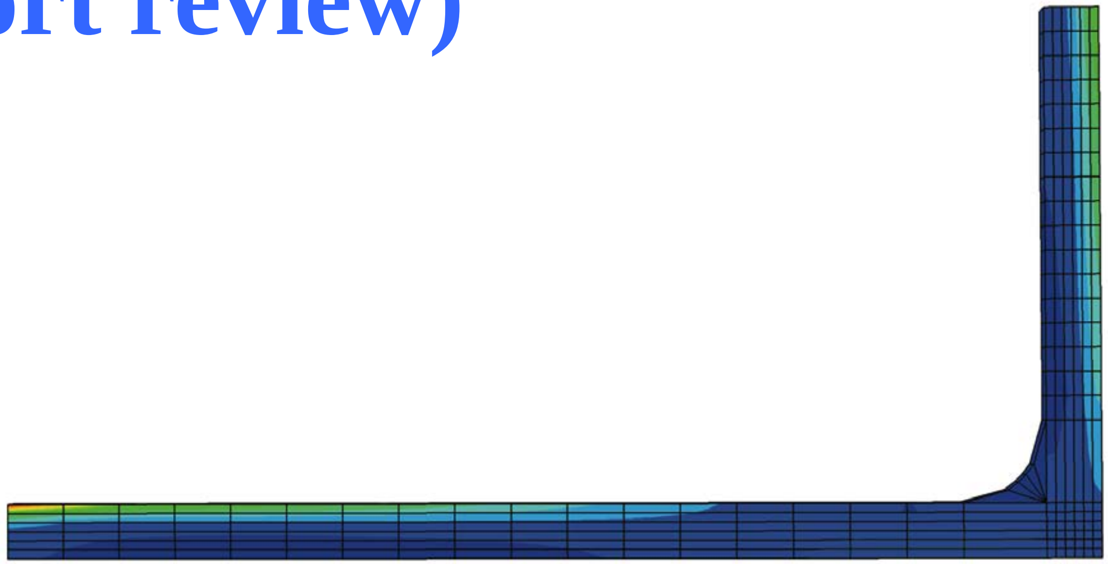






Chapter 1

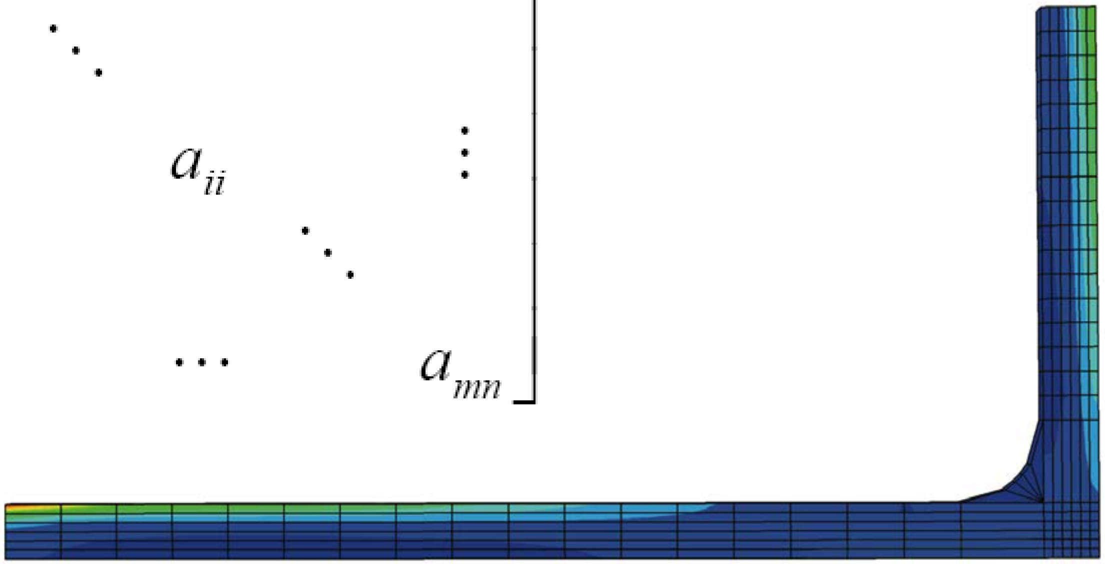
Fundamental Mathematical Concepts (short review)



Matrices

A matrix is an array of ordered numbers. A general matrix consists of m·n numbers arranged in m rows and n columns, thus the matrix is of order m x n (m by n). When we have only one row (m = 1) or one column (n = 1), **A** is also called a vector

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1i} & & a_{1n} \\ \vdots & \ddots & & & \\ a_{i1} & & a_{ii} & & \vdots \\ \vdots & & & \ddots & \\ a_{m1} & & \cdots & & a_{mn} \end{bmatrix}$$

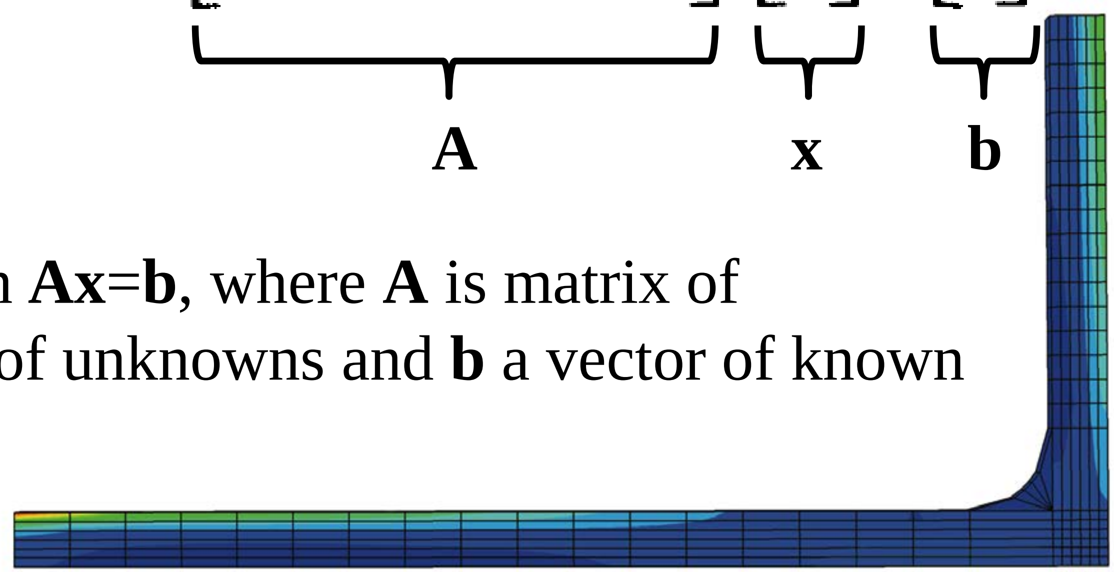


Matrices

When dealing with systems of linear equations, a matrix formulation proves highly advantages:

$$
\begin{aligned}
5x_1 - 4x_2 + x_3 &= 0 \\
-4x_1 + 6x_2 - 4x_3 + x_4 &= 1 \\
x_1 - 4x_2 + 6x_3 - 4x_4 &= 0 \\
x_2 - 4x_3 + 5x_4 &= 0
\end{aligned}
\;\longrightarrow\;
\underbrace{\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}
$$

This results in an equation **Ax**=**b**, where **A** is matrix of coefficients, **x** is a vector of unknowns and **b** a vector of known quantities.



Basic Matrix Operations

Scalar multiplication:

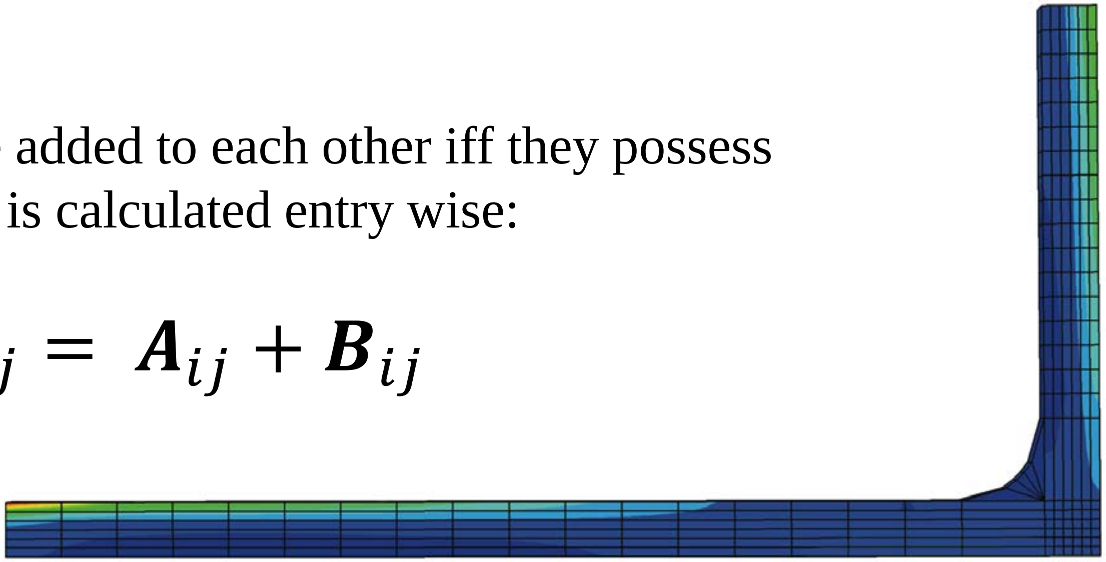
A matrix $\boldsymbol{A}$ is multiplied by a scalar value c such that $c\boldsymbol{A}$. This is achieved by multiplying each entry of $\boldsymbol{A}$ by c:

$$(c\boldsymbol{A})_{ij} = c \cdot \boldsymbol{A}_{ij} \qquad \text{where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

Addition:

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ may be added to each other iff they possess the same order. The sum $\boldsymbol{A}+\boldsymbol{B}$ is calculated entry wise:

$$(\boldsymbol{A}+\boldsymbol{B})_{ij} = \boldsymbol{A}_{ij} + \boldsymbol{B}_{ij}$$



Basic Matrix Operations

Transposition:

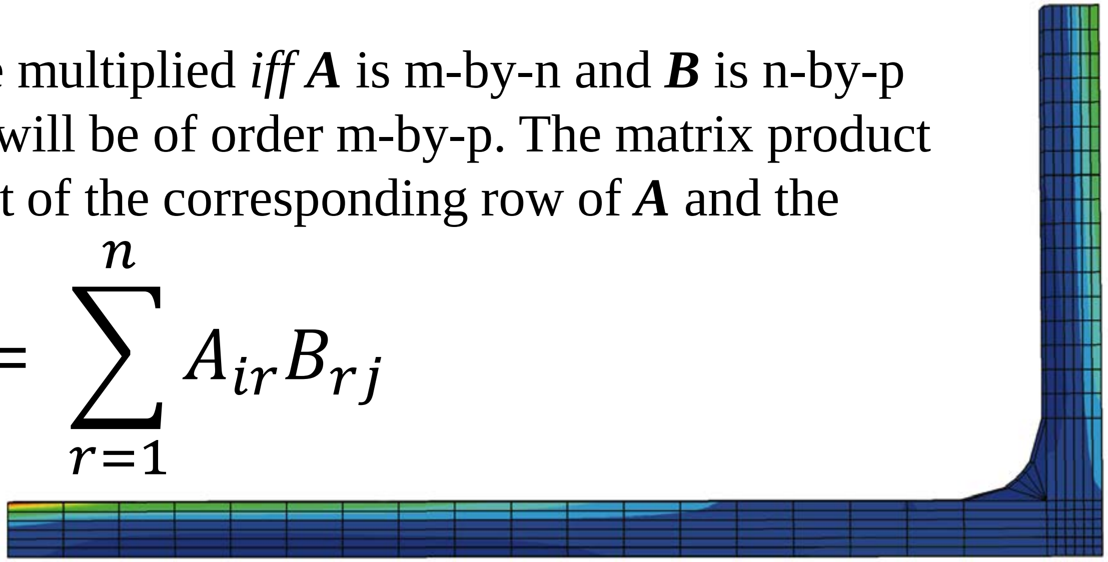
The transpose of a matrix ***A*** denoted by $\boldsymbol{A}^{\boldsymbol{T}}$ is obtained by interchanging The rows and columns pf a matrix:

$$(\boldsymbol{A}^T)_{ij} = \boldsymbol{A}_{ji}$$

Multiplication:

Two matrices ***A*** and ***B*** may be multiplied *iff* ***A*** is m-by-n and ***B*** is n-by-p such that the resulting matrix will be of order m-by-p. The matrix product ***AB*** is given by the dot product of the corresponding row of ***A*** and the column of ***B.***

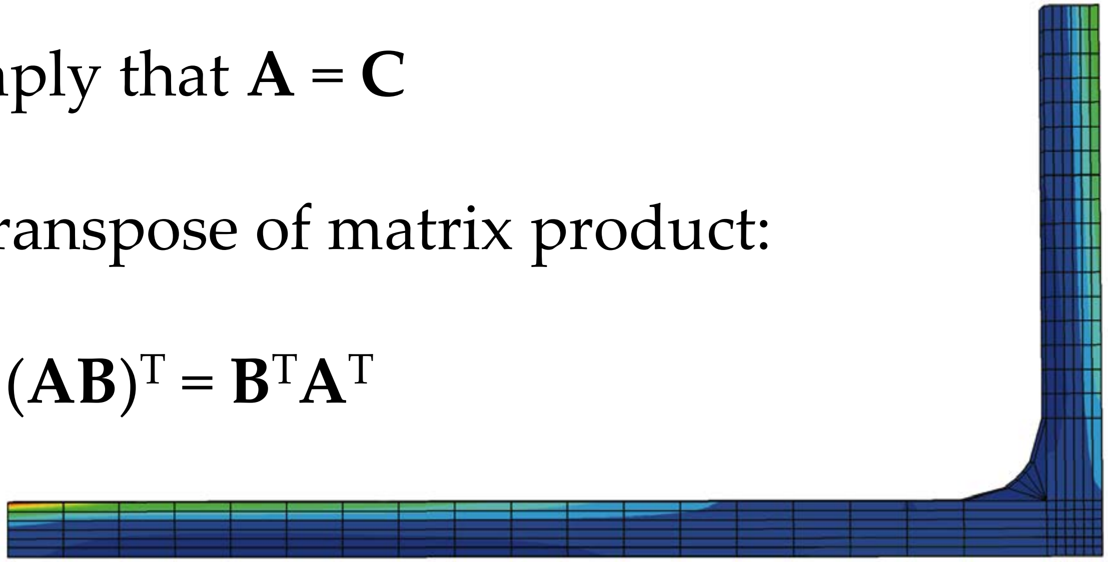
$$[\boldsymbol{AB}]_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$



Rules of Matrix Operations

- Commutative law does not hold, i.e. $\mathbf{AB} \neq \mathbf{BA}$
- Distributive law does hold, i.e. $\mathbf{E} = (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$
- Associative law does hold, i.e. $\mathbf{G} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$
- $\mathbf{AB} = \mathbf{CB}$ does not imply that $\mathbf{A} = \mathbf{C}$
- Special rule for the transpose of matrix product:

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$



Special Square Matrices

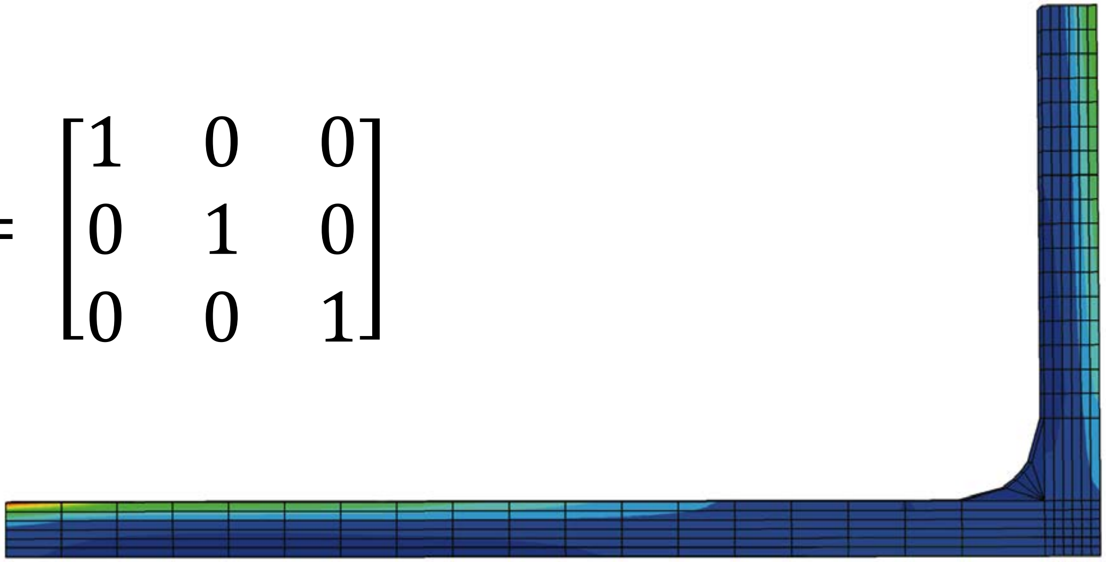
Square matrix:

A matrix is said to be square if m = n

Identity matrix:

The identity matrix is a square matrix with entries on the diagonal equal to 1 while all others are equal to 0. Any square matrix ***A*** multiplied by the identity matrix ***I*** of equal order returns the unchanged matrix ***A.***

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Special Square Matrices

Diagonal Matrix D:

All other entries but those on the diagonal equal to zero

$$\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix},$$

Upper Triangular Matrix U:

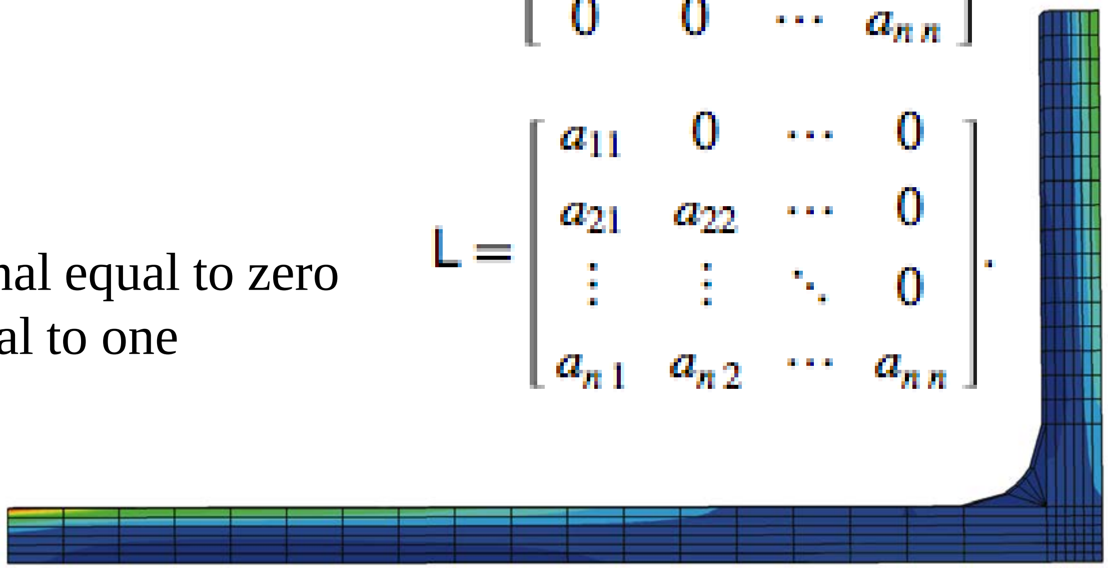
- All entries below the diagonal equal to zero
- Entries on the diagonal equal to one

$$\mathsf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Lower Triangular Matrix L:

- All entries above the diagonal equal to zero
- Entries on the diagonal equal to one

$$\mathsf{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$



Special Matrices

Symmetric Matrix:

A symmetric matrix is a square matrix the satisfies $\boldsymbol{A}^T = \boldsymbol{A}$

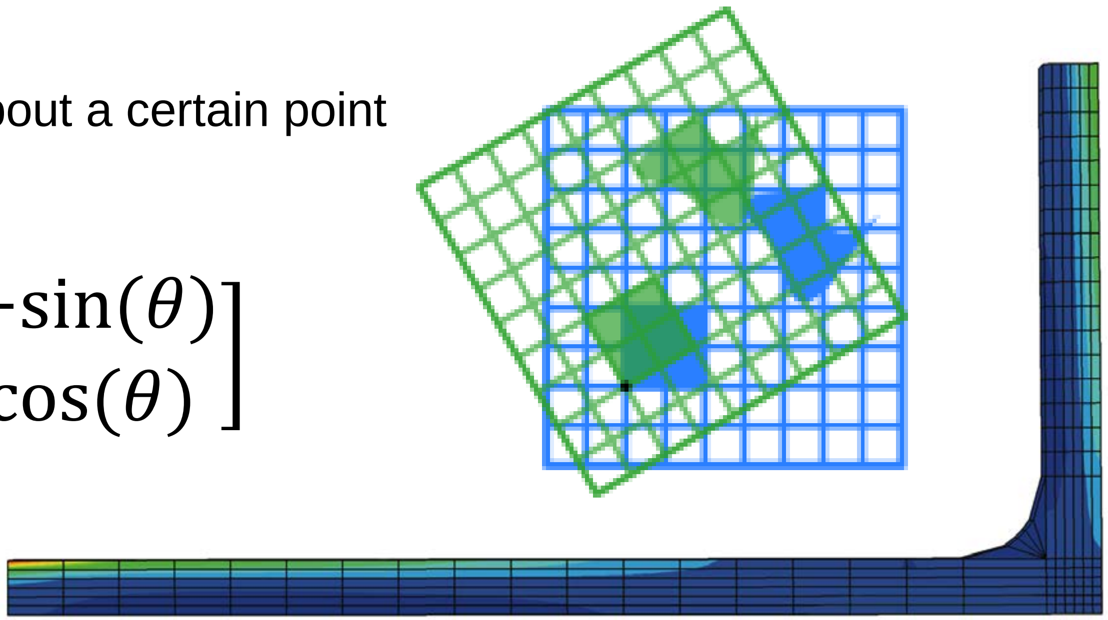
Sparse Matrix:

A matrix with mostly/many zero entries

Rotation Matrix R:

- Used to rotate quantities about a certain point
- In 2D it is given as follows:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



Special Matrices

Banded Matrix:

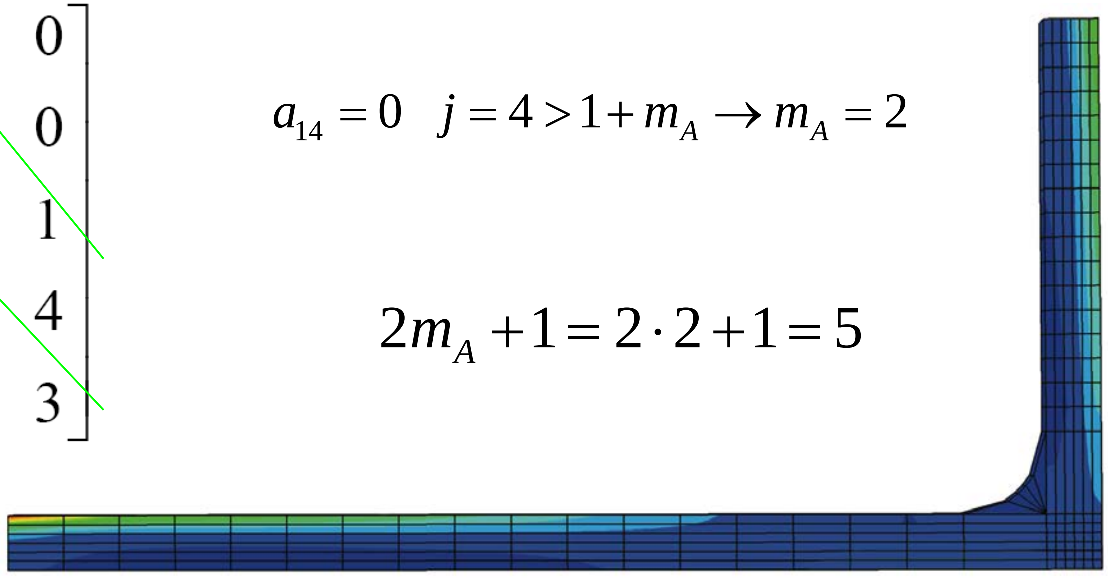
- For symmetric banded matrix **A** we have $a_{ij} = 0$ for $j > i+m_A$, $2m_A+1$ being the bandwidth
- If the half-bandwidth, m_A, of a matrix is zero, we have nonzero elements only on the diagonal of the matrix and denote it as a diagonal matrix (for example, unit matrix).

m_A

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 \\ 1 & 4 & 5 & 6 & 1 \\ 0 & 1 & 6 & 7 & 4 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix}$$

$$a_{14} = 0 \quad j = 4 > 1 + m_A \rightarrow m_A = 2$$

$$2m_A + 1 = 2 \cdot 2 + 1 = 5$$



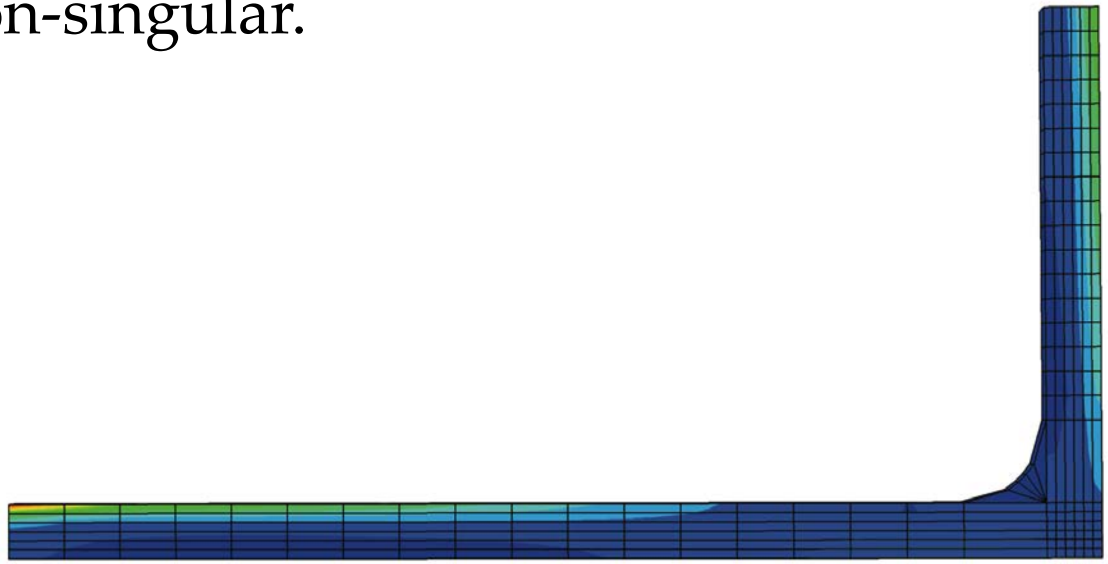
Matrix Inversion

The inverse of a matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$

If a matrix is invertible then there is

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

and $\mathbf{A}$ is said to be non-singular.



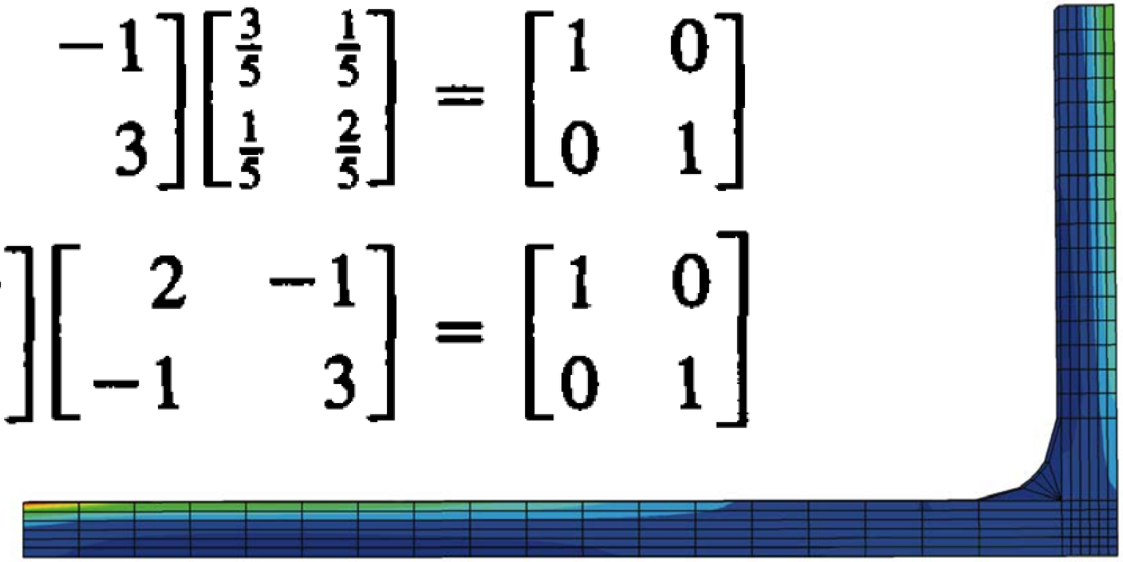
Matrix Inversion

Inversion: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

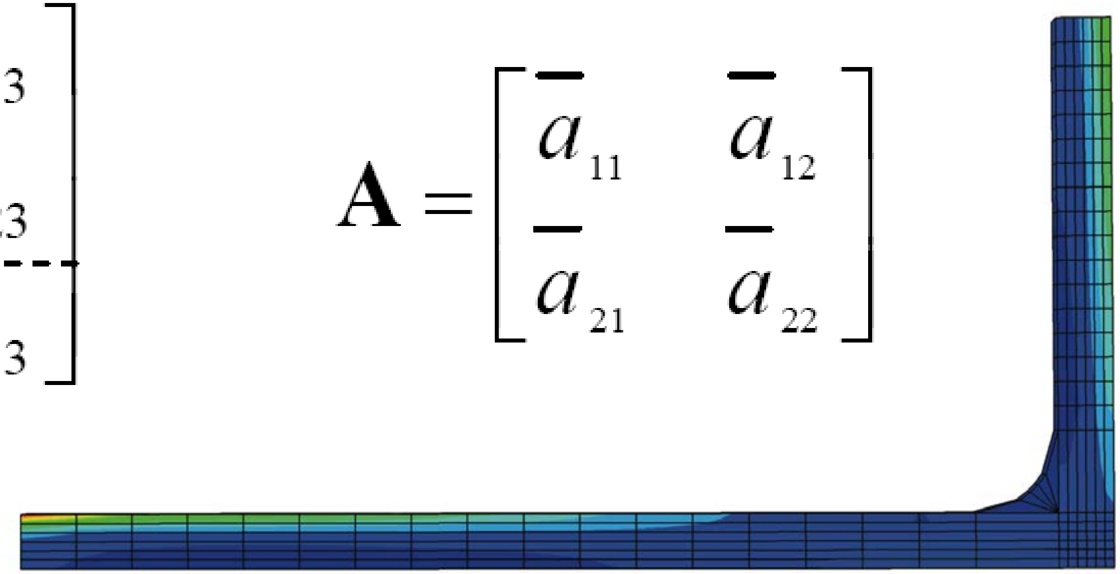


Sub Matrices

- Matrices can be subdivided to facilitate matrix manipulations
- Partitioning lines must run completely across the original matrix

$$\mathbf{A} = \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]$$

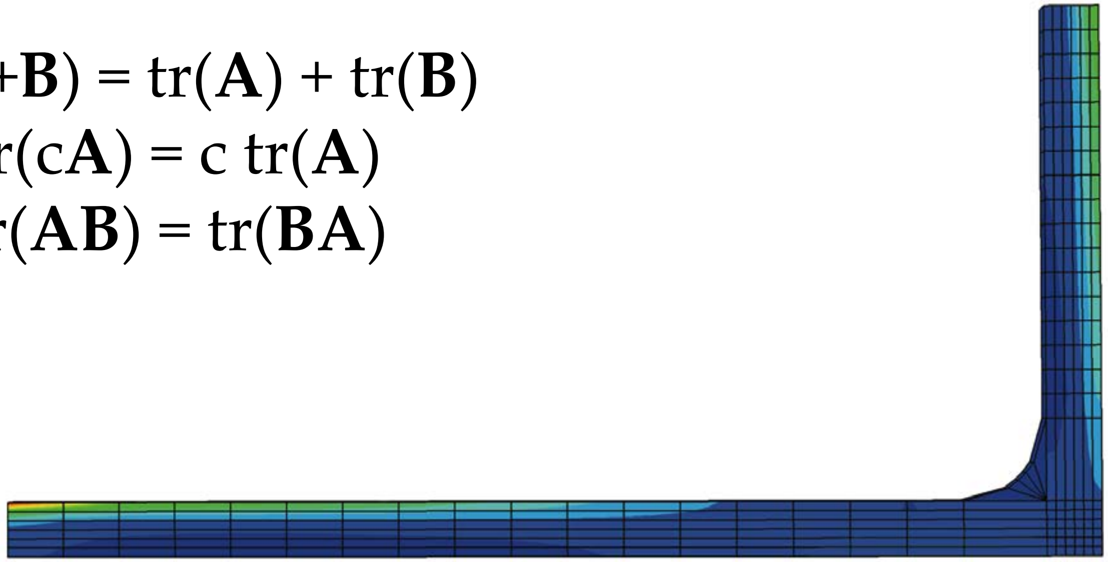
$$\mathbf{A} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}$$



The Trace of a Matrix

- The trace of a matrix **A** is defined only if **A** is a square matrix (nxn)
- The trace of a matrix is a scalar value: $tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$
- Some rules:

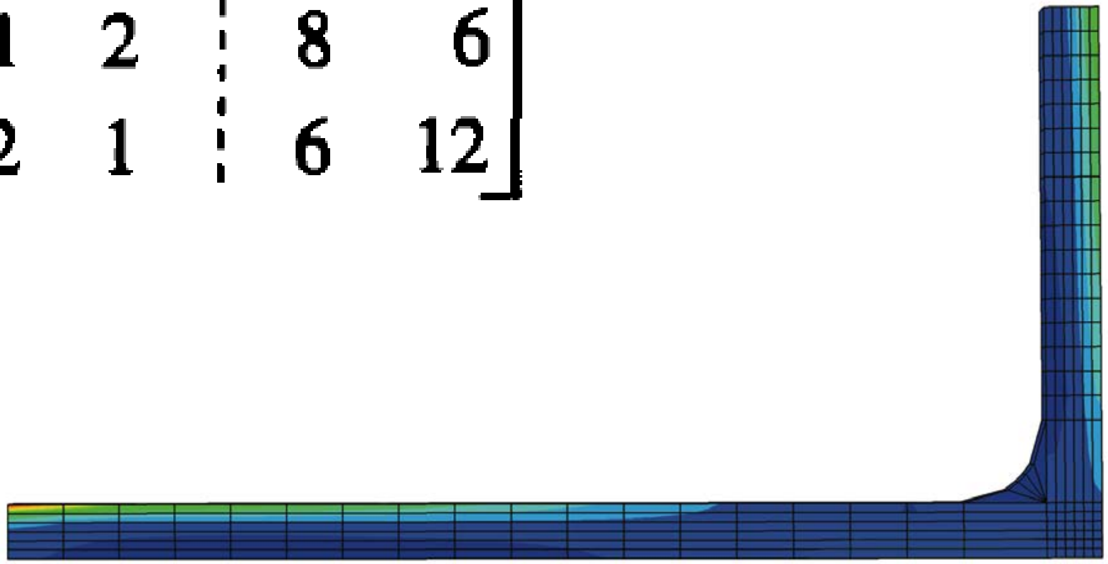
$$\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$
$$\mathrm{tr}(c\mathbf{A}) = c\ \mathrm{tr}(\mathbf{A})$$
$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$



The Trace of a Matrix

- The trace of a matrix **A,** tr(**A**) = 4+6+8+12=30

$$\mathbf{A} = \left[\begin{array}{cc:cc} 4 & 3 & 1 & 2 \\ 3 & 6 & 2 & 1 \\ \hdashline 1 & 2 & 8 & 6 \\ 2 & 1 & 6 & 12 \end{array}\right]$$



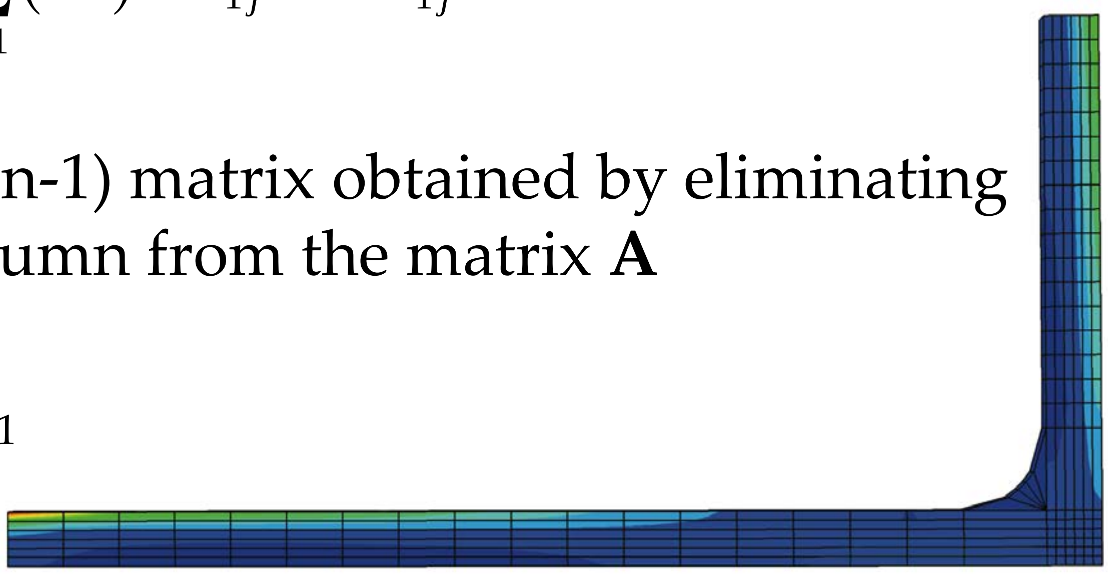
The Determinant of a Matrix

- The determinant of a matrix **A** is defined only if **A** is a square matrix (nxn)
- The determinant of a matrix is a scalar value and is obtained by means of the recurrence formula:

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$$

- where $\mathbf{A}_{1j}$ is the (n-1)x(n-1) matrix obtained by eliminating the 1^{st} row and the j^{th} column from the matrix **A**

- if $\mathbf{A}=[a_{11}]$ then $\det\mathbf{A}=a_{11}$



The Determinant of a Matrix

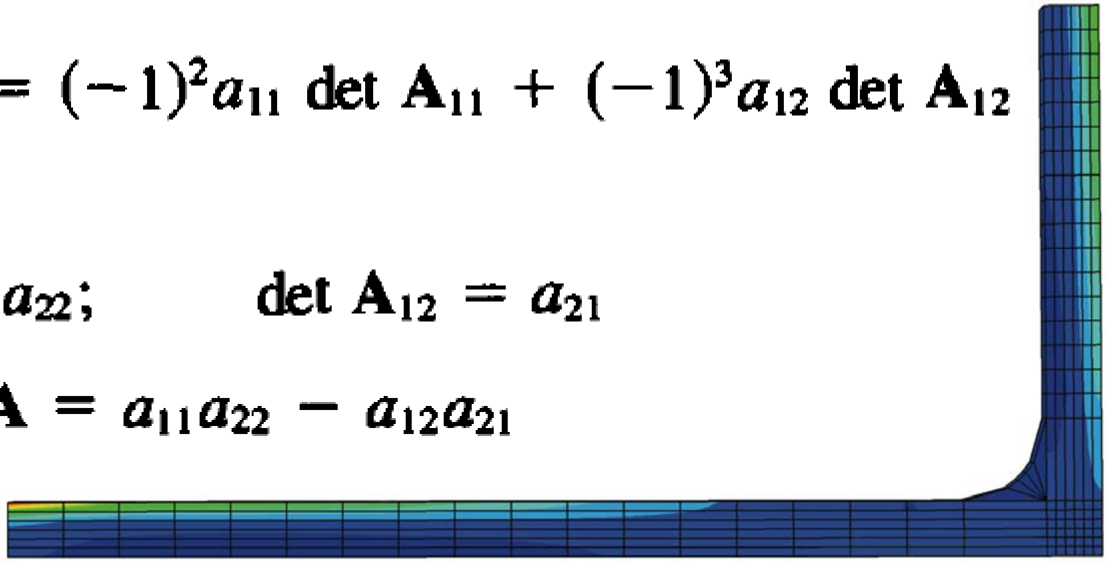
The determinant of a matrix is a scalar value and is obtained by means of the recurrence formula:

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det \mathbf{A} = (-1)^2 a_{11} \det \mathbf{A}_{11} + (-1)^3 a_{12} \det \mathbf{A}_{12}$$

$$\det \mathbf{A}_{11} = a_{22}; \qquad \det \mathbf{A}_{12} = a_{21}$$

$$\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$$



The Determinant of a Matrix

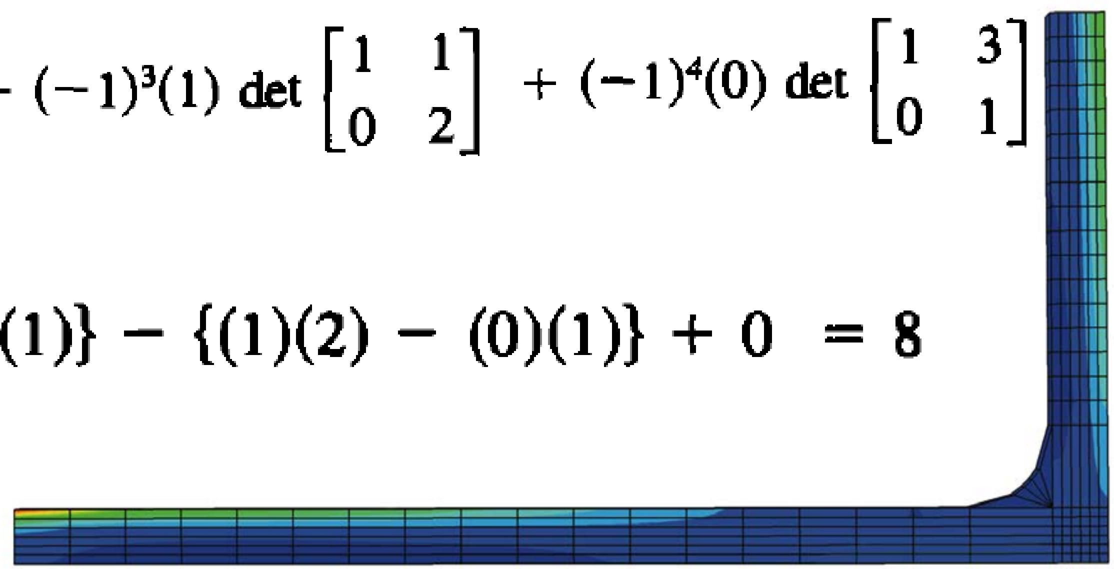
The determinant of a matrix using the recurrence formula along the first row (2 1 0):

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\det \mathbf{A} = (-1)^2(2) \det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} + (-1)^3(1) \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + (-1)^4(0) \det \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\det \mathbf{A} = (2)\{(3)(2) - (1)(1)\} - \{(1)(2) - (0)(1)\} + 0 = 8$$



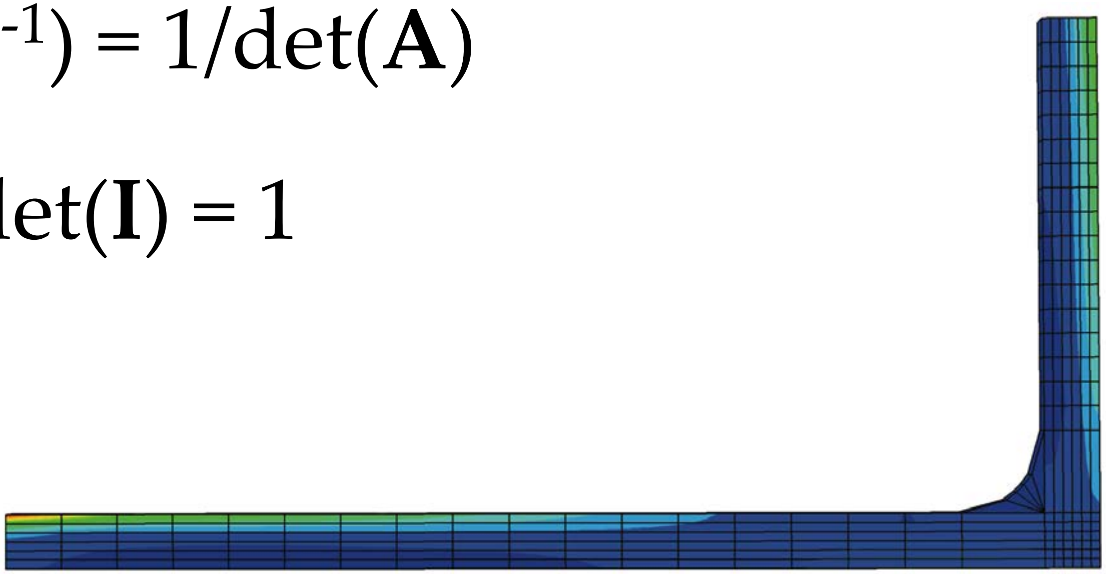
The Determinant of a Matrix

Some useful operations with determinants:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\,\det(\mathbf{B})$$

$$\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$$

$$\det(\mathbf{I}) = 1$$



The Determinant of a Matrix

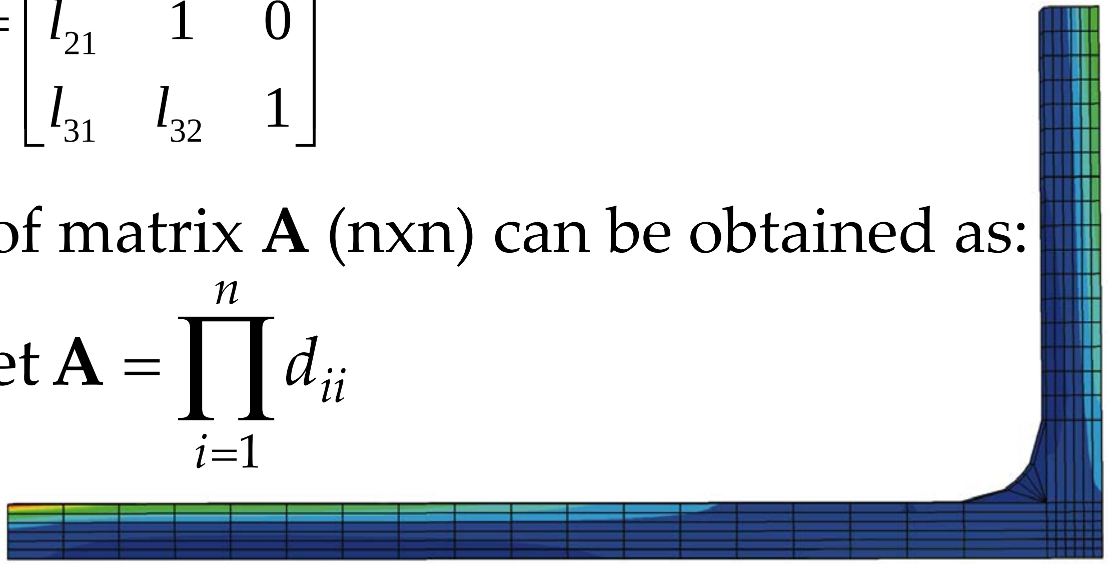
- It is convenient to decompose a symmetric matrix **A** by so called LDL decomposition (Cholesky): $\mathbf{A}=\mathbf{LDL}^{\mathrm{T}}$

- **L** is a lower triangular matrix with all diagonal elements equal to 1 and **D** is a diagonal matrix with components d_{ii}

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

- Thus the determinant of matrix **A** (nxn) can be obtained as:

$$\det \mathbf{A} = \prod_{i=1}^{n} d_{ii}$$

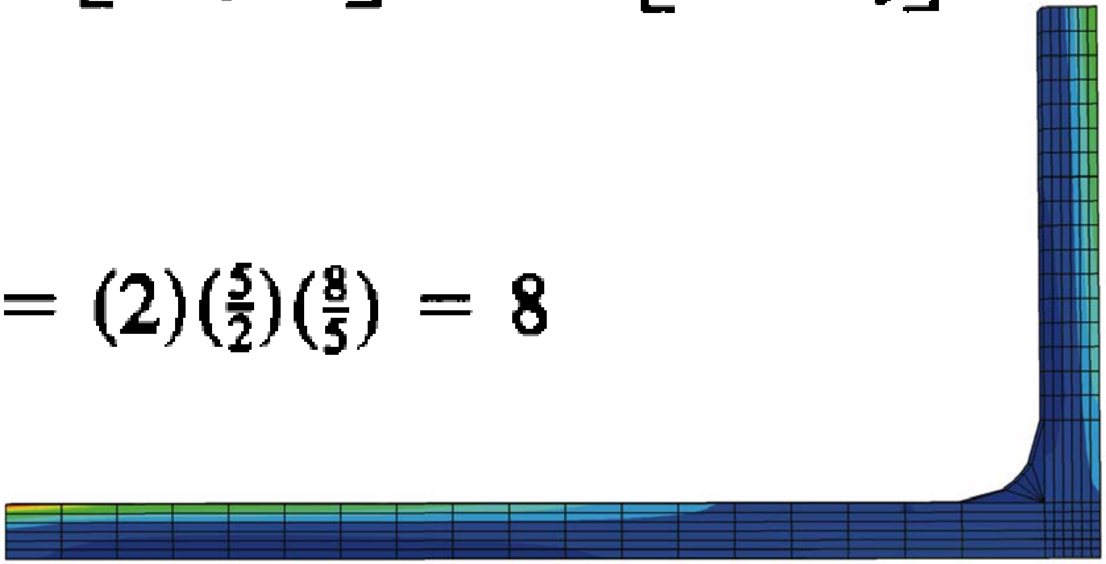


The Determinant of a Matrix

LDL decomposition: $\mathbf{A}=\mathbf{LDL}^{T}$

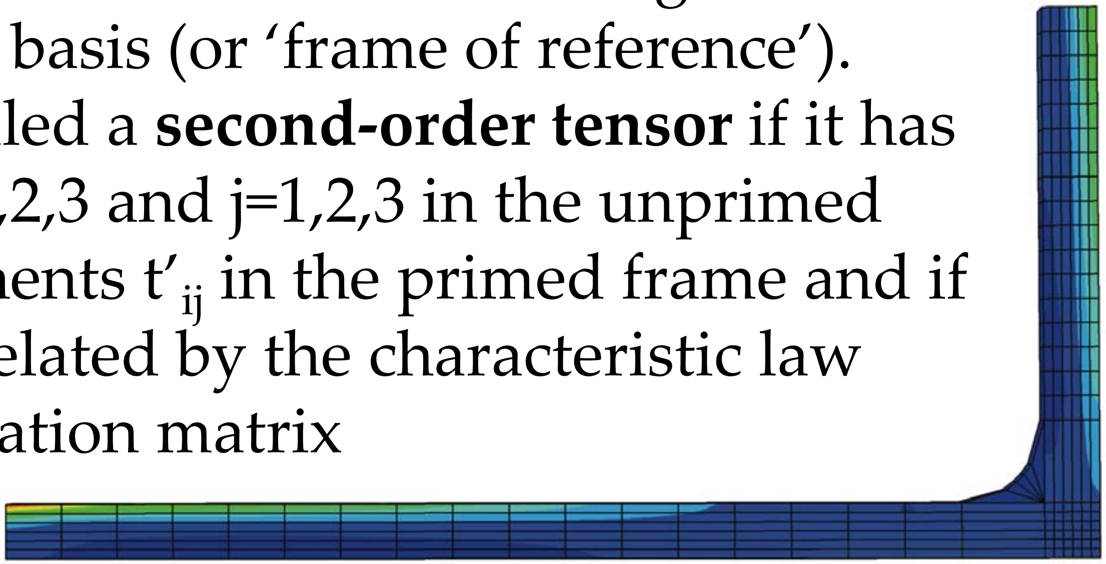
$$\mathbf{A}=\begin{bmatrix}2 & 1 & 0\\ 1 & 3 & 1\\ 0 & 1 & 2\end{bmatrix} \qquad \mathbf{L}=\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & \frac{2}{5} & 1\end{bmatrix} \qquad \mathbf{D}=\begin{bmatrix}2 & 0 & 0\\ 0 & \frac{5}{2} & 0\\ 0 & 0 & \frac{8}{5}\end{bmatrix}$$

$$\det \mathbf{A} = (2)\left(\tfrac{5}{2}\right)\left(\tfrac{8}{5}\right) = 8$$



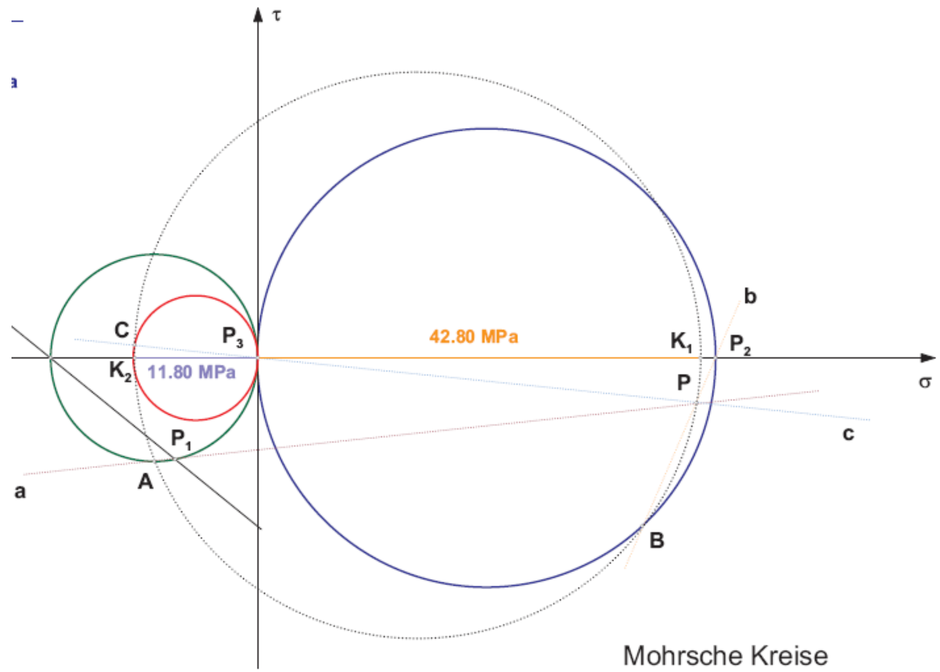
Tensors

- A set of quantities that obey certain transformation laws relating the bases in one generalized coordinate system to those of another
- A tensor consists of an array of a certain order (for example: tensor of order 0 is a scalar, tensor of order 1 is a vector)
- Each tensor has a transformation law detailing the response of a change of basis (or ‘frame of reference’).
- Bathe: An entity is called a **second-order tensor** if it has nine components t_{ij}, i=1,2,3 and j=1,2,3 in the unprimed frame and nine components t'_{ij} in the primed frame and if these components are related by the characteristic law $t'_{ij}=p_{ik}p_{jl}t_{kl}$, P being a rotation matrix



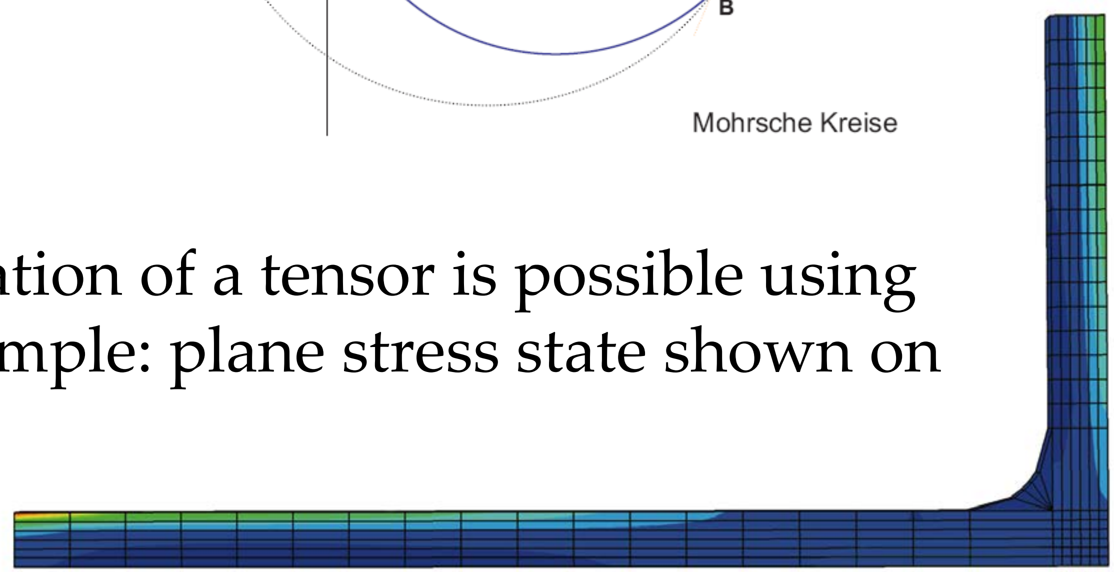
Stress Tensors

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ & & \sigma_{zz} \end{pmatrix}$$



Mohrsche Kreise

A graphical representation of a tensor is possible using Mohr's circles (for example: plane stress state shown on figure above)



Variational Calculus

- Variational operator – δ
- Variations (of deformation) are small enough not to disturb the equilibrium and are consistent with the geometric constraint of the system
- Some rules:

$$\delta\left(\frac{du}{dx}\right)=\frac{d}{dx}(\delta u)$$

$$\delta\int_0^a u\,dx=\int_0^a \delta u\,dx$$

$$\delta(F_1 \pm F_2)=\delta F_1 \pm \delta F_2$$

$$\delta(F_1F_2)=\delta(F_1)F_2+F_1\delta(F_2)$$

$$\delta\left(\frac{F_1}{F_2}\right)=\frac{\delta(F_1)F_2-F_1\delta(F_2)}{F_2^2}$$

$$\delta F^n = nF^{n-1}\delta F$$