Method of Frobenius

General Considerations

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Outline

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The DE

- Our DE is $L[y] = x^2 \frac{d^2y}{dx^2} + x^2 p(x) \frac{dy}{dx} + x^2 q(x) y = 0$.
- We will assume that the singular point $x = 0$ is a regular singular point.
- Since $x = 0$ is a R.S.P., we know that we can expand $xp(x)$ and $x^2 q(x)$ as convergent Taylor series about $x = 0$.
- We set $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$.

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Inserting the Series into the DE

- We compute the first two derivatives of $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the usual way. We then use the Taylor series for the coefficients to obtain

$$
\begin{aligned}
L[y] = &\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \left(\sum_{m=0}^{\infty} p_m x^m\right) \cdot \\
&\left[\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}\right] + \left(\sum_{m=0}^{\infty} q_m x^m\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) \\
=&[r(r-1)+p_0 r+q_0]a_0 x^r + \\
&\{[(1+r)r+p_0(1+r)+q_0]a_1+(rp_1+q_1)a_0\}x^{r+1}+\cdots \\
&+\{[(n+r)(n+r-1)+p_0(n+r)+q_0]a_n + \\
&\left.\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]a_k\right\} x^{n+r}+\cdots
\end{aligned}
$$

Simplification

- The expression for $L[y]$ on the previous slide can be simplified by letting

$$F(r) = r(r-1) + p_0 r + q_0.$$

Then

$$L[y] = a_0 F(r) t x^r + [a_1 F(1+r) + (r p_1 + q_1) a_0] x^{1+r} + \cdots$$
$$+ a_n F(n+r) x^{n+r} + \left\{ \sum_{k=0}^{n-1} [(k+r) p_{n-k} + q_{n-k}] a_k \right\} x^{n+r}$$
$$+ \cdots$$

- We now set the coefficient of each power of x equal to zero. The results are...

Outline

Coefficients

- We have, first of all,

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

the indicial equation.

- Also, we have

$$F(n+r)a_n = -\sum_{k=0}^{n-1} [(k+r)p_{n-k} + q_{n-k}]a_k, \qquad n \geq 1$$

- Note that the indicial equation is a quadratic equation in r. It's roots determine the values r_1 and r_2 for which there *may* be solutions. The second equation shows that, in general, a_n depends on r and all of the preceding coefficients.

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The Recurrence Relation

- We can solve the recurrence relation $F(n+r)a_n = -\sum_{k=0}^{n-1}[(k+r)p_{n-k} + q_{n-k}]a_k$ for a_n provided that $F(1+r), F(2+r), \ldots, F(n+r)$ are not zero.
  - If $F(n+r)$ vanishes for some positive integer n, then $n+r$ must be a root of the indicial equation.
  - Hence, if the indicial equation has two real roots r_1, r_2 with $r_1 > r_2$ and $r_1 - r_2$ *not* equal to an integer, then our DE has two solutions; one corresponding to each value of r obtained from the indicial equation.
  - It can be shown that these solutions converge wherever the series for $xp(x)$ and $x^2q(x)$ converge.
- What happens when the roots to the indicial equation differ by a positive integer or are equal?

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  - If $F(n+r)$ vanishes for some positive integer n, then $n+r$ must be a root of the indicial equation.
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  - It can be shown that these solutions converge wherever the series for $xp(x)$ and $x^2q(x)$ converge.
- What happens when the roots to the indicial equation differ by a positive integer or are equal?

Equal Roots

- When the roots to the indicial equation are equal, we have only one solution of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n.$$

  - One can show that there is a second linearly independent solution of the form

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=0}^{\infty} b_n x^n$$

    where the coefficients b_n need to be calculated - a sometimes difficult problem.
  - We remark, however, that the second solution is often rejected on the grounds that there is a logarithmic singularity at $x = 0$ - many physical applications don't want such a singularity.

Roots Differing by a Positive Integer

- Here we have $r_1 = r_2 + N$ for some positive integer N. In this case we have a solution $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$, but it may not be possible to find a second solution of this form when $r = r_2$.
    - Why? It is because $F(r_2 + N) = F(r_1) = 0$ and the recurrence relation becomes

    $$0 \cdot a_N = -\sum_{k=0}^{n-1} [(k+r)p_{n-k} + q_{n-k}] a_k$$

    when $n = N$. This equation cannot be satisfied for any choice of a_N, if the right side of this equation is nonzero. In this case, the second solution has the form

    $$y_2(x) = y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

Roots Differing by a Positive Integer, II

- What if $-\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]a_k=0$? Then it's clear that a_N is arbitrary and we can obtain a second solution of the form
$$y=x^{r_2}\sum_{n=0}^{\infty}b_nx^n.$$
This can be seen via the example on the next slide(s).

Example: Roots Differing by a Positive Integer

Example

Use the method of Frobenius to solve
$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - (x^2 + 5/4)y = 0.$

Solution: It is clear that $x = 0$ is a R.S.P. (Check!)

- We let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and we obtain the following, after computing the derivatives and adjusting the series:
  - Indicial Equation: $r^2 - 2r - \frac{5}{4} = 0$, so that $r_1 = \frac{5}{2}$ and $r_2 = -\frac{1}{2}$. These roots differ by a positive integer.
  - We also obtain the condition $\left[(r+1)r - (r+1) - \frac{5}{4}\right] a_1 = 0$ and
  - the recurrence relation

$$\left[(n+r)(n+r-1) - (n+r) - \frac{5}{4}\right] a_n - a_{n-2} = 0.$$

- We will always be able to find a solution corresponding to the larger root and we proceed to do this next.

Example, Continued

- Let $r = \frac{5}{2}$. Then
  - $4a_1 = 0$ and so $a_1 = 0$.
  - The recurrence relation is

$$a_n = \frac{a_{n-2}}{n(n+3)}, \quad n \geq 2.$$

  - We obtain, via induction, that all of the odd indexed coefficients vanish and that

$$a_{2n} = \frac{a_0}{2^n n! \left[5 \cdot 7 \cdot 9 \cdots (2n+3)\right]}, \quad n \geq 1.$$

- Our solution is therefore

$$y_1(x) = a_0 x^{5/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! \left[5 \cdot 7 \cdot 9 \cdots (2n+3)\right]}\right].$$

Example, Continued

- We now consider $r = -\frac{1}{2}$. Let us assume (hopefully, but without justification!) that there is a second solution of the standard form. Letting $r = -\frac{1}{2}$ results in $-2a_1 = 0$ and

$$a_n = \frac{a_{n-2}}{n(n-3)}, \quad n \geq 2, n \neq 3.$$

- Some thought shows that a_3 is arbitrary! Also, we have $a_2 = -a_0/2$,

$$a_{2n} = \frac{-a_0}{2^n n! \, [3 \cdot 5 \cdot 7 \cdots (2n-3)]}, \quad n \geq 3$$

and

$$a_{2n+1} = \frac{a_3}{2^{n-1}(n-1)! \, [5 \cdot 7 \cdot 9 \cdots (2n+1)]}, \quad n \geq 2.$$

Example, Continued

- Our solution in this case is therefore

$$y_2 = a_0 x^{-1/2}\left\{1 - \frac{x^2}{2} - \frac{x^4}{2\cdot 4} - \sum_{n=3}^{\infty} \frac{x^{2n}}{2^n n!\,[3\cdot 5\cdot 7\cdots(2n-3)]}\right\} +$$

$$a_3 x^{-1/2}\left\{x^3 + \sum_{n=2}^{\infty} \frac{x^{2n+1}}{2^{n-1}(n-1)!\,[5\cdot 7\cdot 9\cdots(2n+1)]}\right\}$$

It turns out that we are free to take $a_3 = 0$ (remember, it's arbitrary) and so we are left with only the first series for the second solution. However, it can be seen that we can find two linearly independent solutions to the DE by using the smaller root alone.

  - A good rule of thumb is to work out the solution corresponding to the smaller root first, in the hope that this smaller root by itself may lead directly to the general solution. Keep in mind, though, that this won't always work.

A Second Example

Example

We consider the DE $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - 1/4)y = 0$.

Proceeding using the method of Frobenius we find the following:

- $[r(r-1) + r - \frac{1}{4}]a_0 = (r^2 - \frac{1}{4})a_0 = 0$ - here we get $r_1 = 1/2$ and $r_2 = -1/2$.
- $[(1+r)^2 - \frac{1}{4}]a_1 = 0$
- $[(n+r)^2 - \frac{1}{4}]a_n = a_{n-2}, \quad n \geq 2$

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Second Example, Larger Root

We set $r = 1/2$ to obtain $a_1 = 0$ and find that the recurrence relation is

$$a_n = \frac{-a_{n-2}}{n(n+1)}, \qquad n \geq 2.$$

This relation along with $a_1 = 0$, tells us that all of the odd coefficients vanish and that

$$a_{2n} = \frac{(-1)^n}{(2n)!(2n+1)}.$$

(We've set $a_0 = 1$.) We obtain from this recurrence relation that

$$y_1 = \frac{\sin t}{\sqrt{t}}.$$

Second Example, Smaller Root

Setting $r = -1/2$, we see that, since $1 + (-1/2) = 1/2$, we cannot find any value for a_1 - it is arbitrary. Set $a_1 = 0$, then. This causes all of the odd coefficients to be zero and the recurrence relation,

$$a_n = \frac{-a_{n-2}}{n(n-1)}, \qquad n \geq 2,$$

gives us that

$$a_{2n} = \frac{(-1)^n}{(2n)!}.$$

Our second solution is therefore

$$y_2 = \frac{\cos t}{\sqrt{t}}.$$

Outline

Frobenius Theorem

Let r_1 and r_2 be roots of the indicial equation with $r_1 \geq r_2$ if they are real. Then the differential equation has two linearly independent solutions on the interval $(0, \rho)$ (ρ is determined by the radius of convergence of the series for $tp(t)$ and $t^2 q(t)$) of the following form:

- If $r_1 - r_2$ is not a positive integer, then

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n, \quad y_2(t) = t^{r_2} \sum_{n=0}^{\infty} a_n(r_2) t^n.$$

- If $r_1 = r_2$, then

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n, \qquad y_2(t) = y_1(t) \ln t + t^{r_1} \sum_{n=0}^{\infty} b_n t^n.$$

Last Conclusion

- If $r_1 = r_2 + N$, a positive integer, then for some constant a (possibly zero),

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n, \qquad y_2(t) = a y_1(t) \ln t + t^{r_2} \sum_{n=0}^{\infty} b_n t^n.$$

Outline

Calculation of Coefficients with Equal Roots

We run into trouble if the indicial equation has equal roots $r_1 = r_2$ because our DE then only has one solution in the form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$. The method of finding a second solution goes as follows. Let us rewrite the series solution as follows:

$$y(t) = y(t,r) = \sum_{n=0}^{\infty} a_n(r) t^{n+r} \tag{1}$$

to emphasize that the solution $y(t)$ depends on our choice of r. Then, using notation developed above in our general discussion of the method,

$$L[y](t,r) = a_0 F(r) t^r + \sum_{n=1}^{\infty} \left\{ a_n(r) F(n+r) + \sum_{k=0}^{n-1} [(k+r)p_{n-k} + q_{n-k}] a_k \right\} t^{n+r}.$$

Calculation of Coefficients with Equal Roots

We now think of r as a continuous variable and determine a_n as a function of r by requiring that the coefficient of t^{n+r} be zero for $n \geq 1$. Thus

$$a_n(r) = \frac{-\sum_{k=0}^{n-1}[(k+r)p_{n-k} + q_{n-k}]a_k}{F(n+r)}. \tag{2}$$

With this choice of $a_n(r)$, we see that

$$L[y](t,r) = a_0 F(r) t^{n+r}. \tag{3}$$

In the case of equal roots, $F(r) = (r - r_1)^2$, so that equation (3) can be written in the form

$$L[y](t,r) = a_0 (r - r_1)^2 t^r.$$

Calculation of Coefficients with Equal Roots

Since $L[y](t, r_1) = 0$, we obtain one solution

$$y_1(t) = t^{r_1}\left(a_0 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right).$$

Observe now, that

$$\begin{aligned}\frac{\partial}{\partial r}L[y](t,r) =& L\left[\frac{\partial y}{\partial r}\right](t,r) \\ =& \frac{\partial}{\partial r}a_0(r-r_1)^2 t^r \\ =& 2a_0(r-r_1)t^r + a_0(r-r_1)^2(\ln t)t^r\end{aligned} \tag{4}$$

also vanishes when $r = r_1$.

Calculation of Coefficients with Equal Roots

Thus the second solution is

$$\begin{aligned} y_2(t) &= \frac{\partial}{\partial r} y_1(t,r)\Big|_{r=r_1} \\ &= \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} a_n(r) t^{n+r}\right]_{r=r_1} \\ &= \left\{\sum_{n=0}^{\infty} a_n(r_1) t^{n+r_1}\right\} \ln t + \sum_{n=0}^{\infty} \frac{da_n}{dr}(r_1) t^{n+r_1} \\ &= y_1(t) \ln t + \sum_{n=0}^{\infty} \frac{da_n}{dr}(r_1) t^{n+r_1} \end{aligned} \tag{5}$$

Calculation of Coefficients when Roots Differ by an Integer

Here we suppose that the zeros of the indicial equation, $r_1 > r_2$, are such that $r_1 - r_2 = N \in \mathbb{N}$. It *may* be the case that there are two solutions of the form

$$x^r \sum_{n=0}^{\infty} a_n x^n. \tag{6}$$

However, there is *always* a solution of the form

$$y = x^{r_1} \sum_{n=0}^{\infty} a_n x^n. \tag{7}$$

A second solution of this form for the other zero of the indicial equation will exist if the recursion relation is well–defined for all n when using the second (smaller) zero r_2. If the recursion relation is not well–defined for the second root, there is a second solution of the form

Calculation of Coefficients when Roots Differ by an Integer

$$y_2(x) = ay_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty}\frac{d}{dr}\left[(r-r_2)a_n(r)\right]_{r=r_2}x^n. \qquad (8)$$

The constant a may or may not be zero but we won't pursue how to calculate its value here.