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Metric Rectification for Perspective Images of Planes

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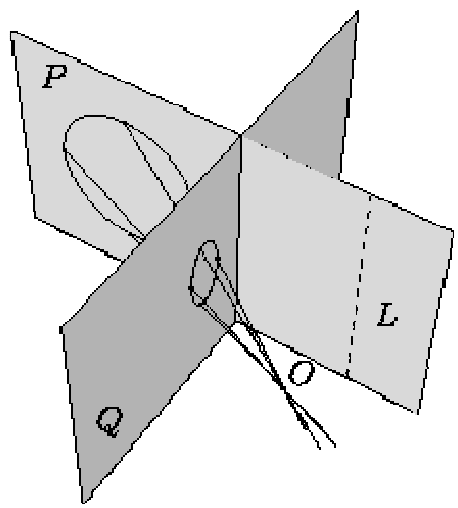
The main purpose of this survey is to understand completely the geometry, constraints and algorithmic implementation for metric rectification of planes. In this survey, I consider the perspective images and thus, using rectification helps me to measure metric properties from a perspective image. Additionally, because I consider perspective images, the concept of projective transformation is important. Thus, I start with defining the projective transformation.

A *projective transformation* is a transformation which is used in projective geometry. I can say that it is the composition of a pair of perspective projections. It helps to understand the change of perceived positions of observed objects if the point of view of the observer changes. Projective transformation maps lines to lines, however it is not necessary to preserve parallelism. Here, it is important to state that projective transformations do not preserve sizes or angles but it preserves incidence and cross-ratio. These two preserved properties are very important in projective geometry. Furthermore projectivity is another name of the projective transformation. We can define any plane projective transformation as an invertible 3 x 3 matrix in homogeneous coordinates. In other words, any invertible 3 x 3 matrix defines a projective transformation of the plane. If projective transformations are not affine, they cannot define on all of the plane. It can only be defined on the complement of a line, and the missing line is mapped to infinity.

An example of a projective transformation can be given by a perspective transformation. If I consider the figure which is below, I can say that a perspective transformation with center ***O***, is mapping the plane ***P*** to the plane ***Q***. The transformation is not defined on the line ***L***, where ***P*** intersects the plane parallel to ***Q*** and going through ***O***.



Here in this example, perspective transformation gives a transformation from one plane to another. However, if we identify the two planes by fixing a Cartesian system in each, we get a projective transformation from the plane to itself.

We learned the general concepts of projective transformation so far. Besides these concepts, there are several applications of plane projective transformation in computer vision, such as: mosaicing and photogrammetry. In the most of these applications, the projective transformation can be determined uniquely if four or more image points in the Euclidean world coordinates are known. Thus, after the transformation is completed, Euclidean measurements can be made on the world plane directly from image measurements. These Euclidean measurements can be angles and lengths. Additionally, the image can be rectified by a projective warping to one which would have been obtained from a fronto-parallel view of the plane.

D. Liebowitz and A. Zisserman [1] did a research for metric rectification for perspective images. I concentrated to this paper in my survey. The researchers showed that providing the Euclidean coordinates of four is not necessary to determine uniquely the projective transformation. The idea is that instead using metric properties on the world plane, length ratio and an angle can be used directly to partially determine the projective transformation up to a particular ambiguity. Although this partial determination requires less information about the world plane to be known, it is sufficient to enable metric measurements of entities on the world plane to be made from their images.

This is a very important contribution and this contribution is the extended and improved version of the Collins and Beveridge' paper: "Matching Perspective Views of Coplanar Strucuters using Projective Unwarping and Similarity Matching [2]." In this paper, the researchers stated that once the vanishing line of the plane is identified, the transformation from world to image plane can be reduced basically to an affinity. Thus, the researchers used this approach to reduce the dimension of the search, from eight to six, in registering satellite images. Because the idea of metric rectification for perspective images of planes constructed on the significant step which is done by Collins and Beveridge, I would like to introduce this paper first. Then I will turn back the metric rectification.

Collins and Beveridge [2] considered the problem of matching perspective views of coplanar structures composed of line segments. They considered both model-to-image and image-to-image correspondence matching. We know that these matching scenarios generally require discovery of an eight parameter projective mapping. However, if the horizon line of the object plane can be found in the image, these problems reduce to a six parameter affine matching problem. The researchers achieve this by using vanishing point analysis which is taken as a significant step in Liebowitz and Zisserman [1]. Besides, if the intrinsic lens parameters of the camera are known, the problem becomes four parameter affine similarity matching.

As stated above, the main point of this paper is that the full perspective matching problem for coplanar structures can often be reduced to a simpler four parameter affine matching problem when the horizon line of the planar structure can be determined in the image. Here, the important point is to know the horizon line, thus the image can be transformed to show how the structure would appear if the camera's line of sight was perpendicular to the object plane. The researchers stated this process as rectification in aerial photogrammetry.

Matching problems involve solving two different things in the same time. These are a discrete correspondence between two sets of features which are model-image or image-image, and an associated transformation that maps one set of features into registration with the other. This can be seen as a match being a correspondence and transformation constitutes matching. As I stated above, the relevant set of transformation in the eight parameter projective transformation group for planar structures under a perspective camera model. We need to pay attention to the restrictive transformations. Because these restrictive transformations can often more easily computed, and this makes the matching easier. In their paper, the researchers considered the Frontal planes as one of these special cases. Considered frontal planes are the planar structures which are viewed head-on with the viewing direction of the camera held perpendicular to the object plane. So, why the frontal planes are special case? Because if we know the intrinsic camera parameters, perspective mapping of a frontal plane to its appearance in the image can be described with just four affine parameters. These parameters are an image rotation angle, a 2D translation vector and an image scale.

Thus, we can say that, generally the perspective projection of a frontal plane is described by a six parameter affine transformation. However, if we are using a calibrated camera and we know its intrinsic lens effects, thus it can be inverted to recover the ideal pinhole projection image. After correction for intrinsic lens effects, the frontal view of an object plane can be described by a four parameter affine similarity mapping. Besides this, there are some arbitrary orientations. Planes which are viewed at an angle, the function mapping object coordinates to image coordinates is not affine. It becomes a more general projective transformation. As we know that the lines which are parallel on a tilted object plane appear to converge in the image plane, intersecting at a vanishing point. Here the researchers considered the vanishing point concept. Besides, two or more vanishing points from different sets of coplanar planar lines form a line in the image called the vanishing line or horizon line of the plane. If we turn back to our frontal planes, for frontal planes, all parallel lines on the object remain parallel in the image. The reason of this is that the image projection of a frontal plane is described by an affine transformation, and this transformation preserves parallelism. Here, we can say that these set of parallel lines in the image intersect in a point at infinity. Additionally, all vanishing points of parallel lines appear at infinity for frontal planes, and the vanishing line passes through them is also said to be at infinity. The result of these considerations is that if we apply a projective mapping to the image which takes the vanishing line of a coplanar structure to the line at infinity, the vanishing points of all lines in the object plane will also appear at infinity. Thus, all parallel lines in the planar structure will appear parallel in the image. So we can understand that the new image is a frontal view of the object plane. By this way, the mapping from object to image can be represented as an affine transformation.

Here the important point to understand that the vanishing line of a frontal plane appears at infinity in the image plane, and besides that is possible to recover a frontal view from the image of a tilted object plane by applying a projective transformation which maps the objects' vanishing line to infinity. However, then another question comes to mind. We see that there is six-dimensional space of projective transformations which map a given line in the image off to infinity. Which one we need to choose as the best? This questions' answer is considered in the rectification section of this paper. The researchers considered a pinhole camera image to solve this problem, and after some considerations they found a result which is a frontal view of the object plane which is a rectified four parameter affine view. Additionally, they stated that even if when the camera lens parameters are not known, this transformation can be used to map a vanishing line to infinity. In this situation, we cannot recover the pure pinhole image and we cannot interpret the position of the vanishing line in the image geometrically in terms of 3D plane orientation. However, the image can be rectified to present some six parameter affine mapping of the frontal object plane.

The last important part in this paper is the correspondence matching. In this part, the researchers considered a two step approach to match a coplanar line segments seen from two arbitrary 3D viewpoints. As a first step, the researchers rectified both sets of line segments. Thus, perspective matching problem becomes a simpler affine matching problem. As a second step, the researchers used a local search matching algorithm to find the optimal affine map and correspondence between the two sets of line segments. Here, if both sets of line segments can be extracted from images, then an image-to-image matching problem can be solved. On the other hand, if one set of segments can be derived from a geometric object model, thus a model-to-image matching problem can be solved.

By introducing the Collins and Beveridge [2] paper, I showed the important step which is used by Liebowitz and Zisserman [1]. The point is that if the vanishing line of the plane can identified, thus the transformation from the world to image can be reduced to an affinity. Liebowitz and Zisserman improved this result in four ways:

1. The researchers showed that affinity can be reduced to a similarity by using known metric information.
2. The researchers showed that an imaged plane can be rectified directly from metric information without identifying the vanishing line first.
3. The researchers described how the metric rectification of a plane constraints the camera internal calibration parameters.
4. For increasing the accuracy of the results, the researchers estimated vanishing points using a Maximum Likelihood Estimator.

Firstly, the researchers stated that by using known metric information the affinity can be reduced to a similarity. We know that once the metric structure recovery can be stratified, we can determine firstly the affine and then metric properties.

If we state the points on the image plane as x, these points are related to the points on the world plane. Thus, if we state the points on the world plane as ***x'***, we can show this relationship as ***x' = Hx*** where the transformation matrix is ***H***. It is important to say that ***x*** and ***x'*** are homogeneous 3-vectors. After that, the researchers showed that transformation matrix can be uniquely decomposed into a concatenation of three matrixes. These matrixes are ***S***, ***A*** and ***P***. Here ***S*** matrix represents the similarity transformation, ***A*** matrix represents affine transformation and ***P*** matrix represents the pure projective transformation. Thus, this relationship can be stated ***H = SAP*** where pure projective transformation is

$$\mathtt{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{pmatrix}$$

Remember that vanishing line of the plane is $\boldsymbol{l_\infty} = \boldsymbol{(l_1,l_2,l_3)^T}$ which has two degrees of freedom. Here, I would like to give information about the vanishing line $\boldsymbol{l_\infty}$ of the plane, because determining the projective transformation matrix depends on the vanishing line of the plane. As we know, vanishing line is considered on the world plane, and can be stated as the image of the line at infinity. Here, $\boldsymbol{l_\infty}$ is important, because parallel lines intersect at the vanishing points in the image and these vanishing points lie on $\boldsymbol{l_\infty}$.

Affine matrix has two degrees of freedom which are represented by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ specify the image of the circular points geometrically. Here it is important to know that the circular points are a pair of complex conjugate points on the line at infinity. These circular points are ***I*** and ***J***. Additionally these are transformed from coordinates of their metric plane to the affine plane. Here, the importance of the circular points is that they are invariant to Euclidean transformations. This means that if we can identify the circular points, metric properties of the plane can be identified too.

$$\mathtt{A} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\alpha}{\beta} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarity transformation matrix has four degrees of freedom;

$$\mathrm{S} = \begin{pmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

Here, ***R*** is a rotation matrix, ***t*** is a translation vector and ***s*** is isotropic scaling.

After determining these concepts, the researchers give the generally known rectification process by applying constraints sequentially on the projective and affine components of the rectification homography. This is a two step rectification process which is firstly starts from projective to affine and secondly continues from affine to metric.

The first rectification process is mainly determining the ***P***. We know that this requires the identification of the vanishing line of the plane. The researchers assumed that the transformation from projective to affine is determined. Because ***P*** is determined, the image can be affine rectified and affine properties such as length ratios on parallel line segments measures. Then, the researchers moved on the recovery of metric geometry. For recovering the metric geometry from affine, affine transformation of the plane which was the matrix ***A*** must be considered. This will restore the angles and length ratios for non-parallel segments. As we know, the affine matrix has the parameters, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We have to provide the constraints on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This is stated that there are three methods. These are

1. A known angle between lines
2. Equality of two unknown angles
3. A known length ratio

I would like to consider these three methods. Firstly, we have to know that in each case it is shown that the constraint is a circle. This is in fact a circle in the complex plane since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are originally real and imaginary components, and the circles may be plotted on the plane with $\boldsymbol{\alpha}$ as the real axis and $\boldsymbol{\beta}$ the imaginary. However, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are real, the complex interpretation is not significant in seeking a solution.

Now, I will consider the method, in a known angle between lines, $\boldsymbol{\Theta}$ is the angle between the lines imaged as $\boldsymbol{l_a}$ and $\boldsymbol{l_b}$ on the world plane. Besides, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lie on the circle with centre

$$(c_\alpha, c_\beta) = (\frac{(a+b)}{2}, \frac{(a-b)}{2}\cot\theta)$$

And we can define the radius as:

$$r = |\frac{(a-b)}{2\sin\theta}|$$

where the line directions $\boldsymbol{a = -l_{a2}/l_{a1}}$ and $\boldsymbol{b = -l_{b2}/l_{b1}}$. If $\boldsymbol{\Theta = \pi / 2}$, the centre will be on the $\boldsymbol{\alpha}$ axis.

Second method is the equality of two unknown angles. Here, the researchers are supposed that the angle between two lines imaged with directions $\boldsymbol{a_1}$, $\boldsymbol{b_1}$ is the same as that between lines imaged with directions $\boldsymbol{a_2}$, $\boldsymbol{b_2}$.

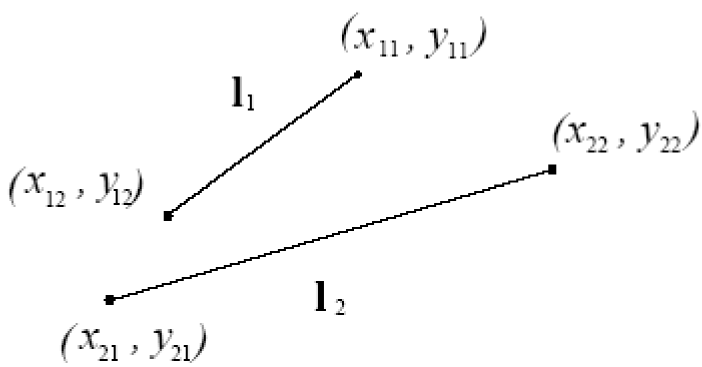
Thus, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lie on the circle with centre on the $\boldsymbol{\alpha}$ axis

$$(c_\alpha, c_\beta) = (\frac{a_1 b_2 - b_1 a_2}{a_1 - b_1 - a_2 + b_2}, 0)$$

And the squared radius

$$r^2 = \left(\frac{a_1 b_2 - b_1 a_2}{a_1 - b_1 - a_2 + b_2}\right)^2 + \frac{(a_1 - b_1)(a_1 b_1 - a_2 b_2)}{a_1 - b_1 - a_2 + b_2} - a_1 b_1$$

Last method is using a known length ratio. Here, the researchers are supposed that the length ratio of the two non-parallel line segments is ***s*** on the world plane. Besides, the situation is illustrated as



Here, the researchers stated that ***α*** and ***β*** lie on the circle with center on the ***α*** axis and can be showed as

$$(c_\alpha, c_\beta) = \left(\frac{(\Delta x_1 \Delta y_1 - s^2 \Delta x_2 \Delta y_2}{\Delta {y_1}^2 - s^2 \Delta {y_2}^2}, 0\right)$$

And radius can be showed as:

$$r = \left|\frac{s(\Delta x_2 \Delta y_1 - \Delta x_1 \Delta y_2)}{\Delta {y_1}^2 - s^2 \Delta {y_2}^2}\right|$$

Here, we are trying to determine ***α*** and ***β***. For to determine ***α*** and ***β***, two independent constraints are always required. Additionally, it is important that the constraints are dependent on line orientation and the same constraint circle results from any parallel line sets.

Before introducing the unstratified rectification process, I would like to give more information about the metric structure recovery which we considered above. Here, I consider a concept from the Oliver Faugeras' paper[3] which is used as the basis approach in D. Liebowitz and A. Zisserman' paper [1] in metric structure recovery. In this paper, the researchers considered the stratification of three-dimensional space for projective, affine and Euclidean structures. My concentration is on the last stratification part which considers three-dimensional space as a Euclidean space. As we know, this part comes after the stratification of three-dimensional space as a projective space and as affine space. This part is the final part in stratification. Here the point is considering the world as a Euclidean space embedded in the affine space which is constructed before. This consideration consists of two main parts

1. Euclidean transformation of the plane which are the absolute points.
2. Euclidean transformation of the space which is the absolute conic.

By understanding these concepts, we can combine the absolute conic and absolute point concepts to our general consideration.

Firstly, I consider the Euclidean transformation of the plane. Here, the point is to specialize the set of affine transformations of the plane. This requires not only preserving the line at infinity, but also need to preserve two special points on that line. The two special points are called absolute points, ***I*** and ***J***. The coordinates of these points is ***(1, ±i, 0)*** where $i = \sqrt{-1}$. Here, we need to turn back to affine transformation of the plane. We know that there is a one-to-one correspondence between the usual affine plane and the projective plane minus the line at infinity. In the affine plane, an affine transformation defines a correspondence ***X → X'***, and this can be expressed in matrix form as

$$\mathbf{X}' = \mathbf{B}\mathbf{X} + \mathbf{b}$$

where ***B*** is a ***2 x 2*** matrix of rank 2 and ***b*** is a ***2 x 1*** vector.

The researchers stated that ***I*** and ***J*** remain invariant depending on ***B*** matrix. Thus,

$$\frac{1}{i} = \frac{b_{11}1 + b_{12}i + b_1 0}{b_{21}1 + b_{22}i + b_2 0},$$

$$\frac{1}{-i} = \frac{b_{11}1 - b_{12}i + b_1 0}{b_{21}1 - b_{22}i + b_2 0},$$

which gives

$$(b_{11} - b_{22})i - (b_{12} + b_{21}) = 0\,,$$

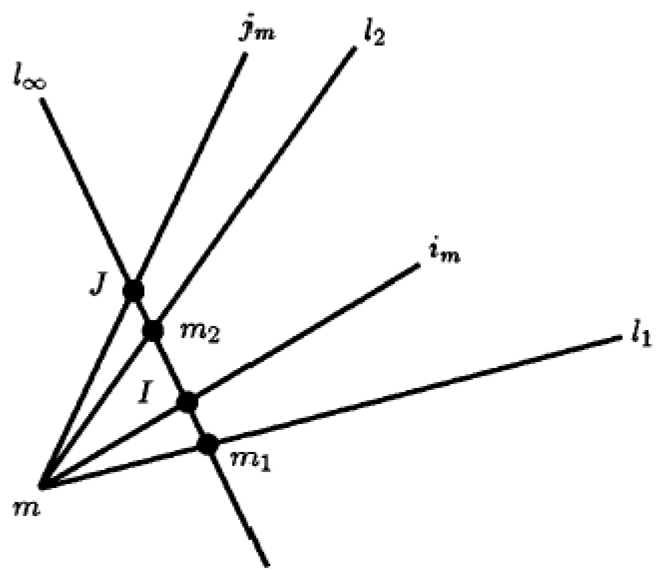
$$-(b_{11} - b_{22})i - (b_{12} + b_{21}) = 0\,.$$

As can be seen that $\boldsymbol{b_{11} - b_{22} = b_{12} + b_{21} = 0}$, Thus we can write

$$\mathbf{X}' = c\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\mathbf{X} + \mathbf{b}$$

where $\boldsymbol{c > 0}$ and $\boldsymbol{0 \leq \alpha \leq 2\pi}$. This class of transformations can be called the class of similitudes. Here, it forms a subgroup of the affine group and therefore of the projective group. The name of this group is the similitude group or the Euclidean transformations group. The affine point which is represented by $\boldsymbol{X}$, first rotated by $\boldsymbol{\alpha}$ around the origin, scaled by $\boldsymbol{c}$, and at last translated by $\boldsymbol{b}$. The researchers are specialized the class of transformations by assuming that $\boldsymbol{c=1}$. Thus they obtained another subgroup which is called the group of rigid displacements.

After that, the researchers gave an application of the use of the absolute points. This really helps to understand the concept. In this example, they showed that how the absolute points can be used to define the angle between two lines. They defined the angle $\boldsymbol{\alpha}$ which is between two lines $\mathbf{l_1}$ and $\mathbf{l_2}$. For defining this angle, they considered their point of intersection $\boldsymbol{m}$ and the two lines $\boldsymbol{i_m}$ and $\boldsymbol{j_m}$ which join $\boldsymbol{m}$ to the absolute points $\boldsymbol{I}$ and $\boldsymbol{J}$. This can be seen from the figure as well.



Then, the angle is given using the Laguerre Formula

$$\alpha = \frac{1}{2i}\log(\{l_1,\ l_2;\ i_m,\ j_m\})$$

This can be considered as an equal result with the cross ratio of the four points $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{m_1}$, and $\mathbf{m_2}$ of intersection of the four lines with the line at infinity $\mathbf{l}_\infty$.

The second important part when considering the three-dimensional space as a Euclidean space is the transformation of the space which considers the absolute conic. Like specializing the set of affine transformations of the plane, we can also specialize the affine transformations of the space. This requires that the affine transformations leave a special conic invariant. They obtained the conic, $\mathbf{\Omega}$, as the intersection of the quadric of equation $\sum_{i=1}^{4} x_i^2 = 0$ with π_∞:

$$\sum_{i=1}^{4} x_i^2 = x_4 = 0\,.$$

They stated that this conic can also be called the absolute conic. It is important to state that in π_∞, $\mathbf{\Omega}$ can be interpreted as a circle which has the radius $\boldsymbol{i} = \sqrt{-1}$. This gives us an imaginary circle. Thus, in the standard projective basis, all its points have complex coordinates. Additionally, if $\boldsymbol{m}$ is a point of $\mathbf{\Omega}$, the complex conjugate point which can be seen as $\overline{m}$ will also on $\mathbf{\Omega}$. The reason is the usage of equations with real coefficients when defining equation of the absolute conic. The researchers also showed that the affine transformations which keep $\mathbf{\Omega}$ invariant can be written as

$$\mathbf{X}' = c\mathbf{C}\mathbf{X} + \mathbf{b}\,.$$

Where $\boldsymbol{c > 0}$ and $\boldsymbol{C}$ is orthogonal, this means it satisfies the equation $\boldsymbol{CC^T = I}$.

After giving some information about Euclidean transformation of the plane and space, we can move the unstratified rectification. We know that the researchers used two step rectification process in the beginning, however then they are considered the unstratified rectification instead of using this two step process. As can be seen from the study, the important point is to determine the parameters of matrixes $\boldsymbol{AP}$. However, instead of using the previous process, it is possible to determine the parameters of $\boldsymbol{AP}$ directly from metric information without first using affine information, such as parallelism, to determine $\boldsymbol{P}$ from the vanishing line. In general, non-linear constraint on the parameters is generated by direct application of the metric constraints. However, if we consider the orthogonal lines, it can be seen that the constraint on the four rectification parameters is linear.

At this point, the parameters are represented by the conic $\boldsymbol{D}$. This is for to obtain a linear constraint. This conic is dual to the circular points. We can define the conic as $\boldsymbol{D} = \boldsymbol{IJ}^T + \boldsymbol{JI}^T$. And this will be 3 x 3 matrix which has a rank two. If we can determine the image of D, thus we can determine the imaged circular points. Additionally, the circular points are imaged on the vanishing line at $((\alpha \pm i\beta)l_3, l_3, -\alpha l_1 - l_2 \pm i\beta l_1)^T$. Additionally if we can determine the circular points, the rectification parameters $\boldsymbol{l_\infty}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be determined.

In the paper, the researchers considered that orthogonal lines are conjugate wrt $\boldsymbol{D}$, this means such as they satisfy $\boldsymbol{l_a^T D l_b = 0}$ for orthogonal lines $\boldsymbol{l_a}$ and $\boldsymbol{l_b}$. Thus, each pair of orthogonal lines places a linear constraint on $\boldsymbol{D}$. Five orthogonal line pairs such as five right angles can be sufficient to determine $\boldsymbol{D}$ linearly, provided lines of more than two orientations are included. In another way $\boldsymbol{D}$ can be determined by four orthogonal line pairs together with the rank two constraints, but this time the solution becomes non-linear.

Here, before starting the implementation details, I would like to consider the general and important points of conics and circular points. Thus, this consideration will help us to understand the reason of defining the conic as $\boldsymbol{D} = \boldsymbol{IJ}^T + \boldsymbol{JI}^T$.

In D. Liebowitz and A. Zisserman' paper [1] which is mainly considered in this survey, the researchers obtained a linear constraint by representing the parameters using conic $\boldsymbol{D}$ which is dual to the circular points. Besides, the conic is defined as $\boldsymbol{D} = \boldsymbol{IJ}^T + \boldsymbol{JI}^T$. For better understanding of this, an investigation of the Euclidean structure is needed [4]. We need a projective encoding of Euclidean structure for recovering the metric information implicit in projective images. The important point in Euclidean structure is the dot product between direction vectors or dually the dot product between normals to hyperplane. These are the different ways of saying the same thing. The former leads to be strafied "hyperplane at infinity + absolute conic" formulation [5], the latter is the "absolute quadratic"[6].

If want to understand the generalities of this concept, we need to consider a $\boldsymbol{k}$-dimensional Euclidean space. If we are dealing with the planar scene and its 2D images, we need to use $\boldsymbol{k}$ as 2. On the other hand, if we are dealing with ordinary 3D space, we need to use $\boldsymbol{k}$ as 3. In this structure, homogeneous Euclidean coordinates, points, displacement vectors and hyperplanes are encoded as homogeneous $\boldsymbol{k+1}$ component column vectors

$\boldsymbol{x} = (\boldsymbol{x},1)^T$, $\boldsymbol{t} = (\boldsymbol{t},0)^T$ and row vectors $\boldsymbol{p} = (\boldsymbol{n}, \boldsymbol{d})$ where $\boldsymbol{x}$, $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{d}$ are the usual $\boldsymbol{k}$-D coordinate vectors of the point, the displacement, the hyperplane normal and hyperplane offset respectively. It is important to state that $\boldsymbol{p} \,.\, \boldsymbol{x} = \boldsymbol{n} \,.\, \boldsymbol{x} + \boldsymbol{d} = 0$ and $\boldsymbol{p} \,.\, \boldsymbol{t} = \boldsymbol{n} \,.\, \boldsymbol{t}$ is satisfied by the points and displacements on the plane respectively. Here, the displacement directions can be attached to the point space as vanishing point or a hyperplane at infinity $\boldsymbol{p}_\infty$ of points at infinity. Finite and infinite points are mixed by projective transformation. In the presence of a projective transformation which is encoded by an arbitrary nonsingular $\boldsymbol{(k+1)*(k+1)}$ matrix $\boldsymbol{T}$, directions which are column vectors and points transform contravariantly, for example if $\boldsymbol{T}$ acting on the left which means that $\boldsymbol{x} \rightarrow \boldsymbol{Tx}$, $\boldsymbol{v} \rightarrow \boldsymbol{Tv}$. hyperplanes which are row vectors transform covariantly for to preserve the point-on-plane relation $\boldsymbol{p} \,.\, \boldsymbol{x} = \boldsymbol{n} \,.\, \boldsymbol{x} + \boldsymbol{d} = 0$, such as $\boldsymbol{T}^{\boldsymbol{1}}$ acting on the right which means $\boldsymbol{p} \rightarrow \boldsymbol{pT}^{\boldsymbol{1}}$.

Another important concept in the Euclidean structure is the absolute quadratic and conic. If we consider the usual Euclidean dot product between hyperplane normals, we face with this equation

$$\boldsymbol{n}_1 \,.\, \boldsymbol{n}_2 = \boldsymbol{p}_1 \,\boldsymbol{\Omega}\, \boldsymbol{p}_2^T$$

where $\boldsymbol{\Omega}$ matrix is stated as

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{I}_{k\times k} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{pmatrix}$$

This matrix is the absolute (hyperplane) quadratic which is symmetric and has rank $\boldsymbol{k}$. Encoding the Euclidean structure in projective coordinates is done by $\boldsymbol{\Omega}$. $\boldsymbol{\Omega}$ transforms like a point in each of its two indices under projective transformation, thus the dot product between plane normal is invariant which means $\mathbf{\Omega} \rightarrow \boldsymbol{T}\,\mathbf{\Omega}\,\boldsymbol{T}^T$ and $\boldsymbol{p}_i \rightarrow \boldsymbol{p}_i\boldsymbol{T}^{-1}$. This gives us that $\boldsymbol{p}_1\,\mathbf{\Omega}\,\boldsymbol{p}_2^T = \boldsymbol{n}_1 \,.\, \boldsymbol{n}_2$ is constant.

If $\mathbf{\Omega}$ is restricted to coordinates on $\boldsymbol{p}_\infty$, it becomes nonsingular. Besides it can be inverted to give the k x k symmetric positive definite absolute conic $\mathbf{\Omega}^*$. Here, the measurement is the dot products between displacement vectors, it is like that $\mathbf{\Omega}$ measures them between hyperplane normals. It is important to state that $\mathbf{\Omega}^*$ is defined only on direction vectors, thus is not defined on finite points, and it has no unique canonical form in terms of unrestricted coordinates, unlike $\mathbf{\Omega}$.

The other important concept in the Euclidean structure is the circular points. The complex conjugate vectors satisfy $\mathbf{x}_\pm \equiv \frac{1}{\sqrt{2}}(\mathbf{x} \pm i\mathbf{y})$ under given any two orthonormal direction vectors such as $\boldsymbol{x}$, $\boldsymbol{y}$. Intuitively, these complex directions lie on the absolute conic, and any complex projective point which lies on the absolute point can be decomposed into two orthogonal direction vectors. These are its real and imaginary parts. If we consider the 2D case, there will be only one such conjugate pair up to complex phase, and Euclidean structure of the plane can be characterized by these circular points. However, it can be harder for numerical purposes. Because of this, if we avoid complex numbers by using the real and imaginary parts of $\boldsymbol{x}$ and $\boldsymbol{y}$, instead of $\boldsymbol{x}_\pm$, the numerical solutions becomes easier. Here, I want to state that the phase freedom in $\boldsymbol{x}_\pm$ corresponds to the 2 x 2 orthogonal mixing freedom of $\boldsymbol{x}$ and $\boldsymbol{y}$.

As a result, we can say that $\mathbf{\Omega}$ is easy to use, however the rank $\boldsymbol{k}$ constraint $\mathbf{det\ \Omega = 0}$ must be handled by constrained optimization**.** The absolute conic $\mathbf{\Omega^*}$ has neither constraint nor gauge freedom, however it has significantly more complicated image projection properties. Thus it can only be defined once the plane at infinity is known and a projective coordinate system on it has been chosen such as by induction from one of the images.

After considering the general and important concepts of conics and circular points, I would like to turn back the D. Liebowitz and A. Zisserman' paper [1]. Here, if we consider their implementation in details, we can see that the researchers consider three important points:

1. Vanishing point estimation
2. Image warping
3. Automatic detection of vanishing points and orthogonal directions.

I will consider these one by one.

Firstly, I consider the vanishing point estimation. As I stated above, projective transformation matrix $\boldsymbol{P}$ consists of the vanishing line of the plane and can be determined by the intersection of two imaged parallel lines. The intersection of two lines, such as $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$, represented as $\boldsymbol{x = l_1 * l_2}$. However, the problem is that there are more than two imaged parallel lines available. Thus, the vanishing point is over constrained. Additionally, a set of line segments may not intersect precisely a point because of the presence of measurement error. Because of these reasons, there are several approaches to estimate the vanishing point. B. Caprile and V. Torre [7] proposed to calculate a weighted sum of all pairwise line intersections.

Collins [8] proposed an approach which is an application of Bayesian statistics to error in projective spaces. However, in this paper, the researchers proposed a different approach which is significantly improves the accuracy of the metric rectification. The researchers defined and implemented a maximum likelihood estimate of the vanishing point. Thus, they minimized the errors by using this approach.

Their approach can be summarized as follows. They supposed that there are $n > 2$ line segments l_i, and they tried to estimate the vanishing point v. Here, they are trying to find the ML estimate of the vanishing point however this involves finding the estimate of the line segments. They stated that isotropic mean zero Gaussian noise on the end points can help to model the error in the fitted line segments. Assuming the end points of l are x^a and x^b, ML estimator minimizes

$$C = \sum_i d_\perp^2(\hat{\mathbf{l}}_i, \mathbf{x}_i^a) + d_\perp^2(\hat{\mathbf{l}}_i, \mathbf{x}_i^b)$$

which is subject to the constraint $v.l_i = 0$ where $d\perp(x,l)$ is the perpendicular image distance between the point x and line l. Additionally, the researchers minimized the cost function by showing that $C(v)$ can be obtained in closed form by given v. Then, they minimized $C(v)$ over v using the Levenberg-Marquart numerical algorithm. They obtained an initial solution for v from the null vector of the matrix $(l_1,l_2,\ldots l_n)$ via singular value decomposition.

The second point which is considered in implementation details is the image warping. Generally, images are wrapped by the inverse homography to each pixel in the target image. Bilinear interpolation is used to determine the density at the source point in the original image. Besides, the researchers stated that if we want to automate the warping and ensure that the convex hull of the original image is correctly mapped into the rectangle of the target image, we have to use oriented projective geometry.

The third point is the automatic detection of vanishing points and orthogonal directions. Detecting two dominant directions of lines in the image which are orthogonal in the plane can be used to achieve automation of the correction process. The researchers obtained the dominant directions by a frequency histogram on line direction with frequency weighted by segment length. Generally, the histogram is bimodal and readily segmented. Here, they used to assumption that the lines in each dominant direction is parallel and a vanishing point is determined by this way. After that they have an affine image and they searched this affine image for dominant directions and lines in the two directions constrained to be right angles. However, this searching provides only one constraint and thus there is an ambiguity in relative scale. Here, the important point is that histogram approach gives good performance when detecting dominant directions of parallel lines in affine images, however it doesn’t give good performance when detecting vanishing point. Thus, for example Hough Transform must be accomplished for more robust vanishing point detection.

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