MHF4U
Advanced Functions
University Preparation

Advanced Functions: Content and Reporting Targets

Mathematical Processes across all strands: Problem Solving, Reasoning and Proving, Reflecting, Selecting Tools and Computational Strategies, Connecting, Representing, and Communicating

Introductory Unit	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6
• Sketch graphs from descriptions of a set of properties, from a description of a scenario, using prior knowledge of: Function notation Vertical line test Key properties of functions: - average rate of change - instantaneous rate of change - zeros - y-intercept/ initial condition - domain and range • Inverse functions • Transformations of functions • Difference tables	• Polynomial Functions – Characteristics embed average and instantaneous rates of change • Solving Equations – embed intervals of increase/ decrease • Transformations • Explore end behaviours • Embed inequalities	• Rational Functions • Embed average and instantaneous rates of change • Solving equations • Embed inequalities • Explore behaviour around asymptotes	• Radian measure and graphing primary and their reciprocal trig functions (using key properties) • Square scale • Rate of change	• Trigonometric Functions • Solving equations • Solving problems • Identities • transformations	• Exponential and Logarithmic Functions • Average and instantaneous rates of change • Exponent laws/laws of logarithms • Include non-Natural number bases • Solve exponential and logarithmic equations	• Consolidate characteristics of functions (compare and contrast) • Composition of functions • Compound functions • Generalize functions

Rationale

Inclusion of the Introductory unit:

- This introductory unit is not to be presented as a review unit. Rather, build conceptual understanding of general functions based on key concepts previously studied.
- Build a framework of key concepts that can be applied to all functions e.g., average rate of change, intervals of increase/decrease, domain/range, zeros

Embedding 'average and instantaneous rates of change' with each type of function:

- The different natures of the average and instantaneous rates of change of various types of functions can be appreciated more deeply by linking them function by function
- A gradual building of this key concept allows re-visiting it as students' thinking matures

Splitting polynomial and rational functions into two units

- The framework of key concepts developed in the Introductory Unit is applied to familiar polynomial functions
- Concepts of average rate of change, and end behaviours can be established with polynomial functions without the added complexity of asymptotes
- Polynomial and rational functions have different properties.
- Concepts of rational functions can be built on known properties of polynomial functions.

Positioning trigonometry after rational functions

- The tangent and reciprocal functions are introduced as applications of concepts of rational functions

Splitting trigonometry into two units

- The initial focus is on introducing radian measure, and revisiting all of the Grade 11 concepts using radians e.g., zeros, period, amplitude, domain/range, phase shift
- Rates of change are connected to graphs and numerical representations in the first trigonometric unit to consolidate graphical properties
- Some consolidation of basic trigonometric facts is needed before students are ready to pose and solve problems that can be modelled by these functions.

Using Unit 6 for Consolidating and Culminating

- Allows for a consolidation of characteristics of functions to compare and contrast the various types of functions in this course
- Uses the characteristics of functions as the basis for combining and composing functions
- Concepts in this unit provides opportunities to generalize functions and properties of functions

Advanced Functions – Planning Tool

P Number of pre-planned lessons (including instruction, diagnostic and formative assessments, summative assessments other than summative performance tasks)

J Number of jazz days of time (instructional or assessment)

T Total number of days

SP Summative performance task (see Assessment – Grade 9 Applied)

Unit	Cluster of Curriculum Expectations	Overall and Specific Expectations	P	J	T	SP
0	• Revisit contexts studied in the Grade 11 Functions course MCR3U using simplifying assumptions, adding precision to the graphical models, and discussing key features of the graphs using prior academic language (e.g., domain, range, intervals of increase/decrease, intercepts, slope) and 'local maximum/minimum,' 'overall maximum/minimum' • Recognize that transformations previously applied to quadratic and trigonometric functions also apply to linear and exponential functions, and to functions in general • Use function notation to generalize relationships between two functions that are transformations of each other and whose graphs are given • Represent key properties of functions graphically and using function notation • Form inverses of functions whose graphs are given, and apply the vertical line test to determine whether or not these inverses are functions	**C1** identify and describe some key features of polynomial *(linear, quadratic, trigonometric, exponential)** functions, and make connections between the numeric, graphical, and algebraic representations of polynomial* functions **D1** demonstrate an understanding of average and instantaneous** rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point** *reviews characteristics of functions already known **to be addressed in Units 1, 2, 3, 4, 5, and 6	6	1	7	
1	• Identify and use key features of polynomial functions • Solve problems using a variety of tools and strategies related to polynomial functions • Determine and interpret average and instantaneous rates of change for polynomial functions	**C1** identify and describe some key features of polynomial functions, and make connections between the numeric, graphical, and algebraic representations of polynomial functions **C3** solve problems involving polynomial and simple rational* equations graphically and algebraically **C4** demonstrate an understanding of solving polynomial and simple rational inequalities* **D1** demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point *to be addressed in Unit 2	15	2	17	

Unit	Cluster of Curriculum Expectations	Overall and Specific Expectations	P	J	T	SP
2	• Identify and use key features of rational functions • Solve problems using a variety of tools and strategies related to rational functions • Determine and interpret average and instantaneous rates of change for rational functions	**C2** identify and describe some key features of the graphs of rational functions, and represent rational functions graphically **C3** solve problems involving polynomial* and simple rational equations graphically and algebraically **C4** demonstrate and understanding of solving polynomial* and simple rational inequalities **D1** demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point * addressed in Unit 1	6	1	7	
3	• Explore, define and use radian measure • Graph primary trigonometric functions and their reciprocals in radians and identify key features of the functions • Solve problems using a variety of tools and strategies related to trigonometric functions • Determine and interpret average and instantaneous rates of change for trigonometric functions	**B1** demonstrate an understanding of the meaning an application of radian measure **B2** make connections between trigonometric ratios and the graphical and algebraic representations of the corresponding trigonometric functions and between trigonometric functions and their reciprocals, and use these connections to solve problems **D1** demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point	9	1	10	
4	• Graph and transform sinusoidal functions using radian measure • Identify domain, range, phase shift, period, amplitude, and vertical shift of sinusoidal functions using radian measures • Develop equations of sinusoidal functions from graphs and descriptions expressed in radian measure • Solve problems graphically that can be modeled using sinusoidal functions • Prove trigonometric identities • Solve linear and quadratic trigonometric equations using radian measures • Make connections between graphic and algebraic representations of trigonometric relationships	**B2** make connections between trigonometric ratios and the graphical and algebraic representations of the corresponding trigonometric functions and between trigonometric functions and their reciprocals, and use these connections to solve problems **B3** solve problems involving trigonometric equations and prove trigonometric identities	11	2	13	

Unit	Cluster of Curriculum Expectations	Overall and Specific Expectations	P	J	T	SP
5	• Develop the understanding that the logarithmic function is the inverse of the exponential function • Simplify exponential and logarithmic expressions using exponent rules • Identify features of the logarithmic function including rates of change • Transform logarithmic functions • Evaluate exponential and logarithmic expressions and equations • Solve problems that can be modeled using exponential or logarithmic functions	**A1** demonstrate an understanding of the relationship between exponential expressions and logarithmic expressions, evaluate logarithms, and apply the laws of logarithms to simplify numeric expressions **A2** identify and describe some key features of the graphs of logarithmic functions, make connections between the numeric, graphical, and algebraic representations of logarithmic functions, and solve related problems graphically **A3** solve problems involving exponential and logarithmic equations algebraically, including problems arising from real-world applications **D1** demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point	12	2	14	
6	• Consolidate understanding of characteristics of functions (polynomial, rational, trigonometric, and exponential) • Create new functions by adding, subtracting, multiplying, or dividing functions • Create composite functions • Determine key properties of the new functions • Generalize their understanding of a function	**D1** demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point **D2** determine functions that result from the addition, subtraction, multiplication, and division of two functions and from the composition of two functions, describe some properties of the resulting functions, and solve related problems **D3** compare the characteristics of functions, and solve problems by modeling and reasoning with functions, including problems with solutions that are not accessible by standard algebraic techniques	11	2	13	
	Summative Performance Tasks				**4**	
	Total Days		**70**	**11**	**85**	

The number of prepared lessons represents the lessons that could be planned ahead based on the range of student readiness, interests, and learning profiles that can be expected in a class. The extra time available for "instructional jazz" can be taken a few minutes at a time within a pre-planned lesson or taken a whole class at a time, as informed by teachers' observations of student needs.

The reference numbers are intended to indicate which lessons are planned to precede and follow each other. Actual day numbers for particular lessons and separations between terms will need to be adjusted by teachers.

Introductory Unit: Advanced Functions

Lesson Outline

Big Picture

Students will:

- revisit contexts studied in the Grade 11 Functions course (MHF4U) using simplifying assumptions, adding precision to the graphical models, and discussing key features of the graphs using prior academic language (e.g., domain, range, intervals of increase/decrease, intercepts, slope) and 'local maximum/minimum,' 'overall maximum/minimum;'
- recognize that transformations previously applied to quadratic and trigonometric functions also apply to linear and exponential functions, and to functions in general;
- use function notation to generalize relationships between two functions that are transformations of each other and whose graphs are given;
- represent key properties of functions graphically and using function notation;
- form inverses of functions whose graphs are given, and apply the vertical line test to determine whether or not these inverses are functions.

Day	Lesson Title	Math Learning Goals	Expectations
1–2	Adding Precision to Graphical Models and Their Descriptions	• From initial simplifying assumptions about a context and the corresponding distance/time graph, introduce the complicating factors in the context and analyse adjustments needed in the graph e.g., swimming laps in a pool; riding a bicycle up a hill, down a hill, on the flat • Use the following academic language to describe changes: speed (rate of change), intervals of increase/decrease, domain/range, overall and local maximum, and overall and local minimum • Graph corresponding speed/time graphs	D1.1, D1.1, D3.1, and setting up C1.2
3	Transformations Across Function Types	• Use function notation to generalize relationships between sets of two congruent functions e.g., $h(x) = f(x) + 2$ to generalize a line and the line shifted 2 units, a parabola and the parabola shifted 2 units up, an exponential function and the exponential function shifted 2 unit up; $f(x) = g(x + 3)$ • Use graphical and numerical representations of the functions • Introduce the concept that lines and exponential functions can be seen through a transformational lens. • Graph $y = f(x) – 3$ from any given $y = f(x)$	Setting up C1.6, A2.3
4	Using Function Notation to Generalize Relationships	• Use function notation to generalize relationships between sets of two functions, one a single transformation of the other e.g., $h(x) = 2f(x)$ to generalize a sinusoidal function and the stretched sinusoidal function, a line and the stretched line, a parabola and the stretched parabola, an exponential function and the stretched exponential function shifted 2 units up; $f(x) = g(x + 3)$ • Use graphical and numerical representations of the functions	Setting up C1.6

Day	Lesson Title	Math Learning Goals	Expectations
5	Representing Key Properties of Functions Graphically and Using Function Notation	• Interpret graphically, values shown in function notation e.g., Graph $y = f(x)$ that has all of the following properties: $f(1) = 2, f(3) = f(-1) = 0, f(0) = 4, f(x) > 0$ for $x < 0$, and $f(x) < 0$ for $x > 0$, domain $x \epsilon R$, range $-4 < y < 4$ • Explore multiple solutions to each of the above, noting the lack of information for determining concavity • Represent critical points and key regions of the graph of a function using functional notation	Setting up C1.7,2.2, A2.1
6	Forming Inverses and Function Testing	• Form inverses of given functions (graphical representations) and determine whether or not the inverse is a function	Setting up A1.1, 2.2
7	Jazz day to summarize	•	

Unit 1: Polynomial Functions

Grade 12

Lesson Outline

Big Picture
Students will: • identify and use key features of polynomial functions; • solve problems using a variety of tools and strategies related to polynomial functions; • determine and interpret average and instantaneous rates of change for polynomial functions.

Day	Lesson Title	Math Learning Goals	Expectations
1	Ch…Ch…Changes	• Make connections between verbal and graphical rates of change. • Make connections between average velocity and slopes of secants. • Calculate and interpret average rates of change of functions arising from real-world applications.	D1.2, 1.3, 1.4, 1.7
2–3	"Secant" Best Is Not Good Enough Freeze Frame	• Make connections between motion data and instantaneous rates of change. • Make connections between instantaneous rates of change and the tangent in context. • Use technology to calculate slopes of secants and tangents at various points along a curve. • Interpret slopes of secants and tangents in context.	D1.1, 1.3, 1.5-1.8
4	Smooth Curves Passing Through Points	• Solve problems involving average and instantaneous rates of change at a point using numerical and graphical methods. • Distinguish situations in which the rate of change is zero, constant, or changing by examining applications.	D1.2, 1.9
5–6	Characteristics of Polynomial Functions	• Investigate and summarize graphical characteristics, e.g., zeros, finite differences, end behaviour, domain and range, increasing/decreasing behaviour, of polynomials functions through numeric, graphical and algebraic representations. • Compare these characteristics for linear, quadratic, cubic and quartic functions.	C1.1, 1.2, 1.3, 1.4
7	In Factored Form	• Make connections between a polynomial function in factored form and the x-intercepts of its graph. • Sketch the graph of polynomial functions, expressed in factored form using the characteristics of polynomial functions. • Determine the equation of a polynomial given a set of conditions, e.g., zeros, end behaviour, and recognize there may be more than one such function	C1.5, 1.7
8		• Investigate transformations applied to $f(x) = x^3$ and $f(x) = x^4$. • Investigate and compare the properties of odd and even functions.	C1.6, 1.9

Day	Lesson Title	Math Learning Goals	Expectations
9–10		• Divide polynomials. • Examine remainders of polynomial division and connect to the remainder theorem. • Make connections between the polynomial function $f(x)$, the divisor $x - a$, the remainder of the division $f(x)/(x-a)$ and $f(a)$ using technology. • Identify the factor theorem as a special case of the remainder theorem. • Factor polynomial expressions in one variable of degree no higher than four.	C3.1, 3.2
11–12		• Solve problems graphically and algebraically using the remainder and factor theorems. • Solve polynomial equations by selecting an appropriate strategy, and verify with technology. • Make connections between the x-intercepts of a polynomial function and the real roots of the corresponding equation. • Use properties of polynomials to fit a polynomial function to a graph or a given set of conditions. • Determine the equation of a particular member of a family of polynomials given a set of conditions, e.g., zeros, end behaviour, point on the graph [See homework and make use of graphs on Day 7.]	C1.8, 3.3, 3.4, 3.7
13–14		• Understand the difference between the solution to an equation and the solution to an inequality. • Solve polynomial inequalities and simple rational inequalities by graphing with technology. • Solve linear inequalities and factorable polynomial inequalities. • Represent the solution to inequalities on a number line or algebraically.	C 4.1, 4.2, 4.3
15–16		• Jazz days	
17		• Summative	

75 min

Math Learning Goals
- Make connections between verbal and graphical rates of change.
- Make connections between the average velocity and slope of the secant line.
- Investigate the average rates of change of two sprinters to determine who was running the fastest during the race.

Materials
- BLM 1.1.1, 1.1.2

Assessment Opportunities

Minds On…

Think/Pair/Share → Matching Activity

Students categorize the rates of change of the graphs as either: zero, constant or changing (BLM 1.1.1). Ask: "Which graphs have a rate of change of zero? How do you know? Which graphs have a constant rate of change? How can you tell? Which graphs have a non-constant rate of changing? How do you know?"

Students match verbal and graphical rates of change (BLM 1.1.1).

Mathematical Process/Connecting/Observation/Mental Note: Conduct a poll to determine correctness of pairs' matches to graph A, e.g., using Clickers or TI-Navigator or by holding up a sheet with the number written on it. Ask a pair with the correct answer to explain their strategy.

Circulate to monitor the progress of students who had incorrect answers.

Use an overhead of BLM 1.1.1 to match graph A to its description. Use the same coloured marker to connect the words, the features of the graph and student strategies. Ask: "Did any pair use a different strategy?" Pairs discuss their strategies and matches for the remaining graphs (B–E). Poll answers to the remaining questions on the overhead.

Highlight that the independent variable for each graph is time.

For each graph students identify the missing dependent variable.

Word Wall
- dependent variable, independent variable, velocity, average rate of change, $\Delta y / \Delta x$, secant

Action!

Whole Class → Teacher Directed Instruction

Provide students with the definition of average rate of change as $\Delta y / \Delta x$ and connect to their experiences with slope; then specifically to the average velocity as $V_{average} = \Delta d / \Delta t$

Pairs → Investigation

Pairs complete Part A of BLM 1.1.2.

Curriculum Expectations/Observation/Mental Note: Circulate to ensure that pairs are getting correct answers. Make a mental note to consolidate misconceptions.

Consolidate Debrief

Whole Class → Sharing

Students with correct understanding of concepts misunderstood by their classmates explain their responses.

Students should have connected the relationship between the average rate of change and the slope of the secant.

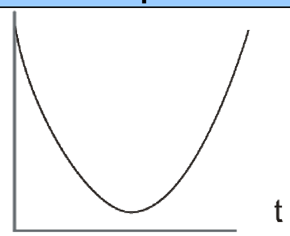
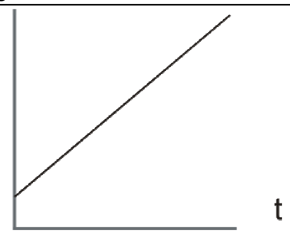
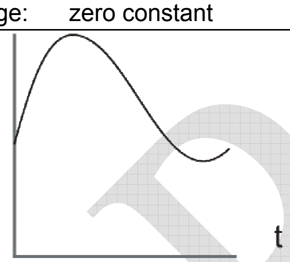
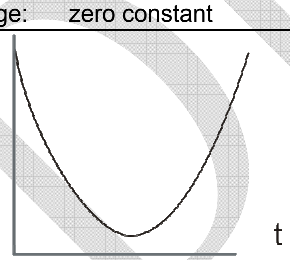
Exploration
Application

Home Activity or Further Classroom Consolidation

Complete Parts B and C of Worksheet 1.1.2.

1.1.1: Ch… Ch… Changes

- Circle the rate of change as zero, constant, or changing for each graph.
- Match the graphs with the descriptions given on the right.
- Be prepared to explain your reasoning.

Graphs
A  Rate of Change: zero constant changing
B  Rate of Change: zero constant changing
C  Rate of Change: zero constant changing
D  Rate of Change: zero constant changing
E 
Rate of Change: zero constant changing

Description
1. A grade 12 student's height over the next 12 months.
2. Money deposited on your 12th birthday grew slowly at first, then more quickly.
3. Andrea walks quickly, slows to a stop, and then speeds up until she is travelling at the same speed as when she started.
4. Over a one-month period the rate of growth for a sunflower is constant.
5. Clara walks quickly and then slows to a stop. She then walks quickly and slows to a second stop. Clara then walks at a pace that is a little slower than when she started.

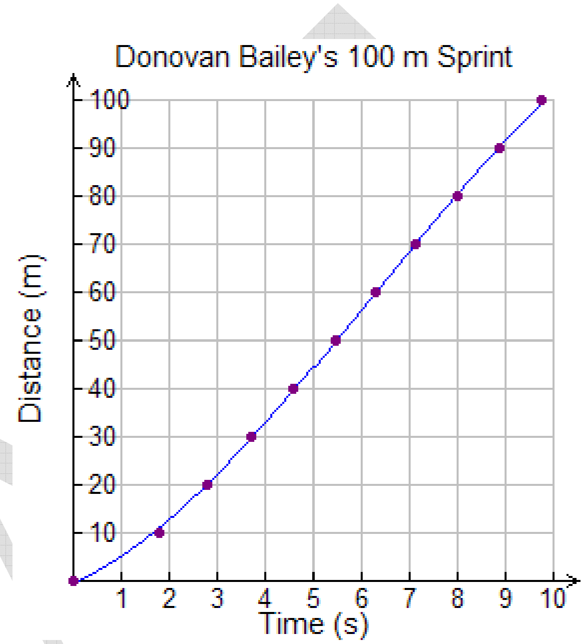
1.1.2: A Race to the Finish Line

During the 1997 World Championships in Athens, Greece, Maurice Greene and Donovan Bailey ran a 100 m race.
(http://hypertextbook.com/facts/2000/KatarzynaJanuszkiewicz.shtml)

Part A

The graph and table below show Donovan Bailey's performance during this 100 m race.

Donovan Bailey's Performance	
Time (s)	**Distance (m)**
0	0
1.78	10
2.81	20
3.72	30
4.59	40
5.44	50
6.29	60
7.14	70
8.00	80
8.87	90
9.77	100



1. a) Calculate Donovan Bailey's average velocity for this 100 m sprint.

$$\text{Average Velocity} = \frac{\textit{change in distance}}{\textit{change in time}} = \frac{\Delta d}{\Delta t}$$

   b) Draw a line from (0,0) to (9.77,100) on the graph above.

   A line passing through at least two different points on a curve is called a secant.

   c) Explain the relationship between your answer to a) and the slope of the secant.

2. a) Draw the secants from (0,0) to (5.44,50) and from (5.44,50) to (9.77,100).

   b) Calculate the average velocities represented by the two secants drawn in a).

   i) ii)

1.1.2: A Race to the Finish Line (continued)

c) Compare Bailey's performance during the first and the second half of the race.

3. Describe the relationship between average velocity and the slope of the corresponding secant.

4. Calculate Bailey's average velocity for each 10 m interval of this 100 m race. Record your answers in the table below.

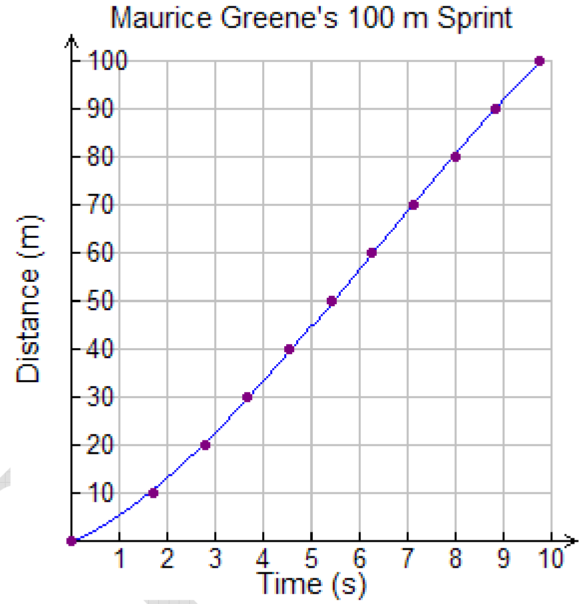
Interval (m)	**Distance Travelled Δd (m)**	**Time Elapsed Δt (s)**	**Average Velocity (m/s)**
0 to 10			
10 to 20			
20 to 30			
30 to 40			
40 to 50			
50 to 60			
60 to 70			
70 to 80			
80 to 90			
90 to 100			

1.1.2: A Race to the Finish Line (continued)

Part B

The graph and table show Maurice Greene's performance during the same 100 m race.

Maurice Greene's Performance	
Time (s)	**Distance (m)**
0	0
1.71	10
2.75	20
3.67	30
4.55	40
5.42	50
6.27	60
7.12	70
7.98	80
8.85	90
9.73	100



Calculate Greene's average velocity for each 10 m interval of this 100 m race. Record your answers in the table below.

Interval (m)	Distance Travelled Δd (m)	Time Elapsed Δt (s)	Average Velocity (m/s)
0 to 10			
10 to 20			
20 to 30			
30 to 40			
40 to 50			
50 to 60			
60 to 70			
70 to 80			
80 to 90			
90 to 100			

Part C

Using your calculations from Parts A and B, describe this 100 m race run by Donovan Bailey and Maurice Greene. Include who was fastest and who was leading at various points during the race.

Unit 1: Day 2: "Secant" Best Is Not Good Enough

Grade 12

75 min

Math Learning Goals

- Make connections between collected motion data to instantaneous rates of change.
- Make connections between average and instantaneous rates of change as they relate to other real world applications.

Materials

- BLM 1.2.1, 1.2.2, 1.2.3
- Overhead of BLM 1.2.2
- graphing calculators
- CBRs
- overhead projector

Assessment Opportunities

Minds On…

Small Groups → Exploration

Students generate the graphs of linear, quadratic and cubic functions using the CBR and graphing calculators (BLM 1.2.1).

Students may have to walk the line several times to match graphs.

Whole Group → Discussion

Discuss strategies students used to replicate the graphs referring to connections between the motion and the average rates of change.

Action!

Small Groups → Investigation

Students use a CBR to gather distance vs. time data with the goal of having the fastest rate of change at t= 1s (BLM 1.2.2).

Learning Skills – Collaborate & Teamwork/Observation/Checklist: Observe students' ability to work collaboratively.

Check to ensure students are smoothing their graphs rather than connecting using line segments.

Consolidate Debrief

Whole Class → Debrief

Ask 2–3 groups who finish first to prepare to share their data with the class, either on the blackboard, overhead projector, Smart Board, Document Camera or chart paper. Ask the group who ran the fastest to prepare to display their graphs including the secant lines.

While examining the numerical data, ask the class:

- How did the average velocities help you to estimate the velocity at 1s?
- How accurate were the estimations?
- What contributed to the accuracy?

Class discussion should include the calculation of average velocities over smaller intervals and how average velocities are used to approximate instantaneous velocities.

Make connections between the numerical and graphical data: Use a straight edge to trace the secant lines for the displayed graph.

Make connections between the slope of the secant lines and the average velocities: On their graphs students name this line the tangent as representing the velocity at $t = 1$.

Word Wall

- instantaneous rate of change,
- tangent

Application
Exploration
Reflection

Home Activity or Further Classroom Consolidation

Complete "Instantaneous Success." (Worksheet--)
Write a journal entry that summarizes your new learning about: secants, tangents, average rates of change, and instantaneous rates of change.

1.2.1: Race Match Challenge

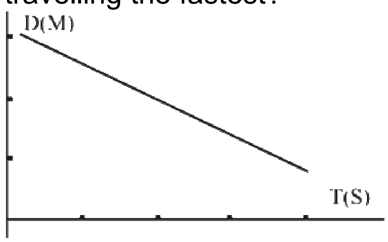
In this activity you will be walking three different walks. The first walk will model a linear function, the second will model a quadratic function, and the third will model a cubic function.

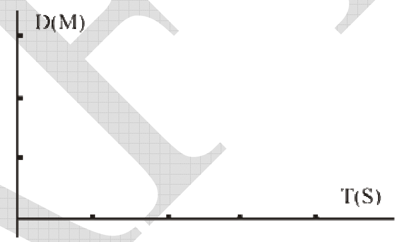
Instructions

1. Attach the CBR to your graphing calculator.
2. Press **APPS** and select **CBL/CBR** by pressing **ENTER**.
3. Press any key to enter the **CBL/CBR** program.
4. Select 3: **RANGER** and press **ENTER** to continue.
5. Select 1: **SETUP/SAMPLE**
6. Bring the arrow (cursor) to the **START NOW** option and press **ENTER**.
7. When you are ready press **ENTER** to start collecting data.

Walk 1 – Linear Walk

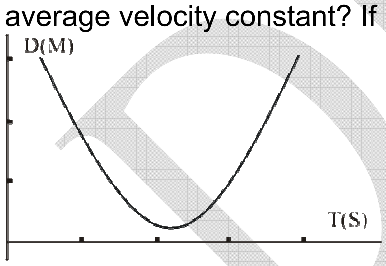
Place yourself 3 metres from the CBR and walk towards the CBR at a constant velocity until you are 0.5 metres away. Your graph should look like the graph below. Draw a sketch of your walk on the second graph below. Is the average velocity constant? If not, over what interval were you travelling the fastest?

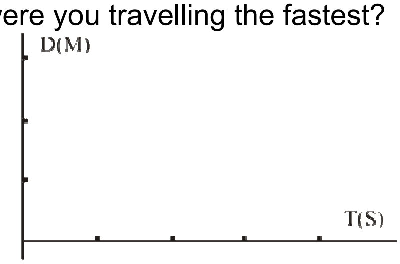




Walk 2 – Quadratic Walk

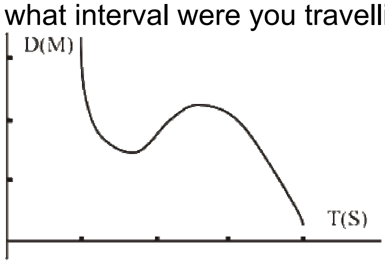
Walk towards the CBR and gradually reduce your walking rate (velocity) until you are 0.5 metres from the CBR then walk away from the CBR gradually increasing your velocity. Your graph should look like the graph below. Draw a sketch of your walk on the second graph below. Is the average velocity constant? If not, over what interval were you travelling the fastest?

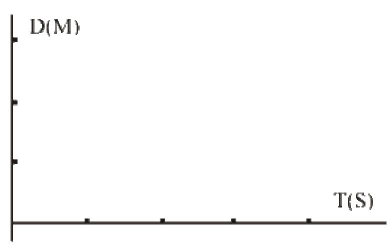




Walk 3 – Cubic Walk

Place yourself 3 metres away from the CBR. Create a walk similar to the one below. Draw a sketch of your walk on the second graph below. Is the average velocity constant? If not, over what interval were you travelling the fastest?

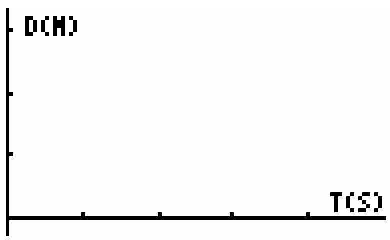




1.2.2: "Secant" Best Is Not Good Enough

You are to determine the fastest student at the 1 second point of the race. Stand 0.5 metres away from the CBR. With your partner, plan your motion so that you will be moving as fast as possible 1 second after the CBR has started. Using your knowledge of average rates of change, determine the fastest person in the class at $t = 1$ s.

1. Attach the CBR to your graphing calculator.
2. Press **APPS** and select **CBL/CBR** by pressing **ENTER**.
3. Press any key to enter the **CBL/CBR** program.
4. Select 3: **RANGER** and press **ENTER** to continue.
5. Select 1: **SETUP/SAMPLE**
6. Bring the arrow (cursor) to the **START NOW** option and press **ENTER**.
7. Have your partner stand 0.5 m away from the CBR. When you are ready press **ENTER** to start collecting data and have your partner run away from the CBR.
8. Draw a sketch of your graph below.



9. Using the cursor keys find two points that you can use to determine the average velocity for the first 2 seconds. Draw a secant on the graph above to represent this average velocity.

10. Use the cursor keys to fill in the following table:

t (s)	0	0.5	0.9	1	1.1	1.5	2
d(m)							

11. a) Calculate the average velocity over the following intervals.

0–1s	0.5–1s	0.9–1s	1–1.1s	1–1.5s	1–2s

b) Using these calculations, estimate the velocity at $t = 1$ s. Does your answer from a) accurately represent the velocity at $t = 1$ s? Explain.

c) How could you improve the estimate from a)?

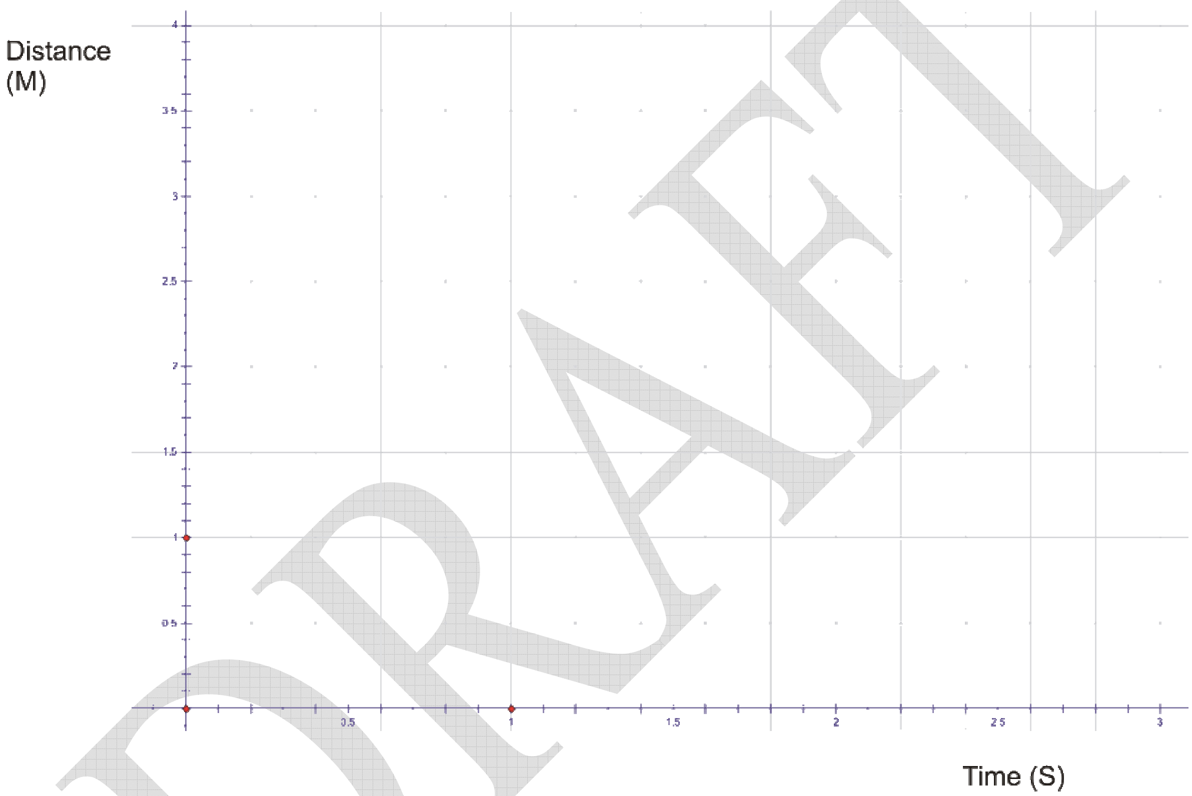
1.2.2: “Secant” Best Is Not Good Enough (continued)

12. a) Use the data points from # 10 to construct a more accurate graph below.

b) Sketch the secant lines that represent the average velocities over each interval found in 11.

c) How does the slope of the secant lines connect to the average velocities?

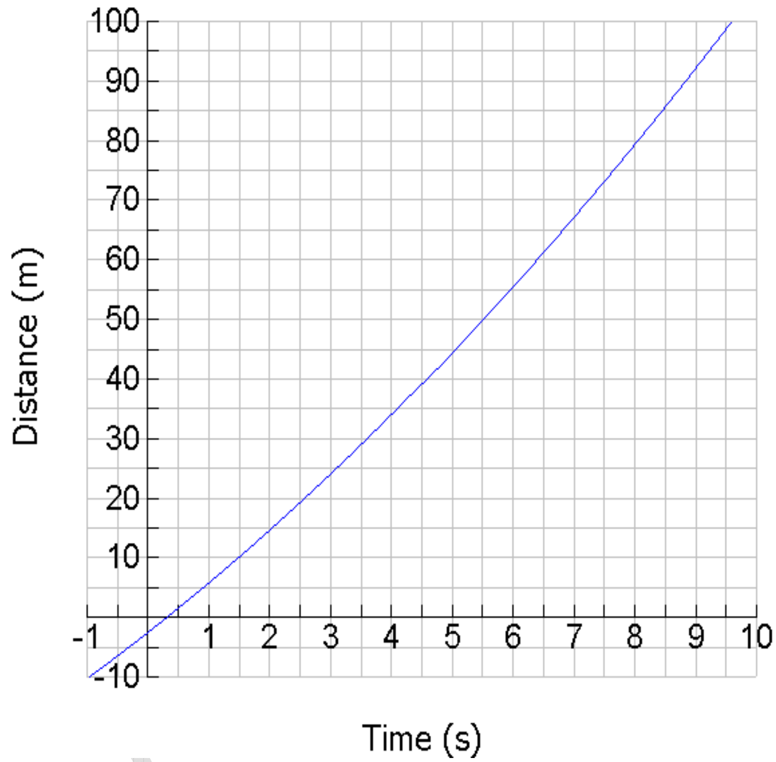
d) Sketch the graph of the line to represent the estimated velocity at t = 1s.



1.2.3: "Instant"aneous Success!

In 1997, Donavan Bailey ran the 100 m sprint in 9.77 seconds. The table below describes his run. One model that describes this run is a quadratic model with an equation of:
s(t) = 0.28t2 + 8.0t – 2.54.

Time (s)	Distance (m)
0	0
1.78	10
2.81	20
3.72	30
4.59	40
5.44	50
6.29	60
7.14	70
8.00	80
8.87	90
9.77	100



1. a) Estimate Donavan Bailey's instantaneous velocity at $t = 6$ s.

   b) Explain why your answer to a) is a good approximation.

   c) Plot a point on the curve at 6 seconds. Draw a line that passes through this point but does not pass through the curve again. **This line is called a tangent to the curve**.

1.2.3: “Instant”aneous Success! (continued)

2. Either by hand or with technology, use the algebraic model $s(t) = 0.28t^2 + 8.0t – 2.54$ to approximate the instantaneous velocity of Donavan Bailey at t = 6 s.

   a) Reset your graphing calculator
   b) Enter the equation of Bailey’s run by pressing Y=
   c) Set the Window using the axes above as a guide.
   d) Press Graph.
   e) Press Trace and then 6 Enter to position the cursor at t = 6 s, record the y-value.
   f) Press Trace and 7 Enter, record the y-value.
   g) Calculate the average velocity over this interval.

Point A		**Point B**		**Average Velocity**
x-value	6	x-value	7	
y-value		y-value		

h) Repeat steps f) and g) above two more times, each time choosing a point closer to t = 6 s. Calculate the average velocities over these intervals.

Point A		**Point B**		**Average Velocity**
x-value	6	x-value		
y-value		y-value		

Point A		**Point B**		**Average Velocity**
x-value	6	x-value		
y-value		y-value		

i) Use the calculations from steps g) and h) to estimate the instantaneous velocity at t = 6 s.

j) Draw the secants (on the graph) that correspond with the three average velocities calculated above. How do the secants compare to the tangent drawn in 1(c)?

75 min

Math Learning Goals

- Make connections between the instantaneous rate of change and the tangent in a real world context.
- Use the Geometer's Sketchpad® to speed up calculations of slopes of secants and estimate slopes of tangents at various points on a given polynomial graph.
- Interpret slopes of secants and tangents in contexts.

Materials

- BLM 1.3.1,1.3.2, 1.3.3
- The Geometers' Sketchpad®

Assessment Opportunities

Minds On…

Small Groups → Sorting Activity

Mathematical Process – Connecting/Observation/Mental Note:

To make sense of their relationships, students sort descriptions into 3 sets – those describing average rates of change, those that are close to instantaneous rates of change, and those that are instantaneous rates of change (BLM 1.3.1).

Whole Class → Discussion

Guide a class discussion focused on misconceptions and different points of view observed in small group activity. Challenge each group to create one unique example of average rate of change, close to average rate of change, and instantaneous rate of change.

Measurements although described as instantaneous are based on a small interval of time, and should be referred to as close to instantaneous.

BLM 1.3.1 Answers:
Average – 1,3,5,7
Close to – 2,4,8
Instant - 6

Action!

Pairs → Investigation

Pairs complete the investigation using The Geometers' Sketchpad (BLM 1.3.2). Pairs work on completing the questions, keeping their own notes (BLM 1.3.2).

Consolidate Debrief

Whole Class → Reflection

Guide student reflection on the use of technology to determine the instantaneous rate of change from the activity.

Given a function, discuss strategies for determining the instantaneous rate of change without technology.

Highlight responses that focus on finding the slope over a very small interval.

Reflection

Home Activity or Further Classroom Consolidation

Complete Worksheet by hand or using technology.

Student licensed take-home version of GSP® is available through the IT department at boards.

1.3.1: Average, Close to Instant or Instant?

1. Some road tolls in the U.S. give speeding tickets based on the time it takes you to travel between exits.

2. A police officer pulls you over for speeding since her radar gun displays 130 km/hr.

3. Canada's population grew at a rate of 0.869% from 2006 to 2007.

4. Roy Halliday's fast ball was measured to have a velocity of 152 km/h.

5. Your parents kept a growth chart from the time you were 1 until you were 5 years old. They have calculated that your growth rate in that period was 9 cm/year.

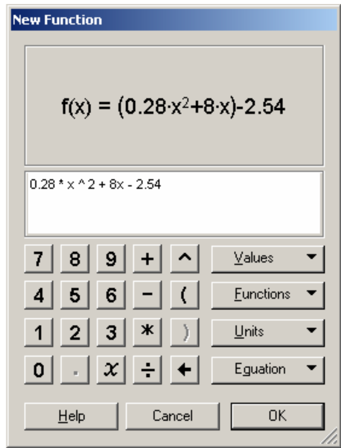
6. In 1996, Hurricane Bertha had wind gusts up to185 km/h. At some times during Hurricane Bertha the wind was gusting at 100 km/h.

7. Water is being poured into a container. The rate in which the water level increases between 0 and 5 seconds of the pour is 7 mm/sec.

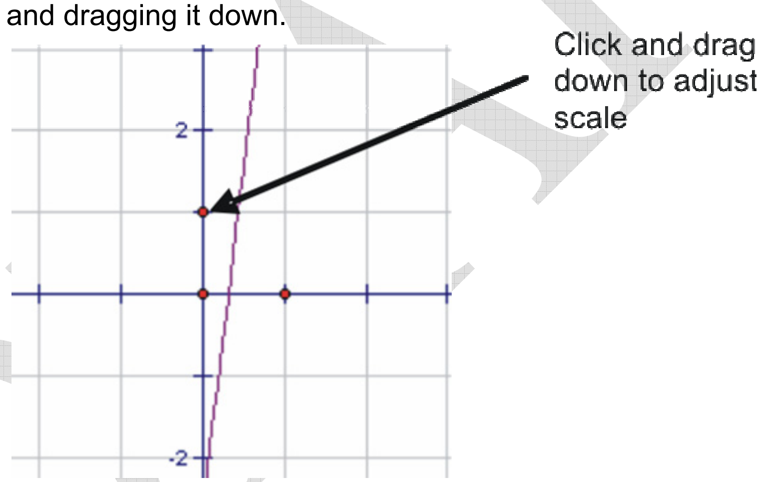
8. A CO_2 probe measures the rate of increase of atmospheric CO_2. The probe reads an increase of 1.7 x 10-8 ppm/sec.

1.3.2: Sketching A Sprinter's Path

1. Open Geometer's Sketchpad.
2. From the menu bar select **Graph → Plot New Function**.
3. Enter the equation given yesterday of Bailey's sprint, $s(t) = 0.28t^2 + 8.0t - 2.54$, then press **OK**. (**Note:** Replace *t* with *x*.)



1. From the menu bar select Graph → **Rectangular Grid.**
2. Change the y-scale of the graph to show a range that includes 0 to 100 m by clicking the red dot on the axis and dragging it down.



6. Select the graph of the function and from the menu bar select **Construct → Point On Function Plot**.
7. Deselect all objects by clicking on a blank part of the grid.
8. Repeat step 6 to create another point on the curve.
9. Using the labelling tool A, label one point A and the other point B.
10. Select point **A** and from the main menu select **Measure → Abscissa (*x*).** This value represents the *x*-value (time) of point **A**.
11. Repeat step 10 using point **B**.
12. Select both points on the curve and select **Construct → Line** from the main menu.
13. Select the line and from the menu bar select **Measure → Slope**. This represents the **slope of the secant**.
14. Find the slope between the *x*-values 6 and 7.

1.3.2: Sketching A Sprinter's Path (continued)

15. Compare the slope from #14 to the average velocity between 6 and 7 seconds and the velocity of Donovan Bailey's run. What do you notice?

16. How is the slope of the secant related to the average velocity?

17. Find the average velocity by dragging both points of Donavan Bailey's run for the intervals given in the table below:

Interval	Average Velocity (m/s)
Between 0.5 and 2 seconds	
Between 0.5 and 1.5 seconds	
Between 0.5 and 1 seconds	
Between 0.5 and 0.6 seconds	

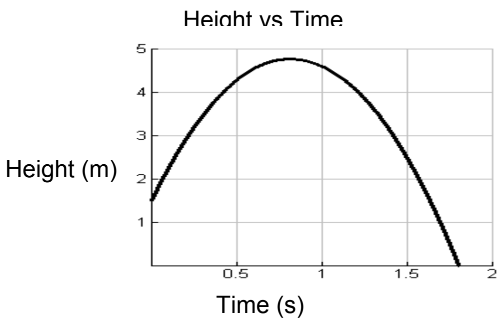
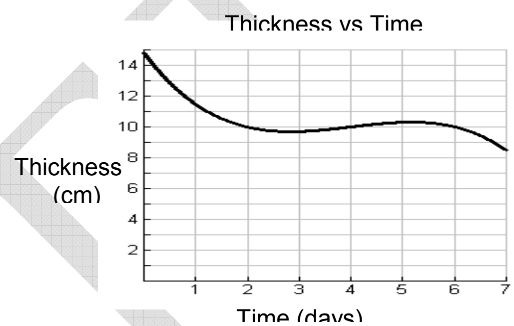
18. Describe a method for determining Bailey's instantaneous velocity at time t.

19. Donavan Bailey's instantaneous velocity was measured during the race at different time values as shown in the table below. Verify if these values are correct using the points you created on the function.

Time (s)	Actual Instantaneous Velocity (m/s)	Calculated Instantaneous Velocity (m/s)
1.78	8.90	
2.81	10.55	
3.72	11.28	
4.59	11.63	
5.44	11.76	
6.29	11.80	
7.14	11.70	
8.00	11.55	
8.87	11.38	
9.77	11.00	

20. Does your calculated data match the actual data? Explain why or why not.

1.3.3: Freeze Frame

1. A height of a shotput can be modelled by the function H(t) = -4.9 t2 + 8t + 1.5, where h is the height in metres and t is the t in seconds.	2. The thickness of the ice on a lake for one week is modelled by the function: T(d) = -0.1d3 + 1.2d2 – 4.4d + 14.8, where T is the thickness in cm and d is the number of days after December 31st. The graph of the function is provided below.
The graph of the function is provided below. 	
	
a) At what point do you think the shot put was travelling the fastest? What factors did you use to make your inference?	a) When do you think the warmest day occurred during the week? Justify your answer.
b) Determine the average rate of change on a short interval near the point you chose in a).	b) Determine the average rate of change on a short interval near the point you chose in a).
c) Estimate the instantaneous rate of change at the point you chose in a).	c) Determine the instantaneous rate of change at the point you chose in a).
d) Were your answers to the average rate of change the same as the instantaneous rate of change, if not why not?	d) Were your answers to the average rate of change the same as the instantaneous rate of change, if not why not?

Unit 1: Day 4: Smooth Curves Passing Through Points

Grade 12

75 min

Math Learning Goals

- Solve problems involving average and instantaneous rates of change at a point using numerical and graphical methods.
- Distinguish situations in which the rate of change is zero, constant, or changing by examining applications.

Materials

- BLM 1.4.1
- Overhead of BLM 1.4.1

Assessment Opportunities

Minds On…

Pairs → Discussion

Students compared their solutions in pairs (BLM 1.3.3).
Circulate to determine that pairs are getting correct answers, and team up pairs as necessary to undo misconceptions.

Literacy Strategy: Placemat

Groups of 4 → Placemat

Students create a placemat with 4 divisions and a circle in the centre. Individually students write their description of the instantaneous rate of change of a function at a point in their section of the placemat. Group consensus is written in the centre of their placemat.

One member of each group shares their statement(s) with the whole class. Discuss as a class common points and any misconceptions about instantaneous rates of change. Groups make changes to their description as needed.

Listen to groups' descriptions of rate of change to form ability groupings for **Action!** section.

Curriculum Expectations/Observation/Mental Note

Action!

Groups of 4 → Investigation

Form homogeneous groups based on observations during **Minds On…** activity - students with a thorough understanding work on Graph B of (BLM 1.4.1); others work on Graph A. Inform students context for graphs comes from BLM 1.3.3.

Select a group that worked on Graph A to share responses from their investigation for question #1and a group to share responses from Graph B. Students copy the information for the graphs they did not work on to complete the remaining questions.

Consolidate Debrief

Whole Class → Sharing

Discuss the remaining questions. Highlight similarities between the signs of the slopes of the tangents between the 2 graphs. Select 2 students to trace the curves with a straight edge to demonstrate the trend in the slopes of the tangents as time changes – bring out connections to the key characteristics of graphs (intervals of increase/decrease, local maximum/minimum).

Journal Entry

Home Activity or Further Classroom Consolidation

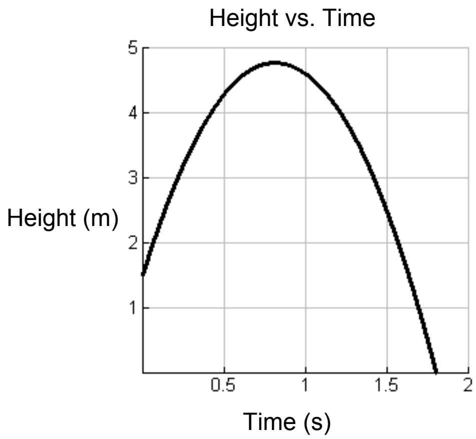
Enter what you know about rates of changes and key characteristics of graphs in your journal.

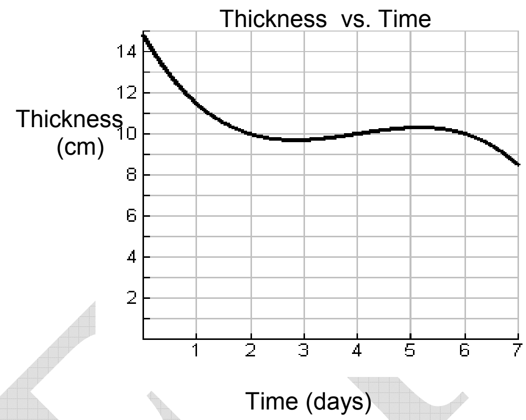
Literacy Strategy: Journals

1.4.1: Tangent Slopes and Graph Characteristics

1. Using the graphs below determine the instantaneous rates of change (m_T) for one of the graphs at the given points.

A.





t	0	0.25	0.5	0.8	1	1.5	1.8
m_T							

t	0	1	2	3	4	5	6	7
m_T								

2. State the domain and the range of the two functions.
   Graph A
   Domain:
   Range:

   Graph B:
   Domain:
   Range:

3 a) Describe the graphical feature (e.g., local maximum/minimum point, interval of increase/decrease), m_T values (e.g., +, -, 0), and, where appropriate, the trend of the slope of the tangent (e.g., changing from positive to zero to negative) as time increases.

Interval	Graphical feature	m_T values	m_T trend (if appropriate)
Graph A: Domain 0-0.8			
Graph A: at 0.8			
Graph A: Domain 0.8–1.75			
Graph B: Domain 0–3			
Graph B: at 3			
Graph B: Domain 3–5			
Graph B: at 5			
Graph B: Domain 5–7			

b) Describe the context where the slope of the tangent is zero. What does it mean?

c) What are the similarities and differences between the slopes of the tangents?

1.4.1: Tangent Slopes and Graph Characteristics (continued)

4. The slope of the secant line is a good estimate of the slope of the tangent. Rod thought that using an interval of 1 second to determine the slope of the secant line in graph A is good enough to use to determine the slope of the tangent. Do you agree or disagree? Justify your reasoning.

5. For Graph A, state the slope of the tangent at 0.5 and 1 second. At which point is the shot put going faster? Explain.

6. Draw three or four curves that have a secant slope of 2 and that pass through the same two points. What inference can be made from this?

7. Anne says that a tangent crosses a curve in one and only one point. Do you agree or disagree? Use Graph B to justify your position.

Unit 1: Day 5: Characteristics of Polynomial Functions — Grade 12

75 min

Math Learning Goals
- Investigate and summarize the graphical characteristics, e.g., zeros, finite differences, end behaviour, domain and range, increasing/decreasing behaviour, of polynomial functions given algebraic, expanded form, and numeric representations.

Materials
- BLM 1.5.1, 1.5.2, 1.5.3
- graphing calculator

Assessment Opportunities

Minds On…

Think/Pair/Share → Concept Attainment Activity

Students compare the data sets and answer questions on BLM 1.5.1 to distinguish between mathematical relationships that are and are not polynomial functions.

During whole class sharing, guide development of a class summary of polynomial functions for the Word Wall, e.g., using a Frayer model with quadrant headings: definition, facts, examples, non-examples.

Sample Definition: A polynomial function is a relationship whose equation can be expressed in the form y = (an expression that is constructed from one or more variables and constants, using only the operations of +, −, ×, and constant positive whole number exponents). For each *x*-value in the domain, there corresponds one and only one element in the range.

Sample Facts:
Exponents on *x* are whole numbers
No variables in denominators
Graphs are smooth continuous curves

Action!

Small Groups or Pairs → Investigation

Students use GSP® files – Quartic Polynomial Investigation, Cubic Polynomial Investigation, and Linear and Quadratic Polynomial Investigation – to manipulate sliders on parameters in the equations.

Alternatively, students use graphing calculators to explore the graphical effects of changing a, b, c, d, and k parameters in equations $y = ax + k$, $y = ax^2 + bc + k$, $y = ax^3 + bx^2 + cx + k$, and $y = ax^4 + bx^3 + cx^2 + dx + k$

Students summarize some of the graphical characteristics (e.g., overall shape, end behaviour, domain and range, increasing/decreasing behaviour, maximum number of zeros; single, double, and triple roots of the corresponding equations) without yet making precise links between the algebraic conditions leading to these various types of roots (BLM 1.5.2).

Learning Skills (Initiative)/Observation/Rating Scale: Observe how the students individually demonstrate initiative as they conduct their group investigations.

Whole Class → Presentations

Different groups use the GSP® files and a Smart board to demonstrate the properties of quartic, cubic, quadratic, and linear polynomial functions, and capture summaries in a Smart Notebook file for class sharing. Alternatively, groups use large chart paper to record their responses.

You may wish to address these questions one at a time.

Consolidate Debrief

Whole Class → Presentations

Students individually hypothesize the answers to questions such as the following:
- What is the effect of changing the degree of the polynomials?
- What is the effect of the leading coefficient being positive or negative?
- What is the maximum number of *x*-intercepts for given a function and how do you know?

Record all hypotheses. Guide a discussion as to how the hypotheses could be tested. Model the testing of the hypotheses using the GSP® file and/or the artefacts of the whole class presentations.

Concept Practice

Home Activity or Further Classroom Consolidation

Complete Worksheet 1.5.3.

1.5.1: Polynomial Concept Attainment Activity

Compare and contrast the examples and non-examples of polynomial functions below. Through reasoning, identify 3 attributes of every polynomial function that distinguish them from non-polynomial functions:

a. ______________________________________

b. ______________________________________

c. ______________________________________

Yes Examples

$y = x$

$y = 2x - 1$

$y = -\frac{2}{5}x$

$y = x^2$

$y = (x - 2)^2 + 1$

$f(x) = -x^2 + x$

$y = -0.2(4x - 3)(x + 3)$

$y = x^3 + 2x^2 - x + 11$

$y = 4$

$h(x) = -x^4 + \frac{1}{2}x^2 - 3$

Non Examples

$y = \sqrt{x}$

$f(x) = 3x^{\frac{1}{2}} - x$

$x = -6$

$x^2 + y^2 = 16$

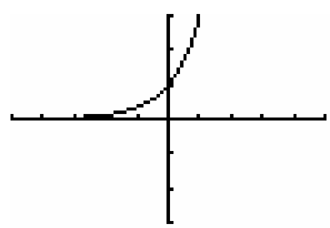
$h(x) = \sqrt[3]{x}$

$y = \sin \beta$

$y = \frac{1}{x - 2}$

$y = 2^x$

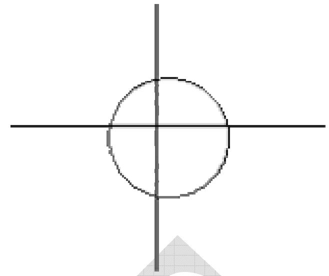
$y = \frac{x - 1}{x^2 - x + 1}$

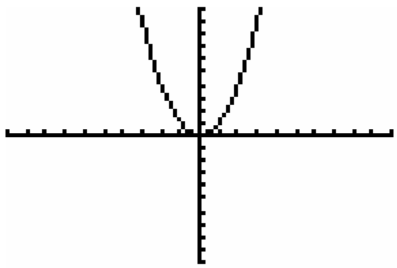


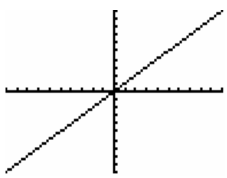
1.5.1: Polynomial Concept Attainment Activity (continued)

$y = -4x^0 + 4$

$y = x(x^2 - 4)(x + 2)$







1.5.2: Graphical Properties of Polynomial Functions

1. Use technology to graph a large number of polynomial functions that illustrate changes in *a*, *b, c, d*, and *k* values in the equations below. Certain patterns will emerge as you try different numerical values for *a, b, c, d*, and *k*. Remember that any of *b, c, d*, and *k* can have a value of 0. Your goal is to see patterns in overall shapes of linear, quadratic, cubic, and quartic polynomial functions, and to work with properties of the functions, rather than to precisely connect the equations and graphs of the functions – the focus of later work. Answer the questions by referring to the graphs drawn by technology.

$y = ax^2 + bx + k$
$a \neq 0$

$y = ax^4 + bx^3 + cx^2 + dx + k$
$a \neq 0$

$y = ax^4 + bx^3 + cx^2 + dx + k$
$a \neq 0$

$y = ax^4 + bx^3 + cx^2 + dx + k$
$a \neq 0$

<table>
<tr><th rowspan="2">Instructions</th><th>Linear</th><th>Quadratic</th><th>Cubic</th><th>Quartic</th></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>Sketch, without axes, the standard curve (i.e., a = 1, and b, c, d, and k all = 0), and state the equation of the curve</td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="5">Sketch the basic shape, then place the x and y axes to illustrate the indicated numbers of zeros. Enter a sample equation for each sketch</td><td>No zeros</td><td>No zeros</td><td>Why is it impossible to have no zeros?</td><td>No zeros</td></tr>
<tr><td>Just 1 zero</td><td>Just 1 zero</td><td>Just 1 zero (Hint: 2 shapes possible)</td><td>Just 1 zero</td></tr>
<tr><td rowspan="3">Explain why there can be no more than 1 zeros</td><td>Exactly 2 zeros</td><td>Exactly 2 zeros</td><td>Exactly 2 zeros (Hint: 2 axes placement possible)</td></tr>
<tr><td rowspan="2">Explain why there can be no more than 2 zeros</td><td>Exactly 3 zeros</td><td>Exactly 3 zeros</td></tr>
<tr><td></td><td>Exactly 4 zeros</td></tr>
</table>

2. a) Read and reflect on the following description of the **end behaviour** of graphs.
Graphs of functions 'come in' from the left and 'go out' to the right as you read the graphs from left to right. A graph can 'come in' high (large-sized negative x-values correspond to large-sized positive y-values) or low (large-sized negative x-values correspond to large-sized negative y-values).
Similarly a graph can 'go out' high (large-sized positive x-values correspond to large-sized positive y-values) or low (large-sized positive x-values correspond to large-sized negative y-values). For example, the standard linear function $y = x$ comes in low and goes out high.

b) Describe the end behaviours of:
i) $y = x^2$ ii) $y = x^3$ iii) $y = x^4$ iv) $y = -x^2$ v) $y = -x^3$ vi) $y = -x^4$

c) Contrast the directions of 'coming in' and 'going out' of the odd degreed polynomial functions (e.g., $y = x$ and $y = x^3$) and the even-degreed functions (e.g., $y = x^2$ and $y = x^4$)

3. Recall that the domain of a function is the set of all x-values in the relationship, and the range is the set of all y-values. When the function is continuous, we cannot list the x and y-values. Rather, we use inequalities or indicate the set of values. E.g., $x > 5$, $y \varepsilon \Re$.
a) State the domain and range of each of the standard polynomial functions
i) $y = x$ ii) $y = x^2$ iii) $y = x^3$ iv) $y = x^4$

b) Sketch an example of each of the following types of functions:
i) quadratic with domain $x \varepsilon \Re$, and range $y \geq 3$.

ii) quartic with domain $x \varepsilon \Re$, and range $y \leq 2$

4. Recall that a function is **increasing** if it rises upward as scanned from the left to the right. Similarly, a function is **decreasing** if it goes downward as scanned from the left to the right. A function can have **intervals of increase** as well as **intervals of decrease**. For example, the standard quartic function $y = x^4$ decreases to $x = 0$, then increases.
a) Sketch a cubic function that increases to $x = -3$, then decreases to $x = 4$, then increases

1.5.2: Graphical Properties of Polynomial Functions (continued)

b) Use graphing technology to graph $y = x^4 - 5x^2 + 4$, sketch the graph, then describe the increasing and decreasing intervals.

5. Compare and contrast the following pairs of graphs:

a) $y = x$ and $y = -x$

b) $y = x^2$ and $y = -x^2$

iii) $y = x^3$ and $y = -x^3$

iv) $y = x^4$ and $y = -x^4$

1.5.3: Numerical Properties of Polynomial Functions

1. Consider the function $y = x$
   a) What type of function is it?
   b) Complete the table of values.
   c) Calculate the first differences.
   d) In this case, the first differences were positive. How would the graph differ if the first differences were negative?)

x	***y***	**First Differences**
–3		►
–2		►
–1		►
0		►
1		►
2		►
3		

2. Consider the function $y = x^2$
   a) What type of function is it?
   b) Complete the table of values.
   c) Calculate the first and second differences.

x	***y***	**Differences**	
		First	Second
–3		►	
–2		►	►
–1		►	►
0		►	►
1		►	►
2		►	►
3			

1.5.3: Numerical Properties of Polynomial Functions

3. Consider the function $y = x^3$
   a) What type of function is it?
   b) Complete the table of values.
   c) Calculate the first, second, and third differences.

x	y	Differences		
		First	Second	Third
–3				
–2				
–1				
0				
1				
2				
3				

4. Consider the function $y = x^3$
   a) What type of function is it?
   b) Complete the table of values.
   c) Calculate the first, second, and third differences.

x	y	Differences			
		First	Second	Third	Fourth
–3					
–2					
–1					
0					
1					
2					
3					

1.5.3: Numerical Properties of Polynomial Functions (continued)

5. a) Summarize the patterns you observe in questions 1–4.

   b) Hypothesize as to whether or not your patterns hold when values for the *b, c, d*, and *k* parameters are not equal to 0 in $y = ax + k$, $y = ax^2 + bc + k$, $y = ax^3 + bx^2 + cx + k$, and $y = ax^4 + bx^3 + cx^2 + dx + k$.

   c) Test your hypothesis on at least 6 different examples. Explain your findings.

Unit 1: Day 6: Characteristics of Polynomial Functions

Grade 12

75 min

Math Learning Goals

- Investigate and summarize the graphical characteristics, e.g., finite differences, domain and range, increasing/decreasing behaviour, local and overall maximums and minimums, of polynomial functions given algebraic, expanded form, and numeric representations.
- Compare these characteristics for linear, quadratic, cubic and quartic functions.

Materials

- BLM 1.6.1
- graphing calculators

Assessment Opportunities

Minds On…

Groups of Three → Card Game

Randomly hand each student a card (BLM 1.6.1). Students search for the algebraic, graphical, and numerical forms of their function. Each group gathers together to record and display its set of 3 representations for the same function.
Each group:

- records the domain and range
- colours increasing intervals of the function blue
- colours decreasing intervals of the function pink

Action!

Whole Class → Developing Academic Language

Point to a local maximum on one of the displayed graphs, and ask students to consider whether that point should be coloured pink or blue. Guide the discussion that an interval of increase changes to an interval of decrease at the point you identified, and that the function is neither increasing nor decreasing at that one point.

Students may see that this is a similar argument to the one that zero is neither positive nor negative.

Introduce the four terms: 'local maximum,' 'local minimum', 'overall maximum,' and 'overall minimum,' pointing at examples on the displayed graphs. Split the class roughly into quarters, assigning each quarter one of these four terms, and asking students in each quarter to define, in their own words, the term assigned to them. Students go to the four corners of the room and share their definitions. Each group creates a definition for its word that is posted on the Word Wall.
Circulate to listen to the math talk.

Groups of 3 → Practice

Form different groups of 3 from the beginning of the lesson. Assign different groups to the different functions displayed earlier. Each group describes its function as linear, quadratic, cubic, or quartic and names its overall and local maximums and minimums. Groups rotate clockwise to the next function and show corrections to any detail they think is wrong.

Curriculum Expectation/Observation/Mental Note: Observe students.

Consolidate Debrief

Whole Class → Discussion

Lead a discussion to summarize the graphical and numerical features of polynomial functions. Challenge students to predict the shapes of graphs of quartic, cubic, quadratic, and linear polynomial functions whose equations or difference tables displayed on the overhead. Use technology to display the graphs to allow students to check their predictions. Ask a volunteer to describe a displayed graph, and another student to provide feedback. Display sketches and ask students to predict the form of the equation, and justify each algebraic feature. Repeat focusing on the types of functions and representations posing most challenges.

Practice Application

Home Activity or Further Classroom Consolidation

Your friend missed class today. Complete a letter to them that describes in detail what was learned today.

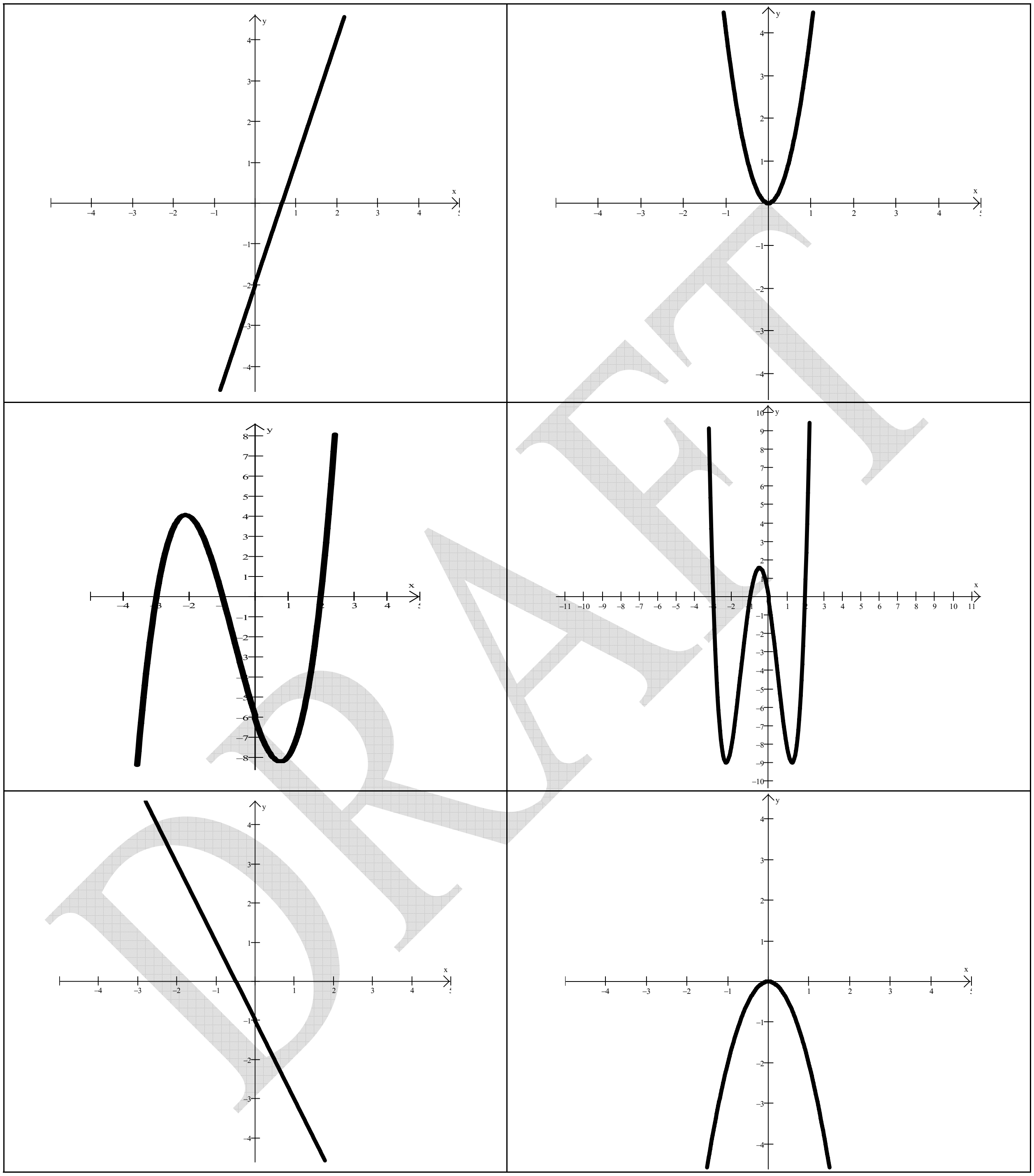
1.6.1: Family Reunion (Match My Graph)

$y = 3x - 2$	$\begin{array}{ccccc} x & -1 & 0 & 1 & 2 \\ y & -5 & -2 & 1 & 4 \end{array}$
$y = 4x^2$	$\begin{array}{ccccc} x & -1 & 0 & 1 & 2 \\ y & 4 & 0 & 4 & 16 \end{array}$
$f(x) = x^3 + 2x^2 - 5x - 6$	$\begin{array}{ccccccc} x & -3 & -2 & -1 & 0 & 1 & 2 \\ y & 0 & 4 & 0 & -6 & -8 & 0 \end{array}$
$y = x^4 + 2x^3 - x^2 - 6x$	$\begin{array}{ccccc} X & -1 & -3 & 0 & 2 \\ y & 0 & 0 & 0 & 0 \end{array}$

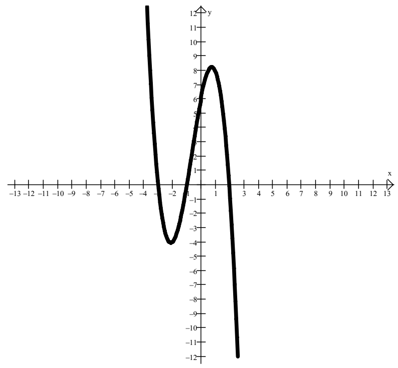
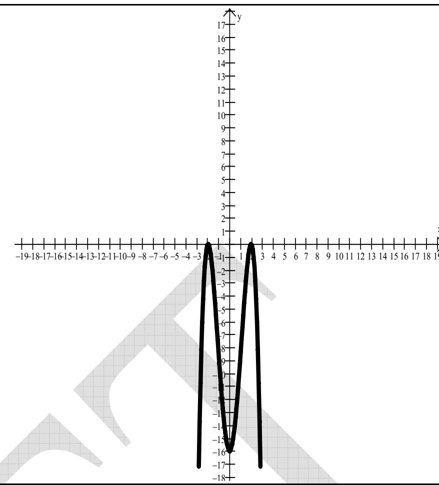
1.6.1: Family Reunion (Match My Graph) (continued)

$y = -2x - 1$	x: –1, 0, 1, 2 y: 1, –1, –3, –5
$y = -2x^2$	x: –1, 0, 1, 2 y: –2, 0, –2, –8
$f(x) = -x^3 - 2x^2 + 5x + 6$	*x*: –3, –2, –1, 0, 1, 2 y: 0, –4, 0, 6, 8, 0
$f(x) = -x^4 + 8x^2 - 16$	*x*: –2, –1, 0, 1, 2, 3 y: 0, –9, –16, –9, 0, –25

1.6.1: Family Reunion (Match My Graph) (continued)



1.6.1: Family Reunion (Match My Graph) (continued)

75 min

Math Learning Goals

- Make connections between a polynomial function in factored form and the x-intercepts of its graph.
- Sketch the graph of polynomial functions, expressed in factored form using the characteristics of polynomial functions.
- Determine the equation of a polynomial given a set of conditions, e.g., zeros, end behaviour, and recognize there may be more than one such function.

Materials

- BLM 1.7.1, 1.7.2, 1.7.3
- graphing calculators/ software/ sketch: GSP_gr12_U1D7
- Day 6 graphs

Assessment Opportunities

Minds On…

Individual → Exploration

Recall with students that $(x-1)^2 = (x-1)(x-1)$, $(x-1)^3 = (x-1)(x-1)(x-1)$ etc. Students complete questions 1, 2, and 3 (BLM 1.7.1).

Take up questions 2 and 3. Possible answer 2: the degree of the polynomial is equal to the number of factors that contain an "x." Possible answers 3: multiply the factored form; multiply the variable term of each factor and logically conclude that this is the degree of the polynomial.

Display sketches of Day 6 polynomial functions. Reference to finite differences and its connection to the degree would be beneficial

Action!

Individual → Investigation

Students observe the sketches of the graphs generated and displayed in **Minds On…** and answer questions 4, 5, and 6. Students check their answers in pairs. Make connection between the x-intercept and its corresponding co-ordinates $(x_I, 0)$. Introduce language to describe shape and post on word wall. Possible answer 4: the graph crosses the x axis when $y = 0$ in the polynomial function. This occurs when each factor is equal to zero. When these mini equations are "solved" the x-intercepts are evident.

Answers 6 respectively: The graph "bounces" at the intercept. The graph has an "inflection point" at the intercept.

Word wall:
- concave
- up/down,
- hills,
- valleys,
- inflection point (informal descriptions in students' own words)

Pairs → Think Pair Share

Students complete BLM 1.7.2 in pairs.

Curriculum Expectation/Observation/Mental Note: Circulate to observe pairs are successful in representing the graph of a polynomial function given the equation in factored form. Make a mental note to consolidate misconceptions.

If students look only at factors of the form $(x-a)$ they may have the misconception that "the x intercept is the opposite of the "a" value. Use some factors of the form $(ax-b)$ to help students understand that the intercept can be found by solving $ax-b=0$

Whole Class → Discussion and Practice

Present a graph of a polynomial function with 3 intercepts and ask: What is a possible equation of this polynomial? How do you know? What are some possible other equations? Do a few other examples including some with double roots and inflection points. Then, state some characteristics of a polynomial function and ask for an equation that satisfies these characteristics (e.g., What is a possible equation of the polynomial function of degree 3 that begins in quadrant 2, ends in quadrant 4, and has x intercepts of -1, ½, and 4.)
Students complete BLM 1.7.3.

This assessment for learning will determine if students are ready to formulate an equation given the characteristics of a polynomial function.

Consolidate Debrief

Pairs to Whole Class → A answers B

Ask: What are the advantages of a polynomial function being given in factored form? Why is it possible that 2 people sketching the same graph could have higher or lower "hills" and "valleys"? How could you get a more accurate idea of where these turning points are located? Explain why the graph the function $y = (x^2-4)(x^2-9)$ is easier or more difficult to graph than $y = (x2 + 4)(x2 + 9)$. How can you identify the y intercept of a function given algebraically and why is this helpful to know?

Strategy: A answers B: Ask a series of questions one at a time, to be answered in pairs, then ask for one or more students to answer, alternating A and B partners

Exploration
Application

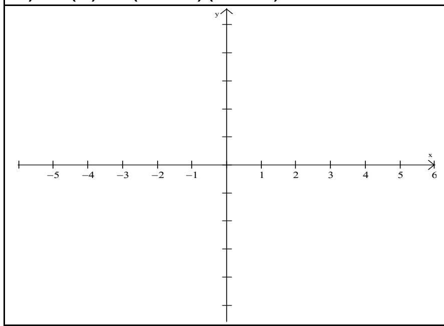
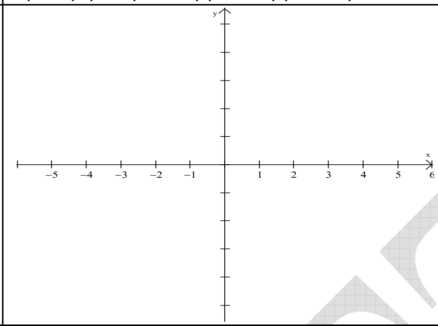
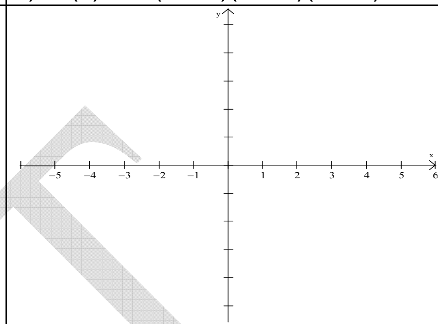
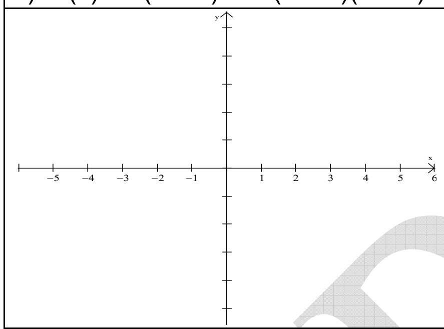
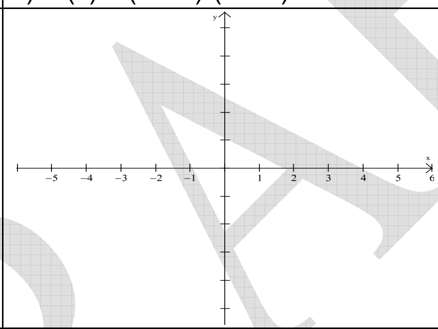
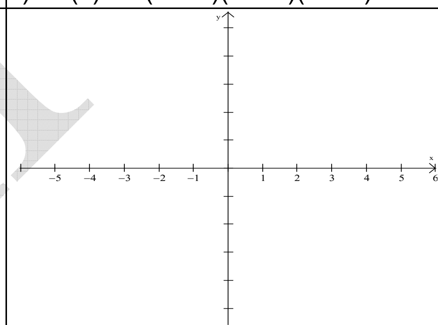
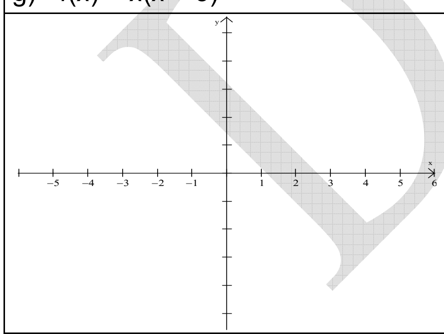
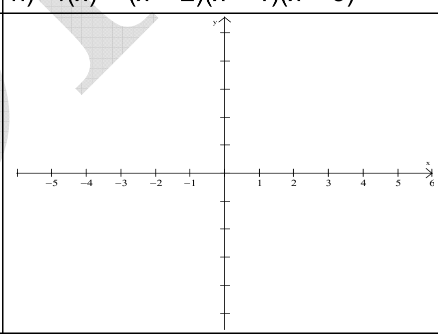
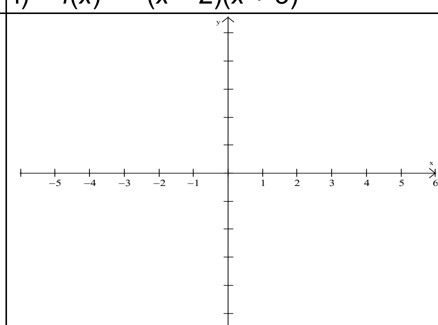
Home Activity or Further Classroom Consolidation

Respond to the question and complete the assigned questions. What information is needed to determine an exact equation for a graph if x intercepts are given?

Assign questions similar to BLM 1.7.1

1.7.1: What Role Do Factors Play?

1. Use technology (graphing calculator, software, GSP_Gr12_U1D7) to determine the graph of each polynomial function. Sketch the graph, clearly identifying the *x*-intercepts.

a) $f(x) = (x-2)(x+1)$	b) $f(x) = (x-2)(x+1)(x+3)$	c) $f(x) = -(x-2)(x+1)(x+3)$
		
Degree of the function: ______ *x*-intercepts: ___________	Degree of the function: ______ *x*-intercepts: ___________	Degree of the function: ______ *x*-intercepts: ___________
d) $f(x) = x(x+1)^2 = x(x+1)(x+1)$	e) $f(x) = (x-2)^2(x+2)^2$	f) $f(x) = x(x-2)(x+1)(x+3)$
		
Degree of the function: ______ *x*-intercepts: ___________	Degree of the function: ______ *x*-intercepts: ___________	Degree of the function: ______ *x*-intercepts: ___________
g) $f(x) = x(x-3)^3$	h) $f(x) = (x+2)(x-1)(x-3)^2$	i) $f(x) = -(x-2)(x+3)^3$
		
Degree of the function: ______ *x*-intercepts: ___________	Degree of the function: ______ *x*-intercepts: ___________	Degree of the function: ______ *x*-intercepts: ___________

2. Compare your graphs with the graphs generated on the previous day and make a conclusion about the degree of a polynomial when it is given in factored form.

3. Explain how to determine the degree of a polynomial algebraically if given in factored form.

1.7.1: What Role Do Factors Play?

4. What connection do you observe between the factors of the polynomial function and the *x*-intercepts? Why does this make sense? (**Hint:** all co-ordinates on the *x* axis have $y = 0$).

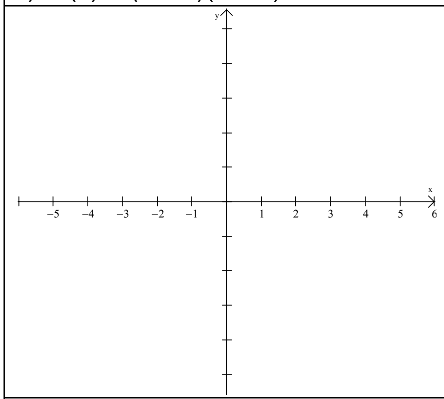
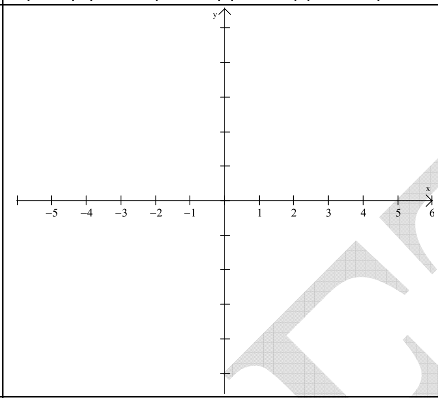
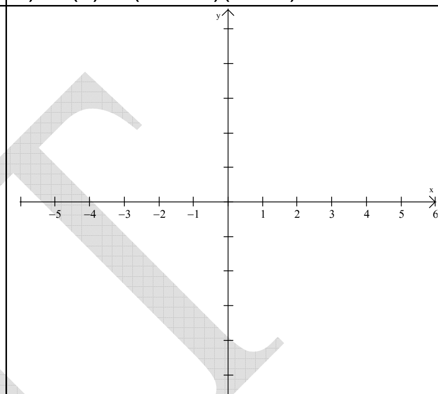
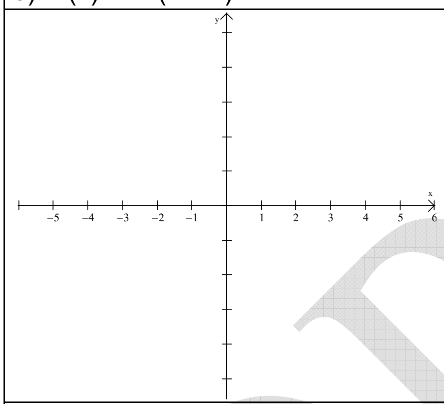
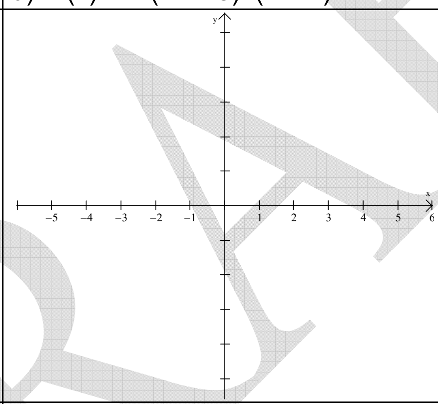
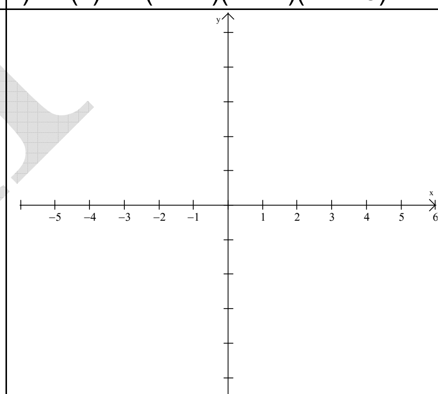
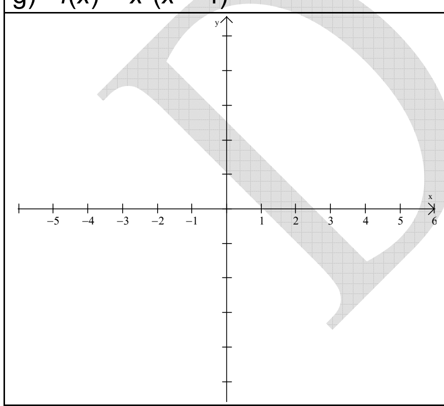
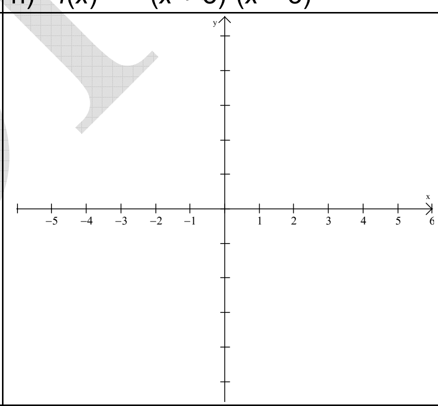
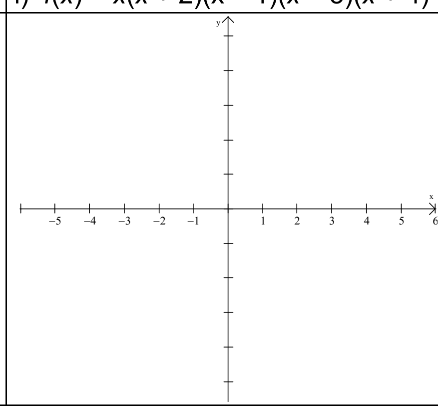
5. Use your conclusions from 4 to state the *x*-intercepts of each of the following. Check by graphing with technology, and correct if necessary.

$f(x) = (x - 3)(x + 5)(x - 1/2)$	$f(x) = (x - 3)(x + 5)(2x - 1)$
x-intercepts: ___________ Does this check? _______	*x-intercepts*: ___________ Does this check? _______
$f(x) = (2x - 3)(2x + 5)(x - 1)(3x - 2)$	
x-intercepts: ___________ Does this check? _______	

6. What do you notice about the graph when the polynomial function has a factor that occurs twice? Three times?

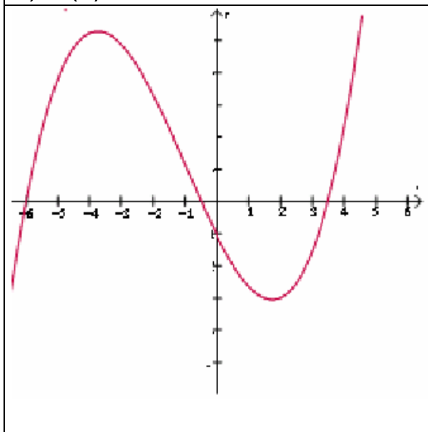
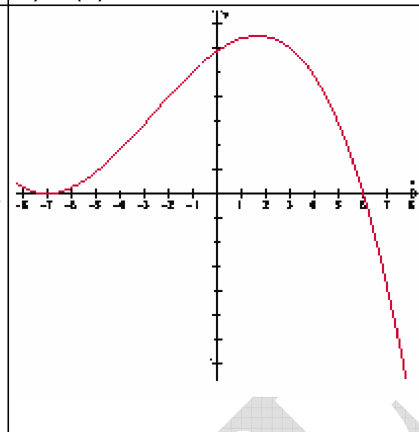
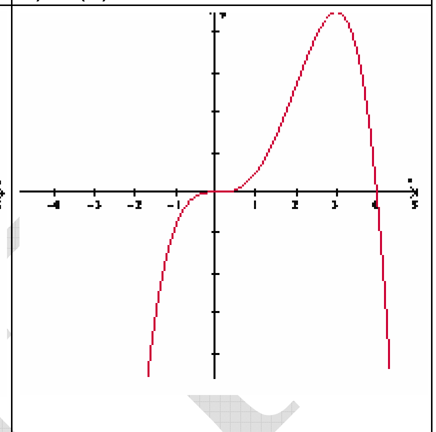
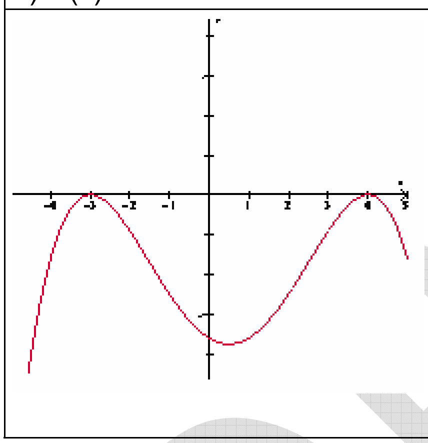
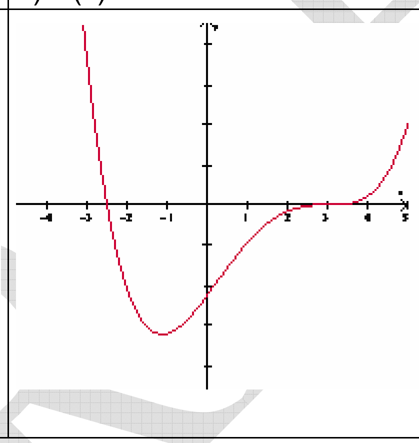
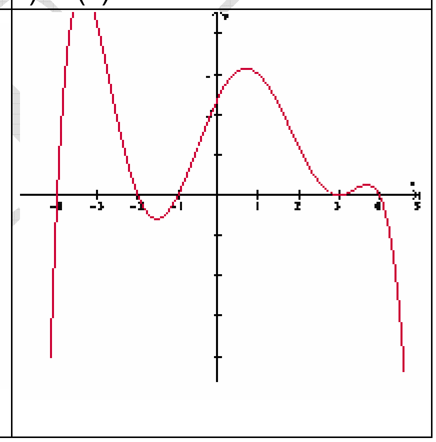
1.7.2: Factoring in our Graphs

Draw a sketch of each graph using the properties of polynomial functions. After you complete each sketch, check with your partner, discuss your strategies, and make any corrections needed.

a) $f(x) = (x – 4)(x + 3)$	d) $f(x) = – (x – 1)(x + 4)(x – ½)$	c) $f(x) = (2x – 1)(x + 1)^2$
		
e) $f(x) = 2x(x – 2)^3$	d) $f(x) = – (2x – 3)2(x + 2)^2$	f) $f(x) = x(x – 2)(x + 1)(2x + 3)$
		
g) $f(x) = x^3(x – 4)$	h) $f(x) = – (x + 3)^2(x – 3)^3$	i) $f(x) = x(x + 2)(x – 1)(x – 3)(x + 4)$
		

1.7.3: What's My Polynomial Name?

1. Determine a possible equation for each polynomial function.

a) $f(x)$ =	b) $f(x)$ =	c) $f(x)$ =
		
d) $f(x)$ =	e) $f(x)$ =	f) $f(x)$ =
		

2. Determine an example of an equation for a function with the following characteristics:

   a) Degree 3, a double root at 4, a root at –3 ______________________________

   b) Degree 4, an inflection point at 2, a root at 5 ______________________________

   c) Degree 3, roots at $\frac{1}{2}, \frac{3}{4}$, -1 ______________________________

   d) Degree 3, starting in quadrant 2, ending in quadrant 4, root at –2 and double root at 3

   ____

   e) Degree 4, starting in quadrant 3, ending in quadrant 4, double roots at –10 and 10__________________________

75 min

Math Learning Goals
- Investigate transformations applied to f(x) = x3 and f(x) = x4.
- Investigate and compare the properties of odd and even functions.

Materials
- BLM 1.8.1, 1.8.2, 1.8.3, 1.8.4, 1.8.5
- Graphing technology
- Flash cards Navigator, or Clickers

Assessment Opportunities

Minds On…

Individual → Anticipation Guide
Students review prior knowledge of transformations using BLM 1.8.1.

Curriculum Expectations/Anticipation Guide/Mental Note: Anticipation Guide responses are collected using Flash cards, technology or hand signals (a = 1 finger, b = 2 fingers, etc.) once completed. Note which students may need additional support.

This diagnostic data can also be collected using TI Navigator or Clickers if available.

For Kinaesthetic Learners – Tai Chi movements could be used as an alternative warm-up/diagnostic. (i.e., move upper body and arms to the left; raise arms and body up; show a vertical stretch by having both arms move in opposite directions from the middle).

Action!

Small Groups → Exploration
Students use their knowledge of transformations to sketch graphs of polynomials and confirm using technology (BLM 1.8.2).

Pairs → Investigation
Discuss with the class what changes to the function are necessary to create a reflection in the y-axis and a reflection in the x-axis.

Students work through BLM 1.8.3 and 1.8.4 with a partner using graphing technology.

Learning Skills/Teamwork/Observation/Checkbric: Observe students' ability to work collaboratively.

Consolidate Debrief

Whole Class → Discussion
Lead a class discussion about what observations can be made from BLM 1.8.3 and 1.8.4 to define and draw out key properties of odd and even functions. Pose questions such as:

Which functions look the same after you reflect them over the y-axis? Both axis? Which functions have $f(-x) = f(x)$? $f(-x) = -f(x)$?

What general conclusions can you make about even and odd functions? Possible answer: Cubic functions with all odd powers in terms will be odd. Quartic functions with all even powers in terms will be even. A mixture of even and odd powers in a function will be neither even nor odd.

This is a good opportunity to also show students how you can test for even and odd numerically. For example if they check $f(2)$ and $f(-2)$ and don't get the same value then the function isn't even.

Exploration Application

Home Activity or Further Classroom Consolidation
Complete worksheet 1.8.5.

1.8.1: What's the Change?

If the graph of a function $y = f(x)$ is provided, choose the most appropriate description of the change indicated.

1. $y = f(x + 3)$ a. shift right b. shift up c. slide left d. slide down	2. $y = f(5x)$ a. vertical compression b. horizontal stretch c. horizontal compression d. shift up
3. $y = -f(x)$ a. reflection about x axis b. reflection about y axis c. shift down d. slide left	4. $y = 2f(x)$ a. horizontal compression and shift right b. horizontal compression and shift left c. vertical stretch and shift right d. horizontal stretch and shift left
5. $y = f(-3x)$ a. horizontal stretch and reflection in x axis b. horizontal stretch and reflection in y axis c. vertical stretch and reflection in x axis d. horizontal stretch and reflection in y axis	6. $y = f(0.5(x + 3))$ a. horizontal compression and shift right b. horizontal compression and shift left c. vertical stretch and shift right d. horizontal stretch and shift left

1.8.1: What's the Change? (continued)

7. $y = f(x) + 5$ a. vertical shift down b. horizontal shift left c. vertical slide up d. horizontal shift right	8. $y = -f(-x)$ a. reflection in x axis and vertical compression b. reflection in y axis and horizontal compression c. reflection in both axes d. vertical and horizontal compression
9. $y = f(-(x + 2))$ a. vertical compression and shift left b. reflection in x axis and shift right c. reflection in y axis and shift left d. reflection in y axis and shift left	10. $y = \frac{1}{2}f(x - 1) + 4$ a. vertical compression, shift right, shift up b. vertical compression, slide left, shift up c. horizontal compression, shift right and shift down d. vertical stretch, shift right, shift up

1.8.2: Transforming the Polynomials

Using your knowledge of transformations and $f(x) = x^3$ or $f(x) = x^4$ as the base graphs, sketch the graphs of the following polynomial functions and confirm using technology.

$f(x) = x^3$	$f(x) = -x^3$	$f(x) = (-x)^3$
$f(x) = (x + 1)^3$	$f(x) = 2x^3 - 1$	$f(x) = -\frac{1}{4}(x - 3)3 + \frac{1}{2}$
$f(x) = x^4$	$f(x) = -x^4$	$f(x) = (-x)^4$
$f(x) = (x - 2)^4$	$f(x) = -\frac{1}{2}x^4 + 3$	$f(x) = (2(x + 3))^4$

Putting it all together:
For $f(x) = a(k(x - d))3 + c$ and $f(x) = a(k(x - d))4 + c$, describe the effects of changing a, k, d and c in terms of transformations

1.8.3: Evens and Odds – Graphically

☐ What transformation reflects a function in the *y*-axis? ______________________

☐ What transformation reflects a function in the *x*-axis? ______________________

For each function given below:

- write the equation that will result in the specified transformation
- enter the equations into the graphing calculator (*to make the individual graphs easier to view change the line style on the calculator)
- confirm the equation provides the correct transformation; adjust if needed
- sketch a graph of the original function and the reflections
- record any observations you have about the resulting graphs

Function	Reflection in y-axis	Followed by reflection in y-axis	Observations
$f(x) = x$	$f(x) =$	$f(x) =$	
$f(x) = x^2$ Insert graph of $f(x) = x^2$	$f(x) = x^2$ Insert graph of $f(x) = x^2$	$f(x) = -x^2$ Insert graph of $f(x) =- x^2$	• a reflection in *y*-axis makes no change in the graph • a reflection in both axes makes a change in the graph
$f(x) = 3^x$	$f(x) =$	$f(x) =$	
$f(x) = x^4$	$f(x) =$	$f(x) =$	

1.8.3: Evens and Odds – Graphically (continued)

Functions	Reflection in *y*-axis	Followed by reflection in *x*-axis	Observations
$f(x) = 2x+1$	$f(x) =$	$f(x) =$	
$f(x) = (x-2)^2$	$f(x) =$	$f(x) =$	
$f(x) = x^3+1$	$f(x) =$	$f(x) =$	
$f(x) = x^4+x^2$	$f(x) =$	$f(x) =$	
$f(x) = x^3 + x$	$f(x) =$	$f(x) =$	

What conclusions can you make between polynomial functions that have symmetry about the *y*-axis? Both axes? Test other functions to confirm your hypothesis.

1.8.4: Evens and Odds – Algebraically

For each of the functions in the table below find the algebraic expressions for $-f(x)$ and $f(-x)$. Simplify your expressions and record any similarities and differences you see in the algebraic expressions. The second one is done for you as an example.

Function	$-f(x)$	$f(-x)$	Observations
$f(x) = x$			
$f(x) = x^2$	$-f(x) = -(x^2)$ $= -x^2$	$f(-x) = (-x)^2$ $= x^2$	• the expressions for $f(-x)$ and $f(x)$ are the same. • the expressions for $-f(x)$ and $f(x)$ are opposites.
$f(x) = x^3$			
$f(x) = x^4$			
$f(x) = 2x+1$			

1.8.4: Evens and Odds – Algebraic (continued)

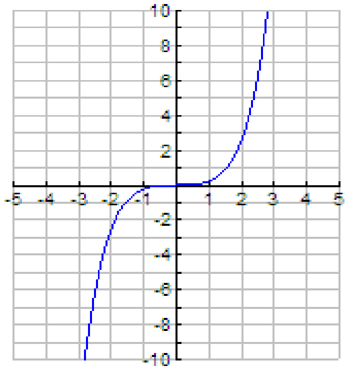
Function	$-f(x)$	$f(-x)$	Observations
$f(x) = (x-2)^2$			
$f(x) = x^3 + 1$			
$f(x) = x^4 + x^2$			
$f(x) = x^3 + x$			

What conclusions can you make between polynomial functions that are the same and different when comparing f(x), – f(x) and f(–x) ? Test other functions to confirm your hypothesis.

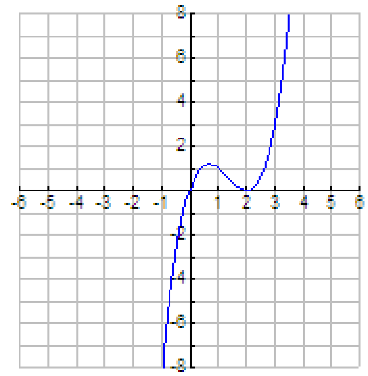
1.8.4: Evens and Odds – Practice

Determine whether each of the functions below is even, odd or neither. Justify your answers.

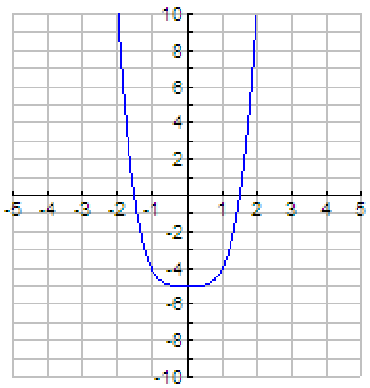
1.



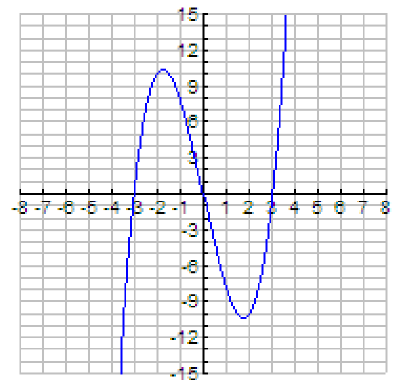
2.



3.



4.



5. $f(x) = 3x^2 + 4$

6. $f(x) = -2x + 5$

7. $f(x) = 2x^2 + 3x$

8. $f(x) = -3x^3 + x$

Unit 2: Rational Functions

Grade 12

Lesson Outline

Big Picture			
Students will: • identify and use key features of rational functions; • solve problems using a variety of tools and strategies related to rational functions; • determine and interpret average and instantaneous rates of change for rational functions.			
Day	**Lesson Title**	**Math Learning Goals**	**Expectations**
1–3		• Investigate and summarize the key features (e.g. zeros, end behaviour, horizontal and vertical asymptotes, domain and range, increasing/decreasing behaviour) of rational functions, and make connection between the graphical and algebraic representations. • Demonstrate an understanding of the relationship between the degrees of numerator and the denominator and the asymptotes. • Sketch the graph of rational functions using its key features.	C 2.1, 2.2, 2.3
4		• Solve problems graphically and algebraically involving applications of polynomial and simple rational functions and equations. • Solve simple rational equations algebraically and verify using technology. • Use properties of simple rational functions to fit a rational function to a graph or a given set of conditions. • Make connections between the x-intercepts of a simple rational function and the real roots of the corresponding function.	C3.5, 3.6, 3.7
5		• Solve problems involving average and instantaneous rates of change at a point using numerical and graphical methods. • Investigate average rates of change near horizontal and vertical asymptotes.	D1.1–1.8
6		• Jazz day	
7		• Summative Assessment	

Unit 3: Trigonometric Functions

Grade 12

Lesson Outline

Big Picture

Students will:
- identify and use key features of rational functions;
- solve problems using a variety of tools and strategies related to rational functions;
- determine and interpret average and instantaneous rates of change for rational functions.

Day	Lesson Title	Math Learning Goals	Expectations
1		• Explore and define radian measure. • Develop and apply the relationship between radian and degrees measure. • Use technology to determine the primary trigonometric ratios, including reciprocals of angles expressed in radians.	B1.1, 1.3
2–3		• Determine the exact values of trigonometric and reciprocal trigonometric ratios for special angles and their multiples using radian measure. • Recognize equivalent trigonometric expressions and verify equivalence with technology.	B1.4, 3.1
4–5		• Graph $f(x) = \sin x$ and $f(x) = \cos x$, using radian measures. • Make connections between the graphs of trigonometric functions generated with degrees and radians. • Graph the reciprocals, using radian measure and properties of rational functions.	B1.2, 1.3, 2.3, C2.1, 2.2
6		• Make connections between the tangent ratio and the tangent function using technology. • Graph the reciprocal trig functions for angles in radians with technology, and determine and describe the key properties. • Understand notation used to represent the reciprocal functions.	B2.2, 2.3, C1.4, 2.1
7–8		• Investigate symmetry of the trigonometric functions and make connections to average and instantaneous rates of change at a point, e.g., examine difference tables, odd, even functions. • Solve problems involving average and instantaneous rates of change at a point using numerical and graphical methods for trigonometric functions.	D1.1–1.9
9		• Jazz day	
10		• Summative	

Unit 4: Functions

Lesson Outline

[Big Picture](#)
Students will: • graph and transform sinusoidal functions using radian measure; • identify domain, range, phase shift, period, amplitude, and vertical shift of sinusoidal functions using radian measure; • develop equations of sinusoidal functions from graphs and descriptions expressed in radian measure; • solve problems graphically that can be modeled using sinusoidal functions; • prove trigonometric identities; • solve linear and quadratic trigonometric equations using radian measure; • make connections between graphic and algebraic representations of trigonometric relationships.

Day	Lesson Title	Math Learning Goals	Expectations
1		• Demonstrate an understanding of transformations of sine and cosine functions using radians. • Sketch the graphs of transformations of the form $y = \sin(x) + c$, $y = \cos(x) + c$, $y = \sin(x - d)$, $y = \cos(x - d)$, $y = \sin(x - d) + c$, $y = \cos(x - d) + c$, $y = a\sin(x)$, $y = a\cos(x)$. • State the domain and range, phase shift, period, amplitude, vertical translation for transformations of sine and cosine functions using radians. • Recognize equivalent trigonometric expressions, such as those involving horizontal translations, by considering the graphs.	B2.4, 2.5, 3.1
2–3		• Demonstrate an understanding of transformations of sine and cosine functions using radians. • Sketch the graphs of transformations of the form $y = \sin k(x)$, $y = \cos k(x)$, $y = \sin(kx)$, $y = \cos(kx)$, $y = \sin(kx - d)$, $y = \cos(kx - d)$. • State the domain and range, phase shift, period, amplitude, and vertical translation for transformations of sine and cosine functions. • Sketch graphs of $y = a\ \sin(k(x - d) + c$ and $y = a\ \cos(k(x - d) + c$ in radians. • Recognize equivalent trigonometric expressions, such as those involving transformations by considering the graphs.	B2.4, 2.5, 3.1
4		• Determine an equation of a sinusoidal function given its graph or descriptions of its properties, in radians. • Recognize that more than one equation can be used to represent the graph of the function.	B2.6, 3.1
5–6		• Pose and solve problems involving real world applications of sinusoidal functions in radians, given a graph or a graph generated with or without technology from its equations.	B2.7, 3.1
7		• Develop an understanding of compound angle formulae through exploration of numeric examples, and using technology. • Use the formulae to determine the exact trigonometric ratios for special angles, e.g. $\sin\left(\frac{\pi}{12}\right)$.	B3.1, 3.2

Day	Lesson Title	Math Learning Goals	Expectations
8		• Demonstrate an understanding that an identity holds true for any value of the independent variable (graph left side and right side of the equation as functions and compare). • Apply a variety of techniques to prove identities.	B3.3
9–10		• Solve linear and quadratic trigonometric equations with and without graphing technology, for real values in the domain from 0 to 2π. • Make connections between graphical and algebraic solutions.	B3.4
11		• Summative	

Unit 5: Exponential and Logarithmic Functions

Grade 12

Lesson Outline

Big Picture			
Students will: • develop the understanding that the logarithmic function is the inverse of the exponential function; • simplify exponential and logarithmic expressions using exponent rules; • identify features of the logarithmic function including rates of change; • transform logarithmic functions; • evaluate exponential and logarithmic expressions and equations; • solve problems that can be modeled using exponential or logarithmic functions.			
Day	**Lesson Title**	**Math Learning Goals**	**Expectations**
1–2		• Explore and describe key features of the graphs of exponential functions (domain, range, intercepts, increasing/decreasing intervals, asymptotes). • Define the logarithm of a number to be the inverse operation of exponentiation, and demonstrate understanding considering numerical and graphical examples. • Using technology, graph implicitly, logarithmic functions with different bases to consolidate properties of logarithmic functions and make connections between related logarithmic and exponential equations (e.g., graph $x = a^y$ using Winplot, Graphmatica or graph a reflection in $y = x$ using GSP®).	A1.1, 1.3, 2.1, 2.2
3		• Investigate the relationship between $y = 10^x$ and $y = b^x$ and how they relate to $y = logx$ and $y = log_b x$. • Make connections between related logarithmic and exponential equations. • Evaluate simple logarithmic expressions. • Approximate the logarithm of a number with respect to any base with technology. • Solve simple exponential equations, by rewriting them in logarithmic form.	A1.1, 1.2, 1.3
4–5		• Explore graphically and use numeric patterning to make connections between the laws of exponents and the laws of logarithms. • Explore the graphs of a variety of logarithmic and exponential expressions to develop the laws of logarithms. • Recognize equivalent algebraic expressions involving logs and exponents. • Use the laws of logarithms to simplify and evaluate logarithmic expressions.	A1.4, 3.1
6–7		• Solve problems involving average and instantaneous rates of change using numerical and graphical methods for exponential and logarithmic functions. • Solve problems that demonstrate the property of exponential functions that the instantaneous rate of change at a point of an exponential function is proportional to the value of the function at that point.	D1.4–1.9

Day	Lesson Title	Math Learning Goals	Expectations
8		• Pose and solve problems using given graphs or graphs generated with technology of logarithmic and exponential functions arising from real world applications.	A2.4
9		• Solve exponential equations by finding a common base. • Solve simple logarithmic equations. • Solve exponential equations by using logarithms.	A3.2, 3.3
10	Log or Rhythm	• Demonstrate understanding of the roles of the parameters c, and d in $y = log_{10}((x - d)) + c$. • Demonstrate understanding of the roles of the parameters a, k in $y = alog_{10}(kx)$.	A2.3
11		• Solve problems of exponential or logarithmic equations algebraically arising from real world applications.	A3.4
12		• Summative Assessment	

75 min

Math Learning Goals

- Investigate the roles of the parameters d and c in functions of the form $y = \log_{10}(x - d) + c$ and the roles of the parameters a and k in functions of the form $y = a \log_{10}(kx)$.
- Connect prior knowledge of transformations in order to graph logarithmic functions.

Materials

- BLM 5.10.1, 5.10.2, 5.10.4, 5.10.5.
- computer access with GSP® or graphing calculators

Assessment Opportunities

Minds On…

Groups of 5 → Activating Prior Knowledge

Provide each group with an envelope containing the squares of one type of function from BLM 5.10.1.

Students match the equation, the graph, and the written description of the transformation.

Each member of the group takes the three pieces of a match and finds members of the other groups that have the same transformation.

Groups summarize the similarities of the transformation notation regardless of the function type. One group member shares with the whole class.

Photocopy BLMs – different colour for each function, cut and placed in envelopes

Students weak in transformations should be given the quadratic transformations.

Note: the scales on the graphs are all 1:1 unless stated otherwise.

Action!

Pairs → Exploration

Direct the students to the appropriate link for the GSP® activity.

Students follow the instructions and record their observations on the Student Observation Log (BLM 5.10.2).

Learning Skills/Teamwork/Checkbric: Circulate to assess how individual students stay on task and help each other complete the investigation.

Log Graphs.gsp

BLM 5.10.3 provides Alternate Graphing Calculator Investigation

Consolidate Debrief

Whole Class → Discussion

Lead the class in a discussion of their Student Observation Log sheet to review the effects of each of the parameters on the transformations of the logarithmic function. Post a summary on class wall.

Curriculum Expectations/Observations/Mental Note: Assess students' communication of transformations of functions orally, visually, and in writing, using precise mathematical vocabulary.

As a fun review and practice of understanding the various transformations, teach students the "tai chi" movements for each transformation (BLM 5.10.4).

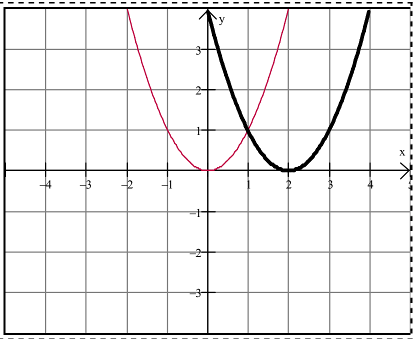
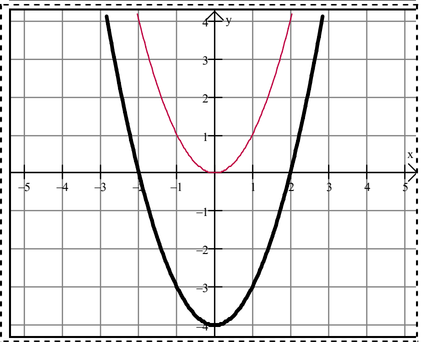
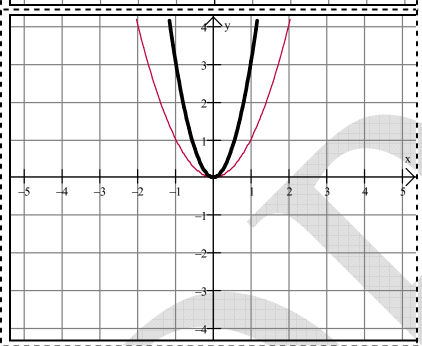
Exploration
Application

Home Activity or Further Classroom Consolidation

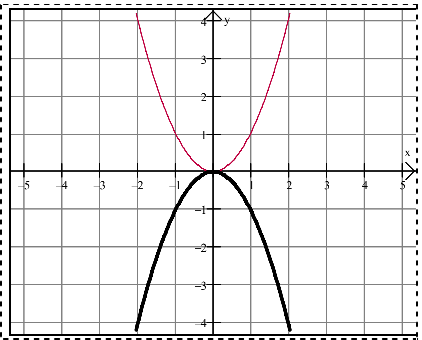
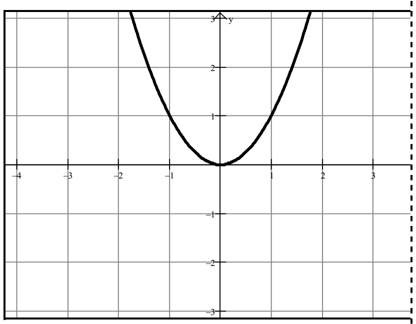
Complete BLM 5.10.5 CSI Math.

Collect next day for assessment.

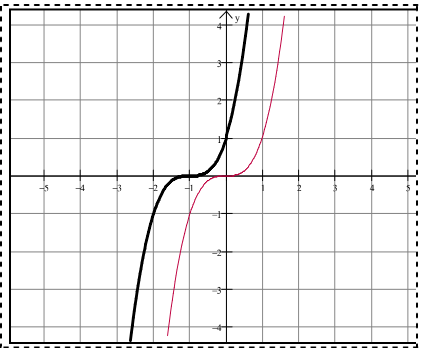
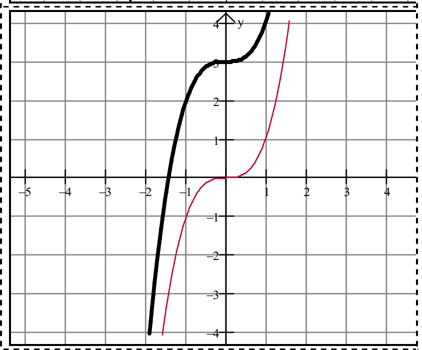
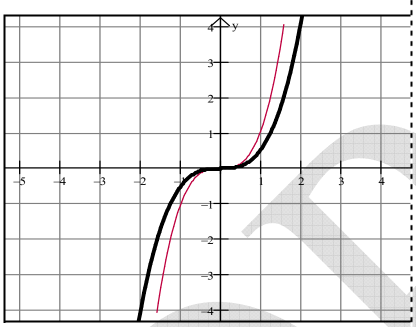
5.10.1: Log or Rhythm

	$f(x) = (x - 2)^2$	Horizontal Translation Right _____
	$y = x^2 - 4$	Vertical Translation Down _____
	$f(x) = 3x^2$	Vertical Stretch Factor _____ from the *x*-axis

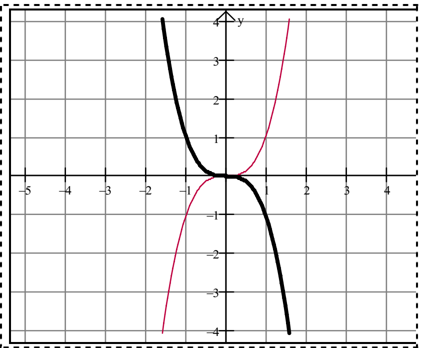
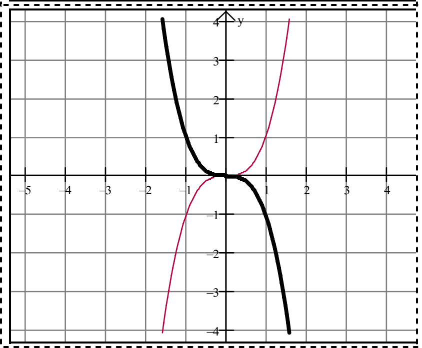
5.10.1: Log or Rhythm (continued)

	$y = -x^2$	Vertical Reflection in the *x*-axis
	$f(x) = (-x)^2$	Horizontal Reflection in the *y*-axis

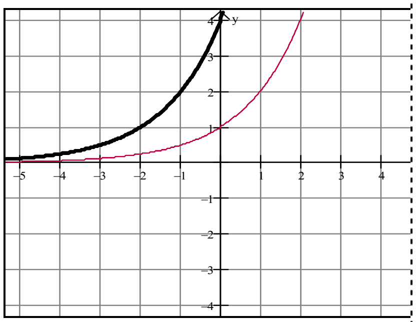
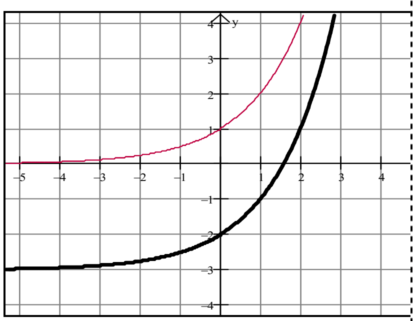
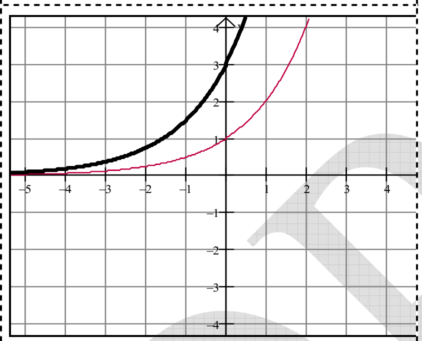
5.10.1: Log or Rhythm (continued)

	$y = (x + 1)^3$	Horizontal Translation Left _____
	$f(x) = x^3 + 3$	Vertical Translation Up _____
	$\frac{1}{2}x^3$	Vertical Compression Factor _____ to the *x*-axis

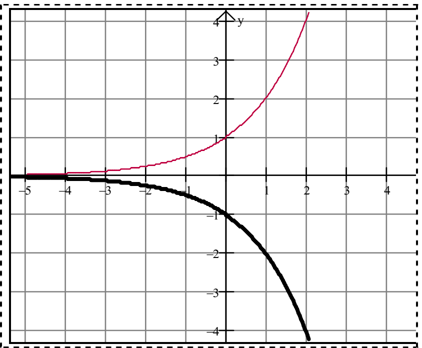
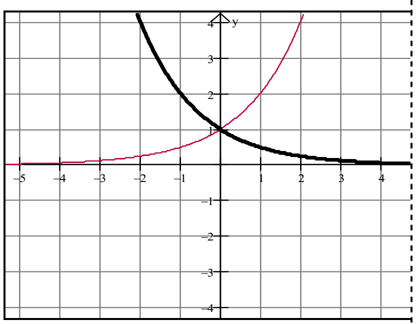
5.10.1: Log or Rhythm (continued)

	$-x^3$	Vertical Reflection in the *x*-axis
	$(-x)3$	Horizontal Reflection in the *y*-axis

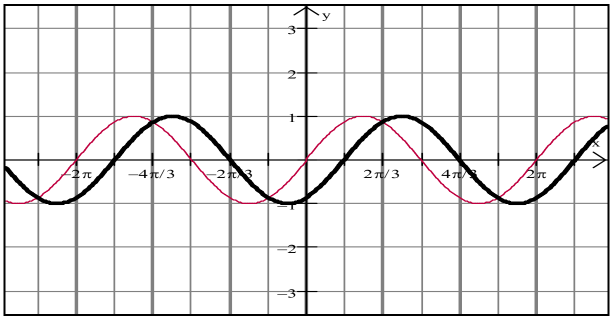
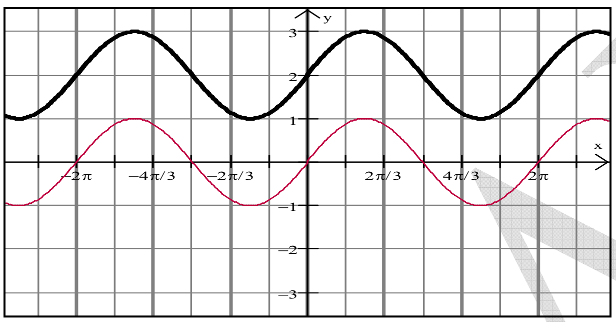
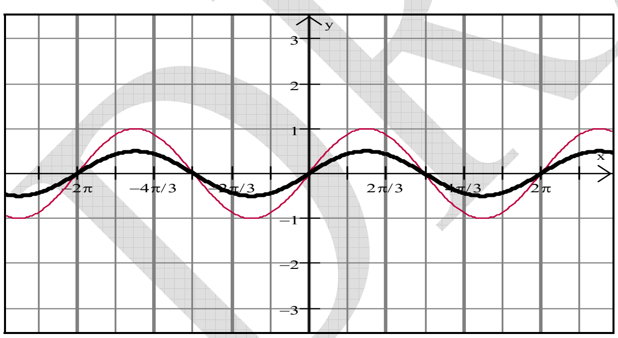
5.10.1: Log or Rhythm (continued)

	$f(x) = 2x + 2$	Horizontal Translation Left ______
	$y = 2x - 3$	Vertical Translation Down ______
	$f(x) = 3(2x)$	Vertical Stretch Factor ______ from the *x*-axis

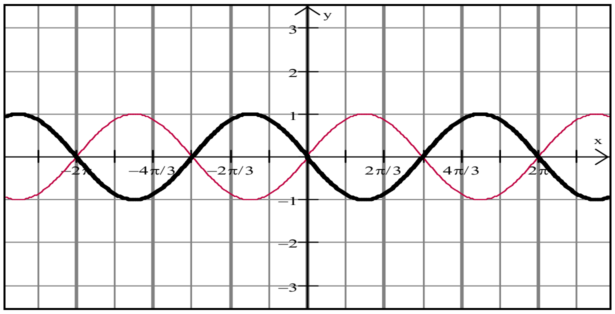
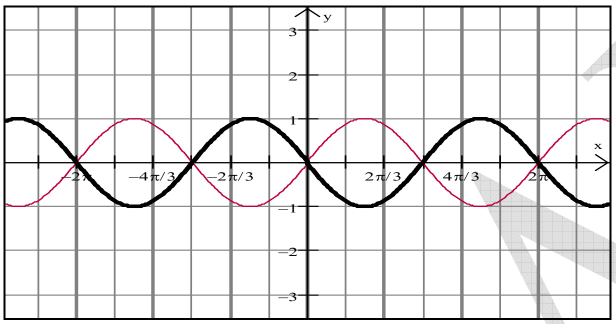
5.10.1: Log or Rhythm (continued)

	$y = -2x$	Vertical Reflection in the *x*-axis
	$f(x) = 2^{-x}$	Horizontal Reflection in the *y*-axis

5.10.1: Log or Rhythm (continued)

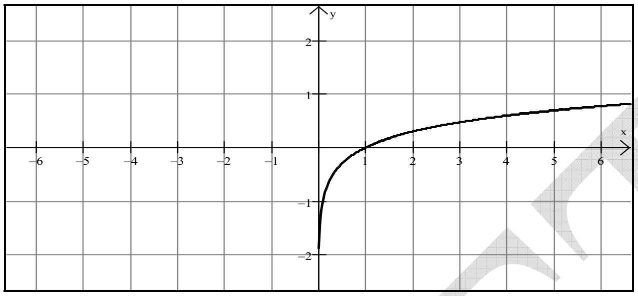
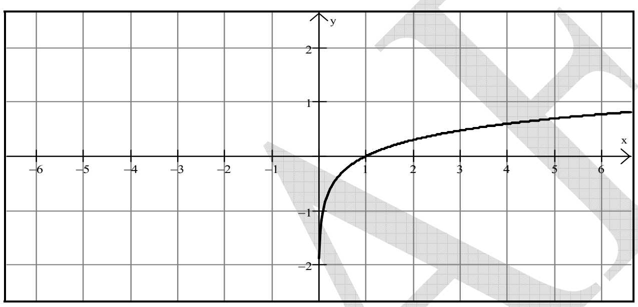
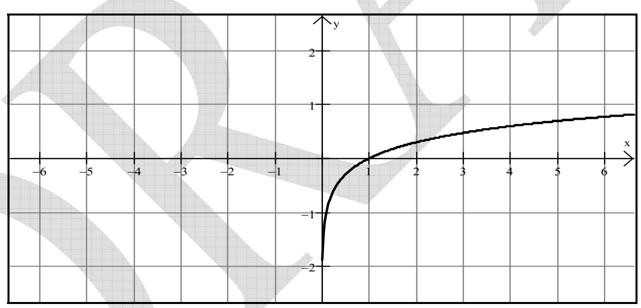
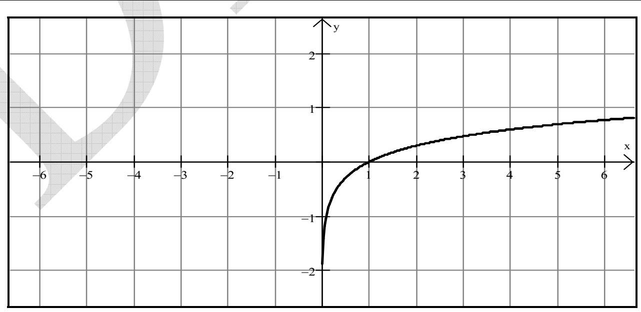
 Each interval on the horizontal axis represents $\frac{\pi}{3}$	$y = \sin(x - \frac{\pi}{3})$	Horizontal Translation Right ______
 Each interval on the horizontal axis represents $\frac{\pi}{3}$	$f(x) = \sin(x) + 2$	Vertical Translation Up ______
 Each interval on the horizontal axis represents $\frac{\pi}{3}$	$y = \frac{1}{2}\sin(x)$	Vertical Compression Factor ______ to the *x*-axis

5.10.1: Log or Rhythm (continued)

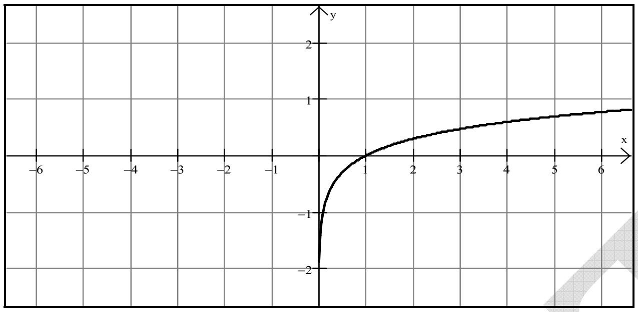
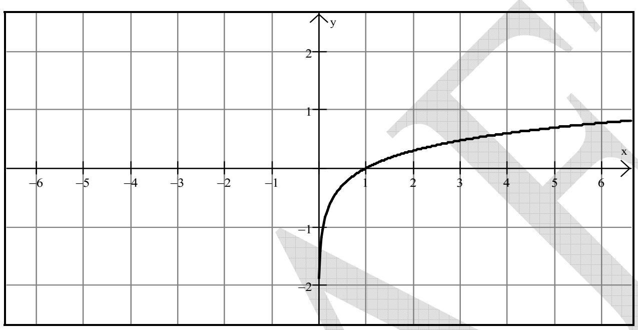
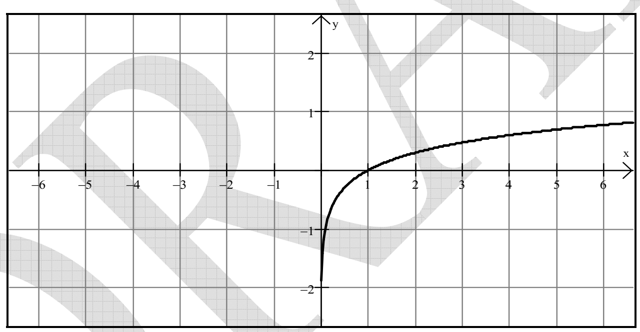
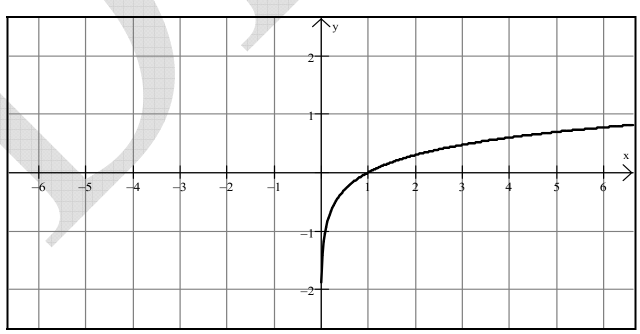
 Each interval on the horizontal axis represents $\frac{\pi}{3}$	$f(x) = -\sin(x)$	Vertical Reflection in the *x*-axis
 Each interval on the horizontal axis represents $\frac{\pi}{3}$	$y = \sin(-x)$	Horizontal Reflection in the *y*-axis

5.10.2: Student Observation Log

As you work through the investigations, use the chart below to record your observations. Record sketches of the transformed graphs and give a brief but accurate description in words of how each parameter affects the graph of $y = \log_{10}x$.

	Graphs	Descriptions
$y = \log_{10} x + c$		
$y = \log_{10} x - c$		
$y = \log_{10} (x + d)$		
$y = \log_{10} (x - d)$		

5.10.2: Student Observation Log (continued)

	Graphs	Descriptions
$y = a\log_{10} x$		
$y = -\log_{10} x$		
$y = \log_{10}(kx)$		
$y = \log_{10}(-x)$		

Special Notes:

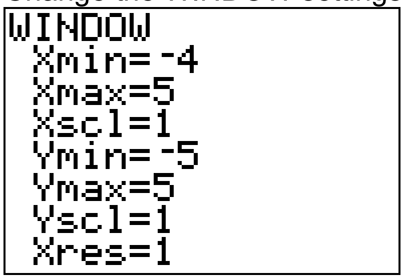
5.10.3: Alternate Graphing Calculator Investigation

Investigation of Transformations of the Logarithmic Function

In this investigation you will explore the roles of the parameters c, and d in the function $y = \log_{10}(x - d) + c$ and of the parameters a and k in $y = a\log_{10}(kx)$. You will record your observations on the Student Observation Log sheet.

Investigation 1: $\quad y = \log_{10}(x) + c$

Change the WINDOW settings to the settings shown in the screen shot below.



Enter the function $y = \log_{10}(x)$ into the equation editor as y_1 and the following three functions in y_2, one at a time, to compare and contrast their graphs:

$f(x) = \log_{10}(x) + 3$

$f(x) = \log_{10}(x) - 2$

$f(x) = \log_{10}(x) + 1.5$

Describe the effect that the parameter 'c' has on the transformations of $y = \log_{10}(x) + c$

- what happens when 'c' is positive?
- what happens when 'c' is negative?
- record your graph and observations on the Student Observation Log.

Investigation 2: $\quad y = \log_{10}(x - d)$

Enter the function $y = \log_{10}(x)$ into the equation editor as y_1 and the following three functions in y_2, one at a time, to compare and contrast their graphs:

$f(x) = \log_{10}(x + 3)$

$f(x) = \log_{10}(x - 2)$

$f(x) = \log_{10}(x + 1.5)$

Describe the effect that the parameter "d' has on the transformations of $y = \log_{10}(x - d)$

- what happens when 'd' is positive?
- what happens when 'd' is negative?
- record your graph and observations on the Student Observation Log.

5.10.3: Alternative Graphing Calculator Investigation (continued)

Investigation 3: $y = a \log_{10}(x)$

Enter the function $y = \log_{10}(x)$ into the equation editor as y_1 and the following three functions in y_2, one at a time, to compare and contrast their graphs:

$f(x) = 3 \log_{10}(x)$

$f(x) = -2 \log_{10}(x)$

$f(x) = \frac{1}{2} \log_{10}(x)$

$f(x) = -0.5 \log_{10}(x)$

Describe the effect that the parameter 'a' has on the transformations of $y = a \log_{10}(x)$

- what happens when 'a' is positive?
- what happens when 'a' is negative?
- record your graph and observations on the Student Observation Log.

Investigation 4: $y = \log_{10}(kx)$

Enter the function $y = \log_{10}(x)$ into the equation editor as y_1 and the following three functions in y_2, one at a time, to compare and contrast their graphs:

$f(x) = \log_{10}(3x)$

$f(x) = \log_{10}(-2x)$

$f(x) = \log_{10}(x)$

$f(x) = \log_{10}(-0.5x)$

Describe the effect that the parameter 'k' has on the transformations of $y = \log_{10}(kx)$

- what happens when 'k' is positive?
- what happens when 'k' is negative?
- record your graph and observations on the Student Observation Log.

5.10.4: Math "Tai Chi"

*Write all of the transformations on the board. Each time you point to a new transformation, students must display the appropriate movement. As the students get better at recognizing each one, a natural development is an elimination game. This makes a great review before a test!

Transformation	Description	Movement
$f(x) + d$	Vertical Translation UP	Slowly raise arms, palms UP
$f(x) - d$	Vertical Translation Down	Slowly lower arms, palms DOWN
$f(x - c)$	Horizontal Translation Right	Slowly motion arms to the RIGHT
$f(x + c)$	Horizontal Translation Left	Slowly motion arms to the LEFT
$af(x),\ a > 0$	Vertical Stretch	Left arm up, right arm down (Pulling an elastic apart vertically)
$af(x),\ a < 0$	Vertical Compression	Start with arms apart, and bring them together (opposite of V. Stretch)
$f(kx),\ k > 0$	Horizontal Stretch	Slowly pull arms apart horizontally (pulling an elastic apart horizontally)
$f(kx),\ k < 0$	Horizontal Compression	Slowly bring palms together (opposite of H. Stretch)
$-f(x)$	Reflection in the x-Axis	Hold arms in upright goal post position and then flip them down at the elbow to point to the floor.
$f(-x)$	Reflection in the y-Axis	Hold arms in the upright goal position and then cross them in front to make an X.

5.10.5: CSI Math

The drama class is writing a murder mystery, and they need some props. The case hinges on the time of death which can be determined by Newton's Law of Cooling. But, they would like to be as accurate as possible, so they have come to Professor Wiggin's math class for a graph that will help them solve the problem.

Your task is to create a graph of the cooling body temperature on chart paper so that it is large enough for an audience to see.

Here is some information to help you complete this task.

A coroner uses a formula derived from Newton's Law of Cooling, a general cooling principle, to calculate the elapsed time since a person has died. The formula is

$$t = -23 \log(T - RT) + 33, \text{ where}$$

- t is the time elapsed in hours since death
- RT is the constant room temperature
- T is the body's measured temperature (°F)

A more accurate estimate of the time of death is found by taking two readings and averaging the calculated time.

Use your knowledge of the logarithmic function to describe the transformations that occur in the equation $t = -23 \log(T - RT) + 33$.

•
•
•
•

Now, using your descriptions of the transformations, draw a 'rough' sketch of the graph, and estimate the window settings that will allow an optimal view of the time in question. Check your sketch using a graphing calculator.

Prepare a full size accurate graph on chart paper for the drama class to use.

Unit 6: Combining Functions

Grade 12

Lesson Outline

Big Picture

Students will:

- consolidate understanding of characteristics of functions (polynomial, rational, trigonometric, and exponential);
- create new functions by adding, subtracting, multiplying, or dividing functions;
- create composite functions;
- determine key properties of the new functions;
- generalize their understanding of a function.

Day	Lesson Title	Math Learning Goals	Expectations
1–2		• Explore real-world problems that involve a combination of functions (dampen springs, drug absorption, pendulum, friction, air resistance). • Solve related problems graphically.	D2.2, 3.3
3		• Investigate graphically and numerically contexts that involve the sum and difference of functions of the same family and determine whether these result in functions of the same family. • Determine through investigation the following properties of the resulting functions: domain, range, maximum, minimum, number of zeros, odd or even, increasing/decreasing behaviours, average rates of change, and instantaneous rates of change at a point.	D2.1, 2.3, 2.8
4		• Graph the sum and difference of functions that are not of the same family. • Determine through investigation the following properties of the resulting functions: domain, range, maximum, minimum, number of zeros, odd or even, increasing/decreasing behaviours, average rates of change, and instantaneous rates of change at a point. • Solve equations and inequalities not accessible by standard algebraic techniques using graphical and numerical methods.	D2.1, 2.3, 2.8, 3.2
5		• Graph the product of any two functions. • Determine through investigation the following properties of the resulting functions: domain, range, maximum, minimum, number of zeros, odd or even, increasing/decreasing behaviours, average rates of change, and instantaneous rates of change at a point.	D2.1, 2.3
6		• Graph the quotient of any two functions. • Determine through investigation the following properties of the resulting functions: domain, range, maximum, minimum, number of zeros, odd or even, increasing/decreasing behaviours, average rates of change, and instantaneous rates of change at a point. • Compare functions on appropriate intervals, e.g., zeroes, equalities, $f/g > 0$.	D2.1, 2.3, 2.8

Day	Lesson Title	Math Learning Goals	Expectations
7		• Connect transformations of functions with composition of functions. • Determine the composition of functions numerically and graphically, and interpret the composition of 2 functions in real-world application. • Explore the composition of a function with its inverse numerically and graphically, and demonstrate with examples the property that the composition of a function and its inverse function maps a number onto itself.	D2.4, 2.7, 2.8
8		• Determine the composition of functions algebraically and state the domain and the range of the composition. • Explore the composition of a function with its inverse algebraically.	D2.5, 2.7
9		• Solve problems that involve the composition of two functions including those arising from real-world applications.	D2.6
10–11	Under Pressure Growing Up Soy Fast!	• Solve problems including those arising from real-world applications by reasoning with functions. • Reflecting on quality of 'fit'.	D3.3
12		• Jazz day	
13		• Summative Assessment	

75 min

Math Learning Goals

- Solve problems including those arising from real-world applications.
- Reasoning with functions.
- Reflecting on quality of 'fit'.

Materials

- BLM 6.10.1
- Computers with GSP® file
- (MHF_U6_L10_V1_DL.gsp)

Assessment Opportunities

Minds On…

Whole Class → Discussion

Present the context of a leaky tire and why it is important to know the tire pressure.

Tire pressure is a measure of the amount of air in a vehicle's tires, in pounds per square inch. If tire pressure is too high, then less of the tire touches the ground. As a consequence, your car will bounce around on the road. And when your tires are bouncing instead of firmly planted on the road, traction suffers and so do your stopping distances. If tire pressure is too low, then too much of the tire's surface area touches the ground, which increases friction between the road and the tire. As a result, not only will your tires wear prematurely, but they also could overheat. Overheating can lead to tread separation — and a nasty accident.

http://cars.cartalk.com/content/advice/tirepressure.html

Think/Pair/Share → Discussion

Individually, students use the data in BLM 6.10.1 Part A to hypothesize a model. Student pairs sketch a possible graph of this relationship. Invite pairs to share their predictions with the entire class.

Lead a discussion about the meaning of "tolerance" in the context of "hitting" a point on the curve.

Some people suggest that traditional two-sided tolerances are analogous to "goal posts" in a football game: It implies that all data within those tolerances are equally acceptable. The alternative is that the best product has a measurement which is precisely on target.

http://en.wikipedia.org/wiki/Tolerance_(engineering)

Action!

Pairs → Investigation

Under Pressure.gsp

Students use BLM 6.10.1 Part B and the GSP® file to manipulate each function model using sliders. They determine which model – linear, quadratic, or exponential best fits the data provided. They conclude by providing an equation that best fits the data. Students discuss with their partner other factors that would limit the appropriateness of each model in terms of the context and record answers on the BLM. Circulate around the computer lab and assist students who may have difficulty working with the GSP® sketch.

Reasoning/Observation /Mental Note: Observe students facility with the inquiry process to determine their preparedness for the homework assignment.

Consolidate Debrief

Whole Class → Discussion

Students present their models for the tire pressure/time relationship and determine which pair found the "best" model for the data. This could be done using GSP® or a Smart Board. Discuss the appropriateness of each model in this context including the need to limit the domain of the function.

Curriculum Expectations/BLM/Anecdotal Feedback: Collect BLMs from each pair and provide feedback on student responses.

Home Activity or Further Classroom Consolidation

Complete the follow up questions in Part C and Part D "Pumped Up" on BLM 6.10.1 (Possible answer: $p = 10\sqrt{0.5x+14}$ 52 pumps).

6.10.1: Under Pressure

Part A – Forming a Hypothesis

A tire is inflated to 400 kilopascals (kPa) and over the next few hours it goes down until the tire is quite flat. The following data is collected over the first 45 minutes.

Time t (min)	Pressure P, (kPa)
0	400
5	335
10	295
15	255
20	255
25	195
30	170
35	150
40	135
45	115

Sketch what you think the graph of tire pressure P against time t should look like

Part B – Testing Your Hypothesis and Choosing a Best Fit Model

Open the sketch Under Pressure.gsp and follow the Instructions on the screen.

Enter your best equations, number of hits, and tolerances in the table below.

	Linear Model $f(x) = mx + b$	Quadratic Model $f(x) = a(x - h)^2 + k$	Exponential Model $f(x) = a \times b^{x-h} + k$
Your Best Equations			
Number of Hits			
Tolerance			

Equation of the Best Fit Model: →

6.10.1: Under Pressure (continued)

Part C – Evaluating Your Model

1. Is the quadratic model a valid choice if you consider the entire domain of the quadratic function and the long term trend of the data in this context?

2. Using each the 3 “best” models, predict the pressure remaining in the tire after 1 hour. How do your predictions compare? Which of the 3 gives the most reasonable prediction? Justify your answer.

3. Using each of the 3 “best” models, determine how long it will take before the tire pressure drops below 23 kPA? [**Note:** The vehicle in question becomes undriveable at that point.]

4. Justify, in detail, why you think the model you obtained is the best model for the data in this scenario. Consider more than number of hits in your answer.

Part D – Pumped Up

Johanna is pumping up her bicycle tire and monitoring the pressure every 5 pumps of the air pump. Her data is shown below. Determine the algebraic model that best represents this data and use your model to determine how many pumps it will take to inflate the tire to the recommended pressure of 65 psi.

Number of Pumps	Tire Pressure (psi)
0	14
5	30
10	36
15	41
20	46
25	49

75 min

Math Learning Goals

- Solve problems including those arising from real-world applications.
- Reasoning with functions.

Materials

- BLM 6.11.1, 6.11.2
- graphing calculator

Assessment Opportunities

Minds On…

Individual → Exploration

Students hypothesize about the effects of limiting fertilizer on the growth of the Glycine Max (commonly known as the Soybean plant), under the 3 given conditions. Give them 5 minutes to sketch their predictions and rationales (BLM 6.11.1 Part One).

Whole Class → Discussion

Identify 4 corners in the room as either – Linear, Polynomial, Exponential, or Sinusoidal. Students choose to go to one of the corners and discuss which group's data best fits this model. Corner members share their reasoning. One student from each group shares the discussion for that model. Students may switch position if they've changed their opinion after all 4 corners have been shared.

The initial discussion of the different models students think would be appropriate is very important to help them properly connect the context to the mathematical characteristics of the functions they have been studying.

Action!

Pairs → Investigation

Students take the data provided in BLM 6.11.1 Part Two, and, using their knowledge of function properties, determine a function model for each scenario in the experiment.

Pairs answer questions about the models they have chosen, in terms of the transformations used, and discuss the relationship between the model and the context of growing soybean plants.

Predictions are made about how well the models they have chosen would describe the continued growth of the plants in the experiment.

A further scenario provides the opportunity to combine two or more functions together.

Mathematical Process Focus: Reasoning and Proving, Reflecting, Connecting.

Reasoning/Observation /Mental Note: Observe students facility with the inquiry process to determine their preparedness for the homework assignment.

Modelling with functions becomes much more powerful when students recognize that rarely does a single function serve as an appropriate model for a real-world problem.

Consolidate Debrief

Pairs/Share/Whole Group → Discussion

Pairs discuss the implications of extending the experiment for another three weeks on the function models that they created with another pair. Also discuss their models for the growth of the plants in the control group and justify why the combinations they chose fit the context.

Ask 2–3 volunteers to present a summary of their discussions.

Consideration
Application

Home Activity or Further Classroom Consolidation

Create a piecewise function by combining several functions and solving to determine appropriate restrictions on the domain of each "piece" of the model. (Worksheet)

The home activity alternatively could be used as an extension for some students.

6.11.1: Growing Up Soy Fast!

Part One

Your biology class is studying the lifecycle of Glycine max (the plant more commonly known as Soybean), and specifically you will be investigating the effects of limiting the amount of food (fertilizer) used for the plants' growth.

- **Group A** will not give their plants any fertilizer for the first week after planting after which they will feed and water the plants normally for the remainder of the study.
- **Group B** will feed and water their plants normally for the first week, then not give any fertilizer for the 2nd week, and then return to the regular amounts of fertilizer and water for the 3rd week of the study.
- **Group C** will feed and water their plants normally for the first week, and then gradually decrease the amount of fertilizer until the end of the study.

Make predictions about the relationship between Day Number (from beginning of study) and plant height (cm) for each of the groups. Sketch your predictions below.

Group A

Sketch	**Rationale**

Group B

Sketch	**Rationale**

Group C

Sketch	**Rationale**

6.11.1: Growing Up Soy Fast! (continued)

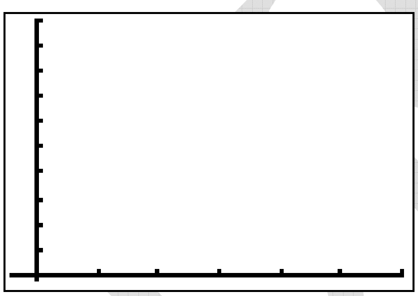
Part Two

The heights of the plants were measured throughout the study and the following data was taken by each group:

Group A		Group B		Group C	
Day	Height (cm)	Day	Height (cm)	Day	Height (cm)
1	4.4	1	3.7	1	3.5
3	5.8	2	9.5	3	19.0
4	6.7	5	24.0	5	23.9
7	10.4	6	26.4	6	25.8
8	12.0	8	28.5	9	31.0
10	15.9	11	30.1	11	33.9
13	24.5	14	33.1	12	35.2
15	32.6	16	37.7	15	39.0
16	37.6	18	45.6	17	41.3
19	57.9	20	58.2	20	44.6
21	77.1	21	66.4	21	45.7

Analysing the Data for Group B

1. Use your graphing calculator to construct a scatterplot for the **Group B** data. Sketch the scatterplot you obtained below. Remember to label your axes.



2. Perform an analysis of the data and, selecting from the functions you have studied, identify the type of function that you think best models it.

    ____

3. Use your knowledge of function properties to determine a function model that best fits the data.

    My function model is: ________________________

Analysing the Data for Group C

4. The data for **Group C** was entered into the graphing calculator and a regression analysis was performed. The function that best models this data is $y = 17\sqrt{0.3x + 3}$. Graph this function on the grid below. Be sure to include all labels.

5. State the domain and range of the model. Use interval notation.

   Domain: $x \in$ ____________ Range: $y \in$ _________________

6. Describe, in words, the transformations needed to obtain $y = 17\sqrt{0.3x + 3}$ from the graph of $y = \sqrt{x}$.

   →

   →

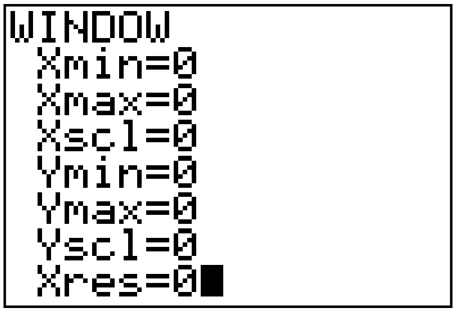
   →

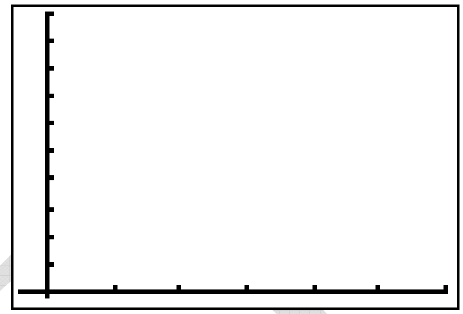
7. Describe what the transformations could mean in the context of this experiment.

6.11.1: Growing Up Soy Fast! (continued)

Analysing the Data for Group A

8. Enter the data for **Group A** into the graphing calculator and construct a scatterplot for the data. Complete both of the screen shots below using the information from your calculator.





9. The type of function that I feel best describes this data is ______________________

10. Use your knowledge of function properties to determine a function model that best fits the data.

    My function model is: ______________________

11. State the following for the model you found above.

    a) Domain (interval notation) ______________________
    b) Range (set notation) ______________________
    c) Maximum (if any) ______________________
    d) Minimum (if any) ______________________
    e) Equation of Asymptotes (if any) ______________________

12. Describe the restrictions to the domain and range (if any), of the function model you have found, that would apply to the real-life application of Soybean plant growth.

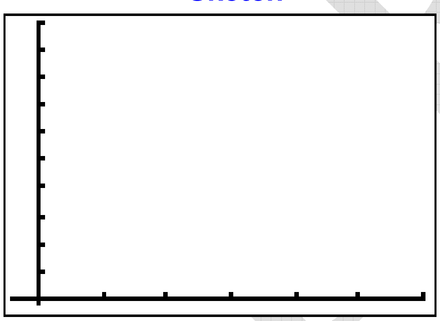
6.11.1: Growing Up Soy Fast! (continued)

Making Predictions from Your Results

13. If each group were to continue to feed and water their plants normally for another three weeks, discuss what changes or modifications (if any) you would need to make on each of the three function models.

14. Another group in your class, **Group D**, acted as the Control Group. They fed and watered their plants as normal during the whole experiment.

    a) What type of function (or combination of functions) do you feel would best model the data that this group would have collected?

Sketch	Type or Combination of Functions and Rationale
	

    b) Discuss what changes or modifications (if any) you would need to make on this function model, if **Group D** was to continue the experiment for another three weeks.

6.11.2: The Chipmunk Explosion

Chipmunk Provincial Park has a population of about 1000 chipmunks. The population is growing too rapidly, due to campers feeding them. To curb the explosive population growth, the park rangers decided to introduce a number of foxes (a natural predator of chipmunks) into the park. After a period of time, the chipmunk population peaked and began to decline rapidly. The following data gives the chipmunk population over a period of 14 months.

Time (months)	Population (1000s)
1	1.410
2	1.970
3	2.260
5	5.100
6	5.920
7	5.890
9	4.070
9.5	3.650
10	3.260
11	2.600
12	2.090
13	1.670
14	1.330

Use graphing technology to create a scatter plot of the data.

Questions

1. Determine a mathematical function model that represents this data. It will be necessary to use more than one type of function. Include the domain over which each type of function applies to the model.

2. Determine when the population reaches a maximum and what the maximum population is.

3. Determine when the population will fall to less than 100 chipmunks.

6.11.1: Solutions

Part Two

The heights of the plants were measured throughout the study and the following data was taken by each group:

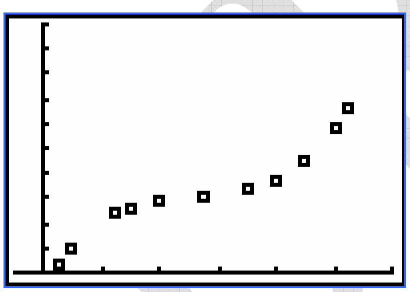
Group A	
Day	**Height (cm)**
1	4.4
3	5.8
4	6.7
7	10.4
8	12.0
10	15.9
13	24.5
15	32.6
16	37.6
19	57.9
21	77.1

Group B	
Day	**Height (cm)**
1	3.7
2	9.5
5	24.0
6	26.4
8	28.5
11	30.1
14	33.1
16	37.7
18	45.6
20	58.2
21	66.4

Group C	
Day	**Height (cm)**
1	3.5
3	19.0
5	23.9
6	25.8
9	31.0
11	33.9
12	35.2
15	39.0
17	41.3
20	44.6
21	45.7

Analysing the Data for Group B

1. Use your graphing calculator to construct a scatterplot for the **Group B** data. Sketch the scatterplot you obtained below. Remember to label your axes.



2. Perform an analysis of the data and, selecting from the functions you have studied, identify the type of function that you think best models it.

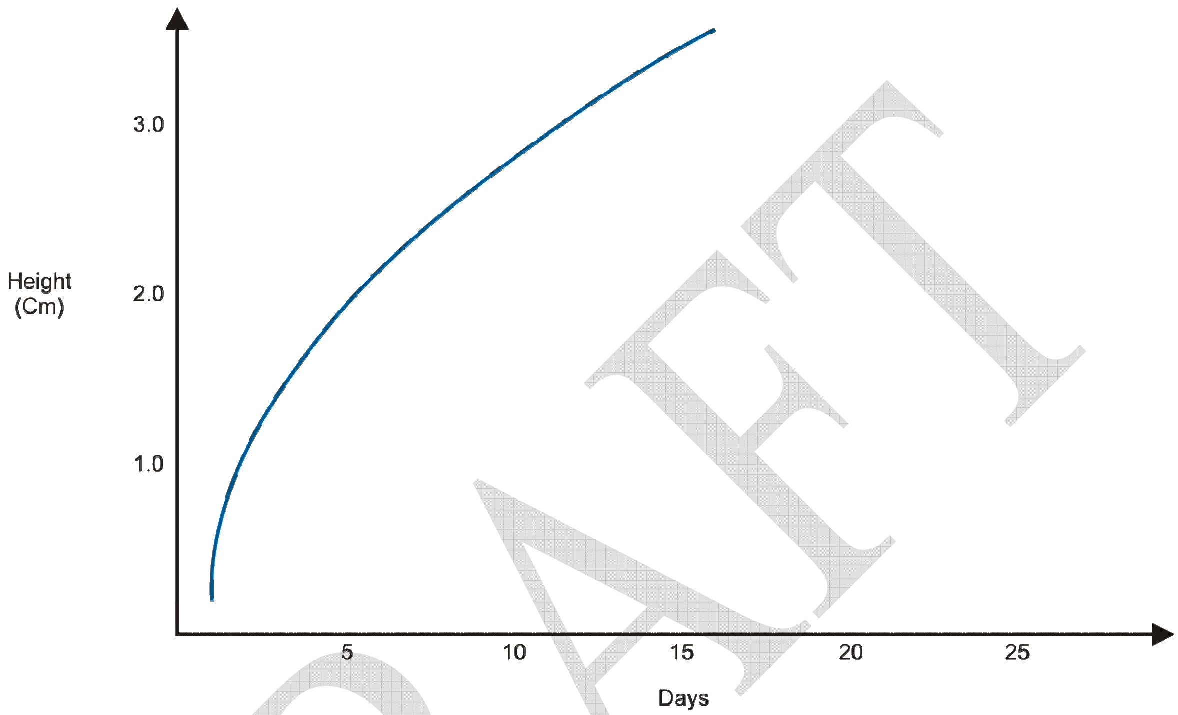
   __________cubic___________

3. Use your knowledge of function properties to determine a function model that best fits the data.

   My function formula is: $y = 0.035(x-10)^3 + 30$

Analysing the Data for Group C

4. The data for Group C was entered into the graphing and a regression analysis was performed. The function that best models this data is $y = 17\sqrt{0.3(x-1)} + 3.5$. Graph this function on the grid below. Be sure to include all labels.



5. State the domain and range of the model. Use interval notation.

   Domain: $x \in$ $[1, \infty)$ Range: $y \in$ $[3.5, \infty)$

6. Describe, in words, the transformations needed to obtain $y = 17\sqrt{0.3(x-1)} + 3.5$ from the graph of $y = \sqrt{x}$.

   → vertical stretch by a factor of 17

   → horizontal stretch by a factor of $0.3^{-1}\left(\frac{10}{3}\right)$

   → vertical shift up 3.5 units; horizontal shift 1 unit right

7. Describe what the transformations could mean in the context of this experiment.

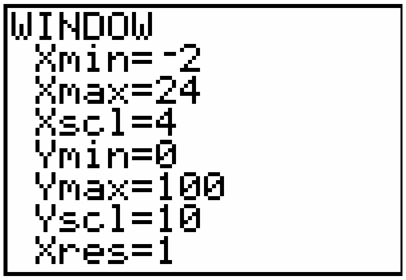
   some examples might be …

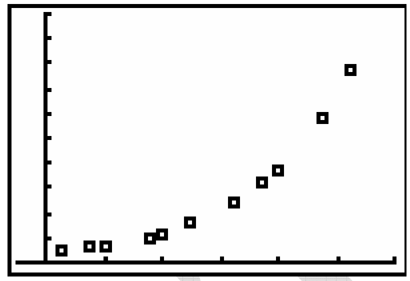
   → the vertical shift represents the height of the plant on the first day a measurement was taken
   → horizontal stretch indicates that the model shows growth beginning on Day 1
   → plants grow more rapidly than the rate at which the basic square root function increases, so vertical stretch factor was needed

6.11.1: Solutions (continued)

Analysing the Data for Group A

8. Enter the data for **Group A** into the graphing calculator and construct a scatterplot for the data. Complete both of the screen shots below using the information from your calculator.





9. The type of function that I feel best describes this data is ___**exponential**___

10. Use your knowledge of function properties to determine a function model that best fits the data.

    My function formula is: $y = 3.8\left(1.15^{x}\right)$

11. State the following for the model you found above.

    a) Domain (interval notation) $x \in (-\infty, \infty)$

    b) Range (set notation) $\{y \in \mathbf{R} \mid y > 0\}$

    c) Maximum (if any) **no maximum value**

    d) Minimum (if any) **minimum value is 0**

    e) Equation of Asymptotes (if any) $\boldsymbol{y = 0}$

12. Describe the restrictions to the domain and range (if any), of the function model you have found, that would apply to the real-life application of Soybean plant growth.

    note that answers will vary … some examples might be

    → the domain would begin at $x = 0$ to correspond to the height of the plant on the day prior to the first observation, and would end shortly after $(x = 21)$ the three week observation period ends since we don't know how much longer the model will accurately predict the growth pattern of the plant

    → the range would begin at $x = 3.8$, since this is the predicted height of the plant on the day prior to the first observation.

6.11.1: Solutions (continued)

Making Predictions from Your Results

13. If each group were to continue to feed and water their plants normally for another three weeks, discuss what changes or modifications (if any) you would need to make on each of the three function models.

    note that answers will vary … some examples might be

    → the model for the plants in Group A and Group B could remain unchanged for another three week period

    → the model for the plants in Group C will change because the plants should experience an increased rate of growth for the 2nd three week period … the model might be adjusted to

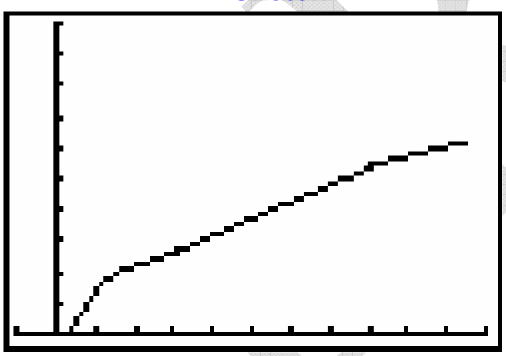
$$y = \begin{cases} 17\sqrt{0.3(x-1)}+3.5, & x \in [0,21] \\ 2.4(1.15x), & x \in (21,42] \end{cases}$$

14. Another group in your class, Group D, acted as the Control Group. They fed and watered their plants as normal during the experiment.

    a) What type of function (or combination of functions) do you feel would best model the data that this group would have collected? Provide proper justification

    note that answers will vary … some examples might be

Sketch



Combination of Functions and Justification

possible combination:

cubic → linear → square root

justification:

initial growth is rapid, then levels off to relatively constant for a period of time until the plant nears maturity, at which time growth rate decreases

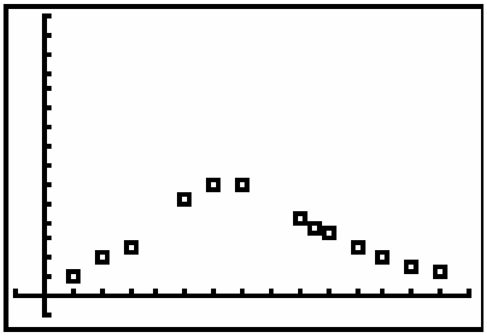
b) Discuss what changes or modifications (if any) you would need to make on each of this function model, if **Group D** was to continue the experiment for another three weeks.

answers will vary … some considerations might be

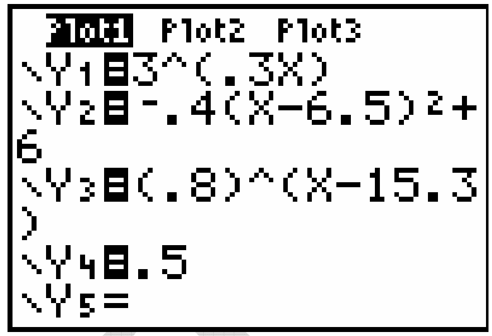
- when will the plant reach maturity and what might its maximum height be

6.11.2: Solutions – Using the TI83+ Graphing Calculator

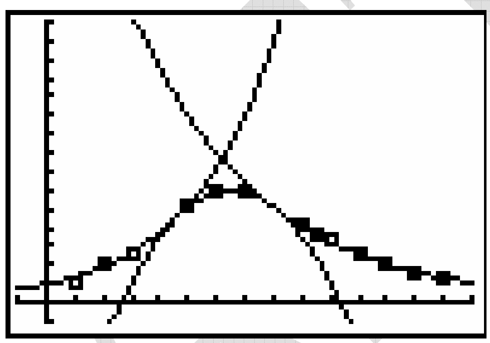
Stat Plot:



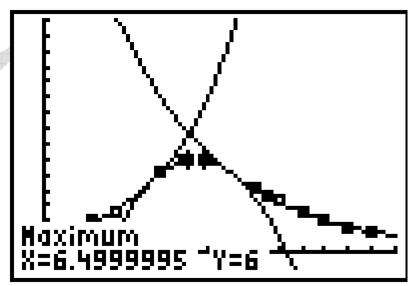
Functions:



1. Stat Plot with Functions:

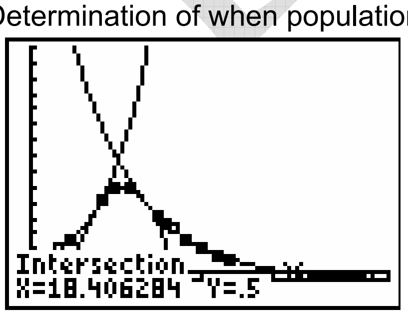


2. Maximum Population:



Maximum population is 6000 at 6.5 months

3. Determination of when population reaches 500:



Population reaches 500 at about 18.4 months.

6.11.2: Solutions – Using GSP®