

## Midterm Exam I, Calculus III, Sample A

1. (10 points) Show that the 4 points  $P_1 = (0, 0, 0)$ ,  $P_2 = (2, 3, 0)$ ,  $P_3 = (1, -1, 1)$ ,  $P_4 = (1, 4, -1)$  are coplanar (they lie on the same plane), and find the equation of the plane that contains them.

**Solution:**  $\mathbf{u} = \overrightarrow{P_1P_2} = \langle 2, 3, 0 \rangle$ ,  $\mathbf{v} = \overrightarrow{P_1P_3} = \langle 1, -1, 1 \rangle$ ,  $\mathbf{w} = \langle 1, 4, -1 \rangle$ , and the scalar triple product is equal to

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = 2 \begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 0,$$

so the volume of the parallelepiped determined by  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  is equal to 0. This means that these vectors are on the same plane. So,  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$  are coplanar.

2. (10 points) Find the equation of the plane that is equidistant from the points  $A = (3, 2, 1)$  and  $B = (-3, -2, -1)$  (that is, every point on the plane has the same distance from the two given points).

**Solution:** The midpoint of the points  $A$  and  $B$  is the point  $C = \frac{1}{2}(3, 2, 1) + (-3, -2, -1) = (0, 0, 0)$ . A normal vector to the plane is given by  $\overrightarrow{AB} = \langle 3, 2, 1 \rangle - \langle -3, -2, -1 \rangle = \langle 6, 4, 2 \rangle$ . So, the equation of the plane is  $6(x - 0) + 4(y - 0) + 2(z - 0) = 0$ , that is,  $3x + 2y + z = 0$ .

3. (6 points) Find the vector projection of  $\mathbf{b}$  onto  $\mathbf{a}$  if  $\mathbf{a} = \langle 4, 2, 0 \rangle$  and  $\mathbf{b} = \langle 1, 1, 1 \rangle$ .

**Solution:** Since  $|\mathbf{a}|^2 = 4^2 + 2^2 = 20$ , the vector projection of  $\mathbf{b}$  onto  $\mathbf{a}$  is equal to

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \frac{\langle 4, 2, 0 \rangle \bullet \langle 1, 1, 1 \rangle}{20} \langle 4, 2, 0 \rangle = \frac{6}{20} \langle 4, 2, 0 \rangle = \frac{3}{5} \langle 2, 1, 0 \rangle.$$

4. (12 points) Consider the curve  $\mathbf{r}(t) = \sqrt{2} \cos t \mathbf{i} + \sin t \mathbf{j} + \sin t \mathbf{k}$ .

- (a) (8 points) Find the unit tangent vector function  $\mathbf{T}(t)$  and the unit normal vector function  $\mathbf{N}(t)$ .  
 (b) (4 points) Compute the curvature  $\kappa$ .

**Solution:** (a)  $\mathbf{r}'(t) = -\sqrt{2} \sin t \mathbf{i} + \cos t \mathbf{j} + \cos t \mathbf{k}$  and  $|\mathbf{r}'(t)| = \sqrt{2 \sin^2 t + \cos^2 t + \cos^2 t} = \sqrt{2}$ . So, the unit tangent vector  $\mathbf{T}(t)$  is equal to

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t \mathbf{i} + \frac{\sqrt{2}}{2} \cos t \mathbf{j} + \frac{\sqrt{2}}{2} \cos t \mathbf{k}.$$

Since  $\mathbf{T}'(t) = -\cos t \mathbf{i} - \frac{\sqrt{2}}{2} \sin t \mathbf{j} - \frac{\sqrt{2}}{2} \sin t \mathbf{k}$  and  $|\mathbf{T}'(t)| = \sqrt{\cos^2 t + \frac{1}{2} \sin^2 t + \frac{1}{2} \sin^2 t} = 1$ , the normal vector is equal to

$$\mathbf{N}(t) = \cos t \mathbf{i} - \frac{\sqrt{2}}{2} \sin t \mathbf{j} - \frac{\sqrt{2}}{2} \sin t \mathbf{k}.$$

(b) One can use the formula  $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$  and  $|\mathbf{r}'(t)| = \sqrt{2}$  from part (a) and  $|\mathbf{T}'(t)| = 1$  from part (b) to get  $\kappa(t) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ . One can also use the formula  $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ . Since  $\mathbf{r}''(t) = -\sqrt{2} \cos t \mathbf{i} - \sin t \mathbf{j} - \sin t \mathbf{k}$ ,  $|\mathbf{r}''(t)| = 2$ , and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{2} \sin t & \cos t & \cos t \\ -\sqrt{2} \cos t & -\sin t & -\sin t \end{vmatrix} = -\sqrt{2} \mathbf{j} + \sqrt{2} \mathbf{k} = \langle 0, -\sqrt{2}, \sqrt{2} \rangle.$$

Then  $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2$  and  $\kappa(t) = \frac{2}{(\sqrt{2})^3} = \frac{\sqrt{2}}{2}$ .

5. (10 points) Find the length of the curve with parametric equation:

$$\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle,$$

between the points  $(1, 0, 1)$  and  $(e^{2\pi}, 0, e^{2\pi})$ .

**Solution:**

$$\mathbf{r}'(t) = \langle e^t, e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t \rangle$$

and

$$|\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{2t}(\sin t + \cos t)^2 + e^{2t}(\sin t - \cos t)^2} = e^t \sqrt{3}.$$

Note that  $\mathbf{r}(0) = \langle 1, 0, 1 \rangle$  and  $\mathbf{r}(2\pi) = \langle e^{2\pi}, 0, e^{2\pi} \rangle$ . So, the length of the curve is equal to

$$\text{length} = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{3} e^t dt = \sqrt{3}(e^{2\pi} - 1).$$

6. (12 points) A spaceship is traveling with acceleration

$$\mathbf{a}(t) = \langle e^t, t, \sin 2t \rangle.$$

At  $t = 0$ , the spaceship was at the origin,  $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ , and had initial velocity  $\mathbf{v}(0) = \langle 1, 0, 0 \rangle$ . Find the position of the spaceship at  $t = \pi$ .

**Solution:** The velocity is equal to

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s) ds = \langle 1, 0, 0 \rangle + \int_0^t \langle e^s, s, \sin 2s \rangle ds \\ &= \langle 1, 0, 0 \rangle + \left\langle \int_0^t e^s ds, \int_0^t s ds, \int_0^t \sin 2s ds \right\rangle \\ &= \langle 1, 0, 0 \rangle + \left\langle e^t - 1, \frac{t^2}{2}, \frac{1}{2}(1 - \cos 2s) \right\rangle = \left\langle e^t, \frac{t^2}{2}, \frac{1}{2}(1 - \cos 2t) \right\rangle. \end{aligned}$$

Since  $\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(s) ds$ ,

$$\begin{aligned} \mathbf{r}(t) &= \int_0^t \left\langle e^s, \frac{s^2}{2}, \frac{1}{2}(1 - \cos 2s) \right\rangle ds = \left\langle \int_0^t e^s ds, \int_0^t \frac{s^2}{2} ds, \frac{1}{2} \int_0^t (1 - \cos 2s) ds \right\rangle \\ &= \left\langle e^t - 1, \frac{t^3}{6}, \frac{t}{2} - \frac{1}{4} \sin 2t \right\rangle. \end{aligned}$$

So,  $\mathbf{r}(\pi) = \langle e^\pi - 1, \frac{\pi^3}{6}, \frac{\pi}{2} \rangle$ .

7. (10 points) Write the equation of the tangent line to the curve with parametric equation

$$\mathbf{r}(t) = \langle \sqrt{t}, 1, t^4 \rangle,$$

at the point  $(1, 1, 1)$ .

**Solution:**  $\mathbf{r}'(t) = \langle \frac{1}{2\sqrt{t}}, 0, 4t^3 \rangle$ . At  $(1, 1, 1)$ ,  $t = 1$  and  $\mathbf{r}'(1) = \langle 1/2, 0, 4 \rangle$ . Thus the parametric equations of the tangent line are

$$x = t/2 + 1, \quad y = 1, \quad z = 4t + 1.$$

8. (12 points) Using cylindrical coordinates, find the parametric equations of the curve that is the intersection of the cylinder  $x^2 + y^2 = 4$  and the cone  $z = \sqrt{x^2 + y^2}$ . (*This problem refers to the material not covered before midterm 1.*)
9. (6 points) Let  $f(x, y) = \sin(x^2 + y^2) + \arcsin(y^2)$ . Calculate:

$$\frac{\partial^2 f}{\partial x \partial y}.$$

(*This problem refers to the material not covered before midterm 1.*)

10. (12 points) (*This problem refers to the material not covered before midterm 1.*) Show that the following limit does not exist:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{7x^2y(x-y)}{x^4 + y^4}$$

Justify your answer. (*This problem refers to the material not covered before midterm 1.*)

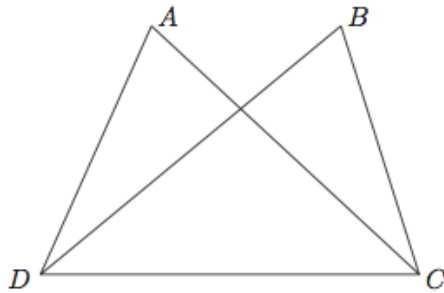
## Midterm Exam I, Calculus III, Sample B

1. (6 Points) Find the center and radius of the following sphere  $x^2 + y^2 + z^2 - 6x + 4z - 3 = 0$ .  
Completing the squares:

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 - 6x + 4z - 3 = (x^2 - 6x + 9) + y^2 + (z^2 + 4z + 4) - 3 - 9 - 4 \\ &= (x - 3)^2 + y^2 + (z + 2)^2 - 16. \end{aligned}$$

So, the equation of the sphere is  $(x - 2)^2 + y^2 + (z - (-2))^2 = 4^2$ , the center is  $(3, 0, -2)$  and radius 4.

2. (6 Points) Write each combination of vectors as a single vector



- (a)  $\overrightarrow{AB} + \overrightarrow{BC}$   
 (b)  $\overrightarrow{AC} - \overrightarrow{BC}$   
 (c)  $\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BA}$ .

**Solution:**

$$(a) \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}, \quad (b) \overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AB}, \quad (c) \overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BA} = \overrightarrow{0}.$$

3. (6 Points) Find the cosine of the angle between the vectors  $\mathbf{a} = \langle 1, 2, 3 \rangle$  and  $\mathbf{b} = \langle 2, 0, -1 \rangle$ .

**Solution:** If  $\vartheta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos \vartheta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\langle 1, 2, 3 \rangle \cdot \langle 2, 0, -1 \rangle}{|\langle 1, 2, 3 \rangle||\langle 2, 0, -1 \rangle|} = -\frac{1}{\sqrt{14}\sqrt{5}} = -\frac{1}{\sqrt{70}}.$$

4. (6 Points) Find the vector projection of  $\mathbf{v}$  onto  $\mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} - \mathbf{k}$  and  $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ .

**Solution:** The vector projection of  $\mathbf{v}$  onto  $\mathbf{u}$  is equal to

$$\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u} = \frac{(2\mathbf{i} - \mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j})}{|2\mathbf{i} - \mathbf{k}|^2} [2\mathbf{i} - \mathbf{k}] = \frac{4}{5} [2\mathbf{i} - \mathbf{k}].$$

5. (7 Points) Find the area the triangle with vertices  $P = (2, 1, 7)$ ,  $Q = (1, 1, 5)$ ,  $R = (2, -1, 1)$ .

**Solution:** Since  $\overrightarrow{PQ} = \langle -1, 0, -2 \rangle$  and  $\overrightarrow{PR} = \langle 0, -2, -6 \rangle$ , and

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -2 \\ 0 & -2 & -6 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k},$$

the area of the triangle  $\triangle PQR$  is equal to

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{(-4)^2 + (-6)^2 + 2^2} = \frac{\sqrt{56}}{2} = \sqrt{14}.$$

6. (5 Points) Show that the line

$$x = 3 + t, \quad y = 1 + 2t, \quad z = 1 - 2t$$

is parallel to the plane

$$2x + 3y + 4z = 5.$$

**Solution:** A line is parallel to the plane if it is perpendicular to a normal vector to the plane. A normal vector to the plane is given by  $\langle 2, 3, 4 \rangle$  and the direction of the line is given by the vector  $\langle 1, 2, -2 \rangle$ . Compute the dot product of these vectors:

$$\langle 1, 2, -2 \rangle \bullet \langle 2, 3, 4 \rangle = 2 + 6 - 8 = 0.$$

So, the line is parallel to the plane.

7. (6 Points) Find a vector parallel to the line of intersection for the two planes

$$x + 2y + 3z = 0 \quad \text{and} \quad x - 3y + 2z = 0.$$

**Solution:** A vector which gives the direction of the line of intersection of these planes is perpendicular to normal vectors to the planes. A normal vector to the first plane  $\langle 1, 2, 3 \rangle$  and a normal vector to the second is given by  $\langle 1, -3, 2 \rangle$ . Then the vector

$$\langle 1, 2, 3 \rangle \times \langle 1, -3, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & -3 & 2 \end{vmatrix} = \langle 13, 1, -5 \rangle$$

is parallel to the line of intersection of these planes.

8. (6 Points) Find cosine of the angle between intersection planes

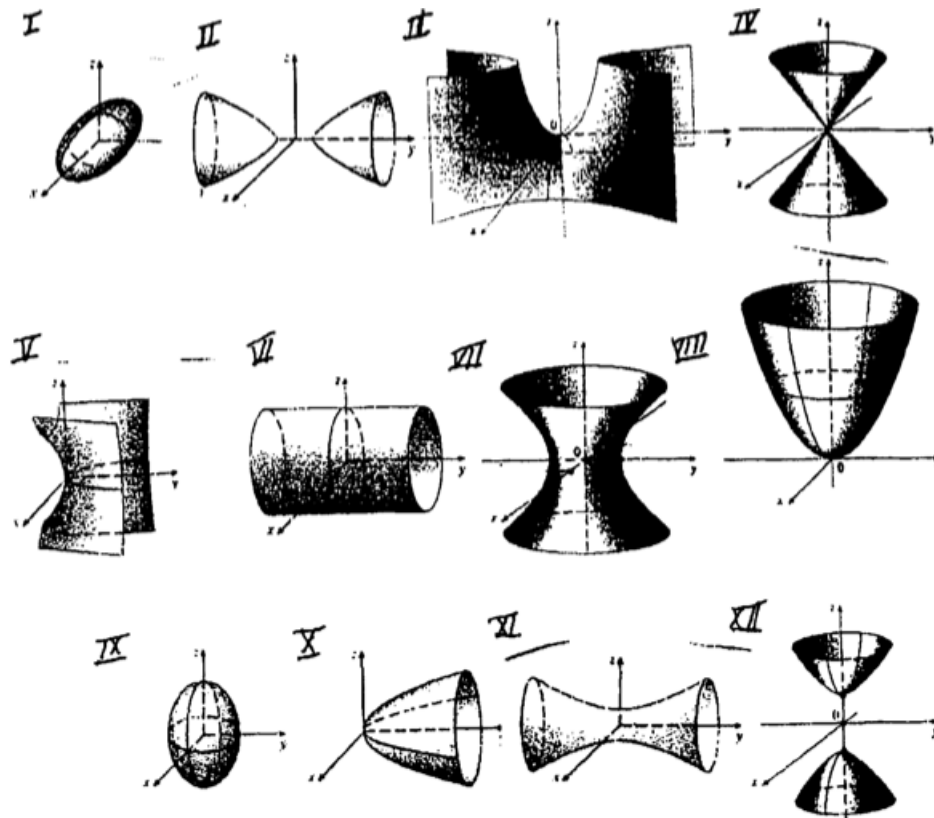
$$2x + y + z = 0 \quad \text{and} \quad 3x - y + 2z = 0.$$

**Solution:** The angle  $\vartheta$  between planes is equal to the angle between normal vectors of the two planes. Since  $\langle 2, 1, 1 \rangle$  is a normal vector to the plane and  $\langle 3, -1, 2 \rangle$  is a normal vector to the second plane,

$$\cos \vartheta = \frac{\langle 2, 1, 1 \rangle \bullet \langle 3, -1, 2 \rangle}{|\langle 2, 1, 1 \rangle| |\langle 3, -1, 2 \rangle|} = \frac{7}{\sqrt{6}\sqrt{14}} = \frac{\sqrt{21}}{6}.$$

9. (6 Points) Match the following equations with their graphs.

- (a)  $z = x^2 - y^2$ .  
 (b)  $x^2 + z^2 - 1 = 0$ .  
 (c)  $x^2 + y^2 + \frac{z^2}{4} = 1$ .



**Solution:** (a) and III, (b) and VI, (c) and IX.

10. (6 Points) Change the point  $(2, -2, 2\sqrt{2})$  in rectangular coordinates to spherical coordinates. (This problem refers to the material not covered before midterm 1.)  
 11. (6 Points) Change the equation  $r - z = 1$  in cylindrical coordinates into rectangular coordinates. (This problem refers to the material not covered before midterm 1.)  
 12. (8 Points) A particle moves with position function  $\mathbf{r}(t) = \langle t, t^2, 3t \rangle$ . Find the tangential component of acceleration.

**Solution:** The  $a_{\mathbf{T}}(t)$  is equal to

$$a_{\mathbf{T}}(t) = \frac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|}.$$

Since  $\mathbf{r}'(t) = \langle 1, 2t, 3 \rangle$  and  $\mathbf{r}''(t) = \langle 0, 2, 0 \rangle$  and  $|\mathbf{r}'(t)| = \sqrt{10 + 4t^2}$ , one has

$$a_{\mathbf{T}} = \frac{\langle 1, 2t, 3 \rangle \bullet \langle 0, 2, 0 \rangle}{\sqrt{10 + 4t^2}} = \frac{4t}{\sqrt{10 + 4t^2}}.$$

13. (8 Points) Consider the curve  $\mathbf{r}(t) = 3 \sin t \mathbf{i} + 4t \mathbf{j} + 3 \cos t \mathbf{k}$ . The unit tangent vector  $\mathbf{T}(t) = \frac{3}{5} \cos t \mathbf{i} + \frac{4}{5} \mathbf{j} - \frac{3}{5} \sin t \mathbf{k}$  is given. Find the curvature.

**Solution:**  $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ . Since  $\mathbf{r}'(t) = 3 \cos t \mathbf{i} + 4 \mathbf{j} - 3 \sin t \mathbf{k}$ ,  $|\mathbf{r}'(t)| = \sqrt{3^2 \cos^2 t + 16 + (-3)^2 \sin^2 t} = 5$ ,  $\mathbf{T}'(t) = -\frac{3}{5} \sin t \mathbf{i} - \frac{3}{5} \cos t \mathbf{k}$ , and  $|\mathbf{T}'(t)| = \frac{3}{5}$ , the curvature is equal to

$$\kappa(t) = \frac{3}{25}.$$

14. (8 Points) Find the length of the curve  $\mathbf{r}(t) = \langle t^2, 2t, \ln t \rangle$  from the point  $(1, 2, 0)$  to the point  $(e^2, 2e, 1)$ .

**Solution:** At the point  $(1, 2, 0)$ ,  $t = 1$  and at the point  $(e^2, 2e, 1)$ ,  $t = e$ . Since  $\mathbf{r}'(t) = \langle 2t, 2, 1/t \rangle$  and

$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 4 + (1/t)^2} = \sqrt{(2t + 1/t)^2} = (2t + 1/t) = \frac{2t^2 + 1}{t},$$

$$\text{length} = \int_1^e \frac{2t^2 + 1}{t} dt = 2 \int_1^e t dt + \int_1^e 1/t dt = e^2 - 1 + 1 = e^2.$$

15. (8 Points) Consider a particle whose acceleration is given by

$$\mathbf{a}(t) = \langle t, t^2, \cos 2t \rangle$$

with initial velocity  $\mathbf{v}(0) = \langle 1, 0, 1 \rangle$ . Find the velocity of the particle as a function of time.

**Solution:**

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s) ds = \langle 1, 0, 1 \rangle + \int_0^t \langle s, s^2, \cos 2s \rangle ds \\ &= \langle 1, 0, 1 \rangle + \left\langle \int_0^t s ds, \int_0^t s^2 ds, \int_0^t \cos 2s ds \right\rangle \\ &= \langle 1, 0, 1 \rangle + \langle t^2/2, t^3/3, \sin 2t/2 \rangle = \langle 1 + t^2/2, t^3/3, 1 + \sin 2t/2 \rangle. \end{aligned}$$

### Midterm Exam I, Calculus III, Sample C

1. (10 Points) Find an equation of the sphere that has center at  $(1, -2, -5)$  and that passes through the origin.

**Solution:** The radius of the sphere is equal to  $r = \sqrt{1^2 + (-2)^2 + (-5)^2} = \sqrt{30}$ . So, the equation of the sphere is given by

$$(x - 1)^2 + (y + 2)^2 + (z + 5)^2 = 30.$$

2. (6 Points) Find a vector  $\mathbf{u}$  in the opposite direction as  $\mathbf{v} = \langle -5, 3, 7 \rangle$ , and has length 6.

**Solution:**  $\mathbf{u} = -6 \frac{\mathbf{v}}{|\mathbf{v}|}$ . Since  $|\mathbf{v}| = \sqrt{(-5)^2 + 3^2 + 7^2} = \sqrt{83}$ ,

$$\mathbf{u} = \left\langle \frac{30}{\sqrt{83}}, -\frac{18}{\sqrt{83}}, -\frac{42}{\sqrt{83}} \right\rangle.$$

3. Let  $A = (1, 0, 0)$ ,  $B = (1, 2, 2)$ ,  $C = (3, 0, 1)$ .

- (a) (6 Points) Find the area of triangle  $ABC$   
(b) (6 Points) Find the equation of the plane passing through  $A$ ,  $B$ , and  $C$  (Write the answer in the form  $ax + by + cz = d$ ).

**Solution:** (a) The area of the triangle  $\triangle ABC$  is equal to  $\frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$ . Since  $\overrightarrow{AB} = \langle 0, 2, 2 \rangle$  and  $\overrightarrow{AC} = \langle 2, 0, 1 \rangle$ ,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \langle 2, 4, -4 \rangle$$

and  $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{2^2 + 4^2 + (-4)^2} = 6$ . So, the area of the  $\triangle ABC$  is equal to 3.

(b) A normal vector to the plane is  $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 2, 4, -4 \rangle$  and the plane is passing through the point  $A = (1, 0, 0)$ . So, the equation of the plane is given by  $2(x - 1) + 4y - 4z = 0$ , i.e.,  $x + 2y - 2z = 1$ .

4. (a) (6 Points) Find parametric equations of the line which passes through the point  $(1, 1, 1)$  and is perpendicular to the plane  $2x + y + z = 0$ .  
(b) (6 Points) Find the equation of the plane passing through the point  $(1, 1, 1)$  and is parallel to the plane  $2x + y + z = 0$ .

**Solution:** (a) A normal vector to the plane is given by  $\langle 2, 1, 1 \rangle$ . So, the parametric equations of the line which passes through the point  $(1, 1, 1)$  are

$$x = 2t + 1, \quad y = t + 1, \quad z = t + 1.$$

(b) A normal vector to the plane is given by  $\langle 2, 1, 1 \rangle$  and since it passes through the point  $(1, 1, 1)$  its equation is given by  $2(x - 1) + (y - 1) + (z - 1) = 0$ , i.e.,  $2x + y + z = 4$ .



5. (3 Points) The ruling (lines on a cylinder) of the cylinder defined by equation  $y = z^2$  is perpendicular to (check one)

$xy$ -plane     $yz$ -plane     $xz$ -plane.

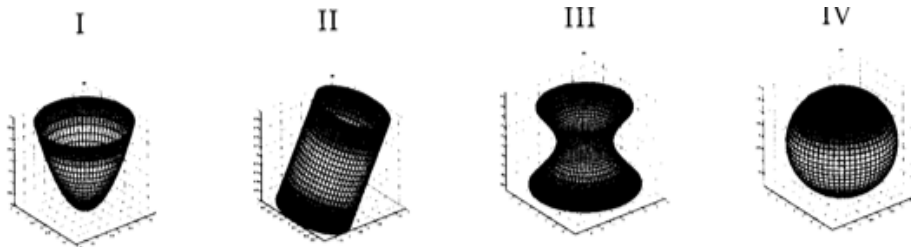
**Solution:** The surface is a parabolic cylinder. The lines parallel to the  $x$ -axis lie on the surface. So, the ruling is perpendicular to the  $yz$ -plane.

6. (5 Points) Change the following cylindrical equation to a rectangular equation, and identify the surface.

$$r^2 + z^2 - 2r \sin \theta = 0.$$

(This problem refers to the material not covered before midterm 1.)

7. (4 Points) The point  $(-1, 1, \sqrt{6})$  is given in rectangular coordinates. Find the cylindrical and spherical coordinates for this point. (This problem refers to the material not covered before midterm 1.)
8. A certain surface is defined by equation  $(y - z)^2 + x^2 = 1$
- (a) (4 Points) Find and sketch the traces on the horizontal planes  $z = -1, 0, 1$
- (b) (4 Points) Which one of the figures best match with this equations? Circle the surface.



**Solution:** (a) For a fixed value of  $z$ , an equation  $(y - z)^2 + x^2 = 1$  is the equation of the circle on the  $xy$ -plane having center at  $(0, z)$  and radius 1. The traces are illustrated in Figure 1. (b) The figure II is the best match for this equation. The surface II is the only surface that has circles of the same size with shifting centers as its horizontal traces.

9. (10 Points) Find symmetric equations of the tangent line to the curve given by the vector function

$$\mathbf{r}(t) = \langle t^2 + 4t, t^3 + 3 \sin t, t^4 + e^{2t} \rangle.$$

at the point  $P = (0, 0, 1)$ .

**Solution:** First find the values of  $t$  for which  $\mathbf{r}(t) = \langle 0, 0, 1 \rangle$ , i.e.,  $t^2 + 4t = 0$ ,  $t^3 + 3 \sin t = 0$ ,

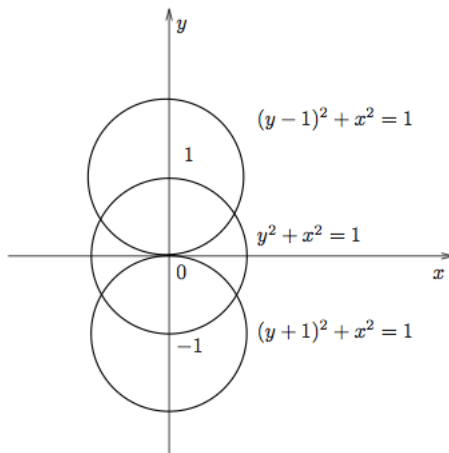


Figure 1: The traces of the surface  $(y - z)^2 + x^2 = 1$  on the horizontal planes  $z = -1, 0, 1$ .

and  $t^4 + e^{2t} = 1$ . The first equation gives  $t = 0$  or  $t = -4$ . Two other equations are satisfied if  $t = 0$ . If  $t = -4$ , then  $(-4)^4 + e^{-2 \cdot 4} > 4^4 > 1$  so that the third equations is not satisfied by  $t = -4$  (the same is true for the second equation). Next  $\mathbf{r}'(t) = \langle 2t + 4, 3t^2 + 3 \cos t, 4t^3 + 2e^{2t} \rangle$  so that  $\mathbf{r}'(0) = \langle 4, 3, 2 \rangle$  and since the line is passing through  $P = (0, 0, 1)$ , the symmetric equations of the line are

$$\frac{x}{4} = \frac{y}{3} = \frac{z - 1}{2}.$$

10. (10 Points) Given the vector function

$$\mathbf{r}(t) = \langle t \sin t + \cos t, \sin t - t \cos t, t^2 \rangle,$$

find the length of its curve for  $0 \leq t \leq 5$ .

Since  $\mathbf{r}'(t) = \langle t \cos t, t \sin t, 2t \rangle$  and  $|\mathbf{r}'(t)| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t + (2t)^2} = \sqrt{5}t$ , the length of the curve is equal to

$$\int_0^5 \sqrt{5}t \, dt = \frac{25\sqrt{5}}{2}.$$

11. Consider a curve

$$x = e^t \sin t, \quad y = e^t \cos t, \quad z = e^t, \quad -2\pi \leq t \leq 2\pi.$$

- (2 Points) Find the unit tangent vector  $\mathbf{T}(t)$ .
- (3 Points) Find the principal unit normal  $\mathbf{N}(t)$ .
- (3 Points) Find the binormal vector  $\mathbf{B}(t)$ .
- (2 Points) Find the equation of the normal plane at  $t = 0$ .

(e) (2 Points) Find the equation of the osculating plane at  $t = 0$ .

**Solution:** (a)  $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle$ . So,  $\mathbf{r}'(t) = \langle e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t, e^t \rangle$  and  $|\mathbf{r}'(t)| = e^t \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 1} = e^t \sqrt{3}$ . Hence the unit tangent vector is equal to

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{3}} \langle \sin t + \cos t, \cos t - \sin t, 1 \rangle.$$

(b)  $\mathbf{T}'(t) = \frac{1}{\sqrt{3}} \langle \cos t - \sin t, -\sin t - \cos t, 0 \rangle$ ,  $|\mathbf{T}'(t)| = \frac{1}{\sqrt{3}} \sqrt{(\cos t - \sin t)^2 + (-\sin t - \cos t)^2} = \frac{\sqrt{2}}{\sqrt{3}}$ . So, the principal unit normal is equal to

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{2}} \langle \cos t - \sin t, -\sin t - \cos t, 0 \rangle.$$

(c) The binormal vector is

$$\begin{aligned} \mathbf{B}(t) = \mathbf{T} \times \mathbf{N}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}}(\sin t + \cos t) & \frac{1}{\sqrt{3}}(\cos t - \sin t) & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}}(\cos t - \sin t) & -\frac{1}{\sqrt{2}}(\sin t + \cos t) & 0 \end{vmatrix} \\ &= \frac{1}{\sqrt{6}}(\sin t + \cos t)\mathbf{i} + \frac{1}{\sqrt{6}}(\cos t - \sin t)\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}. \end{aligned}$$

(d) The vector  $\mathbf{r}'(0) = \langle 1, 1, 1 \rangle$  is normal to the normal plane at  $t = 0$ . Also  $\mathbf{r}(0) = \langle 0, 1, 1 \rangle$  so that the plane is passing through the point  $(0, 1, 1)$ . Thus, the equation of the normal plane at  $t = 0$  is

$$x + (y - 1) + (z - 1) = 0, \quad \text{i.e., } x + y + z = 2.$$

(e) The osculating plane at  $t = 0$  contains vectors  $\mathbf{T}(0)$  and  $\mathbf{N}(0)$ , so a normal vector to the osculating plane is parallel to  $\mathbf{T}(0) \times \mathbf{N}(0) = \mathbf{B}(0) = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k} = \frac{1}{\sqrt{6}}\langle 1, 1, -2 \rangle$ . A simpler normal vector is  $\langle 1, 1, -2 \rangle$ . An equation of the osculating plane at  $t = 0$  is

$$x + (y - 1) - 2(z - 1) = 0, \quad \text{i.e., } z + y - 2x = -1.$$

12. (8 Points) A particle moves with position function

$$\mathbf{r}(t) = \langle e^t, \sqrt{2}t, e^{-t} \rangle$$

Find the tangential and the normal components  $a_{\mathbf{T}}(t)$  and  $a_{\mathbf{N}}(t)$  of acceleration.

**Solution:** The tangential component  $a_{\mathbf{T}}(t)$  and normal component  $a_{\mathbf{N}}(t)$  of acceleration ( $t$ ) are given by

$$a_{\mathbf{T}}(t) = \frac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \quad \text{and} \quad a_{\mathbf{N}}(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

Since  $\mathbf{r}'(t) = \langle e^t, \sqrt{2}, -e^{-t} \rangle$  and  $\mathbf{r}''(t) = \langle e^t, 0, e^{-t} \rangle$  and  $|\mathbf{r}'(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ , one has

$$a_T = \frac{\langle e^t, \sqrt{2}, -e^{-t} \rangle \bullet \langle e^t, 0, e^{-t} \rangle}{\sqrt{e^{2t} + 2 + e^{-2t}}} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + 2 + e^{-2t}}} = \frac{e^{2t} - e^{-2t}}{\sqrt{(e^t + e^{-t})^2}} = \frac{e^{2t} - e^{-2t}}{e^t + e^{-t}} = e^t - e^{-t}.$$

Since

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & \sqrt{2} & -e^{-t} \\ e^t & 0 & e^{-t} \end{vmatrix} = \sqrt{2}e^{-t}\mathbf{i} - 2\mathbf{j} - \sqrt{2}e^t\mathbf{k} = \sqrt{2}\langle e^{-t}, -\sqrt{2}, -e^t \rangle$$

and  $|\sqrt{2}\langle e^{-t}, 0, e^t \rangle| = \sqrt{2}\sqrt{e^{-2t} + 2e^{2t}} = \sqrt{2}\sqrt{(e^{-t} + e^t)^2} = \sqrt{2}(e^{-t} + e^t)$ , one gets

$$a_N = \frac{\sqrt{2}(e^{-t} + e^t)}{e^t + e^{-t}} = \sqrt{2}.$$

## Midterm Exam I, Calculus III, Sample D

1. (10 Points) Find an equation for the plane consisting of all points that are equidistant from  $P = (-1, 2, -3)$  and  $Q = (4, -2, 2)$ .

**Solution:** The midpoint of  $P$  and  $Q$  lies on the plane and is equal to  $\frac{1}{2}(-1, 2, -3) + (4, -2, 2) = \frac{1}{2}(3, 0, -1)$ . A normal vector to the plane is given by  $\langle 4, -2, 2 \rangle - \langle -1, 2, -3 \rangle = \langle 5, -4, 5 \rangle$ . So, an equation of the plane is given by  $5(x - 3/2) - 4(y - 0) + 5(z + 1/2) = 0$ , that is,  $5x - 4y + 5z = 5$ .

2. (8 Points) Find all values of  $x$  such that the vectors  $\langle x, -3, x \rangle$  and  $\langle -5, 2, x \rangle$  are orthogonal.

**Solution:** The  $\langle x, -3, x \rangle$  and  $\langle -5, 2, x \rangle$  are orthogonal if the dot product  $\langle x, -3, x \rangle \cdot \langle -5, 2, x \rangle = -5x - 6 + x^2 = 0$ . The equation  $x^2 - 5x - 6 = 0$  has two solutions  $x_1 = \frac{5 - \sqrt{(-5)^2 - 4 \cdot (-6)}}{2} = -1$  and  $x_2 = \frac{5 + \sqrt{(-5)^2 - 4 \cdot (-6)}}{2} = 6$ . So, the vectors are orthogonal if  $x = -1$  or  $x = 6$ .

3. (8 Points for each part) For each of the following pairs of planes  $P_1, P_2$ , determine whether  $P_1$  and  $P_2$  are parallel or intersect. If the planes are parallel, *explain and find the distance between them*; if the planes intersect, *find the line of intersection*.

$$\begin{aligned} P_1 : \quad x + 2y - 4z &= 2 \\ P_2 : \quad -2x - 4y + 8z &= 1 \end{aligned} \tag{1}$$

$$\begin{aligned} P_1 : \quad x + 2y - 4z &= 2 \\ P_2 : \quad 2x + y + z &= 1 \end{aligned} \tag{2}$$

**Solution:** (1) The normal vectors to  $P_1$  and  $P_2$  are equal to  $\mathbf{n}_1 = \langle 1, 2, -4 \rangle$  and  $\mathbf{n}_2 = \langle -2, -4, 8 \rangle$ , respectively. Since  $\langle -2, -4, 8 \rangle = (-2) \cdot \langle 1, 2, -4 \rangle$ , the vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are parallel and so the planes are parallel. To find the distance between  $P_1$  and  $P_2$ , put, for example,  $x = 0$  and  $y = 0$  in the equation for  $P_1$  to get  $z = -1/2$ . So the point  $(0, 0, -1/2)$  lies on the plane  $P_1$ . Now calculate the distance between the point  $(0, 0, -1/2)$  and the plane  $P_2$ . Recall that given the plane, then the distance of  $(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  is equal to

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

So, in our case the distance  $(0, 0, -1/2)$  to  $P_2$  is equal to

$$D = \frac{|-1/2 - 1|}{\sqrt{(-2)^2 + 1^2 + 1^2}} = \frac{3}{2\sqrt{6}} = \frac{\sqrt{6}}{4}.$$

(2) The normal vectors to  $P_1$  and  $P_2$  are equal to  $\mathbf{n}_1 = \langle 1, 2, -4 \rangle$  and  $\mathbf{n}_2 = \langle 2, 1, 1 \rangle$ . There is no number  $c$  so that  $\mathbf{n}_1 = c\mathbf{n}_2$ . So,  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are not parallel the planes intersect along the line  $L$ . The line is perpendicular to  $\mathbf{n}_1$  and  $\mathbf{n}_2$ . Hence the vector

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 2, -4 \rangle \times \langle 2, 1, 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -4 \\ 2 & 1 & 1 \end{vmatrix} = \langle 6, -9, -3 \rangle$$

is parallel to the line  $L$ . One needs a point on  $L$ . Setting  $z = 0$  in both equation gives the point where  $L$  intersects the  $xy$ -plane. One gets  $x + 2y = 2$  and  $2x + y = 1$  whose solution is  $x = 0$  and  $y = 1$ . So,  $(0, 1, 0)$  lies on  $L$ . The parametric equations of  $L$  are given by

$$x = 6t, \quad y = -9t + 1, \quad z = -3t.$$

(a) (6 Points) Sketch a graph of the surface

$$x^2 + \frac{y^2}{4} - z = -1.$$

Find and label the points at which the surface intersects the  $x$ -axis.

(b) (6 Points) Find the equation of the curve of intersection  $C$  of the surface in part (a) and the plane  $y = -2$ .

**Solution:** (a) and (b). The surface given by  $x^2 + \frac{y^2}{4} - z = -1$  is an elliptic paraboloid with vertex at the point  $(0, 0, 1)$ . The curve of the intersection of the surface with the plane  $y = -2$  is a parabola with equation  $z = x^2 + 2$ .

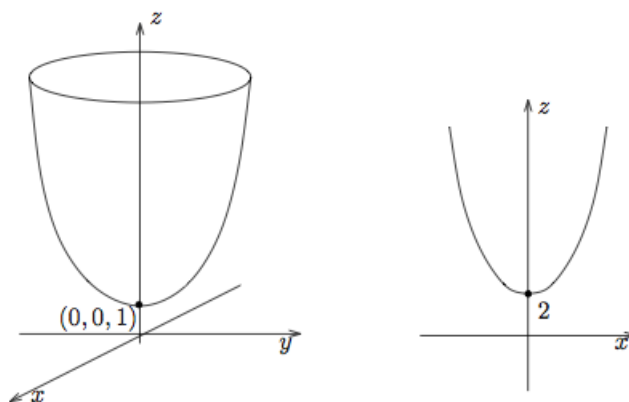


Figure 2: Elliptic paraboloid  $x^2 + y^2/4 - z = -1$ . The surface intersects the  $z$  axis at the point  $(0, 0, 1)$ . The intersection of the surface with the plane  $y = -2$  is the parabola with equation  $z = x^2 + 2$ .

4. Let  $C$  be the space curve traced by the vector-valued function

$$\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + 3t \mathbf{k}.$$

- (a) (8 Points) Find the equation of the line tangent to the  $C$  at the point  $(4, 0, 0)$ .
- (b) (8 Points) Find the curvature  $\kappa(t)$  at the point  $(-4, 0, 3\pi)$ .
- (c) (8 Points) Find the unit tangent vector  $\mathbf{T}(t)$ , the unit normal vector  $\mathbf{N}(t)$ , and binormal vector  $\mathbf{B}(t)$  at the point  $(-4, 0, 3\pi)$ .

(d) (8 Points) Find the length of the curve from  $(4, 0, 0)$  to  $(4, 0, 6\pi)$ .

**Solution:** (a) At  $(4, 0, 0)$ ,  $t = 0$ . Since  $\mathbf{r}'(t) = \langle -4 \sin t, 4 \cos t, 3 \rangle$ ,  $\mathbf{r}'(0) = \langle 0, 4, 3 \rangle$ . So, the parametric equations of the tangent line to the curve  $C$  at the point  $(4, 0, 0)$  are

$$x = 4, \quad y = 4s, \quad z = 3s.$$

(b)  $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ . Since  $\mathbf{r}''(t) = \langle -4 \cos t, -4 \sin t, 0 \rangle$ ,  $|\mathbf{r}''(t)| = 4$ , and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 \sin t & 4 \cos t & 3 \\ -4 \cos t & -4 \sin t & 0 \end{vmatrix} = 12 \sin t \mathbf{i} - 12 \cos t \mathbf{j} + 16 \mathbf{k}.$$

Then  $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$ . Since  $|\mathbf{r}'(t)| = \sqrt{16 + 9} = 5$ ,  $\kappa(t) = \frac{20}{5^3} = \frac{4}{25}$ .

(c) At the point  $(-4, 0, 3\pi)$ ,  $t = \pi$ . Since  $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{5} \langle -4 \sin t, 4 \cos t, 3 \rangle$ ,  $\mathbf{T}(\pi) = \frac{1}{5} \langle 0, -4, 3 \rangle$ . Since  $\mathbf{T}'(t) = \frac{1}{5} \langle -4 \cos t, -4 \sin t, 0 \rangle$  and  $|\mathbf{T}'(t)| = 4/5$ , the unit normal vector is equal to

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{4} \langle -4 \cos t, -4 \sin t, 0 \rangle = \langle -\cos t, -\sin t, 0 \rangle \quad \text{and} \quad \mathbf{N}(\pi) = \langle 1, 0, 0 \rangle.$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \end{vmatrix} = \frac{3}{5} \mathbf{j} + \frac{4}{5} \mathbf{k}.$$

(d) At  $(4, 0, 0)$ ,  $t = 0$ , and at  $(4, 0, 6\pi)$ ,  $t = 2\pi$ . Since  $|\mathbf{r}'(t)| = 5$ ,

$$\text{length} = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} 5 dt = 10\pi.$$

5. Let

$$\mathbf{v}(t) = \langle 4t - 3, 3e^t, 4 \cos 2t \rangle$$

be the velocity vector of a particle at time  $t$ .

(a) (6 Points) Find the acceleration vector  $\mathbf{a}(t)$ .

(b) (6 Points) Find the position vector  $\mathbf{r}(t)$  if the particle has initial position  $\mathbf{r}(0) = \langle 0, 1, 2 \rangle$ .

**Solution:** (a)  $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 4, 3e^t, -8 \sin 2t \rangle$ .

(b)

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(s) ds = \langle 0, 1, 2 \rangle + \left\langle \int_0^t (4s - 3) ds, \int_0^t 3e^s ds, \int_0^t 4 \cos 2s ds \right\rangle \\ &= \langle 0, 1, 2 \rangle + \langle 2t^2 - 3t, 3e^t - 3, 2 \sin 2t \rangle = \langle 2t^2 - 3t, 3e^t - 2, 2 \sin 2t + 2 \rangle. \end{aligned}$$

6. (10 Points) A particle moves with position function

$$\mathbf{r}(t) = \langle t^2 + 1, t, 3t - 1 \rangle.$$

Find the tangential and normal components of acceleration at the point  $(2, 1, 2)$ .

**Solution:** The tangential component  $a_{\mathbf{T}}(t)$  and normal component  $a_{\mathbf{N}}(t)$  are given by

$$a_{\mathbf{T}}(t) = \frac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \quad \text{and} \quad a_{\mathbf{N}}(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

At the point  $(2, 1, 2)$ ,  $t = 1$ . Since  $\mathbf{r}'(t) = \langle 2t, 1, 3 \rangle$  and  $\mathbf{r}''(t) = \langle 2, 0, 0 \rangle$ ,  $\mathbf{r}'(1) = \langle 2, 1, 3 \rangle$ ,  $\mathbf{r}''(1) = \langle 2, 0, 0 \rangle$  and  $|\mathbf{r}'(1)| = \sqrt{4 + 1 + 9} = \sqrt{14}$ . So,

$$a_T = \frac{\langle 2, 1, 3 \rangle \bullet \langle 2, 0, 0 \rangle}{\sqrt{14}} = \frac{4}{\sqrt{14}}.$$

Since

$$\langle 2, 1, 3 \rangle \times \langle 2, 0, 0 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 2 & 0 & 0 \end{vmatrix} = 6\mathbf{j} - 2\mathbf{k}$$

and  $|6\mathbf{j} - 2\mathbf{k}| = \sqrt{36 + 4} = 2\sqrt{10}$ , one gets

$$a_N = \frac{2\sqrt{10}}{\sqrt{14}}.$$