Mixing Business Cards in a Box

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I. Abstract

How long does it take to physically mix cards in a box into a random permutation?

II. Introduction

Imagine you want to run a lottery with a number of names, and you do this by writing names business cards, putting them in a box, and then shaking it for a certain amount of time. Then reach in and draw out the cards. For fairness of the lottery, you hope that cards are now in a random order. In practice, however, depends on how many cards are there in total, it might need a really long time to mix all the cards. Therefore, we conduct some physical experiments to estimate how the number of shakes needed to mix grows with the number of cards.

Throughout the following analysis, let:

n := number of cards in the box;

p(i) := the final position of the card that started at position i.

III. Experiment

1. Materials

i. Business card (89*51mm, coated paper) [Figure 1]

ii. Cardboard box [Figure 2]

Outside dimension (length × width × height): 310 × 235 × 242 mm.

Inside dimension (length × width × height): 298 × 222 × 230 mm.[1]

[1] Therefore, theoretically, this box can fit around 9060 business cards.

Figure 1

Figure 2

2. Mixing Procedure

i. Label the cards sequentially from 1 to n and put them in increasing order into the right bottom corner of the box.

ii. Since the bottom of the box has dimension equals 310 * 235 mm, when shaking it, restrict to a rectangular area with each side equals to three times of the length corresponding side of the box (i.e. 930 * 705mm).

iii. Move the box forward and back, from left to right and then rotate along the side of length equals 310 mm. The movement in each direction is repeated with a certain amount of times.

iv. Gather all the cards together to the right bottom corner again by tilting up the left upper corner of the box and shake it against the table 10 times.[2]

v. Draw the cards out of the box and record their post-orders.

3. Data collection

i. Dataset 1:

Take 100 cards (total height around 37 mm, total weight 300g), use the method suggested above, with 10, 20, 40, 60 times of shaking in each direction, and record the post-order of the cards. Repeat each experiment 10 times.

ii. Dataset 2:

Take 200 cards (total height around 74 mm, total weight 600g), use the

[2] In this way, drawing the card out of the box is predetermined once the random shaking process has been done, leaving shaking procedure the box in step3 the random factor in the whole experiment.

method suggested above, with 20, 40, 60[3] times of shaking in each direction, and record the post-order of the cards. Repeat each experiment 10 times.

IV. Theory

This random mixing process could be modeled as a Markov process. Consider each permutation as a state in the Markov chain, then there are totally n! states in the state space. Since, theoretically, we can get from any state to any other state, the chain is irreducible. Therefore, all states are positive recurrent and there must exist a stationary distribution among the states. If the mixing process is unbiased, then the stationary distribution should be the uniform distribution, by which we would be able to conclude that the card would finally be in random order.

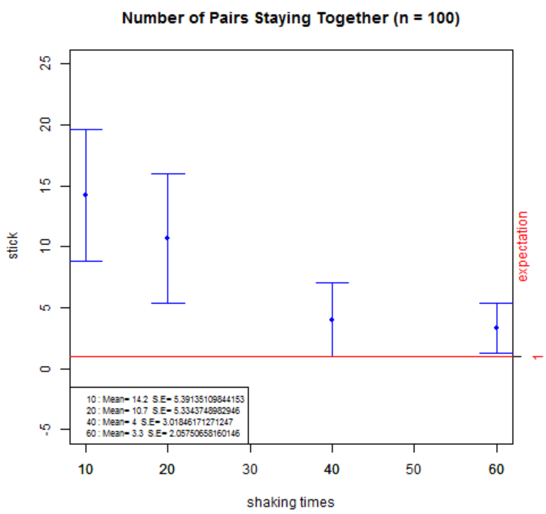
As the number of cards increases from 100 to 200, there will be more states in the Markov process, so we would expect the time it takes for the distribution to converge to be longer.

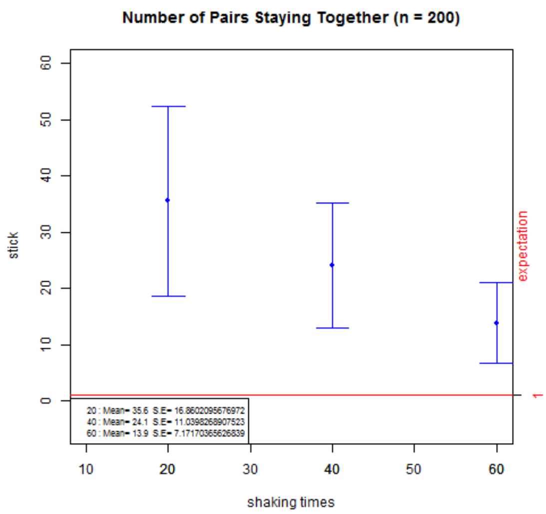
V. Statistics of the Data

In order to assess whether a given permutation is random, we can evaluate the sequence from several aspects, including the number of consecutive pairs, the number of top-half cards that are still in the top-half, the correlation between starting position and final position, etc. The following 8 statistics is provided to see whether the stationary distribution is uniformly distributed. The expectation values are calculated based on uniform distribution.

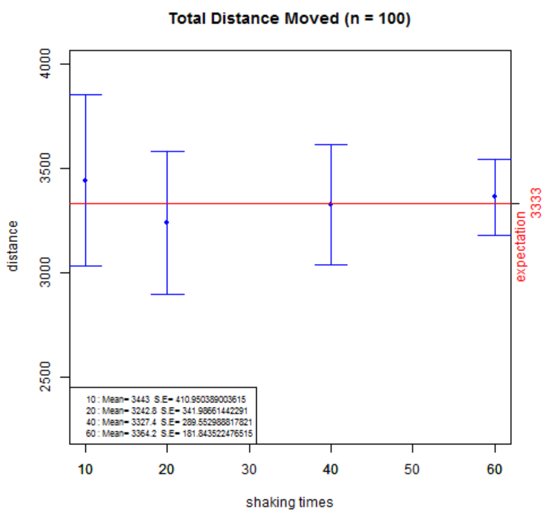
[3] Due to the time sake, 10 times shaking is skipped for n = 200.

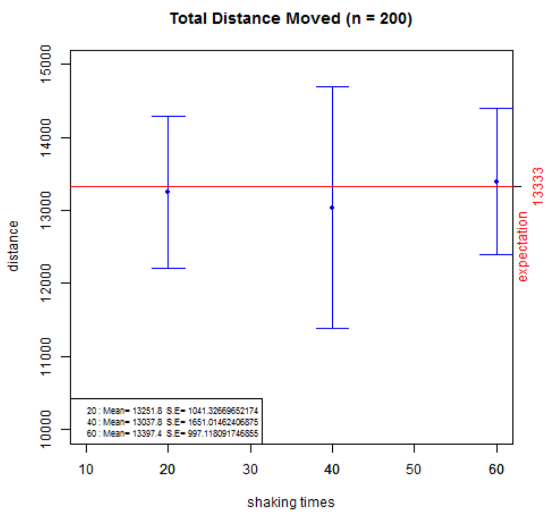
1. Staying together: $\sum_{i=1}^{n-1} \mathbb{1}\{p(i+1) = p(i) + 1\}$



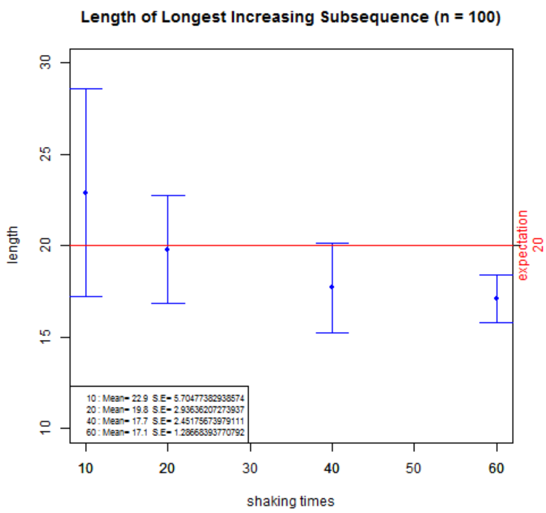


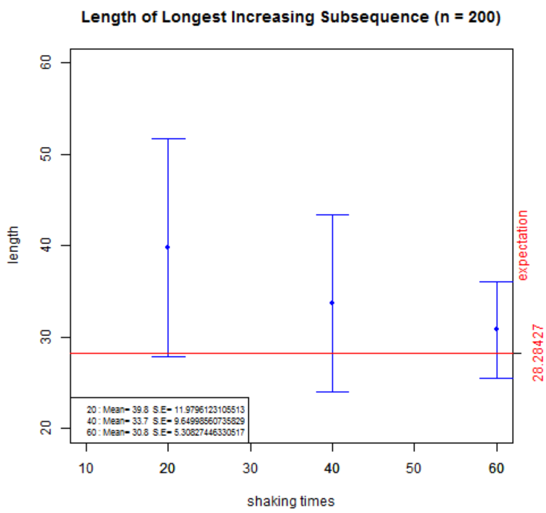
2. Distance moved: $\sum_{i=1}^{n} |p(i) - i|$



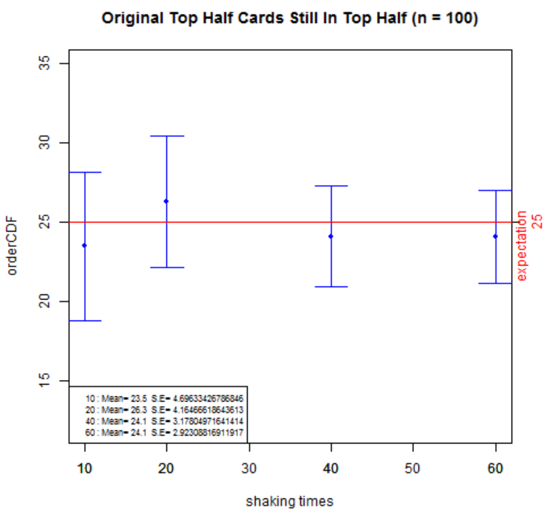


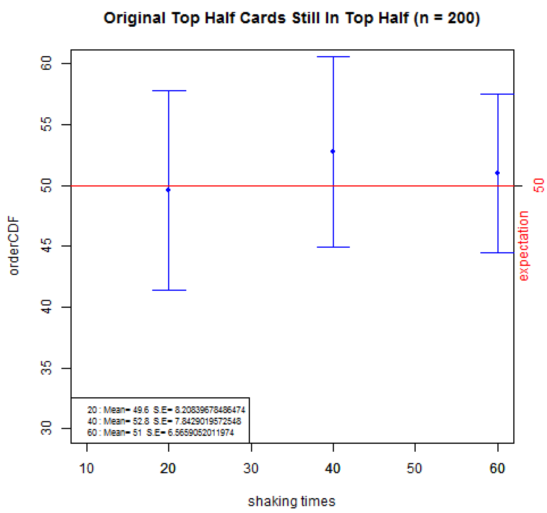
3. Length of the longest increasing subsequence in the post-order





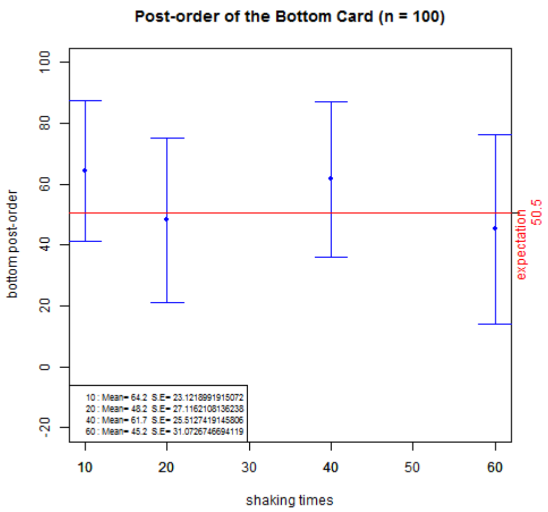
4. Staying in top half: $\sum_{i=1}^{n/2} \mathbb{1}\{p(i) \leq n/2\}$

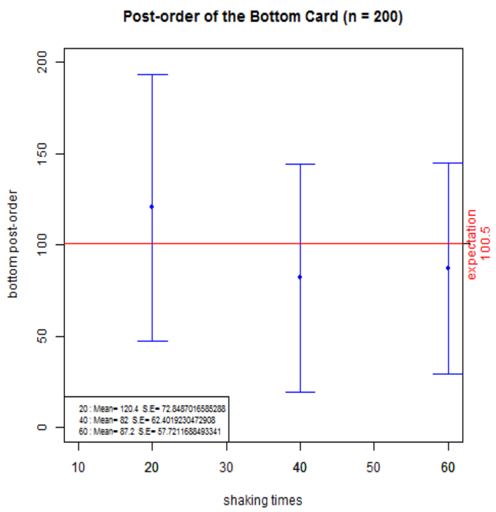




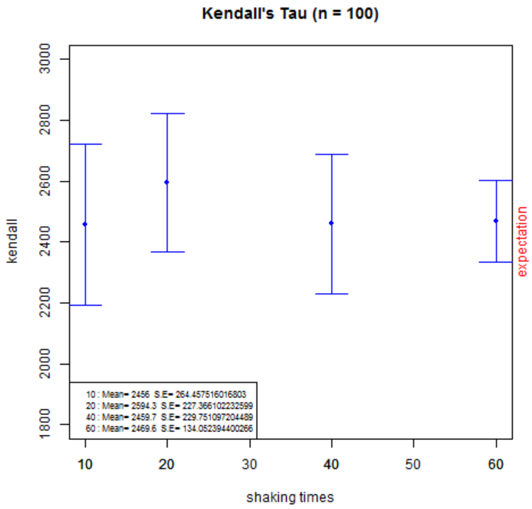
5. Final position of the top card: p(1)

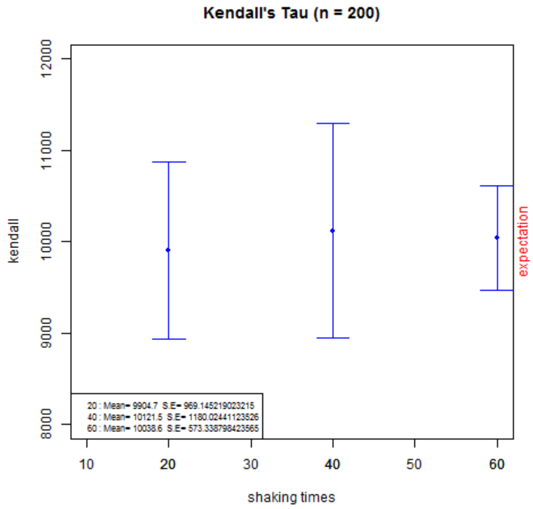
6. Final position of the bottom card: p(n)



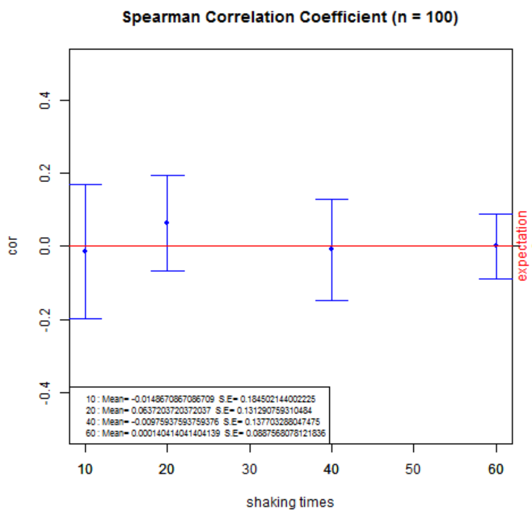


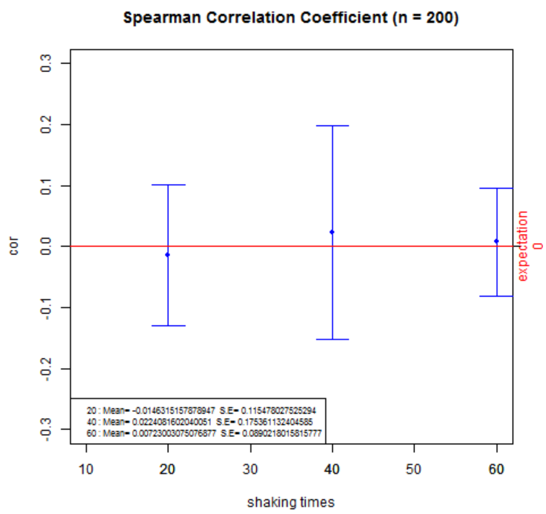
7. Kendall's Tau: $\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{1}\{p(i) < p(j)\}$





8. Spearman correlation coefficient: The usual correlation applied to pairs (i, p(i)).





VI. Analysis

We can see from the previous plots that the distribution of the post-order do seem to converge to the expected uniform distribution. Some statistics, such as number of cards staying together, converge really quickly as time increases. The final position of the top and bottom card is not a robust measure of the randomness of data and we can hardly see any pattern from those plots. It is not surprising as it involves only one data in each experiment and the number of experiment is limited.

Another result could be seen from the plot is that the convergence rate to the stationary distribution decreases as the number of states increases, which confirm our prediction. It also makes sense physically as the number of cards doubles, the volume of the cards also doubles, so the space in the box for the cards to move randomly is reduced. In n = 100 case, when shaking time is 60, the distribution is already close to the uniform distribution, whereas more shakings are needed in n = 200 case to reach the stationary distribution.

VII. Further Discussion

Beside the number of cards and the shaking time, there are, of course, other factors, such as the size of the box or the texture of the card, could also determine whether the card would be in the random ordering. In practice, since these experiments are conducted by human beings, the settings of each experiment, including the velocity to move the box, the force being used to shake the box and the condition of the cards (whether they are brand new or have been worn out), are not exactly the same.

VIII. Conclusion

The mixing method provided in this experiment is likely to put cards in to a random order provided shaking times in each direction is large enough. Therefore, if we have 100 candidates for the lottery, following this mixing procedure with shaking times equals to 60 in each direction would put the cards close to a uniform distribution. If we have around 200 people in the lottery, then more shaking time is needed in order to make the lottery a fair game.