MODEL ORDER REDUCTION IN STRUCTURAL DYNAMICS

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Abstract. *Frequency response analysis in structural dynamics usually requires solving large dynamical systems of the form $(-\omega^2\mathbf{M} + i\omega\mathbf{D} + \mathbf{K})\mathbf{u}(\omega) = \mathbf{f}(\omega)$, which result from a FE discretization. A straightforward solution of big systems requires a high computational cost; therefore several Model Order Reduction (MOR) techniques have been developed in the last decades to obtain faster and efficient results. Between them interpolatory approaches have gained importance for solving second order dynamical systems. This work presents and compares ten MOR techniques which are suitable for structural dynamics problems. These are: Guyan-Irons Reduction, Improved Reduction System, Dynamic Reduction, Real Modal Analysis, Complex Modal Analysis, Craig-Bampton Method, and Interpolatory MOR methods like Multi-point Padé Approximation, the Krylov-based Galerkin Projection and the Derivative-based Galerkin Projection. A brief summary of the theoretical background is presented for each method. A first numerical example shows the applicability for damped systems and a second example shows suitability of the Interpolatory MOR methods for industrial applications, using data from a commercial FE software (ANSYS®).*

1 INTRODUCTION

The equation of motion resulting from a finite element discretization of a mechanical system is given by

$$\mathbf{M}\ddot{\boldsymbol{u}}(t) + \mathbf{D}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \boldsymbol{f}(t) \tag{1}$$

Applying a Fourier transformation to eq. (1), with $\mathbf{u}(\omega) = \mathcal{F}\big(\boldsymbol{u}(t)\big)$ and $\mathbf{f}(\omega) = \mathcal{F}\big(\boldsymbol{f}(t)\big)$ leads to

$$(-\omega_j{}^2\mathbf{M} + i\omega_j\mathbf{D} + \mathbf{K})\mathbf{u}(\omega_j) = \mathbf{f}(\omega_j) \quad j = 1, 2, 3...n_\omega \tag{2}$$

which is the equation of motion in the frequency domain, where $\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{N \times N}$ are the mass, damping and stiffness matrix respectively, $\mathbf{f} \in \mathbb{R}^{N \times 1}$ is the load vector (force vector) and $\mathbf{u} \in \mathbb{R}^{N \times 1}$ is the displacement vector. n_ω is the number of times that $\mathbf{u}(\omega)$ is computed in eq. (2) in the scope of frequency response analysis. If n_ω and N are large numbers solving eq.(2) for $\mathbf{u}(\omega_j)$ becomes computationally expensive, in these cases one may search for a reduced-order model which would lead to a lower computational time. Such a reduced-order model is achieved using a suitable MOR technique. The main idea of MOR techniques is to find a vector space spanned by the columns of $\mathbf{V} \in \mathbb{C}^{N \times n_r}$, with $n_r \ll N$, which maps a reduced set of degrees of freedom (dofs) $\mathbf{u}_r \in \mathbb{C}^{n_r \times 1}$ into the original set of degrees of freedom $\mathbf{u}$, such that

$$\mathbf{u} \approx \tilde{\mathbf{u}} = \mathbf{V}\mathbf{u}_r \tag{3}$$

where $\tilde{\mathbf{u}}$ is the approximation of $\mathbf{u}$. In the following the approximation will not be labeled anymore. By substituting the approximation of $\mathbf{u}$ given by eq. (3) in eq. (2), there exists an error $\mathbf{e}_r$ defined by

$$\mathbf{e}_r = (-\omega^2\mathbf{M} + i\omega\mathbf{D} + \mathbf{K})\mathbf{V}\mathbf{u}_r - \mathbf{f} \tag{4}$$

This error is not contained in the Ran($\mathbf{V}$), therefore it is said to be in the null space of $\mathbf{V}$, i.e. the transpose of $\mathbf{V}$ is orthogonal to $\mathbf{e}_r$,

$$\mathbf{V}^H\big((-\omega^2\mathbf{M} + i\omega\mathbf{D} + \mathbf{K})\mathbf{V}\mathbf{u}_r - \mathbf{f}\big) = 0. \tag{5}$$

Where H is used for the conjugate transpose of a matrix. Eq. (5) is equivalent to performed Galerkin projection [14] or applied the least square method to the overestimated system of equations resulting from substituting eq. (3) in eq. (2). From eq. (5) the following reduced order model is achieved:

$$(-\omega_j{}^2\mathbf{M}_r + i\omega_j\mathbf{D}_r + \mathbf{K}_r)\mathbf{u}_r(\omega_j) = \mathbf{f}_r(\omega_j) \tag{6}$$

where $\mathbf{M}_r, \mathbf{D}_r, \mathbf{K}_r \in \mathbb{C}^{n_r \times n_r}, \mathbf{f}_r \in \mathbb{C}^{n_r \times 1}$ are defined by

$$\begin{aligned} \mathbf{M}_r &= \mathbf{V}^H\mathbf{M}\mathbf{V} \quad \mathbf{D}_r = \mathbf{V}^H\mathbf{D}\mathbf{V} \quad \mathbf{K}_r = \mathbf{V}^H\mathbf{K}\mathbf{V} \\ \mathbf{f}_r &= \mathbf{V}^H\mathbf{f} \end{aligned} \tag{7}$$

For the construction of that matrix $\mathbf{V}$ different approaches have been proposed in the field of structural mechanics. A classical approach to construct $\mathbf{V}$ is using a modal decomposition; this idea was in the mathematics community since the 18th century and it uses the superposition principle, which was stated by Bernoulli in 1753 [1]. With the development of the finite element method (FEM) the modal decomposition has been extensively used to reduce large numerical models leading to a set of decoupled equations, whose solutions are straightforward. In 1965 Guyan and Irons presented the first condensation method, Guyan-Irons Reduction [2, 3]. A

new paradigm was introduced in 1968 by Craig and Bampton [6], the Craig-Bampton method which introduced the concept of sub structuring, has allowed the achievement of computing large complex systems and is suitable for parallelization. A similar procedure to that of the Guyan-Irons reduction but for dynamical analysis was introduced in 1978 by Leung [4], the dynamic condensation (Dynamic Reduction) which allows to reduce a system about a frequency of interest. In 1991 Blair et. al. [19] presented an iterative process to the already established Improved Reduction System method presented by O'Callahan [17] in 1989. From system and control field the moment matching method, i.e. the Krylov Subspace method was reformulated in 1989 by Craig and Su [7] to solve second order dynamical systems, thus the reduced-order model preserves the second order form of the original model, avoiding the structural dynamics model be transformed into the first order formulation (state space form), which destroys the meaning of the original model. It turns out that the Krylov subspace method results in a very efficient method in computational cost and its parallelization could be easily achieved. In this contribution ten MOR techniques are reviewed and classified as follows

1. Modal Decomposition Methods

    - Real Modal Analysis (RMA)
    - Complex Modal Analysis (CMA)

2. Condensation Methods

    - Guyan-Irons Reduction (GR)
    - Improved Reduction System (IRS)
    - Dynamic Reduction (DR)

3. Component Mode Synthesis Methods

    - Craig-Bampton Method (CB)

4. Interpolatory Methods

    - Multi-point Padé Approximation (mP)
    - Krylov-based Galerkin Projection (KGP)
    - Derivative-based Galerkin Projection (DGP)

The general procedure of a frequency response analysis using MOR techniques is summarized as follows:

1. Define a frequency interval $\Delta_\omega = [\omega_L \ \omega_R]$

2. Define a set of reduced dofs or generalized coordinates $\mathbf{u}_r$ and construct the basis for the matrix $\mathbf{V}$, see secs. 2 - 5, such that $\mathbf{V}\mathbf{u}_r$ approximates the physical coordinates $\mathbf{u}$.

3. Compute the reduced force vector and the reduced matrices of eq. (7)

4. Compute $\mathbf{u}_r(\omega)$ in eq. (6) and use eq.(3) to get $\mathbf{u}$ in the interval Δ_ω.

    (a) Define n_ω frequency points in Δ_ω, those frequency points are denominated ω_j.

(b) Compute $\mathbf{u}_r(\omega_j)$ n_ω times, i.e. for each ω_j

$$\mathbf{u}_r(\omega_j) = \left(-\omega_j^2 \mathbf{M}_r + i\omega_j \mathbf{D}_r + \mathbf{K}_r \right)^{-1} \mathbf{f}_r \quad j = 1, 2, 3, ..., n_\omega \tag{8}$$

(c) Project the vector $\mathbf{u}_r$ into the Ran($\mathbf{V}$) to get the solution $\mathbf{u}(\omega_j)$ of the original model using (3):

$$\mathbf{u}(\omega_j) = \mathbf{V}\mathbf{u}_r(\omega_j) \tag{9}$$

2 MOR: Modal Decomposition Methods

This section presents an overview of the classical modal decomposition methods

- Real Modal Analysis

- Complex Modal Analysis

2.1 Real modal analysis

Real Modal Analysis uses a set of natural vibration forms (mode shapes) of the full-model as the vector space which contains $\mathbf{u}$, i.e. the displacement vector $\mathbf{u}(\omega)$ is approximated by the set of n_r natural modes using the expansion theorem [15]

$$\mathbf{u}(\omega) = \sum_{j=1}^{n_r} \boldsymbol{\phi}_j \eta_j(\omega)$$
$$= [\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_3 \cdots \boldsymbol{\phi}_{n_r}] \boldsymbol{\eta}$$
$$= \boldsymbol{\Phi}\boldsymbol{\eta} \tag{10}$$

where the modal matrix $\boldsymbol{\Phi} \in \mathbb{R}^{N \times n_r}$ and the vector of modal coordinates $\boldsymbol{\eta} \in \mathbb{R}^{n_r \times 1}$ are $\mathbf{V}$ and $\mathbf{u_r}$ in eq. (3) respectively.

The modal matrix $\boldsymbol{\Phi}$ is obtained from the free vibration problem of eq. (2), which appears when $\mathbf{f}$ and $\mathbf{D}$ are equal to zero. Then a generalized eigenvalue problem is formulated:

$$(\mathbf{K} - \omega_j^2 \mathbf{M})\boldsymbol{\phi}_j = 0 \qquad j = 1, 2, 3, ..., n_r \tag{11}$$

where the eigenvalue $\lambda_j = \omega^2$ is the jth natural frequency of the system and the eigenvector $\boldsymbol{\phi}_j \in \mathbb{R}^{N \times 1}$ is the jth mode shape, which is normalized for convenience. The mode shapes are mutually orthogonal to the mass and stiffness matrices, i.e.

$$\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_i = \begin{cases} = 0 & \text{for} \quad j \neq i \quad \text{orthogonality} \\ = m_j & \text{for} \quad j = i \end{cases} \tag{12}$$

$$\boldsymbol{\phi}_j^T \mathbf{K} \boldsymbol{\phi}_i = \begin{cases} = 0 & \text{for} \quad j \neq i \quad \text{orthogonality} \\ = k_j & \text{for} \quad j = i \end{cases} \tag{13}$$

If the damping matrix $\mathbf{D}$ is defined to be proportional to the mass and stiffness matrices, the mode shapes are also mutually orthogonal to the damping matrix $\mathbf{D}$. Thus, due to the orthogonality property of the mode shapes of the modal matrix $\boldsymbol{\Phi}$, the reduced-order model given by eq. (6) results in a set of n_r decoupled equations of the form

$$(-\omega^2 m_j + i\omega c_j + k_j)\eta_j(\omega) = \boldsymbol{\phi}_j^T \mathbf{f}(\omega) \quad j = 1, 2, 3, \ldots, n_r \tag{14}$$

2.2 Complex Modal Analysis

In case of non-proportional damping a reduced decoupled system is obtained using a complex modal analysis, i.e. using complex eigenvectors and eigenvalues. Therefore the second order ordinary differential equations given by eq. (1) is expressed as a 2N-dofs first order differential equations, i. e. the state-space formulation

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}f(t) \tag{15}$$

where the state vector $z \in \mathbb{R}^{2N \times 1}$, the state matrix $\mathbf{A} \in \mathbb{R}^{2N \times 2N}$, the input matrix $\mathbf{B} \in \mathbb{R}^{2N \times N}$, and the input vector $f \in \mathbb{R}^{2N \times 1}$ are defined by

$$z = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \tag{16}$$

Considering the free vibration problem, i.e. the external force vector f equal to zero and using a solution of the form $z = \sum_{k=1}^{n_r} e^{\lambda_k} \hat{\phi}_k$ leads to

$$\lambda_k \hat{\phi}_k = \mathbf{A}\hat{\phi}_k \qquad k = 1, 2, ..., n_r \tag{17}$$

which is the standard eigenvalue problem. $\mathbf{A}$ is a real and nonsymmetric matrix, therefore the $2N$ eigenvalues must either be real or they must occur in complex conjugate pairs and the modal matrix is composed by N eigenvectors and their N complex conjugates. The right eigenvectors (eigenvectors of $\mathbf{A}$) $\hat{\phi}_j$ and $\hat{\phi}_k$ are not orthogonal with respect to the matrix $\mathbf{A}$. Nevertheless the eigenvalues of $\mathbf{A}$ and the eigenvalues of $\mathbf{A}^T$ are the same. Furthermore the right eigenvectors $\hat{\phi}_k$, with $k = 1, 2, ..., n_r$, are biorthogonal with respect to the matrix $\mathbf{A}$ to the left eigenvectors $\hat{\psi}_j$ (eigenvectors of $\mathbf{A}^T$), with $j = 1, 2, ..., n_r$, [16]. For convenience the normalization of the eigenvectors is done such that

$$\hat{\psi}_j^T \hat{\phi}_k = \delta_{kj}, \quad j, k = 1, 2, ..., n_r \tag{18}$$

holds true. Then n_r right eigenvectors of $\mathbf{A}$ are used as a basis for a vector space which contains z approximately, i.e. the expansion theorem permits to represent the state vector z as a combination of the right eigenvectors of $\mathbf{A}$

$$z = \sum_{k=1}^{n_r} \hat{\phi}_k \xi_k(t) \tag{19}$$

or in the frequency domain

$$\mathbf{z} = \sum_{k=1}^{n_r} \hat{\phi}_k \hat{\xi}_k(i\omega) \tag{20}$$

where $\hat{\xi}_k(i\omega) = \mathcal{F}(\xi_k(t))$. Taking Fourier transform of eq. (15) and using eq.(20) leads to

$$i\omega\hat{\xi}_k(i\omega) = \lambda_k \hat{\xi}_k(\omega) + \hat{\psi}^T \mathbf{B}f(i\omega) \tag{21}$$

or

$$\hat{\xi}_k(i\omega) = \frac{\hat{\psi}^T \mathbf{B}f(i\omega)}{i\omega - \lambda_k} \tag{22}$$

where the set of modal coordinates $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, .., \hat{\xi}_{n_r}]^T \in \mathbb{C}^{n_r \times 1}$, and the set of right eigenvectors $\hat{\Phi} = [\hat{\phi}_1, \hat{\phi}_2, ..., \hat{\phi}_{n_r}] \in \mathbb{C}^{2N \times n_r}$ are the reduced set of dofs $\mathbf{u}_r$ and the matrix $\mathbf{V}$ in eq. (3) respectively and the set of left eigenvectors $\hat{\Psi} = [\hat{\psi}_1, \hat{\psi}_2, ..., \hat{\psi}_{n_r}] \in \mathbb{C}^{2N \times n_r}$ is $\mathbf{V}^H$ in eq. (5) and (7).

3 MOR: Condensation Methods

This section addresses the formulation of the following condensation methods:

- Guyan-Irons Reduction

- Improved Reduction System

- Dynamic Reduction

The main approach of the condensation methods is to have a linear dependence between dofs, i.e. if a set of *dependent* dofs, $\mathbf{u}_d \in \mathbb{R}^{(N-n_r)\times 1}$, can be expressed as linear dependent of a set of *active* dofs, $\mathbf{u}_a \in \mathbb{R}^{n_r \times 1}$, then a condensation method can be applied to the system, which leads to a reduced-order model. Assuming the above statement is valid for a structural system, the total displacement vector is divided in two groups:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_a \end{bmatrix} \tag{23}$$

then the linear relation between $\mathbf{u}_a$ and $\mathbf{u}_d$ is created through the linear transformation matrix $\mathbf{T}$ as follows:

$$\mathbf{u}_d = \mathbf{T}\mathbf{u}_a \tag{24}$$

Thus, the total displacement vector is defined by

$$\mathbf{u} = \begin{bmatrix} \mathbf{T} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_a \tag{25}$$

where $[\mathbf{T} \ \mathbf{I}]^T$ and the set of active dofs $\mathbf{u}_a$ are $\mathbf{V}$ and the reduced vector of dofs $\mathbf{u}_r$ in eq. (3) respectively.

3.1 Guyan-Irons Reduction

The Guyan-Iron Reduction [2, 3] neglects the inertial and the damping contributions in eq. (2), i.e. it considers just the static case, for this reason it is called static condensation. Then the static equation system is divided as follows

$$\begin{bmatrix} \mathbf{K}_{dd} & \mathbf{K}_{da} \\ \mathbf{K}_{ad} & \mathbf{K}_{aa} \end{bmatrix} \begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_a \end{bmatrix} = \begin{bmatrix} \mathbf{f}_d \\ \mathbf{f}_a \end{bmatrix} \tag{26}$$

The linear relation between dofs given by eq. (24) is obtained by solving for $\mathbf{u}_d$ in the first row of eq. (26) and is given by

$$\mathbf{u}_d = \mathbf{T}_g \mathbf{u}_a + \mathbf{u}_e \tag{27}$$

where $\mathbf{T}_g$ is the Guyan linear transformation matrix defined by

$$\mathbf{T}_g = -\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} \tag{28}$$

and $\mathbf{u}_e = \mathbf{K}_{dd}^{-1}\mathbf{f}_d$ is the Guyan Reduction-error which would emerge if $\mathbf{f}_d$ is not the zero vector. It is also interpreted as a correction factor which guaranties that the exact solution of the full-order model is recovered when static systems are considered. Thus the displacement vector $\mathbf{u}$ can be written as a function of the active dofs $\mathbf{u}_a$:

$$\begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_a \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_a + \begin{bmatrix} \mathbf{K}_{dd}^{-1}\mathbf{f}_d \\ \mathbf{0} \end{bmatrix} \tag{29}$$

Usually, the set of depended dofs are the force-free ones, i.e. $\mathbf{f}_d = \mathbf{0}$. From (29) the Guyan projection matrix is given by

$$\mathbf{V}_g = \begin{bmatrix} \mathbf{T}_g \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} \\ \mathbf{I} \end{bmatrix} \tag{30}$$

$\mathbf{V}_g$ and $\mathbf{u}_a$ are the matrix $\mathbf{V}$ and the reduced set $\mathbf{u}_r$ in eq. (3) respectively. Therefore the reduced-order model, eq. (6), is given by:

$$(-\omega^2\mathbf{M}_g + i\omega\mathbf{D}_g + \mathbf{K}_g)\mathbf{u}_a = \mathbf{f}_g \tag{31}$$

Applying equations given by (7) the following Guyan reduced matrices are obtained:

$$\begin{aligned}
\mathbf{M}_g = \mathbf{V}_g^T\mathbf{M}\mathbf{V}_g &= \begin{bmatrix} (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{dd} & \mathbf{M}_{da} \\ \mathbf{M}_{ad} & \mathbf{M}_{aa} \end{bmatrix} \begin{bmatrix} (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) \\ \mathbf{I} \end{bmatrix} \\
&= \mathbf{M}_{aa} + (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T\mathbf{M}_{dd}(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) + \mathbf{M}_{ad}(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) + (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T\mathbf{M}_{da} \\
&= \mathbf{M}_{aa} + \mathbf{T}_g^T\mathbf{M}_{dd}\mathbf{T}_g + \mathbf{M}_{ad}\mathbf{T}_g + \mathbf{T}_g^T\mathbf{M}_{da}
\end{aligned} \tag{32}$$

$$\begin{aligned}
\mathbf{D}_g = \mathbf{V}_g^T\mathbf{D}\mathbf{V}_g &= \begin{bmatrix} (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{dd} & \mathbf{D}_{da} \\ \mathbf{D}_{ad} & \mathbf{D}_{aa} \end{bmatrix} \begin{bmatrix} (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) \\ \mathbf{I} \end{bmatrix} \\
&= \mathbf{D}_{aa} + (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T\mathbf{D}_{dd}(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) + \mathbf{D}_{ad}(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) + (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T\mathbf{D}_{da} \\
&= \mathbf{D}_{aa} + \mathbf{T}^T\mathbf{D}_{dd}\mathbf{T}_g + \mathbf{D}_{ad}\mathbf{T}_g + \mathbf{T}_g^T\mathbf{D}_{da}
\end{aligned} \tag{33}$$

$$\begin{aligned}
\mathbf{K}_g = \mathbf{V}_g^T\mathbf{K}\mathbf{V}_g &= \begin{bmatrix} (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{dd} & \mathbf{K}_{da} \\ \mathbf{K}_{ad} & \mathbf{K}_{aa} \end{bmatrix} \begin{bmatrix} (-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) \\ \mathbf{I} \end{bmatrix} \\
&= \mathbf{K}_{aa} + (\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T\mathbf{K}_{dd}\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} + \mathbf{K}_{ad}(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da}) - (\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})^T\mathbf{K}_{da} \\
&= \mathbf{K}_{aa} + \mathbf{T}_g^T\mathbf{K}_{da}
\end{aligned} \tag{34}$$

$$\mathbf{f}_g = \mathbf{V}_g^T\mathbf{f} = \begin{bmatrix} \mathbf{T}_g^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}_d \\ \mathbf{f}_a \end{bmatrix} = \mathbf{T}_g^T\mathbf{f}_d + \mathbf{f}_a = \mathbf{f}_a \tag{35}$$

Usually, to save computational time only the first term of the right hand side of eqs. (32) and (33) are considered to define $\mathbf{M}_{aa}$ and $\mathbf{D}_{aa}$ respectively.

3.2 Improved Reduction System (IRS)

In the derivation of Guyan-Irons transformation matrix the inertial terms of eq. (2) were excluded, in the IRS [17] procedure they are considered to have an *improved* solution. Then (2) is subdivided in blocks having the dependent set of dofs force-free, i.e. $\mathbf{f}_d = \mathbf{0}$:

$$\left(-\omega^2 \begin{bmatrix} \mathbf{M}_{dd} & \mathbf{M}_{da} \\ \mathbf{M}_{ad} & \mathbf{M}_{aa} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{dd} & \mathbf{K}_{da} \\ \mathbf{K}_{ad} & \mathbf{K}_{aa} \end{bmatrix}\right) \begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_a \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_a \end{bmatrix} \tag{36}$$

The first row of (36) is given by

$$(\mathbf{K}_{dd} - \omega^2\mathbf{M}_{dd}+)\mathbf{u}_d + (\mathbf{K}_{da} - \omega^2\mathbf{M}_{da})\mathbf{u}_a = \mathbf{0} \tag{37}$$

Therefore the linear transformation defined by eq. (24) is given by

$$\mathbf{u}_d = -(\mathbf{K}_{dd} - \omega^2\mathbf{M}_{dd})^{-1}(\mathbf{K}_{da} - \omega^2\mathbf{M}_{da})\mathbf{u}_a \tag{38}$$

$(\mathbf{K}_{dd} - \omega^2\mathbf{M}_{dd})^{-1}$ can be approximated by the first three terms of the Taylor series about $\omega_0 = 0$, i.e. using the binomial series [18] until the third term

$$
\begin{aligned}
\mathbf{u}_d &= -(\mathbf{K}_{dd}^{-1} + \omega^2\mathbf{K}_{dd}^{-1}\mathbf{M}_{dd}\mathbf{K}_{dd}^{-1})(\mathbf{K}_{da} - \omega^2\mathbf{M}_{da})\mathbf{u}_a \\
&= -(\mathbf{K}_{dd}^{-1} + \omega^2\mathbf{K}_{dd}^{-1}\mathbf{M}_{dd}\mathbf{K}_{dd}^{-1})(\mathbf{K}_{da} - \omega^2\mathbf{M}_{da})\mathbf{u}_a \\
&= -\left(\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} - \omega^2\mathbf{K}_{dd}^{-1}\mathbf{M}_{da} + \omega^2\mathbf{K}_{dd}^{-1}\mathbf{M}_{dd}\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} - \omega^4\mathbf{K}_{dd}^{-1}\mathbf{M}_{dd}\mathbf{K}_{dd}^{-1}\mathbf{M}_{da}\right)\mathbf{u}_a \\
&= \left(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da} + \mathbf{K}_{dd}^{-1}\left(\mathbf{M}_{da} + \mathbf{M}_{dd}(-\mathbf{K}_{dd}^{-1}\mathbf{K}_{da})\right)\omega^2 + \mathcal{O}(\omega^4)\right)\mathbf{u}_a
\end{aligned} \tag{39}
$$

The frequency dependency in eq. (39) is eliminated using the following expression which comes from eq. (31) in case of free vibration of the reduced-Guyan model:

$$\omega^2\mathbf{u}_a = \mathbf{M}_g^{-1}\mathbf{K}_g\mathbf{u}_a \tag{40}$$

Substituting eq. (40) in eq. (39), the set $\mathbf{u}_d$ is given by

$$\mathbf{u}_d = \mathbf{T}_{irs}\mathbf{u}_a \tag{41}$$

where $\mathbf{T}_{irs}$ is given by

$$\mathbf{T}_{irs} = \mathbf{T}_g + \boldsymbol{\tau}_{irs} \tag{42}$$

where $\mathbf{T}_g$ is given by eq. (28) and the inertial-corrective factor $\boldsymbol{\tau}_{irs}$ is given by

$$\boldsymbol{\tau}_{irs} = \mathbf{K}_{dd}^{-1}(\mathbf{M}_{da} + \mathbf{M}_{dd}\mathbf{T}_g)\mathbf{M}_g^{-1}\mathbf{K}_g \tag{43}$$

Thus the IRS projection matrix is given by:

$$\mathbf{V}_{irs} = \begin{bmatrix} \mathbf{T}_{irs} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{irs} + \mathbf{T}_g \\ \mathbf{I} \end{bmatrix} \tag{44}$$

A better IRS projection matrix can be achieved by an iterative procedure which was introduced by Blair et al. [19]. The reduced system matrices that Guyan reduction provides are used in (43) as updating matrices to produce the IRS transformation matrix. Thus, it is possible to update again the matrices to be used in an iterative scheme. The substitution of the reduced Guyan matrices $\mathbf{M}_g^{-1}$ and $\mathbf{K}_g$ in (43) by the new updated IRS matrices results in

$$
\begin{aligned}
\mathbf{T}_{irs,i} &= \mathbf{K}_{dd}^{-1}(\mathbf{M}_{da} + \mathbf{M}_{dd}\mathbf{T}_g)\mathbf{M}_{irs,i-1}^{-1}\mathbf{K}_{irs,i-1} + \mathbf{T}_g \\
&= \boldsymbol{\tau}_{irs,i} + \mathbf{T}_g
\end{aligned} \tag{45}
$$

The ith IRS projection matrix is now given by

$$\mathbf{V}_{irs,i} = \begin{bmatrix} \mathbf{T}_{irs,i} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{irs,i} + \mathbf{T}_g \\ \mathbf{I} \end{bmatrix} \tag{46}$$

with:

$$\mathbf{M}_{irs,0}^{-1} = \mathbf{M}_g^{-1} \tag{47}$$

$$\mathbf{K}_{irs,0} = \mathbf{K}_g \tag{48}$$

The reduced-order model, eq. (6), using the IRS projection matrix defined in eq. (46) is given by

$$(-\omega^2\mathbf{M}_{irs,i} + i\omega\mathbf{D}_{irs,i} + \mathbf{K}_{irs,i})\mathbf{u}_a = \mathbf{f}_{irs,i} \tag{49}$$

These reduced matrices and vector are computed analogously to eqs. (32) - (35).

3.3 Dynamic Reduction

The dynamic reduction [4] is obtained considering the undamped version of eq. (2) which is given by

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{u}(\omega) = \mathbf{f}(\omega) \tag{50}$$

Subdividing eq. (50) in blocks yields

$$\begin{bmatrix} \mathbf{Z}_{dd} & \mathbf{Z}_{da} \\ \mathbf{Z}_{ad} & \mathbf{Z}_{aa} \end{bmatrix} \begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_a \end{bmatrix} = \begin{bmatrix} \mathbf{0}_d \\ \mathbf{f}_a \end{bmatrix} \tag{51}$$

where

$$\mathbf{Z}(\omega) = -\omega^2\mathbf{M} + \mathbf{K} \tag{52}$$

Solving the first row of (51) for $\mathbf{u}_d$ yields

$$\mathbf{u}_d = -\mathbf{Z}_{dd}(\omega)^{-1}\mathbf{Z}_{da}(\omega)\mathbf{u}_a \tag{53}$$

Thus the displacement vector $\mathbf{u}$ can be written as a function of the active dofs $\mathbf{u}_a$ and a chosen frequency ω as follows:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_a \end{bmatrix} \mathbf{u}_a = \mathbf{V}_{dy}\mathbf{u}_a \tag{54}$$

where

$$\mathbf{V}_{dy} = \begin{bmatrix} \mathbf{T}_{dy} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} -\mathbf{Z}_{dd}^{-1}(\omega)\mathbf{Z}_{da}(\omega) \\ \mathbf{I} \end{bmatrix} \tag{55}$$

$\mathbf{u}_a$ and $\mathbf{V}_{dy}$ are the reduced vector $\mathbf{u}_r$ and the matrix $\mathbf{V}$ in eq. (3) respectively. Therefore the reduced-order model , eq. (6), is given by

$$(-\omega^2\mathbf{M}_{dy} + i\omega\mathbf{D}_{dy} + \mathbf{K}_{dy})\mathbf{u}_a = \mathbf{f}_{dy} \tag{56}$$

The reduced matrices and reduced force vector are obtained using eq. (7) analogously to eqs. (32)-(35).

4 MOR: Component-Mode Synthesis Methods

The component-mode synthesis approach consists in the subdivision of a mechanical model into substructures using the following classification for the dofs [15], see fig. 1:
$\mathcal{I}$: interior coordinates (i as subscript in matrix notation).
$\mathcal{R}$: rigid-body coordinates (r as subscript in matrix notation).
$\mathcal{E}$: excess boundary coordinates (e as subscript in matrix notation).
$\mathcal{B} = \mathcal{R} + \mathcal{E}$: boundary coordinates ($b$ as subscript in matrix notation) with the assumption that dofs with external loads are assumed to be boundary dofs.
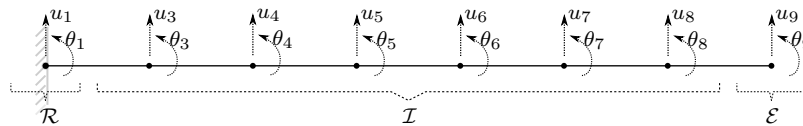

Figure 1: FE discretization of a catiliver beam.

The equation of motion of an undamped sub-structure, identified by the superscript α, is given by

$$\mathbf{M}^{(\alpha)}\ddot{\boldsymbol{u}}^{(\alpha)} + \mathbf{K}^{(\alpha)}\boldsymbol{u}^{(\alpha)} = \boldsymbol{f}^{(\alpha)} + \boldsymbol{r}^{(\alpha)} \tag{57}$$

where $\mathbf{M}^\alpha$, $\mathbf{K}^\alpha$, $\boldsymbol{u}^\alpha$ and $\boldsymbol{f}^\alpha$ are the mass matrix, stiffness matrix, displacement vector and force vector at the component label α respectively. $\boldsymbol{r}^\alpha$ is a force vector which contains the reaction forces on the component due to its connection to adjacent components at the boundary dofs.

Component Modes

Eq. (57) may be written in the following partitioned form

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ib} \\ \mathbf{M}_{bi} & \mathbf{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_i \\ \ddot{\boldsymbol{u}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{0}_i \\ \boldsymbol{f}_b + \boldsymbol{r}_b \end{bmatrix} \tag{58}$$

Fixed-Interface Normal Modes

The fixed interfaced normal modes are a set of n_i modes shapes resulting by solving the following eigenvalue problem of the internal dofs of eq. (58), while the boundary dofs are fixed, see fig. 2,

$$\left[\mathbf{K}_{ii} - \omega_k^2 \mathbf{M}_{ii}\right] \boldsymbol{\phi}_k = 0 \qquad k = 1, 2, \ldots, n_i \tag{59}$$

The eigenvectors $\boldsymbol{\phi}_k$ are gathered in the internal modal matrix $\boldsymbol{\Phi}_{ii} \in \mathbb{R}^{n_i \times 1}$. Therefore the fixed interfaced normal modes for a substructure α is given by

$$\boldsymbol{\Phi}_i = \begin{bmatrix} \boldsymbol{\Phi}_{ii} \\ \boldsymbol{0}_{bi} \end{bmatrix} \tag{60}$$

where $\boldsymbol{\Phi}_i \in \mathbb{R}^{N_\alpha \times 1}$. N_α is the total number of dofs of the structure or substructure.
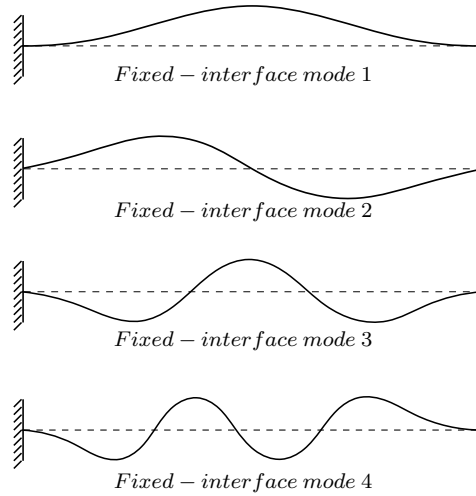
Figure 2: Four component fixed-interface normal modes of a cantiliver beam.

Interface Constraint Modes

Interface constraint modes result when a unit displacement is applied in one dof in the set of boundary dofs, while the remaining dofs of that set are restrained, and conserving the remaining dofs force-free, see fig. 3. The constraint modes are defined by solving the following static equation for $[\boldsymbol{\Psi}_{ib} \ \mathbf{I}_{bb}]^T$:

$$\begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}_{ib} \\ \mathbf{I}_{bb} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{ib} \\ \mathbf{R}_{bb} \end{bmatrix} \qquad (61)$$

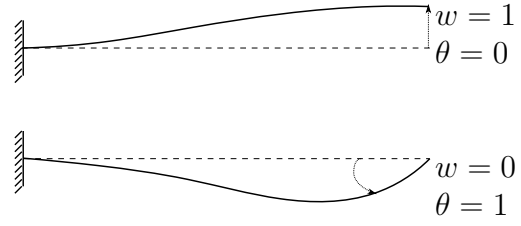Thus the interface constraint mode matrix $\mathbf{\Psi}_b$ is given by

$$\mathbf{\Psi}_b = \begin{bmatrix} \mathbf{\Psi}_{ib} \\ \mathbf{I}_{bb} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} \\ \mathbf{I}_{bb} \end{bmatrix} \qquad (62)$$

Figure 3: Interface-constraint modes in the boundary dofs of a cantilever beam.

Those constraint modes are stiffness-orthogonal to all of the fixed-interface normal modes [15].

4.1 The Craig-Bampton Method

The Craig-Bampton transformation [6, 8] consists of transforming a set of physical coordinates made up of internal dofs $\boldsymbol{u}_i$ and dofs at the boundary $\mathbf{u}_b$ in terms of a hybrid set which is compound of a set of modal coordinates at the interior $\boldsymbol{\eta}_i$ and physical coordinates $\boldsymbol{u}_b$, at the boundary. The Craig-Bampton hybrid coordinates $\boldsymbol{u}_{cb} = [\boldsymbol{\eta}_i \ \ \boldsymbol{u}_b]^T$ are then related to the physical coordinates $\boldsymbol{u} = [\boldsymbol{u}_i \ \ \boldsymbol{u}_b]^T$ as follows

$$\begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_b \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_i & \mathbf{\Psi}_b \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_i \\ \boldsymbol{u}_b \end{bmatrix} \qquad (63)$$

where $\mathbf{\Phi}_i$ and $\mathbf{\Psi}_b$ are given by eqs. (60) and (62) respectively. Substituting those equations in (63)

$$\begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_b \end{bmatrix} = \mathbf{V}_{cb}\mathbf{u}_{cb} \qquad (64)$$

where

$$\mathbf{V}_{cb} = \begin{bmatrix} \mathbf{\Phi}_{ii} & -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} \\ \mathbf{0}_{bi} & \mathbf{I}_{bb} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{ii} & \mathbf{\Psi}_{ib} \\ \mathbf{0}_{bi} & \boldsymbol{I}_{bb} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_i & \mathbf{\Psi}_c \end{bmatrix} \qquad (65)$$

is the Craig-Bampton projection matrix. $\mathbf{V}_{cb}$ and $\mathbf{u}_{cb}$ are the $\mathbf{V}$ matrix and the $\mathbf{u}_r$ vector in eq. (3) respectively. After defining the Craig-Bampton Transformation Matrix above, it follows the transformation of the equation of motion from the set of physical coordinates to a set of coordinates consisting of physical coordinates at the subset of the boundary dofs and modal coordinates at interior dofs.

Using the the Craig-Bampton projection matrix eq. (65) in eq. (7) the reduced order model in eq. (6) is obtained [8] and given by

$$\left(-\omega^2 \begin{bmatrix} \mathbf{m} & \mathbf{M}_{kb} \\ \mathbf{M}_{bk} & \mathbf{M}_{bb} \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{bb} \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{u}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_b \end{bmatrix} \qquad (66)$$

where

$$\mathbf{m} = \Phi_i^T \mathbf{M} \Phi_i = \Phi_{ii}^T \mathbf{M}_{ii} \Phi_{ii}$$

$$\mathbf{M}_{ib} = \Phi_i^T \mathbf{M} \Psi_c = \Phi_{ii}^T \left[\mathbf{M}_{ib} + \mathbf{M}_{ii} \Psi_{ib} \right]$$

$$\mathbf{M}_{bi} = \Psi_c^T \mathbf{M} \Phi_i = \left[\mathbf{M}_{bi} + \Psi_{ib}^T \mathbf{M}_{ii} \right] \Phi_{ii}$$

$$\mathbf{M}_{bb} = \Psi_c^T \mathbf{M} \Psi_c = \mathbf{M}_{bb} + \mathbf{M}_{bi} \Psi_{ib} + \Psi_{ib}^T \mathbf{M}_{bi} + \Psi_{ib}^T \mathbf{M}_{ii} \Psi_{ib}$$

$$\mathbf{k} = \Phi_i^T \mathbf{K} \Phi_i = \Phi_{ii}^T \mathbf{K}_{ii} \Phi_{ii} = \mathbf{m} \omega_E^2$$

$$\mathbf{K}_{ib} = \Phi_i^T \mathbf{K} \Psi_c = \Phi_{ii}^T \left[\mathbf{K}_{ib} - \mathbf{K}_{ib} \right] = 0$$

$$\mathbf{K}_{bi} = \Psi_c^T \mathbf{K} \Phi_i = \left[\mathbf{K}_{bi} - \mathbf{K}_{ib} \right] \Phi_{ii} = 0$$

$$\mathbf{K}_{bb} = \Psi_c^T \mathbf{K} \Psi_c = \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}$$

$$\mathbf{c} = \Phi_i^T \mathbf{D} \Phi_i = \Phi_{ii}^T \mathbf{D}_{ii} \Phi_{ii} = 2\zeta \mathbf{m} \omega_E$$

The mass matrix $\mathbf{M}_{bb}$ and the stiffness matrix $\mathbf{K}_{bb}$ are reduced to the boundary nodes in the same way as they are reduced using the Guyan-Irons reduction. $\mathbf{m}$, $\mathbf{c}$ and $\mathbf{k}$ are the modal mass, modal damping and modal stiffness matrix of the interior dofs respectively. $\Psi_c^T \mathbf{D} \Psi_c$, $\Psi_c^T \mathbf{D} \Phi_i$ and $\Phi_i^T \mathbf{D} \Psi_c$ are considered equal to $\mathbf{0}$. Thus only the submatrix $2\zeta \mathbf{m} \omega_E$ has significance [8].

5 MOR: Interpolatory Methods

The standard formulation of the second order dynamical system, (2), in the frequency domain is given by

$$(s^2 \mathbf{M} + s \mathbf{D} + \mathbf{K}) \mathbf{u}(s) = \mathbf{B} \mathrm{f} \mathrm{f}(s)$$
$$\mathbf{y}(s) = \mathbf{C} \mathbf{u}(s) \tag{67}$$

where $s = i\omega$, $\mathbf{B}$ and $\mathbf{C}$ can be considered equal to the identity matrix $\mathbf{I}$ and the function $\mathrm{f}(s)$ equal to one in case of a response having the same form of the excitation force, i.e. the input vector is constant. The transfer function of eq. (67) is defined by

$$\mathcal{H} = \mathbf{C}(s^2 \mathbf{M} + s \mathbf{D} + \mathbf{K})^{-1} \mathbf{B} \mathrm{f} \tag{68}$$

The main approach of an interpolatory MOR method is to match the first ν_p terms of the Taylor expansion of eq. (68) about ω_p with $p = 0, 1, 2, ..., P$, i.e. about P frequency points in a frequency interval $\Delta_\omega = [\omega_L, \ \omega_R]$, which is the same as to find an approximated vector $\mathbf{u}_\nu$ which matches the first ν terms of the Taylor expansion of the vector $\mathbf{u}$ about the same ω_p frequency points in the context of structural mechanics problems, i.e.

$$\mathbf{u}(s) = \mathbf{u}_\nu(s) + \mathcal{O}\left(\prod_{p=1}^{P} (\Delta s_p)^{\nu_p} \right) \tag{69}$$

where $\Delta s_p = s - s_p$ and ν_p is the order of the Taylor series expansion at the interpolation frequency ω_p. Thus $\mathbf{u}_\nu^{(k)}(s_p) = \frac{d^k}{ds^k}\big(\mathbf{u}(s)\big)$. Therefore the main task for interpolatory MOR methods is to find a vector space that contains the first ν moments (see, sub-sec. 5.2) of $\mathcal{H}$. In the case of the multi-point Padé approximation such an approach is simplified to find the coefficients of the Padé approximation $[L/M]$ of a single dof, which implies to match the first $L + M + 1$ moments of the transfer function, eq. (68). Due to the matching process and the analogy to the Padé approximant those methods are also known as moment matching methods or Padé type methods [9, 10]. Here a similar terminology as in [14] was adopted.

5.1 Multi-Padé approximation

Every dof $u_j(s)$, $j = 1, 2, 3, ..., N$, of the displacement vector can be approximated by a Padé approximant about a frequency point ω_0. The Padé approximant of the jth dof has the form

$$u_j(\omega) = [L/M] = \frac{p_0 + p_1 \Delta\omega_0 + \cdots + p_L \Delta\omega_0^L}{1 + q_1 \Delta\omega_0 + \cdots + q_M \Delta\omega_0^M} + \mathcal{O}(\Delta\omega_0^{L+M+1})$$

$$= \frac{P_j(\omega)}{Q_j(\omega)} + \mathcal{O}(\Delta\omega_0^{L+M+1}) \tag{70}$$

where $\omega = \omega_0 + \Delta\omega$, $P_j(\omega)$ and $Q_j(\omega)$ is a polynomial of degree L and M respectively. By Cross-multiplying eq. (70) and disregarding the error of the right hand side leads to

$$u_j(\omega)Q_j(\omega) = P_j(\omega) \tag{71}$$

Successively differentiating eq. (71) with respect to ω the following equation is obtained [13]

$$\sum_{r=0}^{k} \left(\frac{k!}{r(k-r)!} u_j^{(k-r)} \sum_{m=r}^{M} \frac{m!}{(m-r)!} q_m \Delta\omega^{m-r} \right) - \sum_{l=k}^{L} \frac{l!}{(l-k)!} p_l \Delta\omega^{l-k} = 0 \tag{72}$$

$$k = 0, 1, \ldots, L + M$$

with

$$q_l = 0 \quad if \quad r > M$$
$$p_k = 0 \quad if \quad k > L \tag{73}$$

where $u_j^k = \left(\frac{d^k}{d\omega^k} (\mathbf{u}(\omega_0)) \right)_j$ and $\Delta\omega = \omega - \omega_0$. Then the coefficients of the polynomials P_j and Q_j can be computed if u_j and its first $L + M$ derivatives are known, i.e. the Padé approximant given by eq.(70) matches the first $L + M + 1$ terms of the corresponding Taylor series about ω_0, therefore the Padé approximant matches the first $L + M + 1$ terms (moments) of a power expansion of the transfer function given by eq. (68) [9].

It is said to be a multi-point Padé Approximation [13] if P frequency points are defined and interpolated in the frequency interval of interest, Δ_ω, where each frequency is denoted by ω_p with $p = 1, 2, 3, ..., P$, using

$$\sum_{r=0}^{k} \left(\frac{k!}{r!(k-r)!} u_{j,\omega_p}^{(k-r)} \sum_{m=r}^{M} \frac{m!}{(m-r)!} q_m \Delta\omega_p^{m-r} \right) - \sum_{l=k}^{L} \frac{l!}{(l-k)!} p_l \Delta\omega_p^{l-k} = 0 \tag{74}$$

$$k = 0, 1, \ldots, \left\lceil \frac{L + M + 1}{P} \right\rceil - 1$$

where $\lceil \ \rceil$ is the ceiling function and $u_{j,\omega_p}^{(k)}$ is the k derivative of the jth dof of the displacement vector $\mathbf{u}$ evaluated at ω_p and is defined by

$$u_{j,\omega_p}^{(k)} = (\mathbf{u}_{\omega_p}^{(k)})_j = \begin{cases} \mathbf{K}_\omega^{-1}\mathbf{f} & k = 0 \\ \mathbf{K}_\omega^{-1}\left((2\omega_p\mathbf{M} - i\mathbf{D})\mathbf{u}_{\omega_p} + \mathbf{f}^{(1)}\right) & k = 1 \\ \mathbf{K}_\omega^{-1}\left(k\mathbf{M}(2\omega_p\mathbf{u}_{\omega_p}^{(k-1)} + (k-1)\mathbf{u}_{\omega_p}^{(k-2)} - ik\mathbf{D}\mathbf{u}_{\omega_p}^{(k-1)}\right) & k \geq 2 \end{cases} \tag{75}$$

where $\Delta\omega_p = \omega_p - \omega_0$ and $\mathbf{K}_\omega = (-\omega_p^2\mathbf{M} + i\omega\mathbf{D} + \mathbf{K})$. Eq. (75) implies to solve P times a generalized impedance problem with $\lceil \frac{L+M+1}{P} \rceil$ different right hand sides.

From eq. (75) it is possible to obtain the coefficients of the Padé polynomials for each component u_j of $\mathbf{u}$, those coefficients are collected in the vector $\mathbf{x} = [p_0\, p_1\, p_2\, ...\, p_L\, q_1\, q_2\, ...\, q_M]^T$ and are found by solving $\mathbf{Ax} = \mathbf{b}$ which is formed as follows:

$$
\begin{bmatrix}
\begin{bmatrix} \cdots & \mathbf{A}_1 & \cdots \end{bmatrix}_{[n_\nu \times n_P]} \\
\vdots \\
\begin{bmatrix} \cdots & \mathbf{A}_p & \cdots \end{bmatrix}_{[n_\nu \times n_P]} \\
\vdots \\
\begin{bmatrix} \cdots & \mathbf{A}_P & \cdots \end{bmatrix}_{[n_\nu \times n_P]}
\end{bmatrix}_{[n_P \times n_P]}
\mathbf{x}_{[n_P \times 1]} =
\begin{bmatrix}
\mathbf{b}_1 \\
\vdots \\
\mathbf{b}_p \\
\vdots \\
\mathbf{b}_P
\end{bmatrix}_{[n_P \times 1]}
\tag{76}
$$

where $\mathbf{A}_p \in \mathbb{C}^{\lceil \frac{(L+M+1)}{P} \rceil \times (L+M+1)}$, $\mathbf{b}_p \in \mathbb{C}^{\lceil \frac{(L+M+1)}{P} \rceil \times 1}$, $p = 1, 2, \ldots, P$, $n_\nu = \lceil \frac{L+M+1}{P} \rceil$ and $n_P = L + M + 1$. $\mathbf{A}$ is a symmetric matrix but ill-conditioned, thus this approach can be used only when few interpolation points are considered [9].

$\mathbf{A}_p$ and $\mathbf{b}_p$ are computed by expanding eq. (74) for $k = 0, 1, \ldots, \lceil \frac{L+M+1}{P} \rceil - 1$, i.e the row $k + 1$ of $\mathbf{A}_p$ is an equation given by

$$
\begin{aligned}
-\tfrac{(k)!}{0!}p_k - \tfrac{(k+1)!}{1!}(\Delta\omega_p)^1 p_{k+1} - \tfrac{(k+2)!}{2!}(\Delta\omega_p)^2\, p_{k+2} - \cdots - \tfrac{L!}{(L-k)!}(\Delta\omega_p)^{L-k}\, p_L \\
+ \Big(\sum_{r=0}^{k} \tfrac{k!}{r!(k-r)!}\tfrac{1!}{(1-r)!}(\Delta\omega)^{1-r}u^{(k-r)} \Big)q_1 \\
+ \Big(\sum_{r=0}^{k} \tfrac{k!}{r!(k-r)!}\tfrac{2!}{(2-r)!}(\Delta\omega)^{2-r}u^{(k-r)} \Big)q_2 \\
\cdots + \Big(\sum_{r=0}^{k} \tfrac{k!}{r!(k-r)!}\tfrac{M!}{(M-r)!}(\Delta\omega)^{M-r}u^{(k-r)} \Big)q_M = u_p^{(k)}
\end{aligned}
\tag{77}
$$

Then the coefficients of the left hand side of eq. (77) are placed in the row $k + 1$ of $\mathbf{A}_p$, and the right hand side is placed at the $k + 1$ position of the vector $\mathbf{b}_p$. It permits to express eq. (74) in a matrix notation of the form $[\mathbf{A}_p]_{[n_\nu \times n_P]}[\mathbf{x}]_{[n_P \times 1]} = [\mathbf{b}_p]_{[n_P \times 1]}$ as follows

$$
\begin{bmatrix}
-1 & -(\Delta\omega_p) & -(\Delta\omega_p)^2 & \cdots & -(\Delta\omega_p)^L & (\Delta\omega_p)u_p & (\Delta\omega_p)^2 u_p & \cdots & (\Delta\omega_p)^M u_p \\
0 & -1 & -2(\Delta\omega_p) & \cdots & L(\Delta\omega_p)^{L-1} & \big(u_p^{(0)}+(\Delta\omega_p)u_p^{(1)}\big) & \big(2(\Delta\omega_p)u_p^{(0)}+(\Delta\omega_p)^2u_p^{(1)}\big) & \cdots & \big(M(\Delta\omega_p)^{M-1}U_p^{(0)}+(\Delta\omega_p)^M u_p^{(1)}\big) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_L \\ q_1 \\ q_2 \\ \vdots \\ q_M
\end{bmatrix}
=
\begin{bmatrix}
u_p^{(0)} \\ u_p^{(1)} \\ \vdots
\end{bmatrix}
\tag{78}
$$

Thus P small systems of equations, which share the same unknown vector $\mathbf{x}$, of the form given by eq.(78) are assembled, i.e they are gathered in the assembled-system of equations given by eq.(76), which permits to compute the coefficients of the multi-point Padé approximation with a very low computational cost.

5.2   Krylov-based Galerkin projection Method

First Order approach

The second order dynamical system in eq. (67) is reformulated in the standard first order form, see sub-sec. 2.2, and given by

$$
\begin{aligned}
s\,\mathbf{z}(s) &= \mathbf{Az}(s) + \mathbf{Bf}(s) \\
\mathbf{y}(s) &= \mathbf{Cz}(s)
\end{aligned}
\tag{79}
$$

Where $\mathbf{z} \in \mathbb{R}^{2N \times 1}$, $\mathbf{A} \in \mathbb{R}^{2N \times 2N}$ and $\mathbf{B} \in \mathbb{R}^{2N \times N}$ are the state vector, the state matrix and the input matrix respectively and are defined by

$$\mathbf{z} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f} \end{bmatrix} \tag{80}$$

The transfer matrix of (79) is given by

$$\mathbf{h}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{81}$$

If $\mathbf{A}$ is non singular, the transfer matrix $\mathbf{h}(s)$ can be approximated about $s_0 = 0$ using the Taylor series as follows

$$\mathbf{h}(s) = \mathbf{C}\big((-\mathbf{A}^{-1}\mathbf{B}) + (\mathbf{A}^{-1})(-\mathbf{A}^{-1}\mathbf{B})s + \cdots + (\mathbf{A}^{-1})^i(-\mathbf{A}^{-1}\mathbf{B})s^i + \cdots\big) \tag{82}$$

For convenience one may define

$$\mathcal{H} = \mathbf{A}^{-1} = \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \qquad \mathbf{b} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{K}^{-1}\mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{83}$$

and

$$\boldsymbol{b}_0 = \mathbf{K}^{-1}\mathbf{f} \quad \mathcal{H}_{11} = -\mathbf{K}^{-1}\mathbf{D} \quad \mathcal{H}_{12} = -\mathbf{K}^{-1}\mathbf{M} \tag{84}$$

The resulting non-negative coefficients of the series (82) are said to be the system's *moments*

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1})^i(\mathbf{A}^{-1}\mathbf{B}) = \mathbf{C}(\mathcal{H})^i\mathbf{B}, \quad i = 0, 1, 2, \ldots \tag{85}$$

where $\mathbf{m}_i \in \mathbb{C}^{2N \times 1}$ is the ith moment of eq. (81). It is clear from eq. (85) that $\mathbf{z}(s)$ of (80) is contained in a vector space spanned by the columns of the Krylov subspace $\mathcal{K}_\nu(\mathcal{H}, \mathbf{b})$ with starting vector $\mathbf{b}$ and matrix $\mathcal{H}$. A simplification of the dimension of the vectors in the Krylov subspace can be done if one notes that

$$\mathbf{m}_0 = \mathbf{b} = \begin{bmatrix} \mathbf{K}^{-1}\mathbf{f} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_0 \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{m}_2 = \mathcal{H}^2\mathbf{b} = \begin{bmatrix} \mathcal{H}_{11}\boldsymbol{b}_1 + \mathcal{H}_{12}\boldsymbol{b}_0 \\ \boldsymbol{b}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_2 \\ \boldsymbol{b}_1 \end{bmatrix}$$

$$\mathbf{m}_1 = \mathcal{H}\mathbf{b} = \begin{bmatrix} \mathcal{H}_{11}\boldsymbol{b}_0 \\ \boldsymbol{b}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_0 \end{bmatrix} \qquad \mathbf{m}_3 = \mathcal{H}^3\mathbf{b} = \begin{bmatrix} \mathcal{H}_{11}\boldsymbol{b}_2 + \mathcal{H}_{12}\boldsymbol{b}_1 \\ \boldsymbol{b}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_3 \\ \boldsymbol{b}_2 \end{bmatrix}$$

therefore the displacement part $\mathbf{u}(s)$ of the vector $\mathbf{z}(s)$ is contained in the column space spanned by $\{\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, ...\}$, i,e. by the Krylov subspace $\mathcal{K}_\nu(\mathcal{H}_{11}, \mathcal{H}_{12}; \boldsymbol{b}_0)$,

$$\mathbf{u}(\omega) \in \mathcal{K}_\nu(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}; \mathbf{K}^{-1}\mathbf{f}) \tag{86}$$

which holds true for low frequency range, because the interpolation point s_0 was chosen equal to zero.

It remains to show how to get the Krylov subspace for $s \neq 0$, such that the matrix $\mathbf{V}$ contains information of many points, i.e. a interpolatory approach over the frequency interval Δ_ω. It is achieved by noting that the moments of $\mathbf{h}(s)$ about $s_0 \neq 0$ are equal to the moments of $\mathbf{h}(s+s_0)$ about zero [12], i.e the transfer function of eq. (81) is given by

$$\begin{aligned} \mathbf{h}(s + s_0) &= \mathbf{C}\big((s + s_0)^2\mathbf{M} + (s + s_0)\mathbf{D} + \mathbf{K}\big)^{-1}\mathbf{B} \\ &= \mathbf{C}\big(s^2\mathbf{M} + s(2s_0\mathbf{M} + \mathbf{D}) + (s_0^2\mathbf{M} + (s_0\mathbf{D} + \mathbf{K}))\big)^{-1}\mathbf{B} \end{aligned} \tag{87}$$

For convenience one may define

$$\mathbf{D}_s = 2s_0\mathbf{M} + \mathbf{D} \text{ and } \mathbf{K}_s = s_0^2\mathbf{M} + s_0\mathbf{D} + \mathbf{K} \tag{88}$$

$\mathbf{D}_s$ and $\mathbf{K}_s$ is the generalized damping and the dynamic impedance respectively. The substitution of $\mathbf{D}$ and $\mathbf{K}$ in the Krylov subspace in eq. (86) by the definitions given by eq. (88) leads to

$$\mathcal{K}_\nu(-\mathbf{K}_s^{-1}\mathbf{D}_s, \ -\mathbf{K}_s^{-1}\mathbf{M}; \mathbf{K}_s^{-1}\mathbf{f}) \tag{89}$$

which approximates the solution $\mathbf{u}$ for any s_0. Using $s = i\omega$ and defining P frequencies as interpolation points in Δ_ω a vector space $\mathbf{V}$ may be defined by

$$\mathbf{V}_{[N\times 3P]} = \bigcup_{p=1}^P \mathcal{K}_{\nu_p}\Big(-(-\omega_p^2\mathbf{M} + i\omega_p\mathbf{D} + \mathbf{K})^{-1}(2i\omega\mathbf{M} + \mathbf{D}),$$
$$-(-\omega_p^2\mathbf{M} + i\omega_p\mathbf{D} + \mathbf{K})^{-1}\mathbf{M}; \ (-\omega_p^2\mathbf{M} + i\omega_p\mathbf{D} + \mathbf{K})^{-1}\mathbf{f}\Big) \tag{90}$$

where ω_p is the interpolation frequency, with $p = 1, 2, ..., P$ and ν_p is the number of matched terms about ω_p of the Taylor expansion of the transfer matrix given by eq.(81). If the dynamic damping is neglected eq. (90) simplifies to

$$\mathbf{V}_{[N\times 3P]} = \bigcup_{p=1}^P \mathcal{K}_\nu\big(-(-\omega_p^2\mathbf{M} + i\omega_p\mathbf{D} + \mathbf{K})^{-1}\mathbf{M}, (-\omega_p^2\mathbf{M} + i\omega_p\mathbf{D} + \mathbf{K})^{-1}\mathbf{f}\big) \tag{91}$$

which is the suitable form for proportional (Rayleigh damping and modal damping) damped systems.

Second order approach

The transfer matrix of the second order dynamical system of eq. (67) is given by eq. (68) and rewritten here for convenience

$$\mathcal{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{Bf} = \mathbf{C}\mathbf{K}_s\mathbf{Bf} \tag{92}$$

The Taylor expansion of eq. (92) is given by

$$\mathcal{H}(s_0 \neq 0) = \sum_k^\infty \frac{1}{k!}\frac{d^{(k)}}{ds^{(k)}}\big(\mathcal{H}(s_0)\big)(\Delta s)^k \tag{93}$$

therefore the moments of eq. (93) are

$$\mathbf{m}_k = \frac{1}{k!}\frac{d^{(k)}}{ds^{(k)}}\big(\mathcal{H}(s_0)\big) \quad k = 0, 1, 2, 3, ..., \nu \tag{94}$$

By solving the first ν moments, eq. (94), of the Taylor series given by eq. (93) and considering $\mathbf{C}$ and $\mathbf{B}$ equal to the identity matrix it is possible to group together their terms such that a Krylov subspace of the form given by (89) is obtained. It is illustrated by solving the first four

moments, using eq. (94), of the Taylor series about s_0, i.e.

$$\mathbf{m}_0 = \mathbf{K}_s^{-1}\mathbf{f} = \boldsymbol{b}_0 \tag{95}$$

$$\mathbf{m}_1 = -\mathbf{K}_s^{-2}(2s_0\mathbf{M} + \mathbf{D})\mathbf{f} = (-\mathbf{K}_s^{-1}\mathbf{D}_s)\boldsymbol{b}_0 \tag{96}$$

$$\mathbf{m}_2 = \mathbf{K}_s^{-3}(2s_0\mathbf{M} + \mathbf{D})^2\mathbf{f} - \mathbf{K}_s^{-2}(\mathbf{M})\mathbf{f}$$
$$= (-\mathbf{K}_s^{-1}\mathbf{D}_s)(-\mathbf{K}_s^{-1}\mathbf{D}_s)\mathbf{b}_0 + (-\mathbf{K}_s^{-1}\mathbf{M})\boldsymbol{b}_0 \tag{97}$$

$$\mathbf{m}_3 = -\mathbf{K}_s^{-4}(2s_0\mathbf{M} + \mathbf{D})^3\mathbf{f} + \frac{8}{6}\mathbf{K}_s^{-3}(2s_0\mathbf{M} + \mathbf{D})(\mathbf{M})\mathbf{f}$$
$$+ \frac{4}{6}\mathbf{K}_s^{-3}(2s_0\mathbf{M} + \mathbf{D})(\mathbf{M})\mathbf{f}$$
$$= (-\mathbf{K}_s^{-1}\mathbf{D}_s)(-\mathbf{K}_s^{-1}\mathbf{D}_s)(-\mathbf{K}_s^{-1}\mathbf{D}_s)\mathbf{b}_0 + (-\mathbf{K}_s^{-1}\mathbf{D}_s)(-\mathbf{K}_s^{-1}\mathbf{M})\boldsymbol{b}_0$$
$$+ (-\mathbf{K}_s^{-1}\mathbf{M})(-\mathbf{K}_s^{-1}\mathbf{D}_s)\boldsymbol{b}_0 \tag{98}$$

These show that the moments $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_\nu$ of the second order dynamical system given by eq. (67) span the Krylov subspace defined by eq. (89) and vice versa.

5.3 Derivative-based Galerkin projection Method

The transfer function of the original second order dynamical system of eq. (67) can be redefined as

$$\mathcal{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B} = \mathbf{C}\mathbf{K}_\omega^{-1}\mathbf{B} \tag{99}$$

$\mathbf{ff}(s)$ of eq. (67) is interpreted as a constant input with $\mathbf{f}(s) = 1$. Analogously the transfer function of a reduced second order system is defined by

$$\mathcal{H}_\nu(s) = \mathbf{C}\mathbf{V}(s^2\mathbf{M_r} + s\mathbf{D}_r + \mathbf{K}_r)^{-1}\mathbf{V}^H\mathbf{B} = \mathbf{C}\mathbf{V}\mathbf{K}_{s,\nu}^{-1}\mathbf{V}^H\mathbf{B} \tag{100}$$

One searches for a $\mathcal{H}_\nu$ which acting in a vector $\mathbf{f}$ approximates the original transfer function acting on the same vector, i.e.

$$\| \mathcal{H}(s_p + \Delta s)\mathbf{f} - \mathcal{H}_\nu(s_p + \Delta s)\mathbf{f} \| \leq c|\Delta s|^h \tag{101}$$

where Δs is sufficient small and $h > 0$. An equivalent expression to eq. (101) is given by

$$\mathcal{H}(s)\mathbf{f} = \mathcal{H}_\nu(s)\mathbf{f} + \mathcal{O}((\Delta s)^\nu) \tag{102}$$

which implies

$$\frac{d^k}{ds^k}(\mathcal{H}) = \frac{d^k}{ds^k}(\mathcal{H}_\nu) \quad for \ \ k = 0, 1, 2, 3, ..., \nu - 1 \tag{103}$$

Thus if $\frac{d^k}{ds^k}((s_p^2\mathbf{M} + 2s_p\mathbf{D} + \mathbf{K})^{-1}\mathbf{f}) \in \mathbf{V}$ for $k = 0, 1, 2, ..., \nu - 1$ then one proofs eq. (101) as follows [11]

$$\|\mathcal{H}(s_p + \Delta s)\mathbf{f} - \mathcal{H}_\nu(s_p + \Delta s)\mathbf{f}\| = \|\mathbf{C}\mathbf{K}_s^{-1}(s_p + \Delta s)\mathbf{f} - \mathbf{C}\mathbf{V}\mathbf{K}_{s,\nu}^{-1}(s_p + \Delta s)\mathbf{V}^H\mathbf{f}\|$$

Using the skew projection $\mathbf{P}_\nu = \mathbf{V}\mathbf{K}_{s,\nu}^{-1}(s + \Delta s)\mathbf{V}^H\mathbf{K}_s(s + \Delta s)$

$$\|\mathbf{C}\mathbf{K}_s^{-1}(s_p + \Delta s)\mathbf{f} - \mathbf{C}\mathbf{V}\mathbf{K}_{s,\nu}^{-1}(s_p + \Delta s)\mathbf{V}^H\mathbf{f}\|$$
$$= \|\mathbf{C}\mathbf{K}_s^{-1}(s_p + \Delta s)\mathbf{f} - \mathbf{C}\mathbf{P}_\nu\mathbf{K}_s^{-1}(s_p + \Delta s)\mathbf{f}\|$$
$$= \|\mathbf{C}(\mathbf{I} - \mathbf{P}_\nu) \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^{(k)}}{ds^{(k)}}\big(\mathbf{K}_s^{-1}(s_p + \Delta s)\big)\Delta s^k\mathbf{f}\|$$
$$= \|\mathbf{C} \sum_{k=\nu}^{\infty} \frac{1}{k!}\frac{d^{(k)}}{ds^{(k)}}\big(\mathbf{K}_s^{-1}(s_p + \Delta s)\big)\Delta s^k\mathbf{f}\|$$
$$= |\Delta s|^\nu \|\mathbf{C} \sum_{k=0}^{\infty} \frac{1}{(k+\nu)!}\frac{d^{(k+\nu)}}{ds^{(k+\nu)}}\big(\mathbf{K}_s^{-1}(s_p + \Delta s)\big)\Delta s^k\mathbf{f}\|$$
$$\le c|\Delta s|^\nu \tag{104}$$

where c is chosen uniformly for all Δs sufficient small. For the second order dynamical system given by eq. (67) the matrix $\mathbf{C}$ is considered equal to the identity matrix and one is interested in the state vector $\mathbf{u}$, therefore taking the vector $\mathbf{f}$ of eq. (102) equal to the force vector, and the fact that $s = i\omega$, eq. (102) holds true if

$$\bigcup_{p=1}^{P} \text{span}\left\{ \mathbf{u}(\omega_p), \frac{d}{d\omega}\big(\mathbf{u}(\omega_p)\big), \cdots, \frac{d^{\nu-1}}{d\omega^{\nu-1}}\big(\mathbf{u}(\omega_p)\big)\right\} \subset \text{Ran}(\mathbf{V}) \tag{105}$$

$\mathbf{u}$ and its derivatives are found by solving eq. (75) for $k = 0, 1, 2, ..., \nu - 1$.

6 Numerical Examples

Two numerical example are addressed in this section. In the first example a frequency response analysis using the ten MOR techniques mentioned in section 1 are applied to a cantilever beam in order to evaluate the quality of the approximation of those methods for proportional and non-proportional damped systems. In the second example a medium-sized structure is investigated in order to evaluate the efficiency in computation time of the Interpolatory MOR methods. In both examples the solutions given by the MOR methods are compared with the solution of the full-order model (DM).

6.1 Cantilever Beam

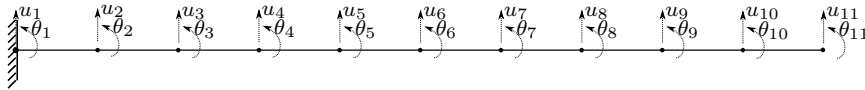A cantilever beam has been discretized in 10 finite elements, as illustrated in fig.4,

Figure 4: FE discretization of a catiliver beam.

The beam has a length of $1m$, cross section of $0.1m \times 0.1m$, modulus of elasticity of $2.068e11\frac{N}{m^2}$ and material density of $7830 \frac{kg}{m^3}$. It is loaded in the vertical direction with a distributed load of $-100 \frac{N}{m}$. $\Delta_\omega = [1, 700]$ and $n_\omega = 700$. In the Real Modal Analysis (RMA) and the complex Modal Analysis (CMA) 20 modes shapes are considered. The condensation methods use the vertical displacement as the active set of dofs, $\mathbf{u}_a$. The Dynamic Reduction

(DR) is done about $\omega_{dy} = 50$ and the IRS reduction method uses 3 iterations. Craig-Bampton method uses 5 fixed-interface normal modes. For the Interpolatory MOR methods $\omega_p = \{1, 30, 80, 110, 300, 500, 700\}$. The response of the dof u_{11} will be compared for a Rayleigh-damped and non-proportional damped beam.

6.1.1 Rayleigh Damped Cantilever Beam

The Rayleigh damping matrix is defined as $\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}$. In this example $a_0 = 0.0002$ and $a_1 = 0.0001$. The ten MOR methods are able to capture the exact solution in amplitude and phase angle, see figs. 5 and 6, even the GR, and IRS which are in general only suitable for a low frequency range.
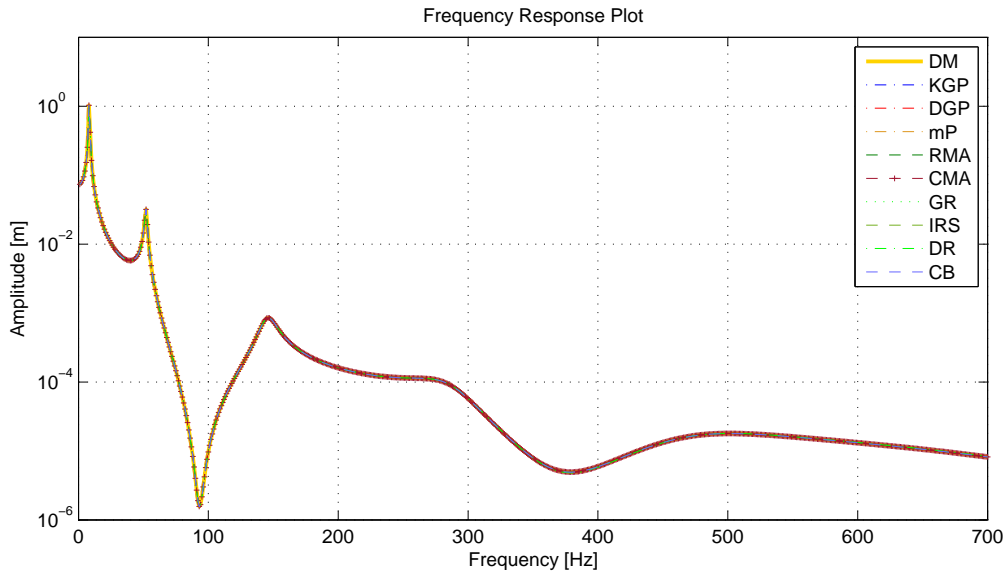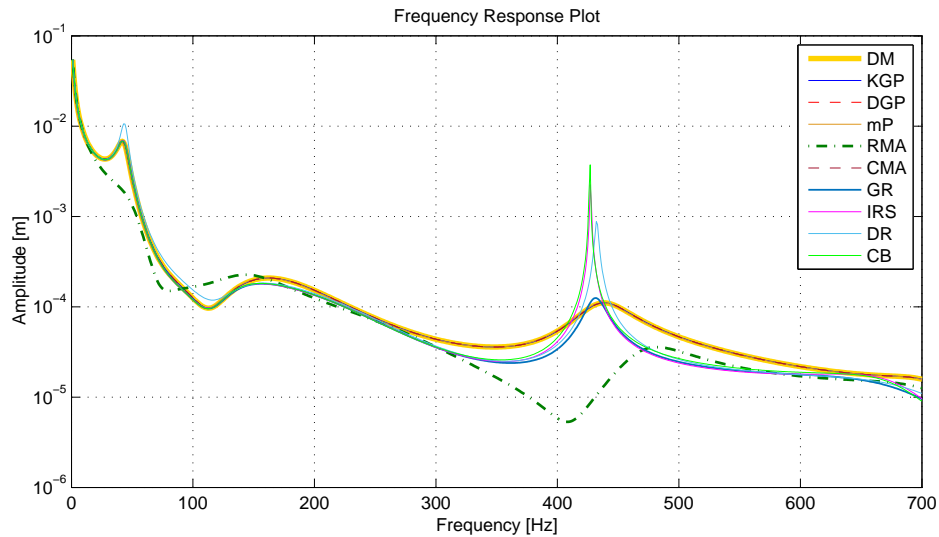

Figure 5: Amplitude of the response for u_{11} of the Rayleigh-damped cantilever beam.
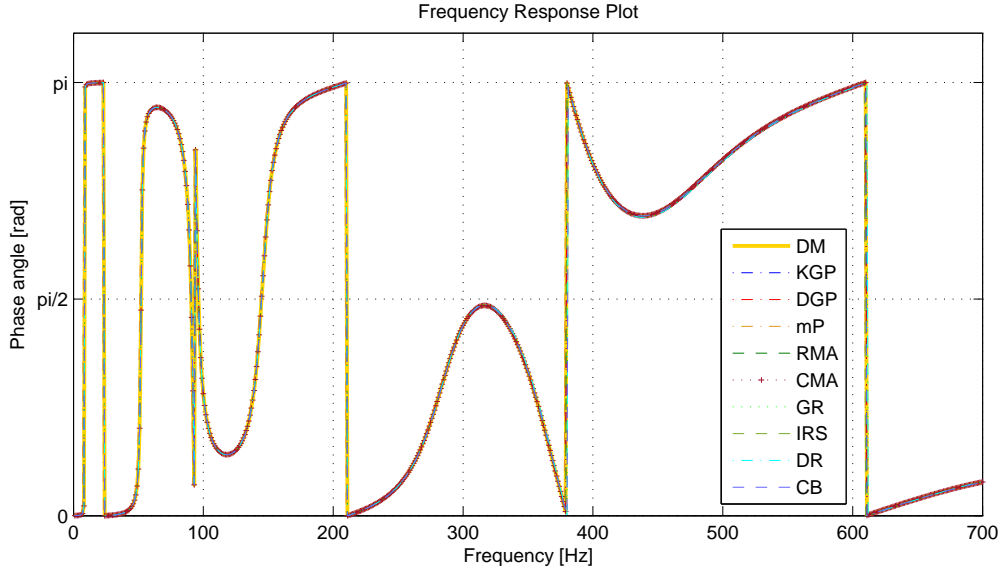
Figure 6: Phase angle of the response for u_{11} of the Rayleigh-damped cantilever beam.

6.1.2   Non-proportional Damped Cantilever Beam

For an arbitrary damped cantilever beam a random dense matrix was defined in MATLAB R2015b® using the command $randn$ with a $seed$ value of 0.1, a mean of 0.15 and a standard deviation of 0.1. That matrix does not have a physical meaning due to the fact that the matrix was not derived according to the assembling of the elements, nevertheless it is important to see how the MOR methods behave for a simple arbitrary damped system like the 10-elements cantilever beam.
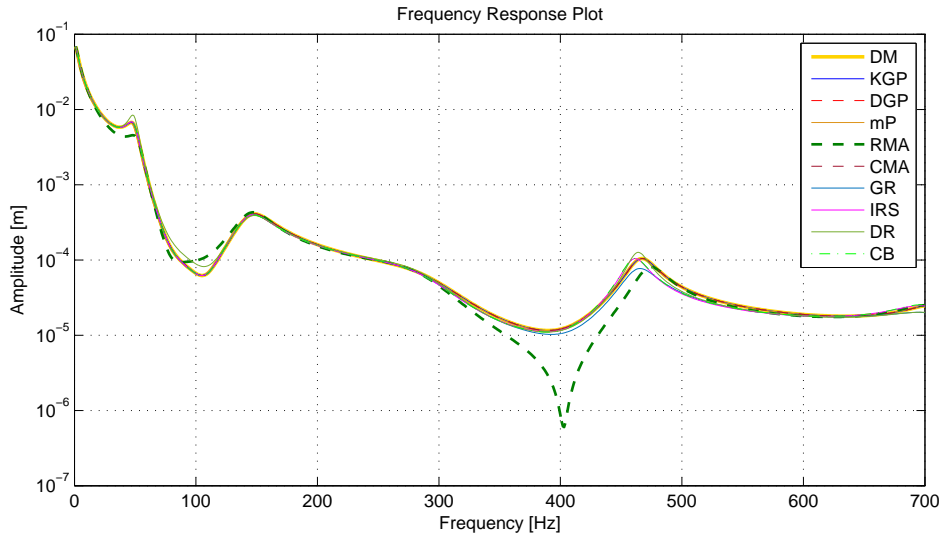
Figure 7: Amplitude of the response for u_{11} of the non-proportional damped cantilever beam.

From figs. 7 and 8 it is clear that the RMA gives a result far from the exact solution, the condensations methods and the CB method do not capture the exact solution any more as the Interpolatory MOR Methods (DGP, KGP and the mP) and the CMA do.
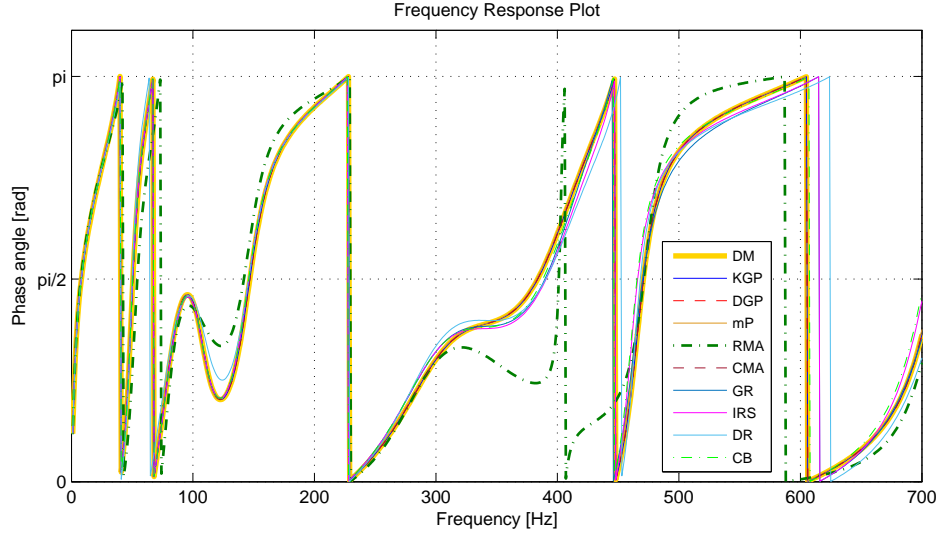
Figure 8: Phase angle of the response for u_{11} of the non-proportional damped cantilever beam.

If the damping increases, the capability of the Interpolatory MOR Methods are more evident. In figs. 9 and 10, the response of u_{11} is shown in the case of an arbitrary damping matrix, created in the same way as before, but with mean value and standard deviation equal to 0.35 and 0.2 respectively. It results in a more evident lack of approximation for the RMA, Condensation Methods and CB Method, in contrast with the Interpolatory MOR Methods and the CMA which lead to the exact solution. Nevertheless the CMA is not suitable for large systems due to the expensive computation of the complex modes, so the Interpolatory MOR Methods remain as a very attractive possibility to be used as standard MOR techniques for frequency response analysis in case of non-proportional damped systems.
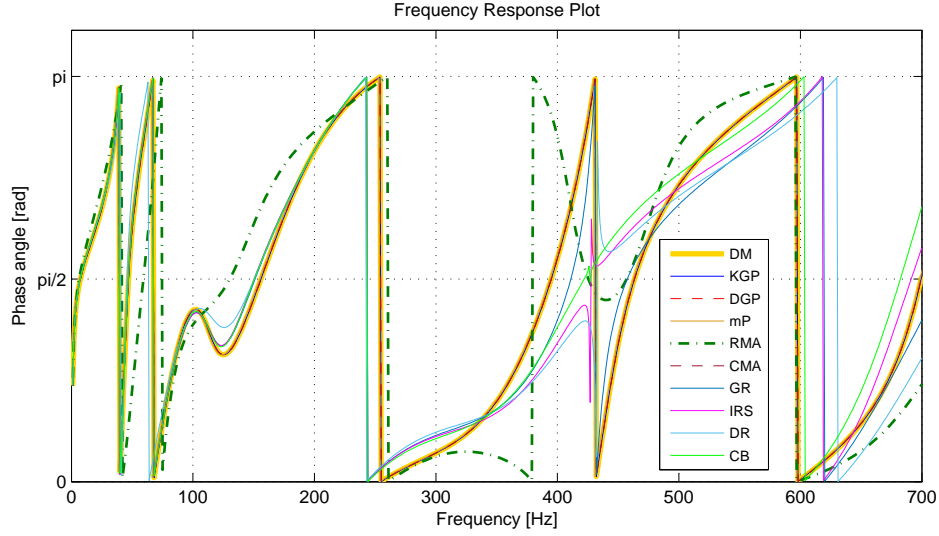

Figure 9: Amplitude of the response for u_{11} of the non-proportional damped cantilever beam

Figure 10: Phase angle of the response for u_{11} of the arbitrary damped cantilever beam

6.2 Solar Panel - Structure

The second numerical example is a solar panel modeling with ANSYS®, see fig. 11. It is a $4\ m \times 12\ m$ structure, built out of beam and shell elements. The beam188 element was used for the frame, with an elastic modulus of $2 \times 10^{11}\ \frac{N}{m^2}$, a Poisson's ratio of 0.3 and mass density of $7850\ \frac{kg}{m^3}$. For the panel the shell181 element was used, with an elastic modulus of $0.7 \times 10^{11}\ \frac{N}{m^2}$, a Poisson's ratio of 0.3 and mass density of $2500\ \frac{kg}{m^3}$. This structure was loaded with a distributed load of $50N$ in vertical direction (Y direction of the global coordinate system). In total, the solar panel contains 9582 dofs, 3 nodes are fixed, i.e. 18 dofs, and has 33 coupling dofs, thus the equation system to be solved contains 9531 unknowns (displacements and rotations).
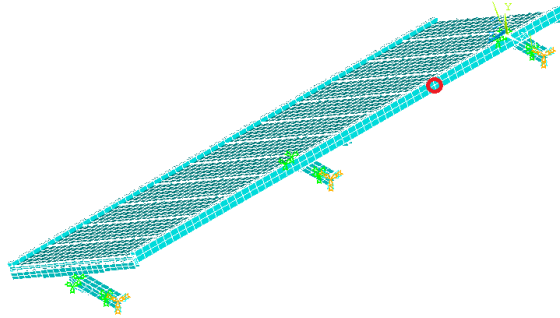

Figure 11: Solar panel modeling in ANSYS®.

For medium- and large-scale structures without going into the sub structuring approach (CB method) the RMA is extensively used in frequency response analysis. The computation of the set of mode shapes is the time-demanding task in the RMA but leads to a reduced-decouple equation system. RMA can be also applied over a large frequency interval $\Delta\omega$, unlike condensation methods which offer good approximation only in a narrow $\Delta\omega$ near to $\omega = 0$ or to the chosen dynamic frequency ω_{dy} in case of DR. For those reasons the computational time of the

Interpolarory MOR are compared to that of the RMA and the DM solving a non-damped solar panel - structure.

6.2.1 Interpolatory MOR Methods and RMA

For this first analysis $\Delta_\omega = [1, 100]$, $n_\omega = 100$, $P = 16$, and ω_p={ 5, 10, 15, 20, 25, 30, 40, 50, 55, 60, 65, 70, 85, 90, 95, 100 }. mP uses $P = 10$ and ω_p={1, 15, 30, 40, 50, 60, 75, 85, 95, 100}. RMA number of modes is equal to 100.

The computation time required for this analysis is shown in tab. 1 and the amplitude of the frequency response for dof UY of the node 153, marked with a circle over the upper beam in fig. 11, is depicted in fig. 12.

$Method$	$c.t.[s]$	$c.t.parallel[s]$
DM	10.330	5.231
mP	3.650	2.153
KGP	3.840	2.560
DGP	3.900	2.249
RMA	1.820	2.316

Table 1: Computation time using DM, mP, KGP, DGP and RMA.

The comparison in computation time (c.t.) shows a faster solution for the parallel implementation of Interpolatory MOR Methods, i.e. the KGP and DGP are 4.5 times faster than the DM runs in the conventional way. The RMA provides also a fast computation for this problem and overlaps the exact solution as KGP and DGP. The mP computation time in tab. 12 is just for the computation of the response of a single dof and fig. 12 shows that the mP-response is slightly shifted from the exact one. For the mP approximation, it is recommended to used a mesh with few interpolation points, e.g. for this example $P < 10$, due to the fact that the matrix $\mathbf{A}$ in eq. (76) is ill-conditioned for large P [9], which yields to wrong coefficients and therefore wrong approximation of the response.
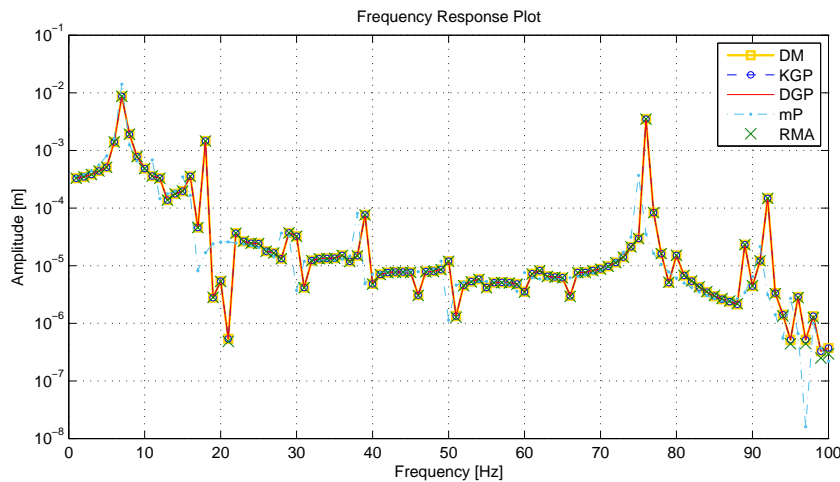

Figure 12: Amplitude of the response using DM, KGP, DGP and RMA.

6.2.2 Interpolatory MOR Methods

In this second investigation $\Delta_\omega = [1, 1000]$, $n_\omega = 3000$, $P = 101$, ω_p={5, 10, 20, 30, 40, 50, ... , 500, 510, 520,...,1000} and RMA number of modes is equal to 500.

The computation time required for this analysis is written in tab. 2 showing that the KGP and DGP (run in parallel) are approximately 18 times faster than the DM and 6 times faster than the DM runs in parallel. Also the KGP and DGP are 2 time faster than the RMA when they are run in parallel.

$Method$	$c.t.[s]$	$c.t.parallel[s]$
DM	274.732	91.884
KGP	42.267	15.522
DGP	39.731	15.507
RMA	38.236	31.439

Table 2: Computation time using DM, mP, KGP, DGP and RMA.

The amplitude of the frequency response for dof UY of the node 153 , marked with a circle over the upper beam in fig. 11, is depicted in figs. 13, 14, 15 and 16.
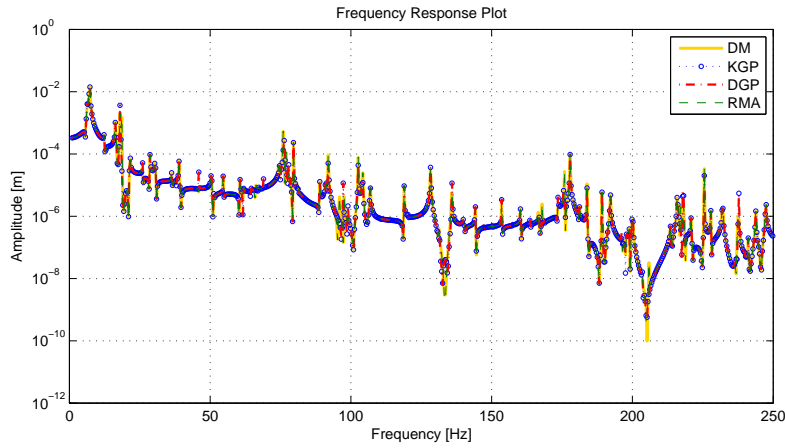

Figure 13: Amplitude of the response using DM, KGP, DGP and RMA: $\Delta_\omega = [1, 250]$.
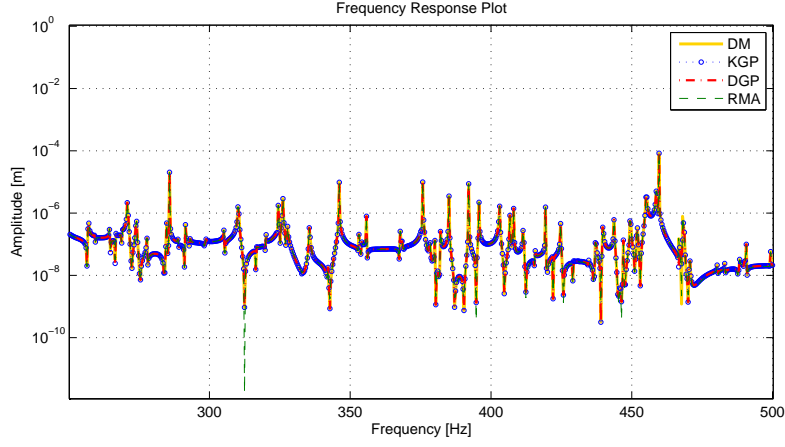
Figure 14: Amplitude of the response using DM, KGP, DGP and RMA: $\Delta_\omega = [250, 500]$.
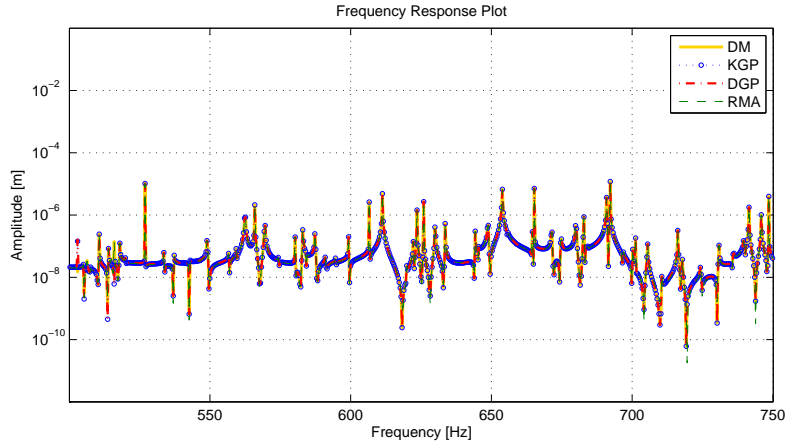

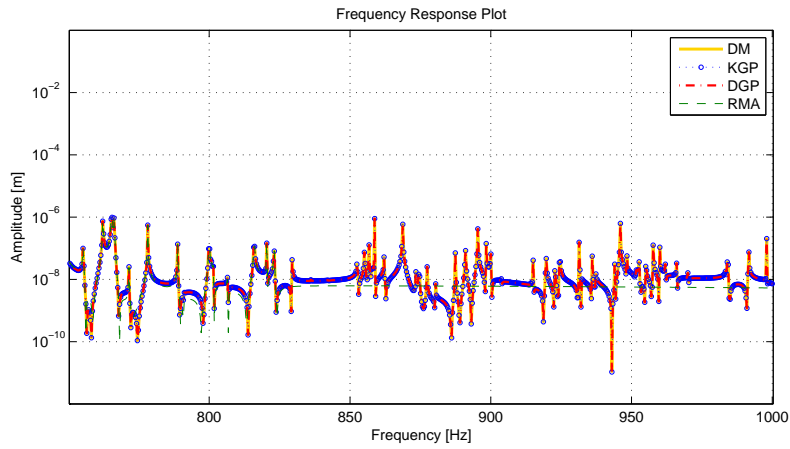Figure 15: Amplitude of the response using DM, KGP, DGP and RMA: $\Delta_\omega = [500, 750]$.

Figure 16: Amplitude of the response using DM, KGP, DGP and RMA: $\Delta_\omega = [750, 1000]$.

The solutions given by the KGP and DGP overlap the exact solution almost everywhere in

the Δ_ω and the RMA using 500 modes is able to capture the exact solution until the 865Hz, see Fig. 16, using almost the same amount of time.

7 CONCLUSIONS

As shown in this work several approaches are available to have a reduced-order model for a second order dynamical system of the form given by eq. (2). Some of them have been extensively used in the structural dynamics fields when $\Delta\omega$ is a narrow band and the damping matrix is proportional to the stiffness and mass matrix. Those are: the GR, IRS and DR. Another classical technique like RMA is suitable when the eigen-frequencies of the modal matrix are the main eigen-frequencies in $\Delta\omega$ which does not limit the interval $\Delta\omega$. In a similar way, the Interpolatory MOR are applicable in a large $\Delta\omega$ with all the interpolatory points contained in that interval.

For non proportional-damped systems the CMA and the Interpolatory MOR are the best choices, but the CMA increases the dimension of the matrices. The mP is applicable in a narrow $\Delta\omega$ or when few interpolation points are needed. The KGP and DGP can be applied to any kind of system defined by eq. (2) over a narrow or large $\Delta\omega$. Thus they are very promising methods to obtain a reduced-order model with cheaper computational cost and good approximation to the exact solution for proportional and non-proportional damped systems.

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