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Modeling Credit Risk in the Jump Threshold Framework

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ABSTRACT *The* jump threshold framework *for credit risk modeling developed by Garreau and Kercheval (2016) enjoys the advantages of both structural and reduced form models. In their paper, the focus is on multi-dimensional default dependence, under the assumptions that stock prices follow an exponential Lévy process (i.i.d. log returns) and that interest rates and stock volatility are constant. Explicit formulas for default time distributions and Basket CDS prices are obtained when the default threshold is deterministic, but only in terms of expectations when the default threshold is stochastic.*

In this paper we restrict attention to the one-dimensional, single-name case in order to obtain explicit closed-form solutions for the default time distribution when the default threshold, interest rate, and volatility are all stochastic. When the interest rate and volatility processes are affine diffusions and the stochastic default threshold is properly chosen, we provide explicit formulas for the default time distribution, prices of defaultable bonds, and CDS premia. The main idea is to make use of the Duffie-Pan-Singleton method of evaluating expectations of exponential integrals of affine diffusions.

KEY WORDS: Credit Risk, Lévy Processes, affine processes.

1. Introduction

There are two main approaches to model credit risk: structural and reduced form. In structural models a company's asset value is specified as some stochastic process, and the default event is defined as some stopping time for this process. A classical example of a structural model is the first passage time model proposed by Black and Cox (1976); the default event is triggered by the firm asset value dropping below a specified default barrier derived, perhaps, from the safety covenants of the bond indenture provisions.

This class of models has several drawbacks. Continuous firm value processes usually lead to predictable default times, which is considered a disadvantage. Analytic tractability tends to be low, requiring the use of numerical approaches to compute default probabilities. Although in the simplest case considered by Black and Cox (1976), the default time distribution in one dimension is known to be inverse Gaus-

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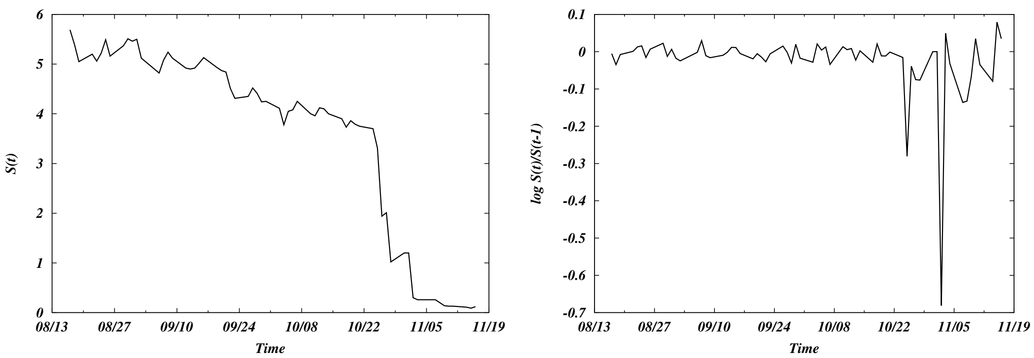
sian, if one is interested in the joint default time distribution of two companies, the formula involves an infinite sum of modified Bessel functions, as discussed in Section 3.6 of (Bielecki and Rutkowski 2013). For three or more companies there is no known formula for the joint default time distribution. Even in one dimension, analytical tractability tends to be a result of sacrificing model flexibility.

The reduced form model framework was developed by Jarrow and Turnbull (1995). Unlike structural models, in reduced form models the firm's financial structure is not an explicit ingredient. Instead, an exogenous default time distribution is specified in terms of a default intensity process. This usually leads to explicit formulas for the default time distribution, and thus prices of credit instruments. The parameters of the intensity process can then be estimated from market quotes.

In these models, the link between a firm's default time and its performance or financial structure is only through the intensity process, which is not observable or clearly linked to market observables. Nevertheless, reduced form models have excellent flexibility and analytical tractability, even in many dimensions. The joint default time distribution of a group of companies can be obtained in closed form by assigning an intensity process to each company. The dependence structure between the default times is modeled through the dependence structure between the intensity processes. Realistic assumptions such as random interest rates can be easily included without sacrificing tractability. In addition, quite general point processes, including self-exciting processes, can be used, for example the generalization of Hawkes processes in (Errais, Giesecke, and Goldberg 2010). That is why in practice reduced form models tend to be more popular than structural models.

Seeking to combine the advantages of both structural and reduced form models, Garreau and Kercheval (2016) introduced what we call the jump threshold models. In this framework, the default barrier to the firm asset value process in the first passage time model is replaced by a default threshold for the jump size of the instantaneous log-return of the stock price. The default event of a company is triggered by the instantaneous log-return of the stock price falling below the (negative) default threshold. This approach is motivated by default events such as that of MF Global, shown in Figure 1. Default occurred not on the day of the largest absolute price drop, but at the largest relative drop a few days later.



(a) MF Global. Stock price. (b) MF Global. Daily log returns.

Figure 1. Bankruptcy of MF Global, October-November 2011. Bankruptcy occurred on Oct. 31, 2011, not at the largest absolute drop in stock price, but at the largest relative drop.

More precisely, let S_t denote the stock price process and S_{t-} its left limit at time

t. Denote by $a(t) < 0$ a (possibly stochastic) default threshold. Then we model the firm's default time τ as

$$\tau = \inf\{t > 0 : \log(S_t/S_{t-}) \leq a(t)\}.$$

Clearly there would be no defaults for a continuous stock price process S_t, so the approach requires us to consider jump processes for the underlying stocks. In (Garreau and Kercheval 2016), exponential Lévy processes are used.

The analytical tractability of a default threshold model is much better than the first passage time model, but is still directly linked to the observable stock price process. Therefore it allows for the use of a consistent set of models to price credit derivatives and options on the same assets.

For any deterministic default threshold $a(t)$, the default time distribution is given explicitly in terms of the tail integral Λ of the Lévy process $Y_t = \log S_t$ according to

$$P(\tau > t) = e^{-\int_0^t \Lambda(a(s))\, ds}. \tag{1}$$

When the default threshold $a(t)$ is stochastic but independent of the jump part of the stock price process, (Garreau and Kercheval 2016) shows

$$P(\tau > t) = E[e^{-\int_0^t \Lambda(a(s))\, ds}]. \tag{2}$$

However, in this generality, they are silent on how to evaluate this expectation.

In higher dimensions, the dynamics of the stock price of each of a group of companies is modeled by a separate exponential Lévy process. The stock price processes can be dependent, and the dependence structure between the default times of the companies is modeled by a Lévy copula. Each company has its own default threshold. The joint default time distribution is known in closed form in any dimensions if the default thresholds are nonrandom, as shown for two dimensions in (Garreau and Kercheval 2016), and known up to expectations for stochastic thresholds independent of the price jumps.

This paper aims to extend and increase the usefulness of results of (Garreau and Kercheval 2016) for the one-dimensional case. The stock price process need not be exponential Lévy, but can have stochastic volatility and incorporate a stochastic interest rate. When the default threshold is stochastic it can depend on the volatility and the interest rate while still yielding explicit formulas for the default time distribution.

We hope that this increased model flexibility will make the default threshold approach more attractive to practitioners.

The most general case we treat is an exponential jump diffusion process with stochastic volatility and interest rate (see Section 4.2). The stock price is given by $S_t = S_0 e^{L_t}$ where

$$\begin{cases} L_t &= \int_0^t \left(R_u - \frac{V_u}{2}\right) du + \int_0^t \sqrt{V_u}\, dW_u^S + Z_t - t\psi(-i) \\ dV_t &= \kappa(\theta - V_t)\, dt + \sigma\sqrt{V_t}\, dW_t^V \\ dR_t &= \gamma(\delta - R_t)\, dt + \eta\sqrt{R_t}\, dW_t^R \end{cases} \tag{3}$$

where W^R is independent of W^S and W^V and Z_t is independent of all three. Here we use a Cox-Ingersoll-Ross (CIR) interest rate model; for comparison we also analyze the case where we substitute a Vasicek interest rate model of the form

$$dR_t = \gamma(\delta - R_t)\,dt + \eta\,dW_t^R.$$

Optionally we may introduce an exogenous default factor X_t as an independent non-negative square root diffusion process

$$dX_t = \alpha(\beta - X_t)\,dt + \xi\sqrt{X_t}\,dW_t^X$$

representing factors external to the stock and interest rate. We can then obtain explicit formulas for the default time distribution when the default threshold is of the form

$$a_t = \Lambda^{-1}(bV_t + cR_t + X_t),$$

where $b > 0$ and $c \geq 0$ are constants measuring the sensitivity of default to volatility and interest rate.

The paper is organized as follows. In Sections 2 and 3 we describe the relevant results of (Garreau and Kercheval 2016), and review standard results on affine diffusions. Section 4 describes our models and presents explicit solutions for the default time distribution, bond prices, credit spreads, and credit default swap spreads. We discuss some illustrative numerical experiments on the significance and sensitivity of the parameters in Section 5. Most of the proofs are postponed to Section 7.

2. Jump Threshold Framework

This section is a brief review of the jump threshold framework results of (Garreau and Kercheval 2016) in one dimension. For background on Lévy processes, see Applebaum (2004), Cinlar (2011), or Sato (1999). We let L_t be a Lévy process adapted to a filtered probability space $(\Omega, P, \mathcal{F})$ and with Lévy measure λ, and define a stock price process by $S_t = S_0 e^{L_t}$. We take P to be the risk neutral measure.

Equity price models based on these *exponential Lévy processes* are commonly studied. Because Lévy processes (and therefore exponential Lévy processes) can have path discontinuities, it becomes possible to define the default time as the first time the price jumps downward by a given minimum percentage:

$$\tau = \inf\{t > 0 \mid \log(S_t/S_{t-}) \leq a(t)\}, \tag{4}$$

where $a(t) < 0$ is called the default threshold and is allowed to be stochastic.

Definition 2.1. Let L_t be a Lévy process and λ its Lévy measure. The *tail integral* of λ is defined by

$$\Lambda(z) = \int_{-\infty}^{z} \lambda(dx) = \lambda((-\infty, z))$$

if $z < 0$ and $\Lambda(z) = \int_z^\infty \lambda(dx)$ if $z > 0$. For our purposes we only need the tail integral for negative z, so we will assume the tail integral is a function $\Lambda : (-\infty, 0) \to (0, \infty)$.

Because λ is a measure, Λ is non-decreasing.

Assume for the moment that the default threshold $a(t)$ is a constant a. By the Lévy-Itô decomposition, we can write $L_t = \mu t + \sigma W_t + Z_t$, where W_t is a standard Brownian motion, and Z_t a pure jump Lévy process that is independent of W_t. Let λ be the Lévy measure of Z_t. Since W_t is continuous, we have $\log(S_t/S_{t-}) = Z_t - Z_{t-}$. The default time is thus the first time Z_t has a jump with size in $(-\infty, a]$. If we denote by J_L the jump measure (Poisson random measure) obtained by the Lévy-Itô decomposition of L_t, then $J_L([x, y] \times [0, t])$ counts the (random) number of jumps that happen between times 0 and t and with size between x and y.

Thus the survival probability can be written as

$$P(\tau > t) = P\left(\int_0^t \int_{-\infty}^a J_L(dxds) = 0\right). \tag{5}$$

Since J_L is a Poisson random measure with Lévy measure $\lambda(dx)ds$, we know $\int_0^t \int_{-\infty}^a J_L(dxds)$ is a Poisson random variable with parameter

$$\int_0^t \int_{-\infty}^a \lambda(dx)ds = t\int_{-\infty}^a \lambda(dx) = t\Lambda(a).$$

This observation leads us to the survival probability formula

$$P(\tau > t) = e^{-t\Lambda(a)}. \tag{6}$$

When $a(t)$ is not constant, similar reasoning leads to the following result, which is a slight generalization of the statement proved in (Garreau and Kercheval 2016), with similar proof.

Theorem 2.2. Let a company's stock price follow an exponential Lévy process $S_t = S_0 e^{L_t}$, with Lévy-Itô decomposition $L_t = \mu t + \sigma W_t + Z_t$. Let Λ be the tail integral of the Lévy measure of Z_t. Define the default time as

$$\tau = \inf\{t > 0 \mid \log(S_t/S_{t-}) \leq a_t\} \tag{7}$$

for some strictly negative default threshold a_t.

(1) If a_t is a deterministic measurable function that is locally bounded below zero, then the survival probability is given by

$$P(\tau > t) = e^{-\int_0^t \Lambda(a_s)\, ds}. \tag{8}$$

(2) If a_t is a measurable predictable stochastic process, independent of Z_t, and with paths locally bounded below zero almost surely, then the survival probability is given by

$$P(\tau > t) = E\left[e^{-\int_0^t \Lambda(a_s)\, ds}\right]. \tag{9}$$

For example, if a_t is strictly negative and continuous, as will be the case with the examples studied in this paper, then it is pathwise locally bounded below zero and the hypothesis of the theorem is satisfied.

3. Modeling with Affine Diffusion Processes

The aim of this paper is to examine models for which the default time distribution (9) can be computed in closed form. Of course this is immediate unless the default threshold is stochastic, in which case we need to be able to compute an expectation. In the case of a single defaultable underlying stock, we show that we can move well beyond exponential Lévy stock price models, as treated in (Garreau and Kercheval 2016), to models with stochastic volatility and stochastic interest rates and still provide explicit solutions for a flexible class of stochastic default thresholds.

Our technique is to make use of affine processes, and a method due to Duffie, Pan, and Singleton (2000) that allows us to find a closed form solution when the integrand in the exponent is an affine process.

An affine diffusion process is a Markov process that satisfies the SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

where the drift term μ is an affine (determininstic) function of X_t, and the diffusion term $\sigma(t, X_t)$ is the square root of an affine function of X_t. That is, the SDE takes the form

$$dX_t = (a(t) + b(t)X_t)dt + \sqrt{c(t) + d(t)X_t}dW_t.$$

Given an affine diffusion process X_t, we are interested in finding a closed form for an expectation of the form

$$E\left[e^{k\int_0^T X_t\,dt}\right]$$

for some constant k.

At this level of generality, we have

Proposition 3.1. Let X_t be an affine diffusion process satisfying

$$dX_t = (a(t) + b(t)X_t)dt + \sqrt{c(t) + d(t)X_t}\,dW_t,$$

where the coefficients a, b, c, d are deterministic, well-behaved (e.g. continuous) functions of t.

Let $f(t, r)$ be the conditional expectation

$$f(t, r) = E\left[e^{k\int_t^T X_s\,ds}\middle| X_t = r\right].$$

Provided that the solution $C(t, T)$ to the terminal value problem

$$\begin{cases} \frac{\partial C}{\partial t} = \frac{d(t)}{2}C^2 - b(t)C + k \\ C(T, T) = 0 \end{cases} \tag{10}$$

exists and is unique, we have

$$f(t,r) = \exp(-rC(t,T) - A(t,T)),$$

where

$$A(t,T) = \int_t^T a(s)C(s,T) - \frac{c(s)}{2}C^2(s,T)\,ds. \tag{11}$$

When the coefficient functions a, b, c, d are constants, (10) is known as a Riccati equation, which can be solved analytically, yielding an explicit formula for $f(t,r)$.

An affine diffusion process that plays an important role in the rest of this paper is the square root diffusion process, which is the solution to the SDE

$$dY_t = \alpha\,(\beta - Y_t)\,dt + \xi\sqrt{Y_t}dW_t,$$

where $\alpha > 0$, $\beta > 0$ and $\xi > 0$ are constants. This process is also known as the Cox-Ingersoll-Ross (CIR) process because it is the short rate process in the CIR interest rate model. Starting with Feller (1951), the square root diffusion process has been well-studied. It is a mean-reverting continuous process because it tends to move towards its long term mean β (with the speed α). It is also nonnegative, which is convenient when modeling the interest rate or the volatility. When $\xi^2 \leq 2\alpha\beta$, the process is strictly positive, or otherwise it occasionally hits zero and become positive again. For more details, see Karatzas and Shreve (2012), Øksendal (2013).

Corollary 3.2. (Shreve 2004) Let Y_t be a square-root diffusion process that satisfies

$$dY_t = \alpha\,(\beta - Y_t)\,dt + \xi\sqrt{Y_t}dW_t$$

for some positive constants α, β and ξ and with initial condition Y_0. Then the expectation $E\left[e^{-\int_0^T Y_t\,dt}\right]$, denoted by $\Psi(\alpha,\beta,\xi,T,Y_0)$ hereafter, is known in closed form as

$$E\left[e^{-\int_0^T Y_t\,dt}\right] \equiv \Psi(\alpha,\beta,\xi,T,Y_0) = e^{-Y_0C(T)-A(T)}, \tag{12}$$

where

$$\begin{aligned} C(T) &= \frac{\sinh(\gamma T)}{\gamma\cosh(\gamma T) + \frac{1}{2}\alpha\sinh(\gamma T)}, \\ A(T) &= -\frac{2\alpha\beta}{\xi^2}\log\left(\frac{\gamma e^{\frac{1}{2}\alpha T}}{\gamma\cosh(\gamma T) + \frac{1}{2}\alpha\sinh(\gamma T)}\right), \\ \gamma &= \frac{1}{2}\sqrt{\alpha^2 + 2\xi^2}. \end{aligned}$$

Furthermore, for $\kappa > 0$,

$$E\left[e^{-\int_0^T \kappa Y_t\,dt}\right] = \Psi(\alpha,\kappa\beta,\sqrt{\kappa}\xi,T,\kappa Y_0). \tag{13}$$

Proof. The first part of the statement appears in Shreve (2004, sec. 6.5), where C is the solution of a Riccati equation. The second part is immediate by considering the process κY_t. □

For comparison, we may also consider the simpler affine model with constant volatility known as the Ornstein-Uhlenbeck process or the Vasicek interest rate model:

Corollary 3.3. (Shreve 2004) Let Y_t be an OU process satisfying

$$dY_t = \alpha\,(\beta - Y_t)\,dt + \xi dW_t,$$

where α, β, ξ are positive constants, and with initial condition Y_0. Then

$$E\left[e^{-\int_0^T Y_t\,dt}\right] \equiv \Upsilon(\alpha,\beta,\xi,T,Y_0) = e^{-Y_0 C(T) - A(T)}, \tag{14}$$

where

$$\begin{aligned} C(T) &= \frac{1}{\alpha}(1-e^{-\alpha T}), \\ A(T) &= \beta\left(T - \frac{1}{\alpha}(1-e^{-\alpha T})\right) \\ &\quad -\frac{\xi^2}{2\alpha^2}\left(T - \frac{2}{\alpha}(1-e^{-\alpha T}) + \frac{1}{2\alpha}(1-e^{-2\alpha T})\right). \end{aligned}$$

Furthermore, for $\kappa > 0$, we have

$$E\left[e^{-\int_0^T \kappa Y_t\,dt}\right] = \Upsilon(\alpha,\kappa\beta,\sqrt{\kappa}\xi,T,\kappa Y_0). \tag{15}$$

4. Stochastic Default Threshold Models with Explicit Default Distribution Functions

To describe the idea in simplest form, we begin with an exponential Lévy model for the stock price,

$$S_t = S_0 e^{L_t},$$

where $L_t = \mu t + \sigma W_t + Z_t$, W_t is Brownian motion with respect to a risk-neutral probability measure, Z_t is a pure jump Lévy process with Lévy measure λ, tail integral Λ, and r is the (constant) risk free interest rate. For the model to be arbitrage-free, we require

$$\mu = r - \frac{\sigma^2}{2} - \psi(-i),$$

where $\psi(u) = \log E[e^{iuZ_1}]$ is the characteristic exponent of Z_t.

Definition 4.1. We say that the jump process Z_t with Lévy measure λ is *suitable* if

(1) Z_t has infinite activity, i.e. $\lambda((-\infty, 0)) = +\infty$, and
(2) $\lambda(I) > 0$ for every non-empty open interval $I \subset (-\infty, 0)$.

We will henceforth restrict attention to suitable Lévy processes. This is a mild restriction, since most commonly used Lévy processes for stock modeling are suitable, e.g. α-stable, variance gamma, CMGY, etc., or can be approximated by a suitable process.

When Z_t is suitable, the tail integral $\Lambda : (-\infty, 0) \to (0, \infty)$ becomes a one-to-one correspondence, and therefore has a unique and well-defined inverse $\Lambda^{-1} : (0, \infty) \to (-\infty, 0)$.

Consider now an independent strictly positive CIR process

$$dX_t = \alpha\,(\beta - X_t)\,dt + \xi\sqrt{X_t}dW_t^X \tag{16}$$

for some constants α, β and ξ that satisfy $\xi^2 \leq 2\alpha\beta$, and where W_t^X and W_t are independent. X_t is intended to represent factors external to the stock price process that may influence the default probability.

If we now define our default threshold process by

$$a_t = \Lambda^{-1}(X_t),$$

then the default time distribution (9) is

$$\begin{aligned} P(\tau > t) &= E\left[e^{-\int_0^t \Lambda(\Lambda^{-1}(X_s))\,ds}\right] \\ &= E\left[e^{-\int_0^t X_s\,ds}\right] \\ &= \Psi(\alpha, \beta, \xi, t, X_0), \end{aligned} \tag{17}$$

where the function Ψ is defined explicitly as before in (12).

This particular form of the default threshold, "lambda-inverse-affine", allows the modeler to introduce extra parameters α, β, ξ that can be fitted to relevant external factors while still allowing for explicit solutions for credit prices.

4.1 *Extension with Stochastic Volatility*

In the exponential Lévy model above, the stock volatility σ is a constant that does not directly influence the default probabilities. Not only does this open the door to considering more general stochastic volatility models, but also to introducing the stock volatility as a factor influencing the default threshold. This allows for models, described next, in which spot volatility is flexibly linked to local default probabilities, as might be the case in real markets.

Assume a constant interest rate r. We model[1] the stock price process by $S_t = S_0 e^{L_t}$, where

$$\begin{cases} L_t &= \int_0^t \left(r - \frac{V_u}{2}\right)\,du + \int_0^t \sqrt{V_u}\,dW_u^S + Z_t - t\psi(-i) \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^V \end{cases}, \tag{18}$$

[1]Without the pure jump process Z_t, this is the same as in the Heston model. With the jump process, this model is similar to the one proposed by Bates (1996), except we are using Lévy processes with infinite activity rather than compound Poisson processes.

$E[dW_t^S dW_t^V] = \rho dt$, Z_t is a suitable pure jump Lévy process with characteristic exponent ψ that is independent of W_t^S and W_t^V, and κ, θ, and σ are positive constants.

Let X_t be a non-negative CIR process independent of V_t and Z_t:

$$dX_t = \alpha\,(\beta - X_t)\,dt + \xi\sqrt{X_t}dW_t^X \tag{19}$$

with $\alpha > 0$, $\beta > 0$, $\xi > 0$ satisfying $\xi^2 \leq 2\alpha\beta$. (Correlation between W^X and W^S is permitted.) For $b > 0$ define the default threshold as

$$a_t = \Lambda^{-1}(bV_t + X_t). \tag{20}$$

The threshold a_t is well-defined and strictly negative since $bV_t + X_t$ is strictly positive. For a default threshold defined this way, the likelihood of default (hazard rate) increases with stock volatility. When $\rho < 0$, the underlying stock price model can also reflect the leverage effect in which a drop in the stock price is correlated to a rise in volatility and hence a rise in the hazard rate. Roughly speaking b measures how sensitive the default threshold is to the volatility of the stock price, and X_t is the part of the default threshold that is not explained by the volatility. In the case $b = 0$ the default threshold is independent of the volatility, and the survival probability reduces to (17).

The survival time probability becomes:

$$\begin{aligned} P(\tau > t) &= E\left[e^{-\int_0^t \Lambda(\Lambda^{-1}(bV_s + X_s))\,ds}\right] \\ &= E\left[e^{-\int_0^t bV_s + X_s\,ds}\right] \\ &= E\left[e^{-\int_0^t bV_s\,ds}\right] E\left[e^{-\int_0^t X_s\,ds}\right] \\ &= \Psi(\kappa, b\theta, \sqrt{b}\sigma, t, bV_0)\Psi(\alpha, \beta, \xi, t, X_0). \end{aligned} \tag{21}$$

The function Ψ is defined in (12).

The parameters of this model can be categorized as

(a) Parameters of the underlying stock price dynamics: $V_0 > 0$, $\kappa > 0$, $\theta > 0$, $\sigma > 0$, $\rho \in (-1, 1)$, and the parameters of the pure jump Lévy process Z_t.

(b) Additional parameters (for X_t) in the default threshold: $b \geq 0$, $X_0 > 0$, $\alpha > 0$, $\beta > 0$ and $\xi > 0$ with $\xi^2 \leq 2\alpha\beta$.

To calibrate this model, we would first calibrate the underlying parameters (a) of the stock price dynamics to option prices quoted in the market. Then, with those fixed, we may calibrate the parameters (b) of the default threshold to prices of credit derivatives such as CDS. Although the correlation ρ and the specific parameters of the jump process Z_t do not appear explicitly in the survival time formula, they will exert an indirect influence by affecting the other stock price parameters V_0, κ, θ, σ in the calibration to market data.

Recall that the credit spread is the interest rate spread above the risk free yield required to match the price of a defaultable bond. The CDS spread is a regular premium payment rate (assumed paid continuously) the buyer of a CDS contract makes to the counterparty in exchange for promise of a default payment in case the reference entity defaults. If the recovery rate is $\bar{\delta}$, the default payment would be

$1-\bar{\delta}$.

With the default time distribution in hand, the following theorem gives formulas for the credit spread and CDS spread in terms of the model parameters.

Theorem 4.2. Let the log-return L_t of a company's stock price follow (18) and define the default threshold process as (20). Let $\bar{\delta}$ be the recovery rate after default. Denote by $B(0,T)$ and $B_d(0,T)$ time zero value of the risk-free bond and the defaultable zero coupon bond with maturity T, respectively. For notational convenience, define

$$\Phi(t) = \Psi(\kappa, b\theta, \sqrt{b}\sigma, t, bV_0)\Psi(\alpha, \beta, \xi, t, X_0)$$

where Ψ is defined in (12).

Then $B_d(0,T)$ is given by

$$\begin{aligned} B_d(0,T) &= B(0,T)\left(\bar{\delta} + (1-\bar{\delta})P(\tau > T)\right) \\ &= B(0,T)\left(\bar{\delta} + (1-\bar{\delta})\Phi(T)\right), \end{aligned} \tag{22}$$

where we use (21) for the second equality. The credit spread is

$$-\frac{1}{T}\log\left(\bar{\delta} + (1-\bar{\delta})\Phi(t)\right), \tag{23}$$

and the CDS spread is

$$(1-\bar{\delta})\frac{1 - e^{-rT}\Phi(T) - r\int_0^T e^{-rt}\Phi(t)\,dt}{\int_0^T e^{-rt}\Phi(t)\,dt},$$

4.2 *Extension with Independent Random Interest Rate*

In this section we extend the default threshold model to take into consideration both random interest rate and stochastic volatility. The default time distribution is also linked to both the interest rate and the volatility, and we still obtain a closed formula for the default time distribution.

4.2.1 CIR interest rate process. The most natural choice for the interest rate is a CIR process. Let Z_t be a suitable pure jump Lévy process with Lévy measure λ, tail integral Λ, and characteristic exponent ψ. The underlying stock price process is modeled by $S_t = S_0 e^{L_t}$, where

$$\begin{cases} L_t &= \int_0^t \left(R_u - \frac{V_u}{2}\right) du + \int_0^t \sqrt{V_u}\, dW_u^S + Z_t - t\psi(-i) \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^V \\ dR_t &= \gamma(\delta - R_t)dt + \eta\sqrt{R_t}dW_t^R \end{cases} \tag{24}$$

and $E[dW_t^S dW_t^V] = \rho dt$, $E[dW_t^R dW_t^V] = E[dW_t^R dW_t^S] = 0$, κ, θ, σ, γ, δ and η are positive constants, and Z_t is independent of W_t^R, W_t^S and W_t^V.

Let X_t be a non-negative CIR process as in (19), independent of V_t, R_t, and Z_t. We model the default threshold by

$$a_t = \Lambda^{-1}(bV_t + cR_t + X_t) \tag{25}$$

where $b \geq 0$, $c \geq 0$ are constants.

Here b and c model the sensitivity of the default threshold to the volatility and the interest rate, respectively. X_t is the part of default threshold not explained by the volatility or interest rate. With this setup the default hazard rate increases with stock volatility and with interest rates. When $c = 0$ and the risk-free interest rate is constant, this model reduces to the previous one.

The survival probability is given by

$$\begin{aligned} P(\tau > t) &= E\left[e^{-\int_0^t \Lambda(\Lambda^{-1}(bV_s + cR_s + X_s))\, ds}\right] \\ &= E\left[e^{-\int_0^t bV_s + cR_s + X_s\, ds}\right] \\ &= E\left[e^{-\int_0^t bV_s\, ds}\right] E\left[e^{-\int_0^t cR_s\, ds}\right] E\left[e^{-\int_0^t X_s\, ds}\right] \\ &= \Psi(\kappa, b\theta, \sqrt{b}\sigma, t, bV_0)\Psi(\gamma, c\delta, \sqrt{c}\eta, t, cR_0)\Psi(\alpha, \beta, \xi, t, X_0), \end{aligned} \tag{26}$$

where again Ψ is defined in (12).

Compared to the previous model, this model has four more parameters: $\gamma > 0$, $\delta > 0$ and $\eta > 0$ for the CIR interest rate model, and $c \geq 0$, the sensitivity of the random interest rate to the default threshold. To calibrate the model, first calibrate the CIR interest rate model to the zero coupon bond yield curve. Then, with γ, δ and η fixed, calibrate other parameters as before, except that now when calibrating the default threshold to the credit derivatives there will be one more parameter c.

Theorem 4.3. Let the log-return L_t of a company's stock price follow (32) with $c \geq 0$ and define the default threshold process as (33). Let $\bar{\delta}$ be the recovery rate after default. As before, define

$$\Phi(t) = \Psi(\kappa, b\theta, \sqrt{b}\sigma, t, bV_0)\Psi(\alpha, \beta, \xi, t, X_0), \tag{27}$$

where Ψ is defined in (12). Also let

$$\Theta(c, t) = \frac{\Psi(\gamma, (c+1)\delta, \sqrt{c+1}\eta, t, (c+1)R_0)}{B(0, t)},$$

where

$$B(0, t) = E\left[e^{-\int_0^t R_t\, dt}\right] = \Psi(\gamma, \delta, \eta, t, R_0) \tag{28}$$

is the time zero value of the risk-free zero coupon bond with maturity T. Then the time zero value of a defaultable zero coupon bond with maturity T is

$$B_d(0, T) = B(0, T)\left(\bar{\delta} + (1 - \bar{\delta})\Phi(T)\Theta(c, T)\right), \tag{29}$$

the credit spread is

$$-\frac{1}{T}\log\left(\bar{\delta}+(1-\bar{\delta})\Phi(T)\Theta(c,T)\right), \tag{30}$$

and the CDS spread is

$$(1-\bar{\delta})\frac{1-B(0,T)P(\tau>t)+\int_0^T B'(0,T)P(\tau>t)\,dt}{\int_0^T B(0,t)P(\tau>t)\,dt}, \tag{31}$$

where $P(\tau > t)$ is given by (26), $B(0,t)$ is given by (28), and $B'(0,t)$ is the derivative of $B(0,t)$ with respect to t. When $c = 0$, $\Theta(0,t) = 1$. The survival probability reduces to the constant interest rate case (21) and the formulas reduce to those of the previous theorem.

4.2.2 Vasicek interest rate process. As an alternative for comparison, analysis for a Vasicek interest rate model is similar. (The OU process is simpler than the CIR process, but has a positive probability of negative values.) The underlying stock price process is modeled by $S_t = S_0 e^{L_t}$, where

$$\begin{cases} L_t &= \int_0^t \left(R_u - \frac{V_u}{2}\right)\,du + \int_0^t \sqrt{V_u}\,dW_u^S + Z_t - t\psi(-i) \\ dV_t &= \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dW_t^V \\ dR_t &= \gamma(\delta - R_t)\,dt + \eta\,dW_t^R \end{cases} \tag{32}$$

and $E[dW_t^S dW_t^V] = \rho dt$, $E[dW_t^R dW_t^V] = E[dW_t^R dW_t^S] = 0$, κ, θ, σ, γ, δ and η are positive constants, and Z_t is independent of W_t^R, W_t^S and W_t^V.

As before, we introduce the exogenous default factor X_t as in (19) and independent of V_t, R_t, and Z_t, and model the default threshold by

$$a_t = \Lambda^{-1}(bV_t + cR_t + X_t) \tag{33}$$

where $b \geq 0$, $c \geq 0$ are constants.

The survival probability is now given by

$$P(\tau > t) \quad = \quad \Psi(\kappa, b\theta, \sqrt{b}\sigma, t, bV_0)\Upsilon(\gamma, c\delta, \sqrt{c}\eta, t, cR_0)\Psi(\alpha, \beta, \xi, t, X_0), \tag{34}$$

where Ψ is defined in (12) and Υ in (14).

The same arguments as for CIR interest rate show that the riskless zero coupon bond price is

$$B(0,t) = E\left[e^{-\int_0^t R_t\,dt}\right] = \Upsilon(\gamma, \delta, \eta, t, R_0), \tag{35}$$

and the defaultable zero coupon bond price is

$$B_d(0,T) = B(0,T)\left(\bar{\delta} + (1-\bar{\delta})\Phi(T)\Theta^*(c,T)\right), \tag{36}$$

where Φ is given in equation (27) and

$$\Theta^*(c,t) = \frac{\Upsilon(\gamma,(c+1)\delta,\sqrt{c+1}\eta,t,(c+1)R_0)}{B(0,t)}. \tag{37}$$

The CDS spread is

$$(1-\bar{\delta})\frac{1-B(0,T)P(\tau>t)+\int_0^T B'(0,T)P(\tau>t)\,dt}{\int_0^T B(0,t)P(\tau>t)\,dt}, \tag{38}$$

where now $P(\tau > t)$ is given by (34) and $B(0,t)$ by (35).

5. Significance and Sensitivity

In this section we examine a few numerical experiments to illustrate parameter significance and sensitivity. It's worth noting that all such experiments are straightforward to implement numerically because no simulation is required – that is the point of obtaining our explicit formulas for probabilities and expectations.

In the largest model with exogenous default factor X_t, stochastic volatility, and stochastic interest rates, the cost of extra flexibility is the inconvenience of calibrating a greater number of model parameters. We have the stock volatility parameters $V_0, \kappa, \theta, \sigma, \rho$; the parameters of the chosen pure jump Lévy process Z; the interest rate parameters γ, δ, η; and the default threshold parameters $b, c, X_0, \alpha, \beta, \xi$. Of course, these parameters would not all be estimated simultaneously. The interest rate parameters could be calibrated independently on bond or bond option prices, then the stock parameters on stock option prices. With these in hand, the default parameters would be fitted to CDS prices or the like. This calibration effort is justified for parameters that significantly impact the model's output.

5.1 *Default Threshold Parameters b, c*

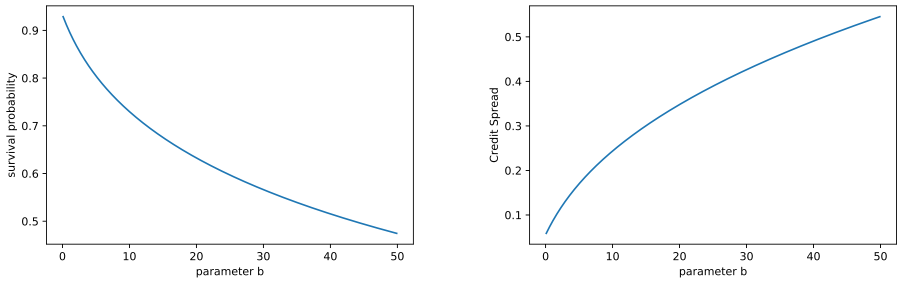
How significant is the introduction of parameters b and c in the default threshold (33)? To examine the dependence on these parameters, we first set baseline values for the ten explicit model parameters:

Table 1. Baseline parameter values

V_0	κ	θ	σ	γ	δ	η	b	c	X_0	α	β	ξ
.04	1.0	.04	1.0	2.0	.01	.01	3	3	.04	25	.04	.1

We also take the (observable) initial interest rate to be $R_0 = 0.01$, and the CDS recovery rate on default to be 0.2. Setting maturity equal to 1 year, we may calculate the one-year survival probability, the credit spread for a one-year bond, and the CDS spread for a one-year contract (continuous premium payment).

Figures 2 and 3 indicate that the parameter b, measuring the sensitivity of the default threshold a_t to the stock variance (or squared volatility) V_t, can play a significant role in the default threshold as reflected in the credit spread and CDS spread. Since $\Lambda^{-1} : (0,\infty) \to (-\infty, 0)$ is an increasing function, $a_t = \Lambda^{-1}(bV_t + cR_t + X_t)$ increases toward zero when the argument $bV_t + cR_t + X_t$ increases, which



(a) Survival probability vs b. (b) Credit spread vs b.

Figure 2. Survival probability and credit spread vs b with other baseline parameters

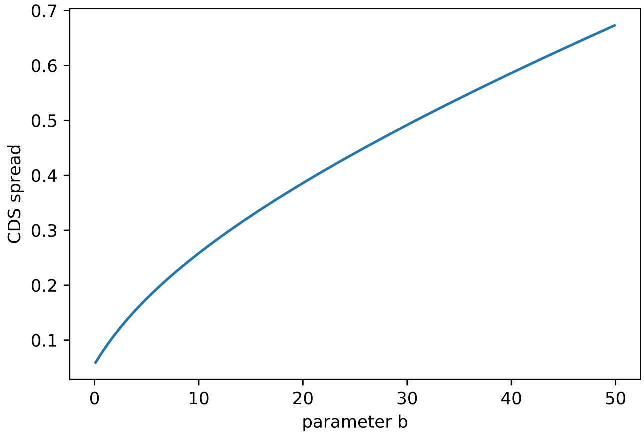


Figure 3. CDS spread vs b with other baseline parameters.

raises the default probability. A larger b magnifies the volatility impact on default probability in the manner illustrated.

The parameter c has a similar impact as the sensitivity of the default threshold to the interest rate R_t. See Figure 4 showing the joint impact of b and c on the one-year credit spread.

5.2 *Model Sensitivity to* κ, θ, σ

It is interesting to perform a sensitivity analysis for some other parameters. For example, if we vary the stock variance mean reversion rate κ and the variance mean level θ, we can plot the one year credit spread as in Figure 5.

We can see from Figure 5 that when the mean level θ is set below the initial variance $V_0 = 0.04$, the credit spread declines against the reversion rate κ. We expect this since a higher mean reversion rate will push the variance down more quickly, and a lower volatility is consistent with a lower default risk. We see the reverse effect when the mean level θ is set above the initial value V_0.

In Figure 6 we see the credit spread plotted against the volatility of variance parameter σ for various values. As σ increases the variance V_t tends to spend more

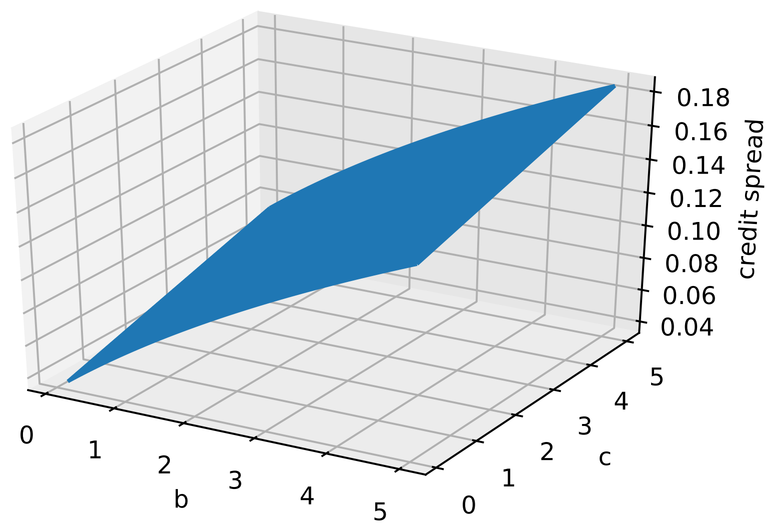


Figure 4. Credit spread vs b and c with other baseline parameters.

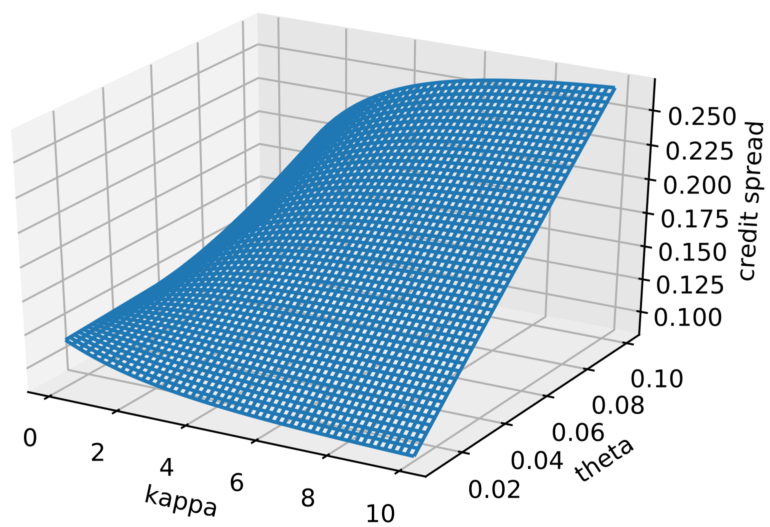


Figure 5. Credit spread vs mean reversion rate κ and mean level θ of the stock variance V_t, with other baseline parameters.

time near zero, pushing the default threshold a_t away from zero and hence reducing the default probability and credit spread. We note that at the baseline value $\sigma = 1$, the variance process V_t will reach zero with positive probability infinitely often.

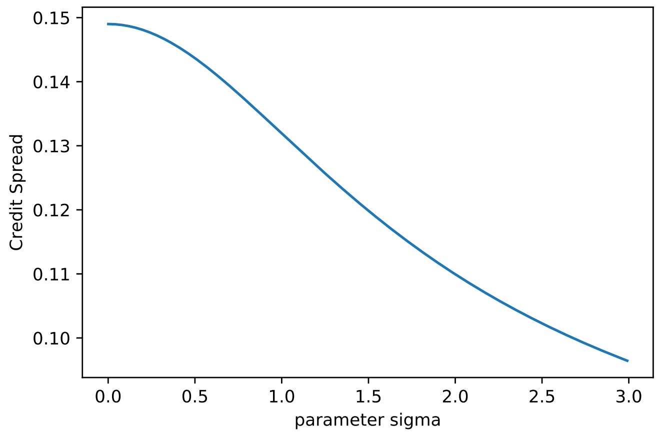


Figure 6. Credit spread vs "volatility of variance" σ with other baseline parameters.

5.3 *Sensitivity to interest rate model: CIR vs Vasicek*

In Figure 7 we compare survival probabilities and riskless bond prices (vs time) for the CIR and Vasicek interest rate models. At the baseline parameter values, the CIR interest rate tends to spend more time near zero (where it's volatility is lower) than does the Vasicek interest rate, leading to a slightly higher survival probability. The riskless zero coupon bond prices, which are independent of default, also reflect this tendency, lower rates correspond to higher bond prices.

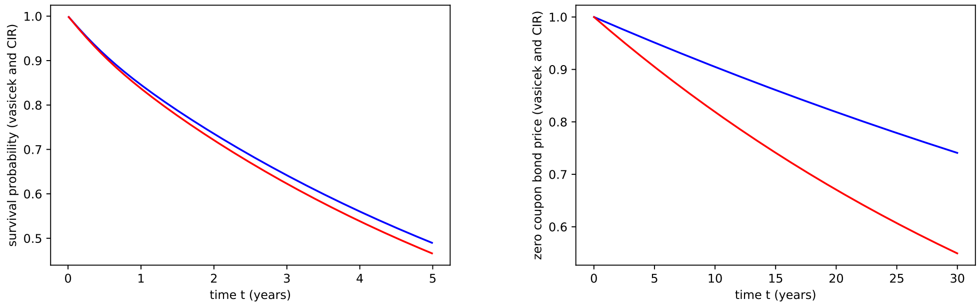
Care must be taken with the volatility parameter η in the Vasicek model since when this grows large the rate R_t spends more time below zero. The result is larger bond prices that can exceed 1, as in Figure 8. By contrast, the CIR model riskless bond price is relatively insensitive to changes in η. Since the interest rate affects the default threshold in the same direction, we see similar sensitivity to η in the credit spread also in Figure 8. A low sensitivity to η can be viewed as an advantage for the CIR interest rate model, in addition to its other advantages: it can be set in advance and removed from the list of parameters requiring estimation.

6. Concluding Remarks

The ingredients of a jump threshold default model are a stock price (jump diffusion) process S_t and a default threshold process a_t, where the default time of the firm is defined by

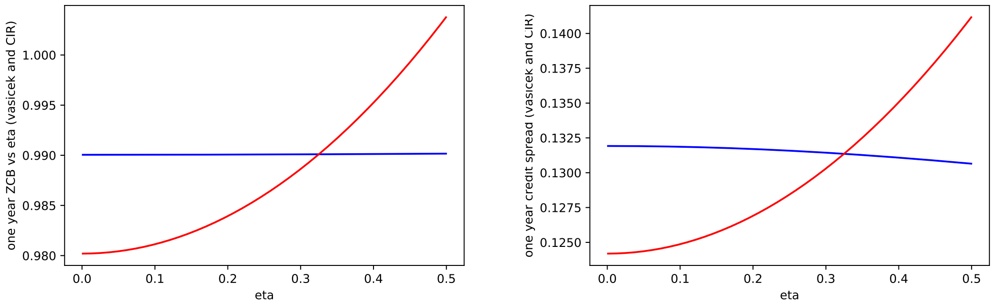
$$\tau = \inf\{t > 0 : \log(S_t/S_{t-}) \leq a(t)\}.$$

The survival probability $P(\tau > t)$ was given by (1) when a_t is non-random and S_t is exponential Lévy. This paper obtains explicit formulas for survival probability,



(a) Survival Probabilities. (b) Riskless zero coupon bond prices.

Figure 7. Default surivival probabilities (up to 5 years) and riskless zero coupon bond prices (up to 30 years) comparing CIR interest rate model (blue) to Vasicek (red) with baseline parameters.



(a) Bond prices. (b) Credit Spread.

Figure 8. One year riskless bond price and credit spread vs. volatility parameter η for CIR (blue) and Vasicek (red).

credit spread, defaultable bond price, and CDS spread for more general models.

In these more general models, the stock price process includes stochastic volatility V_t, and stochastic underlying interest rates R_t, where the log jumps are given by an infinite activity Lévy process L_t. The default threshold a_t may be stochastic and driven by the endogenous factors V_t and R_t and by an exogenous stochastic factor X_t via $a_t = \Lambda^{-1}(bV_t + cR_t + X_t)$, where Λ is the (increasing) tail integral of L. This provides the modeler the ability to incorporate a variety of relevant factors into the default model without sacrificing tractability. The main idea is to make use of affine processes

$$dY_t = (a(t) + b(t)Y_t)dt + \sqrt{c(t) + d(t)Y_t}\, dW_t,$$

for which it is known how to compute expectations of the form $E\left[e^{k\int_0^T X_t\,dt}\right]$ by explicitly solving an appropriate PDE.

A primary interest of the jump threshold approach is to model multi-dimensional dependent defaults for a basket of firms using Lévy copulas, as begun in Garreau and Kercheval (2016). The contribution of this paper to the one-dimensional case is another step toward solving the multi-dimensional problems explicitly.

7. Proofs

A standard proof of Theorem 3.1 involves a PDE for f, which can be obtained by the Feynman-Kac theorem. The version below, and its proof, can be found in Pham (2009).

Theorem 7.1 (Feynman-Kac). Let X_t be the unique solution of the SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

h a continuous function on $\mathbb{R}$, and q a continuous function on $[0, T] \times \mathbb{R}$. Define

$$g(t, x) = E\left[e^{-\int_t^T q(t,X_u)\,du} h(X_T) \middle| X_t = x\right].$$

Then $g(t, x)$ is the unique solution of the PDE

$$g_t(t, x) + \mu(t, x)g_x(t, x) + \frac{1}{2}\sigma^2(t, x)g_{xx}(t, x) - q(t, x)g(t, x) = 0$$

with terminal condition $g(T, x) = h(x)$.

Proof of Proposition 3.1
By the Feynman-Kac theorem, f is the solution of the PDE

$$f_t + (a(t) + b(t)r)f_r + \frac{(c(t) + d(t)r)}{2} f_{rr} + krf = 0.$$

with the terminal condition $f(T, r) = 1$. We guess the solution is of the form $f(t, r) = \exp(-rC(t, T) - A(t, T))$. Denote $C' = \partial C/\partial t$ and $A' = \partial A/\partial t$. Since

$$f_t = (-rC' - A')f, \qquad f_r = -Cf, \qquad f_{rr} = C^2 f,$$

the PDE can be rewritten as

$$\begin{cases} -rC' - A' - (a(t) + b(t)r)C + \frac{C^2}{2}(c(t) + d(t)r) + kr = 0 \\ \exp(-rC(T, T) - A(T, T)) = 1 \end{cases}.$$

The second equation gives $-rC(T, T) - A(T, T) = 0$, which holds for all r. Thus $A(T, T) = C(T, T) = 0$. Rearrange the terms in the first equation to get

$$r\left(-C' - b(t)C + \frac{d(t)}{2}C^2 + k\right) + \left(-A' - a(t)C + \frac{c(t)}{2}C^2\right) = 0.$$

Since this also holds for all r, both parentheses should be zero. The first parentheses gives us the ODE (10) for C. Once C is given analytically, A can be obtained by integrating the second parentheses with respect to t. With the terminal condition $A(T, T)$, one can easily verify the formula given in (11).

Proof of Theorems 4.2 and 4.3.
Theorem 4.2 is a special case of Theorem 4.3 when $R_t = r$ is a constant and $c = 0$, so it suffices to prove Theorem 4.3.

First we examine the price $B_d(0,T)$ of the defaultable bond. At maturity T, the holder of one unit of defaultable bond receives $\bar{\delta}$ dollars if the company defaults before T, and one dollar if there is no default event. The payoff is thus $1_{\{\tau>T\}}+\bar{\delta}\cdot 1_{\{\tau\leq T\}}=\bar{\delta}+(1-\bar{\delta})1_{\{\tau>T\}}$. The time zero defaultable bond price is the expected discounted payoff under the risk-neutral measure

$$\begin{aligned} B_d(0,T) &= E\left[e^{-\int_0^T R_t\,dt}\left(\bar{\delta}+(1-\bar{\delta})1_{\{\tau>T\}}\right)\right] \\ &= \bar{\delta}B(0,T)+(1-\bar{\delta})E\left[e^{-\int_0^T R_t\,dt}1_{\{\tau>T\}}\right]. \end{aligned} \tag{39}$$

The default time τ is now defined as the first time the log-return of S_t hits the default threshold $a_t=\Lambda^{-1}(bV_t+cR_t+X_t)$. Let $\mathcal{G}_t$ be the filtration generated by $\{V_t,R_t,X_t\}$. We apply the tower property and rewrite the last expectation above as

$$E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{\tau>T\}}\middle|\mathcal{G}_T\right]\right].$$

Conditional on $\mathcal{G}_T$ the default threshold $a_t=\Lambda^{-1}(bV_t+cR_t+X_t)$ is a deterministic function for $t\in[0,T]$, and the event $\{\tau>T\}$ can be replaced by $\{Y=0\}$, where Y is a Poisson distributed random variable with conditional mean

$$\int_0^T \Lambda(\Lambda^{-1}(bV_t+cR_t+X_t))\,dt=\int_0^T bV_t+cR_t+X_t\,dt,$$

that is

$$\begin{aligned} &E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{\tau>T\}}\middle|\mathcal{G}_T\right]\right]=E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{Y=0\}}\middle|\mathcal{G}_T\right]\right], \\ &Y\sim \text{Poi}\left(\int_0^T bV_t+cR_t+X_t\,dt\right). \end{aligned}$$

If we define two independent Poisson random variables Y_1,Y_2 with conditional distributions

$$Y_1\sim \text{Poi}\left(\int_0^T cR_t\,dt\right),\qquad Y_2\sim \text{Poi}\left(\int_0^T bV_t+X_t\,dt\right),$$

then Y has the same distribution as Y_1+Y_2. Thus we can write

$$E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{\tau>T\}}\middle|\mathcal{G}_T\right]\right]=E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{Y_1+Y_2=0\}}\middle|\mathcal{G}_T\right]\right].$$

Since the Poisson distribution is nonnegative, the event $\{Y_1+Y_2=0\}$ is equivalent to $\{Y_1=0\}\cap\{Y_2=0\}$ and we can factor the indicator function $1_{\{Y_1+Y_2=0\}}$ as $1_{\{Y_1=0\}}1_{\{Y_2=0\}}$ and write

$$\begin{aligned} E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{\tau>T\}}\middle|\mathcal{G}_T\right]\right] &= E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{Y_1=0\}}1_{\{Y_2=0\}}\middle|\mathcal{G}_T\right]\right] \\ &= E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{Y_1=0\}}\middle|\mathcal{G}_T\right]E\left[1_{\{Y_2=0\}}\middle|\mathcal{G}_T\right]\right]. \end{aligned}$$

The last equation holds because Y_2 is independent of R_t and Y_1.

The expectation with Y_2 is

$$\begin{aligned} E\left[E\left[1_{\{Y_2=0\}}\middle|\mathcal{G}_T\right]\right] &= E\left[e^{-\int_0^T bV_t+X_t\,dt}\right] \\ &= \Psi(\kappa, b\theta, \sqrt{b}\sigma, t, bV_0)\Psi(\alpha,\beta,\xi,t,X_0) = \Phi(t), \end{aligned}$$

the same formula as (21).

For the other expectation we have

$$\begin{aligned} E\left[E\left[e^{-\int_0^T R_t\,dt}1_{\{Y_1=0\}}\middle|\mathcal{G}_T\right]\right] &= E\left[e^{-\int_0^T R_t\,dt}E\left[1_{\{Y_1=0\}}\middle|\mathcal{G}_T\right]\right] && (40)\\ &= E\left[e^{-\int_0^T R_t\,dt}e^{-\int_0^T cR_t\,dt}\right] && (41)\\ &= E\left[e^{-\int_0^T (c+1)R_t\,dt}\right] && (42)\\ &= \Psi(\gamma,(c+1)\delta,\sqrt{c+1}\eta,t,(c+1)R_0) && (43)\\ &= B(0,T)\Theta(c,t), && (44) \end{aligned}$$

recalling that

$$\Theta(c,t) = \frac{\Psi(\gamma,(c+1)\delta,\sqrt{c+1}\eta,t,(c+1)R_0)}{B(0,t)}.$$

Hence

$$\begin{aligned} B_d(0,T) &= \bar{\delta}B(0,T) + (1-\bar{\delta})E\left[e^{-\int_0^T R_t\,dt}1_{\{\tau>T\}}\right] && (45)\\ &= \bar{\delta}B(0,T) + (1-\bar{\delta})\Phi(t)B(0,T)\Theta(c,t) && (46)\\ &= B(0,T)\left(\bar{\delta} + (1-\bar{\delta})\Phi(t)\Theta(c,t)\right), && (47) \end{aligned}$$

the desired result.

The credit spread is the difference of the yields of the defaultable bond and the risk-free bond, that is

$$\begin{aligned} -\frac{1}{T}\left(\log B_d(0,T) - \log B(0,T)\right) &= -\frac{1}{T}\log\frac{B_d(0,T)}{B(0,T)} \\ &= -\frac{1}{T}\log\left(\bar{\delta} + (1-\bar{\delta})\Phi(t)\Theta(c,t)\right). && (48) \end{aligned}$$

To compute the CDS spread, recall that, for any $t>0$,

$$B(0,t) = E\left[e^{-\int_0^t R_t\,dt}\right] = \Psi(\gamma,\delta,\eta,t,R_0).$$

Since Ψ is a differentiable function of t, denote by $B'(0,t)$ the derivative of $B(0,t)$ with respect to t. (In the special case of constant interest rate $R_t = r$, we have $B(0,t) = e^{-rt}$ and $B'(0,t) = -re^{-rt}$.)

To compute the CDS spread (premium), assume the recovery rate is $\bar{\delta}$, so the default payment should be $(1-\bar{\delta})$. There is no initial payment of a CDS contract, so the present value of all the cash flow generated by the premium payment should be equal to the present value of the default payment. Specifically, let c be the premium rate paid continuously, T the maturity of the CDS and R_t the risk free interest rate. Then the present value of the premium payment is

$$c \int_0^T B(0,t) P(\tau > t)\, dt,$$

and the present value of the default payment

$$(1-\bar{\delta}) \int_0^T B(0,s)\, dP(\tau < s) = (1-\bar{\delta}) \left(1 - B(0,T)P(\tau > T) + \int_0^T B'(0,t) P(\tau > t)\, dt \right),$$

where an integration by parts is used to make the expression more explicit. Equating the two present values, we obtain

$$c = (1-\bar{\delta}) \frac{1 - B(0,T)P(\tau > T) + \int_0^T B'(0,t)P(\tau > t)\, dt}{\int_0^T B(0,t)P(\tau > t)\, dt}.$$

We have now derived formulas for the price of a defaultable bond, the credit spread, and the CDS spread, all in terms of the explicit default time distribution $P(\tau > t)$ given by (26). This completes the proof.

8. Acknowledgement

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