Modeling Mechanical Systems

Dr. Nhut Ho
ME584

Agenda

- Idealized Modeling Elements
- Modeling Method and Examples
- Lagrange’s Equation
- Case study: Feasibility Study of a Mobile Robot Design
- Matlab Simulation Example

- Active learning: Pair-share exercises, case study

Idealized Modeling Elements

Inductive storage

Figure: 02-01-01UNT02

Electrical inductance

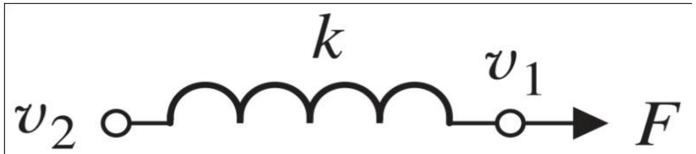


Figure: 02-01-02UNT02

Translational spring

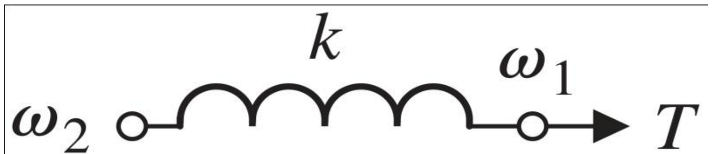


Figure: 02-01-03UNT02

Rotational spring

Figure: 02-01-04UNT02

Fluid inertia

Capacitive Storage

Figure: 02-01-05UNT02

Electrical capacitance

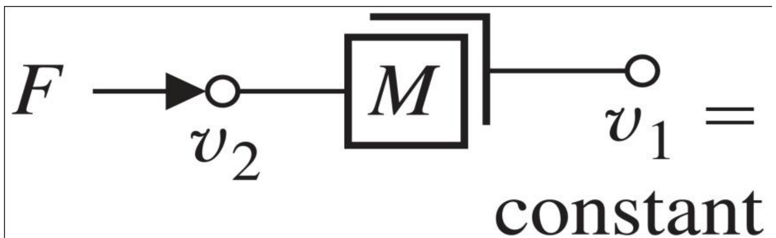


Figure: 02-01-06UNT02

Translational mass

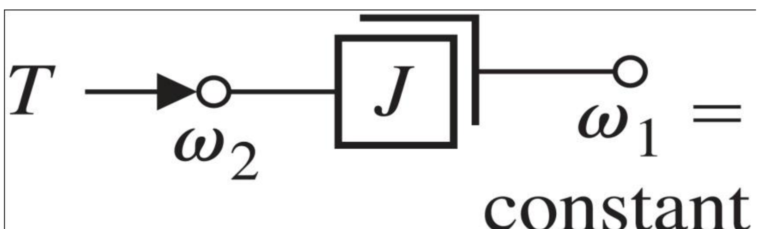


Figure: 02-01-07UNT02

Rotational mass

Energy dissipators

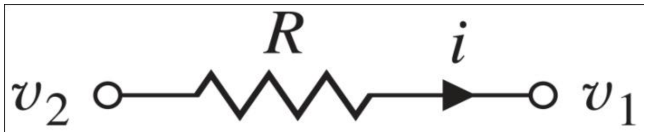


Figure: 02-01-10UNT02

Electrical resistance

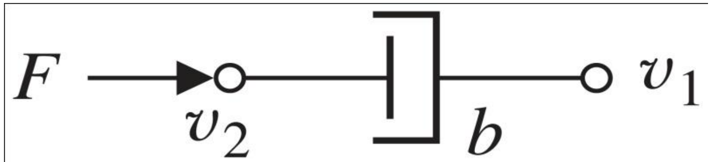


Figure: 02-01-11UNT02

Translational damper

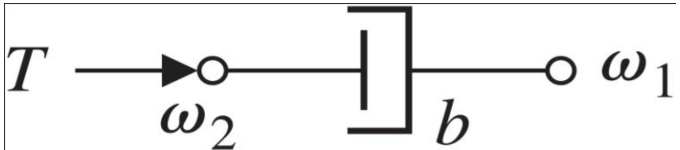
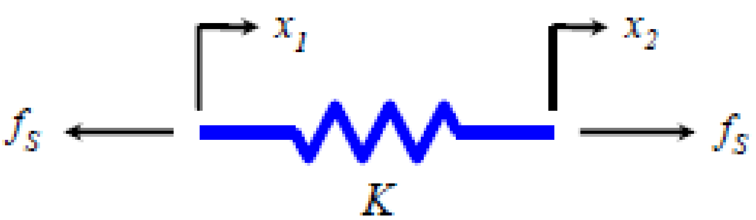


Figure: 02-01-12UNT02

Rotational damper

Springs

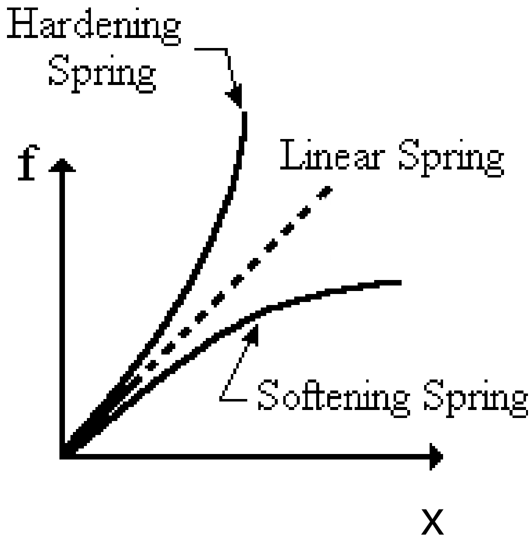
- Stiffness Element
- Stores potential energy



$$f_S = K(x_2 - x_1)$$

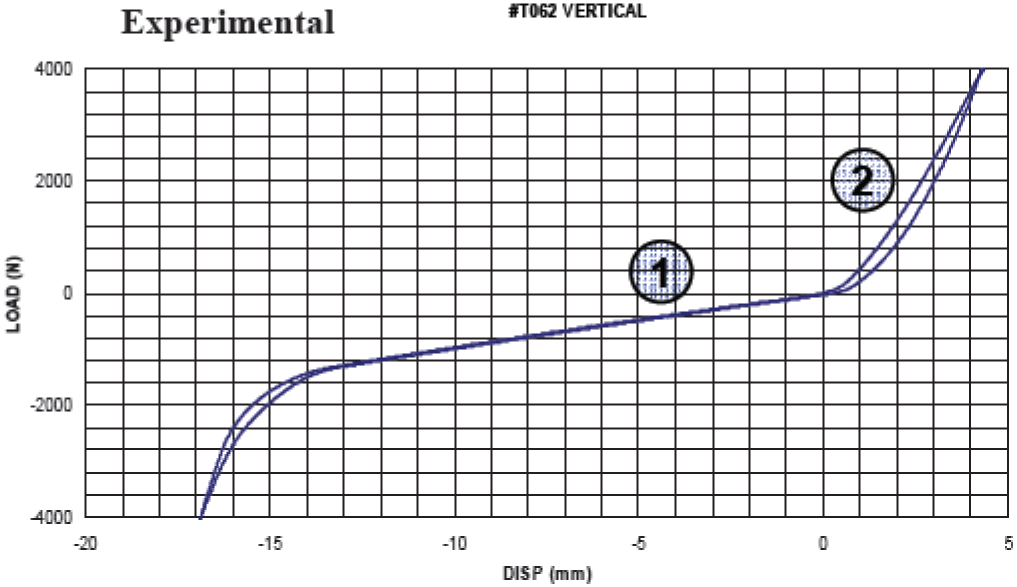
– Idealization
  - Massless
  - No Damping
  - Linear

– Reality
  - 1/3 of the spring mass may be considered into the lumped model.
  - In large displacement operation springs are *nonlinear*.

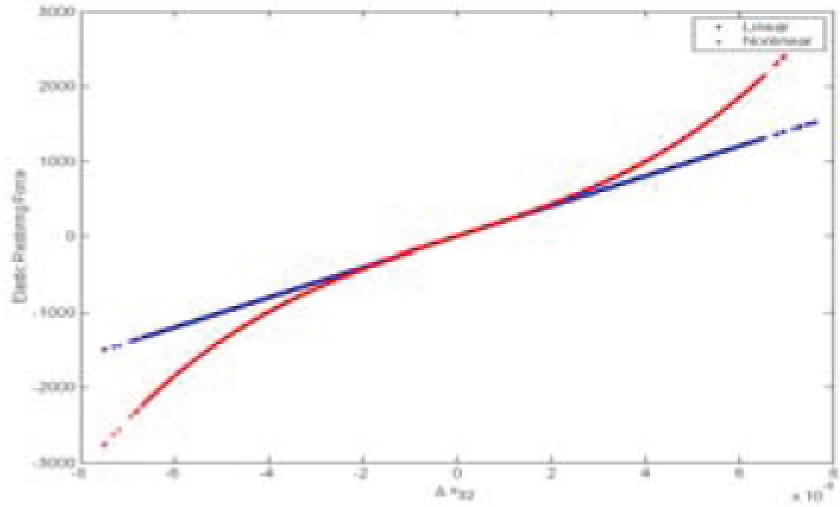


Actual Spring Behavior

Experimental



Analytical



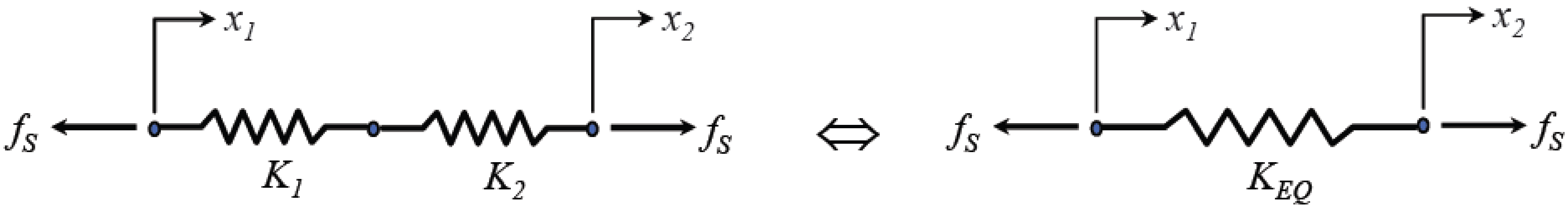
$$\text{Restoring force} = \left(K + \mu \Delta x^2\right)\Delta x$$

1 $\dfrac{\text{Small motions}}{\text{for isolation}} \approx K$

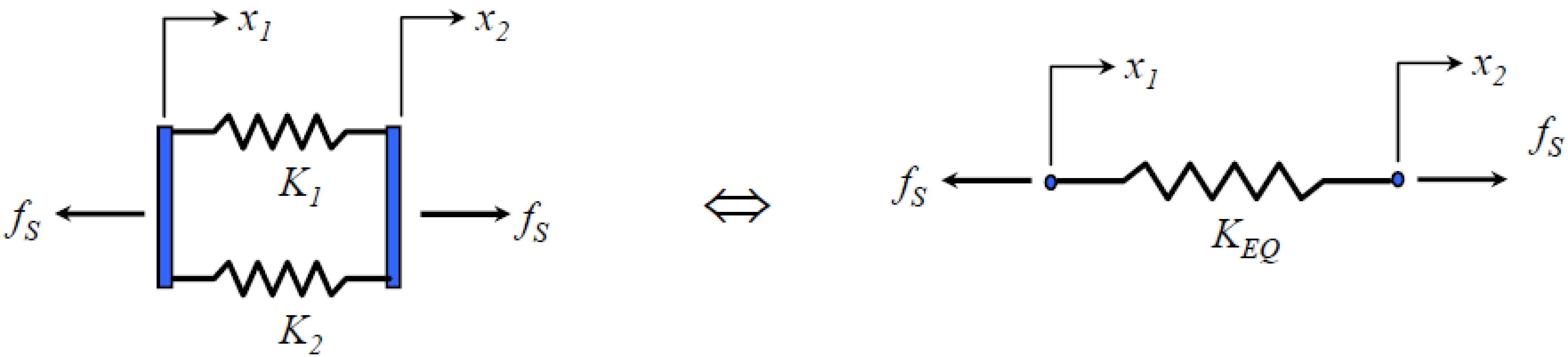
2 $\dfrac{\text{Large motions}}{\text{for static loads}} = K + \mu \Delta x^2$

Spring Connections

- Spring in series: $K_{EQ}=K_1K_2/(K_1+K_2)$

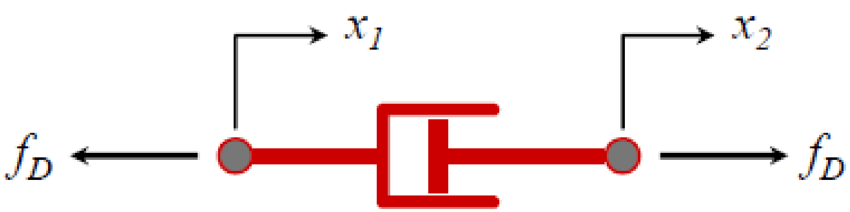


- Spring in parallel: $K_{EQ}=K_1+K_2$



Dampers and Mass

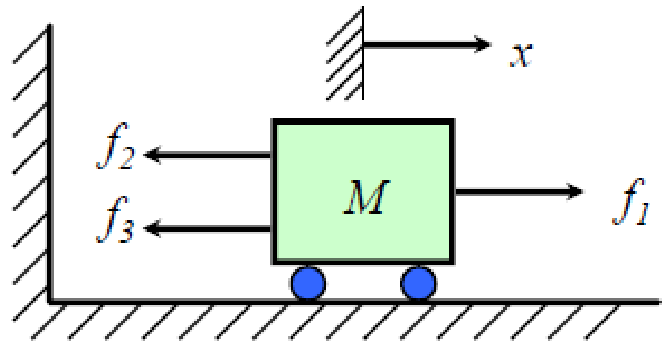
– Friction Element



$$f_D = B(\dot{x}_2 - \dot{x}_1) = B(v_2 - v_1)$$

– Dissipate Energy

– Inertia Element

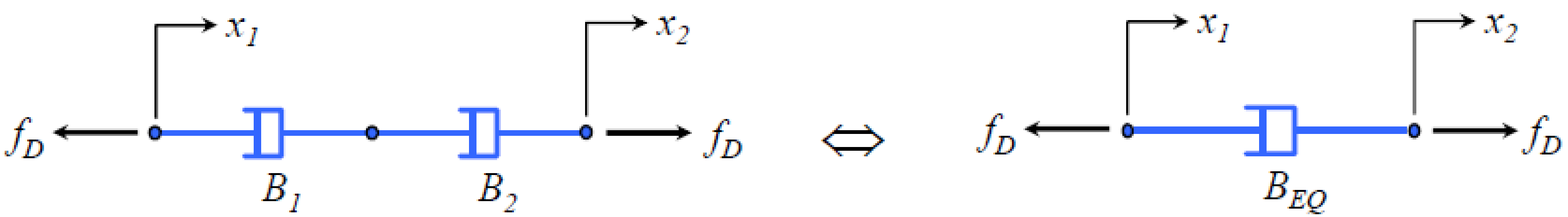


$$M\ddot{x} = \sum_i f_i = f_1 - f_2 - f_3$$

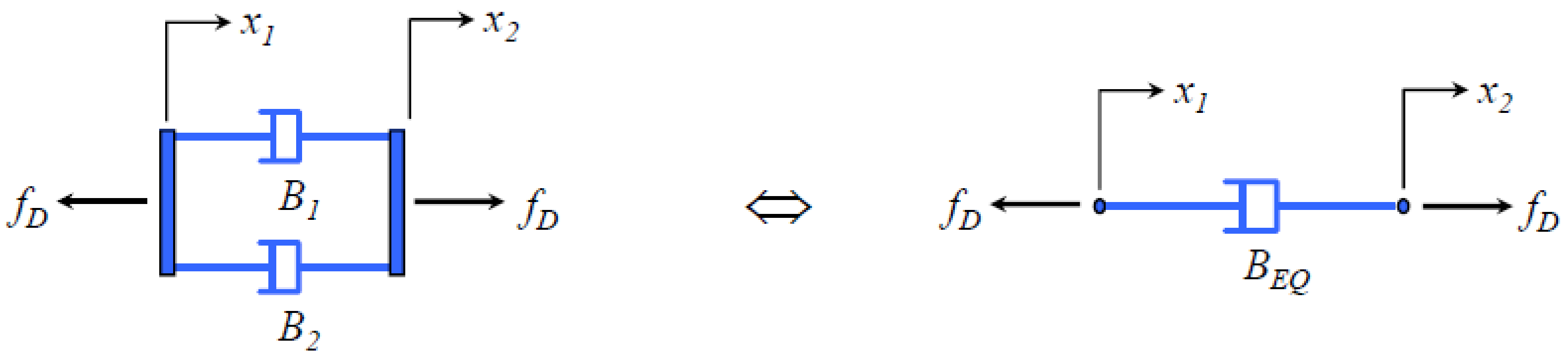
– Stores Kinetic Energy

Dampers Connections

- Dampers in series: $B_{EQ}=B_1B_2/(B_1+B_2)$



- Dampers in parallel: $B_{EQ}=B_1+B_2$



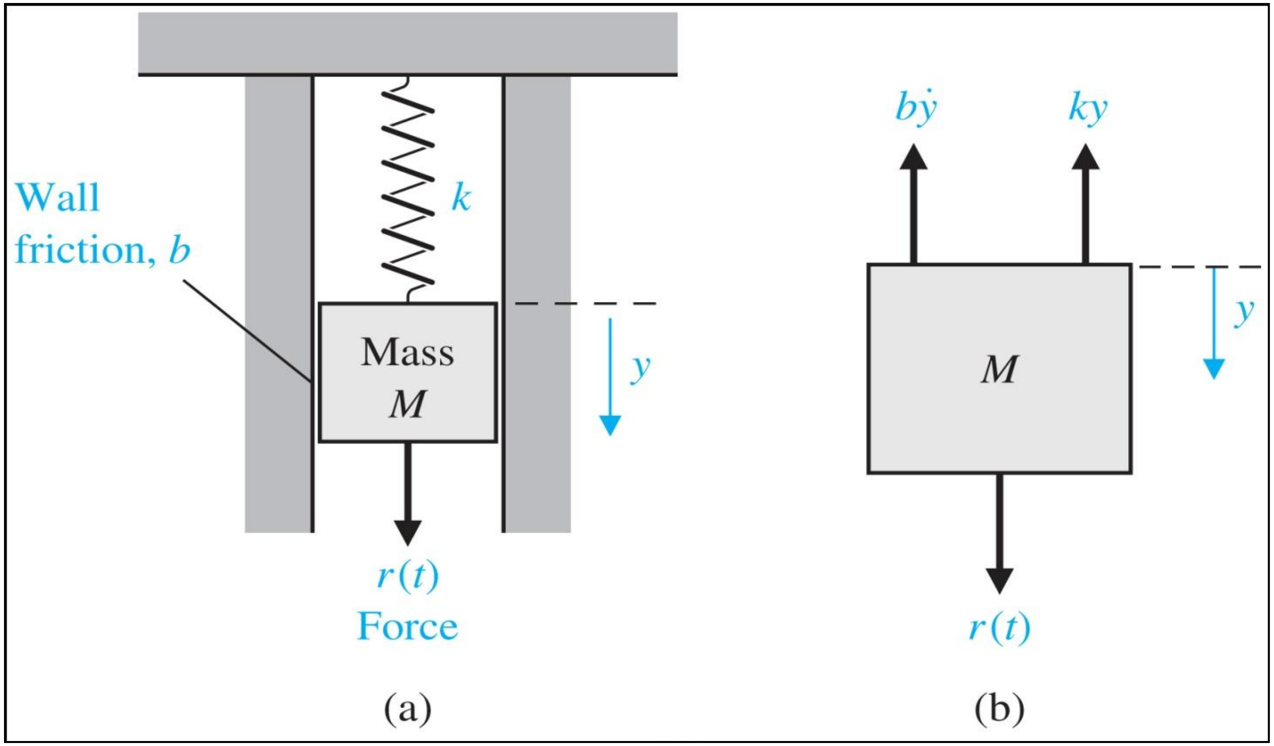
Modeling Mechanical Systems

Modeling Methods

- State assumptions and their rationales
- Establish inertial coordinate system
- Identify and isolate discrete system elements (springs, dampers, masses)
- Determine the minimum number of variables needed to uniquely define the configuration of system (subtract constraints from number of equations)
- Free body diagram for each element
- Write equations relating loading to deformation in system elements
- Apply Newton's 2^{nd} Law:
  - **F = ma** for translation motion
  - **T = Iα** for rotational motion

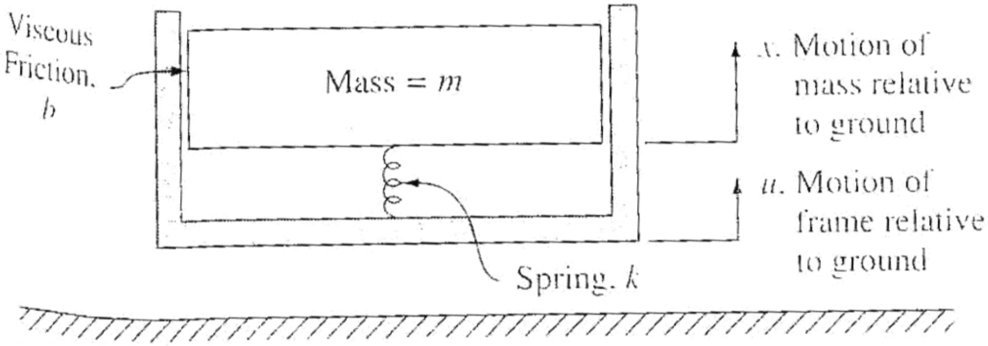
Example 1: Automobile Shock Absorber

Spring-mass-damper **Free-body diagram**



$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

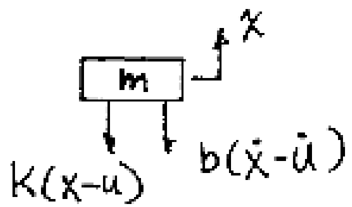
Example 2: Mechanical System



- Draw a free body diagram, showing all forces and their directions
- Write equation of motion and derive transfer function of response x to input u

Example 2: Mechanical System

a. - b. Free body:



c. Equation of: Using Newton's second law:

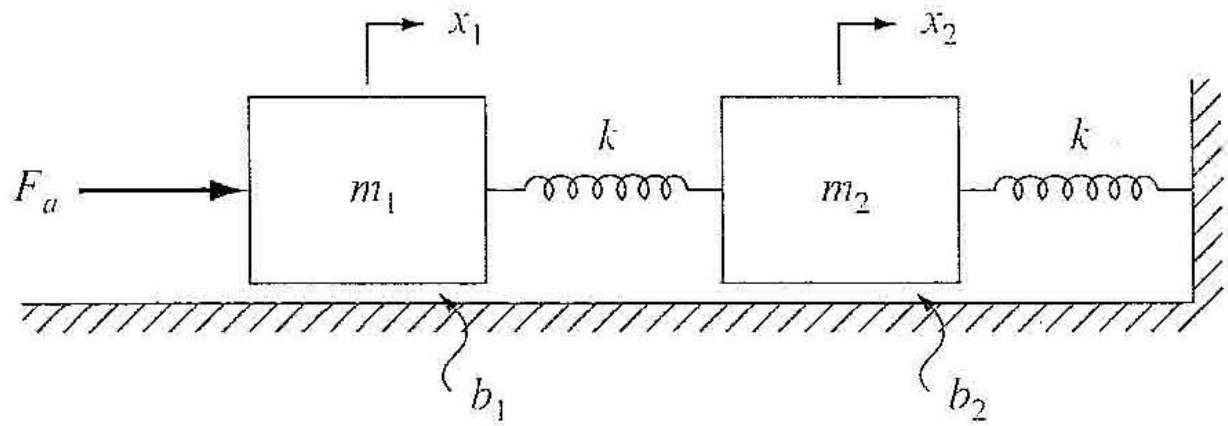
$$\sum F_x = m\ddot{x} \qquad -b(\dot{x}-\dot{u})-k(x-u)=m\ddot{x}$$

$$m\ddot{x}+b\dot{x}+kx = +b\dot{u}+ku$$

In D-operator notation: $[mD^2+bD+k]x=[bD+k]u$

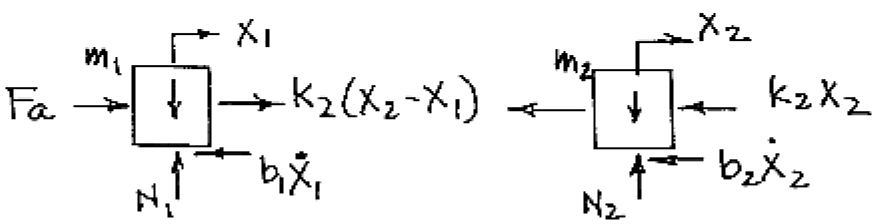
The transfer function is: $\dfrac{x}{u}=\dfrac{bD+k}{mD^2+bD+k}$

Example 3: Two-Mass System



- Derive the equation of motion for x_2 as a function of F_a. The indicated damping is viscous.

Example 3: Two-Mass System



Equations of motion for the two masses:

$$F_a + k_2\,(x_2 - x_1) - b_1\dot{x}_1 = m_1\ddot{x}_1$$

$$-k_2(x_2 - x_1) - k_2x_2 - b_2\dot{x}_2 = m_2\ddot{x}_2$$

Or

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_2x_1 - k_2x_2 = F_a$$

$$[m_1D^2 + b_1D + k_2]x_1 - k_2x_2 = F_a$$

and

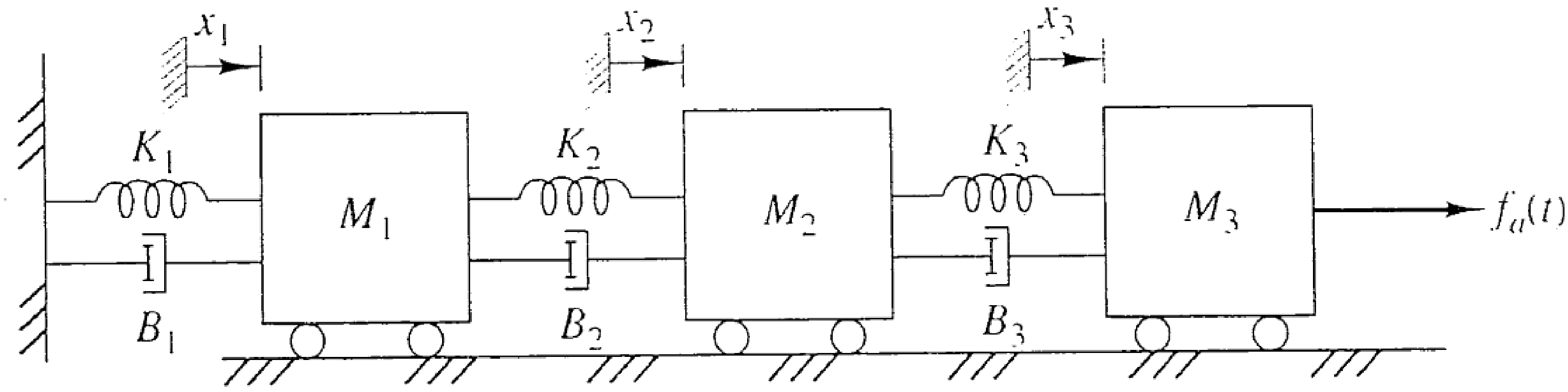
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + 2k_2x_2 - k_2x_1 = 0$$

$$-k_2x_1 + [m_2D^2 + b_2D + 2k_2]x_2 = 0$$

Use Cramer's rule to solve for x_2

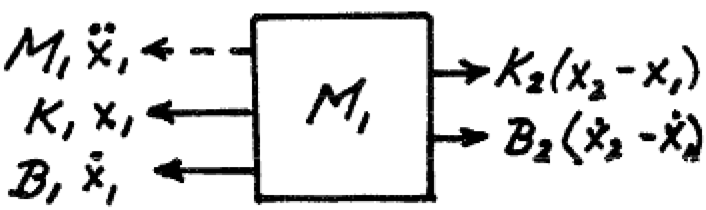
$$x_2 = \frac{k_2F_a}{[m_1D^2 + b_1D + k_2][m_2D^2 + b_2D + 2k_2] - k_2^2}$$

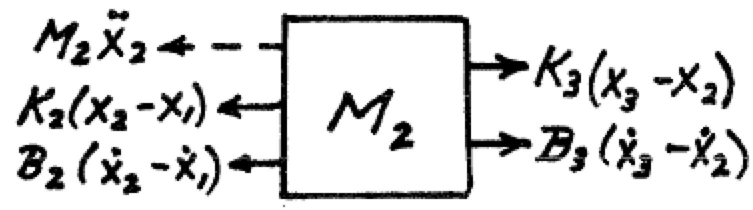
Example 4: Three-Mass System



- Draw the free-body-diagram for each mass and write the differential equations describing the system

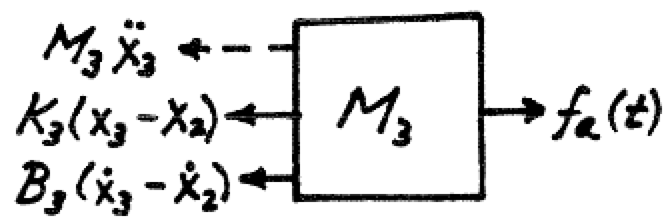
Example 4: Three-Mass System





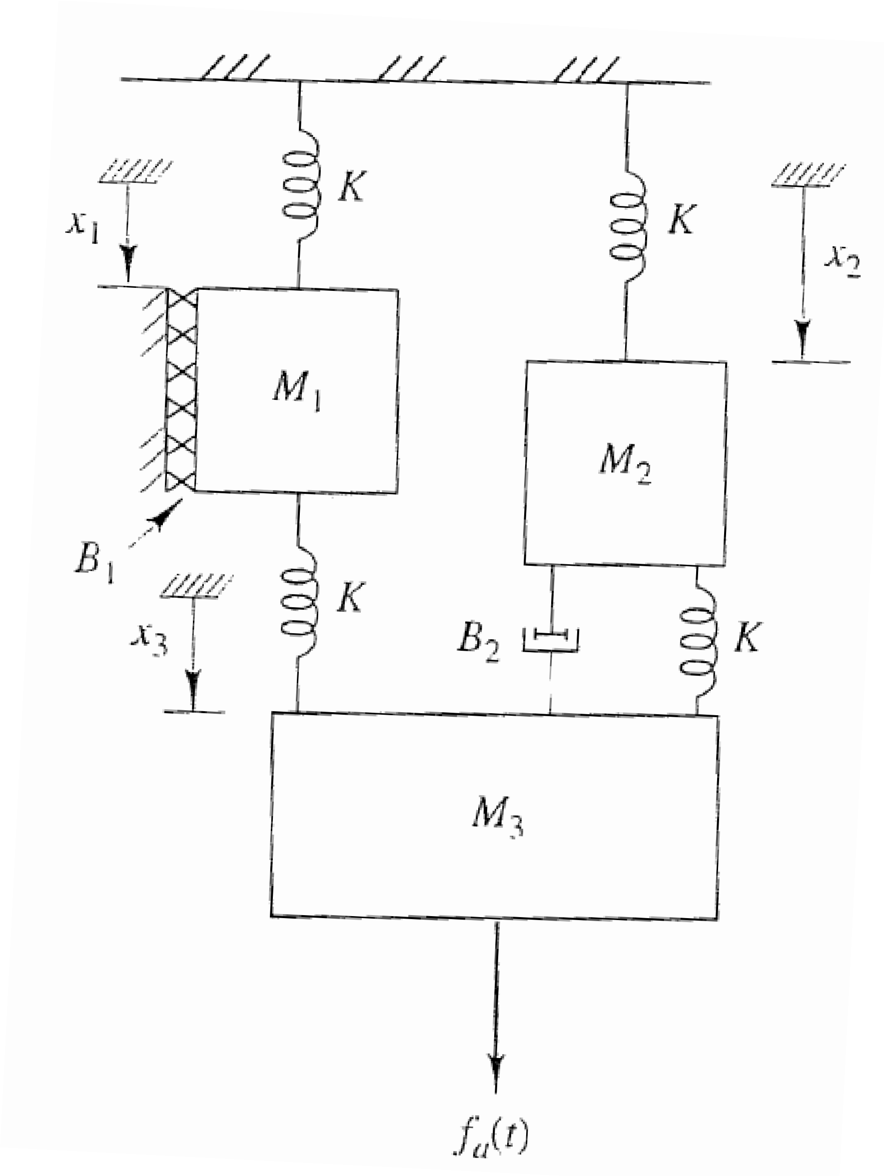
Summing the forces shown on each of the free-body diagrams and collecting terms, we get

$$
\begin{aligned}
M_1\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 &= 0 \\
-B_2\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + (K_2 + K_3)x_2 & \\
-B_3\dot{x}_3 - K_3x_3 &= 0 \\
-B_3\dot{x}_2 - K_3x_2 + M_3\ddot{x}_3 + B_3\dot{x}_3 + K_3x_3 &= f_a(t)
\end{aligned}
$$



Example 5: Pair-Share Exercise

- All springs are identical with constant K
- Spring forces are zero when $x_1=x_2=x_3=0$
- Draw FBDs and write equations of motion
- Determine the constant elongation of each spring caused by gravitational forces when the masses are stationary in a position of static equilibrium and when $f_a(t) = 0$.



Example 5: Pair-Share Exercise:

(a) Summing the forces shown on each of the free-body diagrams and collecting terms, we obtain

$$\begin{aligned} M_1\ddot{x}_1 + B_1\dot{x}_1 + 2Kx_1 - Kx_3 &= M_1 g \\ M_2\ddot{x}_2 + B_2\dot{x}_2 + 2Kx_2 - B_2\dot{x}_3 - Kx_3 &= M_2 g \\ -Kx_1 - B_2\dot{x}_2 - Kx_2 + M_3\ddot{x}_3 + B_2\dot{x}_3 + 2Kx_3 &= M_3 g + f_a(t) \end{aligned}$$

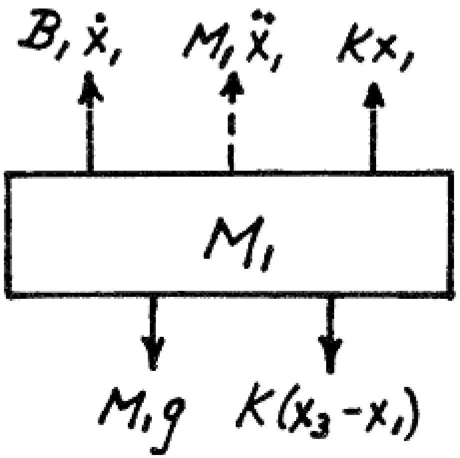
(b) Letting $f_a(t) = 0$, replacing x_1, x_2, and x_3 by the constant displacements x_{1_0}, x_{2_0}, and x_{3_0}, and noting that all the derivatives of these constant displacements are zero, we have the following three algebraic equations.

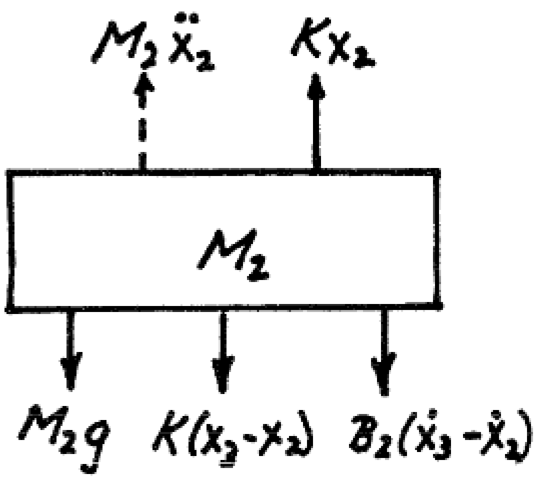
$$2x_{1_0} - x_{3_0} = \frac{M_1 g}{K}, \quad 2x_{2_0} - x_{3_0} = \frac{M_2 g}{K},$$

$$\text{and} \quad -x_{1_0} - x_{2_0} + 2x_{3_0} = \frac{M_3 g}{K}$$

Solving these equations simultaneously gives

$$\begin{aligned} x_{1_0} &= (3M_1 + M_2 + 2M_3)\frac{g}{4K} \\ x_{2_0} &= (M_1 + 3M_2 + 2M_3)\frac{g}{4K} \\ x_{3_0} &= (M_1 + M_2 + 2M_3)\frac{g}{2K} \end{aligned}$$



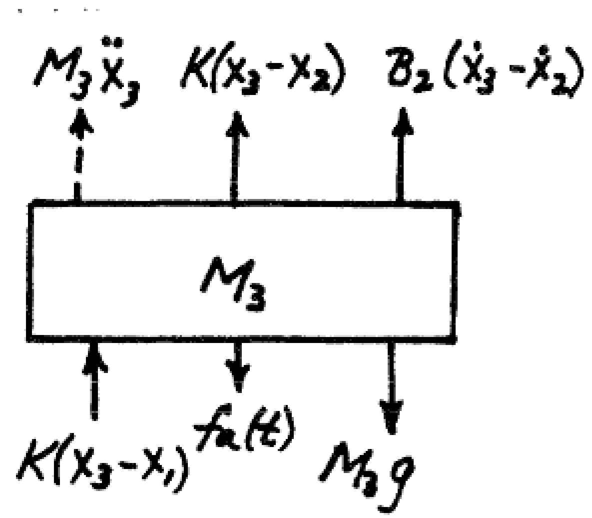


Example 5: Pair-Share Exercise:

The four spring elongations are x_{1_0}, x_{2_0}, and

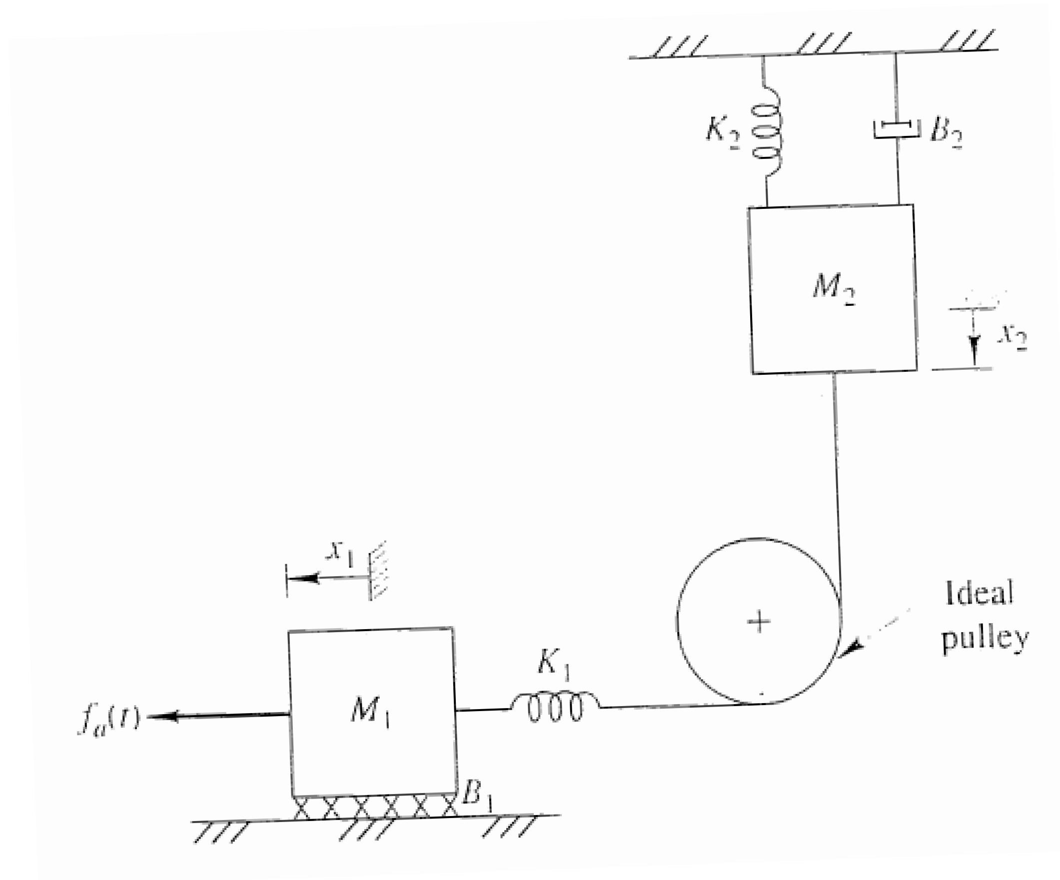
$$x_{3_0} - x_{1_0} = (-M_1 + M_2 + 2M_3)\frac{g}{4K}$$
$$x_{3_0} - x_{2_0} = (M_1 - M_2 + 2M_3)\frac{g}{4K}$$

Note that the elongations are not affected by the vicous damping coefficients B_1 and B_2.



Example 6: Pair-Share Exercise

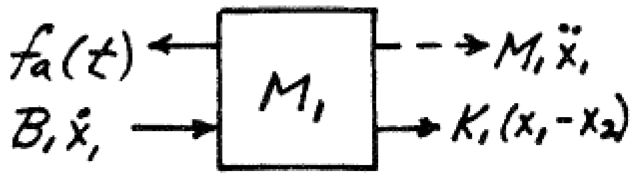
- Assume that the pulley is ideal
  - No mass and no friction
  - No slippage between cable and surface of cylinder (i.e., both move with same velocity)
  - Cable is in tension but does not stretch
- Draw FBDs and write equations of motion
- If pulley is not ideal, discuss modeling modifications

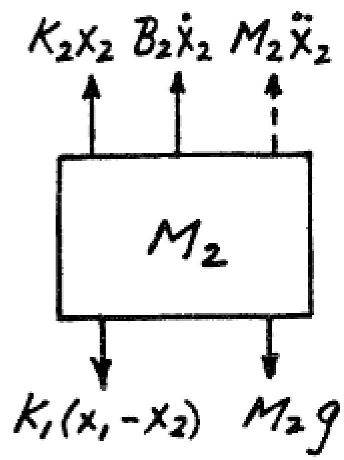


Example 6: Pair-Share Exercise

When drawing the free-body diagrams, note that the downward force of the cable on M_2 is the same as the force of the cable to the right on M_1 because of the pulley. Summing the forces shown on each of the diagrams and collecting terms, we get

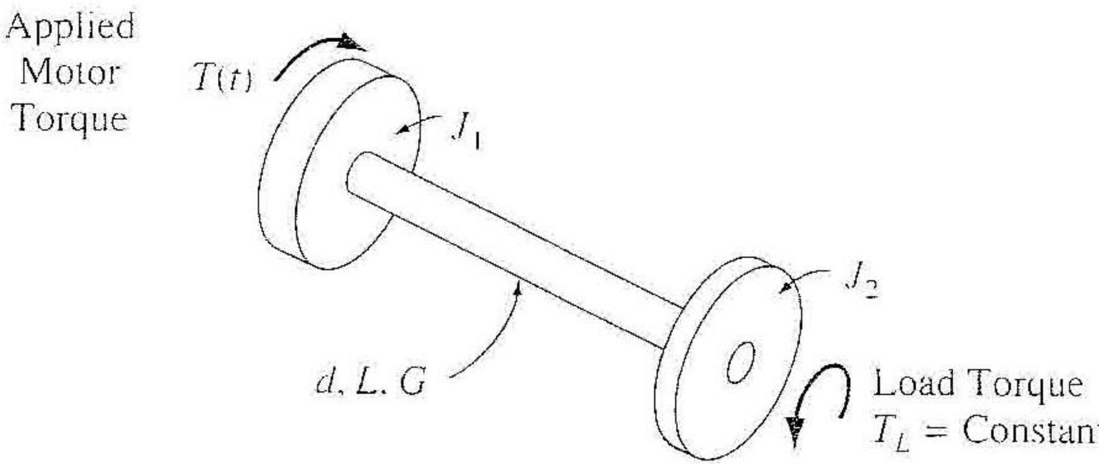
$$\begin{aligned} M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 &= f_a(t) \\ -K_1x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + (K_1 + K_2)x_2 &= M_2g \end{aligned}$$





- Pulley is not ideal
  - Add rotation mass and friction
  - Model the slippage behaviors
  - Add spring to model cable

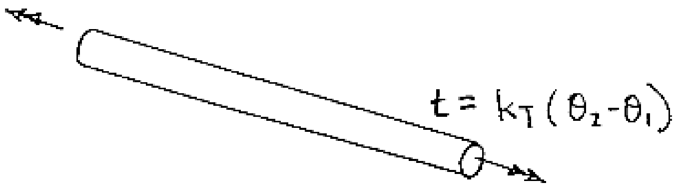
Example 7: Electric Motor



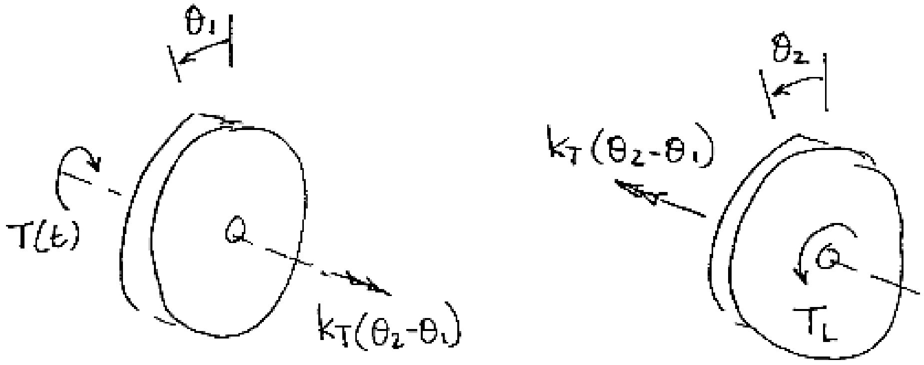
- An electric motor is attached to a load inertia through a flexible shaft as shown. Develop a model and associated differential equations (in classical and state space forms) describing the motion of the two disks J1 and J2.
- Torsional stiffness is given in Appendix B

Example 7: Electric Motor

The torsional stiffness of the shaft is given by: $k_T = \dfrac{GJ}{L}$



And torque in the shaft by: $k_T(\theta_2 - \theta_1)$



For disk 1: $\sum M_x = J_1 \ddot{\theta}_1$ $\quad -T(t) + k_T(\theta_2 - \theta_1) = J_1 \ddot{\theta}_1$

Example 7: Electric Motor

or

$$J_1\ddot{\theta}_1 + k_T\theta_1 - k_T\theta_2 = -T(t)$$

For disk 2 $\qquad \sum M_x = J_2\ddot{\theta}_2 \qquad T_L - k_T(\theta_2 - \theta_1) = J_2\ddot{\theta}_2$

or

$$J_2\ddot{\theta}_2 + k_T\theta_2 - k_T\theta_1 = T_L$$

Classical form: Convert the two second-order equations into a single fourth-order equation. Using the D-operator notation:
From the second equation:

$$\theta_2 = \frac{T_L + k_T\theta_1}{J_2D^2 + k_T}$$

Substitute into the first:

$$[J_2D^2 + k_T][J_1D^2 + k_T]\theta_1 - k_T{}^2\theta_1 = -[J_2D^2 + k_T]T(t) + k_T T_L$$

Example 7: Electric Motor

For the state-space form, let:

$$x_1 = \dot{\theta}_1 \qquad \dot{x}_1 = \ddot{\theta}_1 \qquad \text{and} \qquad x_2 = \theta_1 \qquad \dot{x}_2 = \dot{\theta}_1 = x_1$$

$$x_3 = \dot{\theta}_2 \qquad \dot{x}_3 = \ddot{\theta}_2 \qquad \text{and} \qquad x_4 = \theta_2 \qquad \dot{x}_4 = \dot{\theta}_2 = x_3$$

Substituting gives:

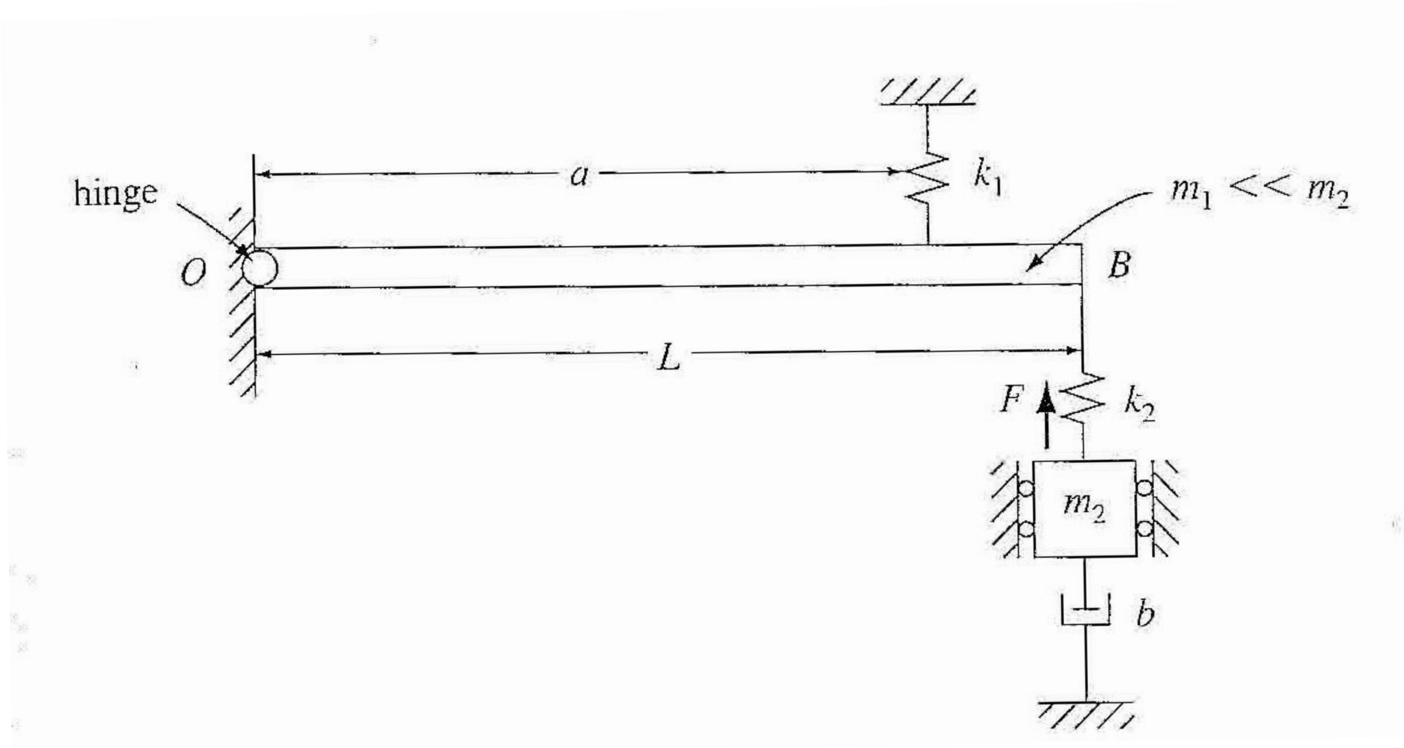
$$\dot{x}_1 = \frac{[-T(t) - k_T x_2 + k_T x_4]}{J_1}$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = \frac{(T_L - k_T x_4 + k_T x_2)}{J_2}$$

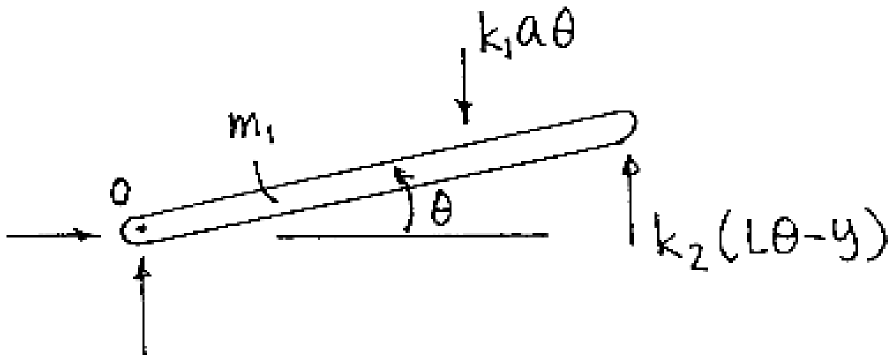
$$\dot{x}_4 = x_3$$

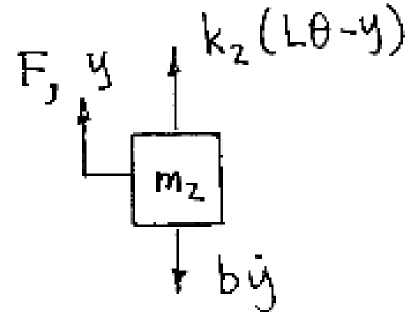
Example 8: Pair-Share Exercise: Copy Machine



- The device from a copying machine is shown. It moves in a horizontal plane. Develop the dynamic model, assuming that mass of bar is negligible compared to attached mass m_2 and angular motions are small. The mass is subjected to a step input F, find an expression for the displacement of point B after the transient motions have died out.

Example 8: Pair-Share Exercise: Copy Machine





For a massless bar: $\sum M_0 = J\ddot{\theta} = 0$ $\quad -k_1 a^2 \theta - k_2 L(L\theta - y) = 0$

For the mass $\sum F_y = m_2 \ddot{y}$ $\quad F + k_2 (L\theta - y) - b\dot{y} = m_2 \ddot{y}$

From the first equation:

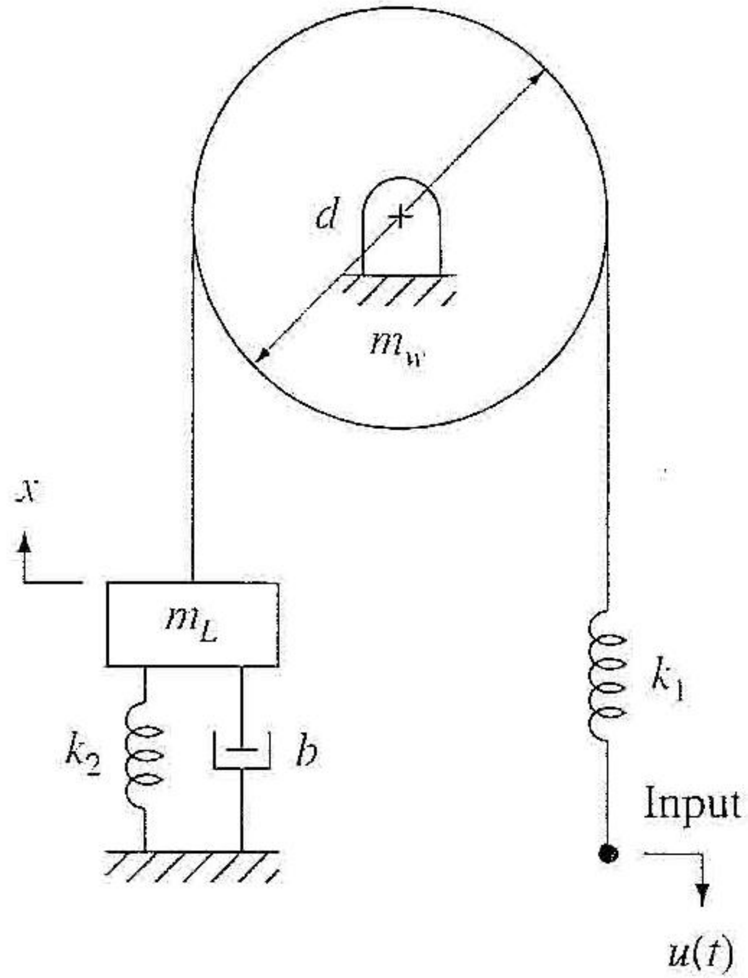
$$\theta = \frac{k_2 L y}{(k_1 a^2 + k_2 L^2)}$$

Substituting into the second:

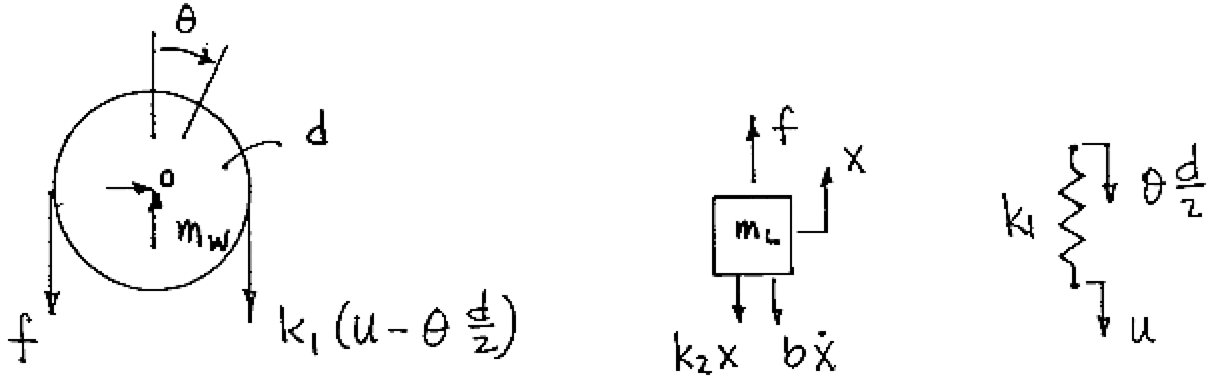
$$m_2 \ddot{y} + b\dot{y} + k_2 y - \frac{k_2^2 L^2}{(k_1 a^2 + k_2 L^2)} y = F$$

Example 9: Mass-Pulley System

- A mechanical system with a rotating wheel of mass m_w (uniform mass distribution). Springs and dampers are connected to wheel using a flexible cable without skip on wheel.
- Write all the modeling equations for translational and rotational motion, and derive the translational motion of x as a function of input motion u
- Find expression for natural frequency and damping ratio



Example 9: Mass-Pulley System



For the mass: $\sum F_x = m_L\ddot{x}$ $\quad f - k_2x - b\dot{x} = m_L\ddot{x}$

For the pulley: $\sum M_0 = J_0\ddot{\theta}$ $\quad \frac{d}{2}k_1(u - \frac{d}{2}\theta) - f\frac{d}{2} = J_0\ddot{\theta}$

Solving for the force f:

$$f = -\frac{\left[J_0\ddot{\theta} - \frac{d}{2}k_1\left(u - \frac{d}{2}\theta\right)\right]}{\frac{d}{2}}$$

Variables x & θ are not independent:

$$x = \frac{d}{2}\theta \quad \text{or} \quad \theta = 2\left(\frac{x}{d}\right)$$

Example 9: Mass-Pulley System

Thus
$$f = -\frac{\left[J_0 \frac{2}{d}\ddot{x} - \frac{d}{2}k_1 u + \left(\frac{d}{2}\right)^2 k_1 \frac{2}{d}x\right]}{\left(\frac{d}{2}\right)}$$

$$f = -J_0\left(\frac{2}{d}\right)^2 \ddot{x} - k_1 u + k_1 x$$

Substitute f into the equation of motion for the mass.

$$-J_0\left(\frac{2}{d}\right)^2 \ddot{x} + k_1 u - k_1 x - k_2 x - b\dot{x} = M_L \ddot{x}$$

Or

$$\left[m_L + J_0\left(\frac{2}{d}\right)^2\right]\ddot{x} + b\dot{x} + (k_1 + k_2)x = k_1 u$$

Example 9: Mass-Pulley System

One standard second-order system form is: $\ddot{\eta} + 2\zeta\omega_n\dot{\eta} + \omega_n^2\eta = 0$

Thus

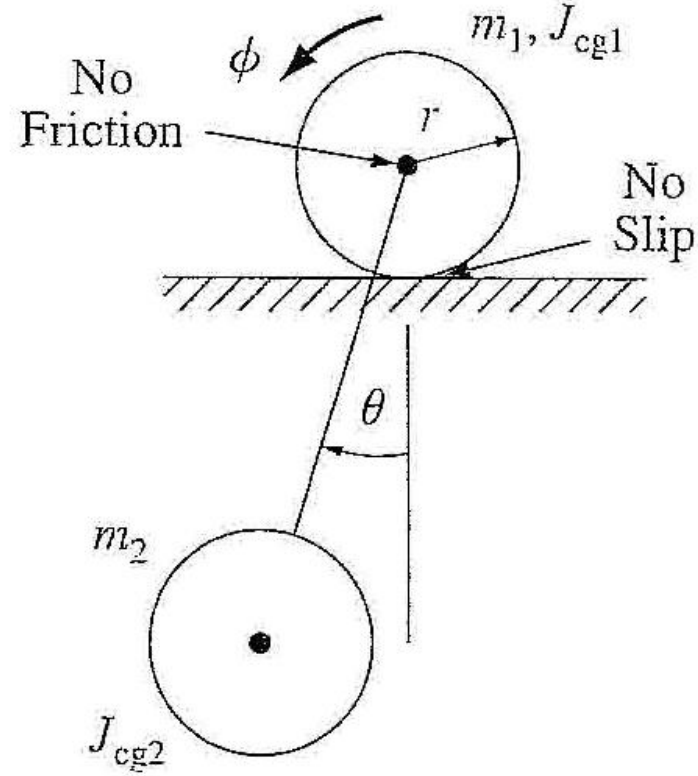
$$\omega_n = \sqrt{\frac{k_1 + k_2}{m_L + J_0\frac{4}{d^2}}} \qquad 2\zeta\omega_n = \frac{b}{m_L + J_0\frac{4}{d^2}}$$

$$\zeta = \left(\frac{\frac{1}{2}b}{m_L \div J_0\frac{4}{d^2}}\right)\sqrt{\frac{m_L + J_0\frac{4}{d^2}}{k_1 + k_2}}$$

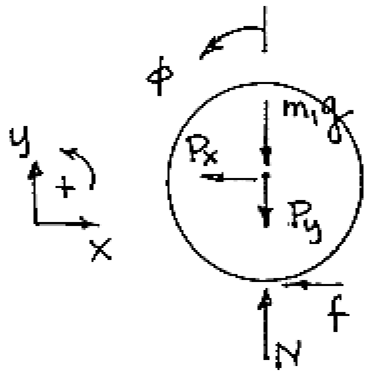
$$\zeta = \frac{b/2}{\sqrt{(k_1 + k_2)\left(m_L + J_0\frac{4}{d^2}\right)}}$$

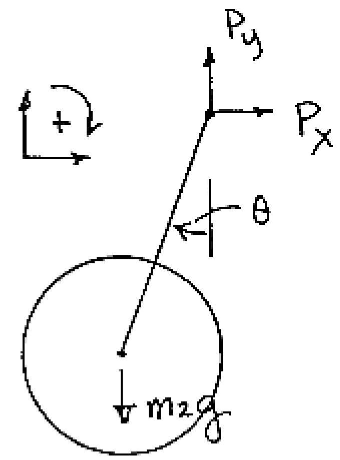
Example 10: Pair-Share Exercise: Double Pendulum

- The disk shown in the figure rolls without slipping on a horizontal plane. Attached to the disk through a frictionless hinge is a massless pendulum of length L that carries another disk. The disk at the bottom of the pendulum cannot rotation relative to the pendulum arm.
- Draw free-body diagrams and derive equations of motion for this system.



Example 10: Pair-Share Exercise:





Using the free body of the wheel:

$$\sum F_x = m_1 \ddot{x} \qquad -f - P_x = m_1 \ddot{x} \qquad \text{(a)}$$

$$\sum F_y = m_1 \ddot{y} \qquad -N - P_y = 0 \qquad \text{(b)}$$

$$\sum M_{cg1} = J_{cg1} \ddot{\phi} \qquad -fr_x = J_{cg1} \ddot{\phi} \qquad \text{(c)}$$

Using the free body of the pendulum:

$$\sum F_x = m_2 \ddot{x} \qquad P_x = m_2 \ddot{x} \qquad \text{(d)}$$

$$\sum F_y = m_2 \ddot{y} \qquad P_y - m_2 g = m_2 \ddot{y} \qquad \text{(e)}$$

$$\sum M_{cg2} = J_{cg2} \ddot{\theta} \qquad P_x L \cos\theta - P_y L \sin\theta = J_{cg2} \ddot{\theta} \qquad \text{(f)}$$

Example 10: Pair-Share Exercise: Double Pendulum

The position of the center of mass of the pendulum is given by

$$\vec{p} = (-r\varphi - L\sin\theta)\vec{i} - (L\cos\theta)\vec{j}$$

Thus the acceleration components of the mass center of the pendulum are:

$$\ddot{x} = -r\ddot{\phi} - L\ddot{\theta}\cos\theta + L\dot{\theta}^2\sin\theta \quad \text{and} \quad \ddot{y} = L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta$$

From (d) and (e)

$$P_x = m_2(-r\ddot{\phi} - L\ddot{\theta}\cos\theta + L\dot{\theta}^2\sin\theta) \quad \text{and} \quad P_y = m_2(L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)$$

Substituting these results into (f) gives:

Example 10: Pair-Share Exercise: Double Pendulum

$$\ddot{\theta} + \frac{m_2 r L \cos\theta}{J_{cg2} + m_2 L^2}\ddot{\phi} + \frac{m_2 g L \sin\theta}{J_{cg2} + m_2 L^2} = 0$$

For the free body of the wheel: $\ddot{x} = -r\phi$ Eqn. (a) becomes:

$f = -P_x + m_1 r\ddot{\phi}$ Substitute into (c) to obtain:

$$\ddot{\theta} + \frac{J_{cg1} + (m_1 + m_2) r^2}{m_2 r L \cos\theta}\ddot{\phi} - \dot{\theta}^2 \tan\theta = 0$$