MODELING WITH

Two teachers use a powerful, challenging tool in their Chinese classrooms to build, ensure, and solidify students' understanding of quantitative relationships.

DIAGRAMS

Meixia Ding

A tape diagram is a tape-like drawing used to illustrate number relationships (CCSSI 2010). In the scenario that follows, the tape diagram was used to model additive comparisons, a challenging concept for young learners (Nunes, Bryant, and Watson 2009). This representation also facilitated a type of mathematical teaching practice embraced by *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014): By actively engaging students in co-construction and purposeful discussions of the tape diagram, meaningful mathematical discourse took place, which enabled students to reason about the quantitative relationship. That is to say, the larger quantity contains the "same as" part as the small quantity and the "more than" part. This modified, condensed version of modeling with tape diagrams was presented in a Chinese second-grade classroom.

KIANKHOON (GIRL); SIRIKORNT (BOY); MYSONDANUBE (TAPE LETTERING)/THINKSTOCK

Scenario: The Jump problem

A word problem is presented on the board:

> Shawn and James had a contest to see who could jump farther. Shawn jumped 75 centimeters. James jumped 23 more centimeters than Shawn. How far did James jump?

The teacher draws the first tape to represent Shawn's jump, suggesting that students draw the second tape underneath to show James's jump. Typical student drawings are then selected for discussion (see **fig. 1**).

Teacher: Why did you draw the second tape longer?

Student: Because James has twenty-three more.

T: Which part of this tape shows twenty-three more? Come here to point it out.
[*Student gestures to the second part of the tape, and the teacher labels it as "23 more."*]

T: [*Pointing to the remaining part*] What does this part mean?

S: Pretending James jumped the same distance as Shawn, the first part shows that; but James actually jumped twenty-three more, so the tape is a little longer.

T: So, James's jump contains which two parts?

S: The part that is same as Shawn and the part that is more than Shawn.

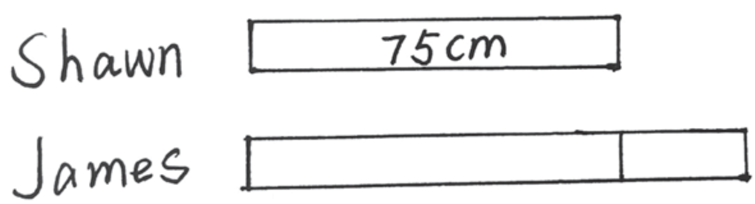
T: Now, can we compose a math expression to find out James's jump?

S: Seventy-five plus twenty-three!

Tape diagrams are a powerful model recommended by the Common Core, but they are relatively new to teachers and students in the United States. When a tape diagram is used to illustrate quantitative relationships (as in the scenario above), the model is an effective support for learning. However, these diagrams are often opaque and used ineffectively in teaching, resulting in difficulties for students and teachers and negativity toward tape diagrams. Vanduzer (2017) reported such difficulties when teaching the Jump problem. Our study aims to support teachers who are less familiar with tape diagrams in learning to use this model productively, as exemplified by two Chinese teachers.

FIGURE 1

This is a typical student drawing of the Jump problem discussed in the scenario.



MEIXIA DING

Mathematics modeling: Tape diagrams

Tape diagrams, also known as *strip diagrams*, *bar models*, *fraction strips*, or *length models* (CCSSI 2010), are linear representations that can be used to effectively model quantitative relationships during problem solving (Ng and Lee 2009). In comparison with the other discrete models, such as counters and cubes, tape diagrams can illustrate structural relationships that transcend the specific problem context (e.g., whole number, fractions, decimals, percentages, and formal algebra). International studies frequently report wide use of these diagrams in East Asian countries, such as Japan (Murata 2008), Singapore (Cai et al. 2005), and China (Ding and Li 2014). The Common Core State Standards for Mathematics (CCSSM) expectations are for U.S. students to use tape diagrams by sixth grade to solve real-world problems involving ratios and proportional relationships. However, to use this model productively, students ought to be exposed to this model in earlier grades so they can gain familiarity with this model and make sense of the embedded structural relationships (Murata 2008). Such structural knowledge can be transferred to new contexts to solve increasingly challenging tasks. For example, students' grasp of multiplicative comparison can lay a foundation for their later learning of ratios and proportions as expected by the Common Core. Promisingly, the current Common Core–based elementary school textbooks do introduce tape diagrams in early grades. For instance, beginning in first grade, the Go Math, Math Expressions, and enVisionmath 2.0 curricula all present tape diagrams named *bar models* when teaching simple word problem solving. This necessitates better support for teachers and students to use this model productively in mathematics classrooms. Our report of the Chinese approach aims to provide such support.

The Chinese approach

Background

The two Chinese teachers, Teacher Yang and Teacher Chen, are mathematics specialists who have more than fifteen years of teaching experience. They have won various teaching awards in China. Each teacher taught two consecutive lessons about comparisons to their second graders. In general, comparison problems involve three quantities: the large, the small, and the difference. Each of the quantities can be treated as an unknown, resulting in three types of comparison problems: finding the difference, finding the large quantity, and finding the small quantity (CCSSI 2010). Previous studies (e.g., Hudson 1983) have noted that a special type of problem, equalizing two quantities, can ease students' path to comparison problems. In the current study, students who had already learned how to find the difference in the first grade were expected to learn how to equalize two quantities in lesson 1 and how to find the large/small quantity in lesson 2. With pictorial illustrations from the textbook, both teachers used the same example task in lesson 1 (see **fig. 2c**) regarding how to equalize Jun's eight beads and Fang's twelve beads. In lesson 2, Yang used a modified textbook example (see **fig. 2d**) that contained two subproblems:

> Ying placed 11 sticks. Hua placed 3 more. Ping displayed 3 less than Ying. How many sticks did Hua or Ping place?

Chen used her own example task:

> Teacher Chen's favorite number is 45. A student's favorite number is 3 bigger or 35 smaller than 45. What is this student's favorite number?

To understand comparison problems, students must first grasp the concepts of "one-to-one correspondence" and "the same as" (Greeno and Riley 1987), which can be illustrated through modeling. Once students can visualize the large quantity as containing two parts—the same-as part (the small quantity) and the more-than part (the difference)—they may understand why "the large quantity = the small quantity + the difference" and "the large quantity – the difference = the small quantity." We report below how Chinese teachers use the tape diagrams to develop this understanding with students.

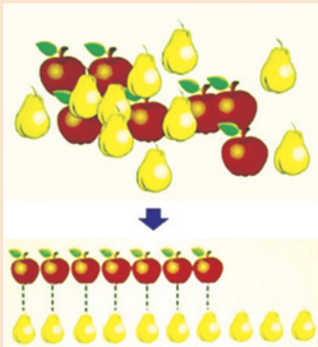
KIANKHOON; MYSONDANUBE (TAPE)/THINKSTOCK

FIGURE 2

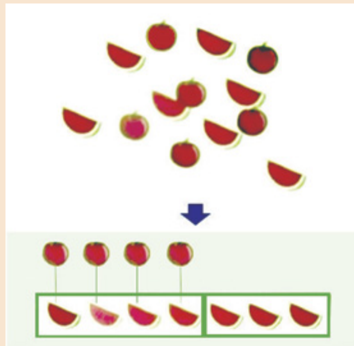
Structural uses of pretapes, such as these examples of moving from pretape to tape diagrams, likely contribute to students' learning. (Translations are under each.)

Review problem

(a) Lesson 1: Equalize two quantities

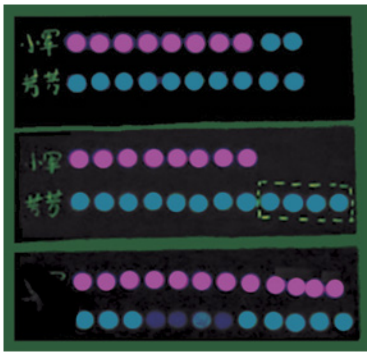


(b) Lesson 2: Find the large/small quantity



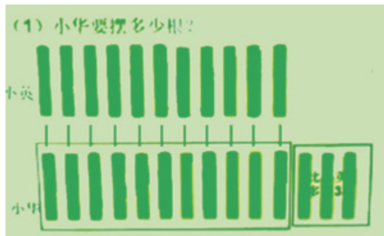
Worked example

(c)



Pretape 1: labeled with "Jun"
Pretape 2: labeled with "Fang"

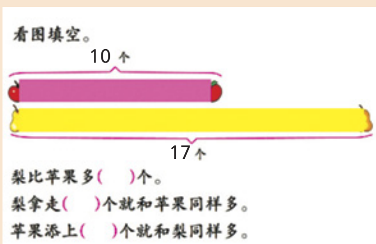
(d)



Pretape 1: labeled with "Ying"
Pretape 2: labeled with "Hua."
The second box is labeled as "3 more sticks than Ying."

Practice problem

(e)

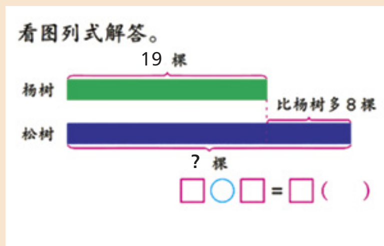


Pear has (_____) more than apple.

Taking away (_____) pears will result in the same number of apples.

Adding (_____) apples will result in the same number of pears.

(f)



Tape 1: 19 poplar trees

Tape 2: Pine trees, 8 more than poplar trees. The question mark is on "pine trees."

GUOSEN ZHANG

Readying for learning: Structural use of pretapes

Across all four lessons, the general move was from pretapes (Murata 2008) to tape diagrams. Pretapes are linear representations but contain discrete objects (e.g., fruits, chips, circles, sticks). According to Murata, pretapes are relatively more concrete and may prepare students to learn tape diagrams. **Figure 2** shows that in both lessons, Yang used pretapes in worked examples (**figs. 2c** and **d**) and used tape diagrams in practice problems (**figs. 2e** and **f**).

In addition to the sequence of moving from pretapes to tape diagrams, both teachers used pretapes with a clear focus on key concepts and quantitative relationships. As **figure 2** indicates, in both lessons, Yang started with a review task involving a mound of fruit, asking, "How can I tell *at a glance* which (type of) fruit has more?" This question prompted students to suggest lining up the different fruits, which formed pretapes. Some students even suggested one-to-one correspondence. On the basis of rearranged pictures, Yang further asked which two parts the large quantity contained (the large part being pears in lesson 1; see **fig. 2a** and the watermelon in lesson 2; see **fig. 2b**). In fact, the watermelon picture in lesson 2 was further boxed into two parts. Such structural uses of pretapes likely contribute to students' learning of tape diagrams.

Building understanding: Progressive co-construction of tape diagrams

During the teaching of worked examples, both teachers spent significant time engaging students in the co-construction of diagrams. Yang used pretape sticks in lesson 2 to model the example task (see **fig. 2d**). After small-group discussions, she invited a student to the board to share her group's modeling strategies.

T: Come here. Tell us what you discussed.

S2: We first place Ying's sticks in the first row. We should place eleven of them.

T: OK, stop. [*The screen shows eleven sticks. She addresses the class.*] Is this what she meant?

Students: Yes.

T: What's next?

S2: Next, you place eleven sticks for Hua in the second row and then add three more.

T: Wait, place eleven sticks. How? Randomly? What should we pay attention to?

S2: Line them up with Ying's. So, one-to-one correspondence.

T: One-to-one correspondence. This means the eleven sticks are actually—[*Students mumble, "Eleven sticks that are the same as Ying's." The screen shows eleven sticks in the second row.*]

T: Are we done?

S2: We need to add three more in the second row for Hua.

T: Why do we need to add three more?

S2: Because the problem says that Hua has three more than Ying.

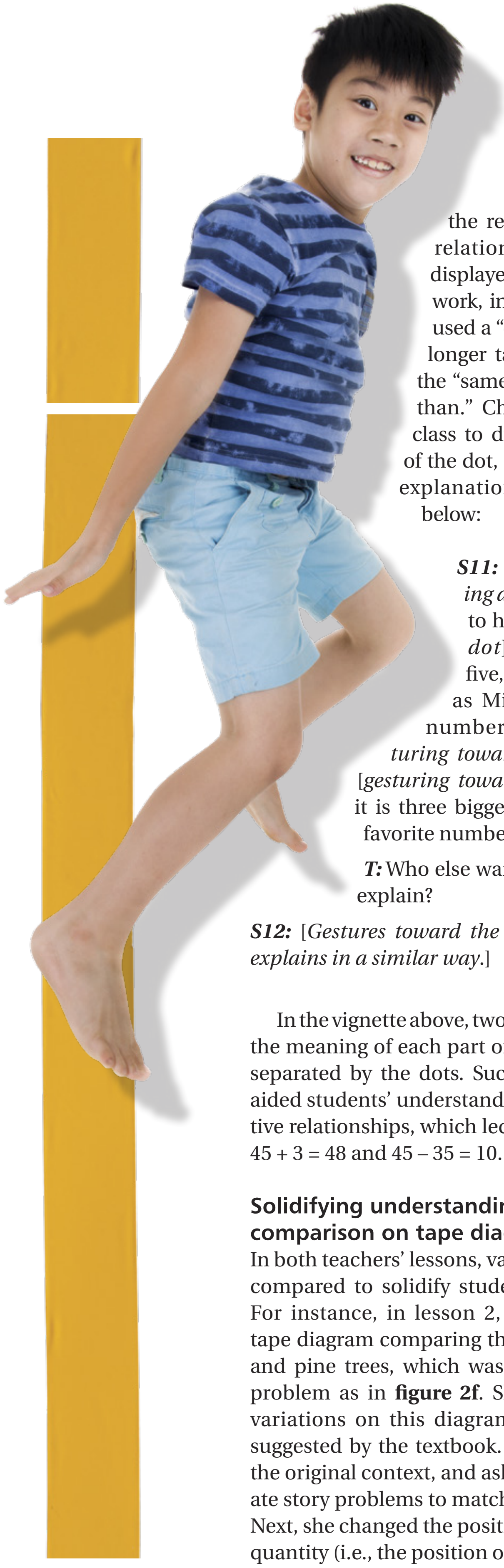
[*The teacher shows three more sticks in the second row.*]

After the pretape was co-constructed, Yang asked the class which two parts the second row contained (see **fig. 2d**). Referring to this pretape, students clearly explained that Hua's sticks contained the "same as" part and the "more than" part, leading toward generation of the numerical solution, 11 + 3 = 14 sticks.

In lesson 2 for the worked example, Chen used tape diagrams rather than pretapes (not illustrated). Similar to the Jump problem scenario, she drew the first tape to represent her favorite number of 45. She then invited the class to co-construct the second tape: "I will draw it with the cursor, and I will stop when you call it." During this process, Chen kept checking with the class if she should stop drawing, and if not, why. Students explained to her that since the student's favorite number was three bigger than hers, the second tape should be a little bit longer than the first tape. Note that Chen only pretended to draw using the cursor. After the collective work, she asked the class to actually draw out the second tape in their seats for both sub-problems, which generated interesting student work for class discussion, which we elaborate on in the next section. In both classrooms, the co-construction of tape diagrams was progressive, which may have decreased the intimidation of tape diagrams and boosted the students' interest in mathematical modeling.

Ensuring understanding: Gesturing and questioning on tape diagrams

During discussion of the tape diagrams, both teachers asked specific questions and requested student gesturing often with the teaching stick to clarify their explanations. In Chen's lesson 2, after the class had co-constructed the tape diagrams, Chen displayed one student's work and asked why the second tape was either longer in the subproblem of "three bigger" or shorter in the subproblem of "thirty-five smaller" than the first tape, thus drawing students' attention to



the relevant quantitative relationships. Next Chen displayed another student's work, in which the student used a "dot" to separate the longer tape into two parts, the "same as" and the "more than." Chen then asked the class to discuss the function of the dot, again eliciting deep explanations, as transcribed below:

> ***S11:*** From here [*pointing at the starting point*] to here [*pointing at the dot*], it means forty-five, and it is the same as Miss Chen's favorite number. From here [*gesturing toward the dot*] to here [*gesturing toward* the end] means it is three bigger than Miss Chen's favorite number.
>
> ***T:*** Who else wants to come up and explain?
>
> ***S12:*** [*Gestures toward the tape diagram and explains in a similar way.*]

In the vignette above, two students explained the meaning of each part of the tape diagrams separated by the dots. Such discussion likely aided students' understanding of the quantitative relationships, which led to the solutions of 45 + 3 = 48 and 45 – 35 = 10.

Solidifying understanding: Variation and comparison on tape diagrams

In both teachers' lessons, varied diagrams were compared to solidify student understanding. For instance, in lesson 2, Chen discussed a tape diagram comparing the number of aspen and pine trees, which was the same practice problem as in **figure 2f**. She then made two variations on this diagram, which were not suggested by the textbook. First, she removed the original context, and asked students to create story problems to match this tape diagram. Next, she changed the position of the unknown quantity (i.e., the position of the question mark on the tape diagram) while retaining the numbers. After the modified tasks were discussed, she displayed both tape diagrams (addition and subtraction) on the same screen for comparison. These variations and comparisons likely promoted students' understanding of comparison problems at a structural level.

Lessons learned: Revisiting the Jump problem

What might we learn from the Chinese approaches in terms of mathematical modeling, particularly with tape diagrams? In this study, it appears that the teachers' structural use of pretapes and engaging students in the co-construction of tape diagrams helped ease students' introduction to this model. The teachers' use of questioning, gesturing, variation, and comparison further develops students' understanding of tape diagrams and the embedded quantitative relationships. These instructional features seem crucial for using tape diagrams productively. Consider, for example, the Jump problem in the scenario. For students who lack experience with tape diagrams, a teacher may start with pretapes (e.g., cubes to model a problem involving smaller numbers: Shawn has 7 apples. James has 3 more apples). Note that the pretapes should *not* be used to find the answer (e.g., counting the cubes to obtain "10 apples"). Rather, they should be used to guide students to understand the key concepts (e.g., the same-as and quantitative relationships: James' apples contain the same-as part and the more-than part). After this, a teacher may present the Jump problem and engage students through progressive co-constructing, questioning, and gesturing on the tape diagram. Furthermore, this problem may then be varied by rewording (e.g., "Shawn jumped 75 centimeters. James jumped 23 *fewer* centimeters than Shawn), by changing the unknown quantity, or by removing the contextual information. Regardless of the changes, the resulting problems may be continuously modeled using the tape diagrams, which may be further compared to deepen student understanding.

Beyond teaching comparison problems, the two exemplary teachers' use of tape diagrams sheds light on the purpose of mathematical modeling in elementary school classrooms. In this

SIRIKORNT; MYSONDANUBE (TAPE)/THINKSTOCK study, the tape diagrams were new for the Chinese second graders; yet students demonstrated capability to reason with this model. This may be partially attributed to teachers' structural use of pretapes and tape diagrams. When pretapes are used to elucidate quantitative relationships, not to find the answers by counting, they may prepare students for quantitative reasoning with tape diagrams. Additionally, it was found that both teachers devoted significant time to mathematical modeling but relatively little time to computational strategies, as the purpose of the diagram is to illustrate the underlying quantitative relationships rather than to aid in computation. Such an approach to tape diagrams offers an opportunity for conceptual, meaning-focused instruction as emphasized by *Principles to Actions* (NCTM 2014), which may provide better support for the mathematical modeling, reasoning, and problem-solving expectations of the Common Core.

Common Core Connections

1.OA.1
2.MD.10

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