Models Involving Interactions between Predator and Prey Populations

Matthew Mitchell

Georgia College and State University

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Abstract

Predator-prey models are used to show the intricate interactions between predator and prey populations. In this project, we will show how these different interactions between the predator and prey populations are possible based on the choice of functional response we make. In general, a functional response is the relationship between the average number of prey eaten by each predator per unit of time versus the density of the prey population. Our choice of functional responses are motivated by the observations of the predator and prey interactions in nature.

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0.1 Introduction to the Predator-Prey Model

Throughout history we have been using mathematics to push other fields of research forward, one of those tools used by mathematicians is the predator-prey model. The predator-prey model is used to understand the interactions between certain predators and their prey within an ecosystem. The model uses a system of nonlinear equations to model these interactions. The general form of the predator-prey model is given by:

$$\begin{cases} \frac{\dot{X}}{X} = g(X) - P(X,Y) \\ \frac{\dot{Y}}{Y} = \sigma(X) - \mu \end{cases} \tag{1}$$

In the model 1 we note that X and Y represent the prey and predator populations respectively. The function $g(X)$ represents the per-capita growth rate of the prey population in the absence of predation. We will assume that $g(X) > 0$ for all $k > x \geq 0$ and $g(k) = 0$ for some $k > 0$, where k is defined to be the carrying capacity of X. The function $P(X,Y)$ is the per-capita consumption rate of the prey. We assume that for a fixed prey population density, the consumption rate of the prey would increase as the predator population density increases. Adversely, for a fixed predator population density, the consumption rate of the prey would decrease as the prey population density increases, for example the likelihood of a certain prey being hunted decreases as the prey population increases. The function $\sigma(X)$ is the per-capita growth rate of the predator population density. We will assume that the per-capita growth rate of the predator population density is positive and is zero when the prey population density is zero. Lastly the parameter μ represents the per-capita death rate of the predator. On graphing the population densities obeying equation 1, with specific choices of $g(X), P(X,Y)$, and $\sigma(X)$, we obtain figure 1 below:

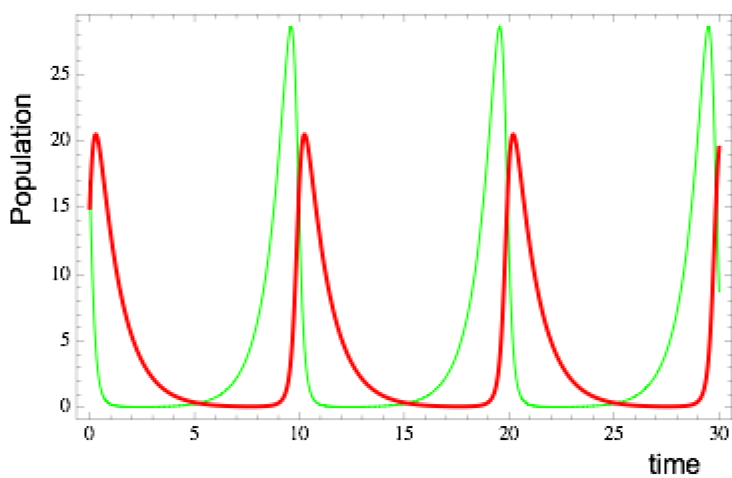


Figure 1: Predator population(red) and Prey population (green)

As we observe in Figure 1 the two populations are exhibiting an oscillatory relationship. In fact we observe that the predator population is lagging behind the prey population.

0.1.1 Introduction to Functional Responses

To understand the predator-prey model (1), we will introduce the concept of functional responses. A functional response is the relationship between the average number of prey eaten by each predator per unit of time as a function of the density of the prey population. The functional response β is given by the equation:

$$\beta = \frac{XP(X,Y)}{Y},$$

where X, Y, and $P(X, Y)$ each represent the prey population density, predator population density, and per-capita consumption rate of the prey respectively. Depending on our choice of β, the predator-prey model can exhibit different dynamics. In this paper, we consider three different types of functional responses. Namely type I, type II, and type III, represented by the following equations:

- Type I: $\beta = \alpha x$
- Type II: $\beta = \frac{\alpha x}{H+x}$
- Type III: $\beta = \frac{\alpha x^2}{H^2+x^2}$.

In these functional responses we have that α is the rate at which the prey is being attacked, and H is associated with the handling time of the prey. The handling time is the time the predator takes to find/hunt prey, consume the prey, be hungry again, etc. In the type I functional response we have a linear function, which implies that as the prey population density increases, the number of prey consumed would increase along with it. The type I model is not seen much in nature, the only examples would be with animals like rabbits, which reproduce quickly enough to support this increased consumption on their population. During the type II functional response we see a gradual increase in prey consumption as the prey population density increases. The consumption rate saturates eventually as the prey population reaches its carrying capacity. Type III functional responses occur in predators which increase their search activity with increasing prey density. Type III is similar to type II except that the predator response to prey is depressed at low prey density. Note that in type III functional response, the concavity of the graph changes. For small values of prey population density there is a decrease in the predation. The concavity of the graph changes again after a threshold, at which there is an increase in predation, with an increase in the prey population. These functional responses are given in the following graphs:

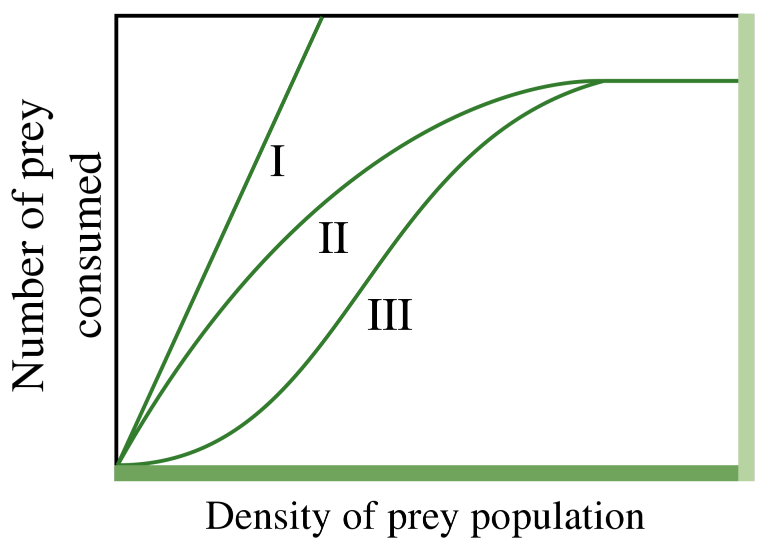


Figure 2: Three types of functional responses

0.2 Preliminary Background

Rewriting equation 1, the general predator-prey model reads as:

$$\begin{cases} \dot{X} = f(X,Y) \\ \dot{Y} = g(X,Y), \end{cases} \tag{2}$$

where

$$\begin{cases} f(X,Y) = X(g(X) - P(X,Y)) \\ g(X,Y) = Y(\sigma(X) - \mu) \end{cases} \tag{3}$$

In order to analyze system 2, we need the following terminologies: the X-nullcline is the set of points where $f(X,Y) = 0$ and the Y-nullcline is the set of points where $g(X,Y) = 0$. Ecologically speaking along the X and Y nullclines, the prey and predator populations remain constant respectively. A fixed or equilibrium point lies at the intersection of the X and Y nullclines. Fixed points are important ecologically because at these points, the population densities do not change with time. For example, if

$$\begin{aligned} \dot{X} &= X(1 - X - Y) \\ \dot{Y} &= Y(X - \mu), \end{aligned} \tag{4}$$

then the X- nullcline is $X = 0$ and $Y = 1 - X$ (represented in blue color in figure 3), and the Y nullcline is $Y = 0$ and $X = \mu$ (represented in green in figure 3). The fixed points are $(0,0), (1,0)$, and $(\mu, 1-\mu)$.

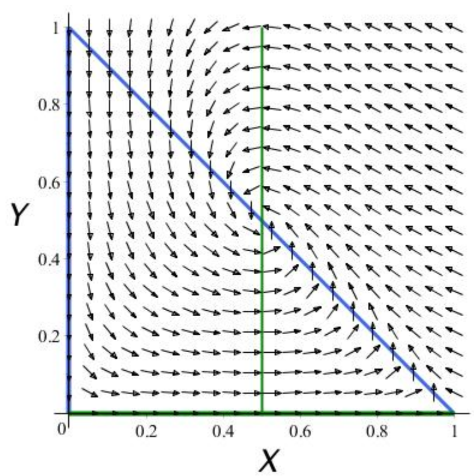


Figure 3: Phase Portrait of system (4), with $\mu = 0.5$

The fixed points allow us to draw insight into what is happening in the system based upon their nature. In the next sub-section we will classify the nature of the fixed points.

0.2.1 Jacobian Matrix

We are interested in analyzing system (2) near its fixed points. Suppose that the fixed point of system (2) is (X^*, Y^*). We define u and v by

$$u = X - X^*,\ v = Y - Y^*$$

Then

$$\dot{u} = \dot{X} = f(X^* + u, Y^* + v)$$

A Taylor series expansion around (X^*, Y^*) yields that

$$\dot{u} = f(X^*, Y^*) + u\frac{\partial f}{\partial X}(X^*, Y^*) + v\frac{\partial f}{\partial Y}(X^*, Y^*) + O(u^2, v^2, uv).$$

Since $f(X^*, Y^*) = 0$, we have that

$$= u\frac{\partial f}{\partial X}(X^*, Y^*) + v\frac{\partial f}{\partial Y}(X^*, Y^*) + O(u^2, v^2, uv)$$

Similarly,

$$\dot{v} = \dot{Y} = f(X^* + u, Y^* + v)$$

A Taylor series expansion around (X^*, Y^*), and the fact that $g(X^*, Y^*) = 0$ yields that

$$\dot{v} = f(X^*, Y^*) + u\frac{\partial f}{\partial X}(X^*, Y^*) + v\frac{\partial f}{\partial Y}(X^*, Y^*) + O(u^2, v^2, uv)$$

$$= u\frac{\partial f}{\partial X}(X^*, Y^*) + v\frac{\partial f}{\partial Y}(X^*, Y^*) + O(u^2, v^2, uv)$$

Thus we obtain that

$$\begin{cases} \dot{u} = u\frac{\partial f}{\partial X}(X^*, Y^*) + v\frac{\partial f}{\partial Y}(X^*, Y^*) + O(u^2, v^2, uv) \\ \dot{v} = u\frac{\partial g}{\partial X}(X^*, Y^*) + v\frac{\partial g}{\partial Y}(X^*, Y^*) + O(u^2, v^2, uv) \end{cases} \tag{5}$$

Note the higher order terms at the end of each equation in system (5) are denoted by $O(u^2, v^2, uv)$. We will ignore these higher order terms before going further with our analysis. We are able to do so because we are only looking at a small area around the fixed point so it follows that the higher order terms are significantly small and can be neglected, giving us a linear system. This process of linearization around each fixed point and studying the linear system allows us to understand the dynamincs around that fixed point. This process is called the linear stability analysis. We can now rewrite our linearized system as the following:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}, \tag{6}$$

where

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial X} & \dfrac{\partial f}{\partial Y} \\ \dfrac{\partial g}{\partial X} & \dfrac{\partial g}{\partial Y} \end{pmatrix}_{(X^*,Y^*)} \tag{7}$$

is defined to be the Jacobian matrix.
The general solution of the linear system (5) is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of the jacobian matrix and $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ are the fundamental solutions of the linear system (5). The eigenvalues λ_1, λ_2 satisfy the characteristic equation $det(J - \lambda I) = 0$. where I is the identity matrix. The characteristic equation can be rewritten as:

$$\lambda^2 - \tau\lambda + \Delta = 0, \tag{8}$$

where τ is the trace of J, and Δ is the determinant of J. The eigenvalues are:

$$\lambda_1 = \tfrac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \lambda_2 = \tfrac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

The nature of the eigenvalue determines the overall structure of the phase portrait. We can use the knowledge of eigenvalues, the trace, and determinant to further classify fixed points.

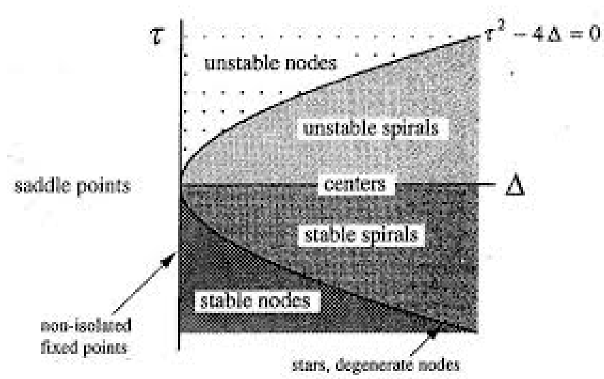


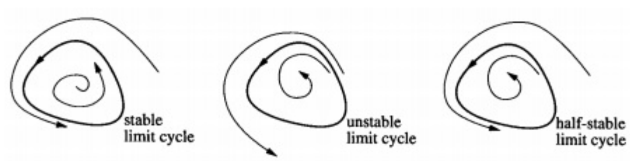
Figure 4: The eigenvalues from system (8) are used to classify the fixed point.

In the diagram above the axes are the trace τ and the determinant Δ of our matrix J. If $\Delta < 0$, the eigenvalues will be real and have opposite signs, therefore the fixed point will be a saddle point. If $\Delta > 0$, the eigenvalues are either real with the same sign, which denote nodes, or complex which will denote spiral and center fixed points. Nodes will satisfy $(\tau)^2 - 4\Delta > 0$ while spirals will satisfy $(\tau)^2 - 4\Delta < 0$. The parabola denoted by the equation $(\tau)^2 - 4\Delta = 0$ is the border between nodes and spirals, and if our fixed point falls on this parabola it will be a degenerate node or a star node.

Now the stability of these spirals and nodes is determined by the trace. When $\tau < 0$, both of the eigenvalues will have negative real parts, so the fixed point will be stable. Naturally unstable spirals and nodes will have $\tau > 0$. Stable centers will occur when $\tau = 0$, where their eigenvalues are purely imaginary.

0.2.2 Limit Cycles

A limit cycle is an isolated closed trajectory. What we mean by isolated is that the neighboring trajectories are not closed, they will either spiral away from or towards our limit cycle.



If we observe that all of the neighboring trajectories are spiraling towards the limit cycle, we say that the limit cycle is stable or that it is attracting. Likewise, if the trajectories are spiraling away from the limit cycle, we say that it is unstable. There are some exceptional cases where we will have that the limit cycle is half-stable.

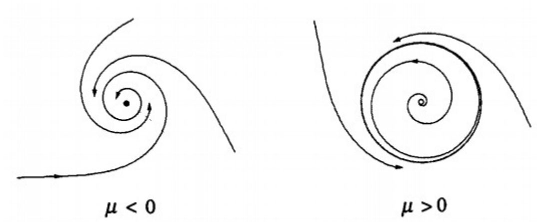
An interesting fact about these limit cycles is that they are inherently nonlinear. What we mean by this is that they cannot occur in linear systems. Although a linear system can have closed orbits, those orbits will not be isolated. Consider $x(t)$ being a periodic solution. It will follow that $cx(t)$ is also a solution, for any constant $c \neq 0$. Therefore $x(t)$ is surrounded by a family of closed orbits.

0.2.3 Bifurcations

Bifurcations occur when the phase portrait changes its topological structure as a parameter is varied. The bifurcation we will observe in this paper is referred to as a Hopf bifurcation. A Hopf bifurcation occurs when two complex conjugate eigenvalues simultaneously cross the imaginary axis into another plane. This loss of stability occurs while varying the parameter in question. There are two types of Hopf bifurcations, namely supercritical Hopf bifurcation and subcritical Hopf bifurcation as described below. The first type of Hopf bifurcation we will discuss is the supercritical hopf bifurcation, consider the system of equations below:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + br^2 \end{cases} \tag{9}$$

In this system above μ controls the stability of the fixed point at the origin, while ω gives the frequency of infinitesimal oscillations, and b determines the dependence of frequency on amplitude for larger amplitude oscillations. Observe now the phase portrait plot for μ above and below zero.



Notice that when $\mu < 0$ the origin is a stable spiral whose rotation depends on the sign of ω. At $\mu = 0$ the origin is still a stable spiral, just a weak one. Lastly, for $\mu > 0$ there is an unstable spiral at the origin and a stable limit cycle born at $r = \sqrt{\mu}$. We say that $\mu = 0$ is the bifurcating value leading to a supercritical Hopf bifurcation.

To analyze this system near the fixed point $(0, 0)$ let's look at the jacobian matrix at the origin for this system.

$$J = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

which has the eigeinvalues of:

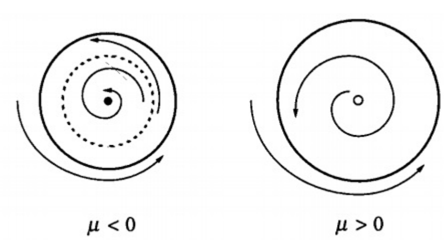
$$\lambda = \mu \pm i\omega$$

It is the case that the eigenvalues cross the imaginary axis from left to right as μ increases from negative to positive.

Our next type of Hopf bifurcation is the subcritical hopf bifurcation. The subcritical hopf bifurcation is much more chaotic compared to it's supercritical counterpart. After the bifurcation occurs trajectories must jump to a distant attractor, which could be a fixed point, another limit cycle, infinity, or in three and higher dimensions a chaotic attractor. Now consider this system below for our subcritical hopf bifurcation.

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \tag{10}$$

The main difference between our two hopf bifurcation cases is that in our subcritical case the r^3 term is destabilizing, which means that it is helping to drive trajectories away from the origin. Now let's observe the phase portrait our system created.



Notice that when $\mu < 0$ there are two attractors, a stable limit cycle and a stable fixed point at the origin. Between these two attractors lies an unstable limit cycle denoted by the dotted line. As μ increases the unstable limit cycle constricts around the fixed point at the origin. A subcritical hopf bifurcation occurs at $\mu = 0$, where the unstable limit cycle shrinks and engulfs the origin, making it unstable. For $\mu > 0$ the large amplitude limit cycle is the only attractor, and it forces solutions to grow into large amplitude oscillations.

0.3 Functional Responses

0.3.1 Type I Functional Response

We will now consider a predator-prey model exhibiting the type I functional response, with a constant reproductive rate of the prey:

$$\begin{cases} \frac{dU}{d\tau} = aU - bUV \\ \frac{dV}{d\tau} = cUV - dV, \end{cases} \tag{11}$$

where

- **a**: represents the natural growth rate of the prey in the absence of predators.
- **b**: represents the predation rate.
- **c**: represents the efficiency and propagation rate of the predator in the presence of prey.
- **d**: represents the natural death rate of the predator in the absence of prey.
- **U**: population of the prey.
- **V**: population of the predator.

We will now introduce the variables u, v, t, δ, and γ as follows:

$$\gamma = \frac{c}{a},\ \delta = \frac{d}{c},\ u = U,\ v = \frac{bV}{a},\ t = a\tau$$

We start by using the chain rule to get dU in terms of du:

$$a\frac{du}{dt} = aU - bUV$$

By implementing our defined terms from above we now get:

$$a\frac{du}{dt} = au - \frac{buva}{b}$$

$$\frac{du}{dt} = u(1-v) \tag{12}$$

Similarly by using the Chain Rule to get dV in terms of dv, we have that:

$$a\frac{d\frac{va}{b}}{dt} = cUV - dV$$

By implementing our defined terms from above we now get:

$$\frac{a^2}{b}\frac{dv}{dt} = \frac{va}{b}(cu - d)$$

$$\frac{dv}{dt} = \frac{cv}{a}(u - \frac{d}{c}) \tag{13}$$

Combining equations (12) and (13), we obtain our model corresponding to type I functional response:

$$\begin{cases} \frac{du}{dt} = u(1-v) \\ \frac{dv}{dt} = \gamma v(u-\delta) \end{cases} \tag{14}$$

Note that $\gamma, \delta > 0$ by the assumptions made on a, c, and d. We are now working towards plotting the phase portrait of this system. First we must find the fixed points of this system, and they are:

$$(0,0) \text{ and } (\delta, 1)$$

The jacobian matrix for our system of equations is as follows:

$$J_1 = \begin{pmatrix} 1-v & -u \\ \gamma v & \gamma u - \gamma\delta \end{pmatrix}$$

We will now evaluate the jacobian matrix at the first fixed point $(0,0)$:

$$J_1\Big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = -\gamma\delta$. Now that we have the eigenvalues for this system we find that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for λ_2. Since our eigenvalues are opposite in sign the fixed point $(0,0)$ would be a saddle point for all values of γ and δ.

We will now evaluate the jacobian matrix at the fixed point $(\delta, 1)$:

$$J_1\Big|_{(\delta,1)} = \begin{pmatrix} 0 & -\delta \\ \gamma & 0 \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = +i\sqrt{\gamma\delta}$ and $\lambda_2 = -i\sqrt{\gamma\delta}$, with the corresponding eigenvectors $\begin{pmatrix} \frac{-\delta}{i\sqrt{\gamma\delta}} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{\delta}{i\sqrt{\gamma\delta}} \\ 1 \end{pmatrix}$ respectively. Since our eigenvalues are purely imaginary, the fixed point $(\delta, 1)$ is a center for our values of γ and δ. Now we are able to plot the phase portrait of the first system:

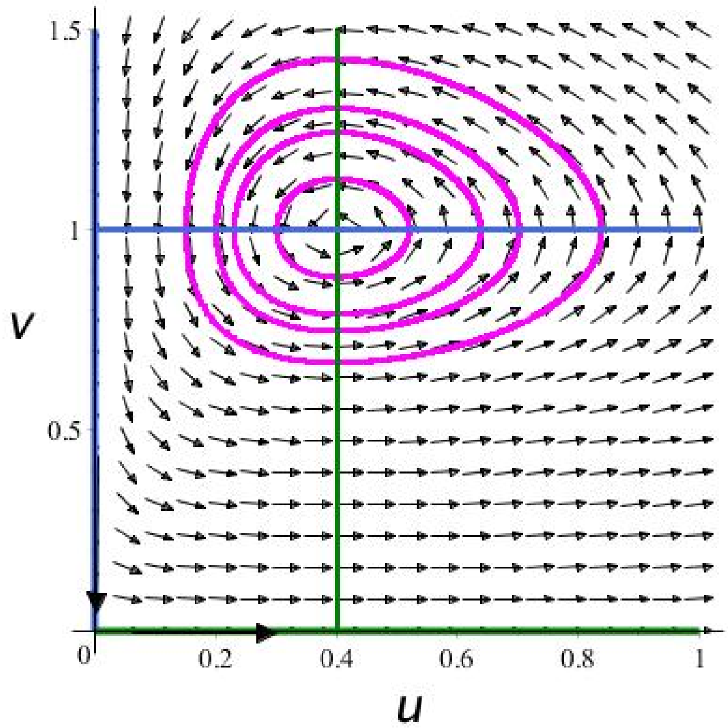


Figure 5: Center at $(\delta, 1)$, and saddle at (0, 0) for $\gamma = 0.5$ and $\delta = 0.4$.

Note that the origin is stable along the v axis, and unstable along the u, so this characteristic gives us a saddle point. While at the center point we have the predator and prey populations oscillating as time increases.

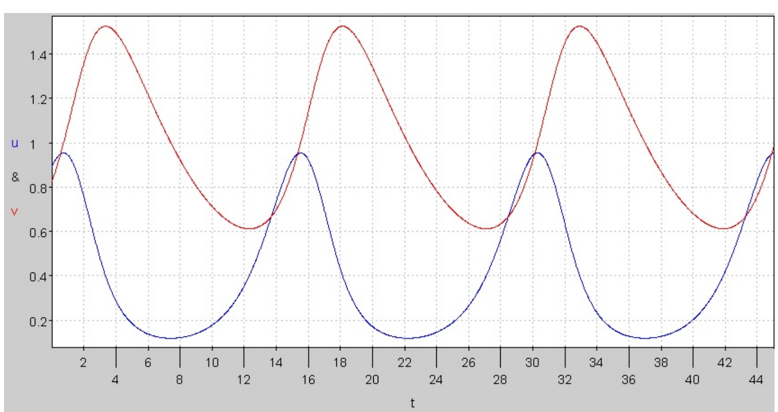


Figure 6: u and v vs t graph using $\gamma = 0.5$ and $\delta = 0.4$.

0.3.2 Type I Functional Response Second Model

We will now consider another predator-prey model exhibiting type I functional response with the assumptions that the reproductive rate of the prey population decreases linearly with its size. The model reads as follows:

$$\begin{cases} \frac{dU}{d\tau} = aU(1 - \frac{U}{K}) - bUV \\ \frac{dV}{d\tau} = cUV - dV \end{cases} \tag{15}$$

where,

- **a**: represents the natural growth rate of the prey in the absence of predators.
- **b**: represents the predation rate.
- **c**: represents the efficiency and propagation rate of the predator in the presence of prey.
- **d**: represents the natural death rate of the predator in the absence of prey.
- **U**: population of the prey.
- **V**: population of the predator.
- **K**: the carrying capacity of the prey population.

We will now introduce the variables u, v, t, δ, and γ as follows:

$$\gamma = \frac{cK}{a},\ \delta = \frac{d}{cK},\ u = \frac{U}{K},\ v = \frac{bV}{a},\ t = a\tau$$

We apply the same method of finding the system of equations as we did for system (11). In doing so we have that the system of equations for the type I functional response second model is:

$$\begin{cases} \frac{du}{dt} = u(1 - u) - uv \\ \frac{dv}{dt} = \gamma v(u - \delta) \end{cases} \tag{16}$$

where $\gamma, \delta > 0$.
We are now working towards plotting the phase portrait of this system. First we must find the fixed points of this system, and they are:

$$(0, 0),\ (\delta, 1 - \delta),\ \text{and}\ (1, 0)$$

The jacobian matrix for our system of equations is:

$$J_2 = \begin{pmatrix} 1 - 2u - v & -u \\ \gamma v & \gamma u - \gamma\delta \end{pmatrix}$$

We will now evaluate the jacobian matrix at the first fixed point $(0, 0)$:

$$J_2\Big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = -\gamma\delta$. Now that we have the eigenvalues for this system we find that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for λ_2. Since our eigenvalues are opposite in sign our fixed point would be a saddle point for all values of γ and δ. We will now evaluate the jacobian matrix at the fixed point $(1, 0)$:

$$J_2\Big|_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & \gamma - \gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = -1$ and $\lambda_2 = \gamma - \gamma\delta$. Now that we have the eigenvalues for this system we find that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for λ_2. Since $\gamma > \gamma\delta$ our eigenvalues will be opposite in sign, thus $(1, 0)$ would be a saddle point for all values of γ and δ. We will now evaluate the jacobian matrix at our last fixed point $(\delta, 1 - \delta)$:

$$J_2\Big|_{(\delta,1-\delta)} = \begin{pmatrix} -\delta & -\delta \\ \gamma - \gamma\delta & 0 \end{pmatrix}$$

We have our eigenvalues for this last jacobian matrix being $\lambda_{1,2} = \frac{1}{2}\delta \pm \frac{1}{2}\sqrt{\delta^2 - 4\delta\gamma + 4\delta^2\gamma}$. We can see that if we vary the values of δ it will change the classification of the fixed point. We will keep $\gamma = 0.5$ as a constant and we will vary the values of δ. Observe the following three phase portraits:

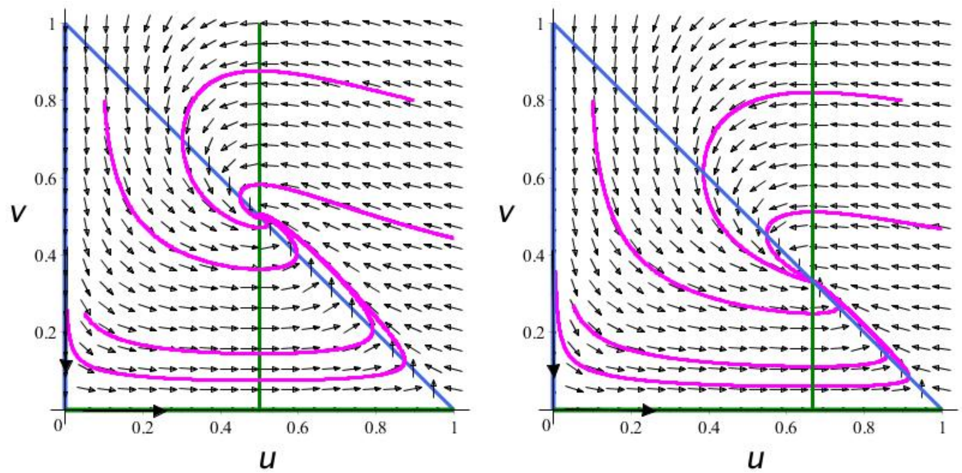


Figure 7: Spiral sink for $\delta = 0.5$, Degenerate node for $\delta = 2/3$.

We obtain a spiral sink when the eigenvalues are complex with negative real parts, while we obtain a degenerate node when we have only one eigenvalue. Observe the last phase portrait of a nodal sink, which occurs when the eigenvlaues are real and negative.

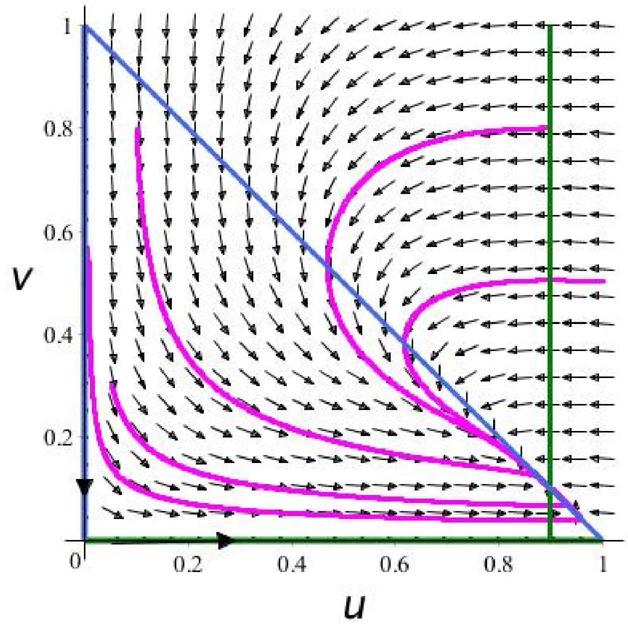


Figure 8: Nodal sink for $\delta = 0.9$.

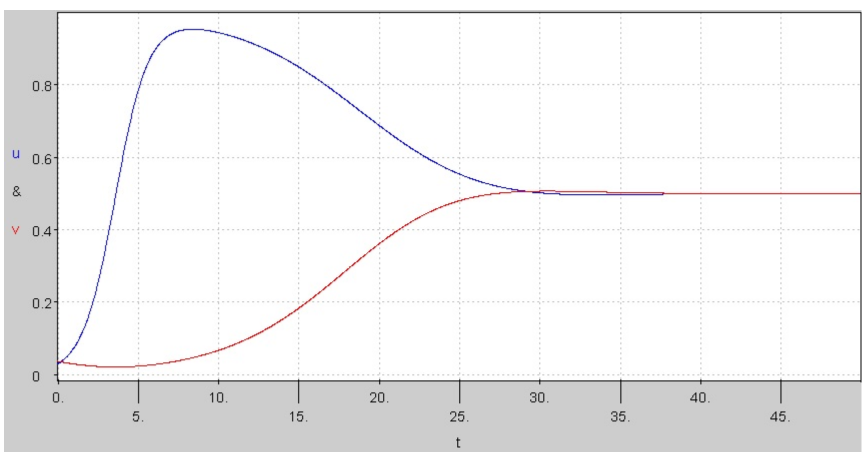


Figure 9: u, v vs t graph using $\gamma = 0.5$, and $\delta = 0.5$

The key difference between our two type I functional responses is that the prey population grows exponentially in the first mode i.e. $u = e^t$ for some $c > 0$, while in the second model, the prey population saturates to its carrying capacity, i.e. $u(t) = \frac{ce^t}{1+ce^t}$ for some $c > 0$.

0.3.3 Type II Functional Response

We will now consider the predator-prey model exhibiting the type II functional response with the assumption that the prey population will increase until it reaches the carrying capacity.

$$\begin{cases} \frac{du}{d\tau} = aU(1 - \frac{U}{K}) - \frac{bUV}{A+U} \\ \frac{dv}{d\tau} = \frac{cUV}{A+U} - dV, \end{cases} \tag{17}$$

where

- **U** represents the population of the prey.
- **V** represents the population of the predator.
- **a**: represents the natural growth rate of the prey in the absence of predators.
- **b**: represents the predation rate.
- **c**: represents the efficiency and propagation rate of the predator in the presence of prey.
- **d**: represents the natural death rate of the predator in the absence of prey.
- **A**: represents the semi-saturation constant, meaning that when the prey density reaches A, the maximal feeding rate of the predator is one-half of its maximum.

We will now introduce the variables u, v, t, δ, γ, and h as follows:

$$\gamma = \frac{c}{a},\ \delta = \frac{d}{c},\ u = \frac{U}{K},\ v = \frac{bV}{aK},\ t = a\tau,\ h = \frac{A}{K} < 1$$

So that under the new variables, system (10) transforms to:

$$\begin{cases} \frac{du}{dt} = u(1-u) - \frac{uv}{h+u} \\ \frac{dv}{dt} = \gamma v(\frac{u}{h+u} - \delta) \end{cases} \tag{18}$$

We are now working towards plotting the phase portrait of this system. First we must find the fixed points of this system, and they are:

$$(0,0),\ (1,0),\ \text{and}\ \left(\frac{\delta h}{1-\delta}, \frac{h(1-\delta h-\delta)}{(1-\delta)^2}\right),\ \text{which exists only if}\ \delta < \frac{1}{h+1}$$

The jacobian matrix for our system of equations is as follows:

$$J_3 = \begin{pmatrix} 1 - 2u - \frac{vh}{(h+u)^2} & \frac{-u}{h+u} \\ \frac{\gamma vh}{(h+u)^2} & \frac{\gamma u}{h+u} - \gamma\delta \end{pmatrix}$$

We will now evaluate the jacobian matrix at the first fixed point $(0,0)$:

$$J_3\Big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = -\gamma\delta$. Now that we have the eigenvalues for this system we find that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for λ_2. Since our eigenvalues are opposite in sign our fixed point would be a saddle point for all values of γ and δ. We will now evaluate the jacobian matrix at the fixed point $(1,0)$:

$$J_3\Big|_{(1,0)} = \begin{pmatrix} -1 & \frac{-1}{h+1} \\ 0 & \gamma - \frac{\gamma}{h+1} - \gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = -1$ and $\lambda_2 = \frac{-\gamma(\delta h+\delta-1)}{(h+1)} > 0$, under the assumption that $\delta < \frac{1}{h+1}$. Now that we have the eigenvalues for this system we find that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} \frac{1}{\gamma\delta h+\gamma\delta-\gamma-h-1} \\ 1 \end{pmatrix}$ for λ_2. Since our eigenvalues are opposite in sign the fixed point would be a saddle point. We will now evaluate the jacobian matrix at the interior fixed point $\left(\frac{\delta h}{1-\delta}, \frac{h(1-\delta h-\delta)}{(1-\delta)^2}\right)$:

$$J_3\Bigg|_{\left(\frac{\delta h}{1-\delta}, \frac{h(1-\delta h-\delta)}{(1-\delta)^2}\right)} = \begin{pmatrix} \frac{\delta(-h-\delta h+1-\delta}{1-\delta} & -\delta \\ \gamma(\delta h+\delta-1) & 0 \end{pmatrix}$$

We have that our eigenvalues for this last jacobian matrix are as follows:

$$\lambda_{1,2} = \frac{\frac{\delta^2 h+2\delta h^2+\delta^2-2\delta-h+1}{h(\delta-1} \pm \sqrt{(\frac{\delta^2 h+2\delta h^2+\delta^2-2\delta-h+1}{h(\delta-1})^2+4\gamma\delta(\delta h+\delta-1)}}{2}$$

We can see that if we vary the values of δ it will change the classification of the fixed point. We will keep $\gamma = 0.5$ and $h = 0.5$ as a constant and we will vary the values of δ. A good way to observe these five points together is if we use a two parameter bifurcation diagram. This diagram will divide the (δ, h) space into different regions based on the stabilty of our coexistence fixed point.

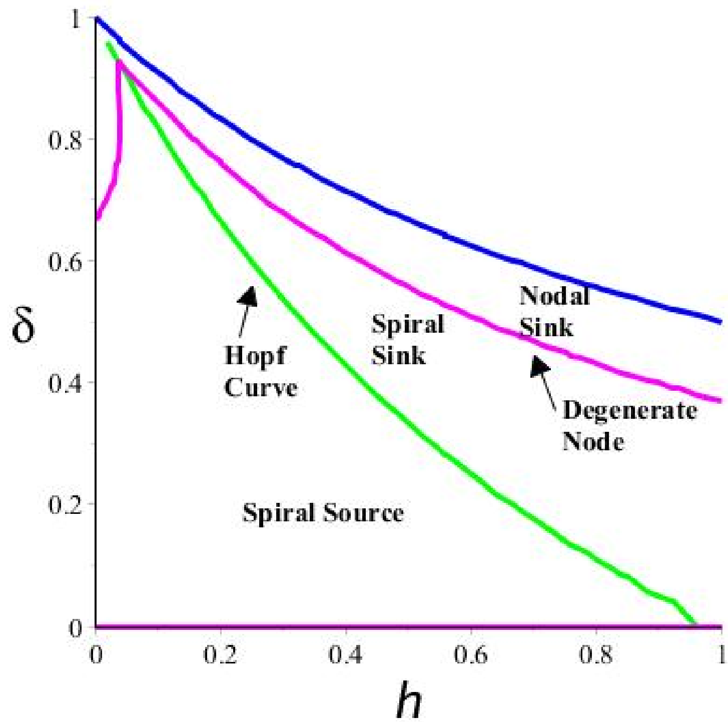


Figure 10: Existance Scenario

We can easily observe the types of changes our phase portraits will undergo as we vary the values of δ while keeping a fixed h value. Observe the next five phase portraits where we met the criteria of $\delta < \dfrac{1}{h+1}$ for our coexistance point to exist.

We have already discussed what it means for the fixed point to be classified as a spiral sink, degenerate node, and a nodal sink. Our last classification we must discuss is a spiral source, which occurs when the eigenvalues are complex with positive real parts.

Fixing $\gamma = 0.5$ and $h = 0.5$ and letting δ vary, we obtain the following phase portraits:

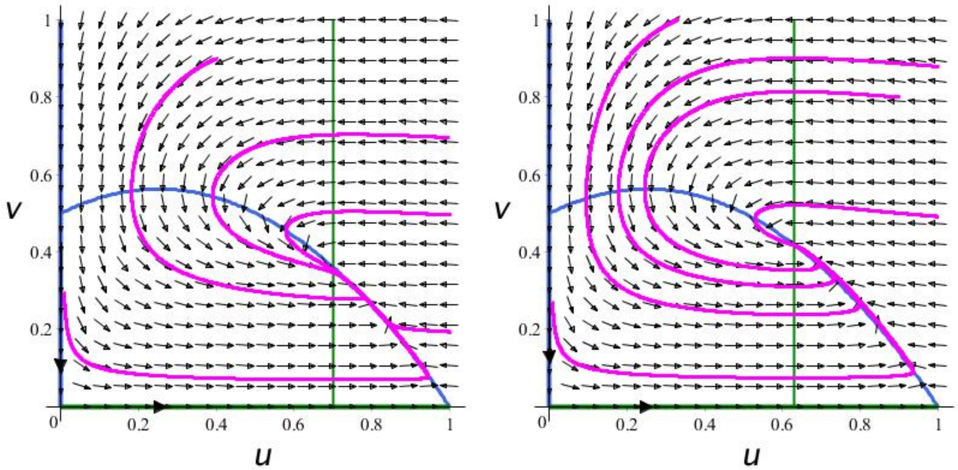


Figure 11: Nodal Sink at $\delta = 0.6$, Degenerate Node at $\delta = 0.5582043783$, for $\gamma = 0.5$, and $h = 0.5$.

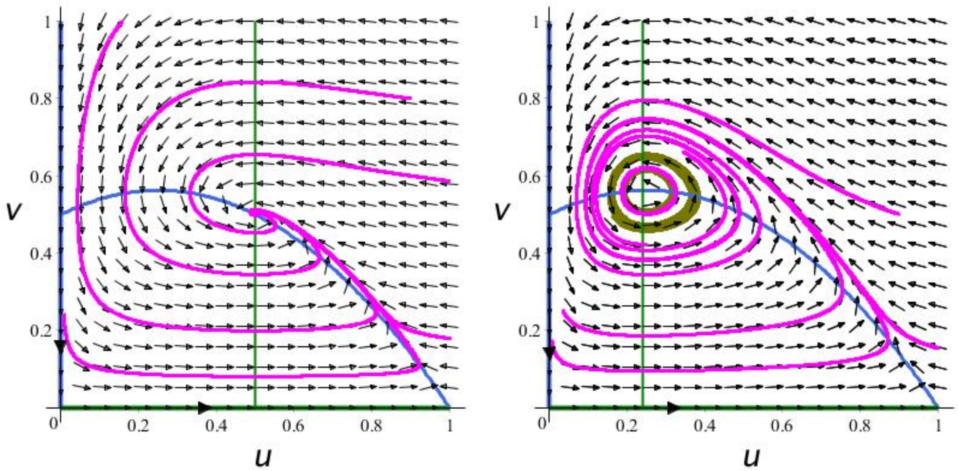


Figure 12: Spiral Sink at $\delta = 0.5$, Hopf Bifurcation at $\delta = 1/3$, A limit cycle is born denoted in olive.

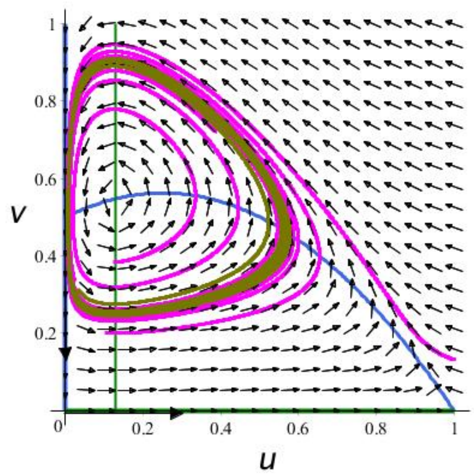


Figure 13: Spiral Source at $\delta = 0.2$.

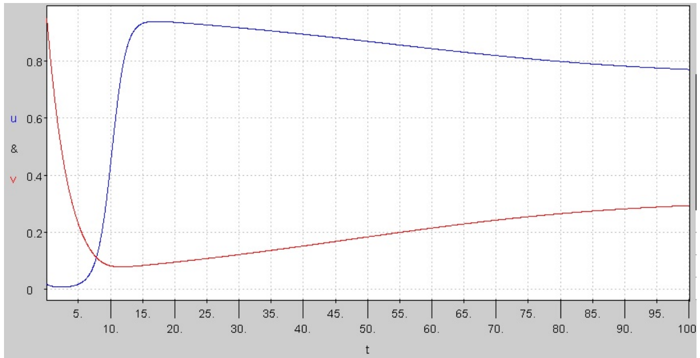


Figure 14: u, v vs t graph using $\gamma = 0.5$, $\delta = 0.6$, and $h = 0.5$.

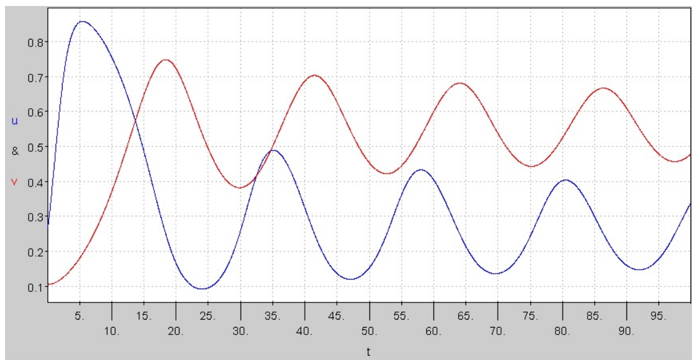


Figure 15: u, v vs t graph using $\gamma = 0.5$, $\delta = \frac{1}{3}$, and $h = 0.5$.

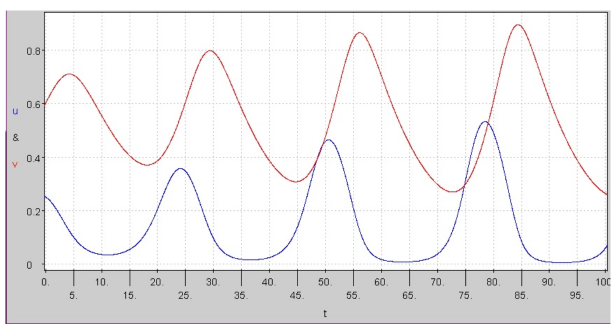


Figure 16: u, v vs t graph using $\gamma = 0.5$, $\delta = 0.2$, and $h = 0.5$.

0.3.4 Type III Functional Response

We will now consider the predator-prey model for the type III functional response:

$$\begin{cases} \frac{du}{d\tau} = aU(1-\frac{U}{K}) - \frac{bU^2V}{A^2+U^2} \\ \frac{dv}{d\tau} = \frac{cU^2V}{A^2+U^2} - dV \end{cases} \tag{19}$$

where,

- **U** represents the population of the prey.
- **V** represents the population of the predator.
- **a**: represents the natural growth rate of the prey in the absence of predators.
- **b**: represents the predation rate.
- **c**: represents the efficiency and propagation rate of the predator in the presence of prey.
- **d**: represents the natural death rate of the predator in the absence of prey.
- **A**: represents the semi-saturation constant, meaning that when the prey density reaches A, the maximal feeding rate of the predator is one-half of its maximum.

We will now introduce the variables u, v, t, δ, γ, and h as follows:

$$\gamma = \frac{c}{a},\ \delta = \frac{d}{c},\ u = \frac{U}{K},\ v = \frac{bV}{aK},\ t = a\tau,\ h = \frac{A}{K} < 1$$

We find the system of equations for our type III functional response by applying the same method that we used earlier in the type I model 1 response.

$$\begin{cases} \frac{du}{dt} = u(1-u) - \frac{u^2v}{h^2+u^2} \\ \frac{dv}{dt} = \gamma v(\frac{u^2}{h^2+u^2} - \delta) \end{cases} \tag{20}$$

We are now working towards plotting the phase portrait of this system. First we must find the fixed points of this system, and they are:

$(0,0)$, $(1,0)$, and $\left(\dfrac{2\sqrt{\delta}h}{\sqrt{1-\delta}}, \dfrac{h(\sqrt{1-\delta}-\sqrt{\delta}h)}{\sqrt{\delta}(1-\delta)}\right)$, note that the third fixed point exists in the positive quadrant if $h < \sqrt{\frac{1-\delta}{\delta}}$

The jacobian matrix for our system of equations is as follows:

$$J_4 = \begin{pmatrix} 1-2u-\frac{2uvh^2}{(h^2+u^2)^2} & -\frac{u^2}{h^2+u^2} \\ \frac{2u\gamma vh^2}{(h^2+u^2)^2} & \frac{\gamma u^2}{h^2+u^2}-\gamma\delta \end{pmatrix}$$

We will now evaluate the jacobian matrix at the first fixed point $(0,0)$:

$$J_4\Big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = -\gamma\delta$. Now that we have the eigenvalues for this system we find that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for λ_2. Since our eigenvalues are opposite in sign our fixed point would be a saddle point for all values of γ and δ. We will now evaluate the jacobian matrix at the fixed point $(1,0)$:

$$J_4\Big|_{(1,0)} = \begin{pmatrix} -1 & \frac{-1}{h^2+1} \\ 0 & \gamma-\frac{\gamma}{h^2+1}-\gamma\delta \end{pmatrix}$$

The eigenvalues for this jacobian matrix are $\lambda_1 = -1$ and $\lambda_2 = \frac{\gamma(\sqrt{1-\delta}-\sqrt{\delta}h)(\sqrt{1-\delta}+\sqrt{\delta}h)}{h^2+1} > 0$ under the assumption that $h < \sqrt{\frac{1-\delta}{\delta}}$. Now that we have the eigenvalues for this system we find that the eigenvectors are of different direction. Since our eigenvalues are opposite in sign the fixed point would be a saddle point for all values of γ and δ. We also note that for our coexistence point to exist our λ_2 must be positive. We will now evaluate the jacobian matrix at our last fixed point $\left(\frac{2\sqrt{\delta}h}{\sqrt{1-\delta}}, \frac{h(\sqrt{1-\delta}-\sqrt{\delta}h)}{\sqrt{\delta}(1-\delta)}\right)$:

$$J_4\Big|_{\left(\frac{2\sqrt{\delta}h}{\sqrt{1-\delta}}, \frac{h(\sqrt{1-\delta}-\sqrt{\delta}h)}{\sqrt{\delta}(1-\delta)}\right)} = \begin{pmatrix} -1+2\delta-2\delta\sqrt{\frac{h^2\delta}{1-\delta}} & -\delta \\ 2\gamma(1-\delta)(1-\sqrt{\frac{h^2\delta}{1-\delta}}) & 0 \end{pmatrix}$$

We have that our eigenvalues for this last jacobian matrix are as follows:

$$\lambda_{1,2} = \frac{-1+2\delta-2\delta\sqrt{\frac{h^2\delta}{1-\delta}} \pm \sqrt{(-1+2\delta-2\delta\sqrt{\frac{h^2\delta}{1-\delta}})^2-8\gamma\delta(1-\delta)(1-\sqrt{\frac{h^2\delta}{1-\delta}})}}{2}$$

We can see that if we vary the values of δ it will change the classification of the fixed point. We will keep $\gamma = 0.5$ and $h = 0.15$ as a constant and we will vary the values of δ. A good way to observe these 4 points together is if we use the two parameter bifurcation diagram.

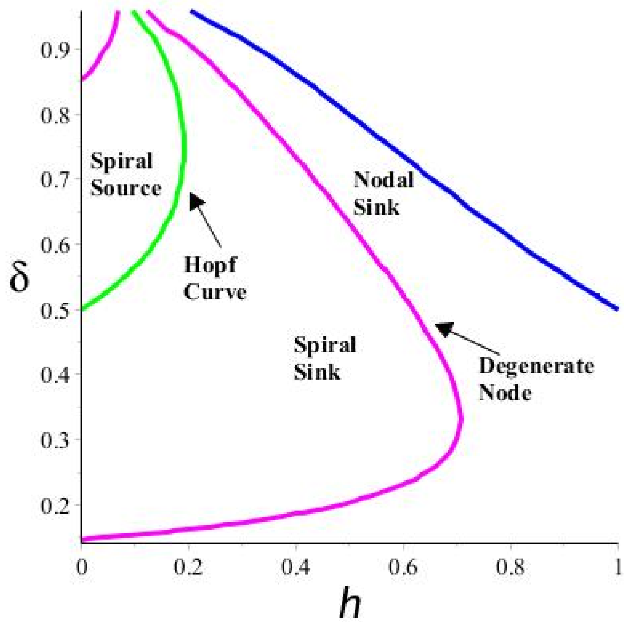


Figure 17: Existance Scenario

We can easily observe the types of changes the phase portrait will undergo as the values of δ are varied. Now observe below the four phase portraits for the type III functional response.

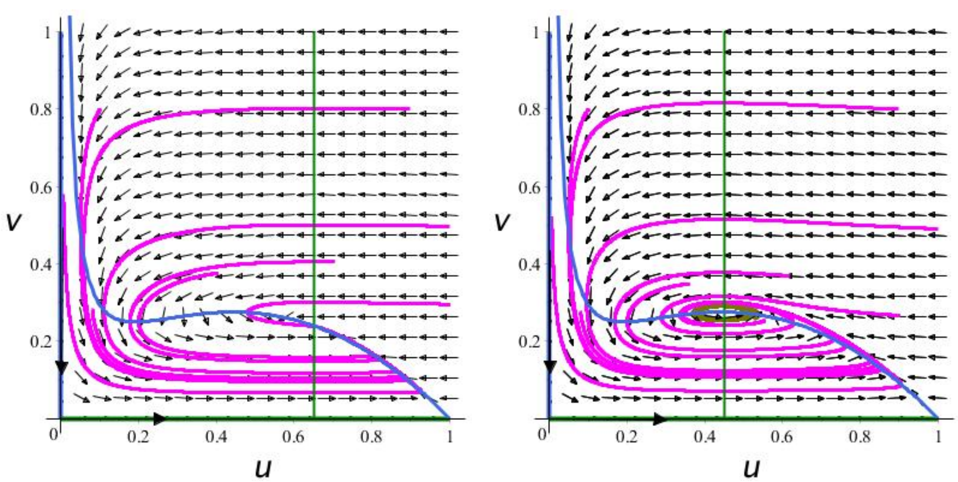


Figure 18: Nodal sink at $\delta = 0.95$, Hopf Bifurcation at $\delta = 0.89698$, where $h = 0.15$.

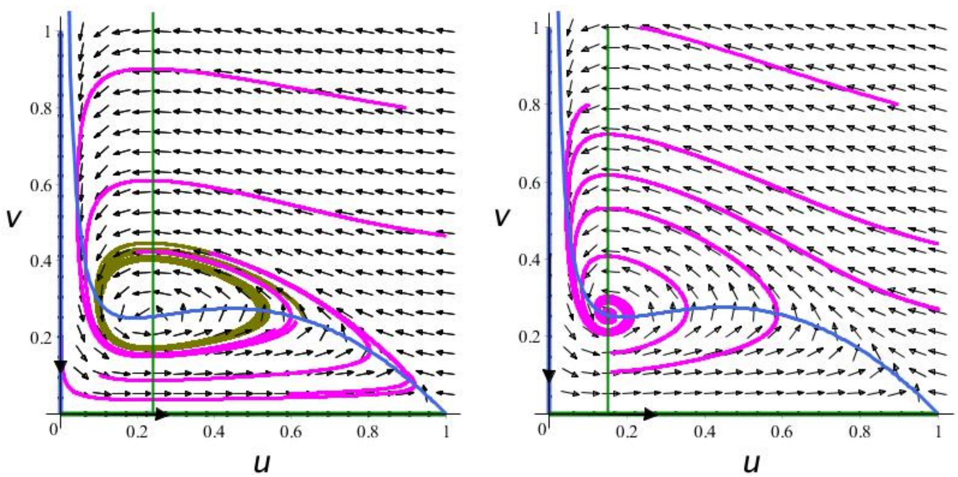


Figure 19: Spiral source at $\delta = 0.7$, Spiral sink at $\delta = 0.5$, where $h = 0.15$. A limit cycle is born, denoted in olive.

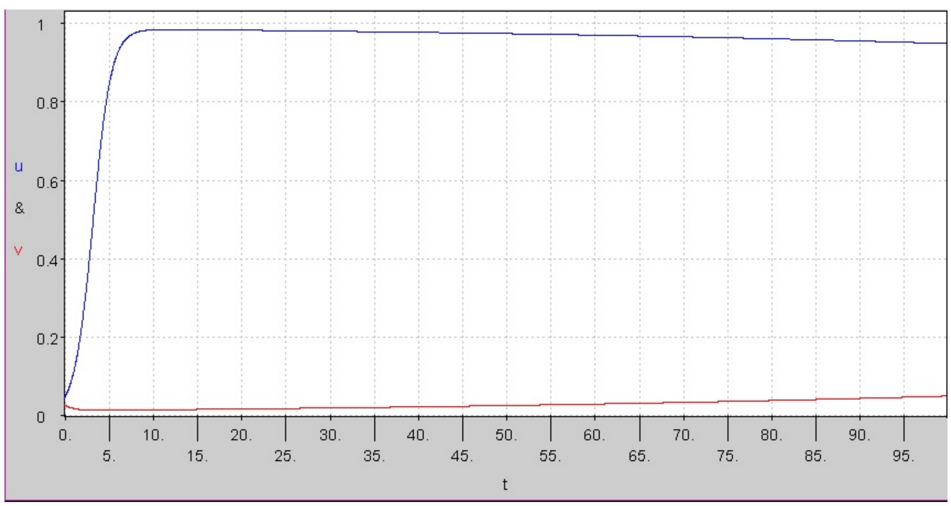


Figure 20: u, v vs t graph using $\gamma = 0.5$, $\delta = 0.95$, and $h = 0.15$.

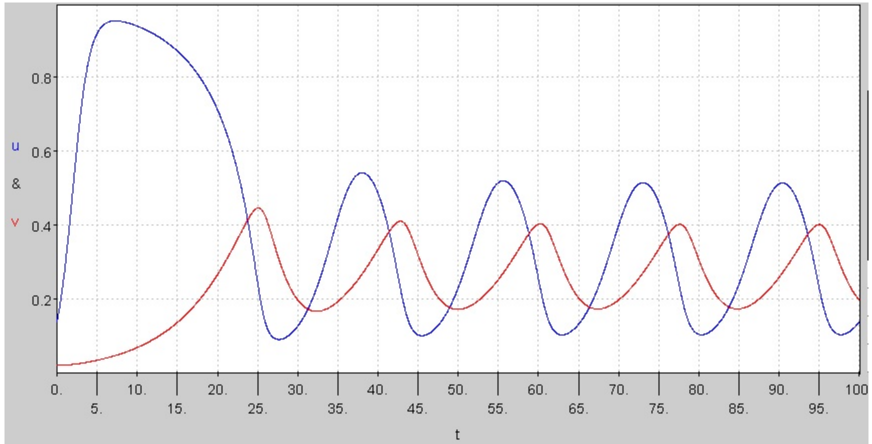


Figure 21: u, v vs t graph using $\gamma = 0.5$, $\delta = 0.7$, and $h = 0.15$.

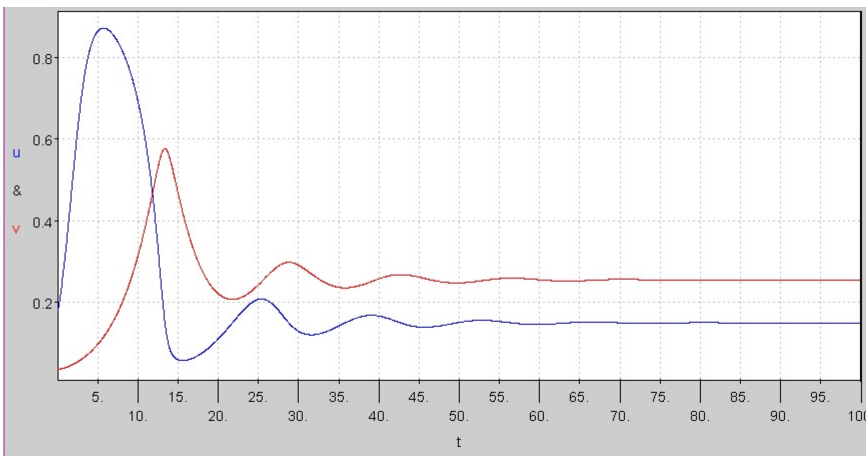


Figure 22: u, v vs t graph using $\gamma = 0.5$, $\delta = 0.5$, and $h = 0.15$.

We will now compare type II and type III functional responses. Notice that as the mortality rate of the predator decreases, in type III, the co-existence equilibrium point loses its stability and the two species can co-exist only in the form of a periodic orbit (see figure 18 (right)). On further decreasing the mortality rate, the co-existence equilibrium point gains back its stability and the two species can coexist in an equilibrium state (see figure 19 (right)). On the other hand, in the type II response, as the mortality rate of the predator decreases, the co-existence equilibrium state loses its stability and the two species can only coexist in the form of a limit cycle(see figure 12 (right). The limit cycle persists at lower values of the predator's mortality rate (see figure 13)

0.4 Conclusion

In this paper, we first considered two different models exhibiting type I functional response. In the first model, where we assume that the prey population grows exponentially in time, the prey and the predator populations co-exist in an oscillatory state. In the second model, where we assume that the prey population saturates to its carrying capacity, the predator and prey populations move towards the co-existence equilibrium state.

Next, we considered models exhibiting type II and type III functional response. In both these cases, we observe a change in the stability of the coexistence equilibrium point take place as the mortality rate of the predator is varied. The coexistence equilibrium state loses its stability through a Hopf bifurcation. Through this bifurcation, limit cycles are born in our phase portraits, which show that the species for some value of δ will live in an oscillatory state.

0.5 References

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