Modern Control systems

Lecture-4 State Space representation of Transfer Function

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Outline

State Space Representations of Transfer function Systems

Many techniques are available for obtaining state space representations of transfer functions.

State space representations in canonical forms

Consider a system defined by,

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u$$

where 'u' is the *input* and 'y' is the *output*. This equation can also be written as,

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

State Space Representations of Transfer function Systems

State space representations in canonical forms

- The process of converting Transfer Function to State-Space form is **NOT** unique.
- Various realizations are possible which are equivalent.(i.e, their properties do not change)
- However, one representation may have advantages over others for a particular task.
- In what follows, we shall present the state space representations of the systems defined by different canonical forms.

Outline

Transfer Function to State Space

Canonical Form I

- We have, $\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$
- Let $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} \times \frac{X(s)}{U(s)}$
- Thus,

$$\frac{Y(s)}{X(s)} = b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

-

$$\therefore u(t) = \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}\frac{dx}{dt} + a_n x$$

$$\frac{d^n x}{dt^n} = u(t) - a_1 \frac{d^{n-1}x}{dt^{n-1}} - \cdots - a_{n-1}\frac{dx}{dt} - a_n x$$

Transfer function to State Space

Canonical Form I

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$; $x_3 = \frac{d^2x}{dt^2} \cdots x_n = \frac{d^{n-1}x}{dt^{n-1}}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

- Case-1: If $m = n - 1 [\therefore \frac{d^m x}{dt^m} = \frac{d^{n-1}x}{dt^{n-1}} = x_n]$

  We know that,

$$\begin{aligned} Y(s) &= X(s)(b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m) \\ \therefore y(t) &= b_0 \frac{d^{n-1}x}{dt^{n-1}} + b_1 \frac{d^{n-2}x}{dt^{n-2}} + \cdots + b_{n-1}\frac{dx}{dt} + b_n x \\ y(t) &= b_0 x_n + b_1 x_{n-1} + \cdots + b_n x_1 \end{aligned}$$

Transfer Function to State Space

Canonical Form I

$$y = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [0]u$$

- <u>Case-2</u>: If $m = n$,

$$\begin{aligned}
\frac{d^m x}{dt^m} &= u(t) - a_1 \frac{d^{n-1}x}{dt^{n-1}} - \cdots - a_{n-1}\frac{dx}{dt} - a_n x \\
y(t) &= b_0(u(t) - a_1 \frac{d^{n-1}x}{dt^{n-1}} - \cdots - a_n x) + b_1 \frac{d^{m-1}x}{dt^{m-1}} + \cdots + b_m x \\
y(t) &= (b_1 - a_1 b_0)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + (b_{n-1}a_{n-1}b_0)\frac{dx}{dt} + (b_n - a_n b_0) + b_0 u
\end{aligned}$$

Transfer Function to State Space

Canonical Form I

$$\therefore y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [b_0]u$$

Transfer function to State Space

Example-1

Consider a transfer function, $G(s) = \frac{5s^2+7s+9}{s^3+8s^2+6s+2}$

Let $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} \times \frac{X(s)}{U(s)} = \frac{5s^2+7s+9}{s^3+8s^2+6s+2}$
Thus,

$$\frac{Y(s)}{X(s)} = 5s^2 + 7s + 9 \quad (1)$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^3 + 8s^2 + 6s + 2} \quad (2)$$

From eq(2), $U(s) = X(s)[s^3 + 8s^2 + 6s + 2]$

$$u(t) = \frac{d^3x}{dt^3} + 8\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x$$

$$\frac{d^3x}{dt^3} = u(t) - 8\frac{d^2x}{dt^2} - 6\frac{dx}{dt} - 2x$$

Transfer function to State Space

Examples on Canonical Form-I: Example-1

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$; $x_3 = \frac{d^2x}{dt^2}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

From eq(1), $\quad Y(s) = X(s)[5s^2 + 7s + 9]$

$$\begin{aligned} y(t) &= 5\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 9x \\ y(t) &= 5x_3 + 7x_2 + 9x_1 \\ y &= \begin{bmatrix} 9 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \end{aligned}$$

Transfer function to State Space

Example-2

Consider a transfer function, $G(s) = \frac{5s^2+7s+9}{s^2+2s+15}$
Let

$$\frac{X(s)}{U(s)} = \frac{1}{s^2+2s+15}$$
$$\frac{d^2x}{dt^2} = -15x - 2\frac{dx}{dt} + u(t)$$

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -15 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Transfer function to State Space

Example-2

$$
\begin{aligned}
\frac{Y(s)}{X(s)} &= 5s^2 + 7s + 9 \\
y(t) &= 5\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 9x \\
&= 5(-15x - 2\frac{dx}{dt} + u(t)) + 7\frac{dx}{dt} + 9x \\
&= -66x - 3\frac{dx}{dt} + 5u(t) \\
y &= \begin{bmatrix} -66 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [5]u
\end{aligned}
$$

Outline

State Space Representations of Transfer function Systems

Canonical Form II

- We have, $\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$

  $\therefore Y(s)[s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n] = U(s)[b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m]$

  Case-1:If $m = n - 1$

- Replacing m with $n-1$ and dividing the equation with s^n and rearranging it, we get,

$$Y(s) = \frac{1}{s}[b_0 U(s) - a_1 Y(s) + \frac{1}{s}[b_1 U(s) - a_2 Y(s) + \frac{1}{s}[\cdots + \frac{1}{s}[b_{n-1}U(s) - a_n Y(s)] \cdots]$$

State Space Representations of Transfer function Systems

Canonical Form II

- Now, let us define the states and output as follows,

$$
\begin{aligned}
X_n(s) &= \frac{1}{s}[b_0U(s) - a_1Y(s) + X_{n-1}(s)] = Y(s) \\
X_{n-1}(s) &= \frac{1}{s}[b_1U(s) - a_2Y(s) + X_{n-2}(s)] \\
X_{n-2}(s) &= \frac{1}{s}[b_2U(s) - a_3Y(s) + X_{n-3}(s)] \\
&\vdots \\
X_1(s) &= \frac{1}{s}[b_{n-1}U(s) - a_nY(s)]
\end{aligned}
$$

State Space Representations of Transfer function Systems

Canonical Form II

- Since $X_n(s) = Y(s)$, replacing $Y(s)$ by $X_n(s)$ and rewriting the state equations, we get

$$\begin{aligned}
X_n(s) &= \frac{1}{s}[b_0U(s) - a_1X_n(s) + X_{n-1}(s)] \\
X_{n-1}(s) &= \frac{1}{s}[b_1U(s) - a_2X_n(s) + X_{n-2}(s)] \\
X_{n-2}(s) &= \frac{1}{s}[b_2U(s) - a_3X_n(s) + X_{n-3}(s)] \\
&\vdots \\
X_1(s) &= \frac{1}{s}[b_{n-1}U(s) - a_nX_n(s)]
\end{aligned}$$

State Space Representations of Transfer function Systems

Canonical Form II

- Applying Inverse Laplace Transformation on both sides, we get

$$
\begin{aligned}
\dot{x}_n &= b_0 u - a_1 x_n + x_{n-1} \\
\dot{x}_{n-1} &= b_1 u - a_2 x_n + x_{n-2} \\
\dot{x}_{n-2} &= b_2 u - a_3 x_n + x_{n-3} \\
&\vdots \\
\dot{x}_2 &= b_{n-2} u - a_n x_n + x_1 \\
\dot{x}_1 &= b_{n-1} u - a_n x_n
\end{aligned}
$$

- The output equation becomes, $y = x_n$

State Space Representations of Transfer function Systems

Canonical Form II

- Summing up all the results into one vector matrix differential equation, we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [0]u$$

State Space Representations of Transfer function Systems

Example of Canonical Form II-Case 1

- Consider a transfer function, $\frac{Y(s)}{U(s)} = G(s) = \frac{5s^2+7s+9}{s^3+8s^2+6s+2}$
- We can see that the order of the numerator is less than that of the denominator and differs by 1. Hence, this falls under Case-1.
- On cross multiplication, we get
  $$Y(s)[s^3 + 8s^2 + 6s + 2] = U(s)[5s^2 + 7s + 9]$$
- dividing the equation with s^3 and rearranging it, we get,

$$Y(s) = \frac{1}{s}(5U(s) - 8Y(s) + \frac{1}{s}(7U(s) - 6Y(s) + \frac{1}{s}(2U(s) - 9Y(s))))$$

State Space Representations of Transfer function Systems

Example of Canonical Form II-Case 1

- $\therefore$ The state equations and output equation can be defined as

$$\begin{aligned}\dot{x}_3 &= 5u - 8x_3 + x_2 = y \\ \dot{x}_2 &= 7u - 6x_3 + x_1 \\ \dot{x}_1 &= 9u - 2x_3\end{aligned}$$

- Summing up all the results and formulating them in the matrix form, we get,

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & -2 \\ 1 & 0 & -6 \\ 0 & 1 & -8\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} + \begin{bmatrix}9 \\ 7 \\ 5\end{bmatrix} u$$

$$y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} + [0]u$$

State Space Representations of Transfer function Systems

Canonical Form II

Case-2:If $m = n$

$$Y(s)[s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n] = U(s)[b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n]$$

- Dividing the equation with s^n and rearranging it, we get,

$$\begin{aligned} Y(s) &= b_0U(s) - a_0Y(s) + \tfrac{1}{s}[b_1U(s) - a_1Y(s) + \tfrac{1}{s}[\cdots + \tfrac{1}{s}[b_{n-1}U(s) - \\ &\quad a_{n-1}Y(s) + \tfrac{1}{s}[b_nU(s) - a_nY(s)]]]\cdots] \end{aligned}$$

State Space Representations of Transfer function Systems

Canonical Form II

- Now, let us define the states and output as follows,

$$
\begin{aligned}
Y(s) &= b_0U(s) + X_n(s) \\
X_n(s) &= \frac{1}{s}[b_1U(s) - a_1Y(s) + X_{n-1}(s)] \\
X_{n-1}(s) &= \frac{1}{s}[b_2U(s) - a_2Y(s) + X_{n-2}(s)] \\
&\vdots \\
X_1(s) &= \frac{1}{s}[b_nU(s) - a_nY(s)]
\end{aligned}
$$

State Space Representations of Transfer function Systems

Canonical Form II

- Substituting $b_0U(s) + X_n(s)$ in the place of $Y(s)$ in state equations, we get,

$$
\begin{aligned}
X_n(s) &= \frac{1}{s}[(b_1 - a_1b_0)U(s) - a_1X_n(s) + X_{n-1}(s)] \\
X_{n-1}(s) &= \frac{1}{s}[(b_2 - a_2b_0)U(s) - a_2X_n(s) + X_{n-2}(s)] \\
&\vdots \\
X_2(s) &= \frac{1}{s}[(b_{n-1} - a_{n-1}b_0)U(s) - a_{n-1}X_{n-1}(s) + X_1(s)] \\
X_1(s) &= \frac{1}{s}[(b_n - a_nb_0)U(s) - a_nX_n(s)]
\end{aligned}
$$

State Space Representations of Transfer function Systems

Canonical Form II

- Applying Inverse Laplace Transformation on both sides, we get

$$
\begin{aligned}
\dot{x}_n &= (b_1 - a_1 b_0)u - a_1 x_n + x_{n-1} \\
\dot{x}_{n-1} &= (b_2 - a_2 b_0)u - a_2 x_n + x_{n-2} \\
\dot{x}_{n-2} &= (b_3 - a_3 b_0)u - a_3 x_n + x_{n-3} \\
&\vdots \\
\dot{x}_2 &= (b_{n-1} - a_{n-1} b_0)u - a_{n-1} x_1 \\
\dot{x}_1 &= (b_n - a_n b_0)u - a_n x_n
\end{aligned}
$$

- The output equation becomes, $y = b_0 u + x_n$

State Space Representations of Transfer function Systems

Canonical Form II

- Summing up all the results into one vector matrix differential equation, we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [b_0] u$$

State Space Representations of Transfer function Systems

Example of Canonical Form II-Case 2

- Consider a transfer function, $\frac{Y(s)}{U(s)} = G(s) = \frac{5s^2+7s+9}{s^2+2s+15}$
- We can see that the order of the numerator is equal to that of the denominator. Hence, this falls under Case-2.
- On cross multiplication, we get
  $$Y(s)[s^2 + 2s + 15] = U(s)[5s^2 + 7s + 9]$$
- dividing the equation with s^2 and rearranging it, we get,

$$Y(s) = 5U(s) + \frac{1}{s}(7U(s) - 2Y(s) + \frac{1}{s}(9U(s) - 15Y(s)))$$

State Space Representations of Transfer function Systems

Example of Canonical Form II-Case 2

- $\therefore$ The state equations and output equation can be defined as

$$\begin{aligned} y &= 5u + x_2 \\ \dot{x}_2 &= 7u - 2y + x_1 \\ \dot{x}_1 &= 9u - 15y \end{aligned}$$

- Substituting $y = 5u + x_2$ in state equations and rearranging them we get,

$$\begin{aligned} \dot{x}_2 &= -3u - 2x_2 + x_1 \\ \dot{x}_1 &= -66u - 15x_2 \end{aligned}$$

- Summing up all the results and formulating them in the matrix form, we get,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -15 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -66 \\ -3 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [5]u \end{aligned}$$

Outline

State Space Representations of Transfer function Systems

Diagonal Canonical Form

Consider the transfer function defined by equation

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Assumption: Denominator polynomial involves only distinct roots.

Therefore, the transfer function can be written as ,

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{(s+p_1)(s+p_2)\cdots(s+p_n)} \\ &= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n} \end{aligned}$$

if $m = n, b_0 =$ constant,
if $m < n, b_0 = 0$

State Space Representations of Transfer function Systems

Continuation of Diagonal Canonical Form...

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & \cdots & 0 \\ 0 & -p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

State Space Representations of Transfer function Systems

Example for Diagonal Canonical Form

- Consider, the system defined by transfer function, $\frac{s+1}{s^2+5s+1}$.
  - Step-1: Rewrite the transfer function as, $\frac{s+1}{(s+3)(s+2)}$
  - Step-2: Split the transfer function into Partial fractions

$$\frac{2}{s+3} + \frac{(-1)}{s+2}$$

Here, $m < n$, $\therefore b_0 = 0$

  - Step-3: Compare the obtained partial fraction form with

$b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$, we get...

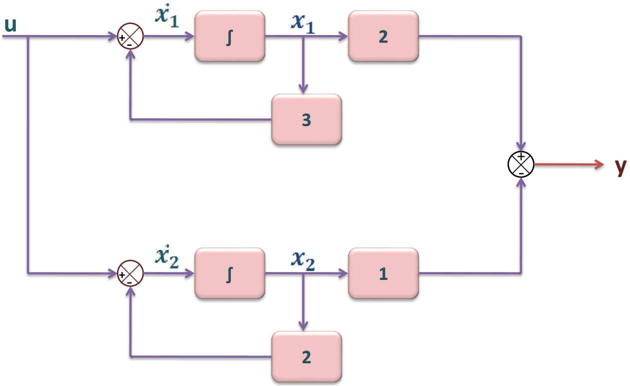
State Space Representations of Transfer function Systems

Example for Diagonal Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

State Space Representations of Transfer function Systems



State Space Representations of Transfer function Systems

Violation of Assumption

- Previously in Diagonal form, the denominator polynomial involves only distinct roots.
- What if the denominator polynomial involves multiple roots ?

Outline

State Space Representations of Transfer function Systems

Violation of Assumption

- Previously in Diagonal form, the denominator polynomial involves only distinct roots.
- What if the denominator polynomial involves multiple roots ?

Jordan Canonical Form

- To deal with multiple roots, Diagonal Canonical Form must be modified to Jordan Canonical form.
- Consider, the equation

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- Let, $p_1 = p_2 = p_3$.

State Space Representations of Transfer function Systems

Continuation of Jordan Canonical Form...

Then the factored form of $\frac{Y(s)}{U(s)}$ becomes

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{(s+p_1)^3(s+p_4)\cdots(s+p_n)}$$

The partial fraction expansion of this last equation becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} \cdots + \frac{c_n}{s+p_n}$$

State Space Representations of Transfer function Systems

Jordan Canonical Form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \vdots\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix} -p_1 & 0 & 0 & \vdots & \cdots & 0\\ 0 & -p_1 & 0 & \vdots & \cdots & 0\\ 0 & 0 & -p_1 & \vdots & \cdots & 0\\ \cdots & \cdots & \cdots & \vdots & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & & \cdots & -p_n \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\\ \vdots\\ 1\end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix}$$

State Space Representations of Transfer function Systems

Example for Jordan Canonical Form

Consider, the system defined by transfer function, $\frac{s+5}{(s^2+2s+1)(s+2)}$.

Step-1: Rewrite the transfer function as, $\frac{s+1}{(s+3)(s+2)}$

Step-2: Split the transfer function into Partial fractions

$$\frac{4}{(s+3)^2} + \frac{(-3)}{s+3} + \frac{3}{s+2}$$

$$\text{Here, } m < n, \therefore b_0 = 0$$

Step-3: Compare the obtained partial fraction form with
$b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} + \cdots + \frac{c_n}{s+p_n}$, we get...

State Space Representations of Transfer function Systems

Example for Jordan Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & \vdots & 0 \\ 0 & -1 & \vdots & 0 \\ \cdots & \cdots & \vdots & \\ 0 & 0 & & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

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