MODERN PHYSICS

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Chapter 2

THE SPECIAL THEORY OF RELATIVITY

This 12-foot tall statue of Albert Einstein is located at the headquarters of the National Academy of Sciences in Washington DC. The page in his hand shows three equations that he discovered: the fundamental equation of general relativity, which revolutionized our understanding of gravity; the equation for the photoelectric effect, which opened the path to the development of quantum mechanics; and the equation for mass-energy equivalence, which is the cornerstone of his special theory of relativity.

Einstein's special theory of relativity and Planck's quantum theory burst forth on the physics scene almost simultaneously during the first decade of the 20th century. Both theories caused profound changes in the way we view our universe at its most fundamental level.

In this chapter we study the special theory of relativity.* This theory has a completely undeserved reputation as being so exotic that few people can understand it. On the contrary, special relativity is basically a system of kinematics and dynamics, based on a set of postulates that are different from those of classical physics. The resulting formalism is not much more complicated than Newton's laws, but it does lead to several predictions that seem to go against our common sense. Even so, the special theory of relativity has been carefully and thoroughly tested by experiment and found to be correct in all its predictions.

We first review the classical relativity of Galileo and Newton, and then we show why Einstein proposed to replace it. We then discuss the mathematical aspects of special relativity, the predictions of the theory, and finally the experimental tests.

2.1 CLASSICAL RELATIVITY

A "theory of relativity" is in effect a way for observers in different frames of reference to compare the results of their observations. For example, consider an observer in a car parked by a highway near a large rock. To this observer, the rock is at rest. Another observer, who is moving along the highway in a car, sees the rock rush past as the car drives by. To this observer, the rock appears to be moving. A theory of relativity provides the conceptual framework and mathematical tools that enable the two observers to transform a statement such as "rock is at rest" in one frame of reference to the statement "rock is in motion" in another frame of reference. More generally, relativity gives a means for expressing the laws of physics in different frames of reference.

The mathematical basis for comparing the two descriptions is called a *transformation*. Figure 2.1 shows an abstract representation of the situation. Two observers

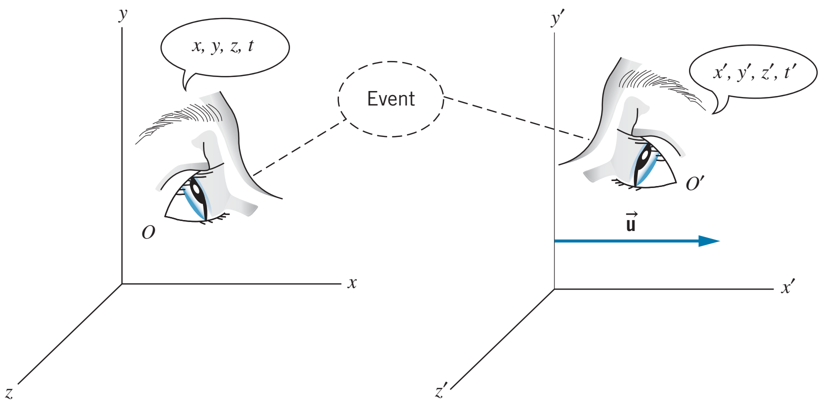


FIGURE 2.1 Two observers O and O' observe the same event. O' moves relative to O with a constant velocity $\vec{\mathbf{u}}$.

*The *general* theory of relativity, which is covered briefly in Chapter 15, deals with "curved" coordinate systems, in which gravity is responsible for the curvature. Here we discuss the *special* case of the more familiar "flat" coordinate systems.

O and O' are each at rest in their own frames of reference but move relative to one another with constant velocity $\vec{\mathbf{u}}$. (O and O' refer both to the observers and their reference frames or coordinate systems.) They observe the same *event,* which happens at a particular point in space and a particular time, such as a collision between two particles. According to O, the space and time coordinates of the event are $x, y, z, t,$ while according to O' the coordinates of the *same event* are x', y', z', t'. The two observers use calibrated meter sticks and synchronized clocks, so any differences between the coordinates of the two events are due to their different frames of reference and not to the measuring process. We simplify the discussion by assuming that the relative velocity $\vec{\mathbf{u}}$ always lies along the common xx' direction, as shown in Figure 2.1, and we let $\vec{\mathbf{u}}$ represent the velocity of O' as measured by O (and thus O' would measure velocity $-\vec{\mathbf{u}}$ for O).

In this discussion we make a particular choice of the kind of reference frames inhabited by O and O'. We assume that each observer has the capacity to test Newton's laws and finds them to hold in that frame of reference. For example, each observer finds that an object at rest or moving with a constant velocity remains in that state unless acted upon by an external force (Newton's first law, the law of inertia). Such frames of reference are called *inertial frames*. An observer in interstellar space floating in a nonrotating rocket with the engines off would be in an inertial frame of reference. An observer at rest on the surface of the Earth is *not* in an inertial frame, because the Earth is rotating about its axis and orbiting about the Sun; however, the accelerations associated with those motions are so small that we can usually regard our reference frame as approximately inertial. (The noninertial reference frame at the Earth's surface does produce important and often spectacular effects, such as the circulation of air around centers of high or low pressure.) An observer in an accelerating car, a rotating merry-go-round, or a descending roller coaster is *not* in an inertial frame of reference!

We now derive the classical or *Galilean* transformation that relates the coordinates x, y, z, t to x', y', z', t'. We assume as a postulate of classical physics that $t = t'$, that is, time is the same for all observers. We also assume for simplicity that the coordinate systems are chosen so that their origins coincide at $t = 0$. Consider an object in O' at the coordinates x', y', z' (Figure 2.2). According to O, the y and z coordinates are the same as those in O'. Along the x direction, O would observe the object at $x = x' + ut$. We therefore have the *Galilean coordinate transformation*

$$x' = x - ut \qquad y' = y \qquad z' = z \tag{2.1}$$

To find the velocities of the object as observed by O and O', we take the derivatives of these expressions with respect to t' on the left and with respect to t on the right (which we can do because we have assumed $t' = t$). This gives the *Galilean velocity transformation*

$$v'_x = v_x - u \qquad v'_y = v_y \qquad v'_z = v_z \tag{2.2}$$

In a similar fashion, we can take the derivatives of Eq. 2.2 with respect to time and obtain relationships between the accelerations

$$a'_x = a_x \qquad a'_y = a_y \qquad a'_z = a_z \tag{2.3}$$

Equation 2.3 shows again that Newton's laws hold for both observers. As long as u is constant ($du/dt = 0$), the observers measure identical accelerations and agree on the results of applying $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.

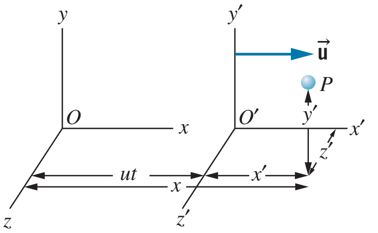


FIGURE 2.2 An object or event at point P is at coordinates x', y', z' with respect to O'. The x coordinate measured by O is $x = x' + ut$. The y and z coordinates in O are the same as those in O'.

Example 2.1

Two cars are traveling at constant speed along a road in the same direction. Car A moves at 60 km/h and car B moves at 40 km/h, each measured relative to an observer on the ground (Figure 2.3*a*). What is the speed of car A relative to car B?

Solution

Let O be the observer on the ground, who observes car A to move at $v_x = 60$ km/h. Assume O' to be moving with car B at $u = 40$ km/h. Then

$$\begin{aligned} v'_x = v_x - u &= 60\,\text{km/h} - 40\,\text{km/h} \\ &= 20\,\text{km/h} \end{aligned}$$

Figure 2.3*b* shows the situation as observed by O'.

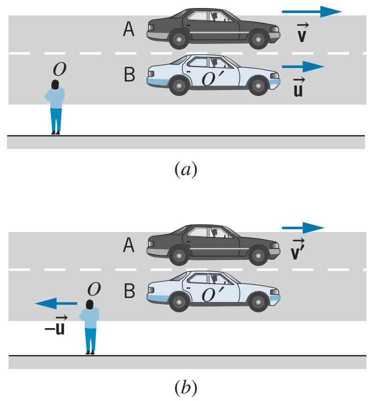


FIGURE 2.3 Example 2.1. (*a*) As observed by O at rest on the ground. (*b*) As observed by O' in car B.

Example 2.2

An airplane is flying due east relative to still air at a speed of 320 km/h. There is a 65 km/h wind blowing toward the north, as measured by an observer on the ground. What is the velocity of the plane measured by the ground observer?

Solution

Let O be the observer on the ground, and let O' be an observer who is moving with the wind, for example a balloonist (Figure 2.4). Then $u = 65$ km/h, and (because our equations are set up with $\vec{\mathbf{u}}$ in the xx' direction) we must choose the xx' direction to be to the north. In this case we know the velocity with respect to O'; taking the y direction to the east, we have $v'_x = 0$ and $v'_y = 320$ km/h. Using Eq. 2.2 we obtain

$$v_x = v'_x + u = 0 + 65\,\text{km/h} = 65\,\text{km/h}$$
$$v_y = v'_y = 320\,\text{km/h}$$

Relative to the ground, the plane flies in a direction determined by $\phi = \tan^{-1}(65\,\text{km/h})/(320\,\text{km/h}) = 11.5^\circ$, or 11.5° north of east.

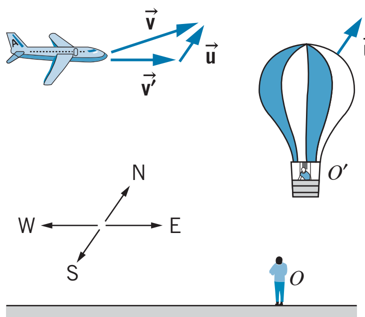


FIGURE 2.4 Example 2.2. As observed by O at rest on the ground, the balloon drifts north with the wind, while the plane flies north of east.

Example 2.3

A swimmer capable of swimming at a speed c in still water is swimming in a stream in which the current is u (which we assume to be less than c). Suppose the swimmer swims upstream a distance L and then returns downstream to the starting point. Find the time necessary to make the round trip, and compare it with the time to swim across the stream a distance L and return.

Solution

Let the frame of reference of O be the ground and the frame of reference of O' be the water, moving at speed u (Figure 2.5*a*). The swimmer always moves at speed c relative to the water, and thus $v'_x = -c$ for the upstream swim. (Remember that u always defines the *positive x* direction.) According to Eq. 2.2, $v'_x = v_x - u$, so $v_x = v'_x + u = u - c$. (As expected, the velocity relative to the ground has magnitude smaller than c; it is also *negative,* since the swimmer is swimming in the negative x direction, so $|v_x| = c - u$.) Therefore, $t_{\text{up}} = L/(c - u)$. For the downstream swim, $v'_x = c$, so $v_x = u + c$, $t_{\text{down}} = L/(c + u)$, and the total time is

$$t = \frac{L}{c+u} + \frac{L}{c-u} = \frac{L(c-u) + L(c+u)}{c^2 - u^2}$$

$$= \frac{2Lc}{c^2 - u^2} = \frac{2L}{c}\frac{1}{1 - u^2/c^2} \qquad (2.4)$$

To swim directly across the stream, the swimmer's efforts must be directed somewhat upstream to counter the effect of the current (Figure 2.5*b*). That is, in the frame of reference of O we would like to have $v_x = 0$, which requires $v'_x = -u$ according to Eq. 2.2. Since the speed relative to the water is always c, $\sqrt{v'^2_x + v'^2_y} = c$; thus $v'_y = \sqrt{c^2 - v'^2_x}$ $= \sqrt{c^2 - u^2}$, and the round-trip time is

$$t = 2t_{\text{across}} = \frac{2L}{\sqrt{c^2 - u^2}} = \frac{2L}{c}\frac{1}{\sqrt{1 - u^2/c^2}} \qquad (2.5)$$

Notice the difference *in form* between this result and the result for the upstream-downstream swim, Eq. 2.4.

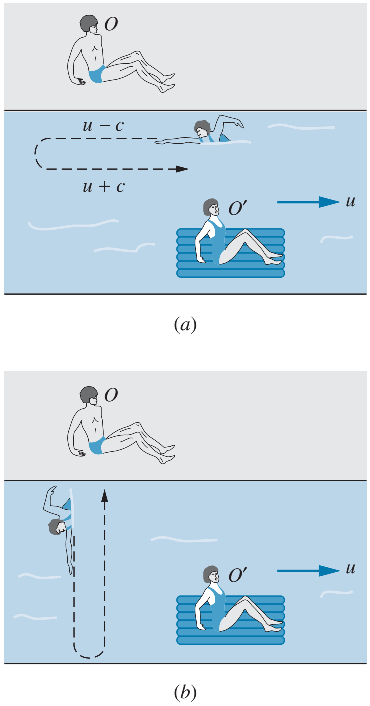
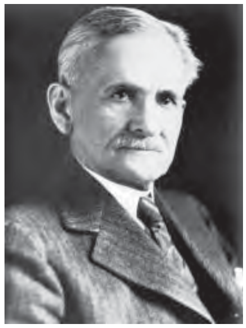


FIGURE 2.5 Example 2.3. The motion of a swimmer as seen by observer O at rest on the bank of the stream. Observer O' moves with the stream at speed u.

2.2 THE MICHELSON-MORLEY EXPERIMENT

We have seen how Newton's laws remain valid with respect to a Galilean transformation that relates the description of the motion of an object in one reference frame to that in another reference frame. It is then interesting to ask whether the same transformation rules apply to the motion of a light beam. According to the Galilean transformation, a light beam moving relative to observer O' in the x' direction at speed $c = 299{,}792{,}458$ m/s would have a speed of $c + u$ relative to O. Direct high-precision measurements of the speed of light beams have become possible in recent years (as we discuss later in this chapter), but in the 19th century it was necessary to devise a more indirect measurement of the speed of light according to different observers in relative motion.

Suppose the swimmer in Example 2.3 is replaced by a light beam. Observer O' is in a frame of reference in which the speed of light is c, and the frame of reference of observer O' is in motion relative to observer O. What is the speed of light as measured by observer O? If the Galilean transformation is correct, we should expect to see a difference between the speed of the light beam according to O and O' and therefore a time difference between the upstream-downstream and cross-stream times, as in Example 2.3.



Albert A. Michelson (1852–1931, United States). He spent 50 years doing increasingly precise experiments with light, for which he became the first U.S. citizen to win the Nobel Prize in physics (1907).

Physicists in the 19th century postulated just such a situation—a preferred frame of reference in which the speed of light has the precise value of c and other frames in relative motion in which the speed of light would differ, according to the Galilean transformation. The preferred frame, like that of observer O' in Example 2.3, is one that is at rest with respect to the medium in which light propagates at c (like the water of that example). What is the medium of propagation for light waves? It was inconceivable to physicists of the 19th century that a wave disturbance could propagate without a medium (consider mechanical waves such as sound or seismic waves, for example, which propagate due to mechanical forces in the medium). They postulated the existence of an invisible, massless medium, called the *ether,* which filled all space, was undetectable by any mechanical means, and existed solely for the propagation of light waves. It seemed reasonable then to obtain evidence for the ether by measuring the velocity of the Earth moving through the ether. This could be done in the geometry of Figure 2.5 by measuring the difference between the upstream-downstream and cross-stream times for a light wave. The calculation based on Galilean relativity would then give the relative velocity $\vec{\mathbf{u}}$ between O (in the Earth's frame of reference) and the ether.

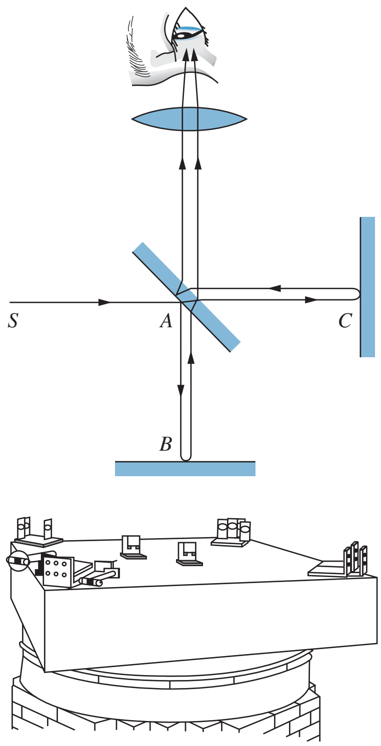


FIGURE 2.6 (Top) Beam diagram of Michelson interferometer. Light from source S is split at A by the half-silvered mirror; one part is reflected by the mirror at B and the other is reflected at C. The beams are then recombined for observation of the interference. (Bottom) Michelson's apparatus. To improve sensitivity, the beams were reflected to travel each leg of the apparatus eight times, rather than just twice. To reduce vibrations from the surroundings, the interferometer was mounted on a 1.5-m square stone slab floating in a pool of mercury.

The first detailed and precise search for the preferred frame was performed in 1887 by the American physicist Albert A. Michelson and his associate E. W. Morley. Their apparatus consisted of a specially designed Michelson interferometer, illustrated in Figure 2.6. A monochromatic beam of light is split in two; the two beams travel different paths and are then recombined. Any phase difference between the combining beams causes bright and dark bands or "fringes" to appear, corresponding, respectively, to constructive and destructive interference, as shown in Figure 2.7.

There are two contributions to the phase difference between the beams. The first contribution comes from the path difference $AB - AC$; one of the beams may travel a longer distance. The second contribution, which would still be present even if the path lengths were equal, comes from the time difference between the upstream-downstream and cross-stream paths (as in Example 2.3) and indicates the motion of the Earth through the ether. Michelson and Morley used a clever method to isolate this second contribution—they rotated the entire apparatus by 90°! The rotation doesn't change the first contribution to the phase difference (because the lengths AB and AC don't change), but the second contribution changes sign, because what was an upstream-downstream path before the rotation becomes a cross-stream path after the rotation. As the apparatus is rotated through 90°, the fringes should change from bright to dark and back again as the phase difference changes. Each change from bright to dark represents a phase change of 180° (a half cycle), which corresponds to a time difference of a half period (about 10^{-15} s for visible light). Counting the number of fringe changes thus gives a measure of the time difference between the paths, which in turn gives the relative velocity u. (See Problem 3.)

When Michelson and Morley performed their experiment, there was no observable change in the fringe pattern—they deduced a shift of less than 0.01 fringe, corresponding to a speed of the Earth through the ether of at most 5 km/s. As a last resort, they reasoned that perhaps the orbital motion of the Earth just happened to cancel out the overall motion through the ether. If this were true, six months later (when the Earth would be moving in its orbit in the opposite direction) the cancellation should not occur. When they repeated the experiment six months later, they again obtained a null result. In no experiment were Michelson and Morley able to detect the motion of the Earth through the ether.

In summary, we have seen that there is a direct chain of reasoning that leads from Galileo's principle of inertia, through Newton's laws with their implicit assumptions about space and time, ending with the failure of the Michelson-Morley experiment to observe the motion of the Earth relative to the ether. Although several explanations were offered for the unobservability of the ether and the corresponding failure of the upstream-downstream and cross-stream velocities to add in the expected way, the most novel, revolutionary, and ultimately successful explanation is given by Einstein's special theory of relativity, which requires a serious readjustment of our traditional concepts of space and time, and therefore alters some of the very foundations of physics.

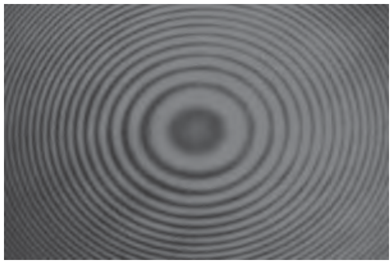


FIGURE 2.7 Interference fringes as observed with the Michelson interferometer of Figure 2.6. When the path length ACA changes by one-half wavelength relative to ABA, all light areas turn dark and all dark areas turn light.

2.3 EINSTEIN'S POSTULATES

The *special theory of relativity* is based on two postulates proposed by Albert Einstein in 1905:

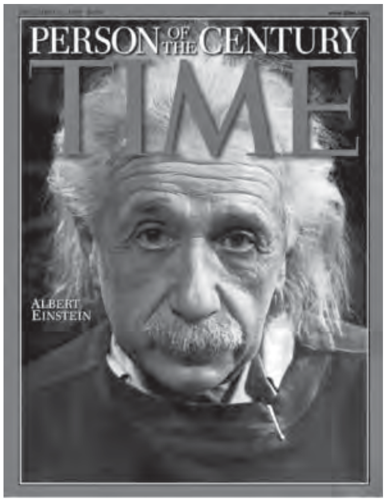
The principle of relativity: *The laws of physics are the same in all inertial reference frames.*

The principle of the constancy of the speed of light: *The speed of light in free space has the same value c in all inertial reference frames.*

The first postulate declares that the laws of physics are absolute, universal, and the same for all inertial observers. Laws that hold for one inertial observer cannot be violated for *any* inertial observer.

The second postulate is more difficult to accept because it seems to go against our "common sense," which is based on the Galilean kinematics we observe in everyday experiences. Consider three observers A, B, and C. Observer B is at rest, while A and C move away from B in opposite directions each at a speed of $c/4$. B fires a light beam in the direction of A. According to the Galilean transformation, if B measures a speed of c for the light beam, then A measures a speed of $c - c/4 = 3c/4$, while C measures a speed of $c + c/4 = 5c/4$. Einstein's second postulate, on the other hand, requires all three observers to measure the same speed of c for the light beam! This postulate immediately explains the failure of the Michelson-Morley experiment—the upstream-downstream and cross-stream speeds are identical (both are equal to c), so there is no phase difference between the two beams.

The two postulates also allow us to dispose of the ether hypothesis. The first postulate does not permit a preferred frame of reference (all inertial frames are equivalent), and the second postulate does not permit only a single frame of reference in which light moves at speed c, because light moves at speed c in *all* frames. The ether, as a preferred reference frame in which light has a unique speed, is therefore unnecessary.



Albert Einstein (1879–1955, Germany-United States). A gentle philosopher and pacifist, he was the intellectual leader of two generations of theoretical physicists and left his imprint on nearly every field of modern physics.

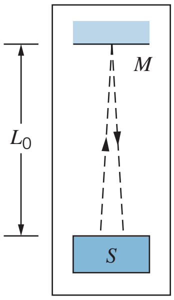


FIGURE 2.8 The clock ticks at intervals Δt_0 determined by the time for a light flash to travel the distance $2L_0$ from the light source S to the mirror M and back to the source where it is detected. (We assume the emission and detection occur at the same location, so the beam travels perpendicular to the mirror).

2.4 CONSEQUENCES OF EINSTEIN'S POSTULATES

Among their many consequences, Einstein's postulates require a new consideration of the fundamental nature of time and space. In this section we discuss how the postulates affect measurements of time and length intervals by observers in different frames of reference.

The Relativity of Time

To demonstrate the relativity of time, we use the timing device illustrated in Figure 2.8. It consists of a flashing light source S that is a distance L_0 from a mirror M. A flash of light from the source is reflected by the mirror, and when the light returns to S the clock ticks and triggers another flash. The time interval between ticks is the distance $2L_0$ (assuming the light travels perpendicular to the mirror) divided by the speed c:

$$\Delta t_0 = 2L_0/c \tag{2.6}$$

This is the time interval that is measured when the clock is at rest with respect to the observer.

We consider two observers: O is at rest on the ground, and O' moves with speed u. Each observer carries a timing device. Figure 2.9 shows a sequence of events that O observes for the clock carried by O'. According to O, the flash is emitted when the clock of O' is at A, reflected when it is at B, and detected at C. In this interval Δt, O observes the clock to move forward a distance of $u\Delta t$ from the point at which the flash was emitted, and O concludes that the light beam travels a distance $2L$, where $L = \sqrt{L_0^2 + (u\Delta t/2)^2}$, as shown in Figure 2.9. Because O observes the light beam to travel at speed c (as required by Einstein's second postulate) the time interval measured by O is

$$\Delta t = \frac{2L}{c} = \frac{2\sqrt{L_0^2 + (u\Delta t/2)^2}}{c} \tag{2.7}$$

Substituting for L_0 from Eq. 2.6 and solving Eq. 2.7 for Δt, we obtain

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} \tag{2.8}$$

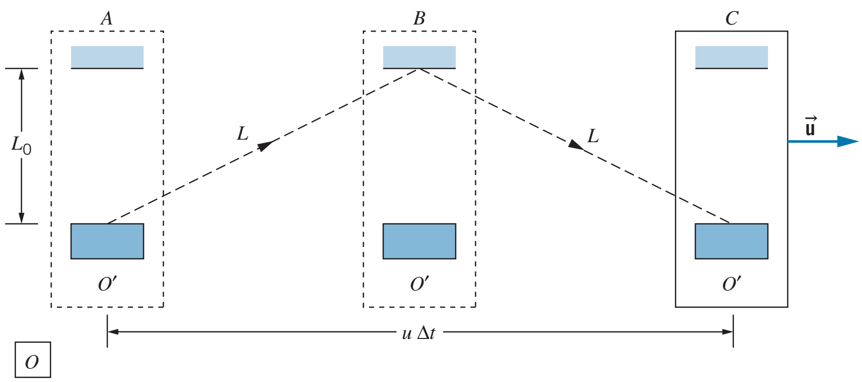


FIGURE 2.9 In the frame of reference of O, the clock carried by O' moves with speed u. The dashed line, of length $2L$, shows the path of the light beam according to O.

According to Eq. 2.8, observer O measures a longer time interval than O' measures. This is a general result of special relativity, which is known as *time dilation*. An observer O' is at rest relative to a device that produces a time interval Δt_0. For this observer, the beginning and end of the time interval occur at the same location, and so the interval Δt_0 is known as the *proper time*. An observer O, relative to whom O' is in motion, measures a longer time interval Δt for the same device. The dilated time interval Δt is always longer than the proper time interval Δt_0, no matter what the magnitude or direction of $\vec{\mathbf{u}}$.

This is a real effect that applies not only to clocks based on light beams but also to time itself; all clocks run more slowly according to an observer in relative motion, biological clocks included. Even the growth, aging, and decay of living systems are slowed by the time dilation effect. However, note that under normal circumstances ($u \ll c$), there is no measurable difference between Δt and Δt_0, so we don't notice the effect in our everyday activities. Time dilation has been verified experimentally with decaying elementary particles as well as with precise atomic clocks carried aboard aircraft. Some experimental tests are discussed in the last section of this chapter.

Example 2.4

Muons are elementary particles with a (proper) lifetime of 2.2 μs. They are produced with very high speeds in the upper atmosphere when cosmic rays (high-energy particles from space) collide with air molecules. Take the height L_0 of the atmosphere to be 100 km in the reference frame of the Earth, and find the minimum speed that enables the muons to survive the journey to the surface of the Earth.

Solution

The birth and decay of the muon can be considered as the "ticks" of a clock. In the frame of reference of the Earth (observer O) this clock is moving, and therefore its ticks are slowed by the time dilation effect. If the muon is moving at a speed that is close to c, the time necessary for it to travel from the top of the atmosphere to the surface of the Earth is

$$\Delta t = \frac{L_0}{c} = \frac{100\,\text{km}}{3.00 \times 10^8\,\text{m/s}} = 333\,\mu\text{s}$$

If the muon is to be observed at the surface of the Earth, it must live for at least 333 μs in the Earth's frame of reference. In the muon's frame of reference, the interval between its birth and decay is a proper time interval of 2.2 μs. The time intervals are related by Eq. 2.8:

$$333\,\mu\text{s} = \frac{2.2\,\mu\text{s}}{\sqrt{1 - u^2/c^2}}$$

Solving, we find

$$u = 0.999978c$$

If it were not for the time dilation effect, muons would not survive to reach the Earth's surface. The observation of these muons is a direct verification of the time dilation effect of special relativity.

The Relativity of Length

For this discussion, the moving timing device of O' is turned sideways, so that the light travels parallel to the direction of motion of O'. Figure 2.10 shows the sequence of events that O observes for the moving clock. According to O, the length of the clock (distance between the light source and the mirror) is L; as we shall see, this length is different from the length L_0 measured *by* O', relative to whom the clock is at rest.

The flash of light is emitted when the clock of O' is at A and reaches the mirror (position B) at time Δt_1 later. In this time interval, the light travels a distance

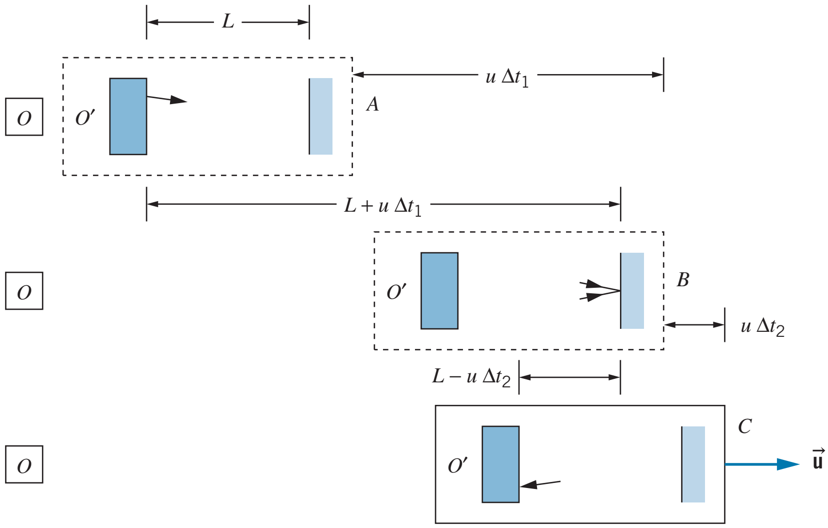


FIGURE 2.10 Here the clock carried by O' emits its light flash in the direction of motion.

$c\,\Delta t_1$, equal to the length L of the clock plus the additional distance $u\,\Delta t_1$ that the mirror moves forward in this interval. That is,

$$c\,\Delta t_1 = L + u\,\Delta t_1 \tag{2.9}$$

The flash of light travels from the mirror to the detector in a time Δt_2 and covers a distance of $c\,\Delta t_2$, equal to the length L of the clock less the distance $u\,\Delta t_2$ that the clock moves forward in this interval:

$$c\,\Delta t_2 = L - u\,\Delta t_2 \tag{2.10}$$

Solving Eqs. 2.9 and 2.10 for Δt_1 and Δt_2, and adding to find the total time interval, we obtain

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c-u} + \frac{L}{c+u} = \frac{2L}{c}\frac{1}{1-u^2/c^2} \tag{2.11}$$

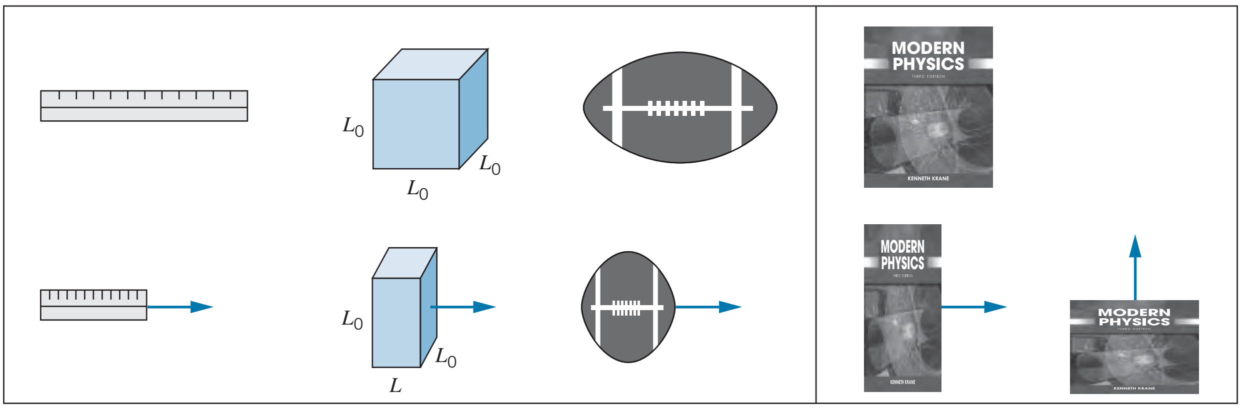


FIGURE 2.11 Some length-contracted objects. Notice that the shortening occurs only in the direction of motion.

From Eq. 2.8,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{2L_0}{c} \frac{1}{\sqrt{1 - u^2/c^2}} \tag{2.12}$$

Setting Eqs. 2.11 and 2.12 equal to one another and solving, we obtain

$$L = L_0\sqrt{1 - u^2/c^2} \tag{2.13}$$

Equation 2.13 summarizes the effect known as *length contraction*. Observer O', who is at rest with respect to the object, measures the *rest length* L_0 (also known as the *proper length,* in analogy with the proper time). All observers relative to whom O' is in motion measure a shorter length, but only along the direction of motion; length measurements transverse to the direction of motion are unaffected (Figure 2.11).

For ordinary speeds ($u \ll c$), the effects of length contraction are too small to be observed. For example, a rocket of length 100 m traveling at the escape speed from Earth (11.2 km/s) would appear to an observer on Earth to contract only by about two atomic diameters!

Length contraction suggests that objects in motion are measured to have a shorter length than they do at rest. The objects do not actually shrink; there is merely a difference in the length measured by different observers. For example, to observers on Earth a high-speed rocket ship would appear to be contracted along its direction of motion (Figure 2.12*a*), but to an observer on the ship it is the passing Earth that appears to be contracted (Figure 2.12*b*).

These representations of length-contracted objects are somewhat idealized. The actual appearance of a rapidly moving object is determined by the time at which light leaves the various parts of the object and enters the eye or the camera. The result is that the object appears distorted in shape and slightly rotated.

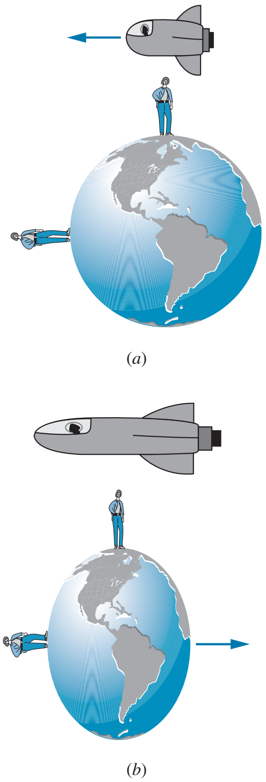


FIGURE 2.12 (*a*) The Earth views the passing contracted rocket. (*b*) From the rocket's frame of reference, the Earth appears contracted.

Example 2.5

Consider the point of view of an observer who is moving toward the Earth at the same velocity as the muon. In this reference frame, what is the apparent thickness of the Earth's atmosphere?

Solution

In this observer's reference frame, the muon is at rest and the Earth is rushing toward it at a speed of $u = 0.999978c$, as we found in Example 2.4. To an observer on the Earth, the height of the atmosphere is its rest length L_0 of 100 km. To the observer in the muon's rest frame, the moving Earth has an atmosphere of height given by Eq. 2.13:

$$L = L_0\sqrt{1 - u^2/c^2}$$

$$= (100\ \text{km})\sqrt{1 - (0.999978)^2} = 0.66\ \text{km} = 660\ \text{m}$$

This distance is small enough for the muons to reach the Earth's surface within their lifetime.

Note that what appears as a *time dilation* in one frame of reference (the observer on Earth) can be regarded as a *length contraction* in another frame of reference (the observer traveling with the muon). For another example of this effect, let's review again the example of the pion decay discussed in Section 1.2. A pion at rest has a lifetime of 26.0 ns. According to observer O_1 at rest in the laboratory frame of reference, a pion moving through the laboratory at a speed of $0.913c$ has a longer lifetime, which can be calculated to be 63.7 ns (using Eq. 2.8 for the time dilation). According to observer O_2, who is traveling through the laboratory at the same velocity as the pion, the pion appears to be at rest and has its proper lifetime of 26.0 ns. Thus O_1 sees a time dilation effect.

O_1 erects two markers in the laboratory, at the locations where the pion is created and decays. To O_1, the distance between those markers is the pion's speed times its lifetime, which works out to be 17.4 m. Suppose O_1 places a stick of length 17.4 m in the laboratory connecting the two markers. That stick is at rest in the laboratory reference frame and so has its proper length in that frame. In the reference frame of O_2, the stick is moving at a speed of $0.913c$ and has a shorter length of 7.1 m, which we can find using the length contraction formula (Eq. 2.13). So O_2 measures a distance of 7.1 m between the locations in the laboratory where the pion was created and where it decayed.

Note that O_1 measures the proper length and the dilated time, while O_2 measures the proper time and the contracted length. The proper time and proper length must always be referred to specific observers, who might not be in the same reference frame. The proper time is always measured by an observer according to whom the beginning of the time interval and the end of the time interval occur at the same location. If the time interval is the lifetime of the pion, then O_2 (relative to whom the pion does not move) sees its creation and decay at the same location and thus measures the proper time interval. The proper length, on the other hand, is always measured by an observer according to whom the measuring stick is at rest (O_1 in this case).

Example 2.6

An observer O is standing on a platform of length $D_0 = 65$ m on a space station. A rocket passes at a relative speed of $0.80c$ moving parallel to the edge of the platform. The observer O notes that the front and back of the rocket simultaneously line up with the ends of the platform at a particular instant (Figure 2.13*a*). (*a*) According to O, what is the time necessary for the rocket to pass a particular point on the platform? (*b*) What is the rest length L_0 of the rocket? (*c*) According to an observer O' on the rocket, what is the length D of the platform? (*d*) According to O', how long does it take for observer O to pass the entire length of the rocket? (*e*) According to O, the ends of the rocket simultaneously line up with the ends of the platform. Are these events simultaneous to O'?

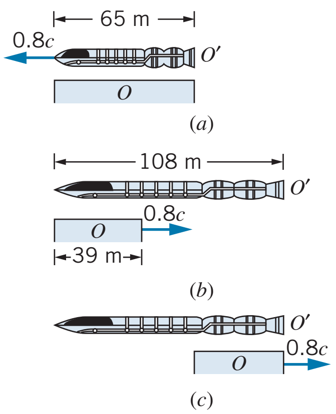


FIGURE 2.13 Example 2.6. (*a*) From the reference frame of O at rest on the platform, the passing rocket lines up simultaneously with the front and back of the platform. (*b, c*) From the reference frame O' in the rocket, the passing platform lines up first with the front of the rocket and later with the rear. Note the differing effects of length contraction in the two reference frames.

Solution

(*a*) According to O, the length L of the rocket matches the length D_0 of the platform. The time for the rocket to pass a particular point is measured by O to be

$$\Delta t_0 = \frac{L}{0.80c} = \frac{65\text{ m}}{2.40 \times 10^8\text{ m/s}} = 0.27\ \mu\text{s}$$

This is a proper time interval, because O measures the interval between two events that occur at the same point in the frame of reference of O (the front of the rocket passes a point, and then the back of the rocket passes the same point).

(*b*) O measures the contracted length L of the rocket. We can find its proper length L_0 using Eq. 2.13:

$$L_0 = \frac{L}{\sqrt{1 - u^2/c^2}} = \frac{65\text{ m}}{\sqrt{1 - (0.80)^2}} = 108\text{ m}$$

(*c*) According to O the platform is at rest, so 65 m is its proper length D_0. According to O', the contracted length of the platform is therefore

$$D = D_0\sqrt{1 - u^2/c^2} = (65\ \text{m})\sqrt{1 - (0.80)^2} = 39\ \text{m}$$

(*d*) For O to pass the entire length of the rocket, O' concludes that O must move a distance equal to its rest length, or 108 m. The time needed to do this is

$$\Delta t' = \frac{108\ \text{m}}{0.80c} = 0.45\ \mu\text{s}$$

Note that this is *not* a proper time interval for O', who determines this time interval using one clock at the front of the rocket to measure the time at which O passes the front of the rocket, and another clock on the rear of the rocket to measure the time at which O passes the rear of the rocket. The two events therefore occur at different points in O' and so cannot be separated by a proper time in O'. The corresponding time interval measured by O for the same two events, which we calculated in part (*a*), *is* a proper time interval for O, because the two events *do* occur at the same point in O.

The time intervals measured by O and O' should be related by the time dilation formula, as you should verify.

(*e*) According to O', the rocket has a rest length of $L_0 = 108$ m and the platform has a contracted length of $D = 39$ m. There is thus no way that O' could observe the two ends of both to align simultaneously. The sequence of events according to O' is illustrated in Figures 2.13*b* and *c*. The time interval $\Delta t'$ in O' between the two events that are simultaneous in O can be calculated by noting that, according to O', the time interval between the situations shown in Figures 2.13*b* and *c* must be that necessary for the platform to move a distance of 108 m − 39 m = 69 m, which takes a time

$$\Delta t' = \frac{69\ \text{m}}{0.80c} = 0.29\ \mu\text{s}$$

This result illustrates the relativity of simultaneity: two events at different locations that are simultaneous to O (the lining up of the two ends of the rocket with the two ends of the platform) *cannot* be simultaneous to O'.

Relativistic Velocity Addition

The timing device is now modified as shown in Figure 2.14. A source P emits particles that travel at speed v' according to an observer O' at rest with respect to the device. The flashing bulb F is triggered to flash when a particle reaches it. The flash of light makes the return trip to the detector D, and the clock ticks. The time interval Δt_0 between ticks measured by O' is composed of two parts: one for the particle to travel the distance L_0 at speed v' and another for the light to travel the same distance at speed c:

$$\Delta t_0 = L_0/v' + L_0/c \tag{2.14}$$

According to observer O, relative to whom O' moves at speed u, the sequence of events is similar to that shown in Figure 2.10. The emitted particle, which travels at speed v according to O, reaches F in a time interval Δt_1 after traveling the distance $v\,\Delta t_1$ equal to the (contracted) length L plus the additional distance $u\,\Delta t_1$ moved by the clock in that interval:

$$v\,\Delta t_1 = L + u\,\Delta t_1 \tag{2.15}$$

In the interval Δt_2, the light beam travels a distance $c\,\Delta t_2$ equal to the length L less the distance $u\,\Delta t_2$ moved by the clock in that interval:

$$c\,\Delta t_2 = L - u\,\Delta t_2 \tag{2.16}$$

We now solve Eqs. 2.15 and 2.16 for Δt_1 and Δt_2, add to find the total interval Δt between ticks according to O, use the time dilation formula, Eq. 2.8, to relate this result to Δt_0 from Eq. 2.14, and finally use the length contraction formula, Eq. 2.13, to relate L to L_0. After doing the algebra, we find the result

$$v = \frac{v' + u}{1 + v'u/c^2} \tag{2.17}$$

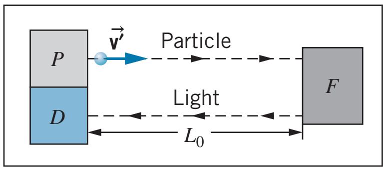


FIGURE 2.14 In this timing device, a particle is emitted by P at a speed v'. When the particle reaches F, it triggers the emission of a flash of light that travels to the detector D.

Equation 2.17 is the *relativistic velocity addition law* for velocity components that are in the direction of u. Later in this chapter we use a different method to derive the corresponding results for motion in other directions.

We can also regard Eq. 2.17 as a velocity transformation, enabling us to convert a velocity v' measured by O' to a velocity v measured by O. The corresponding classical law was given by Eq. 2.2: $v = v' + u$. The difference between the classical and relativistic results is the denominator of Eq. 2.17, which reduces to 1 in cases when the speeds are small compared with c. Example 2.7 shows how this factor prevents the measured speeds from exceeding c.

Equation 2.17 gives an important result when O' observes a light beam. For $v' = c$,

$$v = \frac{c+u}{1+cu/c^2} = c \tag{2.18}$$

That is, when $v' = c$, then $v = c$, *independent of the value of* u. All observers measure the same speed c for light, exactly as required by Einstein's second postulate.

Example 2.7

A spaceship moving away from the Earth at a speed of 0.80c fires a missile parallel to its direction of motion (Figure 2.15). The missile moves at a speed of 0.60c relative to the ship. What is the speed of the missile as measured by an observer on the Earth?

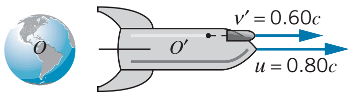


FIGURE 2.15 Example 2.7. A spaceship moves away from Earth at a speed of 0.80c. An observer O' on the spaceship fires a missile and measures its speed to be 0.60c relative to the ship.

Solution

Here O' is on the ship and O is on Earth; O' moves with a speed of $u = 0.80c$ relative to O. The missile moves at speed $v' = 0.60c$ relative to O', and we seek its speed v relative to O. Using Eq. 2.17, we obtain

$$v = \frac{v'+u}{1+v'u/c^2} = \frac{0.60c + 0.80c}{1+(0.60c)(0.80c)/c^2}$$

$$= \frac{1.40c}{1.48} = 0.95c$$

According to classical kinematics (the numerator of Eq. 2.17), an observer on the Earth would see the missile moving at $0.60c + 0.80c = 1.40c$, thereby exceeding the maximum relative speed of c permitted by relativity. You can see how Eq. 2.17 brings about this speed limit. Even if v' were $0.9999\ldots c$ and u were $0.9999\ldots c$, the relative speed v measured by O would remain less than c.

The Relativistic Doppler Effect

In the classical Doppler effect for sound waves, an observer moving relative to a source of waves (sound, for example) detects a frequency different from that emitted by the source. The frequency f' heard by the observer O is related to the frequency f emitted by the source S according to

$$f' = f\frac{v \pm v_O}{v \mp v_S} \tag{2.19}$$

where v is the speed of the waves in the medium (such as still air, in the case of sound waves), v_S is the speed of the source *relative to the medium*, and v_O is the speed of the observer *relative to the medium*. The upper signs in the numerator and denominator are chosen whenever S moves toward O or O moves toward S, while the lower signs apply whenever O and S move away from one another.

The classical Doppler shift for motion of the source differs from that for motion of the observer. For example, suppose the source emits sound waves at $f = 1000$ Hz. If the source moves at 30 m/s toward the observer who is at rest in the medium (which we take to be air, in which sound moves at $v = 340$ m/s), then $f' = 1097$ Hz, while if the source is at rest in the medium and the observer moves toward the source at 30 m/s, the frequency is 1088 Hz. Other possibilities in which the relative speed between S and O is 30 m/s, such as each moving toward the other at 15 m/s, give still different frequencies.

Here we have a situation in which it is not the relative speed of the source and observer that determines the Doppler shift—it is the speed of each with respect to the medium. This cannot occur for light waves, since there is no medium (no "ether") and no preferred reference frame by Einstein's first postulate. We therefore require a different approach to the Doppler effect for light waves, an approach that does not distinguish between source motion and observer motion, but involves only the relative motion between the source and the observer.

Consider a source of waves that is at rest in the reference frame of observer O. Observer O' moves relative to the source at speed u. We consider the situation from the frame of reference of O', as shown in Figure 2.16. Suppose O observes the source to emit N waves at frequency f. According to O, it takes an interval $\Delta t_0 = N/f$ for these N waves to be emitted; this is a proper time interval in the frame of reference of O. The corresponding time interval to O' is $\Delta t'$, during which O moves a distance $u\,\Delta t'$. The wavelength according to O' is the total length interval occupied by these waves divided by the number of waves:

$$\lambda' = \frac{c\,\Delta t' + u\,\Delta t'}{N} = \frac{c\,\Delta t' + u\,\Delta t'}{f\,\Delta t_0} \tag{2.20}$$

The frequency according to O' is $f' = c/\lambda'$, so

$$f' = f\frac{\Delta t_0}{\Delta t'}\frac{1}{1 + u/c} \tag{2.21}$$

and using the time dilation formula, Eq. 2.8, to relate $\Delta t'$ and Δt_0, we obtain

$$f' = f\frac{\sqrt{1 - u^2/c^2}}{1 + u/c} = f\sqrt{\frac{1 - u/c}{1 + u/c}} \tag{2.22}$$

This is the formula for the *relativistic Doppler shift,* for the case in which the waves are observed in a direction parallel to $\vec{\mathbf{u}}$. Note that, unlike the classical formula, it does *not* distinguish between source motion and observer motion; the

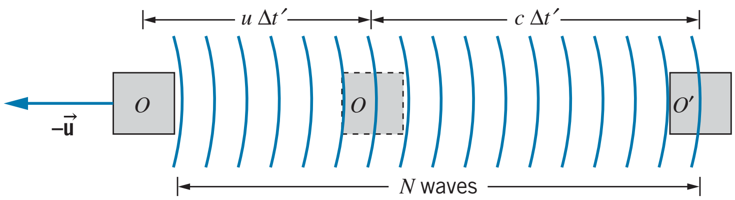


FIGURE 2.16 A source of waves, in the reference frame of O, moves at speed u away from observer O'. In the time $\Delta t'$ (according to O'), O moves a distance $u\,\Delta t'$ and emits N waves.

relativistic Doppler effect depends only on the relative speed u between the source and observer.

Equation 2.22 assumes that the source and observer are separating. If the source and observer are approaching one another, replace u by $-u$ in the formula.

Example 2.8

A distant galaxy is moving away from the Earth at such high speed that the blue hydrogen line at a wavelength of 434 nm is recorded at 600 nm, in the red range of the spectrum. What is the speed of the galaxy relative to the Earth?

Solution

Using Eq. 2.22 with $f = c/\lambda$ and $f' = c/\lambda'$, we obtain

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1-u/c}{1+u/c}}$$

$$\frac{c}{600\,\text{nm}} = \frac{c}{434\,\text{nm}}\sqrt{\frac{1-u/c}{1+u/c}}$$

Solving, we find

$$u/c = 0.31$$

Thus the galaxy is moving away from Earth at a speed of $0.31c = 9.4 \times 10^7\,\text{m/s}$. Evidence obtained in this way indicates that nearly all the galaxies we observe are moving away from us. This suggests that the universe is expanding, and is usually taken to provide evidence in favor of the Big Bang theory of cosmology (see Chapter 15).

2.5 THE LORENTZ TRANSFORMATION

We have seen that the Galilean transformation of coordinates, time, and velocity is not consistent with Einstein's postulates. Although the Galilean transformation agrees with our "common-sense" experience at low speeds, it does not agree with experiment at high speeds. We therefore need a new set of transformation equations that replaces the Galilean set and that is capable of predicting such relativistic effects as time dilation, length contraction, velocity addition, and the Doppler shift.

As before, we seek a transformation that enables observers O and O' in relative motion to compare their measurements of the space and time coordinates of the same event. The transformation equations relate the measurements of O (namely, x, y, z, t) to those of O' (namely, x', y', z', t'). This new transformation must have several properties: It must be linear (depending only on the first power of the space and time coordinates), which follows from the homogeneity of space and time; it must be consistent with Einstein's postulates; and it must reduce to the Galilean transformation when the relative speed between O and O' is small. We again assume that the velocity of O' relative to O is in the positive xx' direction.

This new transformation consistent with special relativity is called the *Lorentz transformation**. Its equations are

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \tag{2.23a}$$

$$y' = y \tag{2.23b}$$

*H. A. Lorentz (1853–1928) was a Dutch physicist who shared the 1902 Nobel Prize for his work on the influence of magnetic fields on light. In an unsuccessful attempt to explain the failure of the Michelson-Morley experiment, Lorentz developed the transformation equations that are named for him in 1904, a year *before* Einstein published his special theory of relativity. For a derivation of the Lorentz transformation, see R. Resnick and D. Halliday, *Basic Concepts in Relativity* (New York, Macmillan, 1992).

$$z' = z \qquad (2.23c)$$

$$t' = \frac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}} \qquad (2.23d)$$

It is often useful to write these equations in terms of *intervals* of space and time by replacing each coordinate by the corresponding interval (replace x by $\Delta x, x'$ by $\Delta x', t$ by $\Delta t, t'$ by $\Delta t'$).

These equations are written assuming that O' moves *away from* O in the xx' direction. If O' moves *toward* O, replace u with $-u$ in the equations.

The first three equations reduce directly to the Galilean transformation for space coordinates, Eqs. 2.1, when $u \ll c$. The fourth equation, which links the time coordinates, reduces to $t' = t$, which is a fundamental postulate of the Galilean-Newtonian world.

We now use the Lorentz transformation equations to derive some of the predictions of special relativity. The problems at the end of the chapter guide you in some other derivations. The results derived here are identical with those we obtained previously using Einstein's postulates, which shows that the equations of the Lorentz transformation are consistent with the postulates of special relativity.

Length Contraction

A rod of length L_0 is at rest in the reference frame of observer O'. The rod extends along the x' axis from x'_1 to x'_2; that is, O' measures the proper length $L_0 = x'_2 - x'_1$. Observer O, relative to whom the rod is in motion, measures the ends of the rod to be at coordinates x_1 and x_2. For O to determine the length of the moving rod, O must make a *simultaneous* determination of x_1 and x_2, and then the length is $L = x_2 - x_1$. Suppose the first event is O' setting off a flash bulb at one end of the rod at x'_1 and t'_1, which O observes at x_1 and t_1, and the second event is O' setting off a flash bulb at the other end at x'_2 and t'_2, which O observes at x_2 and t_2. The equations of the Lorentz transformation relate these coordinates, specifically,

$$x'_1 = \frac{x_1 - ut_1}{\sqrt{1 - u^2/c^2}} \qquad x'_2 = \frac{x_2 - ut_2}{\sqrt{1 - u^2/c^2}} \qquad (2.24)$$

Subtracting these equations, we obtain

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}} - \frac{u(t_2 - t_1)}{\sqrt{1 - u^2/c^2}} \qquad (2.25)$$

O' must arrange to set off the flash bulbs so that the flashes appear to be simultaneous to O. (They will *not* be simultaneous to O', as we discuss later in this section.) This enables O to make a simultaneous determination of the coordinates of the endpoints of the rod. If O observes the flashes to be simultaneous, then $t_2 = t_1$, and Eq. 2.25 reduces to

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}} \qquad (2.26)$$

With $x'_2 - x'_1 = L_0$ and $x_2 - x_1 = L$, this becomes

$$L = L_0\sqrt{1 - u^2/c^2} \qquad (2.27)$$

which is identical with Eq. 2.13, which we derived earlier using Einstein's postulates.

Velocity Transformation

If O observes a particle to travel with velocity v (components v_x, v_y, v_z), what velocity v' does O' observe for the particle? The relationship between the velocities measured by O and O' is given by the *Lorentz velocity transformation*:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} \tag{2.28a}$$

$$v'_y = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} \tag{2.28b}$$

$$v'_z = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} \tag{2.28c}$$

By solving Eq. 2.28*a* for v_x, you can show that it is identical to Eq. 2.17, a result we derived previously based on Einstein's postulates. Note that, in the limit of low speeds ($u \ll c$), the Lorentz velocity transformation reduces to the Galilean velocity transformation, Eq. 2.2. Note also that $v'_y \neq v_y$, even though $y' = y$. This occurs because of the way the Lorentz transformation handles the time coordinate.

We can derive these transformation equations for velocity from the Lorentz coordinate transformation. By way of example, we derive the velocity transformation for $v'_y = dy'/dt'$. Differentiating the coordinate transformation $y' = y$, we obtain $dy' = dy$. Similarly, differentiating the time coordinate transformation (Eq. 2.23*d*), we obtain

$$dt' = \frac{dt - (u/c^2)dx}{\sqrt{1 - u^2/c^2}}$$

So

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{[dt - (u/c^2)\,dx]/\sqrt{1 - u^2/c^2}} = \sqrt{1 - u^2/c^2}\,\frac{dy}{dt - (u/c^2)\,dx}$$

$$= \sqrt{1 - u^2/c^2}\,\frac{dy/dt}{1 - (u/c^2)\,dx/dt} = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - uv_x/c^2}$$

Similar methods can be used to obtain the transformation equations for v'_x and v'_z. These derivations are left as exercises (Problem 14).

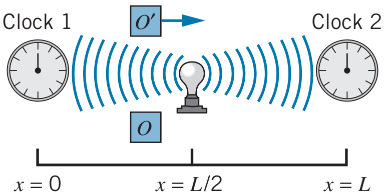


FIGURE 2.17 A flash of light, emitted from a point midway between the two clocks, starts the two clocks simultaneously according to O. Observer O' sees clock 2 start ahead of clock 1.

Simultaneity and Clock Synchronization

Under ordinary circumstances, synchronizing one clock with another is a simple matter. But for scientific work, where timekeeping at a precision below the nanosecond range is routine, clock synchronization can present some significant challenges. At very least, we need to correct for the time that it takes for the signal showing the reading on one clock to be transmitted to the other clock. However, for observers who are in motion with respect to each other, special relativity gives yet another way that clocks may appear to be out of synchronization.

Consider the device shown in Figure 2.17. Two clocks are located at $x = 0$ and $x = L$. A flash lamp is located at $x = L/2$, and the clocks are set running when they receive the flash of light from the lamp. The light takes the same interval of time to reach the two clocks, so the clocks start together precisely at a time $L/2c$ after the flash is emitted, and the clocks are exactly synchronized.

Now let us examine the same situation from the point of view of the moving observer O'. In the frame of reference of O, two events occur: the receipt of a light signal by clock 1 at $x_1 = 0, t_1 = L/2c$ and the receipt of a light signal by clock 2 at $x_2 = L, t_2 = L/2c$. Using Eq. 2.23*d*, we find that O' observes clock 1 to receive its signal at

$$t_1' = \frac{t_1 - (u/c^2)x_1}{\sqrt{1 - u^2/c^2}} = \frac{L/2c}{\sqrt{1 - u^2/c^2}} \tag{2.29}$$

while clock 2 receives its signal at

$$t_2' = \frac{t_2 - (u/c^2)x_2}{\sqrt{1 - u^2/c^2}} = \frac{L/2c - (u/c^2)L}{\sqrt{1 - u^2/c^2}} \tag{2.30}$$

Thus t_2' is smaller than t_1' and clock 2 appears to receive its signal earlier than clock 1, so that the clocks start at times that differ by

$$\Delta t' = t_1' - t_2' = \frac{uL/c^2}{\sqrt{1 - u^2/c^2}} \tag{2.31}$$

according to O'. Keep in mind that this is *not* a time dilation effect—time dilation comes from the *first* term of the Lorentz transformation (Eq. 2.23*d*) for t', while the lack of synchronization arises from the *second* term. O' observes *both* clocks to run slow, due to time dilation; O' *also* observes clock 2 to be ahead of clock 1.

We therefore reach the following conclusion: two events that are simultaneous in one reference frame are not simultaneous in another reference frame moving with respect to the first, unless the two events occur at the same point in space. (If $L = 0$, Eq. 2.31 shows that the clocks are synchronized in all reference frames.) Clocks that appear to be synchronized in one frame of reference will not necessarily be synchronized in another frame of reference in relative motion.

It is important to note that this clock synchronization effect does not depend on the *location* of observer O' but only on the *velocity* of O'. In Figure 2.17, the location of O' could have been drawn far to the left side of clock 1 or far to the right side of clock 2, and the result would be the same. In those different locations, the propagation time of the light signal showing clock 1 starting will differ from the propagation time of the light signal showing clock 2 starting. However, O' is assumed to be an "intelligent" observer who is aware of the locations where the light signals showing the two clocks starting are received relative to the locations of the clocks. O' corrects for this time difference, which is due only to the propagation time of the light signals, and *even after making that correction* the clocks still do not appear to be synchronized!

Although the location of O' does not appear in Eq. 2.31, the *direction* of the velocity of O' is important—if O' is moving in the opposite direction, the observed starting order of the two clocks is reversed.

Example 2.9

Two rockets are leaving their space station along perpendicular paths, as measured by an observer on the space station. Rocket 1 moves at $0.60c$ and rocket 2 moves at $0.80c$, both measured relative to the space station. What is the velocity of rocket 2 as observed by rocket 1?

Solution

Observer O is the space station, observer O' is rocket 1 (moving at $u = 0.60c$), and each observes rocket 2, moving (according to O) in a direction perpendicular to rocket 1. We take this to be the y direction of the reference frame of O. Thus O observes rocket 2 to have velocity components $v_x = 0, v_y = 0.80c$, as shown in Figure 2.18*a*.

We can find v'_x and v'_y using the Lorentz velocity transformation:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} = \frac{0 - 0.60c}{1 - 0(0.60c)/c^2} = -0.60c$$

$$v'_y = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$$

$$= \frac{0.80c\sqrt{1 - (0.60c)^2/c^2}}{1 - 0(0.60c)/c^2} = 0.64c$$

Thus, according to O', the situation looks like Figure 2.18*b*.

The speed of rocket 2 according to O' is $\sqrt{(0.60c)^2 + (0.64c)^2} = 0.88c$, less than c. According to the Galilean transformation, v'_y would be identical with v_y, and thus the speed would be $\sqrt{(0.60c)^2 + (0.80c)^2} = c$. Once again, the Lorentz transformation prevents relative speeds from reaching or exceeding the speed of light.

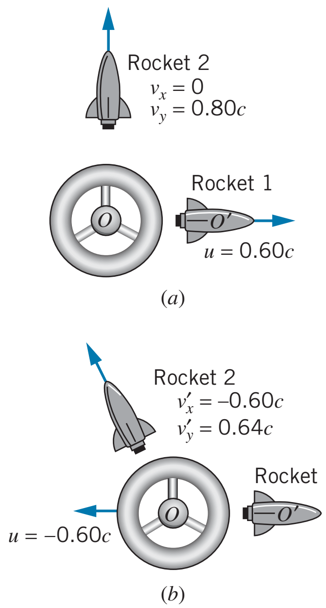


FIGURE 2.18 Example 2.9. (*a*) As viewed from the reference frame of O. (*b*) As viewed from the reference frame of O'.

Example 2.10

In Example 2.6, two events that were simultaneous to O (the lining up of the front and back of the rocket ship with the ends of the platform) were not simultaneous to O'. Find the time interval between these events according to O'.

Solution

According to O, the two simultaneous events are separated by a distance of $L = 65$ m. For $u = 0.80c$, Eq. 2.31 gives

$$\Delta t' = \frac{uL/c^2}{\sqrt{1 - u^2/c^2}}$$

$$= \frac{(0.80)(65\text{ m})/(3.00 \times 10^8\text{ m/s})}{\sqrt{1 - (0.80)^2}} = 0.29\ \mu\text{s}$$

which agrees with the result calculated in part (*e*) of Example 2.6.

2.6 THE TWIN PARADOX

We now turn briefly to what has become known as the twin paradox. Suppose there is a pair of twins on Earth. One, whom we shall call Casper, remains on Earth, while his twin sister Amelia sets off in a rocket ship on a trip to a distant planet. Casper, based on his understanding of special relativity, knows that his sister's clocks will run slow relative to his own and that therefore she should be younger than he when she returns, as our discussion of time dilation would suggest. However, recalling that discussion, we know that for two observers in relative motion, *each* thinks the *other's* clocks are running slow. We could therefore study this problem from the point of view of Amelia, according to whom Casper and the Earth (accompanied by the solar system and galaxy) make a round-trip journey away from her and back again. Under such circumstances, she will think it is her brother's clocks (which are now in motion relative to her own) that are running slow, and will therefore expect her brother to be younger than she when they meet again. While it is possible to disagree over whose clocks are running slow relative to his or her own, which is merely a problem of frames of reference, when Amelia returns to Earth (or when the Earth returns to Amelia), all observers must agree as to which twin has aged less rapidly. This is the paradox—each twin expects the other to be younger.

The resolution of this paradox lies in considering the asymmetric role of the two twins. The laws of special relativity apply only to inertial frames, those moving relative to one another at constant velocity. We may supply Amelia's rockets with sufficient thrust so that they accelerate for a very short length of time, bringing the ship to a speed at which it can coast to the planet, and thus during her outward journey Amelia spends all but a negligible amount of time in a frame of reference moving at constant speed relative to Casper. However, in order to return to Earth, she must decelerate and reverse her motion. Although this also may be done in a very short time interval, Amelia's return journey occurs in a completely different inertial frame than her outward journey. It is Amelia's jump from one inertial frame to another that causes the asymmetry in the ages of the twins. Only Amelia has the necessity of jumping to a new inertial frame to return, and therefore *all observers will agree* that it is Amelia who is "really" in motion, and that it is her clocks that are "really" running slow; therefore she is indeed the younger twin on her return.

Let us make this discussion more quantitative with a numerical example. We assume, as discussed above, that the acceleration and deceleration take negligible time intervals, so that all of Amelia's aging is done during the coasting. For simplicity, we assume the distant planet is at rest relative to the Earth; this does not change the problem, but it avoids the need to introduce yet another frame of reference. Suppose the planet to be 6 light-years distant from Earth, and suppose Amelia travels at a speed of $0.6c$. Then according to Casper it takes his sister 10 years (10 years $\times 0.6c = 6$ light-years) to reach the planet and 10 years to return, and therefore she is gone for a total of 20 years. (However, Casper doesn't know his sister has reached the planet until the light signal carrying news of her arrival reaches Earth. Since light takes 6 years to make the journey, it is 16 years after her departure when Casper sees his sister's arrival at the planet. Four years later she returns to Earth.) From the frame of reference of Amelia aboard the rocket, the distance to the planet is contracted by a factor of $\sqrt{1-(0.6)^2} = 0.8$, and is therefore 0.8×6 light-years = 4.8 light-years. At a speed of $0.6c$, Amelia will measure 8 years for the trip to the planet, for a total round trip time of 16 years. Thus Casper ages 20 years while Amelia ages only 16 years and is indeed the younger on her return.

We can confirm this analysis by having Casper send a light signal to his sister each year on his birthday. We know that the frequency of the signal as received by Amelia will be Doppler shifted. During the outward journey, she will receive signals at the rate of

$$(1/\text{year})\sqrt{\frac{1-u/c}{1+u/c}} = 0.5/\text{year}$$

During the return journey, the Doppler-shifted rate will be

$$(1/\text{year})\sqrt{\frac{1+u/c}{1-u/c}} = 2/\text{year}$$

Thus for the first 8 years, during Amelia's trip to the planet, she receives 4 signals, and during the return trip of 8 years, she receives 16 signals, for a total of 20. She receives 20 signals, indicating her brother has celebrated 20 birthdays during her 16-year journey.

Spacetime Diagrams

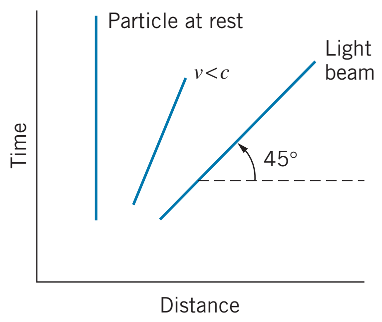


FIGURE 2.19 A spacetime diagram.

A particularly helpful way of visualizing the journeys of Casper and Amelia uses a *spacetime* diagram. Figure 2.19 shows an example of a spacetime diagram for motion that involves only one spatial direction.

In your introductory physics course, you probably became familiar with plotting motion on a graph in which distance appeared on the vertical axis and time on the horizontal axis. On such a graph, a straight line represents motion at constant velocity; the slope of the line is equal to the velocity. Note that the axes of the spacetime diagram are switched from the traditional graph of particle motion, with time on the vertical axis and space on the horizontal axis.

On a spacetime diagram, the graph that represents the motion of a particle is called its *worldline*. The *inverse* of the slope of the particle's worldline gives its velocity. Equivalently, the velocity is given by the tangent of the angle that the worldline makes with the *vertical* axis (rather than with the horizontal axis, as would be the case with a conventional plot of distance *vs*. time). Usually, the units of x and t are chosen so that motion at the speed of light is represented by a line with a 45° slope. A vertical line represents a particle that is at the same spatial locations at all times—that is, a particle at rest. Permitted motions with constant velocity are then represented by straight lines between the vertical and the 45° line representing the maximum velocity.

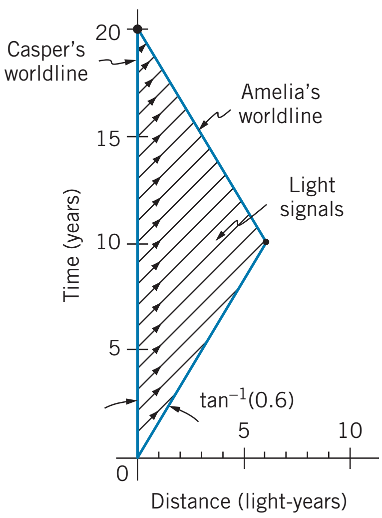


FIGURE 2.20 Casper's spacetime diagram, showing his worldline and Amelia's.

Let's draw the worldlines of Casper and Amelia according to Casper's frame of reference. Casper's worldline is a vertical line, because he is at rest in this frame (Figure 2.20). In Casper's frame of reference, 20 years pass between Amelia's departure and her return, so we can follow Casper's vertical worldline for 20 years.

Amelia is traveling at a speed of $0.6c$, so her worldline makes an angle with the vertical whose tangent is 0.6 (31°). In Casper's frame of reference, the planet visited by Amelia is 6 light-years from Earth. Amelia travels a distance of 6 light-years in a time of 10 years (according to Casper) so that $v = 6$ light-years/10 years $= 0.6c$.

The birthday signals that Casper sends to Amelia at the speed of light are represented by the series of 45° lines in Figure 2.20. Amelia receives 4 birthday signals during her outbound journey (the 4th arrives just as she reaches the planet) and 16 birthday signals during her return journey (the 16th is sent and received just as she returns to Earth).

It is left as an exercise (Problems 22 and 24) to consider the situation if it is Amelia who is sending the signals.

2.7 RELATIVISTIC DYNAMICS

We have seen how Einstein's postulates have led to a new "relative" interpretation of such previously absolute concepts as length and time, and that the classical concept of absolute velocity is not valid. It is reasonable then to ask how far this revolution is to go in changing our interpretation of physical concepts. Dynamical quantities, such as momentum and kinetic energy, depend on length, time, and velocity. Do classical laws of momentum and energy conservation remain valid in Einstein's relativity?

Let's test the conservation laws by examining the collision shown in Figure 2.21*a*. Two particles collide elastically as observed in the reference frame of O'. Particle 1 of mass $m_1 = 2m$ is initially at rest, and particle 2 of mass $m_2 = m$ is moving in the negative x direction with an initial velocity of $v'_{2\mathrm{i}} = -0.750c$. Using the classical law of momentum conservation to analyze this collision, O' would calculate the particles to be moving with final velocities $v'_{1\mathrm{f}} = -0.500c$ and $v'_{2\mathrm{f}} = +0.250c$. According to O', the total initial and final momenta of the particles would be:

$$p'_\mathrm{i} = m_1 v'_{1\mathrm{i}} + m_2 v'_{2\mathrm{i}} = (2m)(0) + (m)(-0.750c) = -0.750mc$$

$$p'_\mathrm{f} = m_1 v'_{1\mathrm{f}} + m_2 v'_{2\mathrm{f}} = (2m)(-0.500c) + (m)(0.250c) = -0.750mc$$

The initial and final momenta are equal according to O', demonstrating that momentum is conserved.

Suppose that the reference frame of O' moves at a velocity of $u = +0.550c$ in the x direction relative to observer O, as in Figure 2.21*b*. How would observer O analyze this collision? We can find the initial and final velocities of the two particles according to O using the velocity transformation of Eq. 2.17, which gives

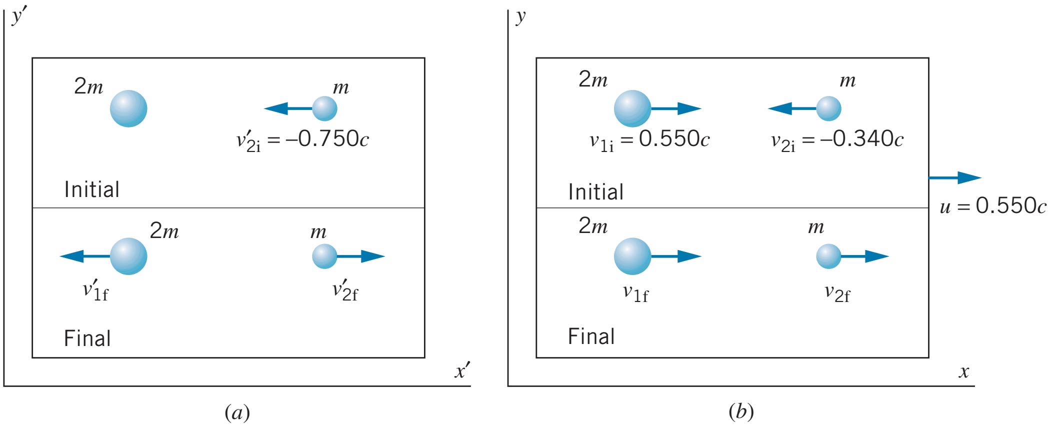


FIGURE 2.21 (*a*) A collision between two particles as observed from the reference frame of O'. (*b*) The same collision observed from the reference frame of O.

the initial velocities shown in the figure and the final velocities $v_{1f} = +0.069c$ and $v_{2f} = +0.703c$. Observer O can now calculate the initial and final values of the total momentum of the two particles:

$$p_i = m_1 v_{1i} + m_2 v_{2i} = (2m)(+0.550c) + (m)(-0.340c) = +0.760mc$$
$$p_f = m_1 v_{1f} + m_2 v_{2f} = (2m)(+0.069c) + (m)(+0.703c) = +0.841mc$$

Momentum is therefore *not conserved* according to observer O.

This collision experiment has shown that that the law of conservation of linear momentum, with momentum defined as $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, does not satisfy Einstein's first postulate (the law must be the same in all inertial frames). We cannot have a law that is valid for some observers but not for others. Therefore, *if we are to retain the conservation of momentum as a general law consistent with Einstein's first postulate, we must find a new definition of momentum*. This new definition of momentum must have two properties: (1) It must yield a law of conservation of momentum that satisfies the principle of relativity; that is, if momentum is conserved according to an observer in one inertial frame, then it is conserved according to observers in all inertial frames. (2) At low speeds, the new definition must reduce to $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, which we know works perfectly well in the nonrelativistic case.

These requirements are satisfied by defining the relativistic momentum for a particle of mass m moving with velocity $\vec{\mathbf{v}}$ as

$$\vec{\mathbf{p}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}} \tag{2.32}$$

In terms of components, we can write Eq. 2.32 as

$$p_x = \frac{mv_x}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad p_y = \frac{mv_y}{\sqrt{1 - v^2/c^2}} \tag{2.33}$$

The velocity v that appears in the denominator of these expressions is always the velocity of the particle as measured in a particular inertial frame. It is not the velocity of an inertial frame. The velocity in the numerator can be any of the components of the velocity vector.

We can now reanalyze the collision shown in Figure 2.21 using the relativistic definition of momentum. The initial relativistic momentum according to O' is

$$p'_i = \frac{m_1 v'_{1i}}{\sqrt{1 - v'^2_{1i}/c^2}} + \frac{m_2 v'_{2i}}{\sqrt{1 - v'^2_{2i}/c^2}} = \frac{(2m)(0)}{\sqrt{1 - 0^2}} + \frac{(m)(-0.750c)}{\sqrt{1 - (0.750)^2}} = -1.134mc$$

The final velocities according to O' are $v'_{1f} = -0.585c$ and $v'_{2f} = +0.294c$, and the total final momentum is

$$p'_f = \frac{m_1 v'_{1f}}{\sqrt{1 - v'^2_{1f}/c^2}} + \frac{m_2 v'_{2f}}{\sqrt{1 - v'^2_{2f}/c^2}}$$
$$= \frac{(2m)(-0.585c)}{\sqrt{1 - (0.585)^2}} + \frac{(m)(0.294c)}{\sqrt{1 - (0.294)^2}} = -1.134mc$$

Thus $p'_i = p'_f$, and observer O' concludes that momentum is conserved. According to O, the initial relativistic momentum is

$$p_i = \frac{m_1 v_{1i}}{\sqrt{1 - v_{1i}^2/c^2}} + \frac{m_2 v_{2i}}{\sqrt{1 - v_{2i}^2/c^2}} = \frac{(2m)(+0.550c)}{\sqrt{1-(0.550)^2}} + \frac{(m)(-0.340c)}{\sqrt{1-(0.340)^2}} = 0.956mc$$

Using the velocity transformation, the final velocities measured by O are $v_{1f} = -0.051c$ and $v_{2f} = +0.727c$, and so O calculates the final momentum to be

$$p_f = \frac{m_1 v_{1f}}{\sqrt{1 - v_{1f}^2/c^2}} + \frac{m_2 v_{2f}}{\sqrt{1 - v_{2f}^2/c^2}} = \frac{(2m)(-0.051c)}{\sqrt{1-(0.051)^2}} + \frac{(m)(+0.727c)}{\sqrt{1-(0.727)^2}} = 0.956mc$$

Observer O also concludes that $p_i = p_f$ and that the law of conservation of momentum is valid. Defining momentum according to Eq. 2.32 gives conservation of momentum in *all* reference frames, as required by the principle of relativity.

Example 2.11

What is the momentum of a proton moving at a speed of $v = 0.86c$?

Solution

Using Eq. 2.32, we obtain

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{(1.67 \times 10^{-27}\ \text{kg})(0.86)(3.00 \times 10^8\ \text{m/s})}{\sqrt{1-(0.86)^2}}$$
$$= 8.44 \times 10^{-19}\ \text{kg} \cdot \text{m/s}$$

The units of kg · m/s are generally not convenient in solving problems of this type. Instead, we manipulate Eq. 2.32 to obtain

$$pc = \frac{mvc}{\sqrt{1 - v^2/c^2}} = \frac{mc^2(v/c)}{\sqrt{1 - v^2/c^2}} = \frac{(938\ \text{MeV})(0.86)}{\sqrt{1-(0.86)^2}}$$
$$= 1580\ \text{MeV}$$

Here we have used the proton's *rest energy* mc^2, which is defined later in this section. The momentum is obtained from this result by dividing by the symbol c (not its numerical value), which gives

$$p = 1580\ \text{MeV/c}$$

The units of MeV/c for momentum are often used in relativistic calculations because, as we show later, the quantity pc often appears in these calculations. You should be able to convert MeV/c to kg · m/s and show that the two results obtained for p are equivalent.

Relativistic Kinetic Energy

Like the classical definition of momentum, the classical definition of kinetic energy also causes difficulties when we try to compare the interpretations of different observers. According to O', the initial and final kinetic energies in the collision shown in Figure 2.21*a* are:

$$K'_i = \tfrac{1}{2} m_1 v'^2_{1i} + \tfrac{1}{2} m_2 v'^2_{2i} = (0.5)(2m)(0)^2 + (0.5)(m)(-0.750c)^2 = 0.281mc^2$$

$$K'_f = \tfrac{1}{2} m_1 v'^2_{1f} + \tfrac{1}{2} m_2 v'^2_{2f} = (0.5)(2m)(-0.500c)^2 + (0.5)(m)(0.250c)^2 = 0.281mc^2$$

and so energy is conserved according to O'. The initial and final kinetic energies observed from the reference frame of O (as in Figure 2.21b) are

$$K_{\mathrm{i}} = \tfrac{1}{2}m_1 v_{1\mathrm{i}}^2 + \tfrac{1}{2}m_2 v_{2\mathrm{i}}^2 = (0.5)(2m)(0.550c)^2 + (0.5)(m)(-0.340c)^2 = 0.360mc^2$$

$$K_{\mathrm{f}} = \tfrac{1}{2}m_1 v_{1\mathrm{f}}^2 + \tfrac{1}{2}m_2 v_{2\mathrm{f}}^2 = (0.5)(2m)(0.069c)^2 + (0.5)(m)(0.703c)^2 = 0.252mc^2$$

Thus energy is *not conserved* in the reference frame of O if we use the classical formula for kinetic energy. This leads to a serious inconsistency—an elastic collision for one observer would not be elastic for another observer. As in the case of momentum, if we want to preserve the law of conservation of energy for all observers, we must replace the classical formula for kinetic energy with an expression that is valid in the relativistic case (but that reduces to the classical formula for low speeds).

We can derive the relativistic expression for the kinetic energy of a particle using essentially the same procedure used to derive the classical expression, starting with the particle form of the work-energy theorem (see Problem 28). The result of this calculation is

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \tag{2.34}$$

Using Eq. 2.34, you can show that both O and O' will conclude that kinetic energy is conserved. In fact, all observers will agree on the applicability of the energy conservation law using the relativistic definition for kinetic energy.

Equation 2.34 looks very different from the classical result $K = \frac{1}{2}mv^2$, but, as you should show (see Problem 32), Eq. 2.34 reduces to the classical expression in the limit of low speeds ($v \ll c$).

The classical expression for kinetic energy also violates the second relativity postulate by allowing speeds in excess of the speed of light. There is no limit (in either classical or relativistic dynamics) to the energy we can give to a particle. Yet, if we allow the kinetic energy to increase without limit, the classical expression $K = \frac{1}{2}mv^2$ implies that the velocity must correspondingly increase without limit, thereby violating the second postulate. You can also see from the first term of Eq. 2.34 that $K \to \infty$ as $v \to c$. Thus we can increase the relativistic kinetic energy of a particle without limit, and its speed will not exceed c.

Relativistic Total Energy and Rest Energy

We can also express Eq. 2.34 as

$$K = E - E_0 \tag{2.35}$$

where the *relativistic total energy* E is defined as

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{2.36}$$

and the *rest energy* E_0 is defined as

$$E_0 = mc^2 \tag{2.37}$$

The rest energy is in effect the relativistic total energy of a particle measured in a frame of reference in which the particle is at rest.

Sometimes m in Eq. 2.37 is called the *rest mass* m_0 and is distinguished from the "relativistic mass," which is defined as $m_0/\sqrt{1 - v^2/c^2}$. We choose not to use relativistic mass, because it can be a misleading concept. Whenever we refer to mass, we always mean rest mass.

Equation 2.37 suggests that mass can be expressed in units of energy divided by c^2, such as MeV/c^2. For example, a proton has a rest energy of 938 MeV and thus a mass of 938 MeV/c^2. Just like expressing momentum in units of MeV/c, expressing mass in units of MeV/c^2 turns out to be very useful in calculations.

The relativistic total energy is given by Eq. 2.35 as

$$E = K + E_0 \tag{2.38}$$

Collisions of particles at high energies often result in the production of new particles, and thus the final rest energy may not be equal to the initial rest energy (see Example 2.18). Such collisions must be analyzed using conservation of total relativistic energy E; kinetic energy will *not* be conserved when the rest energy changes in a collision. In the special example of the elastic collision considered in this section, the identities of the particles did not change, and so kinetic energy was conserved. In general, collisions do not conserve kinetic energy—it is the relativistic total energy that is conserved in collisions.

Manipulation of Eqs. 2.32 and 2.36 gives a useful relationship among the total energy, momentum, and rest energy:

$$E = \sqrt{(pc)^2 + (mc^2)^2} \tag{2.39}$$

Figure 2.22 shows a useful mnemonic device for remembering this relationship, which has the form of the Pythagorean theorem for the sides of a right triangle.

When a particle travels at a speed close to the speed of light (say, $v > 0.99c$), which often occurs in high-energy particle accelerators, the particle's kinetic energy is much greater than its rest energy; that is, $K \gg E_0$. In this case, Eq. 2.39 can be written, to a very good approximation,

$$E \cong pc \tag{2.40}$$

This is called the *extreme relativistic approximation* and is often useful for simplifying calculations. As v approaches c, the angle in Figure 2.22 between the bottom leg of the triangle (representing mc^2) and the hypotenuse (representing E) approaches 90°. Imagine in this case a very tall triangle, in which the vertical leg (pc) and the hypotenuse (E) are nearly the same length.

For massless particles (such as photons), Eq. 2.39 becomes exactly

$$E = pc \tag{2.41}$$

All massless particles travel at the speed of light; otherwise, by Eqs. 2.34 and 2.36 their kinetic and total energies would be zero.

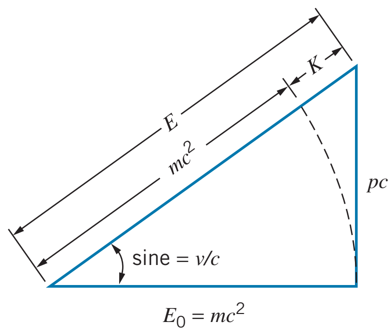


FIGURE 2.22 A useful mnemonic device for recalling the relationships among E_0, p, K, and E. Note that to put all variables in energy units, the quantity pc must be used.

Example 2.12

What are the kinetic and relativistic total energies of a proton ($E_0 = 938\text{ MeV}$) moving at a speed of $v = 0.86c$?

Solution

In Example 2.11 we found the momentum of this particle to be $p = 1580\text{ MeV}/c$. The total energy can be found from Eq. 2.39:

$$E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{(1580\text{ MeV})^2 + (938\text{ MeV})^2}$$
$$= 1837\text{ MeV}$$

The kinetic energy follows from Eq. 2.35:

$$K = E - E_0$$
$$= 1837\text{ MeV} - 938\text{ MeV}$$
$$= 899\text{ MeV}$$

We also could have solved this problem by finding the kinetic energy directly from Eq. 2.34.

Example 2.13

Find the velocity and momentum of an electron ($E_0 = 0.511\text{ MeV}$) with a kinetic energy of 10.0 MeV.

Solution

The total energy is $E = K + E_0 = 10.0\text{ MeV} + 0.511\text{ MeV} = 10.51\text{ MeV}$. We then can find the momentum from Eq. 2.39:

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(10.51\text{ MeV})^2 - (0.511\text{ MeV})^2}$$
$$= 10.5\text{ MeV}/c$$

Note that in this problem we could have used the extreme relativistic approximation, $p \cong E/c$, from Eq. 2.40. The error we would make in this case would be only 0.1%.

The velocity can be found by solving Eq. 2.36 for v.

$$\frac{v}{c} = \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = \sqrt{1 - \left(\frac{0.511\text{ MeV}}{10.51\text{ MeV}}\right)^2}$$
$$= 0.9988 \tag{2.42}$$

Example 2.14

In the Stanford Linear Collider electrons are accelerated to a kinetic energy of 50 GeV. Find the speed of such an electron as (*a*) a fraction of c, and (*b*) a difference from c. The rest energy of the electron is $0.511\text{ MeV} = 0.511 \times 10^{-3}\text{GeV}$.

Solution

(*a*) First we solve Eq. 2.34 for v, obtaining

$$v = c\sqrt{1 - \frac{1}{(1 + K/mc^2)^2}} \tag{2.43}$$

and thus

$$v = c\sqrt{1 - \frac{1}{[1 + (50\text{ GeV})/(0.511 \times 10^{-3}\text{ GeV})]^2}}$$
$$= 0.999\,999\,999\,948c$$

Calculators cannot be trusted to 12 significant digits. Here is a way to avoid this difficulty. We can write Eq. 2.43 as $v = c(1 + x)^{1/2}$, where $x = -1/(1 + K/mc^2)^2$. Because $K \gg mc^2$, we have $x \ll 1$, and we can use the binomial expansion to write $v \cong c(1 + \frac{1}{2}x)$, or

$$v \cong c\left[1 - \frac{1}{2(1 + K/mc^2)^2}\right]$$

which gives

$$v \cong c(1 - 5.2 \times 10^{-11})$$

This leads to the same value of v given above.

(b) From the above result, we have

$$\begin{aligned} c - v &= 5.2 \times 10^{-11} c \\ &= 0.016\ \text{m/s} \\ &= 1.6\ \text{cm/s} \end{aligned}$$

Example 2.15

At a distance equal to the radius of the Earth's orbit (1.5×10^{11} m), the Sun's radiation has an intensity of about 1.4×10^3 W/m^2. Find the rate at which the mass of the Sun is decreasing.

Solution

If we assume that the Sun's radiation is distributed uniformly over the surface area $4\pi r^2$ of a sphere of radius 1.5×10^{11} m, then the total radiative power emitted by the Sun is

$$\begin{aligned} &4\pi(1.5 \times 10^{11}\text{m})^2(1.4 \times 10^3\ \text{W/m}^2) \\ &= 4.0 \times 10^{26}\ \text{W} = 4.0 \times 10^{26}\ \text{J/s} \end{aligned}$$

By conservation of energy, we know that the energy lost by the Sun through radiation must be accounted for by a corresponding loss in its rest energy. The change in mass Δm corresponding to a change in rest energy ΔE_0 of 4.0×10^{26} J each second is

$$\Delta m = \frac{\Delta E_0}{c^2} = \frac{4.0 \times 10^{26}\ \text{J}}{9.0 \times 10^{16}\ \text{m}^2/\text{s}^2} = 4.4 \times 10^9\ \text{kg}$$

The Sun loses mass at a rate of about 4 billion kilograms per second! If this rate were to remain constant, the Sun (with a present mass of 2×10^{30} kg) would shine "only" for another 10^{13} years.

2.8 CONSERVATION LAWS IN RELATIVISTIC DECAYS AND COLLISIONS

In all decays and collisions, we must apply the law of conservation of momentum. The only difference between applying this law for collisions at low speed (as we did in Example 1.1) and at high speed is the use of the relativistic expression for momentum (Eq. 2.32) instead of Eq. 1.2. The law of conservation of momentum for relativistic motion can be stated in exactly the same way as for classical motion:

In an isolated system of particles, the total linear momentum remains constant.

In the classical case, kinetic energy is the only form of energy that is present in elastic collisions, so conservation of energy is equivalent to conservation of kinetic energy. In inelastic collisions or decay processes, the kinetic energy does not remain constant. Total energy is conserved in classical inelastic collisions, but we did not account for the other forms of energy that might be important. This missing energy is usually stored in the particles, perhaps as atomic or nuclear energy.

In the relativistic case, the internal stored energy contributes to the rest energy of the particles. Usually rest energy and kinetic energy are the only two forms of energy that we consider in atomic or nuclear processes (later we'll add the energy of radiation to this balance). A loss of kinetic energy in a collision is thus accompanied by a gain in rest energy, but the total relativistic energy (kinetic energy + rest energy) of all the particles involved in the process doesn't change. For example, in a reaction in which new particles are produced, the loss in kinetic energy of the original reacting particles gives the increase in rest energy of the product particles. On the other hand, in a nuclear decay process such as alpha decay, the initial nucleus gives up some rest energy to account for the kinetic energy carried by the decay products.

The law of energy conservation in the relativistic case is:

In an isolated system of particles, the relativistic total energy (kinetic energy plus rest energy) remains constant.

In applying this law to relativistic collisions, we don't have to worry whether the collision is elastic or inelastic, because the inclusion of the rest energy accounts for any loss in kinetic energy.

The following examples illustrate applications of the conservation laws for relativistic momentum and energy.

Example 2.16

A neutral K meson (mass 497.7 MeV$/c^2$) is moving with a kinetic energy of 77.0 MeV. It decays into a pi meson (mass 139.6 MeV$/c^2$) and another particle of unknown mass. The pi meson is moving in the direction of the original K meson with a momentum of 381.6 MeV$/c$. (*a*) Find the momentum and total relativistic energy of the unknown particle. (*b*) Find the mass of the unknown particle.

Solution

(*a*) The total energy and momentum of the K meson are

$$E_{\mathrm{K}} = K_{\mathrm{K}} + m_{\mathrm{K}}c^2 = 77.0\,\mathrm{MeV} + 497.7\,\mathrm{MeV} = 574.7\,\mathrm{MeV}$$

$$\begin{aligned} p_{\mathrm{K}} &= \frac{1}{c}\sqrt{E_{\mathrm{K}}^2 - (m_{\mathrm{K}}c^2)^2} \\ &= \frac{1}{c}\sqrt{(574.7\,\mathrm{MeV})^2 - (497.7\,\mathrm{MeV})^2} \\ &= 287.4\,\mathrm{MeV}/c \end{aligned}$$

and for the pi meson

$$\begin{aligned} E_\pi &= \sqrt{(cp_\pi)^2 + (m_\pi c^2)^2} \\ &= \sqrt{(381.6\,\mathrm{MeV})^2 + (139.6\,\mathrm{MeV})^2} \\ &= 406.3\,\mathrm{MeV} \end{aligned}$$

Conservation of relativistic momentum ($p_{\mathrm{initial}} = p_{\mathrm{final}}$) gives $p_{\mathrm{K}} = p_\pi + p_{\mathrm{x}}$ (where x represents the unknown particle), so

$$\begin{aligned} p_{\mathrm{x}} &= p_{\mathrm{K}} - p_\pi = 287.4\,\mathrm{MeV}/c - 381.6\,\mathrm{MeV}/c \\ &= -94.2\,\mathrm{MeV}/c \end{aligned}$$

and conservation of total relativistic energy ($E_{\mathrm{initial}} = E_{\mathrm{final}}$) gives $E_{\mathrm{K}} = E_\pi + E_{\mathrm{x}}$, so

$$\begin{aligned} E_{\mathrm{x}} &= E_{\mathrm{K}} - E_\pi = 574.7\,\mathrm{MeV} - 406.3\,\mathrm{MeV} \\ &= 168.4\,\mathrm{MeV} \end{aligned}$$

(*b*) We can find the mass by solving Eq. 2.39 for mc^2:

$$\begin{aligned} m_{\mathrm{x}}c^2 &= \sqrt{E_{\mathrm{x}}^2 - (cp_{\mathrm{x}})^2} \\ &= \sqrt{(168.4\,\mathrm{MeV})^2 - (94.2\,\mathrm{MeV})^2} \\ &= 139.6\,\mathrm{MeV} \end{aligned}$$

Thus the unknown particle has a mass of 139.6 MeV$/c^2$, and its mass shows that it is another pi meson.

Example 2.17

In the reaction $K^- + p \rightarrow \Lambda^0 + \pi^0$, a charged K meson (mass 493.7 MeV/c^2) collides with a proton (938.3 MeV/c^2) at rest, producing a lambda particle (1115.7 MeV/c^2) and a neutral pi meson (135.0 MeV/c^2), as represented in Figure 2.23. The initial kinetic energy of the K meson is 152.4 MeV. After the interaction, the pi meson has a kinetic energy of 254.8 MeV. (*a*) Find the kinetic energy of the lambda. (*b*) Find the directions of motion of the lambda and the pi meson.

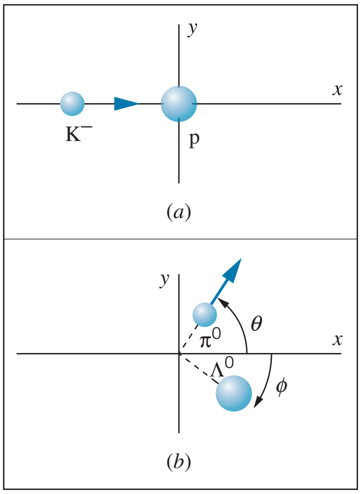


FIGURE 2.23 Example 2.17. (*a*) A K^- meson collides with a proton at rest. (*b*) After the collision, a π^0 meson and a Λ^0 are produced.

Solution

(*a*) The initial and final total energies are

$$E_{\text{initial}} = E_K + E_p = K_K + m_K c^2 + m_p c^2$$
$$E_{\text{final}} = E_\Lambda + E_\pi = K_\Lambda + m_\Lambda c^2 + K_\pi + m_\pi c^2$$

In these two equations, the value of every quantity is known except the kinetic energy of the lambda. Using conservation of total relativistic energy, we set $E_{\text{initial}} = E_{\text{final}}$ and solve for K_Λ:

$$\begin{aligned} K_\Lambda &= K_K + m_K c^2 + m_p c^2 - m_\Lambda c^2 - K_\pi - m_\pi c^2 \\ &= 152.4\text{ MeV} + 493.7\text{ MeV} + 938.3\text{ MeV} \\ &\quad - 1115.7\text{ MeV} - 254.8\text{ MeV} - 135.0\text{ MeV} \\ &= 78.9\text{ MeV} \end{aligned}$$

(*b*) To find the directional information we must apply conservation of momentum. The initial momentum is just that of the K meson. From its total energy, $E_K = K_K + m_K c^2 = 152.4\text{ MeV} + 493.7\text{ MeV} = 646.1\text{ MeV}$, we can find the momentum:

$$\begin{aligned} p_{\text{initial}} = p_K &= \frac{1}{c}\sqrt{(E_K)^2 - (m_K c^2)^2} \\ &= \frac{1}{c}\sqrt{(646.1\text{ MeV})^2 - (493.7\text{ MeV})^2} \\ &= 416.8\text{ MeV}/c \end{aligned}$$

A similar procedure applied to the two final particles gives $p_\Lambda = 426.9$ MeV/c and $p_\pi = 365.7$ MeV/c. The total momentum of the two final particles is $p_{x,\text{final}} = p_\Lambda \cos\theta + p_\pi \cos\phi$ and $p_{y,\text{final}} = p_\Lambda \sin\theta - p_\pi \sin\phi$. Conservation of momentum in the x and y directions gives

$$p_\Lambda \cos\theta + p_\pi \cos\phi = p_{\text{initial}} \quad \text{and} \quad p_\Lambda \sin\theta - p_\pi \sin\phi = 0$$

Here we have two equations with two unknowns (θ and ϕ). We can eliminate θ by writing the first equation as $p_\Lambda \cos\theta = p_{\text{initial}} - p_\pi \cos\phi$, then squaring both equations and adding them. The resulting equation can be solved for ϕ:

$$\begin{aligned} \phi &= \cos^{-1}\left(\frac{p_{\text{initial}}^2 + p_\pi^2 - p_\Lambda^2}{2 p_\pi p_{\text{initial}}}\right) \\ &= \cos^{-1}\left(\frac{(416.8\text{ MeV}/c)^2 + (365.7\text{ MeV}/c)^2 - (426.9\text{ MeV}/c)^2}{2(365.7\text{ MeV}/c)(416.8\text{ MeV}/c)}\right) \\ &= 65.7^\circ \end{aligned}$$

From the conservation of momentum equation for the y components, we have

$$\begin{aligned} \theta &= \sin^{-1}\left(\frac{p_\pi \sin\phi}{p_\Lambda}\right) \\ &= \sin^{-1}\left(\frac{(365.7\text{ MeV}/c)(\sin 65.7^\circ)}{426.9\text{ MeV}/c}\right) = 51.3^\circ \end{aligned}$$

Example 2.18

The discovery of the antiproton $\overline{\mathrm{p}}$ (a particle with the same rest energy as a proton, 938 MeV, but with the opposite electric charge) took place in 1956 through the following reaction:

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{p} + \mathrm{p} + \overline{\mathrm{p}}$$

in which accelerated protons were incident on a target of protons at rest in the laboratory. The minimum incident kinetic energy needed to produce the reaction is called the *threshold* kinetic energy, for which the final particles move together as if they were a single unit (Figure 2.24). Find the threshold kinetic energy to produce antiprotons in this reaction.

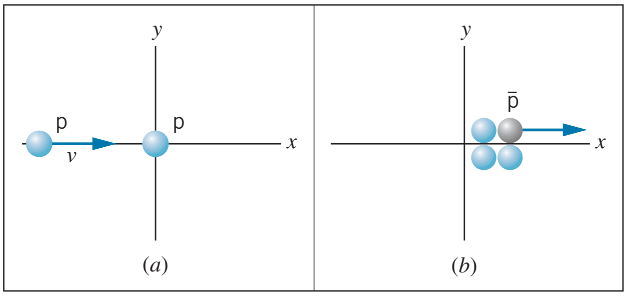


FIGURE 2.24 Example 2.18. (*a*) A proton moving with velocity v collides with another proton at rest. (*b*) The reaction produces three protons and an antiproton, which move together as a unit.

Solution

This problem can be solved by a straightforward application of energy and momentum conservation. Let E_{p} and p_{p} represent the total energy and momentum of the incident proton. Thus the initial total energy of the two protons is $E_{\mathrm{p}} + m_{\mathrm{p}}c^2$. Let E'_{p} and p'_{p} represent the total energy and momentum of *each* of the four final particles (which move together and thus have the same energy and momentum). We can then apply conservation of total energy:

$$E_{\mathrm{p}} + m_{\mathrm{p}}c^2 = 4E'_{\mathrm{p}}$$

and conservation of momentum:

$$p_{\mathrm{p}} = 4p'_{\mathrm{p}}$$

We can write the momentum equation as $\sqrt{E_{\mathrm{p}}^2 - (m_{\mathrm{p}}c^2)^2} = 4\sqrt{E_{\mathrm{p}}'^2 - (m_{\mathrm{p}}c^2)^2}$, so now we have two equations in two unknowns (E_{p} and E'_{p}). We eliminate E'_{p}, for example by solving the energy conservation equation for E'_{p} and substituting into the momentum equation. The result is

$$E_{\mathrm{p}} = 7m_{\mathrm{p}}c^2$$

from which we can calculate the kinetic energy of the incident proton:

$$K_{\mathrm{p}} = E_{\mathrm{p}} - m_{\mathrm{p}}c^2 = 6m_{\mathrm{p}}c^2 = 6(938\,\mathrm{MeV}) = 5628\,\mathrm{MeV} = 5.628\mathrm{GeV}$$

The Bevatron accelerator at the Lawrence Berkeley Laboratory was designed with this experiment in mind, so that it could produce a beam of protons whose energy exceeded 5.6 GeV. The discovery of the antiproton in this reaction was honored with the award of the 1959 Nobel Prize to the experimenters, Emilio Segrè and Owen Chamberlain.

2.9 EXPERIMENTAL TESTS OF SPECIAL RELATIVITY

Because special relativity provided such a radical departure from the notions of space and time in classical physics, it is important to perform detailed experimental tests that can clearly distinguish between the predictions of special relativity and those of classical physics. Many tests of increasing precision have been done since the theory was originally presented, and in every case the predictions of special relativity are upheld. Here we discuss a few of these tests.

Universality of the Speed of Light

The second relativity postulate asserts that the speed of light has the same value c for all observers. This leads to several types of experimental tests, of which we discuss two: (1) Does the speed of light change with the direction of travel? (2) Does the speed of light change with relative motion between source and observer?

The Michelson-Morley experiment provides a test of the first type. This experiment compared the upstream-downstream and cross-stream speeds of light and concluded that they were equal within the experimental error. Equivalently, we may say that the experiment showed that there is no preferred reference frame (no ether) relative to which the speed of light must be measured. If there is an ether, the speed of the Earth through the ether is less than 5 km/s, which is much smaller than the Earth's orbital speed about the Sun, 30 km/s. We can express their result as a difference Δc between the upstream-downstream and cross-stream speeds; the experiment showed that $\Delta c/c < 3 \times 10^{-10}$.

To reconcile the result of the Michelson-Morley experiment with classical physics, Lorentz proposed the "ether drag" hypothesis, according to which the motion of the Earth through the ether caused an electromagnetic drag that contracted the arm of the interferometer in the direction of motion. This contraction was just enough to compensate for the difference in the upstream-downstream and cross-stream times predicted by the Galilean transformation. This hypothesis succeeds only when the two arms of the interferometer are of the same length. To test this hypothesis, a similar experiment was done in 1932 by Kennedy and Thorndike; in their experiment, the lengths of the interferometer arms differed by about 16 cm, the maximum distance over which light sources available at that time could remain coherent. The Kennedy-Thorndike experiment in effect tests the second question, whether the speed of light changes due to relative motion. Their result was $\Delta c/c < 3 \times 10^{-8}$, which excludes the Lorentz contraction hypothesis as an explanation for the Michelson-Morley experiment.

In recent years, these fundamental experiments have been repeated with considerably improved precision using lasers as light sources. Experimenters working at the Joint Institute for Laboratory Astrophysics in Boulder, Colorado, built an apparatus that consisted of two He-Ne lasers on a rotating granite platform. By electronically stabilizing the lasers, they improved the sensitivity of their apparatus by several orders of magnitude. Again expressing the result as a difference between the speeds along the two arms of the apparatus, this experiment corresponds to $\Delta c/c < 8 \times 10^{-15}$, an improvement of about 5 orders of magnitude over the original Michelson-Morley experiment. In a similar repetition of the Kennedy-Thorndike experiment using He-Ne lasers, they obtained $\Delta c/c < 1 \times 10^{-10}$, an improvement over the original experiment by a factor of 300. [See A. Brillet and J. L. Hall, *Physical Review Letters* **42,** 549 (1979); D. Hils and J. L. Hall, *Physical Review Letters* **64,** 1697 (1990).] A considerable improvement in the Kennedy-Thorndike type of experiment has been made possible by comparing the oscillation frequency of a crystal with the frequency of a hydrogen maser (a maser is similar to a laser, but it uses microwaves rather than visible light). The experimenters measured for nearly one year, looking for a change in the relative frequencies as the Earth's velocity changed. No effect was observed, leading to a limit of $\Delta c/c < 2 \times 10^{-12}$. [See P. Wolf et al., *Physical Review Letters* **90**, 060402 (2003).]

Another way of testing the second question is to measure the speed of a light beam emitted by a source in motion. Suppose we observe this beam along the direction of motion of the moving source, which might be moving toward us or away from us. In the rest frame of the source, the emitted light travels at speed c. We can express the speed of light in our reference frame as $c' = c + \Delta c$, where Δc is zero according to special relativity ($c' = c$) or is $\pm u$ according to classical physics ($c' = c \pm u$ in the Galilean transformation, depending on whether the motion is toward or away from the observer).

In one experiment of this type, the decay of pi mesons (pions) into gamma rays (a form of electromagnetic waves traveling at c) was observed. When pions (produced in laboratories with large accelerators) emit these gamma rays, they are traveling at speeds close to the speed of light, relative to the laboratory. Thus if Galilean relativity were valid, we should expect to find gamma rays emitted in the direction of motion of the decaying pions traveling at a speed c' in the laboratory of nearly $2c$, rather than always with c as predicted by special relativity. The observed laboratory speed of these gamma rays in one experiment was $(2.9977 \pm 0.0004) \times 10^8$ m/s when the decaying pions were moving at $u/c = 0.99975$. These results give $\Delta c/c < 2 \times 10^{-4}$, and thus $c' = c$ as expected from special relativity. This experiment shows directly that an object moving at a speed of nearly c relative to the laboratory emits "light" that travels at a speed of c relative to both the object *and* the laboratory, giving direct evidence for Einstein's second postulate. [See T. Alvager et al., *Physics Letters* **12**, 260 (1964).]

Another experiment of this type is to study the X rays emitted by a binary pulsar, a rapidly pulsating source of X rays in orbit about another star, which would eclipse the pulsar as it rotated in its orbit. If the speed of light (in this case, X rays) were to change as the pulsar moved first toward and later away from the Earth in its orbit, the beginning and end of the eclipse would not be equally spaced in time from the midpoint of the eclipse. No such effect is observed, and from these observations it is concluded that $\Delta c/c < 2 \times 10^{-12}$, in agreement with predictions of special relativity. These experiments were done at $u/c = 10^{-3}$. [See K. Brecher, *Physical Review Letters* **39**, 1051 (1977).]

A different type of test of the limit by which the speed of light changes with direction of travel can be done using the clocks carried aboard the network of Earth satellites that make up the Global Positioning System (GPS). By comparing the readings of clocks on the GPS satellites with clocks on the ground at different times of day (as the satellites move relative to the ground stations), it is possible to test whether the change in the direction of travel affects the apparent synchronization of the clocks. No effect was observed, and the experimenters were able to set a limit of $\Delta c/c < 5 \times 10^{-9}$ for the difference between the one-way and round-trip speeds of light. [See P. Wolf and G. Petit, *Physical Review A* **56**, 4405 (1997).]

Time Dilation

We have already discussed the time dilation effect on the decay of muons produced by cosmic rays. Muon decay can also be studied in the laboratory. Muons can be produced following collisions in high-energy accelerators, and the decay of the muons can be followed by observing their decay products (ordinary electrons). These muons can either be trapped and decay at rest, or they can be placed in a beam and decay in flight. When muons are observed at rest, their decay lifetime is $2.198\ \mu s$. (As we discuss in Chapter 12, decays generally follow an exponential law. The lifetime is the time after which a fraction $1/e = 0.368$ of the original muons remain.) This is the *proper lifetime,* measured in a frame of reference in which the muon is at rest. In one particular experiment, muons were trapped in a ring and circulated at a momentum of $p = 3094$ MeV$/c$. The decays in flight occurred with a lifetime of 64.37 μs (measured in the laboratory frame of reference). For muons of this momentum, Eq. 2.8 gives a dilated lifetime of (see Problem 43) 64.38 μs, which is in excellent agreement with the measured value and confirms the time dilation effect. [See J. Bailey et al., *Nature* **268**, 301 (1977).]

Another similar experiment was done with pions. The proper lifetime, measured for pions at rest, is known to be 26.0 ns. In one experiment, pions were observed in flight at $u/c = 0.913$, and their lifetime was measured to be 63.7 ns. (Pions decay to muons, so we can follow the exponential radioactive decay of the pions by observing the muons emitted as a result of the decay.) For pions moving at this speed, the expected dilated lifetime is in exact agreement with the measured value, once again confirming the time dilation effect. [See D. S. Ayres et al., *Physical Review D* **3**, 1051 (1971).]

The Doppler Effect

Confirmation of the relativistic Doppler effect first came from experiments done in 1938 by Ives and Stilwell. They sent a beam of hydrogen atoms, generated in a gas discharge, down a tube at a speed u, as shown in Figure 2.25. They could simultaneously observe light emitted by the atoms in a direction parallel to u (atom 1) and opposite to u (atom 2, reflected from the mirror). Using a spectrograph, the experimenters were able to photograph the characteristic spectral lines from these atoms and also, on the same photographic plate, from atoms at rest. If the classical Doppler formula were valid, the wavelengths of the lines from atoms 1 and 2 would be placed at symmetric intervals $\Delta\lambda_1 = \pm\lambda_0(u/c)$ on either side of the line from the atoms at rest (wavelength λ_0), as in Figure 2.25*b*. The relativistic Doppler formula, on the other hand, gives a small additional asymmetric shift $\Delta\lambda_2 = +\frac{1}{2}\lambda_0(u/c)^2$, as in Figure 2.25*c* (computed for $u \ll c$, so

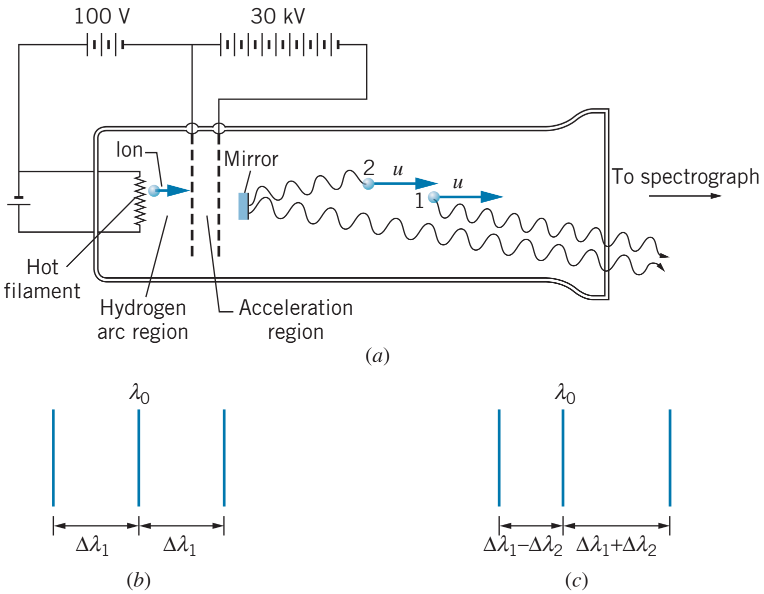


FIGURE 2.25 (*a*) Apparatus used in the Ives-Stilwell experiment. (*b*) Line spectrum expected from classical Doppler effect. (*c*) Line spectrum expected from relativistic Doppler effect.

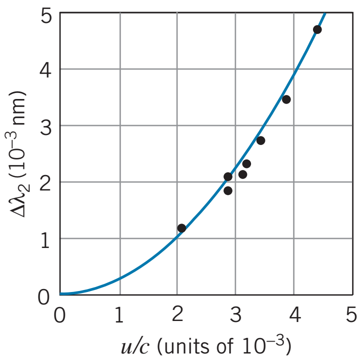


FIGURE 2.26 Results of the Ives-Stilwell experiment. According to classical theory, $\Delta\lambda_2 = 0$, while according to special relativity, $\Delta\lambda_2$ depends on $(u/c)^2$. The solid line, which represents the relativistic formula, gives excellent agreement with the data points.

that higher-order terms in u/c can be neglected). Figure 2.26 shows the results of Ives and Stilwell for one of the hydrogen lines (the blue line of the Balmer series at $\lambda_0 = 486$ nm). The agreement between the observed values and those predicted by the relativistic formula is impressive.

Recent experiments with lasers have verified the relativistic formula at greater accuracy. These experiments are based on the absorption of laser light by an atom; when the radiation is absorbed, the atom changes from its lowest-energy state (the ground state) to one of its excited states. The experiment consists essentially of comparing the laser wavelength needed to excite atoms at rest with that needed for atoms in motion. One experiment used a beam of hydrogen atoms with kinetic energy 800 MeV (corresponding to $u/c = 0.84$) produced in a high-energy proton accelerator. An ultraviolet laser was used to excite the atoms. This experiment verified the relativistic Doppler effect to an accuracy of about 3×10^{-4}. [See D. W. MacArthur et al., *Physical Review Letters* **56**, 282 (1986).] In another experiment, a beam of neon atoms moving with a speed of $u = 0.0036c$ was irradiated with light from a tunable dye laser. This experiment verified the relativistic Doppler shift to a precision of 2×10^{-6}. [See R. W. McGowan et al., *Physical Review Letters* **70**, 251 (1993).] A more recent study used two tunable dye lasers parallel and antiparallel to a beam of lithium atoms moving at $0.064c$. The results of this experiment agreed with the relativistic Doppler formula to within a precision of 2×10^{-7}, improving on the best previous results by an order of magnitude. [See G. Saathoff et al., *Physical Review Letters* **91**, 190403 (2003).]

Relativistic Momentum and Energy

The earliest direct confirmation of the relativistic relationship for energy and momentum came just a few years after Einstein's 1905 paper. Simultaneous measurements were made of the momentum and velocity of high-energy electrons emitted in certain radioactive decay processes (nuclear beta decay, which is discussed in Chapter 12). Figure 2.27 shows the results of several different investigations plotted as p/mv, which should have the value 1 according to classical physics. The results agree with the relativistic formula and disagree with the classical one. Note that the relativistic and classical formulas give the same

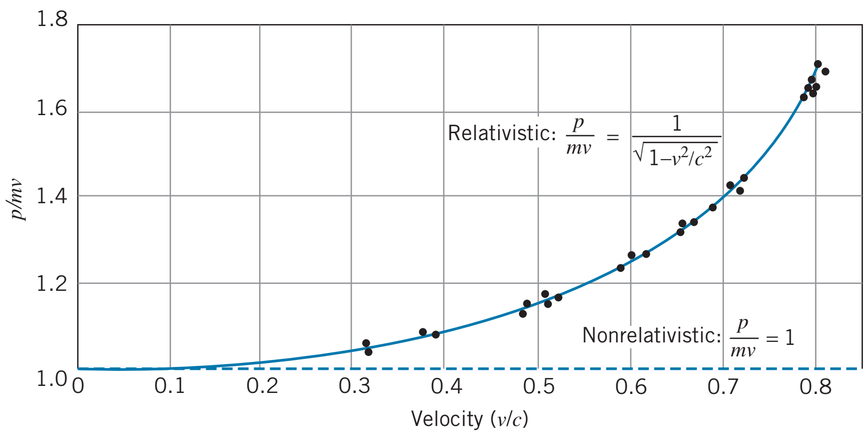


FIGURE 2.27 The ratio p/mv is plotted for electrons of various speeds. The data agree with the relativistic result and not at all with the nonrelativistic result ($p/mv = 1$).

results at low speeds, and in fact the two cannot be distinguished for speeds below 0.1 c, which accounts for our failure to observe these effects in experiments with ordinary laboratory objects.

Other more recent experiments, in which the kinetic energies of fast electrons were measured, are shown in Figure 2.28. Once again, the data at high speeds agree with special relativity and disagree with the classical equations. In a more extreme example, experimenters at the Stanford Linear Accelerator Center measured the speed of 20 GeV electrons, whose speed is within 5×10^{-10} of the speed of light (or about 0.15 m/s less than c). The measurement was not capable of this level of precision, but it did determine that the speed of the electrons was within 2×10^{-7} of the speed of light (60 m/s). [See Z. G. T. Guiragossian et al., *Physical Review Letters* **34**, 335 (1975).]

Nearly every time the nuclear or particle physicist enters the laboratory, a direct or indirect test of the momentum and energy relationships of special relativity is made. Principles of special relativity must be incorporated in the design of the high-energy accelerators used by nuclear and particle physicists, so even the construction of these projects gives testimony to the validity of the formulas of special relativity.

For example, consider the capture of a neutron by an atom of hydrogen to form an atom of deuterium or "heavy hydrogen." Energy is released in this process, mostly in the form of electromagnetic radiation (gamma rays). The energy of the gamma rays is measured to be 2.224 MeV. Where does this energy come from?

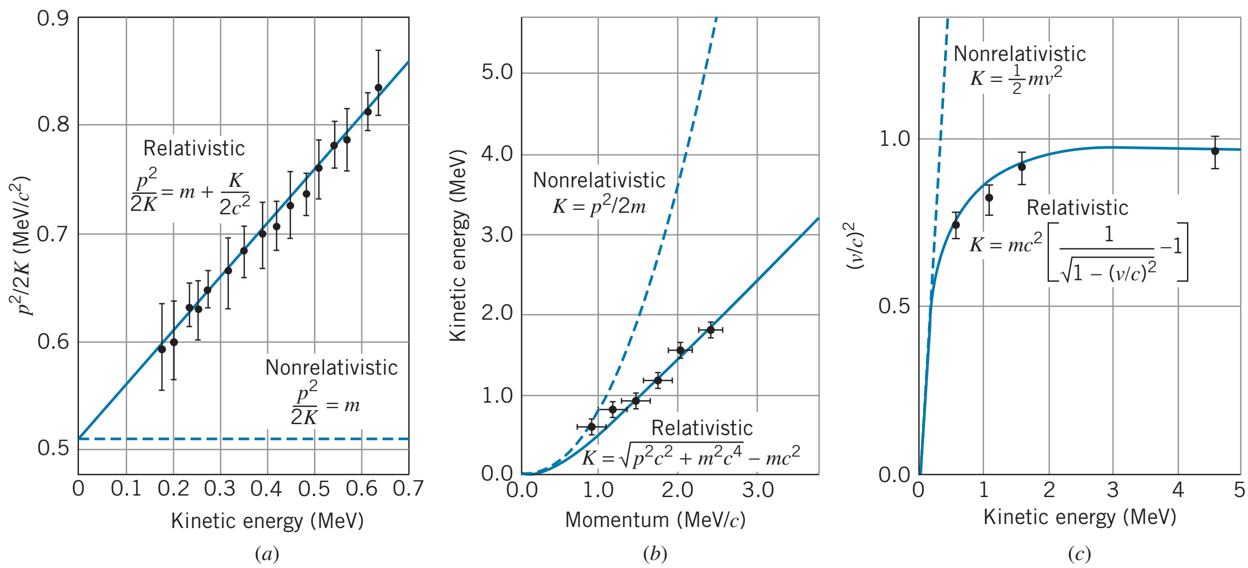


FIGURE 2.28 Confirmation of relativistic kinetic energy relationships. In (*a*) and (*b*) the momentum and energy of radioactive decay electrons were measured simultaneously. In these two independent experiments, the data were plotted in different ways, but the results are clearly in good agreement with the relativistic relationships and in poor agreement with the classical, nonrelativistic relationships. In (*c*) electrons were accelerated to a fixed energy through a large electric field (up to 4.5 million volts, as shown) and the velocities of the electrons were determined by measuring the flight time over 8.4 m. Notice that at small kinetic energies ($K \ll mc^2$), the relativistic and nonrelativistic relationships become identical. [*Sources*: (*a*) K. N. Geller and R. Kollarits, *Am. J. Phys.* **40**, 1125 (1972); (*b*) S. Parker, *Am. J. Phys.* **40**, 241 (1972); (*c*) W. Bertozzi, *Am. J. Phys.* **32**, 551 (1964).].

It comes from the difference in mass when the hydrogen and neutron combine to form deuterium. The difference between the initial and final masses is:

$$\begin{aligned}\Delta m &= m(\text{hydrogen}) + m(\text{neutron}) - m(\text{deuterium})\\ &= 1.007825\ \text{u} + 1.008665\ \text{u} - 2.014102\ \text{u} = 0.002388\ \text{u}\end{aligned}$$

The initial mass of hydrogen plus neutron is greater than the final mass of deuterium by 0.002388 u. The energy equivalent of this change in mass is

$$\Delta E = (\Delta m)c^2 = 2.224\,\text{MeV}$$

which is equal to the energy released as gamma rays.

Similar experiments have been done to test the $E = mc^2$ relationship by measuring the energy released as gamma rays following the capture of neutrons by atoms of silicon and sulfur, and comparing the gamma-ray energies with the difference between the initial and final masses. These experiments are consistent with $E = mc^2$ to a precision of about 4×10^{-7}. [See S. Rainville et al., *Nature* **438**, 1096 (2006).]

Twin Paradox

Although we cannot perform the experiment to test the twin paradox as we have described it, we can do an equivalent experiment. We take two clocks in our laboratory and synchronize them carefully. We then place one of the clocks in an airplane and fly it around the Earth. When we return the clock to the laboratory and compare the two clocks, we expect to find, if special relativity is correct, that the clock that has left the laboratory is the "younger" one—that is, it will have ticked away fewer seconds and appear to run behind its stationary twin. In this experiment, we must use very precise clocks based on the atomic vibrations of cesium in order to measure the time differences between the clock readings, which amount to only about 10^{-7} s. This experiment is complicated by several factors, all of which can be computed rather precisely: the rotating Earth is *not* an inertial frame (there is a centripetal acceleration), clocks on the surface of the Earth are *already* moving because of the rotation of the Earth, and the *general* theory of relativity predicts that a change in the gravitational field strength, which our moving clock will experience as it changes altitude in its airplane flight, will also change the rate at which the clock runs. In this experiment, as in the others we have discussed, the results are entirely in agreement with the predictions of special relativity. [See J. C. Hafele and R. E. Keating, *Science* **177**, 166 (1972).]

In a similar experiment, a cesium atomic clock carried on the space shuttle was compared with an identical clock on the Earth. The comparison was made through a radio link between the shuttle and the ground station. At an orbital height of about 328 km, the shuttle moves at a speed of about 7712 m/s, or $2.5 \times 10^{-5}c$. A clock moving at this speed runs slower than an identical clock at rest by the time dilation factor. For every second the clock is in orbit, it loses 330 ps relative to the clock on Earth; equivalently, it loses about 1.8 μs per orbit. These time intervals can be measured with great precision, and the predicted asymmetric aging was verified to a precision of about 0.1%. [See E. Sappl, *Naturwissenschaften* **77**, 325 (1990).]

Chapter Summary

		Section
Galilean relativity	$x' = x - ut, v_x' = v_x - u$	2.1
Einstein's postulates	(1) The laws of physics are the same in all inertial frames. (2) The speed of light has the same value c in all inertial frames.	2.3
Time dilation	$\Delta t = \dfrac{\Delta t_0}{\sqrt{1 - u^2/c^2}}$ (Δt_0 = proper time)	2.4
Length contraction	$L = L_0\sqrt{1 - u^2/c^2}$ (L_0 = proper length)	2.4
Velocity addition	$v = \dfrac{v' + u}{1 + v'u/c^2}$	2.4
Doppler effect (source and observer separating)	$f' = f\sqrt{\dfrac{1 - u/c}{1 + u/c}}$	2.4
Lorentz transformation	$x' = \dfrac{x - ut}{\sqrt{1 - u^2/c^2}}$, $y' = y,\ z' = z,$ $t' = \dfrac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}}$	2.5

		Section
Lorentz velocity transformation	$v_x' = \dfrac{v_x - u}{1 - v_x u/c^2}$, $v_y' = \dfrac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$, $v_z' = \dfrac{v_z\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$	2.5
Clock synchronization	$\Delta t' = \dfrac{uL/c^2}{\sqrt{1 - u^2/c^2}}$	2.5
Relativistic momentum	$\vec{\mathbf{p}} = \dfrac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$	2.7
Relativistic kinetic energy	$K = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$	2.7
Rest energy	$E_0 = mc^2$	2.7
Relativistic total energy	$E = K + E_0 = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$	2.7
Momentum-energy relationship	$E = \sqrt{(pc)^2 + (mc^2)^2}$	2.7
Extreme relativistic approximation	$E \cong pc$	2.7
Conservation laws	In an isolated system of particles, the total momentum and the relativistic total energy remain constant.	2.8

Questions

1. Explain in your own words what is meant by the term "relativity." Are there different theories of relativity?
2. Suppose the two observers and the rock described in the first paragraph of Section 2.1 were isolated in interstellar space. Discuss the two observers' differing perceptions of the motion of the rock. Is there any experiment they can do to determine whether the rock is moving in any absolute sense?
3. Describe the situation of Figure 2.4 as it would appear from the reference frame of O'.
4. Does the Michelson-Morley experiment show that the ether does not exist or that it is merely unnecessary?
5. Suppose we made a pair of shears in which the cutting blades were many orders of magnitude longer than the handle. Let us in fact make them so long that, when we move the handles at angular velocity ω, a point on the tip of the blade has a tangential velocity $v = \omega r$ that is greater than c. Does this contradict special relativity? Justify your answer.
6. Light travels through water at a speed of about 2.25×10^8 m/s. Is it possible for a particle to travel through water at a speed v greater than 2.25×10^8 m/s?

7. Is it possible to have particles that travel at the speed of light? What does Eq. 2.36 require of such particles?
8. How does relativity combine space and time coordinates into spacetime?
9. Einstein developed the relativity theory after trying unsuccessfully to imagine how a light beam would look to an observer traveling with the beam at speed c. Why is this so difficult to imagine?
10. Explain in your own words the terms *time dilation* and *length contraction*.
11. Does the Moon's disk appear to be a different size to a space traveler approaching it at $v = 0.99c$, compared with the view of a person at rest at the same location?
12. According to the time dilation effect, would the life expectancy of someone who lives at the equator be longer or shorter than someone who lives at the North Pole? By how much?
13. Criticize the following argument. "Here is a way to travel faster than light. Suppose a star is 10 light-years away. A radio signal sent from Earth would need 20 years to make the round trip to the star. If I were to travel to the star in my rocket at $v = 0.8c$, to me the distance to the star is contracted by $\sqrt{1-(0.8)^2}$ to 6 light-years, and at that speed it would take me 6 light-years/$0.8c = 7.5$ years to travel there. The round trip takes me only 15 years, and therefore I travel faster than light, which takes 20 years."
14. Is it possible to synchronize clocks that are in motion relative to each other? Try to design a method to do so. Which observers will believe the clocks to be synchronized?
15. Suppose event A causes event B. To one observer, event A comes before event B. Is it possible that in another frame of reference event B could come before event A? Discuss.
16. Is mass a conserved quantity in classical physics? In special relativity?
17. "In special relativity, mass and energy are equivalent." Discuss this statement and give examples.
18. Which is more massive, an object at low temperature or the same object at high temperature? A spring at its natural length or the same spring under compression? A container of gas at low pressure or at high pressure? A charged capacitor or an uncharged one?
19. Could a collision be elastic in one frame of reference and inelastic in another?
20. (*a*) What properties of nature would be different if there were a relativistic transformation law for electric charge? (*b*) What experiments could be done to prove that electric charge does *not* change with velocity?

Problems

2.1 Classical Relativity

1. You are piloting a small airplane in which you want to reach a destination that is 750 km due north of your starting location. Once you are airborne, you find that (due to a strong but steady wind) to maintain a northerly course you must point the nose of the plane at an angle that is 22° west of true north. From previous flights on this route in the absence of wind, you know that it takes you 3.14 h to make the journey. With the wind blowing, you find that it takes 4.32 h. A fellow pilot calls you to ask about the wind velocity (magnitude and direction). What is your report?
2. A moving sidewalk 95 m in length carries passengers at a speed of 0.53 m/s. One passenger has a normal walking speed of 1.24 m/s. (*a*) If the passenger stands on the sidewalk without walking, how long does it take her to travel the length of the sidewalk? (*b*) If she walks at her normal walking speed on the sidewalk, how long does it take to travel the full length? (*c*) When she reaches the end of the sidewalk, she suddenly realizes that she left a package at the opposite end. She walks rapidly back along the sidewalk at double her normal walking speed to retrieve the package. How long does it take her to reach the package?

2.2 The Michelson-Morley Experiment

3. A shift of one fringe in the Michelson-Morley experiment corresponds to a change in the round-trip travel time along one arm of the interferometer by one period of vibration of light (about 2×10^{-15} s) when the apparatus is rotated by 90°. Based on the results of Example 2.3, what velocity through the ether would be deduced from a shift of one fringe? (Take the length of the interferometer arm to be 11 m.)

2.4 Consequences of Einstein's Postulates

4. The distance from New York to Los Angeles is about 5000 km and should take about 50 h in a car driving at 100 km/h. (*a*) How much shorter than 5000 km is the distance according to the car travelers? (*b*) How much less than 50 h do they age during the trip?
5. How fast must an object move before its length appears to be contracted to one-half its proper length?
6. An astronaut must journey to a distant planet, which is 200 light-years from Earth. What speed will be necessary if the astronaut wishes to age only 10 years during the round trip?

7. The proper lifetime of a certain particle is 100.0 ns. (*a*) How long does it live in the laboratory if it moves at $v = 0.960c$? (*b*) How far does it travel in the laboratory during that time? (*c*) What is the distance traveled in the laboratory according to an observer moving with the particle?
8. High-energy particles are observed in laboratories by photographing the tracks they leave in certain detectors; the length of the track depends on the speed of the particle and its lifetime. A particle moving at $0.995c$ leaves a track 1.25 mm long. What is the proper lifetime of the particle?
9. Carry out the missing steps in the derivation of Eq. 2.17.
10. Two spaceships approach the Earth from opposite directions. According to an observer on the Earth, ship *A* is moving at a speed of $0.753c$ and ship *B* at a speed of $0.851c$. What is the velocity of ship *A* as observed from ship *B*? Of ship *B* as observed from ship *A*?
11. Rocket *A* leaves a space station with a speed of $0.826c$. Later, rocket *B* leaves in the same direction with a speed of $0.635c$. What is the velocity of rocket *A* as observed from rocket *B*?
12. One of the strongest emission lines observed from distant galaxies comes from hydrogen and has a wavelength of 122 nm (in the ultraviolet region). (*a*) How fast must a galaxy be moving away from us in order for that line to be observed in the visible region at 366 nm? (*b*) What would be the wavelength of the line if that galaxy were moving toward us at the same speed?
13. A physics professor claims in court that the reason he went through the red light ($\lambda = 650$ nm) was that, due to his motion, the red color was Doppler shifted to green ($\lambda = 550$ nm). How fast was he going?

2.5 The Lorentz Transformation

14. Derive the Lorentz velocity transformations for v'_x and v'_z.
15. Observer *O* fires a light beam in the *y* direction ($v_y = c$). Use the Lorentz velocity transformation to find v'_x and v'_y and show that O' also measures the value *c* for the speed of light. Assume that O' moves relative to *O* with velocity *u* in the *x* direction.
16. A light bulb at point *x* in the frame of reference of *O* blinks on and off at intervals $\Delta t = t_2 - t_1$. Observer O', moving relative to *O* at speed *u*, measures the interval to be $\Delta t' = t'_2 - t'_1$. Use the Lorentz transformation expressions to derive the time dilation expression relating Δt and $\Delta t'$.
17. A neutral K meson at rest decays into two π mesons, which travel in opposite directions along the *x* axis with speeds of $0.828c$. If instead the K meson were moving in the positive *x* direction with a velocity of $0.486c$, what would be the velocities of the two π mesons?
18. A rod in the reference frame of observer *O* makes an angle of 31° with the *x* axis. According to observer O', who is in motion in the *x* direction with velocity *u*, the rod makes an angle of 46° with the *x* axis. Find the velocity *u*.
19. According to observer *O*, two events occur separated by a time interval $\Delta t = +0.465\ \mu$s and at locations separated by $\Delta x = +53.4$ m. (*a*) According to observer O', who is in motion relative to *O* at a speed of $0.762c$ in the positive *x* direction, what is the time interval between the two events? (*b*) What is the spatial separation between the two events, according to O'?
20. According to observer *O*, a blue flash occurs at $x_b = 10.4$ m when $t_b = 0.124\ \mu$s, and a red flash occurs at $x_r = 23.6$ m when $t_r = 0.138\ \mu$s. According to observer O', who is in motion relative to *O* at velocity *u*, the two flashes appear to be simultaneous. Find the velocity *u*.

2.6 The Twin Paradox

21. Suppose the speed of light were 1000 mi/h. You are traveling on a flight from Los Angeles to Boston, a distance of 3000 mi. The plane's speed is a constant 600 mi/h. You leave Los Angeles at 10:00 A.M., as indicated by your wristwatch and by a clock in the airport. (*a*) According to your watch, what time is it when you land in Boston? (*b*) In the Boston airport is a clock that is synchronized to read exactly the same time as the clock in the Los Angeles airport. What time does that clock read when you land in Boston? (*c*) The following day when the Boston clock that records Los Angeles time reads 10:00 A.M., you leave Boston to return to Los Angeles on the same airplane. When you land in Los Angeles, what are the times read on your watch and on the airport clock?
22. Suppose rocket traveler Amelia has a clock made on Earth. Every year on her birthday she sends a light signal to brother Casper on Earth. (*a*) At what rate does Casper receive the signals during Amelia's outward journey? (*b*) At what rate does he receive the signals during her return journey? (*c*) How many of Amelia's birthday signals does Casper receive during the journey that he measures to last 20 years?
23. Suppose Amelia traveled at a speed of $0.80c$ to a star that (according to Casper on Earth) is 8.0 light-years away. Casper ages 20 years during Amelia's round trip. How much younger than Casper is Amelia when she returns to Earth?
24. Make a drawing similar to Figure 2.20 showing the worldlines of Casper and Amelia from Casper's frame of reference. Divide the world line for Amelia's outward journey into 8 equal segments (for the 8 birthdays that Amelia celebrates). For each birthday, draw a line that represents a light signal that Amelia sends to Casper on her birthday. Do the same for Amelia's return journey. (*a*) According to Casper's time, when does he receive the signal showing Amelia celebrating her 8th birthday after leaving Earth? (*b*) How long does it take for Casper to receive the signals showing Amelia celebrating birthdays 9 through 16?

2.7 Relativistic Dynamics

25. (*a*) Using the relativistically correct final velocities for the collision shown in Figure 2.21*a* ($v'_{1f} = -0.585c, v'_{2f} = +0.294c$), show that relativistic kinetic energy is conserved

according to observer O'. (*b*) Using the relativistically correct final velocities for the collision shown in Figure 2.21*b* ($v_{1f} = -0.051c, v_{2f} = +0.727c$), show that relativistic kinetic energy is conserved according to observer O.

26. Find the momentum, kinetic energy, and total energy of a proton moving at a speed of $0.756c$.
27. An electron is moving with a kinetic energy of 1.264 MeV. What is its speed?
28. The work-energy theorem relates the change in kinetic energy of a particle to the work done on it by an external force: $\Delta K = W = \int F\,dx$. Writing Newton's second law as $F = dp/dt$, show that $W = \int v\,dp$ and integrate by parts using the relativistic momentum to obtain Eq. 2.34.
29. For what range of velocities of a particle of mass m can we use the classical expression for kinetic energy $\frac{1}{2}mv^2$ to within an accuracy of 1%?
30. For what range of velocities of a particle of mass m can we use the extreme relativistic approximation $E = pc$ to within an accuracy of 1%?
31. Use Eqs. 2.32 and 2.36 to derive Eq. 2.39.
32. Use the binomial expansion $(1+x)^n = 1 + nx + [n(n-1)/2!]x^2 + \cdots$ to show that Eq. 2.34 for the relativistic kinetic energy reduces to the classical expression $\frac{1}{2}mv^2$ when $v \ll c$. This important result shows that our familiar expressions are correct at low speeds. By evaluating the first term in the expansion beyond $\frac{1}{2}mv^2$, find the speed necessary before the classical expression is off by 0.01%.
33. (*a*) According to observer O, a certain particle has a momentum of 817 MeV/c and a total relativistic energy of 1125 MeV. What is the rest energy of this particle? (*b*) An observer O' in a different frame of reference measures the momentum of this particle to be 953 MeV/c. What does O' measure for the total relativistic energy of the particle?
34. An electron is moving at a speed of $0.81c$. By how much must its kinetic energy increase to raise its speed to $0.91c$?
35. What is the change in mass when 1 g of copper is heated from 0 to 100°C? The specific heat capacity of copper is 0.40 J/g·K.
36. Find the kinetic energy of an electron moving at a speed of (*a*) $v = 1.00 \times 10^{-4}c$; (*b*) $v = 1.00 \times 10^{-2}c$; (*c*) $v = 0.300c$; (*d*) $v = 0.999c$.
37. An electron and a proton are each accelerated starting from rest through a potential difference of 10.0 million volts. Find the momentum (in MeV/c) and the kinetic energy (in MeV) of each, and compare with the results of using the classical formulas.
38. In a nuclear reactor, each atom of uranium (of atomic mass 235 u) releases about 200 MeV when it fissions. What is the change in mass when 1.00 kg of uranium-235 is fissioned?

2.8 Conservation Laws in Relativistic Decays and Collisions

39. A π meson of rest energy 139.6 MeV moving at a speed of $0.906c$ collides with and sticks to a proton of rest energy 938.3 MeV that is at rest. (*a*) Find the total relativistic energy of the resulting composite particle. (*b*) Find the total linear momentum of the composite particle. (*c*) Using the results of (*a*) and (*b*), find the rest energy of the composite particle.
40. An electron and a positron (an antielectron) make a head-on collision, each moving at $v = 0.99999c$. In the collision the electrons disappear and are replaced by two muons ($mc^2 = 105.7$ MeV), which move off in opposite directions. What is the kinetic energy of each of the muons?
41. It is desired to create a particle of mass 9700 MeV/c^2 in a head-on collision between a proton and an antiproton (each having a mass of 938.3 MeV/c^2) traveling at the same speed. What speed is necessary for this to occur?
42. A particle of rest energy mc^2 is moving with speed v in the positive x direction. The particle decays into two particles, each of rest energy 140 MeV. One particle, with kinetic energy 282 MeV, moves in the positive x direction, and the other particle, with kinetic energy 25 MeV, moves in the negative x direction. Find the rest energy of the original particle and its speed.

2.9 Experimental Tests of Special Relativity

43. In the muon decay experiment discussed in Section 2.9 as a verification of time dilation, the muons move in the lab with a momentum of 3094 MeV/c. Find the dilated lifetime in the laboratory frame. (The proper lifetime is 2.198 μs.)
44. Derive the relativistic expression $p^2/2K = m + K/2c^2$, which is plotted in Figure 2.28*a*.

General Problems

45. Suppose we want to send an astronaut on a round trip to visit a star that is 200 light-years distant and at rest with respect to Earth. The life support systems on the spacecraft enable the astronaut to survive at most 20 years. (*a*) At what speed must the astronaut travel to make the round trip in 20 years of spacecraft time? (*b*) How much time passes on Earth during the round trip?
46. A "cause" occurs at point 1 (x_1, t_1) and its "effect" occurs at point 2 (x_2, t_2). Use the Lorentz transformation to find $t'_2 - t'_1$, and show that $t'_2 - t'_1 > 0$; that is, O' can never see the "effect" coming before its "cause."
47. Observer O sees a red flash of light at the origin at $t = 0$ and a blue flash of light at $x = 3.26$ km at a time $t = 7.63\ \mu$s. What are the distance and the time interval between the flashes according to observer O', who moves relative to O in the direction of increasing x with a speed of $0.625c$?

Assume that the origins of the two coordinate systems line up at $t = t' = 0$.

48. Several spacecraft (A, B, C, and D) leave a space station at the same time. Relative to an observer on the station, A travels at $0.60c$ in the x direction, B at $0.50c$ in the y direction, C at $0.50c$ in the negative x direction, and D at $0.50c$ at 45° between the y and negative x directions. Find the velocity components, directions, and speeds of B, C, and D as observed from A.
49. Observer O sees a light turn on at $x = 524$ m when $t = 1.52\ \mu$s. Observer O' is in motion at a speed of $0.563c$ in the positive x direction. The two frames of reference are synchronized so that their origins match up ($x = x' = 0$) at $t = t' = 0$. (*a*) At what time does the light turn on according to O'? (*b*) At what location does the light turn on in the reference frame of O'?
50. Suppose an observer O measures a particle of mass m moving in the x direction to have speed v, energy E, and momentum p. Observer O', moving at speed u in the x direction, measures v', E', and p' for the same object. (*a*) Use the Lorentz velocity transformation to find E' and p' in terms of m, u, and v. (*b*) Reduce $E'^2 - (p'c)^2$ to its simplest form and interpret the result.
51. Repeat Problem 50 for the mass moving in the y direction according to O. The velocity u of O' is still along the x direction.
52. Consider again the situation described in Section 2.6. Amelia's friend Bernice leaves Earth at the same time as Amelia and travels in the same direction at the same speed, but Bernice continues in the original direction when Amelia reaches the planet and turns her ship around. (*a*) From Bernice's frame of reference, Casper is moving at a velocity of $-0.60c$. Draw Casper's worldline in Bernice's frame of reference. (*b*) Casper celebrates 20 birthdays during Amelia's journey. In Bernice's frame of reference, how long does it take for Casper to celebrate 20 birthdays? (*c*) In Bernice's frame of reference, draw a worldline representing Amelia's outbound journey to the planet. (*d*) Calculate Amelia's velocity during her return journey as observed from Bernice's frame of reference, and draw a worldline showing Amelia's return journey. Amelia's and Casper's worldlines should intersect when Amelia return to Earth. (*e*) Divide Casper's worldline into 20 segments, representing his birthdays. He sends a light signal to Amelia on each birthday. Amelia receives a light signal from Casper just as she arrives at the planet. On which birthday did Casper send this signal? (*f*) Amelia sends Casper a light signal on her 8th birthday. Draw a line on your diagram representing this light signal. When does Casper receive this signal?
53. Electrons are accelerated to high speeds by a two-stage machine. The first stage accelerates the electrons from rest to $v = 0.99c$. The second stage accelerates the electrons from $0.99c$ to $0.999c$. (*a*) How much energy does the first stage add to the electrons? (*b*) How much energy does the second stage add in increasing the velocity by only 0.9%?
54. A beam of 1.35×10^{11} electrons/s moving at a speed of $0.732c$ strikes a block of copper that is used as a beam stop. The copper block is a cube measuring 2.54 cm on edge. What is the temperature increase of the block after one hour?
55. An electron moving at a speed of $v_i = 0.960c$ in the positive x direction collides with another electron at rest. After the collision, one electron is observed to move with a speed of $v_{1f} = 0.956c$ at an angle of $\theta_1 = 9.7^\circ$ with the x axis. (*a*) Use conservation of momentum to find the velocity (magnitude and direction) of the second electron. (*b*) Based only on the original data given in the problem, use conservation of energy to find the speed of the second electron.
56. A pion has a rest energy of 135 MeV. It decays into two gamma ray photons, bursts of electromagnetic radiation that travel at the speed of light. A pion moving through the laboratory at $v = 0.98c$ decays into two gamma ray photons of equal energies, making equal angles θ with the original direction of motion. Find the angle θ and the energies of the two gamma ray photons.
57. Consider again the decay described in Example 2.16 and determine the energies of the two pi mesons emitted in the decay of the K meson by first making a Lorentz transformation to a reference frame in which the initial K meson is at rest. When a K meson at rest decays into two pi mesons, they move in opposite directions with equal and opposite velocities, so they share the decay energy equally. Find the energies and velocities of the two pi mesons in the K meson's rest frame. Then transform back to the lab frame to find their kinetic energies.