fiziks

Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICS & PHYSICAL SCIENCES

Modern Physics

(IIT-JAM/JEST/TIFR/M.Sc Entrance)

Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

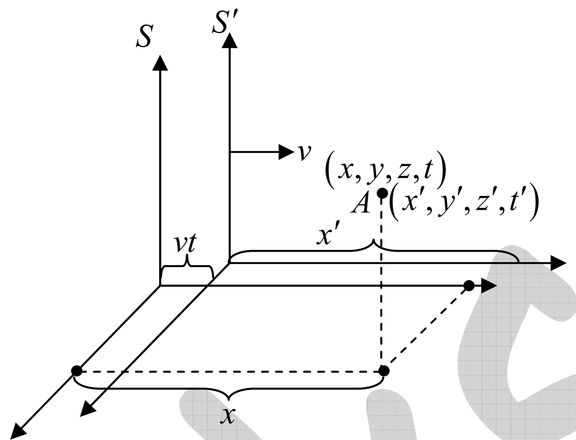
Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

Content Modern Physics

Nuclear Physics

1. Special Theory of Relativity

1.1Galilean Transformations



A frame S' which is moving with constant velocity v relative to an inertial frames S, which is itself inertial.

$$\vec{r}\,' = \vec{r} - \vec{v}t, \qquad x' = x - vt,\ y' = y,\ z' = z,\ t' = t$$

The above transformation of co-ordinates from one inertial frame to another and they are referred as Galilean transformations.

And inverse Galilean transformation is given by

$$x = x' + vt, \qquad y = y',\ z = z',\ t = t'$$

The velocity transformation is given

$$\vec{r} = \vec{r}\,' + \vec{v}t \Rightarrow \frac{d\vec{r}}{dt} = \vec{v} + \frac{d\vec{r}\,'}{dt} \Rightarrow \vec{u} = \vec{v} + \vec{u}\,'$$

The acceleration transformation is given $\frac{d^2r}{dt^2} = \frac{d^2r'}{dt^2}$

It is found that acceleration measured on both frame is same. So it is inertial frame.

When velocity transformation is analyzed as $u' = c$ where c is velocity of light.

$$u = c + v.$$

It is seen velocity of light is depended on the reference frame which is physically not accepted. So for high velocity $v \approx c$ Galilean transformation is not adequate.

So for velocity $u \approx c$ there is need for different transformation, which is given by Lorentz transformation.

1.2 Lorentz Transformation

1.2.1 Postulates of Special Theory of Relativity

(i) There is no universal frame of reference pervading all of space, so there is no such thing as "absolute motion".

(ii) The law of physics are the same in all frames of reference moving at constant velocity to one another.

(iii) The speed of light in free space has the same value for all inertial observers.

1.2.2 Derivation of Lorentz Transformation

Lorentz transformation have to such that

(a) It is linear in x and x' so that a single event in frame S corresponds to a single event in frame S'.

(b) For lower velocity it reduces to Galilean transformation.

(c) The inverse transformation exists.

Let us assume

$$x' = k(x - vt),\ x = k(x' + vt'),\ y' = y \text{ and } z' = z$$

Put the value $x' = k(x - vt)$ in $x = k(x' + vt)$ one will get

$$x = k^2(x - vt) + kvt' \text{ and } t' = kt + \left(\frac{1-k^2}{kv}\right)x$$

Now, $x = ct$ in S frame and $x' = ct'$ in S' frame

$$k(x - vt) = ckt + \left(\frac{1-k^2}{kv}\right)cx$$

Solving these equation for k then

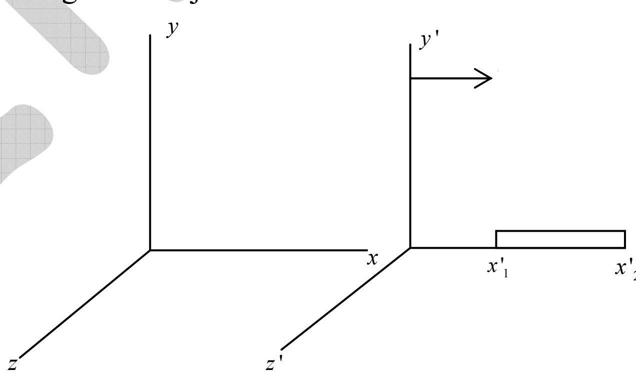
$$k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz Transformation	Inverse Lorentz Transformation
$x' = \dfrac{x - vt}{\sqrt{1-\dfrac{v^2}{c^2}}}$	$x = \dfrac{x' + vt}{\sqrt{1-\dfrac{v^2}{c^2}}}$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = \dfrac{t - \dfrac{vx}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}$	$t = \dfrac{t' + \dfrac{vx'}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}$

1.3 Consequences of Lorentz Transformation

1.3.1 Length Contraction

In order to measure the length of an object in motion, relative to observer, the position of two end points recorded simultaneously, the length of object in direction of motion appeared smaller to observer.



$l_0 = x'_2 - x'_1,\ l = x_2 - x_1$

$x'_2 - x'_1 = \gamma(x_1 - vt) - \gamma(x_2 - vt)$

$(x'_2 - x'_1) = \gamma(x_2 - x_1)$

$(x'_2 - x'_1) = \dfrac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$

$(x_2 - x_1) = \sqrt{1-\dfrac{v^2}{c^2}}\ (x'_2 - x'_1) \Rightarrow l = l_o\sqrt{1-\dfrac{v^2}{c^2}}$

Thus $l < l_0$, this means that the length of rod as measured by an observer relative to which rod is in motion, is smaller than its proper length.

Such a contraction of length in direction of motion relative to observer is called Lorentz Fitzgerald contradiction.

1.3.2 Time Dilation

When two observers are in relative uniform motion and uninfluenced by any gravitational mass, the point of view of each will be that the other's (moving) clock is ticking at a *slower* rate than the local clock. The faster the relative velocity, the greater the magnitude of time dilation. This case is sometimes called special relativistic time dilation.

A clock being at rest in the S' frame measures the time t_2' and t_1' of two events occurring at a fixed position x'. The time interval Δt measures from S frame appears slow (Δt_0) from S' frame i.e. to the observer the moving clock will appear to go slow.

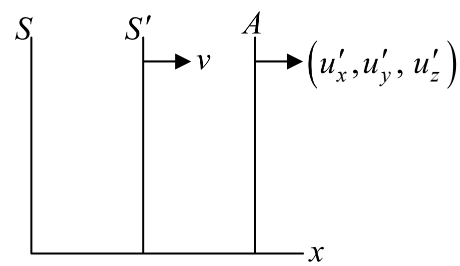
$$\Delta t' = t'_2 - t'_1 = \Delta t_0 \Rightarrow \Delta t = t_2 - t_1 = \gamma\left(t'_2 + \frac{vx'}{c^2}\right) - \gamma\left(t_1 + \frac{vx'}{c^2}\right)$$

$$\Rightarrow \Delta t = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\Delta t_0 \rightarrow$ Proper time, $\Delta t \rightarrow$ time interval measured from S frame $\Delta t > \Delta t_0$

1.4 Relative Velocity

There is one inertial frame S, S' is another inertial frame moving with respect to S in x-direction and A is another inertial frame which is moving with respect to S' with velocity component $\left(u'_x, u'_y, u'_z\right)$.



So, $u'_x = \frac{dx'}{dt'}$ $\quad u'_y = \frac{dy'}{dt'}$ $\quad u'_z = \frac{dz'}{dt'}$

The velocity component of A from S frame is given by

$$u_x = \frac{dx}{dt}, \qquad u_y = \frac{dy}{dt}, \qquad u_z = \frac{dz}{dt}$$

From the inverse Lorentz Transformation,

$$x = \gamma\left(x' + vt'\right), \; y = y', z = z', \; t = \gamma\left(t' + \frac{vx'}{c^2}\right) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Differentiating both sides, we get

$$dx = \gamma\left(dx' + vdt'\right), \qquad dy = dy', \qquad dz = dz', \qquad dt = \gamma\left(dt' + \frac{vdx'}{c^2}\right)$$

$$u_x = \frac{dx}{dt} = \gamma\left(\frac{dx' + vdt'}{dt' + \frac{vdx'}{c^2}}\right) = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \Rightarrow u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}$$

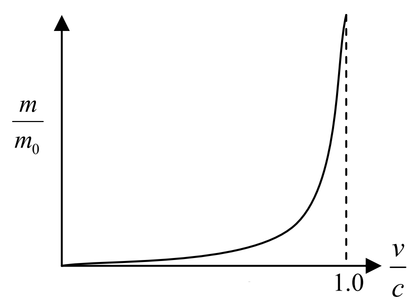
$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \Rightarrow u_y = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \Rightarrow u_z = \frac{u'_z}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}$$

Similarly $u'_x = \dfrac{u_x - v}{1 - \frac{vu_x}{c^2}}, \quad u'_y = \dfrac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \quad u'_z = \dfrac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$

1.5 Relativistic Mass

The mass of a body moving at the speed v relative to an observer is larger than its mass when at rest relative to the observer by the factor $\dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$.



Thus $$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is rest mass of body and m is observed mass.

Relativistic Momentum

$$p = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

1.6 Relativistic Second Law of Motion

$$F=\frac{dp}{dt}=\frac{d}{dt}(mv)=\frac{d}{dt}\left(\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

1.7 Relativistic Energy

Einstein suggested if m is relativistic mass of body then relativistic energy E is given by

$$E=mc^2 \quad \text{and} \quad E=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Rest Energy

If rest mass of particle is m_0 then rest mass energy is given by m_oc^2

Relativistic Kinetic Energy

$$K=mc^2-m_0c^2 \qquad \Rightarrow K=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}-m_0c^2$$

Relationship between total Energy and Momentum

$$E=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad p=\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow E^2=m_0^2c^4+p^2c^2$$

$$\Rightarrow E=\left(m_0^2c^4+p^2c^2\right)^{1/2}$$

1.8 The Doppler Effect in Light

The Doppler Effect in sound evidently varies depending on whether the source, or the observer, or both are moving, which appears to violate the principle of relativity: all that should count is the relative motion of source and observer. But sound waves occur only in a material medium such as air or water, and this medium is itself a frame of reference with respect to which motions of source and observer are measurable. Hence there is no contradiction. In the case of light, however, no medium is involved and only relative motion of source and observer is meaningful. The Doppler Effect in light must therefore differ from that in sound.

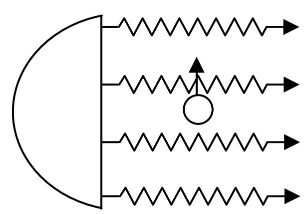
We can analyze the Doppler Effect in light by considering a light source as a clock that ticks ν_0 times per second and emits a wave of light with each tick. We will examine the three situations shown in figure given below.

1.8.1 Transverse Doppler Effect in Light

The observer is moving perpendicular to a line between him and the light source. The proper time between ticks is $t_0 = 1/\nu_0$, so between one tick and the next time

$$t = t_0 / \sqrt{1 - v^2 / c^2}$$

elapses in the reference frame of the observer.

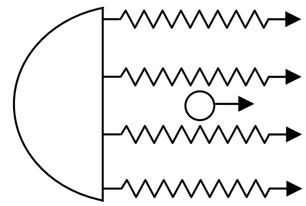


The frequency he finds is accordingly

$$\nu(\text{transverse}) = \frac{1}{t} = \frac{\sqrt{1 - v^2 / c^2}}{t_0}$$

The observed frequency ν is always less than the source frequency ν_0.

1.8.2 Observer and Source Moving Apart



The observer is receding from the light source. Now the observer travels the distance vt away from the source between ticks, which mean that the light wave from a given tick takes vt/c longer to reach him than the previous one. Hence the total time between the arrivals of successive waves is

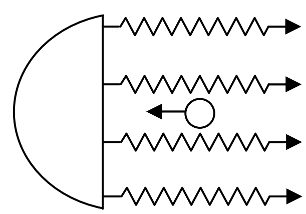
$$T = t + \frac{vt}{c} = t_0 \frac{1+v/c}{\sqrt{1-v^2/c^2}} = t_0 \frac{\sqrt{1+v/c}\sqrt{1+v/c}}{\sqrt{1+v/c}\sqrt{1-v/c}} = t_0\sqrt{\frac{1+v/c}{1-v/c}}$$

and the observed frequency is

$$\nu(\text{receding}) = \frac{1}{T} = \frac{1}{t_0}\sqrt{\frac{1-v/c}{1+v/c}} = \nu_0\sqrt{\frac{1-v/c}{1+v/c}}$$

the observed frequency ν is lower than the source frequency ν_0. Unlike the case of sound waves, which propagate relative to a material medium, it makes no difference whether the observer is moving away from the source or the source is moving away from the observer.

1.8.3 Observer and Source Moving Together



The observer is approaching the light source. The observer here travels the distance vt toward the source between ticks, so each light wave takes vt/c less time to arrive than the previous one. In this case $T = t - vt/c$ and the result is

$$\nu(\text{approaching}) = \nu_0\sqrt{\frac{1+v/c}{1-v/c}}$$

The observed frequency is higher than the source frequency. Again, the same formula holds for motion of the source toward the observer.

1.9 Four Vectors and Relativistic Invariance

- Four position vector $ds = (dx, dy, dz, icdt)$
- Four velocity vector $\frac{ds}{dt} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{icdt}{d\tau}\right) \quad u = \gamma\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{icdt}{dt}\right)$

$$u = \gamma(u_x, u_y, u_z, ic) \quad \Rightarrow u = \gamma(\vec{u}, ic)$$

- Four momentum – Four Energy vector

$$P = \gamma(m_0 u, im_0 c) = (m\vec{u}, imc) = \left(\vec{P}, \frac{imc^2}{c}\right) \Rightarrow \vec{p} = \left(\vec{p}, \frac{iE}{c}\right)$$

- Four Force: $F = \gamma\left(\vec{F}, ic\frac{dm}{dt}\right)$

Four Dimensional Space Time Continuums

The square of interval is represented as

$$S_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2$$

$$S_{12}^2 = r_{12}^2 - c^2(t_2 - t_1)^2$$

(i) Space like Intervals: Time separation between the two events is less than the time taken by light in covering the distance between them.

$$\frac{r_{12}}{c} > (t_2 - t_1)$$

(ii) Time like intervals: Time separation between two events is more than the time taken by light in covering the distance between them

$$\frac{r_{12}}{c} < (t_2 - t_1)$$

(iii) Light-like intervals: Time separation between two events is equal to time taken by light in covering the distance between them

$$\frac{r_{12}}{c} = (t_2 - t_1)$$

Example: Determine the length and the orientation of a rod of length l_0 in a frame of reference which is moving with v velocity in a direction making θ angle with rod.

Solution:

Proper length of the rod in the direction of moving frame $l_{x_0} = l_0 \cos\theta$, the length measured in the moving frame $l_x = l_{0_x} \cos\theta\sqrt{1-\frac{v^2}{c^2}}$ and $l_y = l_0 \sin\theta$

$$|l| = \sqrt{l_0^2 \cos^2\theta\left(1-\frac{v^2}{c^2}\right) + l_0^2 \sin^2\theta} = l_0\sqrt{\cos^2\theta\left(1-\frac{v^2}{c^2}\right) + \sin^2\theta}$$

$$\tan\theta' = \frac{l_y}{l_x} = \frac{l_0 \sin\theta}{l_0 \cos\theta\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \tan\theta' = \gamma\tan\theta \Rightarrow \theta' = \tan^{-1}\left(\gamma\tan\theta\right)$$

Example: A cube of density ρ_o in rest frame is moving with velocity v with respect to observer parallel to one of its edge. What is density measured by observer?

Solution:

In rest frame $\frac{m_0}{V_0} = \rho_0$ and from moving frame $\rho = \frac{\text{mass}}{\text{volume}}$ where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

and relativistic volume $V = \left(l_0 l_0\right)\cdot l_0\sqrt{1-\frac{v^2}{c^2}} = V_0\sqrt{1-\frac{v^2}{c^2}}$.

$$\rho = \frac{\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}{V_0\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{V_0\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2} \Rightarrow \rho = \frac{\rho_0}{1-\frac{v^2}{c^2}} \qquad \because \frac{m_0}{V_0} = \rho_0$$

Example: A spaceship is moving away from the earth with velocity 0.5c fires a rocket whose velocity relative to space is 0.5c (a) Away from earth (b) Towards the earth. Calculate velocity of the rocket as observed from the earth in two cases.

Solution:

(a) $u' = 0.5c,\ v = 0.5c \qquad \Rightarrow u = \dfrac{u' + v}{1 + \dfrac{u'v}{c^2}} = 0.8c$

(b) $u' = -0.5c,\ v = 0.5c \Rightarrow u = \dfrac{-0.5c + 0.5c}{1 - \dfrac{(0.5c)(0.5c)}{c^2}} = 0$

Example: Show that the rest mass of particle of momentum p and kinetic energy T given by $m_0 = \dfrac{p^2c^2 - T^2}{2Tc^2}$

Solution: $E = E_K + E_0 \Rightarrow E = T + m_0c^2$

$$E^2 = p^2c^2 + m_0^2c^4 \Rightarrow \quad m_0 = \frac{p^2c^2 - T^2}{2Tc^2}$$

Example: A π-meson at rest mass m_π decays into μ-meson of mass m_μ and neutrino of mass m_ν . Find the total energy of μ-meson.

Solution: $E_\pi = m_\pi c^2$, $E_\mu^2 = p_\mu^2c^2 + m_\mu^2c^4$, $\quad E_\nu^2 = p_\nu^2c^2 + m_\nu^2c^4$

From conservation of energy $\mathrm{E}_\pi = \mathrm{E}_\mu + \mathrm{E}_\nu$ and from conservation of momentum $P_\nu = -P_\mu = P$

Using $E_\mu^2 - E_\nu^2 = \left(m_\mu^2 - m_\nu^2\right)c^4$ and $E_\pi = E_\mu + E_\nu$

One will get $\dfrac{E_\mu^2 - E_\nu^2}{E_\mu + E_\nu} = \dfrac{\left(m_\mu^2 - m_\nu^2\right)c^2}{m_\pi} \Rightarrow E_\mu - E_\nu = \dfrac{\left(m_\mu^2 - m_\nu^2\right)c^2}{m_\pi}$

$$E_\mu = \frac{1}{2m_\pi}\left[m_\pi^2 + m_\mu^2 - m_\nu^2\right]c^2$$

Example: The rate of a clock in spaceship Suryashakti is observed from earth to be $\frac{3}{5}$ at the rate of the clock on earth.

(a) What is the speed of spaceship Suryashakti relative to earth?

(b) If rate of clock in spaceship Akashganga is observed from earth to be $\frac{5}{13}$ at the rate of the clocks on earth. If both Aakashganga and Suryashakti are moving in same direction relative to someone on earth, then what is the speed of Aakashganga relative to Suryashakti?

Solution: (a)

From the time dilation $\Delta t_1 = \dfrac{\Delta t_{0_1}}{\sqrt{1-\frac{v_1^2}{c^2}}} \Rightarrow \dfrac{5}{3} = \dfrac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} \Rightarrow \dfrac{25}{9} = \dfrac{1}{\left(1-\frac{v_1^2}{c^2}\right)} \Rightarrow v_1 = \dfrac{4}{5}c$

Velocity of Suryashakti is $\frac{4}{5}c$

(b) Similarly

$$\Delta t_2 = \frac{\Delta t_{02}}{\sqrt{1-\frac{v_2^2}{c^2}}} \Rightarrow \frac{13}{5} = \frac{1}{\sqrt{1-\frac{v_2^2}{c^2}}} \Rightarrow \frac{5}{13} = \sqrt{1-\frac{v_2^2}{c^2}} \Rightarrow \frac{25}{169} = 1-\frac{v_2^2}{c^2} \Rightarrow v_2 = \frac{12}{13}c$$

Velocity of Akashganga with respect to Suryashakti is given $v = \dfrac{v_2 - v_1}{1-\frac{v_2 v_1}{c^2}}$

Hence both are moving in same direction $= \dfrac{\frac{4}{5}c - \frac{12}{13}c}{1-\frac{4}{5}\frac{12}{13}} = -\dfrac{8}{17}c$

Velocity of Aakashganga with respect to Suryashakti is $-\frac{8}{17}c$

Example: In the laboratory frame a particle P at rest mass m_0 is moving in the positive x-direction with speed of $\frac{5c}{19}$. It approaches an identical particle Q moving in the negative x-direction with a speed of $\frac{2c}{5}$.

(a) What is speed of the particle P in the rest frame of the particle Q.

(b) What is Energy of the particle P in the rest frame of the particle Q.

Solution: (a) $u'_x = \frac{5}{19}c$, $v = \frac{2}{5}c \quad \Rightarrow u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{3}{5}c$

(b) $v_1 = \frac{3c}{5} \quad \Rightarrow E = \frac{m_0 c^2}{\sqrt{1 - \frac{v_1^2}{c^2}}} = \frac{5}{4} m_0 c^2$

Example: The mass m of moving particle is $\frac{2m_0}{\sqrt{3}}$, where m_o is its rest mass. Then what is linear momentum of particle?

Solution: mass $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \frac{2m_0}{\sqrt{3}} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow v = \frac{c}{2}$

$$P = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} v = \frac{m_0}{\sqrt{1 - \frac{1}{4}}} \times \frac{1}{2} c \Rightarrow P = \frac{m_0 c}{\sqrt{3}}$$

Example: A distant galaxy in constellation Hydra is receding from the earth at 6.12×10^7 m/s by how much is a green spectral line of wavelength 500 nm emitted by this galaxy shifted towards the red at spectrum.

Solution: $\lambda = \lambda_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \quad \Rightarrow \lambda = 500\sqrt{\frac{1 + 0.204}{1 - 0.204}} = 615 nm \quad \because v = 0.204c, \quad \lambda_0 = 500$

which is orange part of spectrum. The shift is $\lambda - \lambda_0 = 115$ nm.

Multiple Choice Questions (MCQ)

Q1. In a system of units in which the velocity of light $c=1$, which of the following is a Lorentz transformation?

(a) $x'=4x, y=y', z'=z, t'=0.25t$

(b) $x'=x-0.5t, y=y', z'=z, t'=t+x$

(c) $x'=1.25x-0.75t, y'=y, z'=z, t'=0.75t-1.25x$

(d) $x'=1.25x-0.75t, y'=y, z'=z, t'=1.25t-0.75x$

Q2. A circle of radius 5 m lies at rest in x - y plane in the laboratory. For an observer moving with a uniform velocity v along the y direction, the circle appears to be an ellipse with an equation

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

The speed of the observer in terms of the velocity of light c is,

(a) $\frac{9c}{25}$ (b) $\frac{3c}{5}$ (c) $\frac{4c}{5}$ (d) $\frac{16c}{25}$

Q3. An electron is moving with a velocity of $0.85c$ in the same direction as that of a moving photon. The relative velocity of the electron with respect to photon is

(a) c (b) $-c$

(c) $0.15c$ (d) $-0.15c$

Q4. The area of a disc in its rest frame S is equal to 1 (in some units). The disc will appear distorted to an observer O moving with a speed u with respect to S along the plane of the disc. The area of the disc measured in the rest frame of the observer O is (c is the speed of light in vacuum)

(a) $\left(1-\frac{u^2}{c^2}\right)^{1/2}$ (b) $\left(1-\frac{u^2}{c^2}\right)^{-1/2}$ (c) $\left(1-\frac{u^2}{c^2}\right)$ (d) $\left(1-\frac{u^2}{c^2}\right)^{-1}$

Q5. A light beam is propagating through a block of glass with index of refraction n. If the glass is moving at constant velocity v in the same direction as the beam, the velocity of the light in the glass block as measured by an observer in the laboratory is approximately

(a) $u=\frac{c}{n}+v\left(1-\frac{1}{n^2}\right)$ (b) $u=\frac{c}{n}-v\left(1-\frac{1}{n^2}\right)$

(c) $u=\frac{c}{n}+v\left(1+\frac{1}{n^2}\right)$ (d) $u=\frac{c}{n}$

Q6. If fluid is moving with velocity v with respect to stationary narrow tube .If light pulse enter into fluid in the direction of flow. What is speed of light pulse measured by observer who is stationary with respect to tube?

(a) c (b) $\frac{c}{n}$ (c) $\frac{c}{n}\left(\frac{1+\frac{nv}{c}}{1+\frac{v}{nc}}\right)$ (d) $\frac{c}{n}\left(\frac{1+\frac{v}{nc}}{1+\frac{nv}{c}}\right)$

Q7. A light beam is emitted at an angle θ_0 with respect to the x' in S frame which is moving with velocity $u\hat{i}$ .Then the angle θ the beam makes with respect to x axis in S frame .

(a) $\theta=\theta_0$ (b) $\frac{u\cos\theta_0}{c}$

(c) $\cos\theta=\frac{\cos\theta_0+\frac{u}{c}}{1+\frac{u}{c}\cos\theta_0}$ (d) $\cos\theta=\frac{1+\frac{u\cos\theta_0}{c}}{\cos\theta_0+\frac{u}{c}}$

Q8. The relativistic form of Newton's second law of motion is

(a) $F=\frac{mc}{\sqrt{c^2-v^2}}\frac{dv}{dt}$ (b) $F=\frac{m\sqrt{c^2-v^2}}{c}\frac{dv}{dt}$

(c) $F=\frac{mc^3}{\left(c^2-v^2\right)^{3/2}}\frac{dv}{dt}$ (d) $F=m\frac{c^2-v^2}{c^2}\frac{dv}{dt}$

Q9. Two particles each of rest mass m collide head-on and stick together. Before collision, the speed of each mass was 0.6 times the speed of light in free space. The mass of the final entity is

(A) $5m/4$ (B) $2m$ (C) $5m/2$ (D) $25\ m/8$

Q10. According to the special theory of relativity, the speed v of a free particle of mass m and total energy E is:

(a) $v = c\sqrt{1-\frac{mc^2}{E}}$

(b) $v = \sqrt{\frac{2E}{m}}\left(1+\frac{mc^2}{E}\right)$

(c) $v = c\sqrt{1-\left(\frac{mc^2}{E}\right)^2}$

(d) $v = c\left(1+\frac{mc^2}{E}\right)$

Q11. The velocity of a particle at which the kinetic energy is equal to its rest energy is (in terms of c, the speed of light in vacuum)

(a) $\sqrt{3}c/2$ (b) $3c/4$ (c) $\sqrt{3/5}c$ (d) $c/\sqrt{2}$

Q12. In the laboratory frame, a particle P of rest mass m_0 is moving in the positive x direction with a speed of $\frac{5}{19}c$. It approaches an identical particle Q, moving in the negative x direction with a speed of $\frac{2}{5}c$. The speed of the particle P in the rest frame of the particle Q is

(a) $\frac{7}{95}c$ (b) $\frac{13}{85}c$ (c) $\frac{3}{5}c$ (d) $\frac{63}{95}c$

Q13. The energy of the particle P in the rest frame of the particle Q is

(a) $\frac{1}{2}m_0c^2$ (b) $\frac{5}{4}m_0c^2$ (c) $\frac{19}{13}m_0c^2$ (d) $\frac{11}{9}m_0c^2$

Q14. If $u(x,y,z,t) = f(x+i\beta y - vt) + g(x - i\beta y - vt)$, where f and g are arbitrary and twice differentiable functions, is a solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{ then } \beta \text{ is}$$

(a) $\left(1-\frac{v}{c}\right)^{1/2}$ (b) $\left(1-\frac{v}{c}\right)$ (c) $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ (d) $\left(1-\frac{v^2}{c^2}\right)$

Q15. A relativistic particle of mass m and velocity $\frac{c}{2}\hat{z}$ is moving towards a wall. The wall is moving with a velocity $\frac{c}{3}\hat{z}$. The velocity of the particle after it suffers an elastic collision is $v\,\hat{z}$ with v equal to

(a) $c/2$ (b) $c/5$ (c) $c/7$ (d) $c/15$

(All the velocities refer to the laboratory frame of reference.)

Q16. The momentum of an electron (mass m) which has the same kinetic energy as its rest mass energy is (c is velocity of light)

(a) $\sqrt{3}mc$ (b) $\sqrt{2}mc$ (c) mc (d) $mc/\sqrt{2}$

Q17. A particle of mass M decays at rest into a mass less particle and another particle of mass m. The magnitude of the momentum of each of these relativistic particles is:

(a) $\frac{c}{2}\sqrt{M^2-4m^2}$ (b) $\frac{c}{2}\sqrt{M^2+4m^2}$

(c) $\frac{c}{2M}\left(M^2-m^2\right)$ (d) $\frac{c}{2M}\left(M^2+m^2\right)$

Q18. Consider a beam of relativistic particles of kinetic energy K at normal incidence upon a perfectly absorbing surface. The particle flux (number of particles per unit area per unit time) is J and each particle has rest mass m_0. The pressure on the surface is

(a) $\frac{JK}{c}$ (b) $\frac{J\sqrt{K\left(K+m_0c^2\right)}}{c}$

(c) $\frac{J\left(K+m_0c^2\right)}{c}$ (d) $\frac{J\sqrt{K\left(K+2m_0c^2\right)}}{c}$

Multiple Select Type Questions (MSQ)

Q19. A particle of mass $m_1 = 3kg$ moving at velocity of $u_1 = +4m/\sec$ along the x axis of frame S, approaches a second Particle of mass $m_2 = 1kg$, moving at velocity $u_2 = -3m/\sec$ along the axis after the collision the m_2 has velocity $U_2 = +3m/\sec$ along the x axis then which of the following is correct

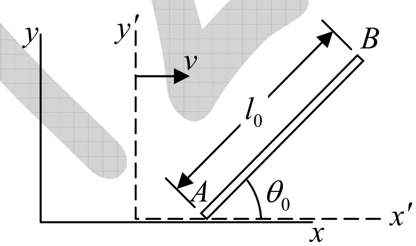
(a) The velocity of m_1 is $U_1 = +2m/\sec$

(b) The momentum of the system before collision and after collision is $9kg - m/\sec$

(c) If the observer S' who has velocity $v = +2m/\sec$ relative to S frame the momentum measured before collision is $+1kg - m/\sec$

(d) If the observer S' who has velocity $v = +2m/\sec$ relative to S frame the momentum measured after collision is $-1kg - m/\sec$

Q20. A rod of length l_0 lies in plane with respect to inertial frame K. The length inclined at angle of θ_0 with respect to horizontal as shown in figure below. If rod is moving in horizontal direction with speed v with respect to observer if $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ .then



(a) The length appeared to observer $\frac{l_0 \cos\theta_0}{\gamma}$

(b) The length of Rod appeared to observer is $l_0\sqrt{1-\frac{v^2\cos^2\theta}{c^2}}$

(c) The angle that rod makes with the horizontal is $\tan^{-1}\left(\gamma \tan\theta_0\right)$

(d) The angle that rod makes with the horizontal is $\tan^{-1}\left(\frac{\tan\theta_0}{\gamma}\right)$

Q21. A train of rest length L_0 and of velocity $v\hat{x}$ is passes a station. At $t'_A = t'_O$, two light pulses are sent from the train's mid point, O, towards the points A and B. At the moment t', two spectators sitting at points A and B receive the light pulses. The

(a)The time to reach light pulse O to A and O to B in the train frame is same and given by $\frac{L_o}{2c}$

(b) The distance traveled by the light pulse from O to A , as seen at the frame of reference of the station is $\frac{L_0}{2}\sqrt{\frac{1+\beta}{1-\beta}}$ where $\beta = \frac{v}{c}$

(c)The distance traveled by the light pulse from O to B, as seen at the frame of reference of the station is $\frac{L_0}{2}\sqrt{\frac{1+\beta}{1-\beta}}$ where $\beta = \frac{v}{c}$

(d) The time needed for the light pulses to travel from point O to point B, as seen in the station's frame. Is $(\Delta t)_B = t_B - t_O = \frac{L_0}{2c}\sqrt{\frac{1-\beta}{1+\beta}}$ where $\beta = \frac{v}{c}$

Q22. The rod of proper length l_0 and two toy train A and B moving with respect table with speed $0.5c$ .rod and toy train A moving in same direction but toy train B moving in opposite direction, then which one of the following observation is correct?

(a) The length of rod measured by observer attached the frame with toy train A is $0.866\ l_0$

(b) The length of rod measured by observer attached the frame table is $.866 l_o$

(c) The length of rod measured by observer attached the frame table is l_o

(d) The length of rod measured by observer attached the frame with toy train B is $0.6\ l_0$

Q23. A particle with mass m and total energy E_0 travels at a constant velocity V which may approach the speed of light. It then collides with a stationary particle with the same mass m, and they are seen to scatter elastically at the relative angle θ with equal kinetic energies. then which of the following are correct.

(a) $\cos\left(\frac{\theta}{2}\right)=\sqrt{\frac{E_0+mc^2}{E_0+3mc^2}}$

(b) $\sin\left(\frac{\theta}{2}\right)=\sqrt{\frac{E_0+mc^2}{E_0+3mc^2}}$

(c) From point of view of classical mechanics $\theta \approx \frac{\pi}{2}$

(d) From point of view of relativistic mechanics $\theta = 0$

Q24. If particle of rest mass m_0 has momentum $2\sqrt{2}m_0c$ then

(a) The velocity of particle is $2\sqrt{2}c$

(b) The velocity of particle is $\frac{2\sqrt{2}c}{3}$

(c) The total energy is $3m_0c^2$

(d) Kinetic energy is $2m_0c^2$

Q25. If The rate of a clock in spaceship "Fiziks" is observed from earth to be $\frac{3}{5}$ of the rate of the clocks on earth then

(a) The speed of spaceship "Fiziks " relative to earth is $\frac{4c}{5}$

(b) The speed of spaceship "Fiziks " relative to earth is $\frac{3c}{5}$

(c) The rate of clock in spaceship "Akashganga" is observed from earth to be $\frac{5}{13}$ of the rate of the clocks on earth. If both Aakashganga and "Fiziks" are moving in the same direction relative to someone on earth, then the speed of Aakashganga relative to "Fiziks" is $\frac{8}{17}c$

(d) The rate of clock in spaceship "Akashganga" is observed from earth to be $\frac{5}{13}$ of the rate of the clocks on earth. If both Aakashganga and "Fiziks" are moving in the opposite direction relative to someone on earth, then the speed of Aakashganga relative to "Fiziks" is $\frac{8}{17}c$

Q26. Consider the decay process $\tau^- \to \pi^- + \nu_\tau$ in the rest frame of the τ^-. The masses of the τ^-, π^- and ν_τ are M_τ M_π and zero respectively. Then which of the following is correct?

(a) The energy of π^- is $\frac{\left(M_\tau^2 + M_\pi^2\right)c^2}{2M_\tau}$

(b) The velocity of π^- is $\left(\frac{M_\tau^2 - M_\pi^2}{M_\tau^2 + M_\pi^2}\right)c$

(c) The velocity of ν_τ is $\left(\frac{M_\pi^2 - M_\tau^2}{M_\tau^2 + M_\pi^2}\right)c$

(d) The magnitude of energy of ν_τ is $\frac{\left(M_\tau^2 - M_\pi^2\right)c^2}{2M_\tau}$

Q27. A particle of rest mass m_o moves with kinetic energy $\frac{m_o c^2}{4}$ disintegrate into two photon with momentum p_1 and p_2 and energy E_1 and E_2 respectively. Assume photon carries momentum p_1 moves in same direction and other photon in opposite direction. Then which of the following is correct

(a) The magnitude of momentum p_1 is $m_o c$

(b) The magnitude of momentum p_2 is $m_o c$

(c) The total kinetic energy of both the photon is $\frac{m_o c^2}{4}$

(d) The value of energy E_2 is $\frac{m_o c^2}{4}$

Q28. A shooter fires a bullet with velocity u in the $\hat{x}$ direction at a target. The target is moving with velocity v in the $\hat{x}$ direction relative to the shooter and is at a distance L from him at the instant the bullet is fired. If $\gamma_1 = 1\sqrt{1 - v^2/c^2}$ and $\gamma_2 = 1\sqrt{1 - u^2/c^2}$

(a) The time that bullet is fired will it take to hit the target in the target's frame of reference is $\gamma_1 L\left(1 - uv/c^2\right)/(u - v)$

(b) The time that bullet is fired will it take to hit the target in the target's frame of reference is $\gamma_2 L\left(1 - uv/c^2\right)/(u - v)$

(c) The time that bullet is fired will it take to hit the target in the bullet's frame of reference is $\gamma_1 L\left(1 - uv/c^2\right)/(v - u)$

(d) The time that bullet is fired will it take to hit the target in the bullet's frame of reference is $\gamma_2 L\left(1 - uv/c^2\right)/(v - u)$

Numerical Answer Type Questions (NAT)

Q29. An Event occurs in S frame at $x = 6\times10^8 m$ and in S' $x' = 6\times10^8 m, t' = 4\sec$ the relative velocity is …………..

Q30. Two events separated by a (spatial) distance $9\times10^9 m$, are simultaneous in one inertial frame. The time interval between these two events in a frame moving with a constant speed 0.8 c (where the speed of light $3\times10^8 m/\sec$) is ……………

Q31. In a certain intertial frame two light pulses are emitted at point $5km$ apart and separated in time by $5\mu\sec$. An observe moving at a speed v along the line joining these points notes that the pulses are simultaneous. Therefore v is………………… c.

Q32. A monochromatic wave propagates in a direction making an angle 60^o with the x-axis in the reference frame of source. The source moves at speed $v = \frac{4c}{5}$ towards the observer. The direction of the ($\cos\theta$) wave as seen by the observer is …………

Q33. A π^0 meson at rest decays into two photons, which move along the x-axis. They are both detected simultaneously after a time, $t = 10$. In an inertial frame moving with a velocity $v = .6c$ in the direction of one of the photons, the time interval between the two detections is ……….. sec

Q34. A rod of proper length l_0 oriented parallel to the x-axis moves with speed $2c/3$ along the x-axis in the S-frame, where c is the speed of the light in free space. The observer is also moving along the x-axis with speed $c/2$ with respect to the S-frame. The length of the rod as measured by the observer is……………… l_0

Q35. If the half-life of an elementary particle moving with speed 0.9c in the laboratory frame is $5\times10^{-8} s$, then the proper half-life is _______________ $\times10^{-8} s.\left(c = 3\times10^8 m/s\right)$

Q36. The muon has mass 105 MeV/c^2 and mean life time 2.2 μs in its rest frame. The mean distance traversed by a muon of energy 315 MeV before decaying is approximately …………. km

Q37. The recently-discovered Higgs boson at the LHC experiment has a decay mode into a photon and a Z boson. If the rest masses of the Higgs and Z boson are $125\ \mathrm{GeV/c^2}$ and $90\ \mathrm{GeV/c^2}$ respectively, and the decaying Higgs particle is at rest, the energy of the photon will approximately be ………………

Q38. A particle of rest mass m has momentum $5mc$ then velocity of the particle is …………… c

Q39. The velocity of a particle at which the kinetic energy is equal to its rest energy is (in terms of c, the speed of light in vacuum) …………………..c

Q40. If particle of rest mass m_0 have kinetic energy is m_0c^2 then total energy is given by …………… m_0c^2

Q41. If particle of rest mass m_0 have kinetic energy is m_0c^2 then velocity is given by …………… c

Q42. A cosmic particle of rest mass m_0 move with speed $.5c$ with respect to Earth .A spaceship moving in opposite direction with same speed $.5c$ The mass of the cosmic particle observed by the observer whose frame is attached to space ship is ………….. m_0

Q43. A particle of rest mass m moving with speed $\frac{c}{2}$ decays into two particles of rest masses $\frac{2}{5}m$ each. The daughter particles move in the same line as the direction of motion of the original particle. The velocities of the daughter particles…………… c

Q44. It is found that pions are radioactive and they are brought to rest half life is measured to be 1.77×10^{-8} sec. A collimated pion beam, leaving the accelerator target at speed of $0.99c$, it is found to drop to half of its original intensity ………….. *meter* from target .

Q45. A rod is moving with a speed of $0.8c$ with respect to stationary observer in a direction at 60^{o} to its own length. The length of the rod observed by the observer is __________ l_o

Solution

MCQ

Ans. 1: (d)

Ans. 2: (c)

Solution: $3 = 5\sqrt{1-\frac{v^2}{c^2}} \Rightarrow v = \frac{4c}{5}$

Ans. 3: (b)

Ans. 4: (a)

Solution: Area of disc from S frame is 1 i.e. $\pi a^2 = 1$ or $\pi a \cdot a = 1$

Area of disc from S' frame is $\pi a \cdot b = \pi a \cdot a\sqrt{1-\frac{u^2}{c^2}} = 1\cdot\sqrt{1-\frac{u^2}{c^2}} = \sqrt{1-\frac{u^2}{c^2}}$

where $b = a\sqrt{1-\frac{u^2}{c^2}}$.

Ans. 5: (a)

Solution: Now $u = \frac{v+\frac{c}{n}}{1+\frac{v\cdot c}{c^2\cdot n}} = \left(v+\frac{c}{n}\right)\left(1+\frac{v}{cn}\right)^{-1} = \left(v+\frac{c}{n}\right)\left(1-\frac{v}{cn}+\frac{v^2}{c^2n^2}\right)$

$$\Rightarrow v-\frac{v^2}{cn}+\frac{v^3}{c^2n^2}+\frac{c}{n}-\frac{v}{n^2}+\frac{v^2}{cn^3} \Rightarrow u = \frac{c}{n}+v\left(1-\frac{1}{n^2}\right)$$

Ans. 6: (c)

Solution: $u' = \frac{c}{n}$ and $u = \frac{u'+v}{1+\frac{u'v}{c^2}}$

$$u = \frac{c}{n}\left(\frac{1+\frac{nv}{c}}{1+\frac{v}{nc}}\right)$$

Ans. 7: (c)

Solution: $u_x = \dfrac{u_x' + v}{1 + u_x' v / c^2} = \dfrac{c\cos\theta_0 + u}{1 + c\cos\theta_0 u / c^2} = \dfrac{c\cos\theta_0 + u}{1 + \cos\theta_0 u / c} \qquad \because u_x' = c\cos\theta_0, \qquad v = u$

$$\cos\theta = \frac{u_x}{c} \Rightarrow \cos\theta = \frac{\cos\theta_0 + \frac{u}{c}}{1 + \frac{u}{c}\cos\theta_0}$$

Ans. 8: (c)

Solution: $P = \dfrac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow F = \dfrac{dP}{dt} = m\dfrac{dv}{dt}\cdot\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} + mv\left(-\dfrac{1}{2}\right)\cdot\dfrac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\cdot\dfrac{-2v}{c^2}\dfrac{dv}{dt}$

$$\Rightarrow F = m\frac{dv}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(1 + \frac{v^2/c^2}{\left(1-\frac{v^2}{c^2}\right)}\right) = \frac{mc^3}{\left(c^2 - v^2\right)^{3/2}}\frac{dv}{dt}$$

Ans. 9: (c)

Solution: From conservation of energy

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = m_1c^2 \Rightarrow \frac{2mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = m_1c^2 \qquad \text{Since } v = 0.6c \Rightarrow m_1 = 5m/2$$

Ans. 10: (c)

Solution: $E = \dfrac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 1 - \dfrac{v^2}{c^2} = \left(\dfrac{mc^2}{E}\right)^2 \Rightarrow \dfrac{v^2}{c^2} = 1 - \dfrac{m^2c^4}{E^2} \Rightarrow v = c\sqrt{1-\left(\dfrac{mc^2}{E}\right)^2}$

Ans. 11: (a)

Solution: $K.E = mc^2 - m_0c^2$, rest mass energy $= m_0c^2$

$$K.E. = \text{rest mass energy} \Rightarrow mc^2 - m_0c^2 = m_0c^2 \Rightarrow mc^2 = 2m_0c^2$$

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}c^2 = 2m_0c^2 \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2 \Rightarrow 4\left(1-\frac{v^2}{c^2}\right) = 1 \Rightarrow 4\frac{v^2}{c^2} = 3 \Rightarrow v = \frac{\sqrt{3}}{2}c$$

Ans. 12: (c)

Solution: $u'_x = \frac{5}{19}c$, $v = \frac{2}{5}c$ $\quad u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{3}{5}c$

Ans. 13: (b)

Solution: $E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{5}{4} m_0 c^2$

Ans.14: (c)

Solution: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \left[f''(x + i\beta y - vt) + g''(x - i\beta y - vt) \right](1 - \beta^2)$

$= \frac{v^2}{c^2} f''(x + i\beta y - vt) + g''(x - i\beta y - vt) \Rightarrow \beta = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$

Ans. 15: (b)

Solution: $v = -\frac{c}{3}\hat{z}$ $\quad u'_x = 0, u'_y = 0, u'_z = \frac{c}{2}$ the speed of particle with respect to wall is

$u_z = \frac{u'_z + v}{1 + \frac{u'_z v}{c^2}} = \frac{c}{5}$

Ans. 16: (a)

Solution: Kinetic energy $T = mc^2$ and $T = E - mc^2$ $\quad E = 2mc^2$

$E^2 = p^2c^2 + m^2c^4$

$p = \sqrt{3}mc$

Ans. 17: (c)

Solution: From conservation of momentum mass less particle and particle of mass m have same momentum p and from conservation of energy. $Mc^2 = \sqrt{p^2c^2 + m^2c^4} + pc$

$p = \frac{c}{2M}\left(M^2 - m^2\right)$

Ans. 18: (d)

Solution: From conservation of energy

$$K+m_oc^2=\sqrt{p^2c^2+m_oc^2} \text{ so momentum } p=\frac{\sqrt{K(K+2m_oc^2}}{c}$$

If particle flux (number of particles per unit area per unit time) is J then pressure

$$P=\frac{F}{A}$$

$$P=\frac{J\sqrt{K\left(K+2m_0c^2\right)}}{c}$$

MSQ

Ans. 19: (a), (b) and (c)

Solution: $m_1u_1+m_2u_2=m_1U_1+m_2U_2=+9kg-m/\sec$ So $U_1=+2m/\sec$

The velocity of particle with respect to observer on S' frame is $u_1'=u_1-v=2m/\sec$

similarly

$u_2'=-5m/\sec$ $U_1'=0m/\sec$ $U_2'=1m/\sec$

Hence S' is inertial frame of reference so $m_1u_1'+m_2u_2'=m_1U_1'+m_2U_2'=1kg-m/\sec$

Ans. 20: (b), (c)

Solution: $\Delta x=\dfrac{l_0\cos\theta_0}{\gamma}, \Delta y=l_0\sin\theta_0$

$$l=\sqrt{(\Delta x)^2+(\Delta y)^2}=l_0\sqrt{1-\frac{v^2\cos^2\theta}{c^2}}$$

$$\tan\theta=\frac{\Delta y}{\Delta x}=\gamma\tan\theta_0$$

Ans. 21: (a), (b) and (d)

Solution: (a) In the train's frame of reference, $\Delta t' = t'_A - t'_O$ where t'_O is the time point when the pulse leaves point O, and t'_A is the time when the light pulse reaches A.

Now, $\Delta t' = t'_A - t'_O = \dfrac{L_0}{2c}$

Notice that the two light pulses are received simultaneously by A and B in the frame of reference of the train.

(b) We define events in S - the station frame, and S' - the train's frame.

Events	S' frame	S frame
Pulse left O	(t'_O, x'_O)	(t_O, x_O)
Pulse reached A	$\left(t'_A, x'_O + \dfrac{L_0}{2}\right)$	(t_A, x_A)
Pulse reached B	$\left(t'_A, x'_O - \dfrac{L_0}{2}\right)$	(t_B, x_B)

Using the Lorentz transformation, $\begin{cases} x = \gamma(x' + \beta c t') \\ t = \gamma\left(t' + \dfrac{\beta}{c} x'\right) \end{cases}$ where $\beta = \dfrac{v}{c}$.

Therefore, the distance the light pulse has to travel in the laboratory frame is:

$$(\Delta x)_A = x_A - x_O = \gamma(x'_A + \beta c t'_A) - \gamma(x'_O + \beta c t'_O) = \gamma(\Delta x' + \beta c \Delta t') = \gamma\left(\frac{L_0}{2} + \beta c \frac{L_0}{2c}\right)$$

$$= \gamma \frac{L_0}{2}(1+\beta) = \frac{L_0}{2}\sqrt{\frac{1+\beta}{1-\beta}}$$

By means of the same treatment, we find: $(\Delta x)_B = x_B - x_O = \dfrac{L_0}{2}\sqrt{\dfrac{1-\beta}{1+\beta}}$

(d) The time needed for the light to travel from O to B in the station's frame of reference is $(\Delta t)_B = t_B - t_O = \gamma\left[t'_B - t'_O + \dfrac{\beta}{c}(x'_B - x'_O)\right] = \gamma\left(\dfrac{L_0}{2c} - \dfrac{\beta}{c}\dfrac{L_0}{2}\right) =$

$$(\Delta t)_B = t_B - t_O = \frac{L_0}{2c}\sqrt{\frac{1-\beta}{1+\beta}}$$

Ans. 22: (b), (d)

Solution: The relative speed between toy train B and rod is zero so no change in length of rod from observer attached with toy train B

The relative speed between rod and table is $u = 0.5c$ so observer attached with table will see length contraction $l = l_0\sqrt{1-\frac{u^2}{c^2}} = .866$

The relative speed between rod and toy train A is $u = \frac{u' + v}{1+\frac{u'v}{c^2}}$ where $u' = 0.5c, v = 0.5c$ so

$u = \frac{4c}{5}$ will see length contraction as $l = l_0\sqrt{1-\frac{u^2}{c^2}} = 0.6\, l_o$

Ans. 23: (a), (c), (d)

Solution: As the elastically scattered particles have the same mass and the same kinetic energy, their momenta must make the same angle $\frac{\theta}{2}$ with the incident direction and have the same magnitude. Conservation of energy and of momentum given

$$mc^2 + E_0 = 2E, \qquad p_0 = 2p\cos\left(\frac{\theta}{2}\right)$$

where E, p are the energy and momentum of each scattered particle. Squaring both sides of the energy equation we have $m^2c^4 + E_0^2 + 2E_0mc^2 = 4\left(p^2c^2 + m^2c^4\right)$

$$E_0^2 + 2E_0mc^2 - 3m^2c^4 = \frac{p_0^2c^2}{\cos^2\left(\frac{\theta}{2}\right)} = \frac{E_0^2 - m^2c^4}{\cos^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{E_0^2 - m^2c^4}{\left(E_0 - mc^2\right)\left(E_0 + 3mc^2\right)}} = \sqrt{\frac{E_0 + mc^2}{E_0 + 3mc^2}}$$

(c) $v << c, E_0 \approx mc^2$ $\quad \cos\left(\frac{\theta}{2}\right) \approx \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$ giving $\theta \approx \frac{\pi}{2}$

(d) $v \to c, E_0 >> mc^2$ $\quad \cos\left(\frac{\theta}{2}\right) \approx 1$ giving $\theta = 0$.

Ans. 24: (b), (c), (d)

Solution: $p = 2\sqrt{2}m_0c = \dfrac{m_0 v}{\sqrt{1-\dfrac{v^2}{c^2}}}$ $\quad v \Rightarrow \dfrac{2\sqrt{2}c}{3}$

$$E^2 = p^2c^2 + m_0^2c^4 = 8m_0^2c^2 + m_0^2c^4 = 9m_0^2c^4 \Rightarrow E = 3m_0c^2$$

Kinetic $T = 3m_0c^2 - m_0c^2 = 2m_0c^2$

Ans. 25: (a), (c)

Solution: For spaceship Fiziks $5 = \dfrac{3}{\sqrt{1-\dfrac{v^2}{c^2}}}$ so $v = \dfrac{4c}{5}$

Speed of Akashganga with respect to earth $13 = \dfrac{12}{\sqrt{1-\dfrac{u^2}{c^2}}}$ $\quad u = \dfrac{12c}{13}$

For Velocity of Akashganga with respect to Fiziks $v = -\dfrac{4c}{5}$ $\quad u_f = \dfrac{u+v}{1+\dfrac{uv}{c^2}} = \dfrac{8c}{17}$

Ans. 26: (a), (b), (d)

Solution: $\tau^- \to \pi^- + \nu_\tau$

From conservation of energy $\mathrm{M_\tau c^2 = E_\pi + E_\nu}$.

$\mathrm{E_\pi^2 = p^2c^2 + M_\pi^2c^4}$ and $\mathrm{E_\nu^2 = p^2c^2}$ since momentum of π^- and ν_τ is same.

$$\mathrm{M_\tau c^2 = E_\pi + E_\nu,\ M_\pi{}^2c^4 = E_\pi^2 - E_\nu^2 \Rightarrow E_\pi - E_\nu = \frac{M_\pi^2c^4}{M_\tau c^2}}$$

$$\mathrm{E_\pi - E_\nu = \frac{M_\pi c^2}{M_\tau}} \text{ and } \mathrm{E_\pi + E_\nu = M_\tau c^2 \Rightarrow E_\pi = \frac{\left(M_\tau^2 + M_\pi^2\right)c^2}{2M_\tau}}. \text{ and } E_\nu = \frac{\left(M_\tau^2 - M_\pi^2\right)c^2}{2M_\tau}.$$

Velocity of π^-, $E_\pi = \dfrac{\left(M_\tau^2 + M_\pi^2\right)c^2}{2M_\tau} = \dfrac{M_\pi c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}$ $\quad \Rightarrow \left(1-\dfrac{v^2}{c^2}\right) = \dfrac{4M_\pi^2M_\tau^2}{\left(M_\tau^2 + M_\pi^2\right)^2}$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \frac{4M_\pi^2M_\tau^2}{\left(M_\tau^2 + M_\pi^2\right)^2} \Rightarrow \frac{v^2}{c^2} = \frac{M_\tau^4 + M_\pi^4 + 2M_\tau^2M_\pi^2 - 4M_\pi^2M_\tau^2}{\left(M_\tau^2 + M_\pi^2\right)^2} \Rightarrow v = \left(\frac{M_\tau^2 - M_\pi^2}{M_\tau^2 + M_\pi^2}\right)c.$$

Ans. 27: (a), (d)

Solution: If kinetic energy of particle is $T = \frac{m_o c^2}{4}$ then total energy is $E = T + m_o c^2 = \frac{5m_0 c^2}{4}$ and

momentum is $E^2 = p^2c^2 + m_0^2 c^4 \Rightarrow p = \frac{3m_o c}{4}$

Then from conservation of momentum $p_1 - p_2 = \frac{3m_o c}{4}$

And from conservation of energy $E_1 + E_2 = p_1 c + p_2 c = \frac{5m_o c^2}{4}$, $p_1 = m_0 c$

Ans. 28: (a), (d)

Solution: The speed of bullet with respect to target $u_t = \frac{v-u}{1-\frac{vu}{c^2}}$ and the bullet have to travel

distance from target reference $L_1 = L\sqrt{1-\frac{v^2}{c^2}} = L\gamma_1$

So time of that bullet hit the target from reference of target $\frac{L_1}{u_t} = \gamma_1 L\left(1-uv/c^2\right)/\left(u-v\right)$

The speed of target with respect to bullet $u_b = \frac{u-v}{1-\frac{vu}{c^2}}$ and the bullet have to travel

distance from target reference $L_2 = L\sqrt{1-\frac{u^2}{c^2}} = L\gamma_2$

So time of that bullet hit the target from reference of target $\frac{L_2}{u_t} = \gamma_2 L\left(1-uv/c^2\right)/\left(u-v\right)$

NAT

Ans. 29: 0

$$x=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow v=0$$

Ans. 30: 40

Solution: $x_2'-x_1'=9\times10^9\,m$ and $t_2'-t_1'=0$. Then

$$t_2-t_1=\left(\frac{t_2'+\frac{v}{c^2}x_2'}{\sqrt{1-\frac{v^2}{c^2}}}\right)-\left(\frac{t_1^1+\frac{v}{c^2}x_1'}{\sqrt{1-\frac{v^2}{c^2}}}\right)\Rightarrow t_2-t_1=\frac{t_2'-t_1'}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{v}{c^2}\frac{\left(x_2'-x_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{v}{c^2}\frac{\left(x_2'-x_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Put $v=0.8c\quad\Rightarrow t_2-t_1\cong 40\sec$

Ans. 31: 0.3

Solution: $\Delta t=0$, $t_2'-t_1'=5\mu s$, $x_2'-x_1'=5km$ $\qquad v=?$

$$t_2-t_1=\frac{t_2'+\left(\frac{-v}{c^2}\right)x_2'}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{t_1'+\left(\frac{-v}{c^2}\right)x_1'}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow\frac{\left[\left(t_2'-t_1'\right)-\frac{v}{c^2}\left(x_2'-x_1'\right)\right]}{\sqrt{1-\frac{v^2}{c^2}}}=0$$

$$\Rightarrow 5\times10^{-6}-\frac{v}{c^2}\times5\times10^3=0\Rightarrow\frac{v}{c^2}=\frac{5\times10^{-6}}{5\times10^3}=10^{-9}\Rightarrow\ v=3\times10^8\times c\times10^{-9}=0.3\ c$$

Ans. 32: 0.928

Solution: $v=\frac{4c}{5}$, $u_x'=c\cos60^\circ=\frac{c}{2}$, $u_y'=c\sin60^\circ=\frac{\sqrt{3}}{2}c$

Now $u_x=\dfrac{\frac{c}{2}+\frac{4}{5}c}{1+\frac{c}{2}\cdot\frac{4c}{5c^2}}=\dfrac{13c}{14}\Rightarrow\cos\theta=\dfrac{13}{14}$

Ans. 33: 15

Solution: $t_1 = t_0\sqrt{\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 10\sqrt{\dfrac{1+0.6}{1-0.6}} = 10\times 2 = 20\sec$,

$$t_2 = t_0\sqrt{\dfrac{1-\frac{v}{c}}{1+\frac{v}{c}}} = 10\sqrt{\dfrac{1-0.6}{1+0.6}} = 10\times\frac{1}{2} = 5\sec \Rightarrow \mathrm{t_1 - t_2} = 15\sec$$

Ans. 34: 0.97

Solution: $u_x = \dfrac{u_x^{'}+v}{1+\dfrac{u_x^{'}v}{c^2}}$ $\quad u_x^{'} = \dfrac{2c}{3}, v = -\dfrac{c}{2}$ $\quad u_x = \dfrac{c}{4}$

$$l = l_0\sqrt{1-\frac{u^2_x}{c^2}} = 0.87\ l_0 \quad l = l_0\sqrt{1-\frac{u^2}{c^2}} = 0.97$$

Ans. 35: 2.18

Solution: $t = \dfrac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$, $\quad t_0 = t\times\sqrt{1-\dfrac{v^2}{c^2}} \quad \Rightarrow t_0 = 5\times 10^{-8}\times\sqrt{.19} \quad \Rightarrow 2.18\times 10^{-8}s$

Ans. 36: 1.86

Solution: Since $E = 315MeV$ and $m_0 = 105\dfrac{MeV}{c^2}$.

$$E = mc^2 \Rightarrow E = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 315 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow 315 = \frac{105}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow v = 0.94c\ .$$

Now, $t = \dfrac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$, $\quad t_0 = 2.2\mu s \Rightarrow t = \dfrac{2.2\times 10^{-6}}{\sqrt{1-\dfrac{8}{9}}} \Rightarrow t = 6.6\ \mu s$

Now the distance traversed by muon is $vt = 0.94c\times 6.6\times 10^{-6} == 1.86km$.

Ans. 37: **30.1**

Solution: $H_B \rightarrow P_H + Z_B$

From conservation of momentum $0 = \vec{P}_1 + \vec{P}_2 \Rightarrow \vec{P}_1 = -\vec{P}_2 \Rightarrow |P_1| = |P_2|$

Now $E_{H_B} = E_{P_H} + E_{Z_B} \Rightarrow E_{P_H} + E_{Z_B} = M_{H_B}c^2$

$E_{P_H}^2 = P_1^2c^2 + 0$ and $E_{Z_B}^2 = P_2^2c^2 + M_{Z_B}^2c^4$

$\Rightarrow \left(E_{Z_B} - E_{P_H}\right)\left(E_{Z_B} + E_{P_H}\right) = M_{Z_B}^2c^4 \quad \because |P_1| = |P_2|$

$$\Rightarrow E_{Z_B} - E_{P_H} = \frac{M_{Z_B}^2c^4}{M_{H_B}c^2} = \frac{M_{Z_B}^2c^2}{M_{H_B}} \quad \because E_{Z_B} + E_{P_H} = M_{H_B}c^2$$

$$\Rightarrow 2E_{P_H} = M_{H_B}c^2 - \frac{M_{z_B}^2c^2}{M_{H_B}} \Rightarrow E_{P_H} = \frac{\left(M^2_{H_B} - M_{z_B}^2\right)c^2}{M_{H_B}}$$

$$\Rightarrow E_{P_H} = \left(\frac{125\times125 - 90\times90}{2\times125}\right)\times\frac{c^4}{c^4} = 30.1GeV$$

Ans. 38: **0.99**

$$\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = 5mc \qquad v = \sqrt{\frac{100}{101}}$$

Ans. 39: **0.866**

Solution: $K.E = mc^2 - m_0c^2$, rest mass energy $= m_0c^2$

$K.E. =$ rest mass energy

$mc^2 - m_0c^2 = m_0c^2$

$mc^2 = 2m_0c^2$

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}c^2 = 2m_0c^2 \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2 \Rightarrow 4\left(1-\frac{v^2}{c^2}\right) = 1 \Rightarrow 4\frac{v^2}{c^2} = 3 \Rightarrow v = \frac{\sqrt{3}}{2}c$$

Ans. 40: **2**

Solution: $E = T + m_0c^2$, $E = m_0c^2 + m_0c^2 = 2m_0c^2$

Ans. 41: **0.866**

Solution: $E = m_0c^2 + m_0c^2 = 2m_0c^2 = \dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}$, $v = \dfrac{\sqrt{3}}{2}c$

Ans. 42: **1.66**

Solution: Speed of cosmic particle with respect $u_x = \dfrac{u_x' + v}{1+\dfrac{u_x' v}{c^2}} = \dfrac{0.5c + 0.5c}{1+\dfrac{(0.5)\times(0.5)c^2}{c^2}} = \dfrac{c}{1+.25} = \dfrac{4c}{5}$,

$$u_y' = 0,\; u_z' = 0 \qquad u = \sqrt{u_x'^2 + u_y'^2 + u_y'^2} = \frac{4c}{5}, m = \frac{m_o}{\sqrt{1-\dfrac{u^2}{c^2}}} = \frac{5m_o}{3}$$

Ans. 43: **0.33**

Solution: Using conservation of momentum

given $m_1 = m, v = \dfrac{c}{2}, m_2 = \dfrac{2m}{5}, v_1 = ?$

$$\frac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}} = 2p_1 = \frac{2.m_1v_1}{\sqrt{1-\dfrac{v_1^2}{c^2}}} \Rightarrow v_1 = \frac{25c}{73} = .33c$$

Ans. 44: **39**

Solution: $\Delta t = \dfrac{\Delta\tau}{\sqrt{1-\dfrac{v^2}{c^2}}}$, $\Delta\tau = 1.77\times10^{-8}\text{ sec}$ $v = 0.99c$,

$$\Delta t = 1.3\times10^{-7}\text{ sec} \qquad v\times\Delta t = 0.99c\times1.3\times10^{-7}\text{ sec} = 39m.$$

Ans. 45: **0.916**

Solution: $l_x = l_{0x}\sqrt{1-\dfrac{v^2}{c^2}} = l_0\cos\theta\sqrt{1-(0.8)^2} \Rightarrow l_x = l_0\times\dfrac{1}{2}\times0.6 = 0.3\,l_0$ and $l_y = l_0\sin\theta = \dfrac{l_0\sqrt{3}}{2}$

New length $l = \sqrt{(0.3l_0)^2 + \left(\dfrac{\sqrt{3}l_0}{2}\right)^2} = l_0\sqrt{0.09+0.75} = 0.916\,l_0$

2. Modern Physics

2.1 Black Body Radiation

A **black body** is an idealized physical body that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence.

A black body in thermal equilibrium (that is, at a constant temperature) emits electromagnetic radiation called black body radiation.

When an object is heated, it radiated electromagnetic energy as result of thermal agitation of electrons in its surface .the intensity of radiation depends on its frequency and on the temperature, the light it emits ranges over the entire spectrum.

An object in thermal equilibrium with its surrounding radiates as much energy it absorbers. A Black body is perfect absorber as well perfect emitter of radiation.

2.1.1 Wien's Distribution Law

Wien's approximation (also sometimes called **Wien's law** or the **Wien distribution law**) is a law of used to describe the spectrum of thermal radiation (frequently called the blackbody function). The equation does accurately describe the short wavelength (high frequency) spectrum of thermal emission from objects, but it fails to accurately fit the experimental data for long wavelengths (low frequency) emission.

The law may be written as

$$I(\nu,T)=\frac{2h\nu^3}{c^2}e^{-\frac{h\nu}{k_BT}}$$

where $I(\nu,T)$ is the amount of energy per unit surface area per unit time per unit solid angle per unit frequency emitted at a frequency ν.

T is the temperature of the black body, h is planks constant, c is the speed of light,

k_B is Boltzmann's constant .

This equation may also be written as $I(\lambda,T)=\frac{2hc^2}{\lambda^5}e^{-\frac{hc}{\lambda k_B T}}$

Where $I(\lambda,T)$ is the amount of energy per unit surface area per unit time per unit solid angle per unit wavelength emitted at a wavelength λ.

2.1.2 Rayleigh's Energy Density Distribution

Total energy per unit volume in the cavity in the frequency interval from ν to $\nu+d\nu$ is given by $$u(\nu)d\nu=\frac{8\pi k_B T}{c^3}\nu^2 d\nu$$

But above relation does not describe the experimental trend at higher frequency. The equation does accurately describe the long wavelength (short frequency) spectrum of thermal emission from objects, but it fails to accurately fit the experimental data for short wavelengths (high frequency) emission.

In term of wavelength the energy density is given by $u(\lambda)d\lambda=\frac{8\pi k_B T}{\lambda^4}d\lambda$

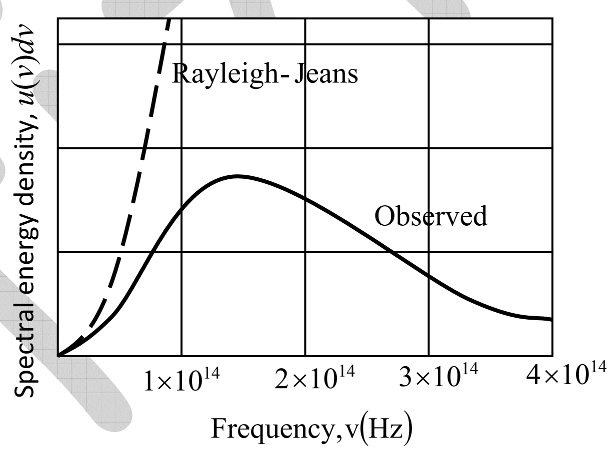


Fig: Comparison of the Rayleigh-Jeans formula for the spectrum of the radiation from a blackbody at 1500 K with the observed spectrum. The discrepancy is known as the ultraviolet catastrophe because it increases with increase frequency. This failure of classical physics led Planck to the discovery that radiation is emitted in quanta whose energy is $h\nu$.

2.1.3 Planks Radiation Formula

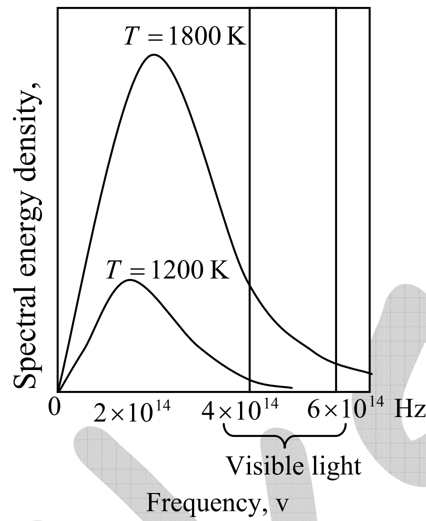


Fig: Blackbody spectra. The spectral distribution of energy in the radiation depends only on the temperature of the body.

With assumption that radiation has discrete energy analogous to oscillator which is given by

$$E_n = nh\nu \text{ where } n = 0,1,2...$$

Total energy per unit volume in the cavity in the frequency interval from ν to $\nu + d\nu$ is given by

$$u(\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/k_BT}-1}$$

where for high frequency (low wavelength) ii will approach to Wien's distribution and for low frequency(high wavelength) it will approach to Raleigh- Jeans formula.

2.2 Photo Electric Effect

In the **photoelectric effect**, electrons are emitted from solids, liquids or gases when they absorb energy from light. Electrons emitted in this manner may be called *photoelectrons*.

The photoelectric effect requires photons with energies from a few electron volts to over 1MeV in high atomic number elements. Study of the photoelectric effect led to important steps in understanding the quantum nature of light and electrons and influenced the formation of the concept of wave-particle duality. It also led to Max Planck's discovery of quantized energy and the Planck Relation $(E = h\nu)$, which links a photon's frequency with its energy. The factor h is known as the Planck constant.

Experimental observation

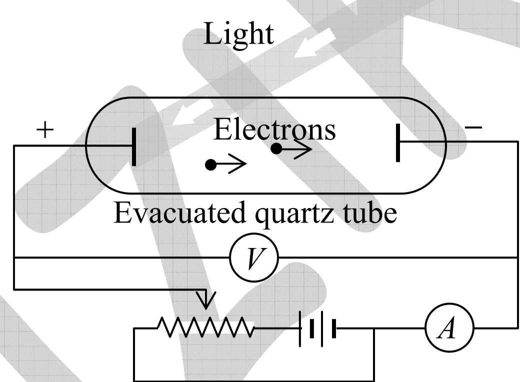


Fig: Experimental observation of the photoelectric effect

1. Because electromagnetic wave is concentrated in photons and not spread out, there should be no delay in the emission of photoelectron as light falls on the matter.
2. All photons of frequency ν have the same energy so changing the intensity of monochromatic light beam will change the number of photoelectrons not their energies.
3. The higher the frequency ν, the greater the photon energy $(E = h\nu)$ and so the more energy the photoelectron have.

2.2.1 Threshold Frequency and Work Function

For a given metal, there exists a certain minimum frequency of incident radiation below which no photoelectrons are emitted. This frequency is called the threshold frequency or critical frequency (ν_0).

There must be minimum energy ϕ for an electron to escape from a particular metal surface; this energy is known as work function which is given by $\phi = h\nu_0$.

Einstein equation of photoelectric effect

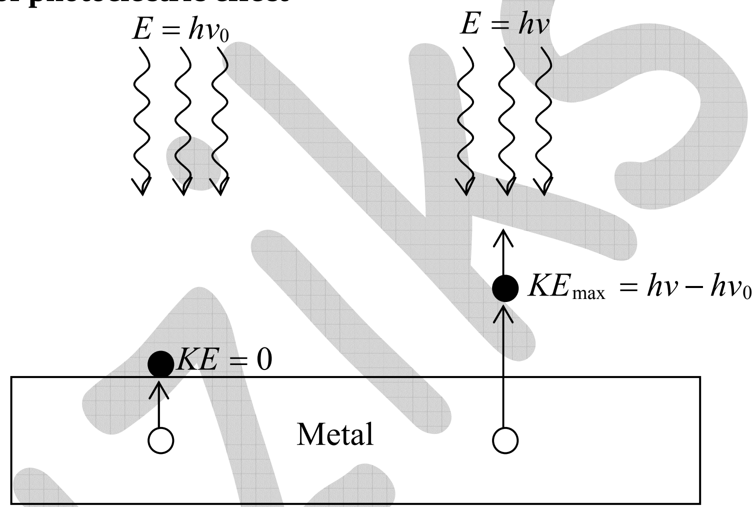


Fig. If the energy $h\nu_0$ (the work function of the surface) is needed to remove an electron from a metal surface, the maximum electron kinetic energy will be $h\nu - h\nu_0$ when light of frequency ν is directed at the surface.

When a metal is irradiated with light, electron may get emitted. Kinetic energy of photoelectron observed when irradiated with a light of frequency $\nu > \nu_o$, where ν_o is threshold frequency is given by $KE_{\max}$ where $KE_{\max} = h\nu - h\nu_0$

This maximum kinetic energy is equivalent to Stopping potential V_s which is energy required to stop electron which contain maximum kinetic energy.

Then $eV_s = h\nu - h\nu_0$ which is known as Einstein equation.

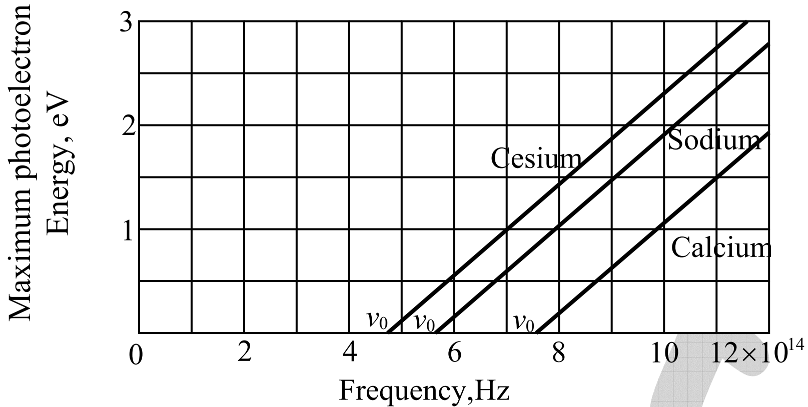


Fig: Maximum photoelectron kinetic energy KE_{max} versus frequency of incident light for three metal surfaces.

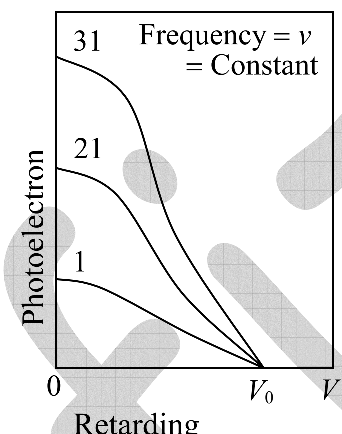


Fig: Photoelectron current is proportional to light intensity I for all retarding voltages. The extinction voltage v_0 which corresponds to the maximum photoelectron energy, is the same for all intensities of light of the same frequency *v*.

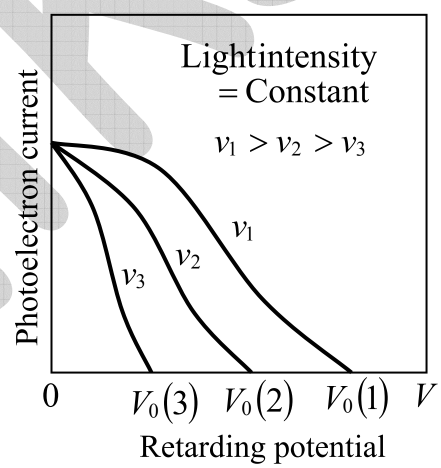


Fig: The extinction voltage V_0, and hence the maximum photoelectron energy, depends on the frequency of the light. When the retarding potential is V = 0, the photoelectron current is the same for light of a given intensity regardless of its frequency.

Example: When light of a given wavelength is incident on metallic surface ,the stopping potential for the photoelectrons is $3.2V$ .if a second light source whose wavelength is double that of the first is used , the stopping potential drop to $0.8V$.

(a) Calculate the wavelength of first radiation

(b) The work function and the cutoff frequency of the metal.

Solution: Let us assume work function of metal is given by ϕ.

For wavelength λ_1 and λ_2 stopping potential is given by $V_{s_1} = 3.2V$ and $V_{s_2} = 0.8$ where $\lambda_2 = 2\lambda_1$.

(a) $eV_{s_1} = \frac{hc}{\lambda_1} - \phi$ ……..(i) $\quad eV_{s_2} = \frac{hc}{\lambda_2} - \phi \Rightarrow eV_{s_2} = \frac{hc}{2\lambda_1} - \phi$ ………...(ii)

Solving equation (i) and (ii) $\lambda_1 = \frac{hc}{2e(V_{s_1} - V_{s_2})} = 2.6 \times 10^{-6}$

(b) Eliminating λ_1 from (i) and (ii) $\phi = e(V_{s_1} - 2V_{s_2}) = 3.84 \times 10^{-19}\, J$

Cutoff frequency $\nu = \frac{\phi}{h} = 5.8 \times 10^{14}$

Example: Ultraviolet light of wavelength $350nm$ and intensity $1\ W/m^2$ is directed at potassium surface.

(a) Find the maximum kinetic of photoelectrons

(b) If 0 .5% of incident photons produce photoelectrons, how many are emitted per second if the potassium surface has area of $1cm^2$

Solution: It is given $hc = 1.24 \times 10^{-6}\, eV.m$ and work function of potassium is $2.2eV$

(a) Energy of photon is $\frac{hc}{\lambda} = \frac{1.24 \times 10^{-6}\, eV.m}{350 \times 10^{-9}} = 3.5eV = 5.68 \times 10^{-19}\, J$

Maximum kinetic energy is given by $E_{\max} = \frac{hc}{\lambda} - \phi = 3.5eV - 2.2eV = 1.3eV$

(b) No. of photon that reach the surface per second is given by

$$n_p = \frac{E/t}{E_p} = \frac{(p/A).A}{E_p} = \frac{(1W/m^2)1 \times 10^{-4} m^2}{5.68 \times 10^{-19}\, J/photon} = 1.76 \times 10^{14}$$

No of photo electron is $n_e = n_p \times 0.005 = 8.8 \times 10^{11}$ photo electrons/second .

Example: In a photoelectric experiment, it was found that the stopping potential decreases from 1.85 V to 0.82 V as the wavelength of the incident light is varied from 300 nm to 400 nm. Calculate the value of the Planck constant from these data.

Solution: The maximum kinetic energy of a photoelectron is $K_{\max} = \dfrac{hc}{\lambda} - \varphi$

and the stopping potential is $V = \dfrac{K_{\max}}{e} = \dfrac{hc}{\lambda e} - \dfrac{\varphi}{e}$

If V_1, V_2 are the stopping potentials at wavelengths λ_1 and λ_2 respectively,

$$V_1 = \frac{hc}{\lambda_1 e} - \frac{\varphi}{e} \quad \text{and} \quad V_2 = \frac{hc}{\lambda_2 e} - \frac{\varphi}{e} \Rightarrow V_1 - V_2 = \frac{hc}{e}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

or, $$h = \frac{e(V_1 - V_2)}{c\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_2}\right)} = \frac{e(1.85V - 0.82V)}{c\left(\dfrac{1}{300\times10^{-9}\, m} - \dfrac{1}{400\times10^{-9}\, m}\right)}$$

$$= \frac{1.03eV}{\left(3\times10^{8}\, m/s\right)\left(\dfrac{1}{12}\times10^{7}\, m^{-1}\right)} = 4.12\times10^{-15}\, eV - s$$

Example: A monochromatic light of wavelength λ is incident on an isolated metallic sphere of radius a. The threshold wavelength is λ_0 which is larger than λ. Find the number of photoelectrons emitted before the emission of photoelectrons will stop.

Solution: As the metallic sphere is isolated, it becomes positively charged when electrons are ejected from it. There is an extra attractive force on the photoelectrons. If the potential of the sphere is raised to V, the electrons should have a minimum energy $\varphi + eV$ to be able to come out. Thus, emission of photoelectrons will stop when

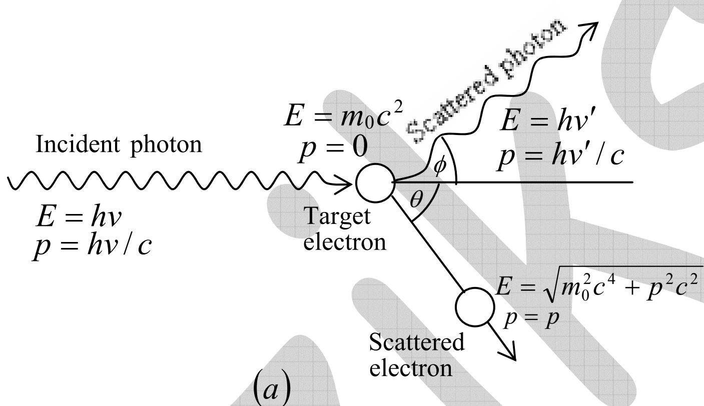
$$\frac{hc}{\lambda} = \varphi + eV = \frac{hc}{\lambda_0} + eV \qquad \text{or, } V = \frac{hc}{e}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

The charge on the sphere needed to take its potential to V is $Q = (4\pi\varepsilon_0 a)V$.

The number of electrons emitted is, therefore, $n = \dfrac{Q}{e}\dfrac{4\pi\varepsilon_0 aV}{e} = \dfrac{4\pi\varepsilon_0 aV}{e}\left(\dfrac{1}{\lambda} - \dfrac{1}{\lambda_0}\right)$

2.3 Compton Scattering

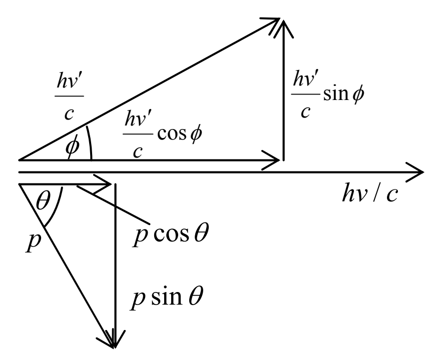
Compton scattering is an inelastic scattering of a photon by a free charged particle, usually an electron. It results in a decrease in energy (increase in wavelength) of the photon (which may be an X-ray or gamma ray photon), called the **Compton Effect**. Part of the energy of the photon is transferred to the scattering electron. **Inverse Compton scattering** also exists, in which a charged particle transfers part of its energy to a photon. This experiment give experimental prove of **particle aspect of photon (light).**



(a)

The scattering of photon of energy $h\nu$ by an electron rest mass m_0. After scattering photon scattered at angle ϕ and electron scattered at angle θ. The scattered photon have energy $h\nu'$ and electron have energy E.

The vector diagram of the momenta and their components of the incident and scattered photon and scattered electron shown in vector diagram



From the conservation of energy

loss in photon energy = gain in electron energy

$$h\nu - h\nu' = KE$$

The momentum of photon is given $p = \frac{E}{c} = \frac{h\nu}{c}$

From conservation of momentum in x direction

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\phi + p\cos\theta$$

Conservation of momentum in y direction $\quad 0=\frac{hv'}{c}\sin\phi-p\sin\theta$

From above two equation

$$p\cos\theta=\frac{hv}{c}-\frac{hv'}{c}\cos\phi \text{ and } p\sin\theta=\frac{hv'}{c}\sin\phi$$

Squaring and adding one will get $p^2c^2=(hv)^2-2(hv)(hv')\cos\phi+(hv')^2$ …….(i)

Kinetic energy of electron is given $hv-hv'==\sqrt{p^2c^2+m_0^2c^4}-m_0c^2$

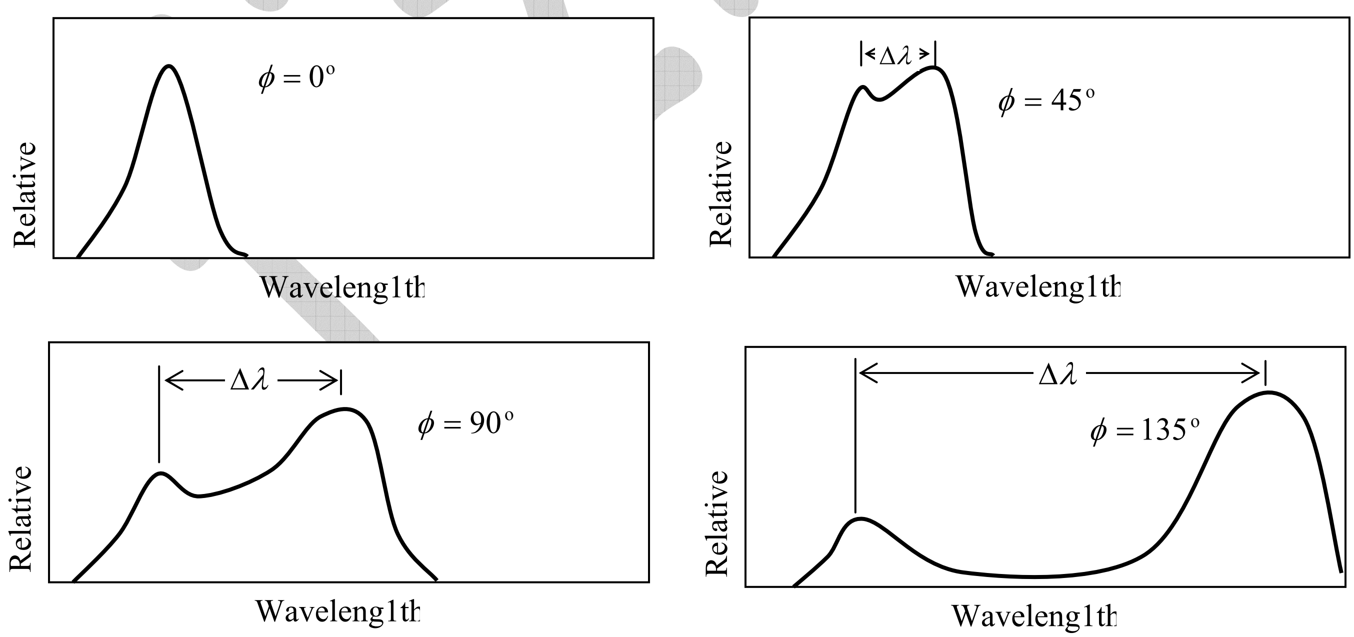
$$p^2c^2=(hv)^2-2(hv)(hv')+(hv')^2+2m_0c^2(hv-hv')\text{…….(ii)}$$

Equating equation (i) and (ii) $2m_0c^2(hv-hv')=2(hv)(hv')(1-\cos\phi)$

$$\lambda'-\lambda=\frac{h}{m_0c}(1-\cos\phi) \text{ where } \frac{v}{c}=\frac{1}{\lambda} \text{ and } \frac{v'}{c}=\frac{1}{\lambda'}$$

$$\lambda'-\lambda=\lambda_C(1-\cos\phi) \text{ where } \lambda_C=\frac{h}{m_0c}=\lambda_c=2.46\times10^{-12}\,m$$

Value of $\Delta\lambda=\lambda'-\lambda$ for different scattering angle ϕ shown in figure



Example: X-ray of wavelength $10.0\,pm$ are scattered from a target

(a) Find the wavelength of the X-rays scattered through 45^0

(b) Find the maximum wavelength present in the scattered X-rays

(c) Find the maximum kinetic energy of the recoil electrons .

It is given that (Where $\lambda_C = \dfrac{h}{m_0 c} = 2.46\times 10^{-12}\,m$)

Solution: (a) $\lambda' - \lambda = \lambda_c(1-\cos\phi)$ $\lambda' = \lambda + \lambda_c(1-\cos\phi)$ $=10.7\,pm$

(b) $\lambda' = \lambda + \lambda_c(1-\cos\phi)$ is maximum for $\phi = \pi$ $\lambda'_{\max} = \lambda + 2\lambda_c = 14.9\,pm$

(c) Kinetic energy of electron is given by $h\nu - h\nu' = KE = \dfrac{hc}{\lambda} - \dfrac{hc}{\lambda'}$

For maximum kinetic energy $KE_{\max} = \dfrac{hc}{\lambda} - \dfrac{hc}{\lambda'_{\max}} = 6.54\times 10^{-15}\,J$

Example: In a Compton scattering prove that $\tan\theta = \dfrac{\sin\phi}{\dfrac{\lambda'}{\lambda} - \cos\phi}$ where θ and ϕ are the angle of recoil of electron and scattering angle of photon .

Solution: From conservation of momentum

We will get $p\sin\theta = \dfrac{h\nu'}{c}\sin\phi$ ....(A)

$p\cos\theta = \dfrac{h\nu}{c} - \dfrac{h\nu'}{c}\cos\phi$ ....(B)

$$\tan\theta = \frac{(h\nu'/c)\sin\phi}{(h\nu/c)-(h\nu'/c)\cos\phi} = \tan\theta = \frac{\sin\phi}{\dfrac{\nu}{\nu'}-\cos\phi} = \tan\theta = \frac{\sin\phi}{\dfrac{\lambda'}{\lambda}-\cos\phi}$$ proved.

Example: Show that the maximum kinetic energy transferred to proton when hit by photon of energy $h\nu$ is $k_{\max}=\dfrac{h\nu}{1+\dfrac{m_pc^2}{2h\nu}}$

Solution: $\dfrac{1}{\nu'}-\dfrac{1}{\nu}=\dfrac{h(1-\cos\phi)}{m_pc^2}$

Kinetic energy $KE=h\nu-h\nu'=h\nu-\dfrac{h\nu}{1+\dfrac{h\nu(1-\cos\phi)}{m_pc^2}}=\dfrac{h\nu}{1+\dfrac{m_pc^2}{h\nu(1-\cos\phi)}}$

For maximum kinetic energy $(\phi=\pi)$, $K_{\max}=\dfrac{h\nu}{1+\dfrac{m_pc^2}{2h\nu}}$

Example: High energy photons ($\gamma-$rays) are scattered from electrons initially at rest. Assume the photons are backscattered and their energies are much larger than the electron's rest-mass energy, $E>>m_ec^2$.

(a) Calculate the wavelength shift

(b) Show that the energy of the scattered photons is half the rest mass energy of the electron, regardless of the energy of the incident photons.

(c) Calculate the electron's recoil kinetic energy if the energy of the incident photons is 150 MeV.

Solution: (a) In the case where the photons backscatter (i.e., $\theta=\pi$), the wave length shift becomes

$$\Delta\lambda=\lambda'-\lambda=\frac{h}{m_ec}(1-\cos\theta)=\frac{2h}{m_ec}$$

The numerical value of $\Delta\lambda$ is easy to obtain by making use of $\hbar c=197.33\times10^{-15}\ MeV\ m$ and $m_ec^2=0.511\ MeV$:

$$\Delta\lambda=\frac{4\hbar}{m_ec^2}=\frac{4\times3.14\times197.33\times10^{-15}\,\text{MeVm}}{0.511\,\text{MeV}}=4.8\times10^{-12}\,\text{m}$$

(b) Since the energy of the scattered photons E' is related to the wavelength λ' by $E' = \frac{hc}{\lambda'}$, equation) yields

$$E' = \frac{hc}{\lambda'} = \frac{hc}{\lambda + 2h/(m_e c)} = \frac{m_e c^2}{m_e c^2 \lambda/(hc) + 2} = \frac{m_e c^2}{m_e c^2 / E + 2},$$

where $E = \frac{\hbar c}{\lambda'}$ is the energy of the incident photons. If $E >> m_e c^2$ we can approximate by

$$E' = \frac{m_e c^2}{2}\left[1 + \frac{2m_e c^2}{E}\right] \approx \frac{m_e c^2}{2} - \frac{(m_e c^2)}{E} \approx \frac{m_e c^2}{2} = 0.25 \text{ MeV}$$

(c) If $E = 15\,MeV$ the kinetic energy of the recoiling electrons can be obtained from the conservation of energy

$$K_e = E - E' \approx 150\,MeV - 0.25\,MeV = 149.75\,MeV$$

2.4 Bohr Atomic Model

In atomic physics, the **Bohr model**, introduced by Niles Bohr in 1913, depicts the atom as small, with a positively charged nucleus surrounded by electrons that travel in circular orbits around the nucleus-similar in structure to the solar system, but with attraction provided by electrostatic forces rather than gravity.

He suggested that electrons could only have certain ***classical*** motions:

1. Electrons in atoms orbit the nucleus.

2. The electrons can only orbit stably, without radiating, in certain orbits (called by Bohr the "stationary orbits"): at a certain discrete set of distances from the nucleus. These orbits are associated with definite energies and are also called energy shells or energy levels. In these orbits, the electron's acceleration does not result in radiation and energy loss as required by classical electromagnetic.

3. Electrons can only gain and lose energy by jumping from one allowed orbit to another, absorbing or emitting electromagnetic radiation with a frequency ν determined by the energy difference of the levels according to the *Planck relation*:

$$\Delta E = E_2 - E_1 = h\nu$$

where h is Planck's constant . The frequency of the radiation emitted at an orbit of period T is as it would be in classical mechanics; it is the reciprocal of the classical orbit period: $\nu = \frac{1}{T}$

2.4.1 Bohr Quantization Rule

The significance of the Bohr model is that the laws of classical mechanics apply to the motion of the electron about the nucleus *only when restricted by a quantum rule*. The angular momentum L is restricted to be an integer multiple of a fixed unit:

$$L = n\hbar \text{ where } n = 1, 2, 3... \text{ and}$$

n is called the principle quantum number , and $\hbar = \frac{h}{2\pi}$. The lowest value of n is 1this gives a smallest possible orbital radius of $0.592 A^0$ known as the **Bohr radius**.

2.4.2 Hydrogen Atom

A **hydrogen atom** is an atom of the chemical element hydrogen. The electrically neutral atom contains a single positively charged proton and a single negatively charged electron bound to the nucleus by the Coulomb force. According to Bohr electron revolve about the nucleus in different quantized circular orbits whose angular momentum is given by $L = n\hbar$ where $n = 1,2,3...$ .The electron is held in a circular orbit by electrostatic attraction. The centripetal force is equal to the Coulomb force.

$$\frac{m_e v^2}{r} = \frac{k_e e^2}{r^2}$$

where m_e is the electron's mass, e is the charge of the electron, $k_e = \frac{1}{4\pi\varepsilon_o}$ is Coulomb's constant and v is velocity of electrons in orbit.

This equation determines the electron's speed at any radius: $v = \sqrt{\frac{k_e e^2}{m_e r}}$

It also determines the electron's total energy at any radius:

$$E = \frac{m_e v^2}{2} - \frac{k_e e^2}{r}$$

Putting the value of v one will get $E = -\frac{k_e e^2}{2r}$

The total energy is negative and inversely proportional to r. This means that it takes energy to pull the orbiting electron away from the proton. For infinite values of r, the energy is zero, corresponding to a motionless electron infinitely far from the proton. The total energy is half the potential energy,

From the quantization the angular momentum

$$L = n\hbar = m_e vr = n\hbar$$

Substituting the expression for the velocity gives an equation for r in terms of n:

$$\sqrt{k_e e^2 m_e r} = n\hbar$$

So that the allowed orbit radius at any n is:

$$r_n = \frac{n^2\hbar^2}{k_e e^2 m_e} \Rightarrow r_1 = 0.53\times 10^{-10}\,m \qquad (\text{For } n = 1)$$

The smallest possible value of r in the hydrogen atom is called the Bohr radius r_1.

The energy of the n^{th} level for any atom is determined by the radius and quantum number:

$$E_n = -\frac{k_e e^2}{2r_n} = -\frac{\left(k_e e^2\right)^2 m_e}{2\hbar^2 n^2} = \frac{-13.6}{n^2}\,eV$$

The combination of natural constants in the energy formula is called the Rydberg energy

R_E which is given by $R_E = \dfrac{\left(k_e e^2\right)^2 m_e}{2\hbar^2}$

This expression is clarified by interpreting it in combinations which form more natural units: $m_e c^2$ is the rest mass energy of the electron $(511\ keV)$.

$\dfrac{k_e e^2}{\hbar c} = \alpha = \dfrac{1}{137}$ is the fine structure constant .

$$R = \frac{1}{2}\left(m_e c^2\right)\alpha^2 = (\ 1.097\times\ 10^7\ m^{-1})$$

For nuclei with Z protons, the energy levels are (to a rough approximation):

$$E_n = -\frac{Z^2 R}{n^2}$$

2.4.3 The Structure and Spectra of Hydrogenic Atoms

Atomic Spectra

The spectrum of atomic hydrogen arises from transitions between its permitted states.

- Each element has a characteristic line spectrum
- When an atomic gas is excited by passing electric current, it emits radiation. The radiation has a spectrum which contains certain specific wavelength, called Emission line spectrum.
- When while light is passed through a gas, gas absorb light of certain wavelength present in its emission spectrum. Resulting spectrum is called Absorption line spectrum.
- The number, intensity and exact wavelength of the lines in the spectrum depend on Temperature, Pressure, Presence of Electric field, Magnetic field, and the motion of the source.

Spectral series

When an electric discharge is passed through gaseous hydrogen, the H_2 molecules dissociate and the energetically excited H atoms that are produced emit light of discrete frequencies, producing a spectrum of a series of 'lines'.

(i) Lyman Series: $\frac{1}{\lambda}=R\left(\frac{1}{1^2}-\frac{1}{n^2}\right)$; $n=2,\ 3,\ 4\ ...$ (In U.V. region)

Where, R is Rydberg constant ($1.097\times 10^7\ m^{-1}$)

(ii) Balmer Series: $\frac{1}{\lambda}=R\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$; $n=3,\ 4,\ 5...$ (In Visible region)

$n=3$ for H_α Line, $n=4$ for H_β Line, $n=5$ for H_γ Line,

(iii) Paschen series: $\frac{1}{\lambda}=R\left(\frac{1}{3^2}-\frac{1}{n^2}\right)$; $n=4,\ 5,\ 6$ (Near Infra Red)

(iv) Bracket Series: $\frac{1}{\lambda}=R\left(\frac{1}{4^2}-\frac{1}{n^2}\right); \quad n=5,\ 6,\ 7...$ (Infra Red)

(v) Pfund Series: $\frac{1}{\lambda}=R\left(\frac{1}{5^2}-\frac{1}{n^2}\right); \quad n=6,\ 7,\ 8...$ (Far Infra Red)

If mass of nucleus is not considered as very heavy then reduce mass will take in to account which is given by $\mu=\frac{m_n m_e}{m_n+m_e}$ where m_e and m_n are mass of electron and nucleus respectively.

- **Correction in Energy due to Reduced Mass**

$$E_n=\frac{\mu(-13.6)}{m_e n^2}$$

- **Correction in Rradius of the orbit**

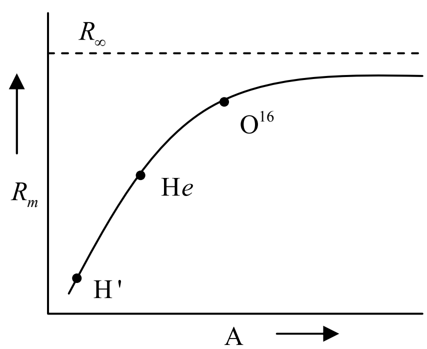
$$r_n=\frac{m_e n^2(a_0)}{\mu}$$

- **Variation of Rydberg constant with respect to Atomic Mass**

R for infinitely heavy nucleus: $R_\infty=\frac{2\pi^2 me^4}{h^3}$

R for nucleus of mass M: $R_m=\frac{2\pi^2 \mu e^4}{h^3}$

$$\Rightarrow R_m=\frac{R_\infty}{1+\frac{m_e}{m_n}}$$



- **Spectral Wavelength for Hydrogen like atoms**

$$\frac{1}{\lambda}=RZ^2\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)$$ where R is Red berg and Z is atomic number.

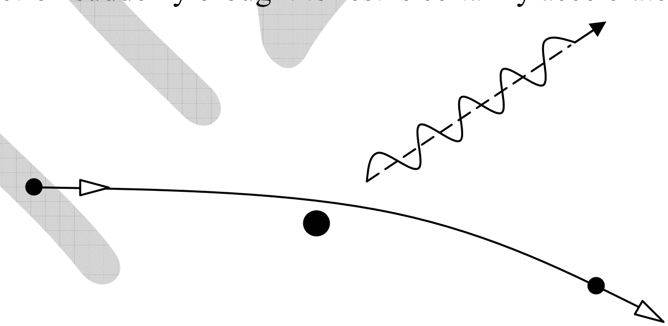
2.4 X-radiation (composed of **X-rays**) is a form of electromagnetic radiation. X-rays have a wavelength in the range of 0.01 to 10 nanometers, corresponding to frequencies in the range (3×10^{16} Hz to 3×10^{19} Hz) and energies in the range 100 eV to 100 keV. The wavelengths are shorter than those of UV rays and longer than of gamma rays.

When the electrons hit the target, X-rays are created by two different atomic processes:

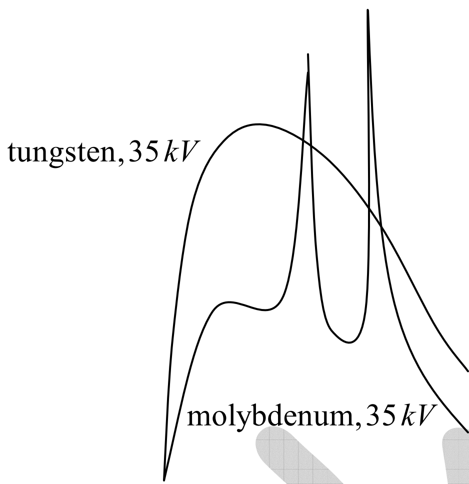
1. **X-ray fluorescence:** If the electron has enough energy it can knock an orbital electron out of the inner electron shell of a metal atom, and as a result electrons from higher energy levels then fill up the vacancy and X-ray photons are emitted. This process produces an emission spectrum of X-rays at a few discrete frequencies, sometimes referred to as the spectral lines.

   The spectral lines generated depend on the target element used and thus are called characteristic lines. Usually these are transitions from upper shells into K shell (called K lines), into L shell (called L lines) and so on.

2. **Bremsstrahlung (breaking radiation):** electromagnetic theory predicts that an accelerated electric charge will radiate electromagnetic waves, and a rapidly moving electron suddenly brought to rest is certainly accelerated.



X- ray spectra of tungsten and molybdenum at $35KV$ accelerating potential.



Analysis:

An electron of initial kinetic energy K is decelerated during an encounter with heavy target nucleus .the electron interacts with the charge nucleus via the coulomb field , transferring momentum to the nucleus . the accompanying deceleration of the electron lead to photon emission . the target nucleus is so massive that the energy it acquires during the collision can safely be neglected .

If K' is the kinetic energy of the electron after the electron then energy of photon is given by $h\nu = K - K'$ and photon (X- ray) wavelength is given by $\frac{hc}{\lambda} = K - K'$

The shortest wavelength photon would be emitted when an electron loses all the kinetic energy in one deceleration process so $K' = 0$ and $\frac{hc}{\lambda_{\min}} = K$

Since $K = eV$ the energy acquired by the electron in being accelerated through the potential difference V so $\frac{hc}{\lambda_{\min}} = K = eV$

And $\lambda_{\min} = \frac{hc}{eV} \Rightarrow \lambda_{\min} = \frac{1.24\times10^{-6}}{V} V.m$

Example: Find the maximum frequency in the radiation from an x-ray machine whose accelerating potential 50,000V.

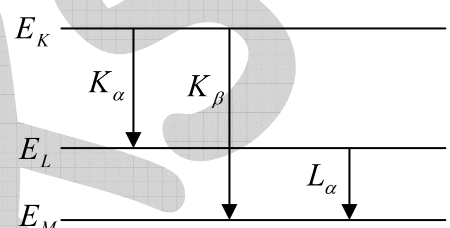
Solution: $\lambda_{\min} = \frac{1.24\times 10^{-6}}{V} V.m \quad \lambda_{\min} = \frac{1.24\times 10^{-6}}{50\times 10^{4} V} V.m = 2.48\times 10^{-11 m}$

$$\nu_{\max} = \frac{c}{\lambda_{\min}} = \frac{3\times 10^{8}}{2.48\times 10^{-11}} = 1.21\times 10^{19}\,Hz$$

Example: Show that the frequency of K_β X-ray of a material equals the sum of the frequencies of K_α and L_α X-rays of the same material.

Solution:

The energy level diagram of an atom with one electron knocked out is shown in figure .



Figure

Energy of K_α X-ray is $E_{K_\alpha} = E_K - E_L$

Energy of K_β X-ray is $E_{K_\beta} = E_K - E_M$,

and Energy of L_α X-ray is $E_{L_\alpha} = E_L - E_M$,

Thus, $E_{K_\beta} = E_{K_\alpha} + E_{L_\alpha}$ or, $h\nu_{K_\beta} = h\nu_{K_\alpha} + h\nu_{L_\alpha}$ or, $\nu_{K_\beta} = \nu_{K_\alpha} + \nu_{L_\alpha}$

2.5 Wave Particle Duality

Wave-Particle Duality: Postulates that all particles exhibit both wave and particle properties. A central concept of quantum mechanics, this duality addresses the inability of classical concepts like "particle" and "wave" to fully describe the behavior of quantum-scale object.

List of experiments which explain particle nature of light wave .

- Photoelectric effect
- Compton Effect
- Pair production

List of experiments which explain wave nature of particle

- Davisson –Germen effect (diffraction due to electrons)
- Young double slit Interference due to electrons

2.5.1 De Broglie Wave

In quantum mechanics, the concept of **matter waves** or **de Broglie waves** reflects the wave–particle duality of matterThe **de Broglie relations** show that the wavelength is inversely proportional to the momentum of a particle and is also called **de Broglie wavelength**.

The wavelength of the wave associated with a particle as given by the de Broglie relation

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

For relativistic case, the mass becomes $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ where m_0 is rest mass and v is velocity of body.

2.5.2 Uncertainty principle

"It is impossible to determine two canonical variable simultaneously for microscopic particle". If q and p_q are two canonical variable then $\Delta q \Delta p_q \geq \dfrac{\hbar}{2}$

where, Δq is the error in measurement at q and Δp_q is error in measurement at p_q and h is Planck's constant $(\hbar \equiv h/2\pi)$.

Important uncertainty relations:

- $\Delta X \cdot \Delta P_X \geq \dfrac{\hbar}{2}$ (X is position and p_x is momentum in x direction)
- $\Delta E \cdot \Delta t \geq \dfrac{\hbar}{2}$ (E is energy and t is time).
- $\Delta L \cdot \Delta \theta \geq \dfrac{\hbar}{2}$ (L is angular momentum, θ is angle measured)

2.5.3 Superposition Principle

According to de Broglie, matter waves are associated with every moving body. These matter waves moves in a group of different waves having slightly different wavelength. The formation of group is due to superposition of individual wave.

Analogy: If $\psi_1(x,t)$ and $\psi_2(x,t)$ are two waves of slightly different wavelength and frequency.

$$\psi_1 = A\sin(kx-\omega t)$$
$$\psi_2 = A\sin[(k+dk)x-(\omega+d\omega)t]$$
$$\psi = \psi_1+\psi_2$$

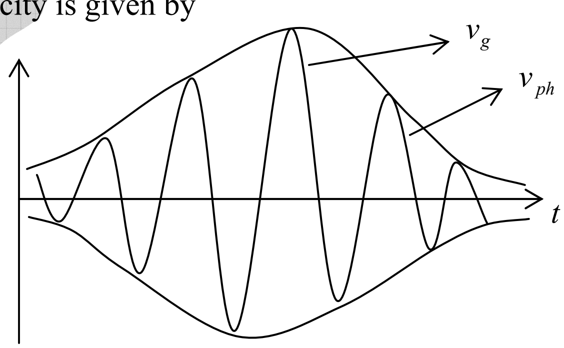
$=2A\cos\left(\frac{dk}{2}-\frac{d\omega t}{2}\right)\sin(kx-\omega t) \rightarrow$ The velocity of individual wave is known as Phase velocity which is given as $v_p = \frac{\omega}{k}$

The velocity of amplitude is given by group velocity v_g which is given by $\frac{d\omega}{dk}$.

$$v_g = \frac{d\omega}{dk}$$

The relationship between group and phase velocity is given by

$$v_g = \frac{d\omega}{dk} = v_p + k\frac{dv_p}{dk}$$
$$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$



Due to superposition of different wave of slightly different wavelength resultant wave moves like a wave packet with velocity equal to group velocity.

Example: Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

Solution: Recall that the energy and momentum of ta relativistic particle are given by

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}, \qquad p = mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Squaring and adding the expressions of E and p, we obtain $E^2 = p^2c^2 + m_0^2c^4$, hence

$$E = c\sqrt{p^2 + m_0^2c^2}$$

Using this relation along with $p^2 + m_0^2c^2/\left(1 - v^2/c^2\right)$ and , we can show that the group velocity is given as follows

$$v_g = \frac{dE}{dp} = \frac{d}{dp}\left(c\sqrt{p^2 + m_0^2c^2}\right) = \frac{pc}{\sqrt{p^2 + m_0^2c^2}} = v$$

The group velocity is thus equal to the speed of the particle,

$$v_g = v$$

The phase velocity can be found from and:

$$v_{ph} = E/p = c\sqrt{1 + m_0^2c^2/p^2}$$

which, when combined with $p = m_0 v/\sqrt{1 - v^2/c^2}$, leads to $\sqrt{1 + m_0^2c^2/p^2} = c/v$,

hence

$$v_{ph} = \frac{E}{p} = c\sqrt{1 + \frac{m_0^2}{p^2}} = \frac{c^2}{v}$$

Example: Use the uncertainty principle to estimate: (a) the ground state radius of the hydrogen atom

(b) the ground state energy of the hydrogen atom

Solution: (a) According to the uncertainty principle, the electron's momentum and the radius of its orbit are related by $rp \sim \hbar$, hence $p \sim \hbar / r$. To find the ground state radius, we simply need to minimize the electron-proton energy

$$E(r) = \frac{p^2}{2m_e} - \frac{e^2}{4\pi\varepsilon_0 r} = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\varepsilon_0 r}$$

with respect to r:

$$0 = \frac{dE}{dr} = -\frac{\hbar^2}{m_e r_0^3} + \frac{e}{4\pi\varepsilon_0 r^2}$$

This leads to the Bohr radius

$$r_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} = 5.3\,nm$$

(b) Inserting we obtain the Bohr energy;

$$E(r_0) = \frac{\hbar^2}{2mr_0^2} - \frac{e^2}{4\pi\varepsilon_0 r_0} = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 = -13.6\,eV$$

The results obtained for r_0 and $E(r_0)$, as shown in , are indeed impressively accurate given the crudeness of the approximation

MCQ (Multiple Choice Questions)

Q1. How is de-Broglie wavelength (λ) of an electron in the nth Bohr orbit related to the radius R of the orbit?

(a) $n\lambda = \pi R$ (b) $n\lambda = \frac{3\pi R}{2}$ (c) $n\lambda = 2\pi R$ (d) $n\lambda = 4\pi R$

Q2. The correct expression for the de-Broglie wavelength λ of a particle (E is the kinetic energy) is

(a) $\lambda = \frac{hc}{\sqrt{\left(E+2m_0c^2\right)}}$ (b) $\lambda = \frac{hc}{\sqrt{E\left(E+m_0c^2\right)}}$

(c) $\lambda = \frac{hc}{\sqrt{E\left(E+2m_0c^2\right)}}$ (d) $\lambda = \frac{hc}{\sqrt{\left(E+m_0c^2\right)}}$

Q3. A particle of mass M at rest decays into two particles of masses m_1 and m_2, having nonzero velocities. The ratio of the de Broglie wavelengths of the particles, λ_1 / λ_2, is

(a) m_1 / m_2 (b) m_2 / m_1 (c) 1 (d) $\sqrt{m_2} / \sqrt{m_1}$

Q4. A proton has kinetic energy E which is equal to that of a photon. The wavelength of photon is λ_2 and that of proton is λ_1. The ratio $\frac{\lambda_2}{\lambda_1}$ is proportional to

(a) E^2 (b) $E^{1/2}$ (c) E^{-1} (d) $E^{-1/2}$

Q5. Electrons with de-Broglie wavelength λ fall on the target in an X-ray tube. The cut-off wavelength of the emitted X-rays is

(a) $\lambda_0 = \frac{2mc\lambda^2}{h}$ (b) $\lambda_0 = \frac{2h}{mc}$ (c) $\lambda_0 = \frac{2m^2c^2\lambda^3}{h^2}$ (d) $\lambda_0 = \lambda$

Q6. In a Compton - scattering experiment, photons with incoming momentum mc (m is the mass of the electron) are scattered at an angle 90°. What is the magnitude of the momentum of the scattered photon?

(a) mc (b) $\frac{mc}{2}$ (c) $\frac{mc}{3}$ (d) $\frac{mc}{4}$

Q7. In photoelectric effect with incoming radiation of frequency v_0 with $hv_0 = 8eV$, electrons of energy $3eV$ are emitted from a metal surface. The energy of the electrons emitted from this surface when radiation with frequency $1.2v_0$ is incident, is:

(a) $4.2eV$ (b) $5.2eV$ (c) $3.6eV$ (d) $4.6eV$

Q8. Diagram shown here corresponds to observations made in photoelectric effect observed with radiation of frequency v, and wavelength λ, resulting in K_{max} as the maximum kinetic energy of photoelectrons. The quantities shown on the x and y-axes in the diagram are:

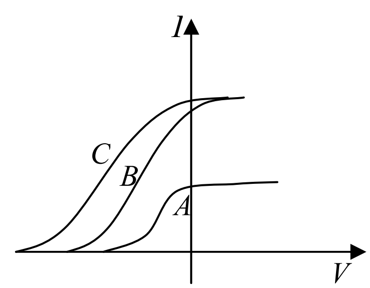
(a) x-axis: λ; y-axis: E_{max} (b) x-axis: E_{max}; y-axis: λ

(c) x-axis: v; y-axis: E_{max} (d) x-axis: E_{max}; y-axis: v

Q9. The maximum kinetic energy of photoelectrons emitted from a surface when photons of energy $6eV$ fall on it is $4eV$. The stopping potential, in volt, is

(a) 2 (b) 4 (c) 6 (d) 10

Q10. In a photoelectric experiment anode potential is plotted against plate current



(a) A and B will have same intensities while B and C will have different frequencies

(b) B and C will have different intensities while A and B will have different frequencies

(c) A and B will have different intensities while B and C will have equal frequencies

(d) B and C will have equal intensities while A and B will have same frequencies.

Q11. In the following electronic transitions in a hydrogen atom, which transition emits the minimum wavelength?

(a) $n = 2$ to $n = 1$ level (b) $n = 3$ to $n = 2$ level

(c) $n = 4$ to $n = 3$ level (d) $n = 5$ to $n = 4$ level

Q12. What is the speed v_n of the electron in the nth Bohr orbit of hydrogen atom, if v_1 is the speed of the electron in the first Bohr orbit?

(a) $v_1 n$ (b) $v_1 n^3$ (c) $\frac{v_1}{n}$ (d) $\frac{v_1}{n^3}$

Q13. If elements with principal quantum number $n > 4$ were not allowed in nature, the number of possible elements would be

(a) 60 (b) 32 (c) 4 (d) 64

Q14. Consider the spectral line resulting from the transition $n = 2 \rightarrow n = 1$ in the atoms and ions given below. The shortest wavelength is produced by

(a) hydrogen atom (b) deuterium atom

(c) singly ionized helium (d) doubly ionised lithium

Q15. A hydrogen atom and a Li^{++} ion are both in the second excited state. If l_H and l_{Li} are their respective electronic angular momenta, and E_H and E_{Li} their respective energies, then

(a) $l_H > l_{Li}$ and $|E_H| > |E_{Li}|$ (b) $l_H = l_{Li}$ and $|\mathrm{E_H}| < |E_{Li}|$

(c) $l_H = l_{Li}$ and $|\mathrm{E_H}| > |E_{Li}|$ (d) $l_H < l_{Li}$ and $|\mathrm{E_H}| < |E_{Li}|$

Q16. $X-$rays are produced in an $X-$ray tube operating at a give accelerating voltage. The wavelength of the continuous $X-$rays has values from

(a) 0 to ∞ (b) $\lambda_{\min}$ to ∞ where $\lambda_{\min} > 0$

(c) 0 to $\lambda_{\max}$ where $\lambda_{\max} < \infty$ (d) $\lambda_{\min}$ to $\lambda_{\max}$ where $0 < \lambda_{\min} < \lambda_{\max} < \infty$

Q17. Given that for an atom with nuclear charge Z_1 the X-ray frequency for transition between two low-lying states is v_1. According to Moseley's law, what is the corresponding frequency v_2 for an atom with nuclear charge Z_2 approximately equal to (ignoring shielding factor)?

(a) $v_2 = v_1 \frac{Z_2}{Z_1}$ (b) $v_2 = v_1 \frac{Z_1}{Z_2}$

(c) $v_2 = v_1 \frac{Z_1^2}{z_2^2}$ (d) $v_2 = v_1 \frac{Z_2^2}{Z_1^2}$

Q18. Which one is the single characteristic of the target element which occurs in Moseley's law for the frequencies of emitted X-rays?

(a) Density
(b) Atomic weight
(c) Atomic number
(d) Spacing between the atomic planes

Q19. For the X-ray spectrum due to transition between $n=2$ and $n=1$ states for large nuclear charge Ze, we have frequencies v_0, v_1, v_2 for $Z_0 = Z_0, Z_1 = Z_0 + 1, Z_2 = Z_0 + 2$ respectively. Moseley's law implies which one of the following equations?

(a) $v_1 = \frac{(v_0 + v_2)}{2}$
(b) $v_1 = (v_0 v_2)^{\frac{1}{2}}$
(c) $\sqrt{v_1} = \frac{\sqrt{v_0} + \sqrt{v_2}}{2}$
(d) $v_1 = \frac{(v_0 v_2)}{(v_0 + v_2)}$

NAT (Numerical Answer Type)

Q20. The de-Broglie wavelengths of a proton and an α - particle are equal. The ratio of their velocities is…………..

Q21. The potential energy of a particle of mass in is given by

$$V(x) = \begin{cases} E_0; & 0 \le x \le 1 \\ 0; & x > 1 \end{cases}$$

λ_1 and λ_2 are the de-Broglie wavelengths of the particle, when $0 \le x \le 1$ and $x > 1$ respectively. 'If the total energy of particle is $2E_0$, then ratio $\frac{\lambda_1}{\lambda_2}$ …………...

Q22. In Compton scattering, an incoming photon of wavelength $\lambda_0 = \frac{h}{2mc}$ ($h =$ Planck's constant, $m =$ mass of electron, $c =$ speed of light) is scattered by an electron at rest. If the photon is scattered backward at angle of 180°, the momentum of the corresponding scattered electron is………….. mc

Q23. In the Compton scattering process, at which scattering angle……… does the maximum energy transfer to the electron occur

Q24. X-ray of energy $50\,keV$ strikes an electron initially at rest. The change in wavelength of the X-ray scattered at angle 90^o is, approximately…………$\times 10^{-12}\,meter$.
(Given, $h = 6.63\times 10^{-34}\,J-s, m = 9.11\times 10^{-31}\,kg$)

Q25. In a photoelectric effect experiment, for radiation with frequency v_0 with $hv_0 = 8eV$, electrons are emitted with energy $2eV$. if the incoming radiation of frequency $1.25v_0$ then the energy of photo electron is ……………….

Q26. A threshold wavelength of a metal is $7000\overset{o}{A}$. The work function is ………….
(Given, velocity of light $c = 3\times 10^8\,m/s$ and Planck's constant $= 6.624\times 10^{-34}\,J-S$)

Q27. The work function of a substance is $4eV$ The longest wavelength of light that can cause photoelectron emission from this substance is approximately…………….$\times 10^{-9}\,m$

Q28. A beam of light has three wavelengths $4144\overset{0}{A}, 4972\overset{0}{A}$ and $6216\overset{0}{A}$ with a total intensity of $3.6\times 10^{-3}\,Wm^{-2}$ equally distributed amongst the three wavelengths. The beam falls normally on an area $1.0cm^2$ of a clean metallic surface of work function $2.3eV$. Assume that there is no loss of light by reflection and that each energetically capable photon ejects one electron. The number of photo electrons liberated in per second ……………..$\times 10^{11}$

Q29. When a beam of $10.6eV$ photons of intensity $2.0\,W/m^2$ falls on a platinum surface of area $1.0\times 10^{-4}\,m^2$ and workfunction $5.6eV$, 0.53% of the incident photons eject photoelectrons. Take1 $eV = 1.6\times 10^{-19}\,J$

(a) The number of photoelectrons emitted per second…………..$\times 10^{11}$.

(b) Minimum energy of photoelectron………………eV.

(c) Maximum energies ………………………eV

Q30. For He^+ which has one electron and a nuclear charge $2e$, then the binding energy of the first excited state with principal quantum number $n = 2$ is ………(eV)

Q31. The shortest wavelength in Lyman series of hydrogen spectra is $91.2\,nm$, the longest wavelength in this series must be ………………nm

Q32. Ionisation potential for a hydrogen atom is $13.6eV$. The ionisation potential for a positronium atom where an electron revolves round a positron, is……….. eV

Q33. An energy of 24.6 eV is required to remove one of the electrons from a neutral helium atom. The energy in (eV) required to remove both the electrons from a neutral helium atom is…………..

Q34. The wavelength of the characteristic X-ray K_α line emitted by a hydrogen like element is $0.32\overset{0}{\text{A}}$. The wavelength of the K_β line emitted by the same element will be……………

Q35. The recoil speed of a hydrogen atom after ii emits a photon in going from $n=5$ state to $n=1$ state is …….m/s.

Q36. As per Bohr model, the minimum energy in eV required to remove an electron from the ground state of doubly ionized Li atom $(Z=3)$ is………..

Q37. K_α wavelength emitted by an atom of atomic number $Z_1 = 11$ is λ. For atomic number $Z_2 =$ ……….. for an atom that emits K_α radiation with wavelength 4λ.

MSQ (Multiple Select Questions)

Q38. Which of the following is correct for Compton effect

(a) The energy of incoming X ray is approximately $20KeV$

(b) The Compton shift is not dependent on energy of incoming X ray

(c) The wavelength of scattered X ray is less than wavelength of incoming X ray

(d) Maximum energy is transfer from X ray to electron at scattering angle 90^0 with horizontal

Q39. For which of the following cases is the de-Brogile wavelength is same?

(a) Particle of mass m, kinetic energy K

(b) Particle of mass $2m$, kinetic energy $2K$

(c) Particle of mass $2m$, kinetic energy $\frac{K}{2}$

(d) particle of $\frac{m}{2}$ and kinetic energy K

Q40. Consider the following statements:

The maximum kinetic energy of a photoelectron depends on:

(a) frequency of incident radiation

(b) nature of photo emitter

(c) intensity of incident radiation

(d) on plate potential

Which of these statements are correct?

Q41. When photons of energy $4.25 eV$ strike the surface of metal A ,, the ejected photoelectrons have maximum kinetic energy $T_A eV$ and de Broglie wavelength λ_A. The maximum kinetic energy of photoelectrons liberated from another metal B by photons of energy $4.70 eV$ is $T_B = (T_A - 1.50) eV$. If the de Broglie wavelength of these photoelectrons is $\lambda_B = 2\lambda_A$, then

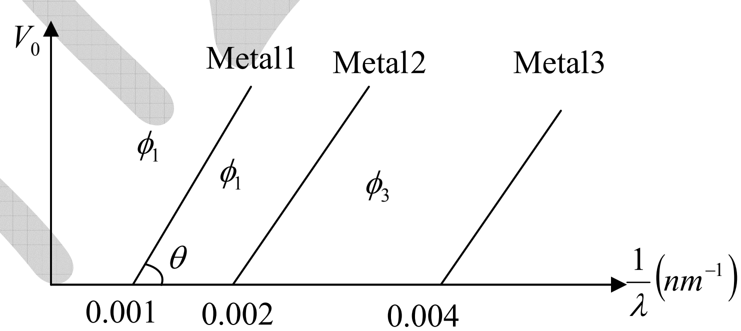
(a) The work function of A is $2.25 eV$

(b) the work function of B is $3.95\ eV$

(c) $T_A = 2.00\ eV$

(d) $T_B = 2.75 eV$

Q42. The graph between the stopping potential (V_0) and $\left(\frac{1}{\lambda}\right)$ is shown in the figure ϕ_1, ϕ_2 and ϕ_3 are work functions. Which of the following is/are correct?



(a) $\phi_1 : \phi_2 : \phi_3 = 1:2:4$

(b) $\phi_1 : \phi_2 : \phi_3 = 4:2:1$

(c) $\tan\theta$ is proportional to $\frac{hc}{e}$ where h is Planck's constant and c is the speed of light

(d) Ultraviolet light can be used to emit photoelectrons from metal 2 and metal 3 only

Q43. In the Bohr model of the hydrogen atom, if kinetic energy is T potential energy is V and total energy is E then which of following is /are correct

(a) $V=\frac{E}{2}$ (b) $V=2E$ (c) $T=-E$ (d) $T=-\frac{E}{2}$

Q44. The transition from the state $n=4$ to $n=3$ in a hydrogen-like atom results in ultraviolet radiation which of the following is /are not Infrared radiation obtained in the transition

(a) $2\rightarrow 1$ (b) $3\rightarrow 2$ (c) $4\rightarrow 2$ (d) $5\rightarrow 4$

Q45. In Bohr's model of the hydrogen atom

(a) the radius of the n^{th} orbit is proportional to n^2

(b) the total energy of the electron in n^{th} orbit is inversely proportional to n^2.

(c) the angular momentum of electron in an n^{th} orbit is an integral multiple of $\frac{h}{2\pi}$.

(d) the magnitude of potential energy of the electron in any orbit is greater than its kinetic energy

Q46. The electron in a hydrogen atom makes a transition $n_1\rightarrow n_2$, where n_1 and n_2 are the principal quantum numbers of the two states. Assume the Bohr model to be valid. The time period of the electron in the initial state is eight times that in the final state. The possible values of n_1 and n_2 are

(a) $n_1=4, n_2=2$ (b) $n_1=8, n_2=2$

(c) $n_1=8, n_2=1$ (d) $n_1=6, n_2=3$

Q47. The shortest wavelength of $X-$rays emitted from an $X-$ ray tube depends on

(a) the current in the tube (b) the voltage applied to the tube

(c) the nature of the gas in tube (d) the atomic number of the target material.

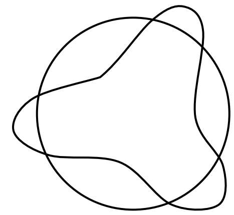
Q48. The potential difference applied to an X - ray tube is increased. As a result, in the emitted radiation

(a) the intensity increases (b) the minimum wavelength increases

(c) the intensity remains unchanged (d) the minimum wavelength decreases

Solutions (MCQ)

Ans. 1: (c)



Solution: According to Bohr Quantization condition, the electron wave can be adjusted around an orbit only when the circumference of the orbit is equal to an integral multiple of the wavelength i.e., $2\pi R = n\lambda$

Ans. 2: (c)

Solution: The de-Broglie wavelength $\lambda = \dfrac{h}{mv}$ (i)

Where relativistic mass $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ $m_0 \rightarrow$ rest mass

$$\Rightarrow \quad \frac{v^2}{c^2} = 1 - \frac{m_0^2}{m^2} \Rightarrow mv = c\sqrt{m^2 - m_0^2} \qquad \text{(ii)}$$

Thus, equation (i) and (ii), we get $\Rightarrow \lambda = \dfrac{hc}{c^2\sqrt{m^2 - m_0^2}}$ (iii)

Now, $c^2\sqrt{m^2 - m_0^2} = \sqrt{c^4\left(m-m_0\right)\left(m+m_0\right)} = \left\{\left(m-m_0\right)\right\}\left\{\left(m-m_0\right)c^2 + 2m_0c^2\right\}^{\frac{1}{2}}$

$= \sqrt{E\left(E + 2m_0c^2\right)}$ [Since $E = \left(m - m_0\right)c^2$]

So, by equation (iii) $\lambda = \dfrac{hc}{\sqrt{E\left(E+2m_0c^2\right)}}$

Ans. 3: (c)

Solution: de-Broglie wavelength $\lambda = \dfrac{h}{p}$ $\therefore \dfrac{\lambda_1}{\lambda_2} = \dfrac{p_1}{P_2}$

Since momentum p is conserved in the decay process, $p_2 = p_1$ $\therefore \dfrac{\lambda_1}{\lambda_2} = 1$

Ans. 4: (d)

Solution: For photon, $E = \dfrac{hc}{\lambda_2}$ or $\lambda_2 = \dfrac{hc}{E}$ (i)

For proton kinetic energy $K = \dfrac{1}{2} m_p v_p^2$ or $\lambda_1 = \dfrac{h}{\sqrt{2m_p K}} = \dfrac{h}{\sqrt{2m_p E}}$ (ii)

From (i) and (ii), $\dfrac{\lambda_1}{\lambda_2} = \dfrac{hc}{E} \times \dfrac{\sqrt{2m_p E}}{h}$ or $\dfrac{\lambda_2}{\lambda_1} = \dfrac{c \times \sqrt{2m_p}}{\sqrt{E}} = c\sqrt{2m_p} \times E^{-1/2}$ or $\dfrac{\lambda_2}{\lambda_1} \propto E^{-1/2}$

Ans. 5: (a)

Solution: Let K be the kinetic energy of the incident electron. Its linear momentum $p = \sqrt{2mK}$.

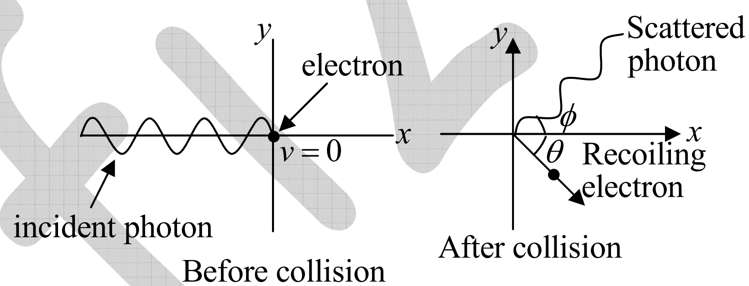
The de-Broglie wavelength is related to the linear momentum as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} \quad \text{or } K = \frac{h^2}{2m\lambda^2}$$

The cut-off wavelength of the emitted $X-$ray is related to the kinetic energy of incident electron as $\dfrac{hc}{\lambda_0} = K = \dfrac{h^2}{2m\lambda^2} \Rightarrow \lambda_0 = \dfrac{2mc\lambda^2}{h}$

Ans. 6: (b)

Solution:



The change in wavelength of scattered photon is given as $\Delta\lambda = \dfrac{h}{m_0 c}(1 - \cos\phi)$ where ϕ is scattered angle. Hence, $\phi = 90^\circ$ (given)

So, $\Delta\lambda = \dfrac{h}{m_0 c}(1 - \cos 90^\circ) = \dfrac{h}{m_0 c}(1 - 0) \Rightarrow \Delta\lambda = \dfrac{h}{m_0 c}$

if λ is wavelength of incident photon then wavelength of photon $= \lambda + \dfrac{h}{mc}$ (i)

But moment of incident photon mc (given) $\Rightarrow \quad \lambda = \frac{h}{mc}$

So, wavelength of scattered photon $= \frac{h}{mc} + \frac{h}{mc}$ by equation (i) $\lambda_s = \frac{2h}{mc}$

Corresponding momentum $= \frac{h}{\lambda_s}$ Momentum $= \frac{h}{\frac{2h}{mc}} = \frac{1}{2}mc$

Ans. 7: (d)

Solution: In photoelectric effect, the energy of electron is given as

$$\frac{mv^2}{2} = h\nu - W \quad \Rightarrow 3eV = 8eV - W$$

work function $W = (8-3)eV$, $W = 5eV$

Now, if energy corresponding to ν_0 is $8eV$, then energy corresponding to $1.2\nu_0$.

$= 1.2 \times 8eV = 9.6eV$

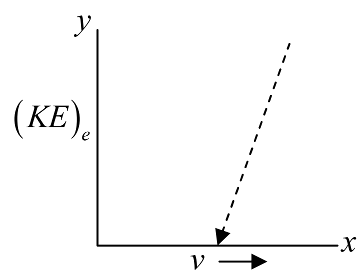
so, energy of electron = $h\nu' - W = (9.6-5)eV = 4.6eV$

Ans. 8: (c)

Solution: If the threshold frequency ν_0 is given as then

$\frac{1}{2}mv^2 = h\nu - h\nu_0$ where h is Planck's constant i.e., (KE) of electron $= h\nu - h\nu_0$

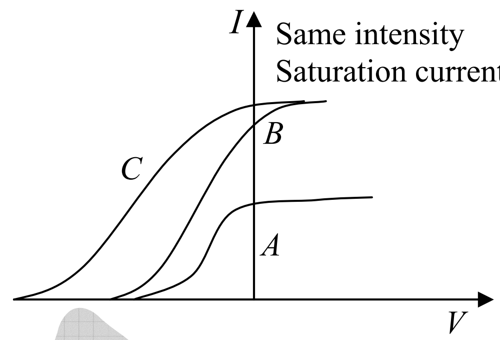
thus, y-axis is KE of electron and x-axis is frequency of incident photon.



Ans. 9: (b)

Solution: Stopping potential is the negative potential which stops the emission of $(K.E)_{max}$ electrons when applied. $\therefore$ Stopping potential = 4 volt

Ans. 10: (d)



Solution: At stopping potential, photoelectric current is zero. It is same for A and B.

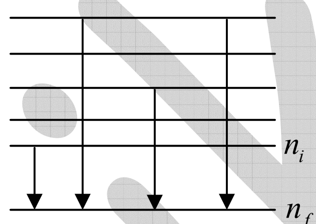
$\therefore A$ and B will have equal frequencies

Saturation current is proportional to intensity. B and C will have equal intensity

Option (d) represents correct answer.

Ans. 11: (a)

Solution: The wavelength in hydrogen atom's transition is given as

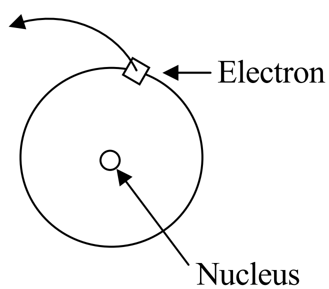


$$\frac{1}{\lambda} = R\left[\frac{1}{n_f^2} - \frac{1}{n_i^2}\right]$$

where R in Rydberg constant clearly λ will be minimum if $n = 2 \rightarrow n = 1$.

Ans. 12: (c)

Solution: The electrostatic force = centripetal force



i.e., $\dfrac{mv^2}{r} = \dfrac{Ze^2}{kr^2}$

$$\Rightarrow \qquad mv^2 r = \frac{Ze^2}{k} \qquad \text{(i)}$$

By Bohr theory $mvr = n\hbar$ (ii)

Dividing equation (i) by, equation (ii), we get

$$v = \frac{Ze^2}{kn\hbar} \Rightarrow v = \frac{1}{n}\left(\frac{Ze^2}{k\hbar}\right) \Rightarrow v \propto \frac{1}{n} \Rightarrow \frac{v_1}{v_2} = \frac{n_2}{n_1} \Rightarrow v_2 = \frac{n_1}{n_2}v_1 \Rightarrow v_n = \frac{v_1}{n}$$

Ans. 13: (a)

Solution: The maximum number of electrons in an orbit are $2n^2$. If $n > 4$ is not allowed, the maximum number of electrons that can lie in first four orbits are

$2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 = 2 + 8 + 18 + 32 = 60$ $\therefore$ Possible elements can be 60.

Ans. 14: (d)

Solution: $\frac{1}{\lambda} = RZ^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$ $\qquad \therefore \frac{1}{\lambda} \propto Z^2$.

λ is shortest if Z is largest. Z is largest for doubly ionised lithium atom $(Z = 3)$ among the given elements.

Hence wavelength for doubly ionised lithium will be the least.

Ans. 15: (b)

Solution: In the second excited state, $n = 3$

$$\therefore l_H = l_{Li} = 3\left(\frac{h}{2\pi}\right) \quad Z_H = 1, Z_{Li} = 3, E \propto Z^2 \therefore |E_{Li}| = 9|E_H| \quad \Rightarrow |E_H| < |E_{Li}|$$

Ans. 16: (b)

Solution: In $X-$ray tube, $\lambda_{\min} = \frac{12375}{V(volt)}$ where $\lambda_{\min}$ is in Å

All wavelengths greater than $\lambda_{\min}$ are found.

Option (b) is correct

Ans. 17: (d)

Solution: Moseley's law states that the square root of the frequency of a K-line is closely proportional to the atomic number of the element and may be expressed as

$\sqrt{v} \propto Z \quad \Rightarrow \quad v = kZ^2$ where k is a constant Thus, $v_1 = kZ_1^2$ and $v_2 = kZ_2^2$

$$\frac{v_2}{v_1} = \frac{Z_2^2}{Z_1^2} \quad \Rightarrow \quad v_2 = v_1\frac{Z_2^2}{Z_1^2}$$

Ans. 18: (c)

Solution: Moseley's law state that the square root of frequency of H_α is directly proportional to the atomic number is $\sqrt{\nu} \propto Z$.

Ans. 19: (c)

Solution: Moseley law states that the square root of frequency for K_α -lines is directly proportional to the atomic number of the element i.e.,

we have given as $\sqrt{\nu} = K(Z-1)$ (1)

$\sqrt{\nu_0} = K(Z_0 - 1)$ (2)

$\sqrt{\nu_1} = K(Z_0 + 1 - 1)$ $\sqrt{\nu_1} = KZ_0$ (3)

$\sqrt{\nu_2} = K(Z_0 + 2 - 1)$ $\sqrt{\nu_2} = K(Z_0 + 1)$ (4)

Adding equations (2) and (4)

$$\sqrt{\nu_0} + \sqrt{\nu_2} = K(Z_0 + 1 + Z_0 - 1) = 2KZ_0 = 2\sqrt{\nu_1} \quad \Rightarrow \frac{\sqrt{\nu_0} + \sqrt{\nu_2}}{2} = \sqrt{\nu_1}$$

NAT

Ans. 20: 4

Solution: Every particle of mass m moving with velocity v is associated with a wave of wavelength given as $\lambda = \dfrac{h}{mv}$ $\quad h \to$ Planck's constant

the wavelength for proton $\quad \lambda_p = \dfrac{h}{m_p v_p}$ (i)

and the wavelength for He $\quad \lambda_\alpha = \dfrac{h}{m_\alpha v_\alpha}$ (ii)

Since, $\lambda_\alpha = \lambda_p \Rightarrow \quad \dfrac{h}{m_p v_p} = \dfrac{h}{m_\alpha v_\alpha} \Rightarrow \dfrac{v_p}{v_\alpha} = \dfrac{m_\alpha}{m_p}$ (iii)

Since, ${}_2He^4$ has four nucleons so $m_\alpha = 4m_p$

Thus, (iii) $= \dfrac{v_p}{v_\alpha} = 4 \quad \Rightarrow v_p : v_\alpha = 4:1$

Ans. 21: 1.414

Solution: de Broglie wavelength $\lambda = \dfrac{h}{\sqrt{2mK}}$ where K denotes kinetic energy of particle

Case (I): $0 \le x \le 1$

Given: potential energy $= E_0$, Given: Total energy $= 2E_0$

$\therefore$ Kinetic energy $= 2E_0 - E_0 = E_0 \therefore \lambda_1 = \dfrac{h}{\sqrt{2mE_0}}$

Case (II): $x > 1$

Given: potential energy $V(x) = 0$, given: Total energy $= 2E_0$

$\therefore$ Kinetic energy $= 2E_0$

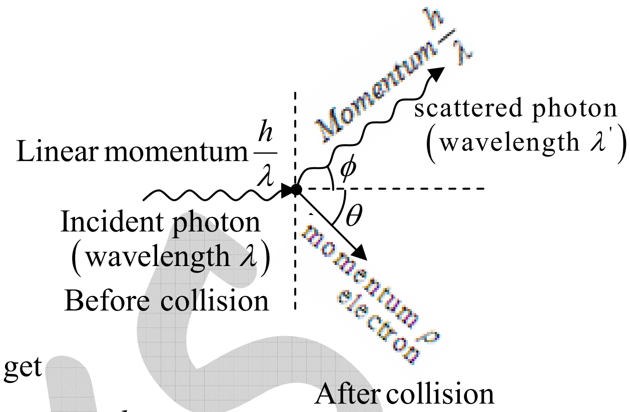
$$\therefore \lambda_2 = \frac{h}{\sqrt{2m(2E_0)}} \quad \therefore \frac{\lambda_1}{\lambda_2} = \sqrt{\frac{2E_0}{E_0}} = \sqrt{2} \quad \therefore \frac{\lambda_1}{\lambda_2} = \sqrt{2}$$

Ans. 22: 2.4

Solution: Applying the conservation law of linear momentum along the horizontal and vertical component, we have

Horizontal $\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\phi + p\cos\theta$ (i)

Vertical component $\frac{h}{\lambda'}\sin\phi = p\sin\theta$ (ii)



By equation $\lambda = \frac{h}{2mc}; \phi = 180^\circ$

Putting these values in equations (i) and (ii), we get

$\Rightarrow \quad 2mc = \frac{h}{\lambda'}\cos 180^\circ + p\cos\theta \quad \Rightarrow 2mc = p\cos\theta - \frac{h}{\lambda'}$ (iii)

by equation (ii) $\Rightarrow \quad 0 = p\sin\theta$ By equations (iii) and (ii), we get $p = \frac{12mc}{5}$

Ans. 23: 3.14

Solution: In Compton effect the range in wavelength is given as $\Delta\lambda = \frac{h}{m_0 c}(1-\cos\phi)$ where

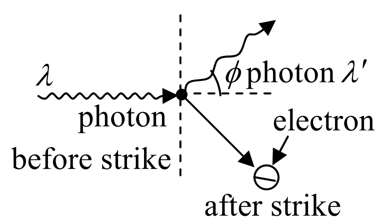
$m_0 =$ electron rest mass, $\phi =$ scattering angle.

The maximum energy is transferred to electron, if $\Delta\lambda$ is maximum

i.e., $(1-\cos\phi)$ is maximum $\Rightarrow \quad \phi = 180^\circ$ Thus, scattering angle is 180° or π.

Ans. 24: 2.4

Solution: When a photon of wavelength λ strike a stationary electron the wavelength of the photon increases. This effect is known as Compton effect.



The change b wavelength is given as $\Delta\lambda = \frac{h}{m_0 c}(1-\cos\phi)$

The change in wavelength does not depend on the energy of the incident photon.

Here, $\phi = 90^\circ \quad \Delta\lambda = \frac{h}{m_0 c}\left(1-\cos 90^\circ\right) = \frac{h}{m_0 c} = 2.4\times 10^{-12}\,m$

Ans. 25: $4\ eV$

Solution: The kinetic energy of photoelectron is given as

$KE = h\nu - W$

where $\nu \to$ frequency of incident photon $W \to$ work function of the substance $h \to$ Planck constant

Now, $h\nu_0 = 8\,eV, KE = 2\,eV$

So, work function $= (8-2)\,eV \quad W = 6\,eV$

Now, if $h\nu_0 = 8\,eV \quad \Rightarrow 1.25\,h\nu_0 = 10\,eV$

Thus, $h\nu = 10\,eV, W = 6\,eV \quad \Rightarrow$ Kinetic energy $= h\nu - W = 10\,eV - 6\,eV = 4\,eV$

Ans. 26: $1.775\,eV$

Solution: The maximum wavelength λ of a photon which can emit electron from a metal is known as threshold wavelength of the metal. The energy corresponding to this threshold wavelength is equal to work function of the metal. Hence, Work function = Threshold energy

$$\Rightarrow \quad = \frac{hc}{\lambda_0} = \frac{6.624\times10^{-34}\times3\times10^{8}}{7000\times10^{-10}} = \frac{12375}{7000}\,eV = 1.775\,eV$$

Ans. 27: 310

Solution: $\lambda_{\max} = \dfrac{hc}{\text{work function}}$

$$\therefore \lambda_{\max} = \frac{\left(6.63\times10^{-34}\right)\times\left(3\times10^{8}\right)}{4\times1.6\times10^{-19}} \quad \text{or } \lambda_{\max} = 310\times10^{-9}\,m$$

Ans. 28: 5.5

Solution: Energy of photon in $eV = \dfrac{12375}{\lambda\left(\overset{0}{A}\right)}$

$$\therefore E_1 = \frac{12375}{4144}\,eV = 2.99\ eV,\ E_2 = \frac{12375}{4972}\,eV = 2.49\,eV\,, \quad E_3 = \frac{12375}{6216}\,eV = 1.99\,eV$$

work function $= 2.3\,eV$

First two wavelengths have energies great than work function of metallic surface. Hence they can eject photoelectrons.

Total intensity $= 3.6\times10^{-3}\,Wm^{-2}$

Number of wavelengths = 3

Intensity per wavelength $= \dfrac{3.6\times10^{-3}}{3} = 1.2\times10^{-3}\;\; Wm^{-2}$ Area $= 10^{-4}\;\; m^2$

$\therefore$ Energy falling per second $= \left(1.2\times10^{-3}\right)\times10^{-4} = 1.2\times10^{-7}\; J/S$

$\therefore$ Let number of photons of first wavelength $= n_1$ and number of photons of second wavelength $= n_2$

$$\therefore\;\; n_1 = \frac{1.2\times10^{-7}}{2.99\times\left(1.6\times10^{-19}\right)} = 2.5\times10^{11} \qquad n_2 = \frac{1.2\times10^{-7}}{2.49\times1.6\times10^{-19}} = 3.0\times10^{11}$$

$\therefore$ Total photons per second $= (2.5+3.0)10^{11} = 5.5\times10^{11}$ $\therefore$ Each capable photon ejects an electron

$\therefore$ Photoelectrons liberated in sec 5.5×10^{11}

Ans. 29: (a) 6.25, (b) 0, (c) 5

Solution: (a) Incident energy $E = 10.6\,eV = 10.6\times\left(1.6\times10^{-19}\right)J = 16.96\times10^{-19}\,J$

$$\frac{\text{Energy incident}}{\text{area}\times\text{time}} = 2\;\; W/m^2$$

$$\therefore \frac{\text{Number of incident photons}}{\text{area}\times\text{time}} = \frac{2}{16.96\times10^{-19}} = 1.18\times10^{18}$$

$$\therefore \frac{\text{Incident photons}}{\text{time}} = \left(1.18\times10^{18}\right)\times\text{area}$$

$$= 1.18\times10^{18}\times\left(1.0\times10^{-4}\right) = 1.18\times10^{14}$$

$$\therefore \frac{\text{Number of photoelectrons}}{\text{time}} = \left(\frac{0.53}{100}\right)\times\left(1.18\times10^{14}\right) \Rightarrow n = 6.25\times10^{11} \qquad \text{(ii)}$$

Minimum energy = zero

Maximum energy $= E_1 -$ work function

or $K_{\max} = (10.6-5.6)eV$ or $K_{\max} = 5.0\,eV$ (iii)

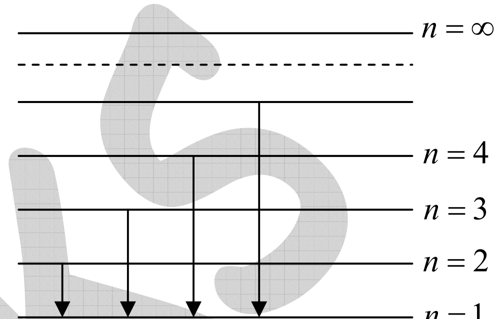
Ans. 30: $13.6eV$

Solution: Since, energy $\propto Z^2$ Energy for $He^+ = (4)\left(\frac{-13.6}{n^2}\right)eV = -\frac{4\times 13.6}{n^2}eV$

The first excited state with principal quantum number $n = 2$

So, binding energy $= -\frac{4\times 13.6}{4} = -13.6\,eV$

Ans. 31: $68.4nm$



Solution: When electron moving in n_i the orbit transited to n_f orbit the frequency of radiation and so wavelength is given by $\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$

where R is Rydberg's constant.

when $n_i = 1$, then the series of spectral lines are known as Lyman series.

Thus, Lyman series is given as $\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n_i^2}\right)$ $\quad (\because n_f = 1)$

$$\Rightarrow \frac{1}{\lambda_{\max}} = \frac{R}{1^2} \Rightarrow \lambda_{\max} = \frac{1}{R} \text{ and } \frac{1}{\lambda_{\min}} = R\left(\frac{1}{1^2} - \frac{1}{2^2}\right) \Rightarrow \frac{1}{\lambda_{\min}} = R\left(1 - \frac{1}{4}\right) \Rightarrow \lambda_{\min} = \left(\frac{4}{3R}\right)$$

By equation (i) and (ii), we have $\frac{\lambda_{\max}}{\lambda_{\min}} = \frac{4}{3} \Rightarrow \lambda_{\max} = \frac{3}{4}\times\lambda_{\min} = \frac{3}{4}\times 91.2\,nm = 68.4\,nm$

Ans. 32: $6.8\,eV$

Solution: The amount of energy required to an electron moving in ground state of the atom, so that electron go off from the atom is called ionisation energy and potential corresponding to this energy is called ionisation potential for hydrogen this is given as $E = 13.6\,eV$

Since, positronium is a system in which electron revolves round the positron hence reduced mass of the positroniuin $\mu = \frac{M_e M_p}{M_e + M_p} = \frac{1}{2}M_e$ $\quad$ (Since $M_p = M_e$)

and E is proportional to reduce mass so for positronium $E = \frac{13.6}{2}eV = 6.8\,eV$

Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

Ans. 33: $79eV$

Solution: When one of the electrons is removed from a neutral helium atom, energy is given by

$$E_n = -\frac{13.6Z^2}{n^2} eV \text{ per atom}$$

For helium ion, $Z = 2$, when doubly ionized

For first orbit, $n = 1 \therefore E_1 = -\frac{13.6}{(1)^2} \times (2)^2 = -54.4eV$

$\therefore$ Energy required removing it $54.4eV$ $\therefore$ Total energy required $= 54.4 + 24.6 = 79eV$

Ans. 34: $0.27 \overset{0}{A}$

Solution: K_α corresponds to: $n = 2$ to $n = 1$ K_β corresponds to: $n = 3$ to $n = 1$

$$\frac{1}{\lambda_\alpha} = R\left[\frac{1}{1} - \frac{1}{4}\right] \text{ or } \frac{1}{0.32\overset{0}{A}} = \frac{3R}{4} \rightarrow \frac{1}{\lambda_\beta} = R\left[\frac{1}{1^2} - \frac{1}{3^2}\right] = \frac{8R}{9}$$

$$\therefore \frac{\lambda_\beta}{0.32\overset{0}{A}} = \frac{3R}{4} \times \frac{9}{8R} \quad \text{or} \quad \lambda_\beta = \frac{27}{32} \times 0.32\overset{0}{A} \text{ or } \lambda_\beta = 0.27\overset{0}{A}$$

Ans. 35: $4.33 \ m/s$

Solution: Linear momentum is conserved in the recoil process.

Momentum of recoil hydrogen atom $= mv$

Momentum of emitted photon $= \frac{\Delta E}{c}$

$$\Delta E = E_5 - E_1 = -13.6\left[\frac{1}{5^2} - \frac{1}{1^2}\right] eV \qquad = (13.6)\left(\frac{24}{25}\right) eV = \frac{13.6 \times 24}{25} \times \left(1.6 \times 10^{-19}\right) J$$

$$\Delta E = 20.8 \times 10^{-18} J \qquad \therefore mv = \frac{\Delta E}{c} \text{ or } \qquad v = \frac{\Delta E}{mc}$$

$$v = \frac{20.8 \times 10^{-18}}{\left(1.67 \times 10^{-27}\right) \times \left(3 \times 10^8\right)} = 4.33 \ m/s$$

Ans. 36: $-122.4eV$

Solution: For hydrogen atom and hydrogen like atoms

$E_n = -\frac{13.6z^2}{n^2}eV$ Therefore, ground state energy of doubly ionized lithium atom

$(Z=3, n=1)$ will be $\therefore E_1 = \frac{-13.6\times(3)^2}{(1)^2} = -13.6\times 9$ or $E_1 = -122.4\,eV$

Ans. 37: 6

Solution: According to Moseley's law,

$f = a^2(Z-b)^2$ where $f = \text{frequency} = \frac{c}{\lambda}$

For K_α line, $b=1$ $\quad\therefore \frac{c}{\lambda} = a^2(z-1)^2$

For one atom, $\frac{c}{\lambda_1} = a^2(Z_1-1)^2$

For other atom, $\frac{c}{\lambda_2} = a^2(Z_2-1)^2$ or $\frac{\lambda_1}{\lambda_2} = \frac{a^2(Z_2-1)^2}{a^2(Z_1-1)^2}$

or $\frac{\lambda}{4\lambda} = \frac{(Z_2-1)^2}{(11-1)^2} = \frac{(Z_2-1)^2}{100}$ or $(Z_2-1)^2 = 25$

or $Z_2 - 1 = 5$ or $Z_2 = 6$

MSQ

Ans. 38: (a) and (b)

Ans. 39: (a), (c) and (d)

Solution: Every moving particle is associated with wave, the wavelength of this wave is called de-Broglie wavelength given as $\lambda = \frac{h}{mv}$ where h is Planck's constant

The kinetic energy and momentum is related as $E = \frac{(p)^2}{2m}$

For constant wavelength momentum should be equal

$$\Rightarrow \quad p = \sqrt{2mE} \quad \Rightarrow \lambda = \frac{h}{\sqrt{2mE}}$$

For $m = m, E = K \quad \lambda_1 = \frac{h}{\sqrt{2mK}}$

$m = 2m, E = 2K \quad \lambda_2 = \frac{h}{2\sqrt{mK}}$

$m = 2m, E = \frac{K}{2} \quad \lambda_3 = \frac{h}{\sqrt{2mK}}$

$m = \frac{m}{2} \quad E = 2K \quad \lambda_4 = \frac{h}{\sqrt{2mK}}$

Ans. 40: (a) and (b)

Solution: When a photon of frequency v incident on a target metal of threshold frequency v_0 and if $v > v_0$ the electron from metal is ejected. These electron are known as photoelectrons and this effect is called photo electric effect.

The kinetic energy of the ejected electron $= hv - hv_0$

$\Rightarrow KE$ depends on incident frequency v and v_0 i.e., nature of photo emitter.

Ans. 41: (a), (b) and (c)

Solution: Consider metal A

Incident energy = work function + Kinetic energy of photoelectrons

$$\therefore 4.25(eV) = W_A + T_A \qquad \text{(i)}$$

Kinetic energy = $\dfrac{P^2}{2m}$ where $P =$ momentum

$$\therefore T_A = \frac{P_A^2}{2m} = \frac{1}{2m}\left(\frac{h}{\lambda_A}\right)^2 \text{, by de Broglie equation} \therefore T_A = \frac{1}{2m}\left(\frac{h}{\lambda_A}\right)^2 \qquad \text{(ii)}$$

consider metal B

$$4.7 = (T_A - 1.5) + W_B \qquad \text{(iii)}$$

Also $$T_B = \frac{1}{2m}\left(\frac{h}{\lambda_B}\right)^2 = \frac{1}{2m}\left(\frac{h}{\lambda_B}\right)^2 \qquad \text{(iv)}$$

From (iv) and (ii), we get

$$\frac{T_B}{T_A} = \left(\frac{\lambda_A}{\lambda_B}\right)^2 \quad \text{or} \quad \frac{T_A - 1.5}{T_A} = \left(\frac{\lambda_A}{\lambda_B}\right)^2 \quad [\because T_B = T_A - 1.5]$$

or $$\frac{T_A - 1.5}{T_A} = \left(\frac{\lambda_A}{2\lambda_A}\right)^2 \quad [\because \lambda_B = 2\lambda_A]$$

or $$4T_A - 6.0 = T_A \text{ or } 3T_A = 6$$

$$T_A = 2.00eV \qquad \text{(v)}$$

from (i), $W_A = 4.25 - T_A$

$$= 4.25 - 2 = 2.25eV$$

From (iii),

$$W_B = 4.7 - (T_A - 1.5) = 4.7 - (2 - 1.5) = 3.95eV \qquad \text{(vii)}$$

Again $$T_B = T_A - 1.5 = 2.25 - 1.5 = 0.5eV \qquad \text{(viii)}$$

Option (a), (b), (c) are correct as depicted in equation (vi) (vii) and (v) above

Ans. 42: (a) and (c)

Solution: According to Einstein's equation, $\frac{hc}{\lambda}-\phi=eV$ where $\phi=$ work function

or $V=\frac{hc}{e\lambda}-\frac{\phi}{e}$ or $V=\left(\frac{hc}{e}\right)\frac{1}{\lambda}-\frac{\phi}{e}$

V and $\left(\frac{1}{\lambda}\right)$ relation represents a straight line.

$\therefore$ Slope of line $=\frac{hc}{e}$ or $\tan\theta=\frac{hc}{e}$

$\therefore$ Option (c) is correct (i)

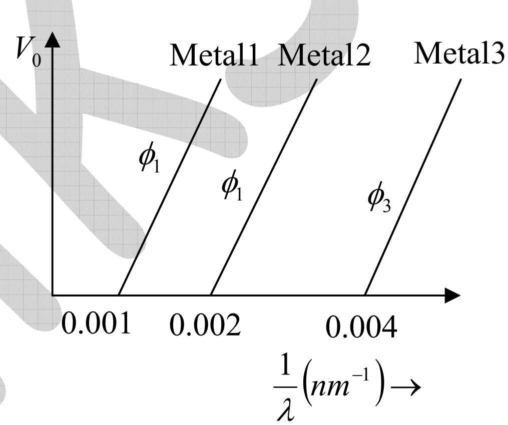
At $V_0=0$

$\phi_1:\phi_2:\phi_3=\frac{hc}{\lambda_{01}}:\frac{hc}{\lambda_{02}}:\frac{hc}{\lambda_{03}}$

$=(0.001\,hc):(0.002\,hc):(0.004\,hc)=1:2:4$

$\therefore$ Option (a) is correct (ii)

Option (b) is obviously incorrect when (a) is correct.



From graph

$$\frac{1}{\lambda_{01}}=0.001\,nm^{-1}\therefore\lambda_{01}=1000nm \text{ for metal 1.}$$

$$\frac{1}{\lambda_{02}}=0.002\,nm^{-1}\therefore\lambda_{02}=500nm \text{ for metal 2}$$

$$\frac{1}{\lambda_{03}}=0.004\,nm^{-1}\therefore\lambda_{03}=250nm \text{ for metal 3.}$$

λ of ultraviolet < 400nm.

The ultraviolet light can be used to emit photoelectrons from metal 1 and metal 2. It cannot emit electrons from metal 3.

Option (d) is incorrect.

Options (a) and (c) are correct.

Ans. 43: (b) and c

Solution: The kinetic energy of an electron in nth orbit of hydrogen atom is

$$K=\frac{me^4}{8\varepsilon_0^2h^2n^2},\quad V=-\frac{e^2}{4\pi\varepsilon_0 r}=E=\frac{-me^{-4}}{8\varepsilon_0^2h^2n^2}$$

The total energy of an electron in nth orbit of hydrogen atom is $E=\frac{-me^{-4}}{8\varepsilon_0^2h^2n^2}\quad \therefore \frac{K}{E}=-1$

Ans. 44: (a), (b) and (c)

Solution: In hydrogen like atoms: $\frac{1}{\lambda}=R\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$

Transition of electron occurs from n_2 to n_1, $\frac{1}{\lambda}$ is proportional to energy

From $n=4$ to $n=3$, ultraviolet radiation is obtained $\frac{1}{\lambda}=R\left(\frac{1}{3^2}-\frac{1}{4^2}\right)=\frac{7R}{144}=0.048R$

(a) $\frac{1}{\lambda}=R\left(\frac{1}{1^2}-\frac{1}{2^2}\right)=\frac{3R}{4}=0.75R$

(b) $\frac{1}{\lambda}=R\left(\frac{1}{2^2}-\frac{1}{3^2}\right)=\frac{5R}{36}=0.14R$

(c) $\frac{1}{\lambda}=R\left(\frac{1}{2^2}-\frac{1}{4^2}\right)=\frac{3R}{16}=0.2R$

(d) $\frac{1}{\lambda}=R\left(\frac{1}{4^2}-\frac{1}{5^2}\right)=\frac{9R}{400}=0.02$

λ is smaller than ultra violet in (a), (b) and (c)

λ is greater than ultra violet in (d). greater the λ, less the energy of radiation

Ans. 45: (a), (b), (c) and (d)

Solution: (a) $r_n\propto n^2$. Option (a) is correct

(b) Total energy of electron $T.E.\quad T.E=\frac{-13.6Z^2}{n^2}$. Option (b) is correct

(c) Angular momentum of electron $=\frac{nh}{2\pi}$. Option (c) is correct

(d) Potential energy of electron $=\left(\frac{-27.2}{n^2}\right)eV$ for hydrogen atom.

Kinetic energy of electrons $=\left(\frac{13.6}{n^2}\right)eV$

$\therefore |P.E.|=2\times|K.E.|\quad \therefore |P.E.|=\frac{27.2}{n^2}$ The option (d) is correct

Ans. 46: (a) and (d)

Solution: According to Bohr model, $r_n \propto n^2$ (i) $v_n \propto \frac{1}{n}$ (ii)

now $T_n = \frac{2\pi}{\omega} = \frac{2\pi r_n}{v_n}$ or $T_n \propto \frac{r_n}{v_n}$ or $T_n \propto \frac{n^2}{1/n} \Rightarrow T_n \propto n^3$

$\therefore \frac{(T_n)_1}{(T_n)_2} = \frac{n_1^3}{n_2^3}$ or $8 = \left(\frac{n_1}{n_2}\right)^3$ or $n_1 = 2n_2$

Option (a): $n_1 = 4,\ n_2 = 2$ It fulfils condition

Option (d): $n_1 = 6, n_2 = 3$ It fulfils condition

Options (a) and (d) are correct

Ans. 47: (b) and (d)

Solution: $X-$rays emitted from an $X-$ray tube depend upon:

(i) The accelerating voltage applied to tube. When accelerated, the electrons acquire greater energy before striking the target.

$X-$rays emitted from target therefore possess greater energy. $X-$rays with shorter wavelength possess greater energy. Hence wavelength of emitted $X-$rays depends on the voltage applied to tube.

(ii) According to Moseley's law, frequency $\nu = a^2 (Z-b)^2$. Frequency depends upon atomic number of target from which $X-$ray are emitted.

Ans. 48: (c) and (d)

Solution: for $X-$ray tube,

$$\lambda_m \left(\overset{0}{A}\right) = \frac{12375}{V}$$

As accelerating voltage is increased, λ_m will decrease.

Number of electrons bombarding the target determine the intensity (or quantity) of emitted radiation. Accelerating voltage does not change the intensity of $X-$rays emitted.

3. Tools and Postulates of Quantum Mechanics

3.1 The Linear Vector Space

A set of vectors ψ, ϕ, χ, … and set of scalars a, b, c defined vector spaces which will follow A rule for vector addition and rule for scalar multiplication

(i) Addition rule:

If ψ and ϕ are vectors of elements of a space, their sum $\psi + \phi$ is also vector of the same space.

(i) Law of Commutative: $\psi + \phi = \phi + \psi$

(ii) Law of Associativity: $(\psi + \phi) + \chi = \psi + (\phi + \chi)$

(iii) Law of Existence of a null vector and inverse vector $\psi + (-\psi) = (-\psi) + \psi = 0$

(ii) Multiplication rule:

- The product of scalar with a vector gives another vector, If ψ and ϕ are two vectors of the space, any linear combination $a\psi + b\phi$ is also a vector of the space, where a and b being scalars.
- Distributive with respect to addition:

$$a(\phi + \psi) = a\phi + a\psi, \quad (a+b)\psi = a\psi + b\psi$$

- Associativity with respect to multiplication of scalars. $a(b\psi) = (ab)\psi$
- For each element ψ there must exist a unitary element I and a no. zero scalar such that $I \cdot \psi = \psi \cdot I = \psi$, $O \cdot \psi = \psi \cdot O = O$

3.1.1 Scalar Product

The scalar product of two functions $\phi(x)$ and $\phi(x)$ is given by $(\psi, \phi) = \int \psi^*(x)\phi(x)dx$

where $\phi(x)$ and $\psi(x)$ are two complex function of variable x. $\phi^*(x)$ and $\psi^*(x)$ are complex conjugate of $\phi(x)$ and $\psi(x)$ respectively.

The scalar product of two function $\phi(x, y, z)$ and $\psi(x, y, z)$ is 3 dimensional is defined or

$$(\psi^*, \phi) = \int \psi^* \phi \, dx\, dy\, dz$$

3.1.2 Hilbert Space

The Hilbert space H consists of a set of vectors ψ , ϕ, χ and set of scalar a,b,c which satisfied the following four properties.

(i) H is a linear space

(ii) H is a linear space that defined scalar product that is strictly positive.

- $(\psi,\phi)=(\phi,\psi)^*$
- $(\psi,a\phi_1+b\phi_2)=a(\psi,\phi_1)+b(\psi,\phi_2)$
- $(\psi,\psi)=|\psi|^2\geq 0$

(iii) H is separable i.e.,

- $\|\psi-\psi_n\|\leq 0$

(iv H is complete $\|\psi-\psi_m\|=0$ when $m\to\infty, n\to\infty$

3.1.3 Dimension And Basis of a Vectors.

Linear independency:

A set of N vectors $\phi_1,\phi_2,\phi_3......\phi_n$, is said to be linearly independent if and only if the solution of the equation. $\sum_{i=1}^{N}a_i\phi_i=0$ is $a_1=a_2=a_3=a_4=0$ other wise $\phi_1,\phi_2,\phi_3......\phi_n$ is said to be linear dependent.

The dimension of a space vector is given by the maximum number of linearly independent vectors the space can have.

The maximum number of linearly independent vectors a space has is N

$\phi_1,\phi_2,\phi_3......\phi_N$, this space is said to be N dimensional. In this case any vector ψ of the vector space can be expressed as linear combination. $\psi=\sum_{i=1}^{N}a_i\phi_i$

Ortho Normal Basis

Two vector ϕ_i,ϕ_j is said to be orthonormal if their scalar product $(\phi_i,\phi_j)=\delta_{i,j}$ where $\delta_{i,j}$ is kronikar delta that means $\delta_{i,j}=0$ where $i\neq j$ and $\delta_{i,j}=1$ if $i=j$

3.1.4 Square Integrable Function

If scalar product $(\psi,\psi)=\int_{-\infty}^{\infty}\psi^*\psi\,dx=\int|\psi(x)|^2\,dx=\alpha$ where α is positive finite number

Then $\psi(x)$ ψ(x) is said to be square integrable.

The square intergable function can be treated as probability distribution function if $\alpha=1$ and ψ is said to be normalized

3.2 Dirac Notation

Dirac introduced what was to become an invaluable notation in quantum mechanics; state vector ψ which is square integrable function by what he called a ket vector $|\psi\rangle$.

And its conjugate ψ by a bra $\langle\psi|$ and scalar product (ϕ,ψ) bra-ket $\langle\phi|\psi\rangle$ (In summery $\psi\rightarrow|\psi\rangle, \psi^*\rightarrow\langle\psi|$ and $(\phi,\psi)=\langle\phi|\psi\rangle$) Where $\langle\phi,\psi\rangle=\int\phi^*(r,t)\psi(r,t)d^3r$.

Properties of kets, bras and bra-kets.

- $(|\psi\rangle)^*=\langle\psi|$
- $(a|\psi\rangle)^*=a^*\langle\psi|$
- $|a\psi\rangle=a|\psi\rangle$
- $\langle a\psi|=a^*\langle\psi|$
- $\langle\phi|\psi\rangle^*=\langle\psi|\phi\rangle$

$$\langle a_1\phi_1+a_2\phi_2|b_1\psi_1+b_2\psi_2\rangle=a_1^*b_1\langle\phi_1|\psi_1\rangle+a_1^*b_2\langle\phi_1|\psi_2\rangle+a_2^*b_1\langle\phi_2|\psi_1\rangle+a_2^*b_2\langle\phi_2|\psi_2\rangle$$

- $|\psi\rangle$ is normalized if $\langle\psi|\psi\rangle=1$
- Schwarz inequality

  $|\langle\psi|\phi\rangle|^2\leq\langle\psi|\psi\rangle\langle\phi|\phi\rangle$, which is analogically derived from $|\vec{A}\cdot\vec{B}|\leq|A|^2|B|^2$
- Triangular inequality $\sqrt{\langle\psi+\phi|\psi+\phi\rangle}=\sqrt{\langle\psi|\psi\rangle}+\sqrt{\langle\phi|\phi\rangle}$
- Orthogonal states $\langle\psi|\phi\rangle=0$
- Orthonormal state $\langle\psi|\phi\rangle=0,\ \langle\psi|\psi\rangle=1,\ \langle\phi|\phi\rangle=1$

- Forbidden quantities: If $|\psi\rangle$ and $|\phi\rangle$ belong to same vector space product of the type $|\psi\rangle\ |\phi\rangle$ and $\langle\phi|\ \langle\psi|$ are forbidden. They are nonsensical.
- If $|\psi\rangle$ and $|\phi\rangle$ belong, however to different vector space then $|\phi\rangle\otimes|\psi\rangle$ represent tensor product of $|\psi\rangle$ and $|\phi\rangle$

3.3 Operator

An operator A is the mathematical rule that when applied to a ket $|\phi\rangle$ will transformed into another $|\psi\rangle$ of the same space and when it acts on a any bra $\langle\chi|$ it transforms it into another bra $\langle\phi|$ that means $A|\phi\rangle=|\psi\rangle$ and $\langle\chi|A=\langle\phi|$

Example of Operator:

Identity operator $I\,|\psi\rangle=|\psi\rangle$

Pairity operator $\pi\,|\psi(r)\rangle=[\psi(-r)]$

Gradiant operator $\nabla\psi(r)$ and Linear momentum operator $P(\psi\rangle=-i\hbar\nabla\psi(\vec{r})$

3.3.1 Linear Operator: A is linear operator if

- $A(\lambda_1|\psi_1\rangle+\lambda_2|\psi_2\rangle=\lambda_1 A|\psi_1\rangle+\lambda_2 A|\psi_2\rangle$
- Product of two linear operator A and B written AB which is defined $(AB)|\psi\rangle=A(B|\psi\rangle)$

3.3.2 Matrix Representation of Operator:

If $|\psi\rangle$ is in orthonormal basis of $|u_i\rangle$ is defined as

$$|\psi\rangle=c_1|u_2\rangle+c_2|u_2\rangle+\ldots|\psi\rangle=\Sigma c_i|u_i\rangle\text{, where }c_i=\langle u_i|\psi\rangle$$

The ket $|\psi\rangle$ is defined as $\begin{pmatrix}\langle u_1|\psi\rangle\\ \langle u_i|\psi\rangle\\ \vdots\end{pmatrix}$ or $|\psi\rangle$ is defined as $\begin{pmatrix}c_1\\ c_2\\ c_3\\ \vdots\end{pmatrix}$ which is column vector.

- The corresponding bra $\langle\psi|$ is defined as $(\langle u_1|\psi\rangle^*\langle u_2|\psi\rangle^*\ldots)$ or $(c_1^*,c_2^*\ldots c_j^*)$ which is row matrix.

- Operator A is represented as Matrix A whose Matrix element A_{ij} is defined as $\langle u_i | A | u_j \rangle$ in $|u_i\rangle$ the basis.
- Transpose of operator A is represented as Matrix A^T whose matrix element is defined as $A_{ij}^T = \langle u_j | A | u_i \rangle$
- Hermitian Adjoint of Matrix A is represent as $A^\dagger$ whose matrix element is defined as $A^\dagger = \langle u_j | A | u_i \rangle$

Hermitian conjugate $A^\dagger$ of matrix A can be find in two step.

Step I: Find transpose of A i.e,.convert row into column ie A^T

Step II: Then take complex conjugate to each element of A^T .

Properties of Hermitian Adjoint $A^\dagger$

- $(A^\dagger)^\dagger = A$
- $(\lambda A)^\dagger = \lambda^* A^\dagger$
- $(A + B)^\dagger = A^\dagger + B^\dagger$
- $(AB)^\dagger = B^\dagger A^\dagger$

3.3.3Eigen Value of Operator:

If A operator is defined such that $A|\psi\rangle = \lambda|\psi\rangle$ then

λ is said to be eigen value and $|\psi\rangle$ is said to be eigen vector corresponding to operator.

3.3.4 Correspondence between Ket and Bra

If $A|\phi\rangle = |\psi\rangle$ then $\langle\phi|A^\dagger = \langle\psi|$ where $A^\dagger$ is Hermitian Adjoint of matrix or operator A.

3.3.5 Hermitian operator: A operator is said to be Hermitian if

$A^\dagger = A$ i.e., Matrix element $\langle u_i | A | u_j \rangle = (\langle u_j | A | u_i \rangle)^*$

- The eigen values of Hermitian matrix is real
- The eigen vectors corresponding to different eigen values are orthogonal.

3.3.6 Commutator: if A and B are two operators then the commentator $[A,B]$ is defined as $AB-BA$

If $[A,B]=0$ then it is said to be A and B operator commute to each other.

Properties of commutator:

- Antisymmetry $[A,B]=-[B,A]$
- Linearity $[A,B+C+D]=[A,B]+[A,C]+[A,D]$
- Distributive : $[AB,C]=[A,B]C+B[A,C]$
- Jacobi Identity $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$

3.3.7 Set of Commuting Observables:

- If two operator A and B commute and if $|\psi\rangle$ is eigen vector o A then $B|\psi\rangle$ is also an eigen vector of A, with the same eigen value.
- If two operator A and B commute and if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigen vector of A with different eigen values then Matrix element $\langle\psi_1|B|\psi_2\rangle$ is zero.
- If two operator A and B commute one can construct an orthonormal basis of state space with eigen vectors common to A and B.

3.3.8 Projection operator: The operator P is said to be operator if

$P^\dagger = P$ $\quad\quad$ $P^2 = P$

- The product of two commutating projection operator P_1 and P_2 is also projection operator.
- The sum of two projection operator is generally not a projection operator.
- The sum of projector $P_1 + P_2 + P_2 + ...$ is projector if P_i P_j are mutually orthogonal.

Example: Prove that $f(x)=x,\ g(x)=x^2,\ h(x)=x^3$ are linear independent

Solution: For linear independency

$$a_1 f(x)+a_2 g(x)+a_3 h(x)=0 \quad \Rightarrow a_1 x+a_2 x^2+a_3 x^3=0$$

Equating the coefficient of x, x^2 and x^3 both side one can get.

$a_1=0 \quad a_2=0 \quad a_3=0$ so f(x), g(x) and h(x) $f(x), g(x)$ are linearly independent.

Example: Prove that vector $\vec{A}=6\hat{i}-9\hat{j}$ and $\vec{B}=-2\hat{i}+3\hat{j}$ are linear dependent.

Solution: $a_1\vec{A}+a_2\vec{B}=0 \Rightarrow a_1(6\hat{i}-9\hat{j})+a_2(-2\hat{i}+3\hat{j})=0 \Rightarrow (6a_1-2a_2)\hat{i}+(-9a_1+3a_2)\hat{j}=0$

$$6a_1-2a_2=0 \quad \text{or} \qquad -9a_1+3a_2=0 \Rightarrow a_1=\frac{a_2}{3}$$

So $\vec{A}$ and $\vec{B}$ are linearly dependent.

Example: If $f(x)=Ae^{-x^2/2}$, $g(x)=Bxe^{-x^2/2}$ then prove that $f(x)$ and $g(x)$ are orthogonal as well as linearly independent.

Solution: For linearly independency: $a_1 f(x)+a_2 g(x)=0 \quad \Rightarrow a_1 e^{-x^2/2}+a_2 xe^{-x^2/2}=0$

$$(a_1+a_2 x)=0 \quad \Rightarrow a_1=0, a_2=0$$

So $f(x)$ and $g(x)$ are linearly independent.

For orthogonality: $\left(f(x)g(x)\right)=\int f^*(x)g(x)dx$

$\Rightarrow (f(x)g(x))=AB\int_{-\infty}^{\infty} xe^{-x^2}dx=0$ Scalar product of $f(x)$ and $g(x)$ is zero orthogonal.

Example: If $f(x)=0 \qquad x<0$

$$f(x)=Ae^{-x/a+ikx} \qquad x>0$$

Then

(a) Find $f^*(x)$

(b) Find value of A such that $f(x)$ is normalized.

Solution: (a) $f(x) = 0 \quad x < 0$

$f^*(x) = 0 \quad x < 0$

$f(x) = Ae^{-x/a+ikx} \quad x > 0$

$f^*(x) = A^* e^{-x/a-ikx} \quad x > 0$

(b) For normalization $\int_{-\infty}^{\infty} f^*(x) f(x) dx = 1$

$$\int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} A^* e^{-x/a+ikx} A e^{-x/a-ikx} dx = 1 \quad \Rightarrow |A|^2 \int_0^\infty e^{-2x/a} dx = 1 \Rightarrow A = \sqrt{\frac{2}{a}}$$

Example: (a) If $|\psi\rangle = A\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$, find $\langle\psi|$

(b) Find the value of A such that $|\psi\rangle$ is normalized.

Solution: (a): If $|\psi\rangle = A\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$ $\quad \langle\psi| = A^*(1 \ \ -i \ \ 0)$

(b) For normalization condition

$$\langle\psi|\psi\rangle = 1$$

$$A^*(1 - i\,0)A\begin{pmatrix}1\\ i\\ 0\end{pmatrix} = 1 \quad |A|^2(1+1+0) = 1$$

$$|A|^2 = \frac{1}{2} \Rightarrow A = \frac{1}{\sqrt{2}}$$

Example: If $|\psi_1\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$ And $|\psi_2\rangle = b_1|\phi_1\rangle + b_2|\phi_2\rangle$

It is given that $\langle\phi_1|\phi_2\rangle = \delta_{ij}$ then

(a) Find condition for $|\psi_1\rangle$ and $|\psi_2\rangle$ are normalized.

(b) Find condition for $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

Solution: (a)

If $|\psi_1\rangle$ is normalized Then $\langle\psi_1|\psi_1\rangle = 1$

$$\left(a_1^*\langle\phi_1| + a_2^*\langle\phi_2|\right)\left(a_1|\phi_1\rangle + a_2|\phi_2\rangle\right) = 1$$

$$|a_1|^2\langle\phi_1|\phi_2\rangle + a_1^*a_2\langle\phi_1|\phi_2\rangle + a_2^*a_1\langle\phi_2|\phi_1\rangle + |a_2|^2\langle\phi_2|\phi_2\rangle = 1$$

It is given that $\langle\phi_1|\phi_1\rangle = 1$, $\langle\phi_2|\phi_2\rangle = 1$, $\langle\phi_1|\phi_2\rangle = 0$, $\langle\phi_2|\phi_1\rangle = 0$,

So $|a_1|^2 + |a_2|^2 = 1$

Similarly for $|\psi_2\rangle$ is normalized

$$\langle\psi_2|\psi_2\rangle = 1 \Rightarrow |b_1|^2\langle\phi_1|\phi_2\rangle + b_1^*b_2\langle\phi_1|\phi_2\rangle + b_2^*b_1\langle\phi_2|\phi_1\rangle + |b_2|^2\langle\phi_2|\phi_2\rangle = 1$$

$\Rightarrow$ $|b_1|^2 + |b_2|^2 = 1$ condition for $|\psi_2\rangle$ is normalized.

(b) For $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

$$\langle\psi_1|\psi_2\rangle = 0 \quad \text{or} \quad \langle\psi_2|\psi_1\rangle = 0$$

$$\left(a_1^*\langle\phi_1| + a_2^*\langle\phi_2|\right)|\left(b_1|\phi_1\rangle + b_2|\phi_2\rangle\right)$$

$$a_1^*b_1\langle\phi_1|\phi_1\rangle + a_1^*b_2\langle\phi_1|\phi_2\rangle + a_2^*b_1\langle\phi_2|\phi_1\rangle + a_2^*b_2\langle\phi_2|\phi_2\rangle$$

$\Rightarrow$ $a_1^*b_1 + a_2^*b_2 = 0$

Similarly from $\langle\psi_2|\psi_1\rangle = 0 \Rightarrow b_1^*a_1 + b_2^*a_2 = 0$

Example: If S operator is defined as

$$S|u_1\rangle = |u_3\rangle \qquad S|u_2\rangle = |u_2\rangle \qquad S|u_3\rangle = |u_1\rangle$$

$\langle u_i|u_j\rangle = \delta_{ij}$ i,j = 1, 2, 3

(a) Construct S matrix

(b) Prove that S is hermitian matrix

Sol.: The Matrix $S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$

Where matrix element $S_{ij} = \langle u_i|S|u_j\rangle$

$S_{11} = \langle u_1 | S | u_1 \rangle = \langle u_1 | u_3 \rangle = 0$ $\quad S_{12} = \langle u_1 | S | u_2 \rangle = \langle u_1 | u_2 \rangle = 0$

$S_{13} = \langle u_1 | S | u_3 \rangle = \langle u_1 | u_3 \rangle = 1$ $\quad S_{21} = \langle u_2 | S | u_1 \rangle = \langle u_2 | u_3 \rangle = 0$

$S_{22} = \langle u_2 | S | u_2 \rangle = \langle u_2 | u_2 \rangle = 1$ $\quad S_{23} = \langle u_2 | S | u_3 \rangle = \langle u_2 | u_1 \rangle = 0$

$S_{31} = \langle u_3 | S | u_1 \rangle = \langle u_3 | u_3 \rangle = 1$ $\quad S_{32} = \langle u_3 | S | u_2 \rangle = \langle u_3 | u_2 \rangle = 0$

$S_{33} = \langle u_3 | S | u_3 \rangle = \langle u_3 | u_1 \rangle = 0$

So S matrix is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad S^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$S = S^\dagger$ so S matrix Hermitian.

Example: If D_x is defined as $\frac{\partial}{\partial x}$ and $\psi(x) = A\sin\frac{n\pi x}{a}$

(a) operate D_x and $\psi(x)$

(b) operate D_x^2 on $\psi(x)$

(c) which one of the above given eigen value problem.

Solution: (a) $D_x\psi(x) = \frac{\partial}{\partial x} A\sin\frac{n\pi x}{a} = A\frac{n\pi}{a}\cos\frac{n\pi x}{a}$

(b) $D_x^2\psi(x) = \frac{\partial^2}{\partial x^2} A\sin\frac{n\pi x}{a} = A\left(-\frac{n^2\pi^2}{a^2}\right)\sin\frac{n\pi x}{a}$

$$D_x^2\psi(x) = -\frac{n^2\pi^2}{a^2} A\sin\frac{n\pi x}{a}$$

(c) when D_x^2 operate on $D_x^2\psi(x) = -\frac{n^2\pi^2}{a^2}\psi(x)$

So operation of $D_x^2(x)$ on $\psi(x) = A\sin\frac{n\pi x}{a}$ give eigen value problem with eigen value $-\frac{n^2\pi^2}{a^2}$

Example: If A operator is given by $A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

(a) find eigen value and eigen vector of A.

(b) normalized there eigen vector.

(c) prove both eigen vector are orthogonal.

Solution: (a) $A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ for eigen value

$$|A-\lambda I|=0$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=0 \qquad \Rightarrow \begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0 \Rightarrow \lambda=1, \text{ and } \lambda=-1,$$

The eigen vector corresponding to $\lambda=1$,

$$A=|u_1\rangle=\lambda|u_1\rangle$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}a\\ b\end{pmatrix}\Rightarrow a=b$$

so eigen vector corresponds to $\lambda=1$, $|u_1\rangle=\begin{pmatrix}a\\ a\end{pmatrix}$

eigen vector corresponds to $\lambda=-1$

$$A|u_2\rangle=\lambda|u_2\rangle$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=-1\begin{pmatrix}a\\ b\end{pmatrix}\Rightarrow a=-b \quad \Rightarrow |u_2\rangle=\begin{pmatrix}a\\ -a\end{pmatrix}$$

(b) For normalised eigen vector.

$$\langle u_1|u_1\rangle=1 \qquad \langle u_2|u_2\rangle=1$$

$$|u_1\rangle=\begin{pmatrix}a\\ a\end{pmatrix} \qquad \langle u_1|=\begin{pmatrix}a & a\end{pmatrix}$$

$$\langle u_1|u_1\rangle=a^2+a^2=1\Rightarrow a=\frac{1}{\sqrt{2}} \qquad |u_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Similarly,

$$|u_2\rangle = \begin{pmatrix} a \\ -a \end{pmatrix} \text{ and } \langle u_2| = (a \quad -a)$$

$$\langle u_2|u_2\rangle = (a \quad -a)\begin{pmatrix} a \\ -a \end{pmatrix} = 1 \quad \Rightarrow a^2 + a^2 = 1 \quad \Rightarrow a = \frac{1}{\sqrt{2}} \;\Rightarrow |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(c) for orthogonality $\langle u_1|u_2\rangle = \langle u_2|u_1\rangle = 0$

$$\frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \qquad \frac{1}{\sqrt{2}}(1 \quad -1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

Example: If momentum operator P_x is defined as $-i\hbar\frac{\partial}{\partial x}$ and position operator X is defined as

$$X\psi(x) = x\psi(x)$$

(a) Find the value of commutator $[\mathrm{X}, \mathrm{P}_x]$

(b) Find the value of $[\mathrm{X}^2, \mathrm{P}_x]$

(c) Find the value of $[\mathrm{X}, \mathrm{P}_x^2]$

Solution: (a): $[X, P_x] = (X\, P_x - P_x X)$

Operate both side with ψ $\quad \Rightarrow XP_x\psi(x) - P_x X\psi(x)$

$$X(-i\hbar)\frac{\partial\psi(x)}{\partial x} - P_x x\psi(x)$$

$$X(-i\hbar)\frac{\partial\psi(x)}{\partial x} + i\hbar\frac{\partial}{\partial x}x\psi(x)$$

$$X(-i\hbar)\frac{\partial\psi(x)}{\partial x} + x\, i\hbar\frac{\partial\psi(x)}{\partial x} + i\hbar\frac{\partial x}{\partial x}\psi(x)$$

$$[X, P_x] = i\hbar\psi(x) \qquad [X, P_x] = i\hbar$$

(b) $[X^2, P_x] = [X \cdot X, P_x] = X[X, P_x] + [X, P_x]X = Xi\hbar + i\hbar X = 2i\hbar X$

(c) $[X, P_x^2] = [X, P_x \cdot P_x]$

$$P_x[X \quad P_x] + [X, P_x]P_x = P_x i\hbar + i\hbar P_x \qquad [X, P_x^2] = 2i\hbar P_x$$

Example: (a) Prove that $P_\psi = |\psi\rangle\langle\psi|$ is projection operator

(b) Operate P_ψ on $|\phi\rangle$

(c) Operate P_ψ on $\langle\phi|$

(d) Operate P_ψ on $|\psi\rangle$ and $\langle\psi|$

(e) Find the eigen value of any projection operator.

Solution: (a) $P_\psi^* = |\psi\rangle\langle\psi = P_\psi$ $\quad P_\psi^2 = P_\psi \cdot \psi$

$= \left(|\psi\rangle\langle\psi|\right)\left(|\psi\rangle\langle\psi|\right)$ $\quad P_\psi^2 = |\psi\rangle\langle\psi| = P_\psi$

So P_ψ is projection operator.

(b) $P_\psi|\phi\rangle = |\psi\rangle\langle\psi|\phi\rangle = \left(\langle\psi|\phi\rangle\right)\cdot|\psi\rangle$

(c) $\langle\phi|P_\psi = \left(\langle\phi|\psi\rangle\right)\langle\psi| = \langle\phi|\psi\rangle\langle\psi|$

(d) $P_\psi|\psi\rangle = |\psi\rangle\langle\psi|\psi\rangle \Rightarrow |\psi\rangle \Rightarrow \langle\psi|\psi\rangle = 1$

$\langle\psi|P_\psi = \langle\psi|\psi\rangle\langle\psi| = \langle\psi|$

(e) $P_\psi^2 = P_\psi$ $\quad P_\psi^2 - P_\psi = 0$ $\quad P_\psi(P_\psi - I) = 0$

$P_\psi = 0,\ P_\psi = I$ $\quad$ so eigen value of $P_\psi = 0$ or 1

Example: If $A = a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $\quad B = b\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

(a) Find the value of [A, B]

(b) Write down eigen vector of B in the basis of eigen vector of A.

Solution (a): [A, B] = AB - BA

$$= a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} b\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - b\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= ab\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - ba\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$ $[A, B] = 0$ so A and B commute.

(b) Eigen vector of A is $|a_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigen value $\lambda_1 = a$

$$|a_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{eigen value } \lambda_2 = a$$

Eigen vector of B is $|b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigen value $\lambda_1 = b$

$$|b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{eigen value } \lambda_{2\partial} = -b$$

$$|b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|a_1\rangle + \frac{1}{\sqrt{2}}|a_2\rangle$$

$$|b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |b_2\rangle = \frac{1}{\sqrt{2}}|a_1\rangle - \frac{1}{\sqrt{2}}|a_2\rangle$$

Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & -3i \\ 0 & 3i & 5 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -i & 3i \\ -i & 0 & i \\ 3i & i & 0 \end{pmatrix}$

(a) find $A^\dagger$ (b) find $B^\dagger$

(c) which one of among A and B have real eigen value.

Solution: (a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & -3i \\ 0 & 3i & 5 \end{pmatrix}$ $A^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & -3i \\ 0 & 3i & 5 \end{pmatrix}$

$A^\dagger = A$ so A is Hermitian.

(b) $B = \begin{pmatrix} 0 & -i & -3i \\ -i & 0 & i \\ 3i & i & 0 \end{pmatrix}$ $B^\dagger = \begin{pmatrix} 0 & -i & -3i \\ i & 0 & -i \\ -3i & -i & 0 \end{pmatrix}$

$B^\dagger = -B$

So it is not Hermitian rather it is Anti-Hermitian.

(c) The eigen value of A matrix is real because A is Hermitian.

3.4 Postulates of Quantum Mechanics

The quantum mechanical postulates enable us to understand.

- how a quantum state is described mathematically at a given time *t*.
- how to calculate the various physical quantities from this quantum state.
- Knowing the system's state at a time *t*, how to find the state at any later time *t*. i.e., how to describe the time evolution of a system.

There are following set of postulates.

Postulate 1: The state of any physical system is specified, at each time *t*, by a state vector $|\psi(t)\rangle$ in the Hilbert space. $|\psi(t)\rangle$ contains all the needed information about the system. Any superposition of state vectors is also a state vector.

Postulate 2: To every measurable quantity A to be called an observable or dynamical variable. There corresponds a linear Hermitian $\hat{A}$ whose eigen vectors form a complete basis. $A|\phi_n\rangle = a_n|\phi_n\rangle$

Postulate 3: The measurement of an observable A many be represented formally by an action of $\hat{A}$ on a state vector $|\psi(t)\rangle$.

The state of the system immediately after the measurement is the normalized projection $\dfrac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$ of $|\psi\rangle$ onto the eigen subspace associate with an.

Postulate 4 (a): When the physical quantity A is measured on a system in the state $|\psi\rangle$ the probability P(a_n) of obtaining the non-degenerate eigen value an of the corresponding observable A is $P(a_n) = \dfrac{|\langle\phi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$ where $A|\phi_n\rangle = a_n|\phi_n\rangle$

Postulate 4 (b): When the physical quantity A is measured on a system in the state $|\psi\rangle$. The probability P(a_n) of the obtaining the eigen value an of the corresponding observable A is

$$P(a_n) = \frac{\sum_{i=1}^{g_n} |\langle \phi_n^i | \psi\rangle|^2}{\langle\psi|\psi\rangle}$$

Where g_n is the degree of degeneracy of an and $|\phi_n^i\rangle$ (i = 1, 2, 3, &, g_n) is orthonormal set of vector which forms a basis in the eigen subspace en associated with eigen value an of A.

Postulate 5: The time evaluation of the state vector $|\psi(t)\rangle$ is governed by schrodinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi\rangle$$

Where H is Hamiltonion of the system.

The solution of schodinger equation must be

(a) Single valued and value must be finite

(b) Continuous

(c) Differentiable

(d) Square integrable.

3.4.1 Expectation Value: The expectation value of operator A is given

- $\langle A\rangle = \dfrac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}$ $\qquad \langle A\rangle = \sum_n \dfrac{a_n|\langle\phi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$

Where $\langle\psi_m|A|\psi_n\rangle = a_n\delta_{mn} \Rightarrow \langle A\rangle = \sum_n a_n P_n(a_n)$

For continuous variable

- $\langle A\rangle = \dfrac{\int_{-\infty}^{\infty} a|\psi(a)|^2\, da}{\int_{-\infty}^{\infty} |\psi(a)|^2\, da} = \int_{-\infty}^{\infty} adP(a)$

- Error in measurement of A is $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$ where $\Delta A \geq 0$

3.4.2 Fourier transformation: Change in basis from one representation to another representation.

$|p\rangle$ is defined as

$$|\,p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i/\hbar p_x}$$

The expansion of $\psi(x)$ in terms of $|p\rangle$ can be written as.

$$\psi(x) = \int_{-\infty}^{\infty} a(p)\,|\,p\rangle dp \quad \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} a(p) e^{ipx/\hbar} dp$$

Where $a(p)$ can be find

$$a(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx$$

In 3D

$$a(p) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int_{-\infty}^{\infty} \psi(r) e^{-i\vec{p}.\vec{r}/\hbar} d^3r$$

Where $a(p)$ being expansion coefficient of $|\,p\rangle$.

- If any function $\psi(x)$ can be expressed as a linear combination of state function ϕ_n

i.e., $\psi(x) = \sum_n c_n \phi_n(x)$ then where $\int \phi_m^* \phi_n dx = \delta_{mn}$ then $c_n = \int \psi_n^*(x) f(x) dx$

which is popularly derived from fourier trick.

Parity operator: The parity operator Π defined by its action on the basis.

$$\Pi\,|\,r\rangle = |-r\rangle \qquad \langle r\,|\,\Pi\,|\,\psi\rangle = \psi(-r)$$

If $\psi(-r) = \psi(r)$ then state have even parity and

If $\psi(-r) = -\psi(r)$ then state have odd parity.

So parity operator have $+1$ and -1 eigen value.

Representation of postulate (4) in continuous basis.

Example: A state function is given by $|\psi\rangle = |\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$ It is given that $\langle \phi_i | \phi_j \rangle = \delta_{ij}$

(a) check $|\psi\rangle$ is normalized or not

(b) write down normalized wavefunction $\langle\psi|$.

(c) It is given $H|\phi_n\rangle = (n+1)\hbar\omega|\phi_n\rangle$ $\quad n = 0, 1, 2, 3, \ldots$

If H will measured on $|\psi\rangle$. what will be measurement with what probability.

(d) Find the expectation value at H i.e., $\langle H\rangle$

(e) Find the error in the measurement in H.

Solution: (a) To check normalization one should verify.

$$\langle \psi | \psi \rangle = 1$$

$$|\psi\rangle = |\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$

$$\langle \psi | \psi \rangle = \langle \phi_1 | \phi_1 \rangle + \langle \phi_1 | \phi_2 \rangle \frac{1}{\sqrt{2}} + \langle \phi_2 | \phi_1 \rangle \frac{1}{\sqrt{2}} + \langle \phi_2 | \phi_2 \rangle \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= 1 + 0 + 0 + \frac{1}{2} = \frac{3}{2}$$

The value of $\langle \psi | \psi \rangle = \frac{3}{2}$ so $|\psi\rangle$ is not normalized.

(b) Now we need to find normalized $|\psi\rangle$ let A be normalization constant.

$$|\psi\rangle = A|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$

$$\langle \psi | \psi \rangle = A^2 + \frac{A^2}{2} = 1 \quad \Rightarrow \frac{3A^2}{2} = 1 \quad \Rightarrow A = \sqrt{\frac{2}{3}}$$

So $\quad |\psi\rangle = \sqrt{\frac{2}{3}}|\phi_1\rangle + \frac{1}{\sqrt{3}}|\phi_2\rangle$

$$\langle \psi | = \langle \phi_1 | \sqrt{\frac{2}{3}} + \langle \phi_2 | \frac{1}{\sqrt{2}}$$

(c) It is given that

$$H|\phi_n\rangle = (n+1)\hbar\omega \qquad n = 0, 1, 2, 3\ldots$$

$$H|\phi_1\rangle = 2\hbar\omega \qquad\qquad H|\phi_2\rangle = 3\hbar\omega$$

When H will measured $|\psi\rangle$ it will measured either $2\hbar\omega$ or $3\hbar\omega$

The probability of measured $2\hbar\omega$ is $P(2\hbar\omega)$ is given by

$$P(2\hbar\omega) = \frac{|\langle\phi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{2}{3} \qquad\qquad P(3\hbar\omega) = \frac{|\langle\phi_2|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{1}{3}$$

So when H will measure state $|\psi\rangle$ the following outcome will come.

Measurement of H on state	: $|\phi_1\rangle$	$|\phi_2\rangle$
Measurement	: $2\hbar\omega$	$3\hbar\omega$
Probability	: 2/3	1/3

(d) $$\langle H\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \sum_n P_n(a_n)a_n$$

$$= 2\hbar\omega\times\frac{2}{3} + 3\hbar\omega\times\frac{1}{3} \qquad \langle H\rangle = \frac{7\hbar\omega}{3}$$

$$\langle H^2\rangle = \frac{\langle\psi|H^2|\psi\rangle}{\langle\psi|\psi\rangle} = \Sigma P_n(a_n)a_n^2 \qquad = \frac{2}{3}\times(2\hbar\omega)^2 + \frac{1}{3}\times(3\hbar\omega)^2$$

$$= \frac{8\hbar^2\omega^2}{3} + \frac{9\hbar^2\omega^2}{3} = \frac{17\hbar^2\omega^2}{3}$$

(e) The error in measurement in H is given as

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} \qquad \langle H^2\rangle = \frac{17\hbar^2\omega^2}{3}$$

$$\langle H\rangle^2 = \left(\frac{7\hbar\omega}{3}\right)^2 = \frac{49\hbar^2\omega^2}{9} \qquad \Delta H = \sqrt{\frac{17}{3} - \frac{49}{9}}\hbar\omega$$

$$\Delta H = \sqrt{\frac{51-49}{9}}\hbar\omega = \frac{\sqrt{2}}{3}\hbar\omega$$

Example: The wave function of a particle is given by $\psi = \left(\frac{B}{\sqrt{2}}\phi_0 + Bi\phi_1 \right)$, where ϕ_0 and ϕ_1 are the normalised eigenfunctions with energy E_0 and E_1 corresponding to ground state and first exited state .

(a) Find the value of B such that ψ is normalised.

(b) What is measurement

(c) What is the probability getting E_1

(d) What is $\langle E \rangle$

Solution: (a) $\langle \psi | = \frac{B}{\sqrt{2}}\phi_0 - Bi\phi_1$

For normalized $\langle \psi | \psi \rangle = 1$

$$\langle \psi | \psi \rangle = \frac{B^2}{2} + B^2 = 1 \Rightarrow B^2 = \sqrt{\frac{2}{3}}$$

Now $|\psi\rangle = \frac{1}{\sqrt{3}}\phi_0 + \sqrt{\frac{2}{3}} i\phi_1$

(b) Measurement are E_0, E_1

(c) Probability getting E_1, $P(E_1) = \frac{|\langle \phi_1 | \psi \rangle|^2}{\langle \psi | \psi \rangle} = \frac{2}{3}$

(d) $\langle E \rangle = \sum_{n=0}^{\infty} E_n P(E_n) = \frac{1}{3} \times E_0 + \frac{2}{3} E_1 = \frac{E_0 + 2E_1}{3}$

Example: (a) Plot $\psi_I(x) = A_1 e^{-|x|} \quad -\infty < x < \infty$

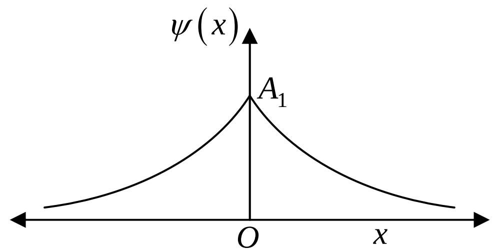
(b) Plot $\psi_{II}(x) = A_2 e^{-x^2} \quad -\infty < x < \infty$

(c) discuss why ψ_I is not solution of Schrödinger wave function rather ψ_{II} is solution of Schrödinger wave function.

Solution: (a) $\psi_I(x) = A_1 e^{+x} x < 0$

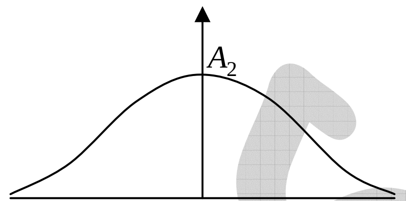
$\psi_{II}(x) = A_1 e^{-x} x > 0$

The plot is given by



(b) $\psi_{II}(x) = A_2 e^{-x^2} \quad -\infty < x < \infty$

The plot is given by



(c) Both the function ψ_I and ψ_{II} are single value, continuous, square integrable by ψ_I is not differentiable of $x = 0$ rather ψ_{II} is differentiable at $x = 0$

So ψ_{II} can be solution of Schrödinger wave function but ψ_I is not solution of Schrödinger wave function.

Example: A time $t = 0$ the state vector $|\psi(0)\rangle$ as $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)$

It is given as Hamiltonian is defined as $H|\phi_n\rangle = n^2 \in_0 |\phi_n\rangle$

(a) What is wave function $|\psi(t)\rangle$ at later time t.

(b) Write down expression of evolution of $|\psi(x, t)|^2$

(c) Find ΔH

(d) Find the value of $\Delta H \cdot \Delta t$

Solution: (a) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{\frac{-i\in_0 t}{\hbar}}|\phi_1\rangle + e^{\frac{-i4\in_0 t}{\hbar}}|\phi_2\rangle\right]$

$|\psi(t)\rangle \propto \left[|\phi_1\rangle + e^{-\omega_{21}t}|\phi_2\rangle\right]$

Where $\omega_{21} = \frac{E_2 - E_1}{\hbar} = \frac{3\in_0}{\hbar}$

(b) Evolution of shape of the wave packet

$$|\psi(x,t)|^2=\frac{1}{2}|\phi_1(x)|^2+\frac{1}{2}|\phi_2(x)|^2+\phi_1\phi_2\cos\omega_{21}t$$

(c) $\Delta H=\left(\langle H^2\rangle-\langle H\rangle^2\right)^{1/2}$

$$\langle H\rangle=\frac{1}{2}E_1+\frac{1}{2}E_2=\frac{5}{2}E_1 \qquad \langle H^2\rangle=\frac{1}{2}E_1^2+\frac{1}{2}E_2^2=\frac{17}{2}E_1^2$$

$$\Delta H=\frac{3}{2}E_1 \qquad \Delta H=\frac{3}{2}\times\in_0$$

(d) $\Delta H=\frac{3}{2}\in_0 \qquad \Delta t=\frac{1}{\Delta\omega_{21}} \qquad \Delta t=\frac{\hbar}{3\in_0} \qquad \Delta H\cdot\Delta t=\frac{3}{2}\in_0\times\frac{\hbar}{3\in_0}=\frac{\hbar}{2}$

$$\Delta H\cdot\Delta t=\frac{\hbar}{2}$$

Example: Consider a one-dimensional particle which is confined within the region $0\le x\le a$ and whose wave function is $\psi(x,t)=\sin\left(\frac{\pi x}{a}\right)e^{i\omega t}$. Find the potential $V(x)$.

Solution: From the fifth postulate.

$$H\psi=i\hbar\frac{\partial\psi}{\partial t} \qquad H=\frac{P^2}{2m}+V(x)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi=i\hbar\frac{\partial\psi}{\partial t}$$

$$\frac{\pi^2\hbar^2}{2ma^2}\sin\frac{\pi x}{a}e^{i\omega t}+V(x)\sin\frac{\pi x}{a}e^{i\omega t}=i\hbar\sin\frac{\pi x}{a}(i\omega)e^{i\omega a}$$

$$\frac{\pi^2\hbar^2}{2ma^2}+V(x)=-\hbar\omega \qquad V(x)=-\hbar\omega-\frac{\pi^2\hbar^2}{2ma^2} \qquad V(x)=-\left(\hbar\omega+\frac{\pi^2\hbar^2}{2ma^2}\right)$$

Example: If eigen value of operator A is 0, $2a_0$, $2a_0$ and corresponding normalized eigen vector is $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-i\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ respectively t system is in state $\frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}$ then

(a) When A is measured on system in state $\frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}$ then what is probability to getting value $0,\ 2a_0$, respectively.

(b) What is expectation value of A?

Solution: Let $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-i\\1\end{pmatrix} \Rightarrow \lambda_1 = 0,\quad |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\1\end{pmatrix} \Rightarrow \lambda_2 = 2a_0$

$\lambda_2 = \lambda_3 = 2a_0$ i.e., $\lambda = 2a_0$ is doubly degenerate.

$$P(0) = \frac{|\langle\phi_1|\psi(t)\rangle|^2}{\langle\psi|\psi\rangle} = \frac{8}{17}$$

$$P(2a_0) = \frac{|\langle\phi_2|\psi(t)\rangle|^2}{\langle\psi|\psi\rangle} + \frac{|\langle\phi_3|\psi(t)\rangle|^2}{\langle\psi|\psi\rangle}$$

$$= \frac{\left[\left(\frac{1}{\sqrt{2}}(0\ \ -i\ \ 1)\right)\frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}\right]^2}{\frac{1}{6}(1\ \ 0\ \ 4)\frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}} + \frac{\left[(1\ \ 0\ \ 0)\frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}\right]^2}{\frac{1}{6}(1\ \ 0\ \ 4)\frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}} = \frac{\frac{1}{2}\times\frac{1}{36}\times 16}{\frac{1}{36}\cdot(1+16)} + \frac{\frac{1}{36}}{\frac{17}{36}}$$

$$= \frac{\frac{2}{9}}{\frac{17}{36}} + \frac{1}{17} = \frac{9}{17} \Rightarrow \langle A\rangle = 0\times\frac{8}{17} + 2a_0\times\frac{9}{17} \Rightarrow \langle A\rangle = \frac{18a_0}{17}\ \text{(Average value)}$$

Example: A free particle which is initially localized in the range $-a < x < a$ is released at time $t = 0$.

$$\psi(x) = \begin{cases} A & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Find (a) A such that $\psi(x)$ is normalized.

(b) Find $\phi(x)$ i.e., wave function in momentum space.

(c) Find $\psi(x,t)$ i.e., wave function after t time.

Solution: (a) $\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = A^2 \int_{-a}^{a} dx = 1 \Rightarrow A = \frac{1}{\sqrt{2a}}$

(b) $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \psi(x) dx = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \int_{-a}^{a} e^{-ikx} = \frac{1}{\sqrt{2\pi}} \frac{\sin ka}{k}$

(c) $\psi(x,t) = \frac{1}{\pi\sqrt{2a}} \int_{-\infty}^{\infty} \frac{\sin ka}{k} e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$

Questions

MCQ (Multiple Choice Questions)

Q1. The quantum mechanical operator for the momentum of a particle moving in one dimension is given by

(a) $i\hbar\frac{d}{dx}$ (b) $-i\hbar\frac{d}{dx}$ (c) $i\hbar\frac{\partial}{\partial t}$ (d) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

Q2. If the distribution function of x is $f(x) = xe^{-x/\lambda}$ over the interval $0 < x < \infty$, the mean value of x is

(a) λ (b) 2λ (c) $\frac{\lambda}{2}$ (d) 0

Q3. If $|\psi_0\rangle$ is written in orthonormal basis of $|\phi_n\rangle$ as $|\psi_0\rangle = \frac{1}{\sqrt{7}}\left[A|\phi_1\rangle + \sqrt{3}|\phi_2\rangle + i|\phi_3\rangle - |\phi_4\rangle\right]$

Then the value of A Such that $|\psi_0\rangle$ is normalized.

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{12}$ (d) -2

Q4. If an operator A is defined as , $A|\phi_n\rangle = (n+1)a_0|\phi_n\rangle$. $n = 1,2,3$ &Where $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$ .If A is measured on state $|\psi_0\rangle = (|\phi_1\rangle + |\phi_2\rangle)$ what is measurement

(a) $2a_0$ (b) $a_0, 2a_0$ (c) $2a_0, 3a_0$ (d) $4a_0$

Q5. If an operator A is defined as, $A|\phi_n\rangle = (n+1)a_0|\phi_n\rangle$. & $n = 1,2,3$ Where $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$. If A is measured on state $|\psi_0\rangle = (2|\phi_1\rangle + 3|\phi_2\rangle)$ what is probability of measurement of $3a_0$

(a) $\frac{1}{4}$ (b) $\frac{4}{13}$ (c) $\frac{9}{13}$ (d) 0

Q6. If an operator A is defined as , $A|\phi_n\rangle = (n+1)a_0|\phi_n\rangle$ & $n = 1,2,3$ Where $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$ .If A is measured on state $|\psi_0\rangle = (2|\phi_1\rangle + 3|\phi_2\rangle)$ what will average value of measurement of A

(a) $5a_0$ (b) $\frac{35a_0}{13}$ (c) $\frac{3a_0}{5}$ (d) $\frac{10a_0}{13}$

Q7. If an operator A is defined as , $A|\phi_n\rangle = na_0|\phi_n\rangle$ $n = 1,2,3$ where $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$ .If A is measured on state $|\psi_0\rangle = (2|\phi_1\rangle + 3|\phi_2\rangle)$ what will average value of measurement of A

(a) $5a_0$ (b) $\frac{35a_0}{13}$ (c) $\frac{3a_0}{5}$ (d) $\frac{22a_0}{13}$

Q8. If potential energy of system is the attractive delta function potential defined as $V(x) = -b\delta(x)$, where $b > 0$, and wave function is defined as $\psi(x) = \begin{Bmatrix} A\cos\frac{\pi x}{2a}, & \text{for} & -a < x < a \\ 0, & & \text{otherwise} \end{Bmatrix}$. The average value of energy is given by

(a) $\langle E\rangle = \frac{\pi^2\hbar^2}{8ma^2} - \frac{b}{a}$ (b) $\langle E\rangle = \frac{\pi^2\hbar^2}{8ma^2} + \frac{b}{a}$

(c) $\langle E\rangle = \frac{\pi^2\hbar^2}{8mb^2} - \frac{b}{a}$ (d) $\langle E\rangle = \frac{\pi^2\hbar^2}{8mb^2} + \frac{b}{a}$

Q9. The wavefunction of a particle is given by $\psi = \left(\frac{1}{\sqrt{2}}\phi_0 - i\phi_1\right)$, where ϕ_0 and ϕ_1 are the normalized eigenfunctions with energies E_0 and E_1 corresponding to the ground state and first excited state, respectively. The expectation value of the Hamiltonian in the state ψ is

(a) $\frac{E_0}{2} + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 - 2E_1}{3}$ (d) $\frac{E_0 + 2E_1}{3}$

Q10. The wave function of a particle is given by $\psi = \left(\frac{1}{\sqrt{2}}\phi_0 + i\phi_1\right)$, where ϕ_0 and ϕ_1 are the normalized eigenfunctions with energies E_0 and E_1 corresponding to the ground state and first excited state, respectively. Then $\langle H^2\rangle$ in the state ψ is

(a) $\frac{E_0^2}{2} + E_1^2$ (b) $\frac{E_0^2}{2} - E_1^2$ (c) $\frac{E_0^2 - 2E_1^2}{3}$ (d) $\frac{E_0^2 + 2E_1^2}{3}$

Q11. The wave function of a particle at time $t = 0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$, where $|u_1\rangle$ and $|u_2\rangle$ are the normalized eigenstates with eigenvalues E_1 and E_2 respectively, $(E_2 > E_1)$. Find the expression $|\psi(t)\rangle$ after later time t

(a) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)e^{-\frac{i(E_1+E_2)t}{2\hbar}}$ (b) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)e^{\frac{i(E_1+E_2)t}{2\hbar}}$

(c) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle e^{\frac{-iE_1t}{\hbar}} + |u_2\rangle e^{\frac{-iE_2t}{\hbar}}\right)$ (d) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle e^{\frac{+iE_1t}{\hbar}} + |u_2\rangle e^{\frac{+iE_2t}{\hbar}}\right)$

Q12. The wave function of a particle at time $t = 0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$, where $|u_1\rangle$ and $|u_2\rangle$ are the normalized eigenstates with eigenvalues $E_1 = E_0$ and $E_2 = 2E_0$ respectively, $(E_2 > E_1)$. The shortest time after which $|\psi(t)\rangle$ will become orthogonal to $|\psi(0)\rangle$ is

(a) $\frac{\pi\hbar}{E_0}$ (b) $\frac{2\pi\hbar}{E_0}$ (c) $\frac{\pi\hbar}{2E_0}$ (d) $\frac{3\pi\hbar}{2E_0}$

Q13. If the expectation value of the momentum is $\langle p\rangle$ for the wavefunction $\psi(x)$, then the expectation value of momentum for the wavefunction $e^{ikx/\hbar}\psi(x)$ is

(a) k (b) $\langle p\rangle - k$ (c) $\langle p\rangle + k$ (d) $\langle p\rangle$

Q14. If a particle is represented by the normalized wave function

$$\psi(x) = \begin{cases} \frac{\sqrt{15}(a^2 - x^2)}{4a^{5/2}} & \text{for } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Find $\langle P^2\rangle$

(a) $\frac{5\hbar^2}{2a^2}$ (b) $\frac{5\hbar^2}{a^2}$ (c) $\frac{2\hbar^2}{5a^2}$ (d) $\frac{\hbar^2}{5a^2}$

Q15. A quantum mechanical particle in has the initial wave function $\psi = \psi_0(x) + \psi_1(x)$, where ψ_0 and ψ_1 are the real wave functions in the ground and first excited state of the Hamiltonian with energy E_0 and E_1 respectively . What is expression of probability density at time t.

(a) $|\psi(x,t)|^2 = |\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\psi_0(x)\psi_1(x)\cos(E_2 - E_1)\frac{t}{\hbar}$

(b) $|\psi(x,t)|^2 = |\psi_0(x)|^2 + |\psi_1(x)|^2 - 2\psi_0(x)\psi_1(x)\cos(E_2 - E_1)\frac{t}{\hbar}$

(c) $|\psi(x,t)|^2 = |\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\psi_0(x)\psi_1(x)\cos(E_2 + E_1)\frac{t}{\hbar}$ (c)

(d) $|\psi(x,t)|^2 = |\psi_0(x)|^2 + |\psi_1(x)|^2 - 2\psi_0(x)\psi_1(x)\cos(E_2 + E_1)\frac{t}{\hbar}$

Q16. A quantum mechanical particle in has the initial wave function $\psi = \psi_0(x) + \psi_1(x)$, where ψ_0 and ψ_1 are the real wave functions in the ground and first excited state of the Hamiltonian with energy E_0 and E_1 respectively . What is expression of probability density at time $t = \pi$ if $E_1 - E_0 = \hbar\omega$ assume $\hbar = 1, \omega = 1$

(a) $(\psi_1(x) - \psi_0(x))^2$

(b) $(\psi_1(x))^2 - (\psi_0(x))^2$

(c) $(\psi_1(x) + \psi_0(x))^2$

(d) $(\psi_1(x))^2 + (\psi_0(x))^2$

Q17. The operator $\left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right)$ is equivalent to

(a) $\frac{d^2}{dx^2} - x^2$

(b) $\frac{d^2}{dx^2} - x^2 + 1$

(c) $\frac{d^2}{dx^2} - x\frac{d}{dx}x^2 + 1$

(d) $\frac{d^2}{dx^2} - 2x\frac{d}{dx} - x^2$

Q18. Suppose Hamiltonian of a conservative system in classical mechanics is $H = \omega xp$, where ω is a constant and x and p are the position and momentum respectively. The corresponding Hamiltonian in quantum mechanics, in the coordinate representation, is

(a) $-i\hbar\omega\left(x\frac{\partial}{\partial x}-\frac{1}{2}\right)$ (b) $-i\hbar\omega\left(x\frac{\partial}{\partial x}+\frac{1}{2}\right)$

(c) $-i\hbar\omega x\frac{\partial}{\partial x}$ (d) $-\frac{i\hbar\omega}{2}x\frac{\partial}{\partial x}$

Q19. The commentator $\left[x^2, p^2\right]$ is

(a) $2i\hbar xp$ (b) $2i\hbar(xp+px)$ (c) $2i\hbar px$ (d) $2i\hbar(xp-px)$

Q20. Given the usual canonical commutation relations, the commentator $[A,B]$ of $A = i\left(xp_y - yp_x\right)$ and $B = \left(yp_z + zp_y\right)$ is

(a) $\hbar\left(xp_z - p_x z\right)$ (b) $-\hbar\left(xp_z - p_x z\right)$

(c) $\hbar\left(xp_z + p_x z\right)$ (d) $-\hbar\left(xp_z + p_x z\right)$

Q21. If $\psi(x) = A\exp\left(-x^4\right)$ is the eigenfunction of a one dimensional Hamiltonian with eigenvalue $E=0$, the potential $V(x)$ (in units where $\hbar = 2m = 1$) is

(a) $12x^2$ (b) $16x^6$ (c) $16x^6+12x^2$ (d) $16x^6-12x^2$

MSQ (Multiple Select Questions)

Q22. If a operator A associated with defined as $A\left|\phi_n\right\rangle = a_n\left|\phi_n\right\rangle$ where $n = 1,2,3,4.....$. if A is measured on state $|\psi\rangle$ then which of the following is /are correct.

(a) The measurement of A on state $|\psi\rangle$ is a_n

(b) The probability of measurement of a_n on state $|\psi\rangle$ is $\frac{\langle\phi_n|\psi\rangle}{\langle\psi|\psi\rangle}$

(c) The expectation value of measurement of A on state $|\psi\rangle$ is $\langle\psi|A|\psi\rangle$

(d) The average value of measurement of A on state $|\psi\rangle$ is $\sum_n a_n \frac{\left|\langle\phi_n|\psi\rangle\right|^2}{\langle\psi|\psi\rangle}$

Q23. The wave function of a particle is given by $\psi = B\left(\frac{1}{\sqrt{2}}\phi_0 + i\phi_1\right)$, where ϕ_0 and ϕ_1 are the orthonormal eigenfunctions with energy E_0 and E_1 corresponding to ground state and first exited state.

(a) The value of B such that ψ is normalized is $\sqrt{\frac{2}{3}}$

(b) The probability of measurement to getting E_1 on state ψ is $\frac{1}{2}$

(c) The average value of energy $\langle E \rangle = \frac{E_0 + E_1}{2}$

(d) The average value of energy $\langle E \rangle = \frac{E_0 + 2E_1}{3}$

Q24. If Hamiltonian H is defined as $H = \varepsilon_0 \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ measure on state $\psi = \begin{pmatrix} 2i \\ 3 \end{pmatrix}$.

(a) if H will measure on state ψ measurement is $5\varepsilon_0$

(b) if H will measure on state ψ measurement is $1\varepsilon_0, 4\varepsilon_0$

(c) Expectation value of H on eigen state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $1\varepsilon_0$

(d) Expectation value of H on eigen state ψ is $\frac{40\varepsilon_0}{13}$

Q25. The wave function ψ of a quantum mechanical system described by a Hamiltonian H can be written as a linear combination of ϕ_1 and ϕ_2 which are eigenfunction of H with eigenvalues E_1 and E_2 respectively at $t = 0$, $\psi_0 = \frac{4}{5}\phi_1 + \frac{3}{5}\phi_2$ and then allowed to evolve with time,

(a) The magnitude of wave function after time $t = T = \frac{h}{2}\left(\frac{1}{E_1 - E_2}\right)$ is $\psi_0 = \frac{4}{5}\phi_1 - \frac{3}{5}\phi_2$

(b) The magnitude of wave function after time $t = T = \frac{h}{2}\left(\frac{1}{E_1 - E_2}\right)$ is $\psi_0 = \frac{4}{5}\phi_1 + \frac{3}{5}\phi_2$

(c) The probability density after time $t = T = \frac{h}{2}\left(\frac{1}{E_1 - E_2}\right)$ is $|\psi(r,t)|^2 = \left(\frac{4}{5}\phi_1 - \frac{3}{5}\phi_2\right)^2$

(d) The probability density after time $t = T = \frac{h}{2}\left(\frac{1}{E_1 - E_2}\right)$ is $|\psi(r,t)|^2 = \left(\frac{4}{5}\phi_1 + \frac{3}{5}\phi_2\right)^2$

Q26. If A state of particle $\psi(r,t=0)=\frac{4}{5}\phi_1+\frac{3}{5}\phi_2$ having energy E_1 and E_2 energy eigen value with ϕ_1 and ϕ_2 are normalized eigenfunction of Hamiltonian H .

(a) then $\psi(r,t)=\frac{4}{5}\phi_1\exp\frac{-iE_1t}{\hbar}+\frac{3}{5}\phi_2\exp\frac{-iE_2t}{\hbar}$

(b) after time $t=\frac{\hbar}{(E_2-E_1)}\cos^{-1}\left(\frac{9}{16}\right)$ $\psi(r,t=0)$ and $\psi(r,t)$ became orthogonal

(c) Average value of energy at time $t=0$ is $\frac{4E_1+3E_2}{5}$

(d) Average value of energy at time $t=t$ is $\frac{16E_1+9E_2}{25}$

Q27. A quantum mechanical state of a system is given by $\psi=Ax(a-x)$ if $0\le x\le a$ and $\psi(x)=0$ other wise then

(a) The value of A is $\sqrt{\frac{30}{a^5}}$ for normalized ψ .

(b) The average value of position on state ψ is $\frac{a}{2}$.

(c) The average value of momentum on state ψ is not equal to zero

(d) The average value of kinetic energy is $\frac{5\hbar^2}{ma^2}$

NAT (Numerical Answer Type)

Q28. If $\psi = \frac{C}{\sqrt{15}}|\phi_1\rangle + \frac{3}{\sqrt{15}}|\phi_2\rangle$ the value of C is ………….. such that ψ is normalized .

Q29. If a state $\psi = C\left[|\phi_1\rangle + (3+4i)|\phi_2\rangle\right]$. Find the value of C such that ψ is normalised.

Q30. A operator is defined as $A\phi_n = (3n-1)a_0\phi_n$ is if A will measure on state ψ which is defined as orthonormal basis of ϕ_n as $\psi = \frac{1}{\sqrt{26}}\phi_1 + \frac{(3+4i)}{\sqrt{26}}\phi_2$. Measurement of A is αa_0 and βa_0 then value of α and β respectively ....and ………. $(\alpha > \beta)$

Q31. A operator is defined as $A\phi_n = (3n-1)a_0\phi_n$ is if A will measure on state ψ which is defined as orthonormal basis of ϕ_n as $\psi = \frac{1}{\sqrt{26}}\phi_1 + \frac{(3+4i)}{\sqrt{26}}\phi_2$. Then probability of measurement of $5a_0$ is ……………

Q32. A operator is defined as $A\phi_n = na_0\phi_n$ is if A will measure on state ψ which is defined as orthonormal basis of ϕ_n as $\psi = \phi_1 + (3-4i)\phi_2$. Then probability of measurement of $2a_0$ is ………

Q33. Assume operator A defined as $A\phi_n = na_0\phi_n$ and operator $B\phi_n = (n+1)b_0\phi_n$ where $n = 1, 2, 3...$ and if A state ψ is defined $\psi = \sqrt{\frac{2}{3}}\phi_1 + \sqrt{\frac{1}{3}}\phi_2$ if some one measure A on state ψ he measure a_0 at the same time there is measurement of operator B the measurement is ……….. b_0

Q34. If state of a system at time $t = 0$ is defined by $\psi = \sqrt{\frac{2}{3}}\phi_1 + \sqrt{\frac{1}{3}}\phi_2$. If Hamiltonian H is defined as $H\phi_n = n^2E_0\phi_n$ after time $t = \frac{\hbar}{\alpha E_0}\cos^{-1} - \frac{1}{2}$ then value of α is …………

Q35. If state of a system at time $t = 0$ is defined by $\psi = \sqrt{\frac{1}{5}}\phi_1 + \sqrt{\frac{4}{5}}\phi_3$ if Hamiltonian H is defined as $H\phi_n = nE_0\phi_n$ where $n = 1, 2, 3..$ after time $t = \frac{\hbar}{\alpha E_0}\cos^{-1} - \beta$ then value of α is …………and β ………

MSQ (Multiple Select Questions)

Q36. Which of the following statement is /are correct about quantum mechanical system

(a) Solution of Schrödinger wave equation must single valued.

(b) Solution of Schrödinger wave equation must be continuous.

(c) Solution of Schrödinger wave equation must be differentiable.

(d) Solution of Schrödinger wave equation must vanish at $x \to \infty$ but $x \to -\infty$ it may be infinite

Q37. Which of the following must be correct about quantum mechanical system?

(a) There is Hermitian operator associated with any physical measurable quantities.

(b) Physical measurable quantities are Eigen values of operator associated with physical measurable quantities which can be real or imaginary.

(c) Eigen values of operator associated with physical measurable quantities must be non degenerate in nature.

(d) Any state of system can be written in orthonormal basis of eigen state .

Q38. Which one of the following is correct about quantum mechanical operator.

(a) If $A(x)$ is operator ψ is unnormalised state then average value of A is $\langle A \rangle = \int_{-\infty}^{\infty} \psi^* A\psi \, dx$

(b) Physical measurable quantities in quantum mechanical system are discrete as well as continuous.

(c) Energy can be measured by Hamiltonian operator which is always discrete.

(d) Momentum operator has always continuous in eigen value .

Q39. If state of a system at time $t=0$ is defined by $\psi=\sqrt{\frac{1}{5}}\phi_1+\sqrt{\frac{4}{5}}\phi_3$ if Hamiltonian H is defined as $H\phi_n=nE_0\phi_n$ where $n=1,2,3..$ then which of following is correct

(a) $|\psi\rangle$ is normalized state .

(b) The measurement of H on state ψ is E_0 and $3E_0$

(c) the probability of measurement of E_0 is $\frac{1}{5}$ and the probability of measurement of $3E_0$ is $\frac{4}{5}$

(d) the average value of H is $\frac{13}{5}E_0$

Q40. If $\begin{aligned}\psi &= Ax(a-x) \qquad & 0\le x\le a\\ &=0 & \text{otherwise}\end{aligned}$, then which of the following is correct

(a) The given function is square integrable

(b) If ψ is normalized then value of $A=\sqrt{\frac{30}{a^5}}$

(c) The average value of position on state ψ is $\frac{a}{2}$

(d) Maximum probability to find the particle is at $x=\frac{a}{2}$

Q41. Which of the following/are correct

(a) $-i\hbar\frac{d}{dx}$ is Hermitian operator .

(b) $\frac{d}{dx}$ is Hermitian operator .

(c) For any operator A the operator $B=AA^{\dagger}$ is Hermitian

(d) $(1+i)AB+(1-i)BA$ is Hermitian irrespective of A and B are Hermitian

Q42. Which of following is correct?

(a) $\begin{aligned}\psi(x) &= Ae^{-x/a} \quad x>0 \\ &= -Ae^{x/a} \quad x<0\end{aligned}$ can be solution of Schrödinger wave equation

(b) $\psi(x) = Ae^{-x^2}$ can be solution of Schrödinger wave equation

(c) $\begin{aligned}\psi(x) &= Axe^{-x} > 0 \\ &= 0 \quad < 0\end{aligned}$ can be solution of Schrödinger wave equation

(d) $\psi(x) = Axe^{-x^2}$ can be solution of Schrödinger wave equation

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Ans. 2: (b)

Solution: $\because$ it is distribution function so $\langle x\rangle = \dfrac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \dfrac{\int_0^{\infty} x.xe^{-\frac{x}{\lambda}}dx}{\int_0^{\infty} xe^{-\frac{x}{\lambda}}dx} \Rightarrow \dfrac{\int_0^{\infty} x^2 e^{\frac{-x}{\lambda}}dx}{\int_0^{\infty} xe^{\frac{-x}{\lambda}}dx} = 2\lambda$

Ans. 3: (a)

Solution: For normalized

$$\langle\psi_0|\psi_0\rangle = 1 \Rightarrow \frac{1}{7}\left[A^2\langle\phi_1|\phi_1\rangle + 3\langle\phi_2|\phi_2\rangle + \langle\phi_3|\phi_3\rangle + \langle\phi_4|\phi_4\rangle\right] = 1$$

$$\frac{1}{7}\left[A^2 + 3 + 1 + 1\right] = 1 \Rightarrow A^2 = 7 - 5 \quad \Rightarrow A = \sqrt{2}$$

Ans. 4: (c)

Solution: Now $|\psi_0\rangle = (|\phi_1\rangle + |\phi_2\rangle)$, $A|\phi_n\rangle = (n+1)a_0|\phi_n\rangle \Rightarrow (n+1)a_0$ is eigen value of A with eigen state $|\phi_n\rangle$ so measurement of A on state $|\psi_0\rangle$ is $2a_0$ and $3a_0$

Ans. 5: (c)

Solution: Now $|\psi_0\rangle = (2|\phi_1\rangle + 3|\phi_2\rangle)$, $A|\phi_n\rangle = (n+1)a_0|\phi_n\rangle$

Now measurement of A on state $|\psi_0\rangle$ is $2a_0$ and $3a_0$ with eigen state $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively

Ans. 6: (b)

Solution: Now $|\psi_0\rangle = (2|\phi_1\rangle + 3|\phi_2\rangle)$, $A|\phi_n\rangle = (n+1)a_0|\phi_n\rangle$ $n = 1, 2, 3$

Now measurement of A on state $|\psi_0\rangle$ is $2a_0$ and $3a_0$ with eigen state $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively

Probability of measurement of $2a_0$ $P(2a_0) = \dfrac{\langle\phi_2|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \dfrac{4}{13}$, Probability of measurement of $3a_0$ $\quad P(3a_0) = \frac{\langle \phi_2 | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{9}{13}$

Average value of measurement of A is

$$\langle A \rangle = \frac{\langle \psi_0 | A | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \sum_n a_n P(a_n) \Rightarrow 2a_0 \times \frac{4}{13} + 3a_0 \times \frac{9}{13} = \frac{35a_0}{13}$$

Ans. 7: (c)

Solution: Now $|\psi_0\rangle = \left(2|\phi_1\rangle + 3|\phi_2\rangle\right)$, $A|\phi_n\rangle = na_0|\phi_n\rangle$

Now measurement of A on state $|\psi_0\rangle$ is a_0 and $2a_0$ with eigen state $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively

Probability of measurement of a_0 $\quad P(a_0) = \frac{\langle \phi_2 | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{4}{13}$,

Probability of measurement of $2a_0$ $\quad P(2a_0) = \frac{\langle \phi_2 | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{9}{13}$

Average value of measurement of A is

$$\langle A \rangle = \frac{\langle \psi_0 | A | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} \sum_n a_n P(a_n) \Rightarrow a_0 \times \frac{4}{13} + 2a_0 \times \frac{9}{13} = \frac{22a_0}{13}$$

Ans. 8: (a)

Solution: $V(x) = -b\delta(x); \qquad b > 0$ and $\quad \psi(x) = \begin{cases} A\cos\frac{\pi x}{2a}; & -a < x < a \end{cases}$

Normalized $\psi = \sqrt{\frac{2}{2a}} \cos\frac{\pi x}{2a}$

$$\langle T \rangle = \int_{-a}^{a} \psi^* \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi \, dx = \frac{\pi^2 \hbar^2}{8ma^2}$$

$$\langle V \rangle = \int_{-a}^{a} \psi^* - b\delta(x) \psi \, dx = \frac{2}{2a}(-b) = -\frac{b}{a}$$

$$\langle E \rangle = \frac{\pi^2 \hbar^2}{8ma^2} - \frac{b}{a}$$

Ans. 9: (d)

Solution: $\psi = \frac{1}{\sqrt{2}}\phi_0 - i\phi_1$ and $\langle H \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{E_0 + 2E_1}{3}$

Ans. 10: (d)

Solution: $\psi = \frac{1}{\sqrt{2}}\phi_0 - i\phi_1$ and $\langle H \rangle = \frac{\langle \psi | H^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \sum_n E_n^2 P(E_n) = \frac{E_0^2 + 2E_1^2}{3}$

Ans. 11: (c)

Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle + |u_2\rangle\right) \Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle e^{\frac{-iE_1 t}{\hbar}} + |u_2\rangle e^{\frac{-iE_2 t}{\hbar}}\right)$

Ans. 12: (a)

Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle + |u_2\rangle\right) \Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle e^{\frac{-iE_1 t}{\hbar}} + |u_2\rangle e^{\frac{-iE_2 t}{\hbar}}\right)$

$|\psi(t)\rangle$ is orthogonal to $|\psi(0)\rangle \Rightarrow \langle \psi(0) | \psi(t) \rangle = 0 \Rightarrow \frac{1}{2}e^{\frac{-iE_1 t}{\hbar}} + \frac{1}{2}e^{\frac{-iE_2 t}{\hbar}} = 0$

$$\Rightarrow e^{\frac{-iE_1 t}{\hbar}} + e^{\frac{-iE_2 t}{\hbar}} = 0 \Rightarrow e^{\frac{-iE_1 t}{\hbar}} = -e^{\frac{-iE_2 t}{\hbar}} \Rightarrow e^{i\frac{(E_2 - E_1)}{\hbar}} = -1$$

$$\Rightarrow \cos\frac{(E_2 - E_1)t}{\hbar} = \cos\pi \Rightarrow t = \frac{\pi\hbar}{E_2 - E_1} = \frac{\pi\hbar}{E_0}$$

Ans. 13: (c)

Solution: $\int_{-\infty}^{\infty} \psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\,dx = \langle p \rangle$

Now

$$\int_{-\infty}^{\infty} e^{-\frac{ikx}{\hbar}}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)e^{\frac{ikx}{\hbar}}\psi(x)\,dx \Rightarrow \int_{-\infty}^{\infty} e^{\frac{-ikx}{\hbar}}\psi^*(x)(-i\hbar)\left[e^{\frac{ikx}{\hbar}}\frac{\partial}{\partial x}\psi(x) + \frac{ik}{\hbar}e^{\frac{ikx}{\hbar}}\psi(x)\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{ikx}{\hbar}}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)e^{\frac{ikx}{\hbar}} + \int_{-\infty}^{\infty} -i\hbar.\frac{ik}{\hbar}e^{\frac{-ikx}{\hbar}}\psi^*(x)\psi(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \psi^*(x)\left[-i\hbar\frac{\partial}{\partial x}\psi(x)\right] + k\int_{-\infty}^{\infty} \psi^*(x)\psi(x) \Rightarrow \langle P \rangle + k$$

Ans. 14: (a)

Solution: $p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \quad \Rightarrow \langle p^2 \rangle = -\hbar^2 \times \frac{15}{16a^5} \int_{-a}^{a} \left(a^2 - x^2\right) \frac{\partial^2}{\partial x^2} \left(a^2 - x^2\right) dx$

$$= -\hbar^2 \times \frac{15}{16a^5} \times (-2) \int_{-a}^{a} \left(a^2 - x^2\right) dx \; = \hbar^2 \times \frac{15}{16a^5} \times 2 \left\{ a^2 \cdot x - \frac{x^3}{3} \right\}_{-a}^{a}$$

$$= \hbar^2 \times \frac{15}{16a^5} \times 2 \left[2a^3 - \frac{2a^3}{3} \right] = \hbar^2 \times \frac{15}{16} \times 2 \times 2a^3 \left[1 - \frac{1}{3} \right] = \frac{15\hbar^2}{4a^2} \times \frac{2}{3} \qquad \langle p^2 \rangle = \frac{5\hbar^2}{2a^2}$$

Ans.15: (a)

Solution: $\psi(x) = \psi_0(x) + \psi_1(x)$

$$\psi(x,t) = \psi_0(x) e^{\frac{-iE_0 t}{\hbar}} + \psi_1(x) e^{\frac{-iE_1 t}{\hbar}}$$

$$\psi(x,t)^* = \psi_0(x)^* e^{\frac{iE_0 t}{\hbar}} + \psi_1(x)^* e^{\frac{iE_1 t}{\hbar}} \qquad \psi_0(x)^* = \psi_0(x), \psi_1(x)^* = \psi_1(x)$$

$$|\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t) = |\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x)\cos(E_1 - E_0)\frac{t}{\hbar}$$

Ans. 16: (a)

Solution: $\psi(x) = \psi_0(x) + \psi_1(x)$

$$\psi(x,t) = \psi_0(x) e^{\frac{-iE_0 t}{\hbar}} + \psi_1(x) e^{\frac{-iE_1 t}{\hbar}}$$

$$\psi(x,t)^* = \psi_0(x)^* e^{\frac{iE_0 t}{\hbar}} + \psi_1(x)^* e^{\frac{iE_1 t}{\hbar}} \qquad \psi_0(x)^* = \psi_0(x), \psi_1(x)^* = \psi_1(x)$$

$$|\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t) = |\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x)\cos(E_1 - E_0)\frac{t}{\hbar}$$

putting $t = \pi$

$$|\psi(x,t)|^2 = |\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x)\cos\pi \qquad \because E_1 - E_0 = \hbar\omega = 1$$

$$|\psi(x,t)|^2 = |\psi_0(x)|^2 + |\psi_1(x)|^2 - 2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x) = \left[\psi_1(x) - \psi_0(x)\right]^2$$

Ans. 17: (b)

Solution: $\Rightarrow\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)f(x)\Rightarrow\left(\frac{d}{dx}-x\right)\left[\frac{d}{dx}f(x)+xf(x)\right]$

$$\Rightarrow\frac{d}{dx}\left[\frac{d}{dx}f(x)+xf(x)\right]-x\frac{d}{dx}f(x)-x^2f(x)$$

$$\Rightarrow\frac{d^2}{dx^2}f(x)+f(x)+x\frac{df(x)}{dx}-x\frac{d}{dx}f(x)-x^2f(x)$$

$$\Rightarrow\frac{d^2}{dx^2}f(x)-x^2f(x)+f(x)=\left(\frac{d^2}{dx^2}-x^2+1\right)f(x)$$

Ans. 18: (b)

Solution: Classically $H=\omega xp$, quantum mechanically H must be Hermitian,

So, $H=\frac{\omega}{2}(xp+px)$ and $H\psi=\frac{\omega}{2}(xp\psi+px\psi)$

$$\Rightarrow H\psi=\frac{\omega}{2}\left(x(-i\hbar)\frac{\partial\psi}{\partial x}+\frac{-i\hbar\partial(x\psi)}{\partial x}\right)=\frac{\omega}{2}(-i\hbar)\left(x\frac{\partial\psi}{\partial x}+x\frac{\partial\psi}{\partial x}+\psi\right)$$

$$\Rightarrow H\psi=\frac{-i\hbar\omega}{2}\left(2x\frac{\partial\psi}{\partial x}+\psi\right)=\frac{-i\hbar\omega}{2}\left(2x\frac{\partial}{\partial x}+1\right)\psi$$

$$\Rightarrow H\psi=-i\hbar\omega\left(x\frac{\partial}{\partial x}+\frac{1}{2}\right)\psi$$

Ans. 19: (b)

Solution: $\left[x^2,p^2\right]=x\left[x,p^2\right]+\left[x,p^2\right]x=xp[x,p]+x[x,p]p+p[x,p]x+[x,p]px$

$$\left[x^2,p^2\right]=xp(i\hbar)+x(i\hbar)p+p(i\hbar)x+(i\hbar)px=2i\hbar(xp+px).$$

Ans. 20: (c)

Solution: $[A,B]=\left[\left(ixp_y-iyp_x\right),\left(yp_z+zp_y\right)\right]$

$$[A,B]=i\left[xp_y,yp_z\right]-i\left[yp_x,yp_z\right]+i\left[xp_y,zp_y\right]-i\left[yp_x,zp_y\right]$$

$$[A,B]=i\left[xp_y,yp_z\right]-0+0-i\left[yp_x,zp_y\right]=i\left[xp_y,yp_z\right]-i\left[yp_x,zp_y\right]$$

$$[A,B]=ix\left[p_y,yp_z\right]+i\left[x,yp_z\right]p_y-iy\left[p_x,zp_y\right]-i\left[y,zp_y\right]p_x$$

$$[A,B] = ix\lfloor p_y, yp_z \rfloor + 0 - 0 - i\lfloor y, zp_y \rfloor p_x = ix\lfloor p_y, yp_z \rfloor - i\lfloor y, zp_y \rfloor p_x$$

$$[A,B] = ix \times (-i\hbar) p_z - izi\hbar \times p_x$$

$$[A,B] = \hbar(xp_z + p_x z)$$

Ans. 21: (d)

Solution: Schrodinger equation $-\nabla^2\psi + V\psi = 0$ (where $\hbar = 2m = 1$ and $E = 0$)

$$-\frac{\partial^2}{\partial x^2}\left(Ae^{-x^4}\right) + VAe^{-x^4} = 0 \Rightarrow -\frac{\partial}{\partial x}\left[e^{-x^4} \times -4x^3\right] + Ve^{-x^4} = 0$$

$$4\left[\left\{3x^2 e^{-x^4} + x^3\left(-4x^3 e^{-x^4}\right)\right\}\right] + Ve^{-x^4} = 0 \Rightarrow 12x^2 e^{-x^4} - 16x^6 e^{-x^4} + Ve^{-x^4} = 0$$

$$\Rightarrow V = 16x^6 - 12x^2$$

MSQ (Multiple Select Questions)

Ans. 22: (a) and (d)

Solution: The expectation value of measurement of A on state $|\psi\rangle$ is

$$\frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} = \sum_n a_n \frac{|\langle\phi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

Ans. 23: (a) and (d)

Solution: (a) For normalized $\langle\psi|\psi\rangle = 1$

$$\langle\psi|\psi\rangle = \frac{B^2}{2} + B^2 = 1 \Rightarrow B = \sqrt{\frac{2}{3}}$$

Now $|\psi\rangle = \frac{1}{\sqrt{3}}\phi_0 + \sqrt{\frac{2}{3}}i\phi_1$ Measurement are E_0, E_1

Probability getting E_1, $P(E_1) = \frac{|\langle\phi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{2}{3}$

Probability getting E_0, $P(E_0) = \frac{|\langle\phi_0|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{1}{3}$

So $\langle E\rangle = \sum_{n=0}^{\infty} E_n P(E_n) = \frac{1}{3} \times E_0 + \frac{2}{3}E_1 = \frac{E_0 + 2E_1}{3}$

Ans. 24: (a), (c) and (d)

Solution: $H = \varepsilon_0 \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$E_1 = \varepsilon_0, E_2 = 4\varepsilon_0$ and corresponding eigenvector $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Hence $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eigen state with eigen value $1\varepsilon_0$,so measurement on state $|\phi_1\rangle$ is $1\varepsilon_0$

but ψ is not eigen state so measurement is both of eigen value $E_1 = \varepsilon_0, E_2 = 4\varepsilon_0$ and

Probability $P(\varepsilon_0) = \dfrac{|\langle \phi_1 | \psi \rangle|^2}{\langle \psi | \psi \rangle} = \dfrac{4}{13}$, Probability $P(4\varepsilon_0) = \dfrac{|\langle \phi_2 | \psi \rangle|^2}{\langle \psi | \psi \rangle} = \dfrac{9}{13}$

$$\langle H \rangle = \varepsilon_0 \times \frac{4}{13} + 4\varepsilon_0 \times \frac{9}{13} = \frac{40\varepsilon_0}{13}$$

Ans. 25: (c)

Solution: $\psi(t=t) = \dfrac{4}{5}\phi_1 \exp -i\dfrac{E_1 t}{\hbar} + \dfrac{3}{5}\phi_2 \exp -i\dfrac{E_2 t}{\hbar}$

$$\psi(t=t) = \exp -i\frac{E_1 t}{\hbar}\left(\frac{4}{5}\phi_1 + \frac{3}{5}\phi_2 \exp -i\frac{(E_1 - E_2)t}{\hbar}\right) \quad \text{put}\, T = \frac{h}{2}\left(\frac{1}{E_1 - E_2}\right)$$

$$\psi(r,t) = \exp -i\frac{E_1 t}{\hbar}\left(\frac{4}{5}\phi_1 - \frac{3}{5}\phi_2\right)$$

$$|\psi(r,t)|^2 = \frac{16}{25}|\phi_1|^2 + \frac{9}{25}|\phi_2|^2 + 2\,\mathrm{Re}\,\phi_1^*\phi_2 \frac{4}{5}\times\frac{3}{5}\cos\frac{(E_2 - E_1)t}{\hbar}$$

$$|\psi(r,t)|^2 = \frac{16}{25}|\phi_1|^2 + \frac{9}{25}|\phi_2|^2 + \mathrm{Re}\,\phi_1^*\phi_2 \frac{4}{5}\times\frac{3}{5}\times 2\cos\frac{(E_2 - E_1)t}{h}\times 2\pi.\frac{h}{2(E_1 - E_2)}$$

$$|\psi(r,t)|^2 = \frac{16}{25}|\phi_1|^2 + \frac{9}{25}|\phi_2|^2 - 2\,\mathrm{Re}\,\phi_1^*\phi_2 \frac{4}{5}.\frac{3}{5}$$

$$|\psi(r,t)|^2 = \left(\frac{4}{5}\phi_1 - \frac{3}{5}\phi_2\right)^2$$

Ans. 26: (a), (b) and (d)

Solution: $\psi(r,t=0)=\frac{4}{5}\phi_1+\frac{3}{5}\phi_2$ from postulates five $\psi(r,t)=\frac{4}{5}\phi_1\exp\frac{-iE_1t}{\hbar}+\frac{3}{5}\phi_2\exp\frac{-iE_2t}{\hbar}$

Now $\langle\psi(r,t=0)|\psi(r,t=t)\rangle=0$

$$\frac{16}{25}\langle\phi_1|\phi_1\rangle\exp\frac{-iE_1t}{\hbar}+\frac{9}{25}\langle\phi_2|\phi_2\rangle\exp\frac{-iE_2t}{\hbar}=0\Rightarrow\frac{16}{25}\exp\frac{-iE_1t}{\hbar}=-\frac{9}{25}\exp\frac{-iE_2t}{\hbar}$$

$$\exp\frac{-i(E_2-E_1)t}{\hbar}=-\frac{9}{16}\Rightarrow\cos\frac{(E_2-E_1)t}{\hbar}=-\frac{9}{16}\Rightarrow t=\frac{\hbar}{(E_2-E_1)}\cos^{-1}\left(\frac{9}{16}\right)$$

Probability of measurement of E_1 and E_2 are $\frac{16}{25},\frac{9}{25}$ respectively at $t=0$ and at $t=t$ from postulates 2 and 3.

Ans. 27: (a,b,d)

Solution: $\int_0^a\psi^*\psi\,dx=1\Rightarrow A^2\int_0^a x^2(a-x)^2dx=1\Rightarrow A=\sqrt{\frac{30}{a^5}}$

$$\langle X\rangle=\frac{\int_0^a\psi^*X\psi\,dx}{\int_0^a\psi^*\psi\,dx}\;\frac{A^2\int_0^a xx^2(a-x)^2dx}{A^2\int_0^a x^2(a-x)^2dx}=\frac{a}{2}$$

$$\langle p_x\rangle=\frac{\int_0^a\psi^*-i\hbar\frac{\partial}{\partial x}\psi}{\int_0^a\psi^*\psi\,dx}=\frac{A^2\int_0^a x(a-x)-i\hbar\frac{\partial}{\partial x}x(a-x)}{A^2\int_0^a x^2(a-x)^2dx}=0$$

$$\langle T\rangle=\left\langle\frac{p_x^2}{2m}\right\rangle=\frac{\int_0^a\psi^*-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi}{\int_0^a\psi^*\psi\,dx}=\frac{A^2\int_0^a x(a-x)-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}x(a-x)}{A^2\int_0^a x^2(a-x)^2dx}=\frac{5\hbar^2}{ma^2}$$

NAT (Numerical Answer Type)

Ans. 28: 2.45

Solution: For ψ is normalised $\langle\psi|\psi\rangle = 1 \Rightarrow \frac{C^2}{15} + \frac{9}{15} = 1 \Rightarrow \frac{C^2}{15} = \frac{6}{15} \Rightarrow C = \sqrt{6}$

Ans. 29: 0.196

Solution: For normalized $\psi \quad \Rightarrow \langle\psi|\psi\rangle = 1$

$$\Rightarrow C^2\left[\langle\phi_1|\phi_1\rangle\right] + (3+4i)(3-4i)\langle\phi_2|\phi_2\rangle = 1$$

$$C^2[1+9+16] = 1 \Rightarrow C = \frac{1}{\sqrt{26}} = .196$$

Ans. 30: 2 and 5

Solution: Measurement of A on any state are eigen value $2a_0, 5a_0$

Ans. 31: 0.96

Solution: Probability getting $P(5a_0) = \frac{\left|\langle\phi_2|\psi\rangle\right|^2}{\langle\psi|\psi\rangle} = \frac{25}{26} = 0.96$

Ans. 32: 0.03

Solution: Probability getting $P(a_0) = \frac{\left|\langle\phi_2|\psi\rangle\right|^2}{\langle\psi|\psi\rangle} = \frac{1}{26} = .03$

Ans. 33: 2

Solution: If A will measure state ψ and get the eigen value a_0 means ψ is projected in the direction of ϕ_1 if again B will measure on state ψ it will measure eigen value associate with ϕ_1 so measurement is $2b_0$.

Ans. 34: 3

Solution: $\psi(t=0) = \sqrt{\frac{2}{3}}\phi_1 + \sqrt{\frac{1}{3}}\phi_2,\ \psi(t=t) = \sqrt{\frac{2}{3}}\phi_1 e^{-\frac{iE_1 t}{\hbar}} + \sqrt{\frac{1}{3}}\phi_2 e^{-\frac{iE_2 t}{\hbar}}$

$\psi(t=0)$ and $\psi(t=t)$ became orthogonal so their scalar product vanish

$$\left(\psi^*(t=0), \psi(t=t)\right) = 0$$

$\frac{2}{3}e^{\frac{-iE_1t}{\hbar}}+\frac{1}{3}e^{\frac{-iE_2t}{\hbar}}=0 \quad \Rightarrow t=\frac{\hbar}{E_2-E_1}\cos^{-1}-\frac{1}{2}$ where $E_2=4E_0$ and $E_1=E_0$ and

$E_2-E_1=3E_0$ comparing with $t=\frac{\hbar}{\alpha E_0}\cos^{-1}\frac{1}{2} \Rightarrow \alpha=3$

Ans. 35: 2 and 0.25

Solution: $\psi=\sqrt{\frac{1}{5}}\phi_1+\sqrt{\frac{4}{5}}\phi_3 \Rightarrow \psi(t=t)=\sqrt{\frac{1}{5}}\phi_1 e^{-\frac{iE_2t}{\hbar}}+\sqrt{\frac{4}{5}}\phi_3 e^{-\frac{iE_3t}{\hbar}}$

$\psi(t=0)$ and $\psi(t=t)$ became orthogonal so their scalar product vanish $\left(\psi^*(t=0),\psi(t=t)\right)=0$

$\frac{1}{5}e^{\frac{-iE_2t}{\hbar}}+\frac{4}{5}e^{\frac{-iE_3t}{\hbar}}=0 \quad t=\frac{\hbar}{E_2-E_1}\cos^{-1}-\frac{1}{4}$ where $E_1=E_0$ and $E_3=3E_0$ and

$E_2-E_1=2E_0$ comparing with $t=\frac{\hbar}{\alpha E_0}\cos^{-1}-\beta \quad \alpha=2 \quad \beta=0.25$

MSQ (Multiple Select Questions)

Ans. 36: (a), (b) and (c)

Solution: Solution of Schrödinger wave equation must single valued, continuous, differentiable and it must vanish at $x\to\infty$ but $x\to-\infty$

Ans. 37: (a) and (d)

Solution: Eigen values of operator associated with physical measurable quantities which can be real it may be non degenerate or degenerate.

Ans. 38: (b)

Ans. 39: (a), (b), (c) and (d)

Solution: $\frac{1}{5}+\frac{4}{5}=1$ so it is normalized, the measurement of H on state ψ is E_0 and $3E_0$ With the probability $\frac{1}{5}$ and $\frac{4}{5}$ and $\langle H\rangle=E_0\times\frac{1}{5}+3E_0\times\frac{4}{5}=\frac{13}{5}E_0$.

Ans. 40: (a), (b), (c) and (d)

Solution: the function is vanish at $x \to \infty \quad x \to -\infty$ so it is square integrable .

$$\int_0^a \psi^* \psi\, dx = 1 \Rightarrow A = \sqrt{\frac{30}{a^5}}\ ,\ \langle x \rangle = \int_0^a \psi^* x \psi\, dx = \frac{a}{2}\ ,|\psi|^2 = \frac{30}{a^5}\left(x(a-x)\right)^2 \text{ is max at } \frac{a}{2}$$

Ans. 41: (a) and (c)

Solution: Any operator A is Hermitian if $A^{\dagger} = A$

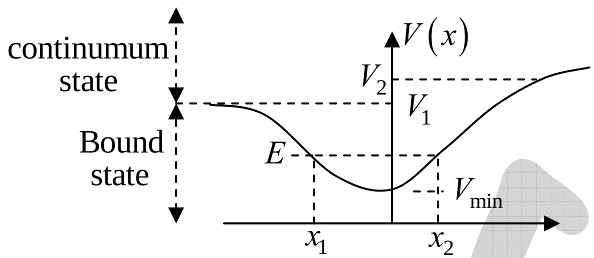
Ans. 42: (b), (c) and (d)

Solution: (a) is not continuous so it can not be solution of Schrödinger wave equation.

4. Application of Quantum Mechanics in Cartesian Coordinate

4.1 One Dimensional System

Properties of one dimensional motion:



(A) Bound states (quantum mechanical discrete spectrum)

Bound states occur whenever the particle cannot move to infinity and particle is confined into limited region.

- From the figure the condition for bound states is $V_{\min} < E < V_1$
- In a one dimensional potential energy level of a bound state system are discrete and non degenerate.
- The wave function ψ_n of one dimensional bound state system has n nodes i.e. ψ_n vanishes n times if n corresponds to $n=0$ the and $n-1$ nodes if $n=1$ corresponds to the ground state.

(B) Continuous spectrum (unbound states)

Unbound states occur in those case where the motion of the system is not confined in above figure.

- $V_1 < E < V_2$: The energy spectrum is continuous and none of the energy eigen values is degenerate.
- $E > V_2$: The energy spectrum is continuous and particles motion is infinite in both directions. And this spectrum is doubly degenerate.

4.2 Current Density (J):

The probability current density is defined as $J_x = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$

The current density can also be given as.

$$J = \rho V$$

Where ρ is probability density. i.e., $\rho = |\psi|^2$ and v is velocity of particle which is $\frac{\hbar k}{m}$ in momentum.

So current density is $J = (\psi^* \psi)\frac{\hbar k}{m} = |\psi|^2 \frac{\hbar k}{m}$

The current density and probability density will be satisfied the continuity equation which is given by $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$

4.3 Free Particle in One Dimension

The potential of free particle is defined as $V(x) = 0; \quad -\infty < x < \infty$

The schrodinger wave function is given by

$$H = E\psi$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x) = E\psi \qquad \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{2mE}{\hbar^2}\psi = 0 \text{ where } \frac{2mE}{\hbar^2} = k^2$$

the solution is given by $\psi = Ae^{ikx} + Be^{-ikx}$ and $E = \frac{\hbar^2 k^2}{2m}$

The energy eigen value of free particle is $E = \frac{\hbar^2 k^2}{2m}$ is continuous and wave function is $\psi_+ = Ae^{ikx}$ and $\psi_- = Ae^{-ikx}$ where ψ_+ moves from positive axis x and ψ_- moves from negative x axis.

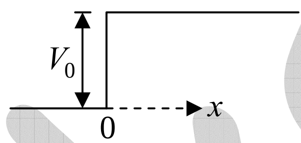
Hence there is two eigen function for energy $E = \frac{\hbar^2 k^2}{2m}$ then wave function is doubly degenerate.

4.4 The Step Potential

If J_i is incident current density , J_r is reflection current density and J_t is transmission current density.then Then reflection coefficient is defined s is de $R = \left|\frac{J_r}{J_i}\right|$ and transmission coefficient is defined as $T = \left|\frac{J_t}{J_i}\right|$

The potential step is defined as

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$



Case I: $E > V$

For $x < 0$. The schrodinger wave is given x, y

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\psi = Ae^{ik_1x} + Be^{-ik_1x} \qquad x < 0$$

Where Ae^{ik_1x} is incoming wave. And Be^{-ik_1x} is reflected wave.

In $x > 0$ the schrodinger wave equation.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m(E-V)}{\hbar^2}\psi = 0 \Rightarrow \frac{d^2\psi}{dx^2} + k_2^2\psi = 0$$

where $k_2^2 = \frac{2m(E-V)}{\hbar^2}$ $\quad \psi_{II} = Ce^{ik_2x} + De^{-ik_2x} \qquad (x > 0)$

$D = 0$ be are no wave reflection reflected in region II i.e., $x > 0$

$\psi_{II} = Ce^{ik_2x}$ which is transmitted wave i.e., $\psi_t = Ce^{ik_2x}$

$$J_i = \frac{\hbar}{2im}\left(\psi_i^*\frac{\partial\psi_i}{\partial x} - \psi_i\frac{\partial\psi_i^*}{\partial x}\right) = \frac{\hbar k_1}{m}|A|^2$$

$$J_r = \frac{\hbar}{2im}\left(\psi_r^*\frac{\partial\psi_r}{\partial x} - \psi_r\frac{\partial\psi_r^*}{\partial x}\right) = \frac{-\hbar k_1}{m}|B|^2$$

$$J_t = \frac{\hbar}{2im}\left(\psi_t^* \frac{\partial \psi_t}{\partial x} - \psi_t \frac{\partial \psi_t^*}{\partial x}\right) = \frac{\hbar k_2}{m}|C|^2$$

$$R = \left|\frac{J_r}{J_i}\right| = \frac{|B|^2}{|A|^2} \quad \text{and} \quad T = \left|\frac{J_t}{J_i}\right| = \frac{K_2}{K_1}\frac{|C|^2}{|A|^2}$$

Using Boundary condition at $x = 0$ i.e.,

Wave must be continuous and differentiable at boundary. So

$$\psi_1(x=0) = \psi_2(x=0) \Rightarrow A + B = C$$

$$\left.\frac{d\psi_1}{dx}\right|_{x=0} = \left.\frac{d\psi_2}{dx}\right|_{x=0} \Rightarrow k_1(A-B) = k_2 C$$

Solution of above two equation

$$\text{B}=\left(\frac{k_1 - k_2}{k_1 + \text{k}_2}\right)\text{A} \quad \text{and} \quad \text{C} = \left(\frac{2\text{k}_1}{k_1 + \text{k}_2}\right)\text{A}$$

$$\text{R}=\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \quad \text{and} \quad \text{T}=\frac{4k_1k_2}{(k_1+k_2)^2} \quad \text{where} \quad R + T = 1$$

Case II: $E < V_0$

Schrodinger wave equation for $0 < x$

$$H\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + 0\psi(x) = E(\psi) \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \Rightarrow \frac{d^2\psi}{dx^2} + k_1^2\psi = 0$$

Solution of the $\psi = Ae^{ikx} + Be^{-ikx}$

If Ae^{ik_1x} is incoming wave $\psi_i = Ae^{ik_1x}$ then Be^{-ik_1x} is reflected wave $\psi_r = Be^{-ik_1x}$

Schrodinger wave equation for $x > 0$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi = 0 \quad \Rightarrow \frac{2m}{\hbar^2}(V_0 - E) = k_2^2$$

$$\frac{d^2\psi}{dx^2} - k_2^2\psi = 0 \Rightarrow \psi = Ae^{k_2x} + Be^{-k_2x}$$

$A = 0$ wave function must vanish at $x \to 0$ then $\psi_t = Be^{-k_2x}$

$J_i = |A|^2 \frac{\hbar k_1}{m}$, $J_r = |B|^2 \frac{\hbar k_2}{m}$ and $J_t = 0$

$R = \frac{J_r}{J_i} = \left|\frac{B}{A}\right|^2$ and $T = \frac{J_t}{J_i} = 0$

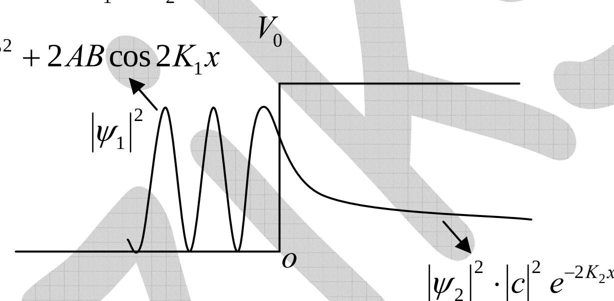
Now we boundary condition.at $x = 0$

$\psi_1(x=0) = \psi_2(x=0) \Rightarrow (\mathrm{A} + \mathrm{B}) = \mathrm{C}$

and $\left.\frac{d\psi_1}{dx}\right|_{x=0} = \left.\frac{d\psi_2}{dx}\right|_{x=0} \Rightarrow (A - B) = \frac{k_2}{ik_1} C$

$B = \frac{k_1 - ik_2}{k_1 + ik_2} A$ and $C = \frac{2k_1}{k_1 + ik_2} A$ so $R = 1$ and $R = 0$

$|\psi_1|^2 = A^2 + B^2 + 2AB\cos 2K_1 x$



where $|\psi_1|^2 = |A^2| + |B^2| + 2AB\cos 2K_1 x$

$|\psi_1|^2 = |c|^2 e^{-2K_2 x}$

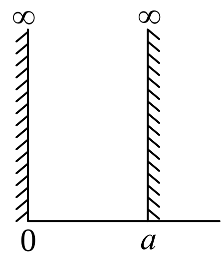
when $E < V_0$, there is finite probability to find the particle at $x > 0$ even if $E < 0$ but current density is zero in region $x > 0$.

Strange part of the problem is that even if transmission coefficient is zero there is finite probability to find the particle $x > 0$.

4.5 Particle in a One Dimensional Box:

The potential of one dimensional box is defined as

$$V(x) = 0 \qquad 0 < x < a$$
$$= \infty \qquad \text{otherwise}$$



Time independent schrodinger wave equation is given as $H\psi = E\psi$ for region $x < 0$ and $x > 0$, $\psi(x) = 0$ because in this region potential is infinity so probability to find the particle in that region is zero.

Schrodinger wave equation in the region $0 < x < a$

$$H\psi = E\psi$$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE\psi}{\hbar^2} \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE\psi}{\hbar^2} = 0$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad \psi(x) = A\sin kx + B\cos kx$$

Now wave function must be continuous at the boundary

So $\quad \psi(0) = \psi(a) = 0 \Rightarrow 0 = A\sin 0 + B\cos 0 = B \Rightarrow B = 0$

$$\psi(x) = A\sin kx$$

$$\psi(a) = 0 \Rightarrow A\sin ka = 0 \Rightarrow ka = n\pi \text{ where } n = 0,1,2,3,....$$

But for $n = 0$ $\psi(x) = 0$ so $n = 0$ is not possible

so $n = 1,2,3,....$ $\quad ka = n\pi$

$$\sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{a} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \text{ where } n = 1,2,3,....$$

E_n is energy eigen value which is discrete.

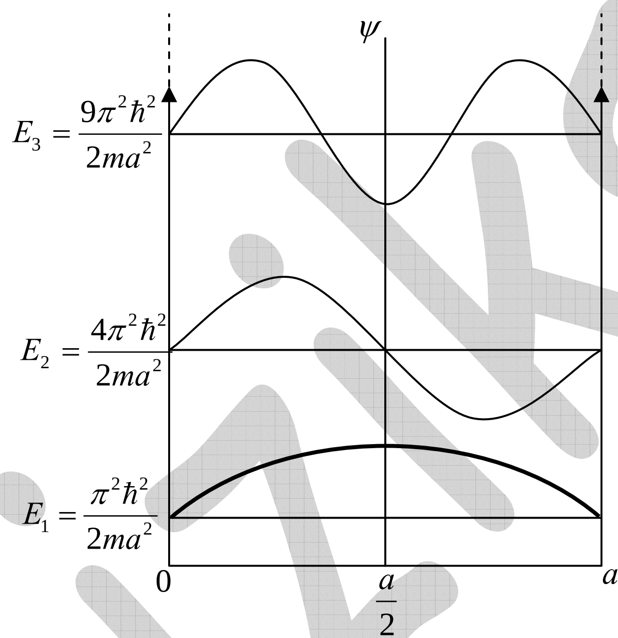
$\psi_n(x) = A\sin\dfrac{n\pi x}{a}$, the value of A can be find with normalization condition which is

$$\int_0^a |\psi_n(x)|^2\, dx = 1 \quad \Rightarrow A = \sqrt{\frac{2}{a}}$$

So energy eigen function is given by $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\qquad n=1,2,3,....$

Energy eigen value is given as $E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$ where $n=1,2,3,....$

The orthonormal condition is given by $\int_0^a \psi_n^*\psi_m dx=\delta_{mn}$

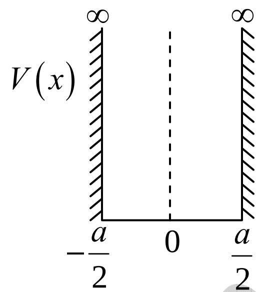


Any function $f(x)$ can be expressed in the term of $\psi_n(x)$.

$$f(x)=\sum_{n=1}^{\infty}c_n\psi_n(x)\text{ where } c_n=\int\psi_n^*(x)f(x)dx$$

4.6 Infinite Symmetric Potential Box

$$V(x) = \infty \qquad x < -\frac{a}{2} \text{ and } \quad x > \frac{a}{2}$$
$$= 0 \qquad -\frac{a}{2} < x < \frac{a}{2}$$



Wave function in region $x < -a/2$ and $x > a/2$ is $\psi(x) = 0$ because wave function is infinite

in region $\frac{-a}{2} < x < \frac{a}{2}$ the solution of Schrödinger wave function is given by.

$\psi = A\sin kx + B\cos kx$ Parity operator will commute with Hamiltonian because so wave function can have either even or odd symmetry .

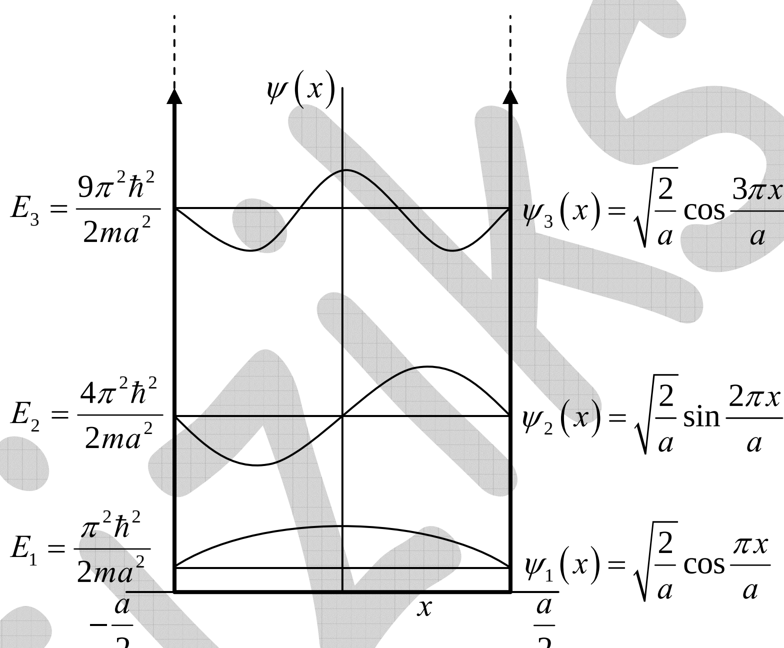
Now wave function must vanish at boundary.

Case I Solution for even symmetry	Case II Solution for odd symmetry
$\psi(-x) = \psi(x)$ $-A\sin kx + B\cos kx = A\sin kx + B\cos kx$ means $A = 0 \Rightarrow \psi(x) = B\cos kx$ Using boundary i.e. $\cos\frac{ka}{2} = 0 \quad \frac{ka}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ $k_n = \frac{n\pi}{a}$ where $n = 1, 3, 5....$ So $\psi_n(x) = B_n \cos k_n x$ normalized wavefunction $\psi_n(x) = \sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a}$ $k_n = \frac{n\pi}{a} = n = 1, 3, 5, \ldots$	$\psi(-x) = -\psi(x)$ $-A\sin kx + B\cos kx = -A\sin kx - B\cos kx$ means $B = 0 \Rightarrow \psi(x) = A\sin kx$ Using boundary i.e. $\sin\frac{ka}{2} = 0 \quad \frac{ka}{2} = \pi, 2\pi, 3\pi.....$ $k_n = \frac{n\pi}{a}$ where $n = 2, 4, 6...$ So $\psi_n(x) = A_n \sin k_n x$ normalized wavefunction $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ $k_n = \frac{n\pi}{a}$ where $n = 2, 4, 6...$

Solution for infinite symmetric box as mentioned above is given by

$$\psi_n = \begin{cases} \sqrt{\dfrac{2}{a}}\cos\dfrac{n\pi x}{a} & n = 1, 3, 5, ... \\ \sqrt{\dfrac{2}{a}}\sin\dfrac{n\pi x}{a} & n = 2, 4, 6, ... \end{cases}$$

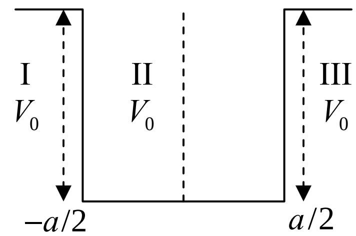
and energy eigen value $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2} \quad n = 1, 2, 3, 4, 5, 6...$



First three eigenstate are shown in the above figure.

4.7 Square Well Finite Potential Box (graphical method)

Square well finite potential box is defined as.



$$V(x) = 0 \qquad -\frac{a}{2} < x < \frac{a}{2}$$

$$= V_0 \qquad \text{otherwise}$$

For Bound state $E < 0$

Schrodinger wave solution in region I, i.e., $x < -\frac{a}{2}$

$$\phi_1(x) = Ae^{\gamma x} + Be^{-\gamma x} \qquad x \to -\infty \text{ wave function must be zero}$$

so B = 0

$$\text{where } \phi_1(x) = Ae^{\gamma x} \qquad x < -\frac{a}{2}$$

Schrondinger wave solution in region II, i.e., $-\frac{a}{2} \le x \le \frac{a}{2}$ is

$$\phi_I(x) = B\cos kx \qquad .$$

$$\phi_{II}(x) = C\sin kx \qquad \text{for odd parity}$$

(potential is symmetric about $x = 0$ so parity must be conserve).

$$\text{where } k^2 = \frac{2mE}{\hbar^2}$$

Schrodinger wave solution in region III i.e., $x > \frac{a}{2}$

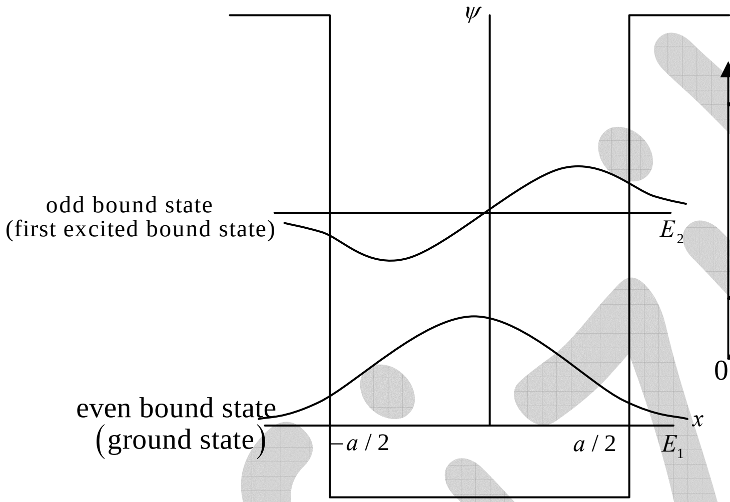
$$\psi = De^{-\gamma x} + Ee^{\gamma x}$$

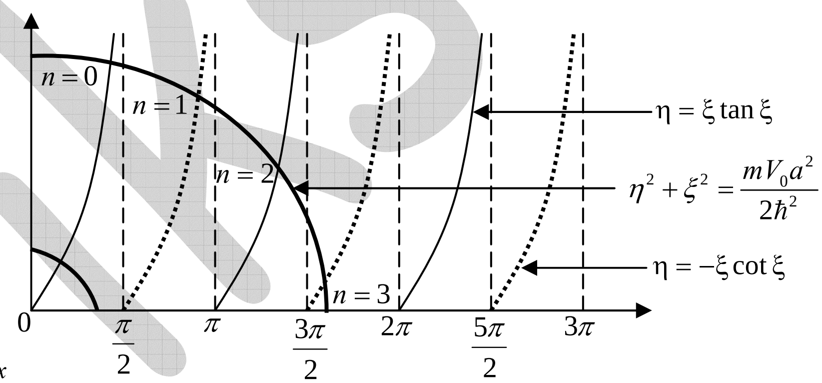
The wave function must vanish at $x \to \infty$ so E = 0

$$\phi_{III}(x) = De^{-\gamma x} \text{ where } \gamma^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

Solution for even parity	Solution for odd parity
$\psi(-x)=\psi(x)\quad \lvert x\rvert>\frac{a}{2}$ $\phi_I(x)=Ae^{\gamma x}$ for $x<-\frac{a}{2}$ $\phi_{II}(x)=B\cos kx$ for $-\frac{a}{2}<x<\frac{a}{2}$ $\phi_{III}(x)=De^{-\gamma x}$ for $x>\frac{a}{2}$ The wave function must be continuous and differentiable at boundary. So $\phi_I\left(x=-\frac{a}{2}\right)=\phi_{II}\left(x=-\frac{a}{2}\right)$ $Ae^{-\frac{\gamma a}{2}}=B\cos\frac{ka}{2}\quad\ldots(X)$ $-\gamma Ae^{-\frac{\gamma a}{2}}=-kB\sin\frac{ka}{2}\quad\ldots(Y)$ Dividing on Y by X one can get. $\gamma=k\tan\frac{ka}{2}$ $\frac{\gamma a}{2}=\frac{ka}{2}\tan\frac{ka}{2}$ $\eta=\frac{\gamma a}{2},\ \xi=\frac{ka}{2},\ \eta^2+\xi^2=\frac{mV_0a^2}{2\hbar^2}$ The even bound state energy can be found by solution of equation $\eta^2+\xi^2=\frac{mV_0a^2}{2\hbar^2}$ and $\eta=\xi\tan\xi$ can be found graphically.	$\psi(-x)=-\psi(x)\quad \lvert x\rvert>\frac{a}{2}$ $\phi_I=Ae^{\gamma x}\quad x<-\frac{a}{2}$ $\phi_{II}=C\sin kx\quad -\frac{a}{2}<x<\frac{a}{2}$ $\phi_{III}=De^{-\gamma x}\quad x>\frac{a}{2}$ The wave function must be continuous and differentiable at boundary. So $\phi_I\left(x=-\frac{a}{2}\right)=\phi_{II}\left(x=-\frac{a}{2}\right)$ $Ae^{-\frac{\gamma a}{2}}=-C\sin\frac{ka}{2}\quad\ldots(X)$ $\Rightarrow A\gamma e^{-\frac{\gamma a}{2}}=Ck\cos\frac{ka}{2}\quad\ldots.(Y)$ $\Rightarrow\frac{\gamma a}{2}=-\frac{ka}{2}\cot\frac{ka}{2}$ $\eta=-\xi\cot\xi\ \eta=\xi\tan\left(\xi+\frac{\pi}{2}\right)$ and $\eta=\frac{\gamma a}{2},\ \xi=\frac{ka}{2},\ \eta^2+\xi^2=\frac{mV_0a^2}{2\hbar^2}$ The even bound state energy can be found by solution of equation $\eta^2+\xi^2=\frac{mV_0a^2}{2\hbar^2}$ and $\eta=-\xi\cot\xi$ can be found graphically.

First even bound state can be found if	First odd bound state can be found if
$\left(\dfrac{mV_0a^2}{2\hbar^2}\right)^{1/2} < \pi$	$\dfrac{\pi}{2} < \dfrac{mV_0a^2}{2\hbar^2} < \dfrac{3\pi}{2}$
Second even bound state can be found	Second odd bound state can be found.
$\pi < \left(\dfrac{mV_0a^2}{2\hbar^2}\right)^{1/2} < 2\pi$	$\dfrac{3\pi}{2} < \dfrac{mV_0a^2}{2\hbar^2} < \dfrac{5\pi}{2}\ldots$





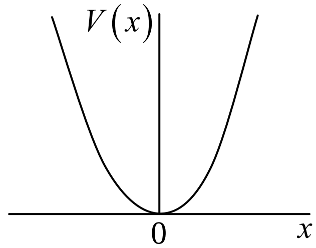
In the table below shown the number of bound states for various range of V_0a^2.

where R denotes the radius.

R	V_0a^2	Even function	Odd function	No. of bound state
$\dfrac{\pi}{2}$	$<\dfrac{\hbar^2\pi^2}{2m}$	1	0	1
$\dfrac{\pi}{2} < R < \pi$	$\dfrac{\hbar^2\pi^2}{2m}$ to $\dfrac{4\pi^2\hbar^2}{2m}$	1	1	2
$\pi < R < \dfrac{3\pi}{2}$	$\dfrac{4\pi^2\hbar^2}{2m}$ to $\dfrac{9\pi^2\hbar^2}{2m}$	2	1	3
$\dfrac{3\pi}{2} < R < 2\pi$	$\dfrac{9\pi^2\hbar^2}{2m}$ to $\dfrac{16\pi^2\hbar^2}{2m}$	2	2	4

4.8 Harmonic Oscillator (Parabolic potential)

The parabolic potential is defined as



$$V(x)=\frac{1}{2}m\omega^2X^2$$

$$H=\frac{P^2}{2m}+\frac{1}{2}m\omega^2X^2$$

The Schrodinger wave function is given.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2X^2\psi=E\psi$$

put $\xi=\sqrt{\frac{m\omega}{\hbar}}x$, $k=\frac{2E}{\hbar\omega}$

$$\frac{d^2\psi}{d\xi^2}=(\xi^2-k)\psi \qquad \ldots(i)$$

For large k. Solution is $\psi(\xi)\approx Ae^{-\frac{\xi^2}{2}}$

The general solution is given by $\psi(\xi)=H_n(\xi)e^{-\frac{\xi^2}{2}}$

So equation (i) reduce to $\frac{d^2H_n}{dx^2}-2\xi\frac{dH_n}{d\xi}+(k-1)H=0$

Solving series solution by putting $H_n(\xi)=\sum_{n=0}^{\infty}a_n\xi^n$

One can find $a_{n+2}=\frac{(2n+1-k)}{(n+1)(n+2)}a_n$

When $k = 2n + 1$ the equation (i) reduce to Hermite polynomial and

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega \qquad \text{where } n = 0, 1, 2, \ldots$$

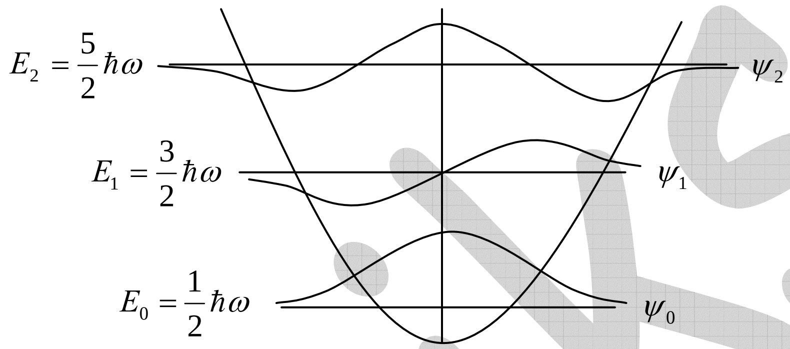
And $$\frac{d^2H_n}{d\xi^2}-2\xi\frac{dH_n}{d\xi}+2nH_n=0 \qquad \ldots(ii)$$

Hermite polynomials and it is given as $H_0=1,\ H_1=2\xi$ and $H_2=4\xi^2-2$

Energy eigen function is $\psi_n(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2}}$

And eigen value is $E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad n = 0, 1, 2, \dots$

Ground state $n = 0 = E = \frac{\hbar\omega}{2}$ is zero point energy



The first three stationary state and corresponding eigen value for the harmonic oscillator.

The eigenfunction $\psi_0(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\xi^2}{2}}$ have energy $E_0 = \frac{\hbar\omega}{2}$

The eigenfunction $\psi_1(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} 2\xi e^{-\frac{\xi^2}{2}}$ have energy $E_1 = \frac{3\hbar\omega}{2}$

The eigenfunction $\psi_2(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^2\ 2!}} \left(4\xi^2 - 2\right) e^{-\frac{\xi^2}{2}}$

$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{8}} \left(4\xi^2 - 2\right) e^{-\frac{\xi^2}{2}}$ have energy $E_2 = \frac{5\hbar\omega}{2}$

Problems on One Dimensional System:

Example: A particle of mass *m* is confined into a box of width a where potential is defined as .

$$V(x)=0 \qquad 0<x<L$$
$$= \infty \qquad \text{otherwise}$$

Find

(a) $\langle X\rangle$ (b) $\langle X^2\rangle$ (c) $\langle P_x\rangle$ (d) $\langle P_x^2\rangle$ (e) $\Delta x\cdot\Delta P_x$

Solution: A normalized wave function in above potential is given by.

$$\phi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{L}$$

(a) $$\langle x\rangle=\int_{-\infty}^{\infty}\phi_n^* x\phi_n dx=\int_0^L\frac{2}{L}k\sin^2\frac{n\pi x}{L}dx=\frac{L}{2}$$

(b) $$\langle x^2\rangle=\int_{-\infty}^{\infty}\phi_n^*(x)x^2\phi_n(x)=\frac{2}{L}\int_0^L x^2\sin^2\frac{n\pi x}{L}dx=\frac{L^2}{3}-\frac{L^2}{2n^2\pi^2}$$

(c) $$\langle P_x\rangle=\int_{-\infty}^{\infty}\phi_n(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\phi_n(x)dx=-\frac{2}{L}(i\hbar)\frac{n\pi}{L}\int_0^L\sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx=0$$

(d) $$\langle P_x^2\rangle=\int_{-\infty}^{\infty}\phi_n^*(x)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\phi_n(x)dx=\frac{n^2\pi^2\hbar^2}{L^2}$$

(e) $$\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2} \qquad \Delta x=L\sqrt{\frac{1}{12}-\frac{1}{2n^2\pi^2}}$$

$$\Delta P_x=\left(\langle P_x^2\rangle-\langle P_x\rangle^2\right)^{1/2} \qquad \Delta P_x=\frac{n\pi\hbar}{L}$$

$$\Delta x\cdot\Delta P_x=n\pi\hbar\sqrt{\frac{1}{12}-\frac{1}{2n^2\pi^2}}$$

Example: A particle of mass m which moves freely inside an infinite potential well of length a has the following initial wave function.

$$\psi(x,0)=\frac{A}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{3}{5a}}\sin\left(\frac{3\pi x}{a}\right)+\frac{1}{\sqrt{5a}}\sin\left(\frac{5\pi x}{a}\right)$$

(a) Find A so that $\psi(x,0)$ is normalized.

(b) If measurement of energy carried out what are the values that will be found and what are the corresponding probability.

(c) Calculate the average energy.

Solution: Particle of mass m confine into a box of width a.

$$\text{So } \phi_n(x) \text{ or } |\phi_n\rangle=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

(a) $$|\psi\rangle=\frac{A}{\sqrt{2}}|\phi_1\rangle+\sqrt{\frac{3}{10}}|\phi_3\rangle+\frac{1}{\sqrt{10}}|\phi_5\rangle$$

$$\langle\psi|\psi\rangle=1 \qquad \frac{A^2}{2}+\frac{3}{10}+\frac{1}{10}=1 \qquad A=\sqrt{\frac{6}{5}}$$

(b) $$|\psi\rangle=\sqrt{\frac{6}{10}}|\phi_1\rangle+\sqrt{\frac{3}{10}}|\phi_3\rangle+\frac{1}{\sqrt{10}}|\phi_5\rangle$$

If energy will be manual on $|\psi\rangle$ state the measurement of $|\psi\rangle$ yields either

$\frac{\pi^2\hbar^2}{2ma^2},\frac{9\pi^2\hbar^2}{2ma^2},\frac{25\pi^2\hbar^2}{2ma^2}$ which is eigen function.

Associate with $|\phi_1\rangle$ $|\phi_3\rangle$ $|\phi_5\rangle$ respectively.

Probability to measures $\frac{\pi^2\hbar^2}{2ma^2}$ is $\frac{|\langle\phi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\frac{3}{5}$

Probability to measures $\frac{9\pi^2\hbar^2}{2ma^2}=\frac{|\langle\phi_3|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\frac{3}{10}$

Probability to measures $\frac{25\pi^2\hbar^2}{2ma^2}=\frac{|\langle\phi_5|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\frac{1}{10}$

(c) Average energy is given by $\langle E\rangle = \Sigma E_i P_i$

$$= \frac{3}{5}\times\frac{\pi^2\hbar^2}{2ma^2} + \frac{3}{10}\times\frac{9\pi^2\hbar^2}{2ma^2} + \frac{1}{10}\frac{25\pi^2\hbar^2}{2ma^2} = \frac{29\pi^2\hbar^2}{10ma^2}$$

Example: Prove that for any normalized wave function of particle in of mass m in one dimensional $\int_{-\infty}^{\infty} J(x)dx = \frac{\hbar}{2im}\left[\int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\psi\, dx - \int_{-\infty}^{\infty}\psi\frac{\partial}{\partial x}\psi^* dx\right]$

Solution: $\frac{\hbar}{2im}\int_{-\infty}^{\infty} J(x)dx = \frac{\hbar}{2im}\left[\int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\psi\, dx - \int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\psi\, dx\right]$

$$= \frac{\hbar}{2im}\left[\frac{1}{-i\hbar}\int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x) - \left(\frac{1}{-i\hbar}\right)\psi^*\left(-i\hbar\right)\frac{\partial\psi(x)}{\partial x}\right]$$

$$\int_{-\infty}^{\infty} J(x)dx = \frac{\hbar}{2im}\frac{2\langle P_x\rangle}{-i\hbar} = \frac{\langle P_x\rangle}{m}$$

Example: Three dimensional wave function is given by $\psi(r) = \left(\frac{A}{r}\right)e^{ikr}$

Find the current density.

Solution: $\psi(r) = \left(\frac{A}{r}\right)e^{ikr}$ $\quad J = \frac{\hbar}{2im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$

where $\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$

$$J = \frac{|A|^2}{r}\frac{\hbar k}{m}\hat{r}$$

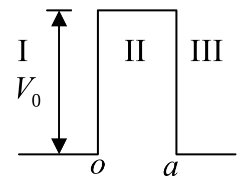
Example: A potential barrier is given as $\begin{array}{ll} V(x) = V_0 & 0 < x < a \\ V(x) = 0 & \text{otherwise} \end{array}$

Prove that the expression of transmission probability for $E < V_0$ is given as.

$$T = \frac{16}{4+\left(\frac{k_2}{k_1}\right)^2}e^{-2k_2L} \quad \text{where } k_1^2 = \frac{2mE}{\hbar^2} \text{ and } k_2^2 = \frac{2m(V_0-E)}{\hbar^2}$$

Solution: The potential is given as $V(x) = V_0 \quad 0 < x < a$

$= 0$ otherwise



Case $E < V_0$

The Schrödinger wave solution in I, II and III is given by

For region I $x < 0$; $\quad \psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$

For region II $0 < x < a$; $\quad \psi_{II}(x) = Ce^{-ik_2x} + De^{k_2x}$

For region III $x > a$; $\quad \psi_{III}(x) = Fe^{ik_1x} + Ge^{-ik_1x}$

Boundary condition $\psi_I = \psi_{II}$ and $\dfrac{d\psi_I}{dx} = \dfrac{d\psi_{II}}{dx}$ at $x = 0$ and at $x = a$

$$\psi_{II} = \psi_{III}; \quad \frac{d\psi_{II}}{dx} = \frac{d\psi_{III}}{dx} \qquad \text{at } x = a$$

$A + B = C + D, \; ik_1A - ik_1B - k_2C + k_2D$

and $Ce^{-k_2a} + De^{k_2a} = Fe^{ik_1a}, \quad -k_2Ce^{-k_2a} + k_2De^{k_2a} = ik_1Fe^{ik_1a}$

The transmission probability is given by

$$\left|\frac{J_T}{J_i}\right| = \frac{|F|^2}{|A|^2} \quad \text{where} \quad \psi_i = Ae^{ik_1x} \text{ and } \psi_t = Fe^{ik_1x}$$

Solving the above four boundary condition

$$\left(\frac{A}{F}\right) = \left[\frac{1}{2} + \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right]e^{(ik_1+k_2)a} + \left[\frac{1}{2} - \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right]e^{-(ik_1-k_2)a}$$

For approximation $V_0 > E$, $\quad \dfrac{k_2}{k_1} - \dfrac{k_1}{k_2} \approx \dfrac{k_2}{k_1} \qquad e^{k_2a} >> e^{-k_2a}$

$$\left(\frac{A}{F}\right) = \left(\frac{1}{2} + \frac{ik_2}{4k_1}\right)e^{(ik_1+k_2)a} \Rightarrow \left(\frac{A^*}{F}\right) = \left(\frac{1}{2} - \frac{ik_2}{4k_1}\right)e^{(-ik_1+k_2)a} \Rightarrow \frac{AA^*}{FF^*} = \left(\frac{1}{4} + \frac{k_2^2}{16k_1^2}\right)e^{2k_2a}$$

$$\Rightarrow T = \frac{FF^*}{AA^*} = \left(\frac{16}{4 + \left(\frac{k_2}{k_1}\right)^2}\right)e^{-2k_2a} \Rightarrow T \approx e^{-2k_2a}$$

Example: Consider a particle of mass m and charge q place in uniform field $E = E_0\hat{i}$. Apart from this a restoring force corresponding to the potential $V_1(x) = \frac{1}{2}m\omega^2X^2$ acts. Find the lowest energy eigen value. Consider the electric field is originated at origin.

Solution: The electric potential energy at the position x will be $-qEx$. So the effective potential is given by

$$V(x) = \frac{1}{2}m\omega^2X^2 - qEx = \frac{1}{2}m\omega^2\left(X - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

So Hamiltonion is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(X - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

Put $X - \frac{qE}{m\omega^2} = X'$ $\quad \Rightarrow H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X'^2 - \frac{q^2E^2}{2m\omega^2}$

So Energy is given by

$$E = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{q^2E^2}{2m\omega^2}$$

Example: A particle of energy 9ev are sent towards a potential step 8eV high.

(a) What is reflection coefficient.

(b) What percentage will transmitted.

Solution: $R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right)^2 = \left(\frac{\sqrt{9} - \sqrt{9-8}}{\sqrt{9} + \sqrt{9-8}}\right)^2 = \left(\frac{3-1}{3+1}\right)^2 = \frac{1}{4}$ $\qquad 1 - T = R$

$= 1 - \frac{1}{4} = \frac{3}{4}$ $\qquad \%T = \frac{3}{4} \times 100 = 75\%$

Example: A particle in the infinite square well has the initial wave function.

$\psi(x,0) = Ax(a - x) \quad (0 \le x \le a)$

(a) Find the value of A such that ψ is normalized.

(b) Write down $\psi(x)$ in the basis of $\phi_n(x)$ where $\phi_n(x)$ is the eigen function of the nth state (wave function) for the system confine into box whose potential is given as

$$V(x)=0 \qquad 0<x<a$$
$$=\infty \qquad \text{otherwise}$$

(c) Write down expression of $\psi(x,t)$

Solution: (a) $\psi(x,0)=Ax(a-x) \quad 0\le x\le a$

For normalization $\int_{-\infty}^{\infty}|\psi(x,0)|^2\,dx=1$

$$=A^2\int_0^a x^2(a-x)^2dx=1 \quad \Rightarrow A=\sqrt{\frac{30}{a^5}}$$

(b) $\psi_n(x)=\Sigma c_n\phi_n(x)$ Where $\psi_n(x)=\sqrt{\frac{30}{a^5}}x(a-x)$

And can be found with Fourier's trick $c_n=\int\psi_n^*\phi(x)dx$

So $$c_n=\sqrt{\frac{2}{a}}\int_0^a\sin\left(\frac{n\pi x}{a}\right)\sqrt{\frac{30}{a^5}}x(a-x)dx$$

$$c_n=\begin{cases}0 & \text{if } n \text{ is even}\\ 8\dfrac{\sqrt{15}}{(n\pi)^3} & \text{if } n \text{ is odd}\end{cases} \qquad \Rightarrow \psi_n(x)=\sum_{n=1,3,...}\frac{8\sqrt{15}}{(n\pi)^3}\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

From Schrodinger wave equation

$$H\phi_n(x)=i\hbar\frac{\partial}{\partial t}\phi_n \qquad \Rightarrow E_n\phi_n(x)=i\hbar\frac{\partial\phi_n}{\partial t} \qquad \Rightarrow \phi_n(x,t)=\phi_n(x,0)e^{-E_nt/\hbar}$$

Where $E_n=\dfrac{n^2\pi^2\hbar^2}{2ma^2}$

So $$\psi(x,t)=\sqrt{\frac{30}{a}}\left(\frac{2}{\pi}\right)^3\sum_{1,3,5,...}\frac{1}{n^3}\sin\left(\frac{n\pi x}{a}\right)e^{-in^2\pi^2\hbar t/2ma^2}$$

Example: $\left|\phi_n\right\rangle$ represent the energy eigen state of a linear harmonic oscillator and If

state $|\psi\rangle=\frac{1}{\sqrt{2}}(|\phi_0\rangle+|\phi_1\rangle)$ of harmonic state of angular ω

(a) If energy is measured what will measurement with what probability

(b) Find average value of energy.

Solution: (a) if energy is measured the measurement is $\frac{\hbar\omega}{2}$ and $\frac{3\hbar\omega}{2}$ with probability $\frac{1}{2}$ and $\frac{1}{2}$

$$\langle E\rangle=\frac{1}{2}\times\frac{\hbar\omega}{2}+\frac{1}{2}\times\frac{3\hbar\omega}{2}=\hbar\omega$$

4.9 Multiple Dimensional Systems

If x, y, z are independent then $\psi(x, y, z)$ can be written as $X(x)Y(y)Z(z)$ and Energy Eigen value can be written as $E_{x,y,z}=E_x+E_y+E_z$

Where $\mathrm{H_x X(x)=E_x X(x)}$, $H_yY(y)=E_yY(y)$, $H_zZ(z)=E_zZ(z)$

and $H=H_x+H_y+H_z$ and $H\psi=E(\psi)$

4.10 Two Dimensional Free Particle

$$\psi(x,y)=\frac{1}{\sqrt{2\pi}}e^{ik_xx}\cdot\frac{1}{\sqrt{2\pi}}e^{ik_yy}\Rightarrow\psi(x,y)=\frac{1}{\sqrt{2\pi}}e^{i(k_xx+k_yy)}$$

$$E=E_x+E_y=\frac{\hbar^2}{2m}\left(k_x^2+k_y^2\right)=\frac{\hbar^2}{2m}k^2\ ,\qquad \left|\vec{k}\right|=\sqrt{k_x^2+k_y^2}\ =\text{constant}$$

So two dimensional free particle is infinitely degenerate.

4.11Three Dimensional Free Particle

The wave function is $\psi(x, y, z)$ is defined as

$$\psi(x,y,z)=\frac{1}{(2\pi)^{3/2}}e^{i(k_xx+k_yy+k_zz)}\qquad E=E_x+E_y+E_z=\frac{\hbar^2}{2m}\left(k_x^2+k_y^2+k_z^2\right)$$

$$E=\frac{\hbar^2}{2m}k^2\qquad k^2=k_x^2+k_y^2+k_z^2$$

so three dimensional free particle is infinitely degenerate.

4.12 Particle in Two Dimensional Box

The two dimensional system is defined as

$$V(x,y)=\begin{cases}0 & 0<x<a, \quad 0<y<a\\ \infty & \text{otherwise}\end{cases}$$

The wave function is given as $\psi(x,y)=\sqrt{\frac{2}{a}}\sin\frac{n_x\pi x}{a}\sqrt{\frac{2}{a}}\sin\frac{n_y\pi y}{a}$

$$\mathrm{E}=\frac{\hbar^2}{2\mathrm{ma}^2}\left(\mathrm{n}_\mathrm{x}^2+\mathrm{n}_\mathrm{y}^2\right) \quad \text{where} \quad n_x=1,2,3..... \quad n_y=1,2,3,......$$

Ground state energy $n_x=1, n_y=1 \;\; E=\frac{2\hbar^2\pi^2}{2ma^2}$

First excited state $n_x=1, n_y=2 \;\; E=\frac{5\hbar^2\pi^2}{2ma^2}$

(Degeneracy of first excited state is doubly degenerate)

Second excited state $n_x=2, n_y=2$, $E=\frac{8\hbar^2\pi^2}{2ma^2}$

Third excited state $n_x=1, n_y=3$, $E=\frac{10\hbar^2\pi^2}{2ma^2}$

$$n_x=3, n_y=3$$

(Third excited state is doubly generate)

4.13 Particle in Three Dimensional Box

$$V(x,y,z)=\begin{cases}0 & 0<x<L_x, \quad 0<y<L_y, 0<z<L_z\\ \infty & \text{otherwise}\end{cases}$$

$$\psi(x,y,z)=\sqrt{\frac{2}{L_x}}\sin\frac{n_x\pi x}{L_x}\sqrt{\frac{2}{L_y}}\sin\frac{n_y\pi y}{L_y}\sqrt{\frac{2}{L_z}}\sin\frac{n_z\pi z}{L_z}$$

$$E_{x,y,z}=\frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right) \text{ where } n_x=1,2,3,...... \quad n_y=1,2,3,......, n_z=1,2,3,......$$

For cubic box $L_x = L_y = L_z = L$ $\quad E_{x,y,z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right)$

where $n_x = 1,2,3,....,\ n_y = 1,2,3,...\ n_z = 1,2,3,....$

Ground state $n_x = 1,\ n_y = 1,\ n_z = 1$ $\quad E = \frac{3\pi^2\hbar^2}{2mL^2}$ Non degenerate

First excited state $n_x = 1,\ n_y = 1,\ n_z = 2$ $\quad E = \frac{6\pi^2\hbar^2}{2mL^2}$ Triple degenerate

$n_x = 1,\ n_y = 2,\ n_z = 1,$ $\quad n_x = 2,\ n_y = 1,\ n_z = 1$

Second excited state $n_x = 2,\ n_y = 2,\ n_z = 1$ $\quad E = \frac{9\pi^2\hbar^2}{2mL^2}$ Triple degenerate

$n_x = 2,\ n_y = 1,\ n_z = 2,$ $\quad n_x = 1,\ n_y = 2,\ n_z = 2$

4.14 Two Dimensional Harmonic Oscillator

$V(x,y) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2$ $\quad \Rightarrow \psi_{n_x,n_y}(x,y) = X_{n_x}(x)Y_{n_y}(y)$

$$E_{n_x,n_y} = E_{n_x} + E_{n_y} = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega$$

$$= (n_x + n_y + 1)\hbar\omega \qquad n_x = 0,1,2,3,.....$$

$$n_y = 0,1,2,3,.....$$

$$= (n+1)\hbar\omega \qquad n = 1,2,3,........$$

Ground state energy $\quad n_x = 0, n_y = 0 \quad E = \hbar\omega$

First excited energy $\quad n_x = 1, n_y = 0 \quad E = 2\hbar\omega$

$n_x = 0, n_y = 1$

Second excited energy $\quad n_x = 2, n_y = 0 \quad E = 3\hbar\omega$

$n_x = 0, n_y = 2$

$n_x = 1, n_y = 1$

So degeneracy of two dimensional harmonic oscillators for n^{th} state is $(n+1)$

Three Dimensional Harmonic Oscillators

$$V(x,y,z)=\frac{1}{2}m\omega_x^2x^2+\frac{1}{2}m\omega_y^2y^2+\frac{1}{2}m\omega_z^2z^2$$

$$\psi_{n_x,n_y,n_z}(x,y,z)=X_{n_x}(x)Y_{n_y}(y)Z_{n_z}(z)$$

$$\mathrm{E}=\left(\mathrm{n_x}+\frac{1}{2}\right)\hbar\omega+\left(\mathrm{n_y}+\frac{1}{2}\right)\hbar\omega+\left(\mathrm{n_z}+\frac{1}{2}\right)\hbar\omega \qquad n_x=n_y=n_z=0,1,2,.........$$

For isotropic harmonic oscillator is $\omega_x=\omega_y=\omega_z$

$$E_{n_x,n_y,n_z}=\left(n_x+n_y+n_z+\frac{3}{2}\right)\hbar\omega=\left(n+\frac{3}{2}\right)\hbar\omega$$

The degeneracy of the isotropic harmonic oscillator is for n$^{\text{th}}$ state is $g_n=\frac{1}{2}(n+1)(n+2)$

where n = 0 corresponding ground state.

Example: If $b=2a$, write down ground, first, second and third excited state energy.

Solution: $E_{n_x,n_y}=\frac{\pi^2\hbar}{2m}\left(\frac{n_x^{\ 2}}{a^2}+\frac{n_y^{\ 2}}{b^2}\right) \Rightarrow E_{n_x,n_y}=\frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^{\ 2}}{a^2}+\frac{n_y^{\ 2}}{(2a)^2}\right) \because b=2a$

$$E_{n_x,n_y}=\frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^{\ 2}}{a^2}+\frac{n_y^{\ 2}}{4a^2}\right)$$

For ground state: $E_{1,1}=\frac{5}{8}\frac{\pi^2\hbar^2}{ma^2}$

For first excited state: $E_{1,2}=\frac{8}{8}\frac{\pi^2\hbar^2}{ma^2}$

For second excited stste $E_{1,3}=\frac{13}{8}\frac{\pi^2\hbar^2}{ma^2}$

For third excited stste $E_{2,1}=\frac{17\pi^2\hbar^2}{8ma^2}$

For fourth excited stste $E_{1,4}=E_{2,2}=\frac{20\pi^2\hbar^2}{8ma^2}$

Example: $V(x,y,z)=0 \qquad 0<x<a, 0<y<b, 0<z<c$

$= \infty \qquad$ Otherwise

If $b=2a$ and $c=3a$, then write down energy eigenvalue for ground state, first excited state and second excited state.

Solution: $E(n_x,n_y,n_z)=\dfrac{\pi^2\hbar^2}{ma^2}\left(n_x^2+\dfrac{n_y^2}{4}+\dfrac{n_z^2}{9}\right)$

For ground state: $n_x=1, n_y=1, n_z=1 \Rightarrow E_{1,1}=\dfrac{49}{36}\times\dfrac{\pi^2\hbar^2}{2ma^2}$

For first excited state: $n_x=1, n_y=1, n_z=2 \Rightarrow E_{1,1,2}=\dfrac{61}{36}\times\dfrac{\pi^2\hbar^2}{2ma^2}$

For second excited state: $n_x=1, n_y=2, n_z=1 \Rightarrow E_{1,2,1}=\dfrac{76}{36}\times\dfrac{\pi^2\hbar^2}{2ma^2}$

For third excited state: $n_x=1, n_y=1, n_z=3 \Rightarrow E_{1,1,3}=\dfrac{81}{36}\times\dfrac{\pi^2\hbar^2}{2ma^2}$

Example: If the potential of two dimensional harmonic oscillator is

$V(x,y)=\dfrac{1}{2}m\omega^2x^2+\dfrac{1}{2}m4\omega^2y^2$, then find energy Eigen value.

Solution: $V(x.y)=\dfrac{1}{2}m\omega^2x^2+\dfrac{1}{2}m(2\omega)^2\,y^2$ i.e. $\omega_1=\omega,\ \omega_2=2\omega$

$$\therefore E_{n_x,n_y}=\left(n_x+\frac{1}{2}\right)\hbar\omega+\left(n_y+\frac{1}{2}\right)2\hbar\omega$$

For ground state: $E_{0,0}=\dfrac{3}{2}\hbar\omega$, For first excited state: $E_{1,0}=\dfrac{5}{2}\hbar\omega$

For second excited state: $E_{0,1}=E_{2,0}=\dfrac{7}{2}\hbar\omega$ (Doubly degenerate)

For third excited state: $E_{3,0}=\dfrac{9}{2}\hbar\omega$

For fourth excited state: $E_{2,1}=E_{4,0}=\dfrac{11}{2}\hbar\omega$ (Doubly degenerate)

Example: If the potential of three dimensional harmonic oscillator is

$$V(x,y,z)=\frac{1}{2}m\omega^2x^2+\frac{1}{2}4m\omega^2y^2+\frac{1}{2}9m\omega^2z^2,$$

then write down unnormalised wavefunction for ground state, first excited state and second excited state

Solution: Wavefunction $\psi_{(n_x,n_y,n_z)}\propto H_{n_x}(x)e^{-\frac{m\omega x^2}{2\hbar}}\cdot H_{n_y}(y)e^{-\frac{m\omega y^2}{2\hbar}}\cdot H_{n_z}(z)e^{-\frac{m\omega z^2}{2\hbar}}$

For ground state: $(n_x,n_y,n_z)=(0,0,0)$, then wavefunction

$$\psi_{(0,0,0)}\propto e^{-\frac{m\omega x^2}{2\hbar}}.e^{-\frac{2m\omega y^2}{2\hbar}}.e^{-\frac{3m\omega z^2}{2\hbar}}$$

For first excited state: $(n_x,n_y,n_z)=(1,0,0)$, then wavefunction

$$\psi_{(1,0,0)}\propto x\ e^{-\frac{m\omega x^2}{2\hbar}}.e^{-\frac{m\omega y^2}{\hbar}}.e^{-\frac{3m\omega z^2}{2\hbar}}$$

For second excited state: $(n_x,n_y,n_z)=(1,0,1),(2,0,0)$, then wavefunction

$$\psi_{(1,0,1)}\propto xz\ e^{-\frac{m\omega x^2}{2\hbar}}.e^{-\frac{m\omega y^2}{\hbar}}.e^{-\frac{3m\omega z^2}{2\hbar}}$$

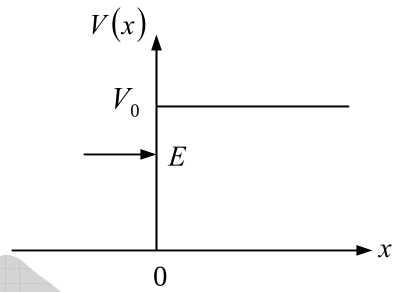
$$\psi_{(2,0,0)}\alpha\left(4x^2-2\right)e^{-\frac{m\omega x^2}{2\hbar}}.e^{-\frac{m\omega y^2}{\hbar}}.e^{-\frac{3m\omega z^2}{2\hbar}}$$

MCQ (Multiple Choice Questions)

Q1. An electron with energy E is incident from left on a potential barrier, given by

$$V(x) = 0 \qquad x < 0$$

$$V(x) = V_0 \qquad x > 0$$



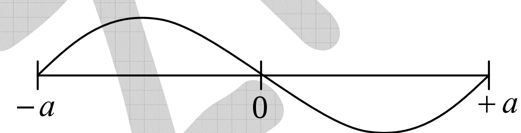
as shown in the figure. For $E < V_0$ the space part of the wavefunction for $x > 0$ is of the form

(a) $\exp \alpha x$ (b) $\exp - \alpha x$ (c) $\exp i\alpha x$ (d) $\exp - i\alpha x$

Q2. The wavefunction of particle moving in free space is given by, $\psi = e^{ikx} + 2e^{-ikx}$ The energy of the particle is

(a) $\frac{5\hbar^2 k^2}{2m}$ (b) $\frac{3\hbar^2 k^2}{4m}$ (c) $\frac{\hbar^2 k^2}{2m}$ (d) $\frac{\hbar^2 k^2}{m}$

Q3. A particle is confined in a one dimensional box with impenetrable walls at $x = \pm a$. Its energy eigenvalue is $2eV$ and the corresponding eigenfunction is as shown below.



The energy for second excited energy of the particle is

(a) $5eV$ (b) $2eV$ (c) $4eV$ (d) $4.5eV$

Q4. A particle of mass m is confined into one dimensional infinite rigid box of width a the quantum state of the system at $t = 0$ given by $|\psi(x,0)\rangle = (|\phi_3\rangle + |\phi_1\rangle)$ the average value of energy on state $|\psi(x,0)\rangle$ is given by

(a) $\frac{5\pi^2\hbar^2}{2ma^2}$ (b) $\frac{5\pi^2\hbar^2}{ma^2}$ (c) $\frac{3\pi^2\hbar^2}{2ma^2}$ (d) $\frac{3\pi^2\hbar^2}{ma^2}$

Q5. A particle of mass m is confined into one dimensional infinite rigid box of width a the quantum state of the system at $t=0$ given by $|\psi(x,0)\rangle=\sqrt{\frac{4}{5}}|\phi_2\rangle+\sqrt{\frac{1}{5}}|\phi_1\rangle$ the probability of measurement of energy $\frac{2\pi^2\hbar^2}{ma^2}$ on state $|\psi(x,0\rangle$ is given by if $|\phi_n\rangle$ is orthonormal eigen state of Hamiltonian

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

Q6. If state of system is define as $|\psi(x,0)\rangle=\sqrt{\frac{4}{5}}|\phi_1\rangle+\sqrt{\frac{1}{5}}|\phi_2\rangle$ for the potential $V(x)=\begin{cases}0 & \text{for } -\frac{a}{2}\le x\le\frac{a}{2}\\ \infty & \text{otherwise}\end{cases}$ where $|\phi_1\rangle,|\phi_2\rangle$ are eigen state of Hamiltonian then which of the following is expression for probability density $|\psi(x,0)|^2$

(a) $\frac{8}{5a}\sin^2\frac{\pi x}{a}+\frac{2}{5a}\sin^2\frac{2\pi x}{a}+\frac{8}{5a}\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}$

(b) $\frac{8}{5a}\cos^2\frac{\pi x}{a}+\frac{2}{5a}\sin^2\frac{2\pi x}{a}+\frac{8}{5a}\cos\frac{\pi x}{a}\sin\frac{2\pi x}{a}$

(c) $\frac{8}{5a}\cos^2\frac{\pi x}{a}+\frac{2}{5a}\cos^2\frac{2\pi x}{a}+\frac{8}{5a}\cos\frac{\pi x}{a}\cos\frac{2\pi x}{a}$

(d) $\frac{8}{5a}\sin^2\frac{\pi x}{a}+\frac{2}{5a}\cos^2\frac{2\pi x}{a}+\frac{8}{5a}\sin\frac{\pi x}{a}\cos\frac{2\pi x}{a}$

Q7. A particle is in the ground state of an infinite square well potential given by,

$$V(x)=\begin{cases}0 & \text{for } -a\le x\le a\\ \infty & \text{otherwise}\end{cases}$$

The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{2}+\frac{1}{\pi}$ (c) $\frac{1}{2}-\frac{1}{\pi}$ (d) $\frac{1}{\pi}$

Q8. A particle in the infinite square well

$$V(x)=\begin{cases}0 & 0<x<a\\ \infty & \text{otherwise}\end{cases}$$

is prepared in a state with the wave function $\psi(x)=\begin{cases}A\sin^3\left(\frac{\pi x}{a}\right) & 0<x<a\\ 0 & \text{otherwise}\end{cases}$

The expectation value of A such that ψ is normalized .

(a) $A=\sqrt{\frac{2}{a}}$ (b) $A=\sqrt{\frac{8}{5a}}$ (c) $A=\sqrt{\frac{16}{5a}}$ (d) $A=\sqrt{\frac{32}{5a}}$

Q9. A particle in the infinite square well

$$V(x)=\begin{cases}0 & 0<x<a\\ \infty & \text{otherwise}\end{cases}$$

is prepared in a state with the wavefunction $\psi(x)=\begin{cases}A\sin^3\left(\frac{\pi x}{a}\right) & 0<x<a\\ 0 & \text{otherwise}\end{cases}$

if energy is measured on state ψ the measurement

(a) $\frac{\hbar^2\pi^2}{2ma^2}$ (b) $\frac{9\hbar^2\pi^2}{2ma^2}$ (c) $\frac{\hbar^2\pi^2}{2ma^2}$ or $\frac{9\hbar^2\pi^2}{2ma^2}$ (d) $\frac{9\pi^2\hbar^2}{10ma^2}$

Q10. A particle is in the ground state of an infinite square well potential given by,

$$V(x)=\begin{cases}0 & \text{for } -a\le x\le a\\ \infty & \text{otherwise}\end{cases}$$

The probability to find the particle in the interval between $-\frac{a}{4}$ and $\frac{a}{4}$ is

(a) $\frac{1}{2}+\frac{\sqrt{2}}{\pi}$ (b) $\frac{1}{4}+\frac{1}{\sqrt{2}\pi}$ (c) $\frac{1}{4}-\frac{1}{\sqrt{2}\pi}$ (d) $\frac{1}{2}-\frac{\sqrt{2}}{\pi}$

Q11. Consider a particle in a one dimensional potential that satisfies $V(x)=V(-x)$. Let $|\psi_0\rangle$ and $|\psi_1\rangle$ denote the ground and the first excited states, respectively, and let $|\psi\rangle=\alpha_0|\psi_0\rangle+\alpha_1|\psi_1\rangle$ be a normalized state with α_0 and α_1 being real constants. The expectation value $\langle x\rangle$ of the position operator x in the state $|\psi\rangle$ is given by

(a) $\alpha_0^2\langle\psi_0|x|\psi_0\rangle+\alpha_1^2\langle\psi_1|x|\psi_1\rangle$ (b) $\alpha_0\alpha_1[\langle\psi_0|x|\psi_1\rangle+\langle\psi_1|x|\psi_0\rangle]$

(c) $\alpha_0^2+\alpha_1^2$ (d) $2\alpha_0\alpha_1$

Q12. A particle of mass m is in potential $V(x)=\begin{cases}0 & -\frac{a}{2}<x<\frac{a}{2}\\ \frac{16\hbar^2}{ma^2} & \text{otherwise}\end{cases}$ then no. of bound state is

(a) One (b) Two (c) Three (d) Infinite

Q13. A quantum mechanical particle in a harmonic oscillator potential has the initial wave function $\psi_0(x)+\psi_1(x)$, where ψ_0 and ψ_1 are the real wavefunctions in the ground and first excited state of the harmonic oscillator Hamiltonian. For convenience we take $m=\hbar=\omega=1$ for the oscillator. What is the probability density of finding the particle at x at time $t=\pi$?

(a) $(\psi_1(x)-\psi_0(x))^2$ (b) $(\psi_1(x))^2-(\psi_0(x))^2$

(c) $(\psi_1(x)+\psi_0(x))^2$ (d) $(\psi_1(x))^2+(\psi_0(x))^2$

Q14. A particle of mass m is confined in the potential $V(x)=\begin{cases}\frac{1}{2}m\omega^2x^2 & \text{for } x>0\\ \infty & \text{for } x\le 0\end{cases}$

Let the wavefunction of the particle be given by $\psi(x)=\frac{1}{\sqrt{5}}\psi_0+\frac{2i}{\sqrt{5}}\psi_1$

where ψ_0 and ψ_1 are the eigenfunctions of the ground state and the first excited state respectively. The expectation value of the energy is

(a) $\frac{31}{10}\hbar\omega$ (b) $\frac{25}{10}\hbar\omega$ (c) $\frac{13}{10}\hbar\omega$ (d) $\frac{11}{10}\hbar\omega$

Q15. The energy eigenvalues of a particle in the potential $V(x)=\frac{1}{2}m\omega^2x^2-ax$ are

(a) $En=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{a^2}{2m\omega^2}$ (b) $En=\left(n+\frac{1}{2}\right)\hbar\omega+\frac{a^2}{2m\omega^2}$

(c) $En=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{a^2}{m\omega^2}$ (d) $En=\left(n+\frac{1}{2}\right)\hbar\omega$

Q16. A particle of mass m is in a cubic box of size a. The potential inside the box $(0\le x<a, 0\le y<a, 0\le z<a)$ is zero and infinite outside. If the particle is in an eigenstate of energy $E=\frac{14\pi\hbar^2}{2ma^2}$, its wavefunction is

(a) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{3\pi x}{a}\sin\frac{5\pi y}{a}\sin\frac{6\pi z}{a}$ (b) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{7\pi x}{a}\sin\frac{4\pi y}{a}\sin\frac{3\pi z}{a}$

(c) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{4\pi x}{a}\sin\frac{8\pi y}{a}\sin\frac{2\pi z}{a}$ (d) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\sin\frac{3\pi z}{a}$

Q17. The energy of the first excited quantum state of a particle in the two-dimensional potential $V(x,y)=\frac{1}{2}m\omega^2\left(x^2+4y^2\right)$ is

(a) $2\hbar\omega$ (b) $3\hbar\omega$ (c) $\frac{3}{2}\hbar\omega$ (d) $\frac{5}{2}\hbar\omega$

Q18
$$\begin{aligned} V(x,y) &= 0 \qquad && 0<x<a, 0<y<b \\ &=\infty && \text{otherwise} \end{aligned}$$

If $b=2a$, the energy of first excited state is given by

(a) $\frac{5}{8}\frac{\pi^2\hbar^2}{ma^2}$ (b) $\frac{\pi^2\hbar^2}{ma^2}$ (c) $\frac{\pi^2\hbar^2}{8ma^2}$ (d) $\frac{13}{8}\frac{\pi^2\hbar^2}{ma^2}$

Q19. A particle of mass m in the potential $V(x,y)=\frac{1}{2}m\omega^2\left(4x^2+y^2\right)$, is in an eigenstate of energy $E=\frac{5}{2}\hbar\omega$. The corresponding un-normalized eigen function is

(a) $y\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2+y^2\right)\right]$ (b) $x\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2+y^2\right)\right]$

(c) $y\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+y^2\right)\right]$ (d) $xy\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+y^2\right)\right]$

Q20. A quantum mechanical particle in a harmonic oscillator potential has the initial wave function $\sqrt{\frac{5}{6}}\psi_0(x)+\sqrt{\frac{1}{6}}\psi_1(x)$, where ψ_0 and ψ_1 are the real wavefunctions in the ground and first excited state of the harmonic oscillator Hamiltonian.if angular frequency of the harmonic oscillator is ω then average value of energy is given by

(a) $\frac{3\hbar\omega}{2}$ (b) $\frac{2\hbar\omega}{3}$ (c) $\frac{8\hbar\omega}{3}$ (d) $\frac{4\hbar\omega}{3}$

NAT (Numerical Answer Type)

Q21. The wavefunction of particle moving in free space is given by, $\psi=e^{ikx}+2e^{-ikx}$. The probability current density for the real part of the wavefunction is …………..in unit of $\frac{\hbar k}{m}$

Q22. The wavefunction of particle of mass m moving in free space is given by, $\psi=4e^{ikx}+e^{-ikx}$. The probability current density for the wavefunction is………..in unit of $\frac{\hbar k}{m}$

Q23. A particle of mass m is confined in one dimensional box of width a if width of the box will increase twice then ratio of energy of bigger to smaller box for given quantum number ………..

Q24. A proton is confined to a cubic box, whose sides have length $10^{-12}\,m$. The minimum kinetic energy of the proton is …………..$\times 10^{-17}\,J$ The mass of proton is $1.67\times 10^{-27}\,kg$ and Planck's constant is $6.63\times 10^{-34}\,Js$.

Q25. A particle in the infinite square well $V(x)=\begin{cases}0 & 0<x<a\\ \infty & \text{otherwise}\end{cases}$ is prepared in a state with the wave function $\psi(x)=\frac{3}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}-\frac{1}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}$ the average value of energy is given by ………….. $\frac{\pi^2\hbar^2}{ma^2}$

Q26. Particle of same mass confined in the other infinite box where potential is defined as

$$V_2(x)=\begin{cases}0 & -\frac{a}{2}<x<\frac{a}{2}\\ \infty & \text{otherwise}\end{cases}$$

The expectation value of position on eigen state is ……………… a

Q27. A particle of mass m is in potential $V(x)=\begin{cases}\infty & 0\le x\\ 0 & 0\le x\le a\\ \frac{16\hbar^2}{ma^2} & x>a\end{cases}$ then no. of bound state is ………………

Q28. Consider the wave function $Ae^{ikr}(r_0/r)$, where A is the normalization constant. For $r=2r_0$, the magnitude of probability current density up to two decimal places, in units of $\left(A^2\hbar k/m\right)$ is…………

Q29. Let ψ_1 and ψ_2 denote the normalized eigenstates for potential is defined as $V_1(x)=\begin{cases}0 & 0<x<a\\ \infty & \text{otherwise}\end{cases}$ of a particle with energy eigenvalues E_1 and E_2 respectively, with $E_2>E_1$. At time $t=0$ the particle is prepared in a state $\Psi(t=0)=\frac{1}{\sqrt{2}}(\psi_1+\psi_2)$

The shortest time T at which $\Psi(t=T)$ will be orthogonal to $\Psi(t=0)$ is $\frac{\hbar\pi}{\alpha}$ then the value of α is given by …………….(unit of $\frac{\pi^2\hbar^2}{2ma^2}$)

Q30. A one dimensional harmonic oscillator is in the superposition of number state $|n\rangle$ given by $$|\psi\rangle=\frac{1}{2}|2\rangle+\frac{\sqrt{3}}{2}|3\rangle$$

The average energy of the oscillator in the given state is ………..$\hbar\omega$.

Q31. The motion of a particle of mass m in one dimension is described by the Hamiltonian $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2+\lambda x$. The difference between the (quantized) energies of the first two levels $\alpha\hbar\omega$ the value of α ………(In the following, $\langle x\rangle$ is the expectation value of x in the ground state.)

Q32. If particle of mass m of two dimension infinite box is defined by

$$\begin{aligned} V(x,y)&=0 && 0<x<a, 0<y<b\\ &=\infty && \text{otherwise}\end{aligned}$$

If $b=2a$, the ratio of first excited to second excited state is …………

Q33. If particle of m is confined into three dimensional box whose potential is defined as

$$\begin{aligned} V(x,y,z)&=0 && 0<x<a, 0<y<b, 0<z<c\\ &=\infty && \text{otherwise}\end{aligned}$$

If $b=2a$ and $c=3a$, then energy eigenvalue first excited state is given by……….. in the unit of $\frac{\pi^2\hbar^2}{ma^2}$.

Q34. A particle of mass m is subjected to a potential

$$V(x,y)=\frac{1}{2}m\omega^2\left(x^2+y^2\right), -\infty \le x \le \infty, -\infty \le y \le \infty$$

The state with energy $4\hbar\omega$ is $g-$fold degenerate. The value of g is …………

Q35. A particle of mass m is confined into two dimensional harmonic oscillator. If un-normalized wave function is given by $y\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2+y^2\right)\right]$, what will be corresponding energy eigen value is ………….$\hbar\omega$

MSQ (Multiple Select Questions)

Q36. A system defined as particle of mass m confined in the infinite square well

$$V_1(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

Where another system defined particle of same mass confined in the other infinite box where potential is defined as $V_2(x) = \begin{cases} 0 & -\frac{a}{2} < x < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$

then which of the following statements is /are correct

(a)The energy eigen value of both is system will same

(b) The energy Eigen function of both the system will same

(c) The expectation value of position is center of potential in both the system.

(d) The eigen functions are either symmetric or anti symmetric about center of potential

Q37. A system defined as particle of mass m confined in the infinite square well

$$V(x) = \begin{cases} 0 & 0 < x < 2a \\ \infty & \text{otherwise} \end{cases}$$

then which of following is/are correct statement

(a) The ground state energy eigen value is $\frac{\pi^2\hbar^2}{2ma^2}$

(b) The ground state eigen function is given by $\sqrt{\frac{1}{a}}\sin\frac{\pi x}{2a}$

(c) The expectation value of position at any eigen state of Hamiltonian is $\frac{a}{2}$

(d) The average kinetic energy if particle in ground state is given by $\frac{\pi^2\hbar^2}{8ma^2}$

Q38. In a one-dimensional harmonic oscillator, ϕ_0, ϕ_1 and ϕ_2 are respectively the ground, first and the second excited states. These three states are normalized and are orthogonal to one another. ψ_1 and ψ_2 are two states defined by $\psi_1 = \phi_0 - 2\phi_1 + 3\phi_2$, $\psi_2 = \phi_0 - \phi_1 + \alpha\phi_2$, where α is a constant

(a)The value of α which ψ_1 is orthogonal to ψ_2 is 1

(b)The value of α which ψ_1 is orthogonal to ψ_2 is -1

(c) For the value of α determined ψ_1 and ψ_2 are orthogonal average value on state ψ_2 is $3\hbar\omega$

(d)) For the value of α determined ψ_1 and ψ_2 are orthogonal average value on state ψ_2 is $\frac{3}{2}\hbar\omega$

Q39. A particle is constrained to move in a truncated harmonic potential well $x > 0$ as shown in the figure and potential is defined as A particle of mass m is confined in the potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{for } -\infty < x < \infty \end{cases}$$

Then which of the following is correct statement

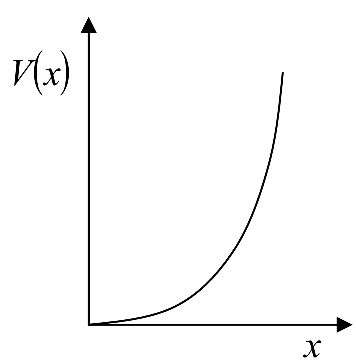
(a) The energy spacing between two consecutive energy levels is constant.

(b) There is finite probability to find the particle outside the harmonic oscillator.

(c) The wave functions are symmetric for n is even

(d) The average value of position at any state is zero.

Q40. A particle is constrained to move in a truncated harmonic potential well $x > 0$ as shown in the figure and potential is defined as a particle of mass m is confined in the potential



$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{for } x > 0 \\ \infty & \text{for } x \le 0 \end{cases}$$

Which one of the following statements is **correct**

(a) The parity of the first excited state is odd

(b) The parity of the ground state is even

(c) the ground state energy is $\frac{1}{2}\hbar\omega$

(d) The first excited state energy is $\frac{7}{2}\hbar\omega$

Q41. If particle of mass m is confined three dimensional box is whose potential is given by

$$V(x,y,z)=0 \qquad 0<x<a, 0<y<b, 0<z<c$$
$$=\infty \qquad \text{otherwise}$$

For $a=b=c$, then which of the following is correct

(a) The ground state eigen value corresponds to $n_x=1, n_y=0, n_z=0$ which is triply degenerate.

(b) The first excited state is triply degenerate.

(c) The average value of position in x direction is $\frac{a}{2}$ on any state ψ_{n_x,n_y,n_z}

(d) One of the normalized eigen state of second excited state is $\sqrt{\frac{8}{a^3}}\sin\frac{2\pi x}{a}.\sin\frac{\pi y}{a}\sin\frac{\pi z}{a}$

Q42. If the potential of two dimensional harmonic oscillator is $V(x,y)=\frac{1}{2}m\omega^2x^2+\frac{1}{2}m4\omega^2y^2$ then which of the following statement is /are correct

(a) The ground state energy eigen value is $\frac{3}{2}\hbar\omega$ it is non degenerate .

(b) The first excited state is $\frac{5}{2}\hbar\omega$ and its degeneracy is 2

(c) The second excited state is $\frac{7}{2}\hbar\omega$ and its degeneracy is 2

(d) The ground state wave function is $\psi(x,y)\propto \exp-\frac{m\omega x^2}{2\hbar}.\exp-\frac{m\omega y^2}{2\hbar}$

Q43. If particle of mass m is confined three dimensional box is whose potential is given by

$$V(x,y,z)=0 \qquad 0<x<a, 0<y<b, 0<z<c$$
$$=\infty \qquad \text{Otherwise}$$

If $a=b=2c$, then write down energy eigenvalue for ground state, first excited state and second excited state.

Then which of the following is /are correct?

(a) The ground state energy eigen value is $\frac{3\pi^2\hbar^2}{4mc^2}$

(b) The first excited state energy is $\frac{9\pi^2\hbar^2}{8mc^2}$

(c) The ground state is non degenerate and first excited state is doubly degenerate.

(d) The one of the normalized wave function corresponds to first excited state is $\sqrt{\frac{2}{c^3}}\sin\frac{\pi x}{c}\sin\frac{\pi y}{2c}\sin\frac{\pi z}{c}$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Solution: $\because E < V_0$, so there is decaying wave function.

Ans. 2: (c)

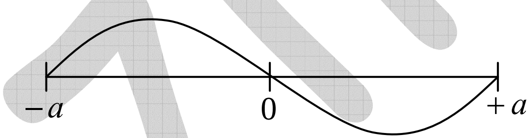
Solution: $H\psi = E\psi$, $H\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{-\hbar^2}{2m}(ik)(ik)e^{ikx} + 2(-ik)(-ik)e^{-ikx}$

$$\Rightarrow H\psi = \frac{\hbar^2 k^2}{2m}\left(e^{ikx} + 2e^{-ikx}\right) = \frac{\hbar^2 k^2}{2m}\psi$$

Ans. 3: (d)

Solution: For A particle is confined in a one dimensional box with impenetrable walls at $x = \pm a$.

The given state



is representation of first exited state whose energy is $2eV$. If E_n is energy of nth state and E_0 is energy of ground state then $E_n = n^2 E_0$.

So $E_2 = 4E_0$ and $E_0 = 0.5eV$ energy of second exited state is $3\times E_0 = 4.5eV$

Ans. 4: (a)

Solution: If some one measure energy on state in normalized $|\psi(x,0)\rangle = \frac{1}{\sqrt{2}}(|\phi_3\rangle + |\phi_1\rangle)$ then measurement is eigen value $9E_0$ and E_0 with probability $\frac{1}{2}$ and $\frac{1}{2}$ respectively where

$$E_0 = \frac{\pi^2\hbar^2}{2ma^2}$$

Then average value of energy $\langle E\rangle = \frac{1}{2}.9E_0 + \frac{1}{2}.E_0 = 5E_0 \Rightarrow \frac{5\pi^2\hbar^2}{2ma^2}$

Ans. 5: (c)

Solution: If some one measure energy on state in normalized $|\psi(x,0)\rangle = \sqrt{\frac{4}{5}}|\phi_2\rangle + \sqrt{\frac{1}{5}}|\phi_1\rangle)$ then measurement is eigen value $4E_0$ and E_0 with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively where $E_0 = \frac{\pi^2\hbar^2}{2ma^2}$ by using formula $P(E_n) = \frac{|\langle\phi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$

Ans. 6: (b)

Solution: $|\phi_1\rangle = \sqrt{\frac{2}{a}}\cos\frac{\pi x}{a}$ and $|\phi_2\rangle = \sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}$

Ans. 7: (b)

Solution: The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

$$= \int_{-a/2}^{a/2}\sqrt{\frac{2}{2a}}\cdot\sqrt{\frac{2}{2a}}\cos\frac{\pi x}{2a}\cdot\cos\frac{\pi x}{2a}dx = \int_{-a/2}^{a/2}\frac{1}{a}\cos^2\frac{\pi x}{2a}dx = \frac{1}{a}\times\frac{1}{2}\left[\int_{-a/2}^{a/2}\left(1+\cos\frac{2\pi x}{2a}\right)dx\right]$$

$$= \frac{1}{2a}\left[x + \frac{a}{\pi}\sin\frac{\pi x}{a}\right]_{-a/2}^{a/2} = \frac{1}{2a}\left[\frac{a}{2}+\frac{a}{2}+\frac{a}{\pi}(1+1)\right] = \frac{1}{2a}\left[a+\frac{2a}{\pi}\right] = \left(\frac{1}{2}+\frac{1}{\pi}\right)$$

Ans. 8: (c)

Solution: $V(x) = \begin{cases} 0 & 0<x<a \\ \infty & \text{otherwise} \end{cases}$, $\psi(x) = \begin{Bmatrix} A\sin^3\left(\frac{\pi x}{a}\right) & 0<x<a \\ 0 & \text{otherwise} \end{Bmatrix}$

$$\psi(x) = A\sin^3\left(\frac{\pi x}{a}\right) = A\frac{3}{4}\sin\frac{\pi x}{a} - A\frac{1}{4}\sin\frac{3\pi x}{a} \quad \because \sin 3A = 3\sin A - 4\sin^3 A$$

$$= A\frac{3}{4}\sin\frac{\pi x}{a} - A\frac{1}{4}\sin\frac{3\pi x}{a} = \frac{A}{4}\left[\sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\times 3\sin\frac{\pi x}{a} - \sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right]$$

$$\Rightarrow \psi(x) = \frac{A}{4}\left[3\sqrt{\frac{a}{2}}\phi_1(x) - \sqrt{\frac{a}{2}}\phi_3(x)\right]$$

$$\langle\psi|\psi\rangle = 1 \Rightarrow 9\frac{a}{32}A^2 + \frac{a}{32}A^2 = 1 \Rightarrow \frac{10a}{32}A^2 = 1 \quad \Rightarrow A = \sqrt{\frac{32}{10a}}$$

Ans. 9: (c)

Solution: $V(x)=\begin{cases}0 & 0<x<a \\ \infty & \text{otherwise}\end{cases}$, $\psi(x)=\begin{cases}A\sin^3\left(\frac{\pi x}{a}\right) & 0<x<a \\ 0 & \text{otherwise}\end{cases}$

$$\psi(x)=A\sin^3\left(\frac{\pi x}{a}\right)=A\frac{3}{4}\sin\frac{\pi x}{a}-A\frac{1}{4}\sin\frac{3\pi x}{a}\quad \because \sin 3A=3\sin A-4\sin^3 A$$

$$=A\frac{3}{4}\sin\frac{\pi x}{a}-A\frac{1}{4}\sin\frac{3\pi x}{a}=\frac{A}{4}\left[\sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\times 3\sin\frac{\pi x}{a}-\sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right]$$

$$\Rightarrow \psi(x)=\frac{A}{4}\left[3\sqrt{\frac{a}{2}}\phi_1(x)-\sqrt{\frac{a}{2}}\phi_3(x)\right]$$

Now, $E_1=\frac{\pi^2\hbar^2}{2ma^2}$, $E_2=\frac{9\pi^2\hbar^2}{2ma^2}\Rightarrow \langle E\rangle = a_n P(a_n)$

Ans. 10: (b)

Solution: The probability to find the particle in the interval between $-\frac{a}{4}$ and $\frac{a}{4}$ is

$$=\int_{-a/4}^{a/4}\sqrt{\frac{2}{2a}}\cdot\sqrt{\frac{2}{2a}}\cos\frac{\pi x}{2a}\cdot\cos\frac{\pi x}{2a}dx=\int_{-a/4}^{a/4}\frac{1}{a}\cos^2\frac{\pi x}{2a}dx=\frac{1}{a}\times\frac{1}{2}\left[\int_{-a/4}^{a/4}\left(1+\cos\frac{2\pi x}{2a}\right)dx\right]$$

$$=\frac{1}{2a}\left[x+\frac{a}{\pi}\sin\frac{\pi x}{a}\right]_{-a/4}^{a/4}=\frac{1}{2a}\left[\frac{a}{4}+\frac{a}{4}+\frac{a}{\pi}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]=\frac{1}{2a}\left[\frac{a}{2}+\frac{\sqrt{2}a}{\pi}\right]=\left(\frac{1}{4}+\frac{1}{\sqrt{2}\pi}\right)$$

Ans. 11: (b)

Solution: Since $V(x)=V(-x)$ so potential is symmetric.

$$\langle\psi_0|x|\psi_0\rangle=0,\ \langle\psi_1|x|\psi_1\rangle=0$$

$$\langle\psi|x|\psi\rangle=\left(\alpha_0\langle\psi_0|+\alpha_1\langle\psi_1|\right)\times\left(\alpha_0|\psi_0\rangle+\alpha_1|\psi_1\rangle\right)=\alpha_0\alpha_1\left[\langle\psi_0|x|\psi_1\rangle+\langle\psi_1|x|\psi_0\rangle\right]$$

Ans. 12: (b)

Solution: We compare the result $\eta^2+\xi^2=\frac{mV_oa^2}{2\hbar^2}=R^2$

Put $V_0=\frac{16\hbar^2}{ma^2}$ $\eta^2+\xi^2=8=R^2$, $R=2\sqrt{2}$ so $\frac{\pi}{2}<R<\pi$ $\eta=\xi\tan\xi$ and $\eta=-\xi\cot\xi$

So there is 2 bound state.

Ans. 13: (a)

Solution: $\psi(x)=\psi_0(x)+\psi_1(x)\Rightarrow\psi(x,t)=\psi_0(x)e^{-i}\frac{E_0t}{\hbar}+\psi_1(x)e^{-i}\frac{E_1t}{\hbar}$

Now probability density at time t

$$|\psi(x,t)|^2=\psi^*(x,t)\psi(x,t)=|\psi_0(x)|^2+|\psi_1(x)|^2+2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x)\cos(E_1-E_0)\frac{t}{\hbar}$$

Putting $t=\pi$

$$|\psi(x,t)|^2=|\psi_0(x)|^2+|\psi_1(x)|^2+2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x)\cos\pi \qquad \because E_1-E_0=\hbar\omega=1$$

$$|\psi(x,t)|^2=|\psi_0(x)|^2+|\psi_1(x)|^2-2\,\mathrm{Re}\,\psi_0^*(x)\psi_1(x)=[\psi_1(x)-\psi_0(x)]^2$$

Ans. 14: (a)

Solution: For half parabolic potential $E_0=\frac{3}{2}\hbar\omega, E_1=\frac{7}{2}\hbar\omega\Rightarrow\langle E\rangle=\frac{1}{5}\times\frac{3}{2}+\frac{4}{5}\times\frac{7}{2}=\frac{31}{10}\hbar\omega$.

Ans. 15: (a)

Solution: Hamiltonian (H) of Harmonic oscillator, $H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2x^2$

Eigen value of this, $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$

But here, $H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2x^2-ax\Rightarrow H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2\left[x^2-\frac{2ax}{m\omega^2}+\frac{a^2}{m^2\omega^4}\right]-\frac{a^2}{2m\omega^2}$

$$H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2\left[x-\frac{a}{m\omega^2}\right]^2-\frac{a^2}{2m\omega^2}\Rightarrow E_n=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{a^2}{2m\omega^2}$$

Ans. 16: (d)

Solution: $E_{n_x,n_y,n_z}=\left(n_x^2+n_y^2+n_z^2\right)\frac{\pi^2\hbar^2}{2ma^2}=\frac{14\pi^2\hbar^2}{2ma^2}$

$\Rightarrow n_x^2+n_y^2+n_z^2=14\Rightarrow n_x=1,n_y=2,n_z=3.$

Ans. 17: (d)

Solution: $V(x,y)=\frac{1}{2}m\omega^2\left(x^2+4y^2\right)=\frac{1}{2}\mathrm{m\omega^2x^2}+\frac{1}{2}\mathrm{m4\omega^2y^2}$, $E=\left(n_x+\frac{1}{2}\right)\hbar\omega+\left(n_y+\frac{1}{2}\right)2\hbar\omega$

For ground state energy $n_x=0,n_y=0\Rightarrow E=\frac{\hbar\omega}{2}+\frac{1}{2}2\hbar\omega=\frac{3\hbar\omega}{2}$

First exited state energy $n_x=1,n_y=0\Rightarrow\frac{3\hbar\omega}{2}+\hbar\omega=\frac{5\hbar\omega}{2}$

Ans. 18: (b)

Solution: $E_{n_x,n_y}=\frac{\pi^2\hbar}{2m}\left(\frac{n_x^{\ 2}}{a^2}+\frac{n_y^{\ 2}}{b^2}\right)\Rightarrow E_{n_x,n_y}=\frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^{\ 2}}{a^2}+\frac{n_y^{\ 2}}{(2a)^2}\right)$ $\qquad\because b=2a$

For ground state: $E_{1,1}=\frac{5}{8}\frac{\pi^2\hbar^2}{ma^2}$ For first excited state: $E_{1,2}=\frac{8}{8}\frac{\pi^2\hbar^2}{ma^2}$

Ans. 19: (a)

Solution: $V(x,y)=\frac{1}{2}m\omega^2\left(4x^2+y^2\right)$, $E=\frac{5}{2}\hbar\omega$

$\Rightarrow V(x,y)=\frac{1}{2}m(2\omega)^2x^2+\frac{1}{2}m\omega^2y^2$

Now, $E_n=\left(n_x+\frac{1}{2}\right)\hbar\omega_x+\left(n_y+\frac{1}{2}\right)\hbar\omega_y=\left(n_x+\frac{1}{2}\right)2\hbar\omega+\left(n_y+\frac{1}{2}\right)\hbar\omega$

$\Rightarrow E_n=\left(2n_x+n_y+\frac{3}{2}\right)\hbar\omega$

$\because E_n=\frac{5}{2}\hbar\omega$ when $n_x=0$ and $n_y=1$.

Ans. 20: (b)

Solution: $\langle E\rangle=\frac{5}{6}.\frac{\hbar\omega}{2}+\frac{1}{6}.\frac{3\hbar\omega}{2}=\frac{8\hbar\omega}{12}=\frac{2\hbar\omega}{3}$

NAT (Numerical Answer Type)

Ans. 21: 0

Solution: The real part of the wave function $\psi_{real} = \cos kx + 2\cos kx$

Current density for real part of wave function = 0

Ans. 22: 15

Solution: Probably current density associated with $4e^{ikx}$ is $J_+ = \frac{16\hbar k}{m}$ and probably density associated with e^{-ikx} $J_- = -\frac{\hbar k}{m}$ so total current density is $J = J_+ + J_- = \frac{15\hbar k}{m}$

Ans. 23: 0.25

Solution: For width a $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ for width $2a$ $E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$

Ratio of energy of bigger to smaller box$=\frac{n^2\pi^2\hbar^2}{2m4a^2} / \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{1}{4} = .25$

Ans. 24: 9.9

Solution: $\frac{3\pi^2\hbar^2}{2ma^2} = 9.9\times10^{-17}$

Ans. 25: 0.9

Solution: $\psi(x) = \frac{3}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} - \frac{1}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a} \Rightarrow \psi(x) = \frac{3}{\sqrt{10}}\phi_1(x) - \frac{1}{\sqrt{10}}\phi_3(x)$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}, \quad E_2 = \frac{9\pi^2\hbar^2}{2ma^2} \Rightarrow \langle E\rangle = a_n P(a_n)$$

Probably $P(E_1) = \frac{|\langle\varphi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{9}{10}$, $P(E_2) = \frac{|\langle\phi_2|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{1}{10}$

$$\langle E\rangle = \frac{9}{10}\times\frac{\pi^2\hbar^2}{2ma^2} + \frac{1}{10}\times\frac{9\pi^2\hbar^2}{2ma^2} \Rightarrow \langle E\rangle = \frac{9\pi^2\hbar^2}{10ma^2}$$

Ans. 26: 0

Solution: $\langle X\rangle = \int_{-\frac{a}{2}}^{-\frac{a}{2}} x|\phi_n|^2\, dx = 0$

Ans. 27: 2

Solution: We compare the result $\eta^2+\xi^2 = \frac{2mV_o a^2}{\hbar^2} = R^2$ and $\eta = -\xi\cot\xi$

Put $V_0 = \frac{16\hbar^2}{ma^2}$ $\eta^2+\xi^2 = 32 = R^2$, $R = 5.6$ so $\frac{3\pi}{2} < R < \frac{5\pi}{2}$. So there is 2 bound state.

Ans. 28: 0.25

Solution: $\vec{J} = |\psi|^2 \frac{\hbar k}{m} = |A|^2 \left|\frac{r_0}{r}\right|^2 \frac{\hbar k}{m} \Rightarrow J = |A|^2 \left|\frac{r_0}{2r_0}\right|^2 \frac{\hbar k}{m} \Rightarrow J = |A|^2 \frac{\hbar k}{4m}$

Ans. 29: $\alpha = 3$

Solution: $\psi(t=0) = \frac{1}{\sqrt{2}}(\psi_1+\psi_2)$ and $\psi(t=T) = \frac{1}{\sqrt{2}} e^{-\frac{iE_1T}{\hbar}}\psi_1 + \frac{1}{\sqrt{2}} e^{-\frac{iE_2T}{\hbar}}\psi_2$

$\int \psi^*(0)\psi(T)\,dx = 0 \Rightarrow \frac{1}{2}e^{-\frac{iE_1}{\hbar}T} + \frac{1}{2}e^{-\frac{iE_2}{\hbar}T} = 0 \Rightarrow e^{-\frac{iE_1}{\hbar}T} = -e^{-\frac{iE_2}{\hbar}T} \Rightarrow e^{i\frac{T}{\hbar}(E_2-E_1)} = -1$

Equate real part $\Rightarrow \cos\left(\frac{T}{\hbar}(E_2-E_1)\right) = -1 \Rightarrow T = \frac{\hbar}{(E_2-E_1)}\cos^{-1}(-1) = \frac{\pi\hbar}{(E_2-E_1)} = \frac{\hbar\pi}{\alpha}$

where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ so $\alpha = E_2 - E_1 = \frac{3\pi^2\hbar^2}{2ma^2}$

Ans. 30: 3.25

Solution: Average energy will $\frac{\frac{1}{4}.\frac{5\hbar\omega}{2} + \frac{3}{4}.\frac{7\hbar\omega}{2}}{\frac{1}{4}+\frac{3}{4}} = 3.25\hbar\omega$

Ans. 31: 1

Solution: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x \Rightarrow V(x) = \frac{1}{2}m\omega^2x^2 + \lambda x$

$$V(x)=\frac{1}{2}m\omega^2\left[x^2+\frac{2}{m\omega^2}\lambda x\right]=\frac{1}{2}m\omega^2\left[x^2+2\cdot x\cdot\frac{\lambda}{m\omega^2}+\frac{\lambda^2}{m^2\omega^4}-\frac{\lambda^2}{m^2\omega^4}\right]$$

$$V(x)=\frac{1}{2}m\omega^2\left(x+\frac{\lambda}{m\omega^2}\right)^2-\frac{\lambda^2}{2m\omega^2} \qquad \therefore\ E_n=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{\lambda^2}{2m\omega^2}$$

$$\Rightarrow E_1-E_0=\frac{3}{2}\hbar\omega-\frac{1}{2}\hbar\omega=\hbar\omega$$

Ans. 32: 1.62

Solution: $E_{n_x,n_y}=\dfrac{\pi^2\hbar}{2m}\left(\dfrac{n_x{}^2}{a^2}+\dfrac{n_y{}^2}{b^2}\right)\Rightarrow E_{n_x,n_y}=\dfrac{\pi^2\hbar^2}{2m}\left(\dfrac{n_x{}^2}{a^2}+\dfrac{n_y{}^2}{(2a)^2}\right)\because b=2a$

For ground state: $E_{1,1}=\dfrac{5}{8}\dfrac{\pi^2\hbar^2}{ma^2}$, For first excited state: $E_{1,2}=\dfrac{8}{8}\dfrac{\pi^2\hbar^2}{ma^2}$

For second excited stste $E_{1,3}=\dfrac{13}{8}\dfrac{\pi^2\hbar^2}{ma^2}$ $\quad \dfrac{E_{1,3}}{E_{1,2}}=\dfrac{13}{8}=1.62$

Ans. 33: 0.84

Solution: For ground state: $n_x=1, n_y=1, n_z=1\Rightarrow E_{1,1,1}=\dfrac{49}{36}\times\dfrac{\pi^2\hbar^2}{2ma^2}$

For first excited state: $n_x=1, n_y=1, n_z=2\Rightarrow E_{1,1,2}=\dfrac{61}{36}\times\dfrac{\pi^2\hbar^2}{2ma^2}$

Ans. 34: 4

Ans. 35: 2.5

Solution: The energy eigen value for two dimensional harmonic oscillator whose potential is

$V(x,y)=\dfrac{1}{2}m\omega_x^2x^2+\dfrac{1}{2}m\omega_y^2y^2$ is given by $E=\left(n_x+\dfrac{1}{2}\right)\hbar\omega_x+\left(n_y+\dfrac{1}{2}\right)\hbar\omega_y$.

If the eigen function is given by $y\exp\left[-\dfrac{m\omega}{2\hbar}\left(2x^2+y^2\right)\right]$ then

$n_x=0, n_y=1, \omega_x=2\omega, \omega_y=\omega$ so $E=\dfrac{5}{2}\hbar\omega$

MSQ (Multiple Select Questions)

Ans. 36: (a), (c) and (d)

Solution: Hence width of the potential is same, so eigen value ,normalization constant will same but eigen function will such that it is either symmetric or anti symmetric about mid center of potential .

Ans. 37: (b) and (d)

Solution: For given potential energy eigen value is $\frac{n^2\pi^2\hbar^2}{8ma^2}$ and eigen function is $\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{2a}$ $n=1,2,3....$ the expectation value of position is center of box ie a.

Ans. 38: (b) and (d)

Solution: For orthogonal condition scalar product For orthogonal condition scalar product

$$(\psi_1,\psi_2)=0 \text{ so } 1+2+3\alpha=0\Rightarrow\alpha=-1$$

$$\psi_2=\phi_0-\phi_1+\alpha\phi_2 \text{ put } \alpha=-1, \langle H\rangle=\frac{\langle\psi_2|H|\psi_2\rangle}{\langle\psi_2|\psi_2\rangle}=\frac{\frac{\hbar\omega}{2}+\frac{3\hbar\omega}{2}+\frac{5\hbar\omega}{2}}{3}=\frac{3}{2}\hbar\omega$$

Ans. 39: (a), (b), (c) and (d)

Solution: The eigen value of harmonic oscillator $\left(n+\frac{1}{2}\right)\hbar\omega$. So spacing between consecutive energy levels are constant i.e. $\hbar\omega$ and there is finite probably to find the particle out side the well where wave function are symmetric of $n=0,2,4,....$ and antisymmetric for $n=1,3,5,....$ so $\langle x\rangle=0$.

Ans. 40: (a) and (d)

Solution: There is only odd parity. Ground state is $\frac{3}{2}\hbar\omega$ and first excited $\frac{7}{2}\hbar\omega$

Ans. 41: (b) and (c).

Solution: For ground state: $n_x = 1, n_y = 1, n_z = 1 \Rightarrow E_{1,1,1} = \frac{3\pi^2\hbar^2}{2ma^2}$

For first excited state: $\left.\begin{Bmatrix} n_x = 1, & n_y = 1 & n_z = 2 \\ n_x = 1, & n_y = 2, & n_z = 1 \\ n_x = 2 & n_y = 1 & n_z = 1 \end{Bmatrix}\right\} \Rightarrow E_{1,1,2} = E_{1,2,1} = E_{2,1,1} = \frac{6\pi^2\hbar^2}{2ma^2}$ (Triply degenerate)

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{8}{a^3}} \sin\frac{n_x\pi x}{a} . \sin\frac{n_y\pi y}{a} \sin\frac{n_z\pi z}{a}$$

second excited state is triply degenerate which can be discussed below .

For second excited state: $\begin{Bmatrix} n_x = 1, & n_y = 2 & n_z = 2 \\ n_x = 2, & n_y = 1, & n_z = 2 \\ n_x = 2 & n_y = 2 & n_z = 1 \end{Bmatrix} \Rightarrow E_{1,2,2} = E_{2,1,2} = E_{2,2,1} = \frac{9\pi^2\hbar^2}{2ma^2}$

Ans. 42: (a) and (c)

Solution: $V(x.y) = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m(2\omega)^2 y^2$ i.e. $\omega_1 = \omega,\ \omega_2 = 2\omega$

$$\therefore E_{n_x,n_y} = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)2\hbar\omega$$

For ground state: $E_{0,0} = \frac{3}{2}\hbar\omega$ and wave function is proportional to

$$\psi(x,y) \propto \exp - \frac{m\omega x^2}{2\hbar} . \exp - \frac{m2\omega y^2}{2\hbar}$$

For first excited state: $E_{1,0} = \frac{5}{2}\hbar\omega$

For second excited state: $E_{0,1} = E_{2,0} = \frac{7}{2}\hbar\omega$ (Doubly degenerate)

Ans. 43: (a), (b), (c) and d)

Solution: $E_{(n_x, n_y, n_z)} = \frac{\pi^2 \hbar^2}{2mc^2}\left(\frac{n_x^2}{4} + \frac{n_y^2}{4} + n_z^2\right)$

For ground state: $n_x = 1, n_y = 1, n_z = 1 \Rightarrow E_{1,1,1} = \frac{6}{4} \times \frac{\pi^2 \hbar^2}{2mc^2}$

For first excited state: $n_x = 1, n_y = 2, n_z = 1$ and $n_x = 2, n_y = 1, n_z = 1$

$\Rightarrow E_{1,2,1} = E_{2,1,1} = \frac{9}{4} \times \frac{\pi^2 \hbar^2}{2mc^2}$ (Doubly degenerate)

$$\psi_{1,2,1} = \sqrt{\frac{2}{2c}}.\sqrt{\frac{2}{2c}}.\sqrt{\frac{2}{c}}.\sin\frac{\pi x}{2c}.\sin.\frac{2\pi y}{2c}\sin\frac{\pi z}{c}$$

$$\psi_{2,1,1} = \sqrt{\frac{2}{2c}}.\sqrt{\frac{2}{2c}}.\sqrt{\frac{2}{c}}.\sin\frac{\pi x}{c}.\sin.\frac{\pi y}{2c}\sin\frac{\pi z}{c}$$

5. Statistical Physics

5.1 Basic Definition, Mathematical Tools and Postulates of Statistical Mechanics

Statistical mechanics is a branch of physics that applies probability theory, which contains mathematical tools for dealing with large population to study of the thermodynamic behavior of systems composed of a large number particles.

It provides a framework for relating the microscopic properties of individual atoms and molecules to the macroscopic bulk properties of materials that can be observed in everyday life.

- **Micro state:** A microstate is a specific microscopic configuration of a thermodynamic system that occupy with a certain property in the course of its thermal fluctuation. The position (x), momentum (p), energy (E), and spin (s, s_z) of individual atom are the example of microstate of system.
- **Macro state:** A macro state refers to macroscopic properties of system such as temperature (T), pressure (P), free energy (F), entropy (S). A macro state is characterized by a probability distribution of possible state across a certain statistical ensemble of all microstates, and distribution describes the probability of finding the system in certain microstate.
- **Accessible state:** Any microstate in which a system can be found without contradicting the macroscopic information available about the system.
- **Statistical Ensemble:** an assembly of large number of mutually non interacting systems, each of which satisfies the same conditions as those known to be satisfied by a particular system under condition. There are three type of ensemble **(a) micro canonical ensemble, (b) canonical ensemble, (c) grand canonical ensemble**. An ensemble is said to be time independent ensemble if number of system exhibiting any particular property is the same at a time.

- **Probability:** The probability p_r of occurrence of an event r in a system is defined with respect to statistical ensemble of N such a systems. If N_r systems in the ensemble exhibit the event r then $p_r = \frac{N_r}{N}$
- **Probability density:** The probability density $\rho(u)$ is defined by the property that $\rho(u)du$ yields the probability of finding the continuous variable u in the range between u and $u + du$.
- **Mean value :** The mean value of u is denoted by $\langle u \rangle$ as defined as $\langle u \rangle = \sum_r p_r u_r$ where the sum is over all possible value values u_r of the variable u and p_r is denotes the probability of occurrence of the particular value u_r .Above definition is for discrete variable .

For continuous variable u ; $\langle u \rangle = \int u\rho(u)du$

- **Dispersions (or variance):** The dispersion of u is defined as $\sigma^2 = \langle (\Delta u)^2 \rangle = \sum_r p_r (u_r - \langle u \rangle)^2$ which is equivalent to $\sigma^2 = \langle (\Delta u)^2 \rangle = \sum_r \; (\langle u^2 \rangle - \langle u \rangle^2)$
- **Stirling formula:** Stirling's approximation (or Stirling's formula) is an approximation for large factorials. It is named after James Stirling.

The formula as typically used in applications is

$$\ln \lfloor N = N \ln N - N$$

5.2 Postulates of statistical mechanics

If an isolated system is found with equal probability in each of its accessible state, it is in equilibrium, which is popularly known as **postulates of equal a priori probabilities**.

Suppose that we were asked to pick a card at random from a well-shuffled pack. It is accepted that we have an equal probability **of picking any card** in the pack. There is nothing which would favor one particular card over all of the others. So, since there are fifty-two cards in a normal pack, we would expect the probability of picking the Ace of Spades, say, to be $\frac{1}{52}$.

We could now place some constraints on the system. For instance, we could **only count red cards**, in which case the probability of picking the Ace of Hearts, say, would be $\frac{1}{52}+\frac{1}{52}=2\times\frac{1}{52}=\frac{1}{26}$, by the same reasoning. In both cases, we have used the principle of equal ***a priori* probabilities**. People really believe that this principle applies to games of chance such as cards, dice.

In statistical mechanics, we treat a many particle system a bit like an extremely large game of cards. Each accessible state corresponds to one of the cards in the pack. The interactions between particles cause the system to continually change state. This is equivalent to constantly shuffling the pack. Finally, an observation of the state of the system is like picking a card at random from the pack. The principle of equal ***a priori* probabilities** then boils down to saying that we have an equal chance of choosing any particular card.

Example: If there are four identical molecule in one dimensional container and it is given that molecule can be found only either right or left end of container .

(a) What are possible configuration and no of ways to arrange these configuration? what are corresponding probability of each configuration?

(b) What is most probable configuration?

(c) If some one is doing the experiment in which he observed molecule position to right of container what is mean value of particle being in right?

(d) How postulates of a priori probability apply on the experiment?

Solution: The number of different ways of arranging N molecules with n on one side and $N-n$ on the other side is given by $\frac{N!}{N!(N-n)!}$, where ! represents the factorial function. The total number of possible ways of arranging the molecules is $2^N=2^4=16$

(a)

Configuration	No. of ways to arrange given configuration	Probability
1. (L,L,L,L)	1	$1\times\frac{1}{16}=\frac{1}{16}$
2. (L,L,L) and (R)	4	$4\times\frac{1}{16}=\frac{4}{16}$
3. (L,L) and(R,R)	6	$6\times\frac{1}{16}=\frac{6}{16}$
4. (L) and (R,R,R)	4	$4\times\frac{1}{16}=\frac{4}{16}$
5. only (R,R,R,R)	1	$1\times\frac{1}{16}=\frac{1}{16}$

(b) Most probable configuration is the one in which half the molecules are on one side and half on the other, i.e. the molecules are uniformly distributed over the space. Most probable configuration is configuration (L,L) and(R,R) which has maximum probability .

(c) $\langle R\rangle=\sum_r p_r R_r \;=0R\times\frac{1}{16}+1R\times\frac{4}{16}+2R\times\frac{6}{16}+3R\times\frac{4}{16}+4R\times\frac{1}{16}=$ 2R

(d) We will now apply a fundamental postulate of statistical mechanics which states that an isolated system which can be in any one of a number of accessible states (=16 in this example) is equally likely to be in any one of these states at equilibrium. Therefore, the probability that the molecules are distributed in any one of these 16 possible ways is simply 1/16. But there are 4 ways in which the molecules can be arranged so that 3 are on the left side and 1 on the right side, and therefore, the probability of finding that configuration is 4/16. Similarly other configuration can be weighted.

Example: Suppose we know 3 particle being spin 1/2 kept into homogeneous magnetic B field at temperature T.

(a) Show all possible microstate and corresponding probability .

(b) Find average value of z component of spin .

(c) If μ_0 is magnetic moment which configuration has maximum energy what is corresponding probability.

Solution: (a)There is total 8 microstate is possible they are

configuration	Microstate	probability
All are in up state	$\uparrow\ \uparrow\ \uparrow$	$\frac{1}{8}$
Two are up and one is down	$\uparrow\ \uparrow\ \downarrow$ $\uparrow\ \downarrow\ \uparrow$ $\downarrow\ \uparrow\ \uparrow$	$\frac{3}{8}$
One up and two down	$\downarrow\ \downarrow\ \uparrow$ $\uparrow\ \downarrow\ \downarrow$ $\downarrow\ \uparrow\ \downarrow$	$\frac{3}{8}$
All three are down	$\downarrow\ \downarrow\ \downarrow$	$\frac{1}{8}$

(b) Average value of $\langle s_z \rangle = \frac{3\hbar}{2}\times\frac{1}{8}+\frac{\hbar}{2}\times\frac{3}{8}+\frac{-\hbar}{2}\times\frac{3}{8}+\frac{-3\hbar}{2}\times\frac{1}{8}=0$

(c) The energy is given by $E=-\mu_0 B$ the magnetic moment of configuration in which all three are down $\downarrow\downarrow\downarrow$ have magnetic moment $-3\mu_0$ so this configuration has maximum energy which is equal to $3\mu_0 B$ the corresponding probability is given by $\frac{1}{8}$

5.3 Ensemble is collection of particle

5.2.1 Micro canonical ensemble

Micro canonical ensemble is theoretical tool used to analyze an **isolated** thermo dynamic system. The microstate of the system has fixed given energy(E), fixed number of particle (N)and fixed volume (V).All accessible micro state has **same probability** .popularly it is known as NVE ensemble .

Schematically the system can be shown as

NVE	NVE	NVE	NVE
NVE	NVE	NVE	NVE
NVE	NVE	NVE	NVE
NVE	NVE	NVE	NVE
NVE	NVE	NVE	NVE
NVE	NVE	NVE	NVE

In above table each cell consider as each microstate energy , volume and no of particle is fixed in each cell.

If Ω is the number of accessible microstates, the probability that a system chosen at random from the ensemble would be in a given microstate is simply $\frac{1}{\Omega}$.

No. of accessible microstate in phase space which has energy between E to $E+dE$ is given by $\Omega = \sum_{sates} = \int \frac{dp.dr}{h^3}$ which is given in term of energy is

$n(E)dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2}dE$ where g is degeneracy of the particle.

5.2.1.1 Entropy:

From the number of accessible microstates, Ω , we can obtain the entropy(S) of the system via $S = k_B \ln \Omega$

where k_B is the Boltzmann constant. or, equivalently,

$\Omega(E,V,N) = e^{\frac{S}{k_B}}$ Ω is equivalent to "micro canonical partition function"

Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

Example: The energy of Einstein oscillator is given by $E_i = n_i h\nu$ if there is N no. of oscillator in the 3 Dimensional system which has total energy E at temperature T.

(a) Write down micro canonical partition function Ω .

(b) What is entropy of the system?

Solution: (a) Ω is no. of microstate which is equivalent to $\Omega = \sum_1^{3N} n_i \Rightarrow \sum_1^{3N} \frac{E_i}{h\nu}$

$$\Omega = \sum_1^{3N} \frac{E_i}{h\nu} \Rightarrow \frac{E}{h\nu} \quad \text{where} \quad \sum_1^{3N} E_i \Rightarrow E$$

Example: If there is N no of particle which have spin $\frac{3}{2}$ which will interact with magnetic field B which are in equilibrium at temperature T

(a) How many no. of microstate for each particle

(b) What is entropy of the system.

Solution: (a) if s=$\frac{3}{2}$ then z component of spin ie. $s_Z = \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}$ so there is 4 microstate for each particle

For the N no. of particle there will be 4^N no of state .

(b) $S = k_B \ln \Omega$, where $\Omega = 4^N$ for given system. so $S = Nk_B \ln 4$

Example: A solid contain N magnetic atoms having spin $\frac{1}{2}$. At sufficiently high temperature each spin is completely random oriented .at sufficiently low temperature all the spin become oriented in same direction let the heat capacity as a function of temperature T by

$$c(T) = a\left(\frac{2T}{T_1} - 1\right) \qquad \text{If } \frac{T_1}{2} < T < T_1$$

$$= 0 \qquad \text{Otherwise}$$

Find the value of "a"

Solution: at very low temperature all spins are oriented in only one direction so there is only one possible microstate for each atoms . hence entropy is $S_1 = 0$, at high temperature all the spin are randomly oriented and they can be either in up or down microstate so there are two microstate for each atom hence for N no of atom entropy is given by $S_2 = Nk_B \ln 2$.

so $S_2 - S_1 = Nk_B \ln 2$ which is determined by theoretical calculation .

now from the given expression of heat capacity we have relation $c = \frac{Tds}{dT}$.

$$S(\infty) - S(o) \Rightarrow S_2 - S_1 = \int_0^{\infty} \frac{c}{T} dT \qquad = \int_{\frac{T_1}{2}}^{T_1} a\left(\frac{2T}{T_1} - 1\right)\frac{dT}{T} \Rightarrow a = \frac{Nk \ln 2}{1 - \ln 2}$$

5.2.2 Canonical Ensemble

The canonical ensemble occurs when a system with fixed volume (V) and number of particle (N) at constant temperature (T) . In other words we will consider an assembly of systems closed to others by rigid, diathermal, impermeable walls. The energy of the microstates can be fluctuate, the system is kept in equilibrium by being in contact with the heat bath at temperature T. It is also referred to as the ***NVT ensemble***

Schematically the system can be shown as

NVT	NVT	NVT	NVT
NVT	NVT	NVT	NVT
NVT	NVT	NVT	NVT
NVT	NVT	NVT	NVT

In above table each cell considers as each microstate temperature, volume and no of particle is fixed in each cell. . Only value of energy is different in different cell which can be exchanged in the process.

5.2.2.1 Partition Function for Canonical Ensemble

According to Gibbs, the probability of finding the system in any of its i[th] state at temperature T where energy of that state is E_i is given by $p(E_i) \propto e^{-E_i\beta}$ where $\beta = \frac{1}{k_B T}$

$p(E_i) = ce^{-E_i\beta}$ where c is proportionality constant . hence $p(E_i)$ is probability then

$$\sum_i p(E_i) = \sum_i ce^{-E_i\beta} = 1 \quad \text{so} \quad c = \frac{1}{\sum_i e^{-E_i\beta}} \quad \Rightarrow c = \frac{1}{\sum_i e^{-E_i\beta}} = \frac{1}{Z}$$

The letter Z stands for the German word *Zustandssumme*, "sum over states" and is popularly known as partition function for canonical ensemble which is given by $Z = \sum_i e^{-E_i\beta}$

In systems with multiple quantum, we can write the partition function in terms of the contribution from energy levels (indexed by i) as follows:

$$Z = \sum_i g_i e^{-E_i\beta} \quad ,$$

where g_j is the degeneracy factor, or number of quantum states which have the same energy level defined by E_i .

In *classical* statistical mechanics, it is not really correct to express the partition function as a sum of discrete terms, as we have done.

In **classical mechanics**, the position and momentum variables of a particle can vary continuously, so the set of microstates is actually uncountable. In this case we must describe the partition function using an integral rather than a sum. For instance, the partition function of a gas of N identical classical particles is

$$Z_N = \frac{1}{\underline{\lfloor N} h^{3N}} \int \exp -\beta H(p_1 ... p_N, x_1 .. x_N) d^3x_1 ... d^3x_N d^3p_1 ... d^3p_N$$

where p_i indicate particle momenta x_i indicate particle positions

d^3 is a shorthand notation serving as a reminder that the p_i and x_i are vectors in three dimensional space, and H is the classical Hamiltonian.

The reason for the (N factorial): However, there is a well-known exception to this rule. If the sub-systems are actually identical particles, in the quantum mechanical sense that they are impossible to distinguish even in principle, the total partition function must be divided by a $N!$ (N factorial):. For simplicity, we will use the discrete form of the partition function in this article. Our results will apply equally well to the continuous form.

The extra constant factor introduced in the denominator was introduced because, unlike the discrete form, the continuous form shown above is not dimensionless. To make it into a dimensionless quantity, we must divide it by where h is some quantity with units of action (usually taken to be **Planck's constant**).

5.2.2.2 Relation Between Macroscopic Variable and Canonical Partition Function Z

- Relation between total energy and partition function for large no for particle average of total energy E is equivalent to average of internal energy U.

$$\langle E\rangle = \langle U\rangle = \sum_i E_i P_i \quad = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \qquad \because \frac{-\partial}{\partial\beta}\ln\left(\sum_i e^{-\beta E_i}\right) = -\frac{\partial \ln Z}{\partial\beta}$$

$$\langle E\rangle = \langle U\rangle = k_B T^2\left(\frac{\partial \ln Z}{\partial T}\right) \qquad \Rightarrow \langle E\rangle = \langle U\rangle = k_B T^2 \frac{1}{Z}\frac{\partial Z}{\partial T}$$

- Relation between partition function and specific heat at constant volume C_V

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

- Relation between partition function and Helmholtz free Energy:

$$\langle F\rangle = \langle U\rangle - T\langle S\rangle \qquad \text{and} \qquad S = -\left(\frac{\partial F}{\partial T}\right)_V$$

$$\Rightarrow U = F - T\left(\frac{\partial F}{\partial T}\right)_V \qquad = -T^2\left(\frac{\partial (F/T)}{\partial T}\right)$$

equating the coefficient of T^2 between relation

$\langle U\rangle = -T^2\left(\frac{\partial(F/T)}{\partial T}\right)$ and so $\langle E\rangle = \langle U\rangle = k_B T^2\left(\frac{\partial \ln Z}{\partial T}\right)$ so $F = -k_B T\ln Z$

- relation between partition function and other thermodynamical variable

once internal energy(U) and Helmholtz free energy(F) is obtained one can find

(a) entropy (S) $S = \frac{U-F}{T}$

(b) pressure (P) $P = -\left(\frac{\partial F}{\partial V}\right)_T$

(c) enthalpy(H) $\mathrm{H} = U + PV$

(d) Gibbs free energy $G = H - TS$

5.2.2.3 Relation Between Entropy and Probability .

$$P_i = \frac{e^{-\beta E_i}}{Z} \quad \Rightarrow \ln P_i = -\beta E_i - \ln Z \quad \Rightarrow F = -k_B T\ln Z \quad \Rightarrow \ln Z = \frac{-F}{k_B T}$$

$$\ln P_i = -\beta E_i + \left(\frac{F}{k_B T}\right) \quad \Rightarrow \ln p_i = \frac{-E_i}{k_B T} + \frac{F}{k_B T} \quad \Rightarrow \langle \ln P_i\rangle = \frac{-\langle E_i\rangle + \langle F\rangle}{k_B T}$$

$$\Rightarrow \langle \ln P_i\rangle = \frac{-U+F}{k_B T} = \frac{-S}{k_B} \quad [\because F = U - TS] \quad \Rightarrow S = -k_B\langle \ln P_i\rangle$$

$$\Rightarrow S = -k_B\sum_i p_i \ln p_i$$

Example: A system in thermal equilibrium has energies 0 and E. Calculate partition function of system.

Then calculate

(i) Helmholtz Free energy (F)

(ii) entropy (S)

(iii) internal energy (U)

(iv) Specific heat at constant volume c_V discuss the trend of specific heat at (a) low temperature and

(b) high temperature

Solution: Let T be the temperature of the system. The partition function Z of the system is

$$Z = e^{0/k_BT} + e^{-E/k_BT} = 1 + e^{-E/k_BT}$$

(i) Free energy F of the system is

$$F = -k_BT\ln Z = -k_BT\ln\left[1 + e^{-E/k_BT}\right]$$

(ii) Entropy S of the system is

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = k_B\ln\left[1 + e^{-E/k_BT}\right] + \frac{E}{T}\frac{1}{e^{E/k_BT}+1}$$

(iii) Internal energy U is

$$U = F + ST$$

$$= -k_BT\ln\left[1 + e^{-E/kT}\right] + k_BT\ln\left[1 + e^{-E/k_BT}\right] + \frac{E}{e^{E/k_BT}+1} \qquad = \frac{E}{e^{E/k_BT}+1}$$

(iv) Specific heat at constant volume c_V is

$$c_V = \left(\frac{\partial U}{\partial T}\right)_{N,V} = k_B\left(\frac{E}{k_BT}\right)^2\frac{e^{E/k_BT}}{\left(e^{E/k_BT}+1\right)^2}$$

(a) At a low temperature $(E/kT) >> 1$, and equation (3.35) reduces to

$$C_V = k_B\left(\frac{E}{k_BT}\right)^2 e^{-E/k_BT}$$

Since with the decrease of T, the function $\left(E/k_BT\right)^2$, therefore

$$C_V \to 0 \qquad \text{when} \qquad T \to 0$$

(b) At a high temperature $(E/k_BT) << 1$ and equation reduces to

$$C_V = k\left(\frac{E}{k_BT}\right)^2\frac{1}{4} \qquad \text{Hence,} \quad C_V \to 0 \qquad \text{when} \quad T \to \infty$$

One Dimensional Free Particle:

Example: The Hamiltonian of one dimensional N free particle is confine in box of length L given

by $E(q,p)=\sum_{i=1}^{N}\left[\frac{p_i^2}{2m}\right]$ write down

(a) Expression of partition function

(b) Internal energy of system

(c) Specific heat at constant volume .

Solution: $Z_N=\frac{1}{h^N}\int \exp\left(-\left\{\sum_{i=1}^{N}\left[\frac{p_i^2}{2m}\right]\right\}\Big/ k_BT\right)\ dq_i\quad dp_i$

$$=\frac{1}{h^N}\prod_{i=1}^{N}\left\{\int_{-\infty}^{\infty} e^{-p_i^2/2mk_BT}dp_i\int_0^L dq_i\right\}$$

For evaluation of the first integral of equation let us put $\frac{p_i^2}{2mk_BT}=u$ and $\frac{p_i dp_i}{mk_BT}=du$

Using equations in the first integral equation we have

$$\int_{-\infty}^{\infty}\exp\left(p_i^2/2mk_BT\right)dp_i=2\int_0^{\infty}\exp\left(-p_i^2/2mk_BT\right)dp_i$$

$=\sqrt{2mk_BT}\int_0^{\infty}e^{-u}u^{-1/2}=\sqrt{2\pi mk_BT}$ and integration of second integral is L

Partition function of one particle is $Z=\frac{1}{h}\left(2\pi mk_BT\right)^{1/2}(L)$

Partition function of N particle is $Z_N=Z^N=\frac{1}{h^N\lfloor N}\left(2\pi mk_BT\right)^{N/2}(L)^N$

(b) the internal energy $\langle E\rangle=\langle U\rangle=k_BT^2\frac{1}{Z}\frac{\partial Z}{\partial T}$

$$\langle E\rangle=\langle U\rangle=\frac{Nk_BT}{2}$$

(c) $\quad C_V=\left(\frac{\partial U}{\partial T}\right)_V=\frac{Nk_B}{2}$

Classical Harmonic Oscillator:

Total energy of the system of N one dimensional classical oscillators is given by

$$E(q,p)=\sum_{i=1}^{N}\left[\frac{p_i^2}{2m}+\frac{1}{2}m\omega^2 q_i^2\right]$$

where q_i and p_i are position and momentum of the i-th oscillator, respectively.

(a) Write down partition

(b) Helmholtz Free energy

(c) Internal energy

(d) Specific heat at constant volume

The partition function of the system is where

$$Z_N=\frac{1}{h^N}\int \exp\left(-\left\{\sum_{i=1}^{N}\left[\frac{p_i^2}{2m}+\frac{1}{2}m\omega^2 q_i^2\right]\right\}\Big/k_BT\right)\ dq_i\quad dp_i$$

$$=\frac{1}{h^N}\prod_{i=1}^{N}\left\{\int_{-\infty}^{\infty} e^{-p_i^2/2mk_BT}dp_i\int_{-\infty}^{\infty} e^{-m\omega^2 q_i^2/2k_BT}dq_i\right\}$$

For evaluation of the first integral of equation let us put $\frac{p_i^2}{2mk_BT}=u$ and $\frac{p_i dp_i}{mk_BT}=du$

Using equations in the first integral equation we have

$$\int_{-\infty}^{\infty}\exp-\left(p_i^2/2mk_BT\right)dp_i=2\int_0^{\infty}\exp\left(-p_i^2/2mk_BT\right)dp_i=\sqrt{2mk_BT}\int_0^{\infty}e^{-u}u^{-1/2}=\sqrt{2\pi mk_BT}$$

For evaluation of the second integral of equation, let us put

$$\frac{m\omega^2 q_i^2}{2k_BT}=u \quad \text{and} \quad \frac{m\omega^2 q_i dq_i}{k_BT}=du$$

$$\int_{-\infty}^{\infty}\exp\left(-m\omega^2 q_i^2/2k_BT\right)dq_i=2\int_0^{\infty}\exp\left(-m\omega^2 q_i/2k_BT\right)\quad dq_i$$

$$=\sqrt{\frac{k_BT}{m\omega^2}}\int_{-\infty}^{\infty}e^{-u}u^{-1/2}du=\sqrt{\frac{2\pi k_BT}{m\omega^2}}$$

$$Z_N=\frac{1}{h^N}\left(2\pi mk_BT\right)^{N/2}\left(\frac{2\pi k_BT}{h\omega}\right)^{N/2}$$

(a) Free energy F of the system is

$$F = -k_B T \ln Z_N = -Nk_B T \ln\left(\frac{2\pi k_B T}{h\omega}\right)$$

Once the free energy of the system is known, we can calculate other thermo dynamical quantities of the system.

(b) Entropy S of the system is

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = Nk_B \ln\left(\frac{2\pi k_B T}{h\omega}\right) + Nk_B$$

(c) Internal energy U is

$$U = F + ST = Nk_B T$$

Thus, the mean energy per oscillator is . $k_B T$

(d) Specific heat at constant volume C_V is

$C_V = \dfrac{\partial U}{\partial T} = Nk_B$ The specific heat at constant volume C_V is independent of the temperature

Quantum Harmonic Oscillator:

Example: In quantum mechanics, energy of an oscillator is quantized and the energy of the N such system is given by

$$E_{n_i} = \sum_{i=1}^{N} \hbar\omega\left(n_i + \frac{1}{2}\right)$$

where n_i is an integer; n_i = 0, 1, 2, 3, ….. then find

(a) The partition function of the system .

(b) entropy

(c) Helmholtz free energy

(d) Internal energy

Solution: (a) Specific heat at constant volume ,also discuss the case for lower temperature and higher temperature.

a) $Z_N = \sum_{n_i} \exp\left(-E_{n_i}/k_BT\right) = \sum_{n_i} \exp\left\{-\sum_{i=1}^{N} \hbar\omega\left(n_i + \frac{1}{2}\right)\Big/ k_BT\right\}$

$$= \prod_{i=1}^{N}\left[\sum_{n_i=0}^{\infty} \exp\left\{-\hbar\omega\left(n_i+\frac{1}{2}\right)\Big/ k_BT\right\}\right]$$

We know that $\sum_{n_i=0}^{\infty} \exp\left\{-\hbar\omega\left(n_i+\frac{1}{2}\right)\Big/ k_BT\right\} = \exp\left(-\hbar\omega/2k_BT\right) + \exp\left(-3\hbar\omega/2k_BT\right)$ ……..

$$= \exp\left(-\hbar\omega/2k_BT\right)\left[\frac{1}{1-\exp\left(-\hbar\omega/k_BT\right)}\right] = \frac{\exp\left(-\hbar\omega/2k_BT\right)}{1-\exp\left(-\hbar\omega/k_BT\right)}$$

Thus, the partition function is

$$Z_N = \left[\frac{\exp\left(-\hbar\omega/2k_BT\right)}{1-\exp\left(-\hbar\omega/k_BT\right)}\right]^N$$

(b) Free energy F of the system is

$$F = -k_BT \ln Z_N = -Nk_BT \ln\left[\frac{\exp\left(-\hbar\omega/2k_BT\right)}{1-\exp\left(-\hbar\omega/k_BT\right)}\right]$$

$$= \frac{N\hbar\omega}{2} + Nk_BT \ln\left[1-\exp\left(-\hbar\omega/k_BT\right)\right]$$

Once the free energy of the system is known, we can calculate other thermodynamical quantities of the system.

(c) Entropy S of the system is

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = -Nk_B \ln\left[1-\exp\left(-\hbar\omega/k_BT\right)\right] + \frac{N\hbar\omega/T}{\exp\left(\hbar\omega/k_BT\right)-1}$$

$$= Nk_B\left\{\left(\frac{\hbar\omega}{k_BT}\right)\frac{1}{\exp\left(\hbar\omega/k_BT\right)-1} - \ln\left[1-\exp\left(-\hbar\omega/k_BT\right)\right]\right\}$$

(d) Internal energy U is $U = F + ST = N\left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega / k_B T) - 1}\right]$

(e) Specific heat at constant volume c_V is

$$C_V = \frac{\partial U}{\partial T} = N\hbar\omega \frac{\exp(\hbar\omega / k_B T)}{\left[\exp(\hbar\omega / k_B T) - 1\right]^2}\left(\frac{\hbar\omega}{k_B T^2}\right) = Nk_B\left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{\exp(\hbar\omega / k_B T)}{\left[\exp(\hbar\omega / k_B T) - 1\right]^2}$$

- At a low temperature, we have $(\hbar\omega / k_B T) >> 1$, and therefore, equation reduces to

$$C_V = Nk_B\left(\frac{\hbar\omega}{k_B T}\right)^2 \exp\left(-\frac{\hbar\omega}{k_B T}\right)$$

Since with the decrease of T, the function $e^{-\hbar\omega / k_B T}$ reduces much faster than the increase of the function $(\hbar\omega / k_B T)^2$, therefore $C_V \to 0$ when $T \to 0$

- At a high temperature, we have $(\hbar\omega / k_B T) << 1$, and therefore, equation reduces to

$$C_V = Nk_B \frac{1 + \hbar\omega / k_B T + ...}{(1 + \hbar\omega / 2k_B T + ...)^2}$$

It gives $C_V \to Nk_B$ when $T \to \infty$ it shows that the classical result for C_V is valid at high temperature.

Example: In quantum mechanics, energy of an oscillator is quantized and the energy of the N such system is given by $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ where n = 0, 1, 2, 3, ….. then

(a) Prove that partition function of the system is $Z_N = \left[2\sinh\frac{\hbar\omega}{2k_B T}\right]^{-N}$

From the expression used in (a) then find

(a) Internal energy

(b) Specific heat at constant volume, also discuss the case for lower temperature and higher temperature.

(c) Helmholtz free energy

(d) Entropy.

Solution:

(a) In quantum mechanics, the energy of an oscillator is $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$. Thus, the quantum mechanical partition function for one oscillator is

$$Z_1 = \sum_n e^{-E_n/k_BT} = \sum_{n=0}^{\infty} e^{-(n+1/2)\hbar\omega/k_BT}$$

$$= \frac{e^{-\hbar\omega/2k_BT}}{1-e^{-\hbar\omega/2k_BT}} = \frac{1}{e^{\hbar\omega/2k_BT}-e^{-\hbar\omega/2k_BT}} = \frac{1}{2\sinh\left(\hbar\omega/2k_BT\right)}$$

Since the partition function Z_N of a system of N independent particles is equal to the product of the partition function Z_1 of individual particle, we have

$$Z_N = Z_1^N = \left[2\sinh\frac{\hbar\omega}{2k_BT}\right]^{-N}$$ Using $\beta = \frac{1}{k_BT}$,

(b) the internal energy U of the system is

$$U = -\frac{\partial}{\partial\beta}\ln Z_N = N\frac{\partial}{\partial\beta}\ln\left[\sinh\frac{\beta\hbar\omega}{2}\right]$$

$$= N\left[\frac{\hbar\omega}{2}\right]\coth\frac{\hbar\omega}{2k_BT} = N\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\right]$$

(c) Specific heat at constant volume C_V is

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{N,\omega} = Nk\frac{e^{\hbar\omega/k_BT}}{\left(e^{\hbar\omega/k_BT}-1\right)^2}\left[\frac{\hbar\omega}{2k_BT}\right]^2$$

(d) The Helmohltz free energy F is

$$F = -k_BT\ln Z_n = Nk_BT\left[2\sinh\frac{\hbar\omega}{2k_BT}\right]$$

$$= N\left[\frac{\hbar\omega}{2}+k_BT\ln\left(1-e^{-\hbar\omega/k_BT}\right)\right]$$

(e) The entropy S is $S = \frac{U-F}{T} = Nk_B\left[\frac{\hbar\omega}{2k_BT}\coth\frac{\hbar\omega}{2k_BT}-\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)\right]$

Quantum Mechanical Treatment of Spin Half Paramagnetic Substance:

Example: Suppose a system comprising N identical particles is placed in a uniform magnetic field H and is kept at a temperature T. When a particle having spin $\frac{1}{2}$ is placed in a magnetic field H, its each energy level splits into two with changes in energies by μH and the particle has a magnetic moment μ or $-\mu$ along the direction of the magnetic field, respectively. Find expressions for internal energy, entropy, specific heat and total magnetic moment M of this system with the help of the canonical distribution.

Solution: As the spins of particles are independent of each other, the partition function of the total system Z_N is equal to the product of the partition functions for spins of individual particle. The partition function for spins of individual particle is

$$Z_i = e^{\mu H/k_B T} + e^{-\mu H/k_B T} = 2\cosh\left(\mu H/k_B T\right)$$

Thus,

$$Z_N = Z_i^N = \left[2\cosh\left(\mu H/k_B T\right)\right]^N$$

The Helmholtz free energy is

$$F = -k_B T \ln Z_N = -Nk_B T \ln\left[2\cosh\left(\mu H/k_B T\right)\right]$$

The entropy is

$$S = \left(-\frac{\partial F}{\partial T}\right)_V = Nk_B\left[\ln\left\{2\cosh\left(\mu H/k_B T\right)\right\} - \left(\mu H/k_B T\right)\tanh\left(\mu H/k_B T\right)\right]$$

Total energy is $U = F + TS = -N\mu H \tanh\left(\mu H/k_B T\right)$

Total magnetic moment is

$$M = -\frac{\partial F}{\partial H} = N\mu \tanh\left(\mu H/k_B T\right)$$

The specific heat at constant volume C_V is

$$C_V = \left[\frac{\partial U}{\partial T}\right]_V = Nk_B\left(\mu H/k_B T\right)^2 \sec h^2\left(\mu H/k_B T\right)$$

Example: If Z is partition of one dimensional harmonic oscillator with energy $\left(n+\frac{1}{2}\right)\hbar\omega$ where

n=0,1,2,3.... at equilibrium temperature T.

(a) what is probability that system has energy $\frac{\hbar\omega}{2}$

(b) what is probability that system has energy lower than $4\hbar\omega$

(c) what is probability that system has energy greater than $4\hbar\omega$

Solution: **(**a) If Z is partition of system what will be probability that system has energy $\frac{\hbar\omega}{2}$

equilibrium temperature T.

$$p(E_i)=\frac{\exp-\frac{E_i}{k_BT}}{Z} \quad , \quad p\left(\frac{\hbar\omega}{2}\right)=\frac{\exp-\frac{\hbar\omega}{2k_BT}}{Z}$$

(b) System has smaller thus energy $4\hbar\omega$ possible energy is $\frac{\hbar\omega}{2},\frac{3\hbar\omega}{2},\frac{5\hbar\omega}{2},\frac{7\hbar\omega}{2},$

so $$p(E<4\hbar\omega)=\frac{\exp-\frac{\hbar\omega}{2k_BT}+\exp-\frac{3\hbar\omega}{2k_BT}+\exp-\frac{5\hbar\omega}{2k_BT}+\exp-\frac{7\hbar\omega}{2k_BT}}{Z}$$

(c)

$$p(E>4\hbar\omega)=1-p(E<4\hbar\omega)$$

$$p(E>4\hbar\omega)=1-\left[\frac{\exp-\frac{\hbar\omega}{2k_BT}+\exp-\frac{3\hbar\omega}{2k_BT}+\exp-\frac{5\hbar\omega}{2k_BT}+\exp-\frac{7\hbar\omega}{2k_BT}}{Z}\right]$$

Example**:** A particle is confined to the region $x \geq 0$ by a potential which increases linearly as $U(x) = u_0 x$. find the mean position of the particle at temperature *T*.

Solution: Partition function is given by $Z = \frac{1}{h}\int e^{-\frac{p^2}{2mk_BT}} dp \int e^{-\frac{u_0 x}{k_B T}} dx$

$$\langle x \rangle = \int x p(x) dx dp_x$$

$$= \frac{\iint x e^{-\frac{p^2}{2mk_BT}} dp\, e^{\frac{u_0 x}{k_B T}} dx}{\iint e^{-\frac{p^2}{2mk_bT}} dp\; e^{-\frac{u_0 x}{k_B T}} dx} \qquad = \frac{\int_0^\infty x e^{-\frac{u_0 x}{k_B T}} dx}{\int_0^\infty e^{-\frac{u_0 x}{k_B T}} dx} \qquad = \frac{\left(\frac{k_B T}{u_0}\right)^2 \int_0^\infty t e^{-t} dt}{\left(\frac{k_B T}{u_0}\right) \int_0^\infty e^{-t} dt} = \frac{k_B T}{u_0}$$

5.2.3 Grand canonical ensemble:

In grand canonical ensemble, each element is in contact with reservoir where exchange of energy and particles is feasible .so in such type of ensemble energy (E) and number of particle (N) of system vary . This is an extension of the canonical but instead the grand canonical ensemble being modeled is allowed to exchange energy and particles with its environment. The **chemical potential** (μ) (or fugacity) is introduced to specify the fluctuation of the number of particles as chemical potential and particle numbers are thermodynamic conjugates. Popularly grand canonical ensemble is also known as T, V, μ. Schematically the grand canonical ensemble can be represented as

T V μ	T V μ	T V μ	T V μ
T V μ	T V μ	T V μ	T V μ
T V μ	T V μ	T V μ	T V μ
T V μ	T V μ	T V μ	T V μ

In above table each cell considers as each microstate temperature, volume and chemical potential which is fixed in each cell. Only value of energy and no of particle is different in different cell which can be exchanged in the process.

Grand canonical partition function is defined :

In grand canonical ensemble for the system of interest having constant value of T, V, μ the partition function in **classical system** is given by

$$Z_\mu = \sum_N \frac{1}{\lfloor N} \frac{1}{h^{3N}} \exp\left(\frac{\mu N}{k_B T}\right) \int \exp\left[\frac{-E_N(q,p)}{k_B T}\right] dqdp$$

here in quantum mechanical system $Z_\mu = \sum_n \sum_N g_n \exp\left[\frac{\mu N - E_{n,N}}{k_B T}\right]$

W**Thermo dynamical quantities in grand canonical ensemble:**

Relation between Helmholtz free energy and grand canonical partition function

According to Gibbs distribution function $\rho(E,N) = \dfrac{\exp\left[\dfrac{\mu N - E_{n,N}}{k_B T}\right]}{Z_\mu}$

If $\Delta\Gamma$ is statistical weight which is equivalent to no of microstate Ω in micro canonical ensemble then $\rho(E,N)\Delta\Gamma = 1$ and $\Delta\Gamma = \dfrac{1}{\rho(E,N)}$

So entropy is given by $S = k_B \ln \Delta\Gamma \Rightarrow k_B \ln \dfrac{1}{\rho(E,N)}$

So entropy $S = k_B \ln\left[\exp\left[\dfrac{-\mu N + E_{n,N}}{k_B T}\right] Z_\mu\right] \Rightarrow \dfrac{E_{n,N}}{T} - \dfrac{\mu N}{T} + k_B \ln Z_\mu$.

$\Rightarrow E_{n,N} - ST - \mu N = -k_B \ln Z_\mu \quad \Rightarrow F - \mu N = -k_B \ln Z_\mu$

Where $F - \mu N$ is popularly known as grand potential popularly represented by Ω.

$\Rightarrow \Omega = -k_B \ln Z_\mu$

- Pressure of the system is $P = -\left(\dfrac{\partial \Omega}{\partial V}\right)_{T,\mu}$
- The entropy of system is $S = -\left(\dfrac{\partial \Omega}{\partial T}\right)_{V,\mu}$

- The average number is $N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{V,T}$
- Helmholtz free energy $F = \mu N - k_B \ln Z_\mu$
- Internal energy $U = F + TS$, $U = \mu N + k_B T^2 \left[\frac{\partial \ln Z_\mu}{\partial T}\right]_{V,\mu}$

5.3 Maxwell-Boltzmann distribution

In statistical mechanics, the **Maxwell–Boltzmann distribution** describes particle speeds in gases, where the particles move freely without interacting with one another, except for very brief elastic collision in which they may exchange momentum and kinetic energy, but do not change their respective states of intermolecular excitation, as a function of the temperature of the system, the mass of the particle, and speed of the particle. Particle in this context refers to the gaseous atoms or molecules – no difference is made between the two in its development and result.

Maxwell –Boltzmann system constituent identical particles who are **distinguishable** in nature and there is **not any restriction** on no of particles which can occupy any energy level. The wave function of particle will not overlap to each other because mean separation of particles is more than the thermal wavelength , which is identified by λ . (where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is defined as the thermal wavelength)

Suppose there are l states with energies, $E_1, E_2, E_3 ....E_l$ and degeneracy $g_1, g_2, g_3 ....g_l$. Respectively, in which the particles are distributed. If there is N numbers of distinguishable particles out of these $n_1, n_2, n_3 ....n_l$ particles is adjusted in energy level $E_1, E_2, E_3 ....E_l$ respectively. So $\sum_{i=1}^{i=l} n_i = N$, $\sum_{i=1}^{i=l} E_i n_i = U$.

The total number of arrangements of the particles in the given distributions is given by

$$W=\frac{\lfloor N}{\lfloor n_1 \lfloor n_2 \lfloor n_3 \ldots\ldots \lfloor n_l}\, g_1^{\,n_1} g_2^{\,n_2} \ldots\ldots g_l^{\,n_l}\ ,\qquad W=\lfloor N \prod_{i=1}^{i=l}\frac{g_i^{\,n_i}}{\lfloor n_i}$$

The Maxwell-Boltzmann distribution law for the particles in the states is

$$n_i=g_i\exp\left(-\alpha-\beta E_i\right)\ ,\ n_i=g_i(\exp-\alpha)(\exp-\beta E_i)$$

After using the values $e^{-\alpha}=\frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}$ where $\beta=\frac{1}{k_B T}$

We get $n_i=g_i\frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} e^{-E_i/k_B T}$

5.3.1 Energy Distribution Function

Energy distribution function $f(E_i)$ is the average number of particles per level in the energy states E_i. Therefore $f\left(E_i\right)=\frac{n_i}{g_i}=\frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} e^{-E_i/k_B T}$

Energy Distribution in Different Dimension

- $f\left(E\right)=\frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} e^{-E/k_B T}$ distribution function in three dimension where V is volume .
- $f\left(E\right)=\frac{N}{A}\left(\frac{h^2}{2\pi m k_B T}\right) e^{-E/k_B T}$ distribution function in two dimension where A is area.
- $f\left(E\right)=\frac{N}{L}\left(\frac{h^2}{2\pi m k_B T}\right)^{\frac{1}{2}} e^{-E/k_B T}$ distribution function in one dimension. Where L is length.

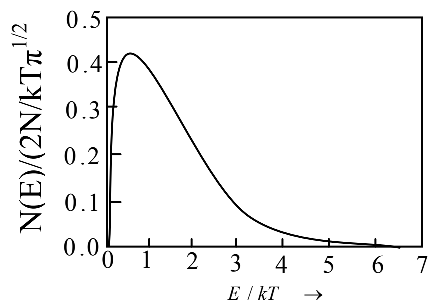
The number of particles $n(E)dE$ having energies in the range from E to $E+dE$ is $n(E)dE = f(E)g(E)dE$ where $f(E)$ is distribution function and $g(E)dE$ is number of level(quantum state) in the range of E to $E+dE$

- $g(E)dE$ in three dimension $g(E)dE = 2\pi V\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} E^{\frac{1}{2}} dE$ where V is volume of three dimensional space
- $g(E)dE$ in two dimension $g(E)dE = \pi A\left(\frac{2m}{h^2}\right) dE$ where A is area of the two dimensional space
- $g(E)dE$ in one dimension $g(E)dE = L\left(\frac{2m}{h^2}\right)^{\frac{1}{2}} E^{\frac{-1}{2}} dE$ where L is area of the one dimensional space

The number of particles $n(E)dE$ having energies in the range from E to $E+dE$ in three dimensional space

$$n(E)dE = \frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} e^{-E/k_BT} 2\pi V\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} E^{1/2} dE$$

$$= \frac{2\pi N}{(\pi k_B T)^{3/2}} E^{1/2} e^{-E/k_BT} dE$$



This is known as the Maxwell-Boltzmann energy distribution law for an-ideal gas.

Where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is defined as the thermal wavelength.

Average Energy

For the Maxwell-Boltzmann energy distribution law, average energy $\langle E\rangle$ of the particles

is $\langle E\rangle = \dfrac{\int_0^\infty E\, n(E)\, dE}{\int_0^\infty n(E)\, dE}$ $\qquad \Rightarrow \langle E\rangle = \dfrac{1}{N}\dfrac{2\pi N}{(\pi k_B T)^{3/2}}\int_0^\infty E^{3/2} e^{-E/k_BT} dE$

Let $E = k_B T x$ and therefore, $dE = k_B T dx$ Then we have

$$E = \frac{2\pi}{(\pi k_B T)^{3/2}}\int_0^\infty (k_B T x)^{3/2} e^{-x} k_B T dx \quad = \frac{2k_B T}{\sqrt{\pi}}\int_0^\infty x^{3/2} e^{-x} dx = \frac{3}{2}k_B T$$

Hence, the average of a particle is $\frac{1}{2}k_B T$ per degree of freedom,

for three degree of freedom it is $\frac{3}{2}k_B T$

Example: Two distinguishable particles have to be adjusted in a state whose degeneracy is three

(a) How many ways the particles can be adjusted?

(b) Show all arrangement .

Solution: (a) $N = 2, n = 2, g = 3$ and no of microstate is $W = \frac{\lfloor N g^n}{\lfloor n} = 9$, 9 ways .

(b) Total no of arrangement for 2 distinguishable in state whose degeneracy is 3.

First level	Second level	Third level
AB	0	0
0	AB	0
0	0	AB
A	B	0
B	A	0
A	0	B
B	0	A
0	A	B
0	B	A

Example: If N no of distinguishable particle is kept into two dimensional box of area A what is average energy at temperature A.

Solution: for two dimensional system $g(E)dE = \pi A\left(\frac{2m}{h^2}\right)dE$ and distribution

Function is given by $f\left(E\right) = \frac{N}{A}\left(\frac{h^2}{2\pi m k_B T}\right)e^{-E/k_BT}$

$$\langle E\rangle = \frac{\int_0^\infty E\,n\left(E\right)dE}{\int_0^\infty n\left(E\right)dE} \quad \text{where } n\left(E\right)dE = f(E)g(E)dE = \frac{N}{A}\left(\frac{h^2}{2\pi m k_B T}\right)e^{-E/k_BT}\pi A\left(\frac{2m}{h^2}\right)dE$$

$$\langle E\rangle = k_B T$$

5.4 Bose Einstein distribution

In quantum statistics, **Bose–Einstein statistics** (or more colloquially **B–E statistics**) is one of two possible ways in which a collection of indistinguishable particles may occupy a set of available discrete energy state . The aggregation of particles in the same state, which is a characteristic of particles obeying Bose–Einstein statistics .who recognized that a collection of identical and **indistinguishable particles** can be distributed in this way.

The Bose–Einstein statistics apply only to those particles not limited to single occupancy of the same state—that is, particles that **do not obey the Pauli exclusion restrictions**. Such particles **have integer values of spin** and are named boson , after the statistics that correctly describe their behavior.

The wave function of particle will overlap to each other because mean separation of particles is less than the thermal wavelength , which is identified by λ . (where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is defined as the thermal wavelength)

Suppose there are l states with energies, $E_1, E_2, E_3.......E_l$ and degeneracy $g_1, g_2, g_3.......g_l$. Respectively, in which the particles are distributed. If there is N numbers of indistinguishable boson particles out of these $n_1, n_2, n_3.......n_l$ particles is adjusted in energy level $E_1, E_2, E_3.......E_l$ respectively. It is given $\sum_{i=1}^{i=l} n_i = N$, $\sum_{i=1}^{i=l} E_i n_i = U$.

The total no. of arrangements of the particles in the given distributions is given by

$$W = \frac{\lfloor n_i + g_i - 1}{\lfloor n_i \lfloor (g_i - 1)} \quad , \quad W = \prod_{i=1}^{i=l} \frac{\lfloor n_i + g_i - 1}{\lfloor n_i \lfloor (g_i - 1)}$$

If n_i and g_i are large numbers , we can omit 1 in comparison to them , so we have $W = \prod_{i=1}^{i=l} \frac{\lfloor n_i + g_i}{\lfloor n_i \lfloor g_i}$

The Bose Einstein distribution of the particle among various states $n_i = \frac{g_i}{e^{(\alpha + \beta E)} - 1}$

The Bose-Einstein energy distribution is

$$f(E) = \frac{n_i}{g_i} = f(E) = \frac{1}{e^{(\alpha + \beta E)} - 1} = \frac{1}{Ae^{\beta E} - 1}$$

where $\beta = \frac{1}{k_B T}$ and $A = e^{\alpha} = e^{-\mu / k_B T} = \frac{V}{N}\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$

Here, μ is chemical potential which is general a function of T. when $A >> 1$, Bose-Einstein gas reduces to the Maxwell-Boltzmann gas. The chemical potential for Bose gas is negative, but for photon gas is zero.

Example: Two indistinguishable boson particles have to be adjusted in a state whose degeneracy is three.

a) How many ways the particles can be adjusted?

b) Show all arrangement.

Solution: (a) $n_i = 2, g_i = 3$ $, W_i = \dfrac{\lfloor n_i + g_i - 1}{\lfloor n_i \lfloor g_i - 1} = 6$ ways

(b) Total no of arrangement for 2 indistinguishable boson particles in state whose degeneracy is 3.

First level	Second level	Third level
AA	0	0
0	AA	0
0	0	AA
A	A	0
0	A	A
A	0	A

Examples: (a) write down distribution function of photon at temperature T ,if average energy in each state is given by $\varepsilon = h\nu$.

(b) what is density of state of photon gas between frequency ν to $\nu + d\nu$

(c) write down expression of no of particle for photon gas at temperature T .

(d) write down expression of average energy for photon gas at temperature T .

Solution: (a) the Bose Einstein distribution is given by $f(E) = \dfrac{1}{e^{(\alpha+\beta E)} - 1} = \dfrac{1}{Ae^{\beta E} - 1}$

where $A = e^{\alpha} = e^{-\mu/k_B T} = \dfrac{V}{N}\left(\dfrac{2\pi m k_B T}{h^2}\right)^{3/2}$, for boson $\mu = 0$. So $f(E) = \dfrac{1}{e^{\beta E} - 1}$ for if average energy in each state is given by $\varepsilon = h\nu$ then $f(E) = \dfrac{1}{e^{\frac{h\nu}{k_B T}} - 1}$

(b) if j is quantum number associate with frequency ν then total no of frequency between ν to $\nu + d\nu$ is same as the number of points between j to $j + dj$. The volume of spherical shell of radius j and thickness dj is $4\pi j^2 dj$.

Hence all three component of j is positive (same as particle in box) and there are two direction of polarization so degeneracy $g = 2$.

So no of standing wave $g(j)dj = (2)\left(\frac{1}{8}\right)4\pi j^2 dj = \pi j^2 dj$

It is given $j = \frac{2L}{\lambda} = \frac{2L\nu}{c}$ and $dj = \frac{2Ld\nu}{c}$ $\quad g(\nu)d\nu = \frac{8\pi L^3\nu^2}{c^3}d\nu$

So density of standing wave in cavity is given by $g(\nu)d\nu = \frac{g(\nu)}{L^3}d\nu$

$$g(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu$$

(c) $N = \int_0^\infty f(E)g(E)dE \Rightarrow \int_0^\infty f(\nu)g(\nu)d\nu$

$$N = \frac{8\pi V}{c^3}\int_0^\infty \frac{\nu^2 d\nu}{e^{\frac{h\nu}{k_BT}} - 1} \quad \text{put } x = \frac{h\nu}{k_BT} \quad N = \frac{8\pi V}{c^3}\left(\frac{k_BT}{hc}\right)^3\int_0^\infty \frac{x^2dx}{e^x - 1}$$

The integral have value

$$N = 1.92V\left(\frac{k_BT}{hc}\right)^3$$

(d) $U = \int_0^\infty Ef(E)g(E)dE \Rightarrow \int_0^\infty h\nu f(\nu)g(\nu)d\nu$

$$U = \frac{8\pi Vh}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{\frac{h\nu}{k_BT}} - 1} \quad \text{put } x = \frac{h\nu}{k_BT}$$

$$U = 8\pi Vc\left(\frac{k_BT}{hc}\right)^4\int_0^\infty \frac{x^3dx}{e^x - 1} \qquad U = \frac{8\pi^5 V k_B^4 T^4}{15c^3h^3}.$$

Example: A system consisting of two boson particles each of which can be any one of three quantum state of respective energies $o, \varepsilon, 3\varepsilon$ is in equilibrium at temperature T .write the expression of partition function.

Two boson can be distributed in three given state with their respective energy level shown in table

S.N.	Energy 0	Energy ε	Energy 3ε	Total energy
1	A A	0	0	0
2	0	AA	0	2ε
3	0	0	AA	6ε
4	A	A	0	ε
5	A	0	A	3ε
6	0	A	A	4ε

$$Z = 1 + \exp(-\beta\varepsilon) + \exp(-2\beta\varepsilon) + \exp(-3\beta\varepsilon) + \exp(-4\beta\varepsilon) + \exp(-6\beta\varepsilon)$$

5.5 Fermi Dirac Distribution

In quantum statistics, Fermi**–Dirac statistics** describes distribution of particles in a system comprising many identical particles that **obey the Pauli Exclusion Principle** (The **Pauli Exclusion Principle** is the quantum mechanical principle that no two identical fermions (particles with half-integer spin) may occupy the same quantum state simultaneously.)

Fermi–Dirac (F–D) statistics applies to identical particles with **half odd integer spin** in a system in thermal equilibrium. Additionally, the particles in this system are assumed to have negligible mutual interaction. This allows the many-particle system to be described in terms of single-particle energy states . The result is the F–D distribution of particles over these states and includes the condition that no two particles can occupy the same state, which has a considerable effect on the properties of the system. Since F–D statistics applies to particles with half-integer spin, these particles have come to be called fermions.

It is most commonly applied to electrons , which are fermions with spin $\frac{1}{2}$

No. of ways W in which n_i indistinguishable particles to place in g_i level with the condition that only one particle or no particle can be placed in each level i.e identical particles that **obey the 5.5.1Pauli exclusion principle. (** it is given It is given $\sum_{i=1}^{i=l} n_i = N$

$\sum_{i=1}^{i=l} E_i n_i = U$.) is given by $W = \sum_{i=1}^{l} \frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i}$

Fermi-Dirac distribution of the particles among various states is given by $n_i = \frac{g_i}{\exp(\alpha + \beta E) + 1}$

So Fermi Dirac distribution $f(E) = \frac{n_i}{g_i} = \frac{1}{e^{(\alpha+\beta E)} + 1} = \frac{1}{Ae^{\beta E} + 1}$

where $\beta = \frac{1}{k_B T}$ and $A = e^{\alpha} = \frac{V}{N}\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$

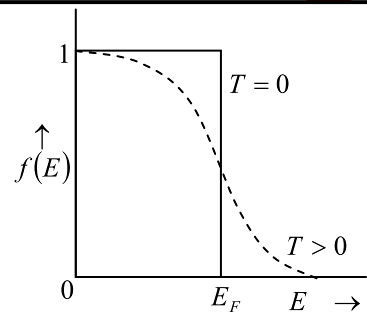
when A.>> 1, Fermi-Dirac gas reduces to the Maxwell-Boltzmann gas. Fermi-Dirac gas is said to be weakly degenerate when A>1, degenerate when A<1 and strongly degenerate when A = 0.**Strongly degenerate Fermi gas** $A < 1$

The Fermi-Dirac energy distribution is

$$f(E) = \frac{1}{\exp(E - \mu)/k_B T + 1}$$

where μ is the chemical potential which is a function of T, i.e; $\mu = \mu(T)$. the gas is strongly degenerate (A = 0) at T = 0. at T = 0, where $\mu = \mu(0) = E_F$. The limiting chemical potential is known as the Fermi energy E_F of the gas and the distribution function can be written as $f(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1}$

Strongly degenerate Fermi gas at T = 0



At T = 0, when $E < E_F$, we have $f(E)=\frac{1}{e^{-\infty}+1}=\frac{1}{0+1}=1$

At T = 0, when $E > E_F$, we have $f(E)=\frac{1}{e^{\infty}+1}=\frac{1}{\infty+1}=0$

Fermi function $f(E)$ versus E at T

The number of energy states in the energy range from E to $E+dE$

$g(E)dE=2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}E^{1/2}dE$ Here, g is the spin degeneracy, $g=(2s+1)$, where s is the spin quantum number of a particle. The number of particles in the energy range from E to E + dE at T = 0 is

$$n(E)dE=\begin{cases} f(E)g(E)dE=2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}E^{1/2}dE & \text{for } E<E_F \\ 0 & \text{for } E>E_F \end{cases}$$

$$N=\int_0^{E_F}n(E)dE \quad =2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{E_F}E^{1/2}\,dE=\frac{4g\pi V}{3}\left(\frac{2m}{h^2}\right)^{3/2}E_F^{3/2}$$

Thus, the Fermi energy is $E_F(0)=\left(\frac{3N}{4\pi gV}\right)^{2/3}\frac{h^2}{2m}$ and the Fermi temperature T_F is defined as $T_F=\frac{E_F}{k_B}$

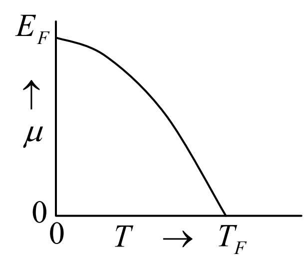


Fig: Variation of chemical potential μ with T.

The Fermi momentum p_F is given by

$$p_F=(2mE_F)^{1/2}=\left(\frac{3N}{4\pi gV}\right)^{1/3}h$$

Total energy of the gas at $T=0$ is

$$U=\int_0^{\infty}En(E)dE=2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{E_F}E^{3/2}dE \quad =\frac{4g\pi V}{5}\left(\frac{2m}{h^2}\right)^{3/2}E_{E_F}^{5/2}=\frac{2g\pi V}{5mh^3}p_F^5$$

Thus, at $T=0$ we have $\frac{U}{N}=\frac{3}{5}E_F$

Example: What is no. of ways if two fermions have to adjust in energy state whose degeneracy is three.

Solution: $g_i = 3$, $n_i = 2$ $\quad W = \dfrac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i} = 3$

Two indistinguishable particles is shown by A,A

Possible selection	1	2	3
1	A	A	0
2	0	A	A
3	A	0	A

Example: Fermions of mass m are kept in two dimensional box of area A at temperature $T = 0$

(a) What is total number of particle if E_F is Fermi energy.

(b) What is the energy of the system if E_F is Fermi energy.

c) Write expression of energy in term of E_F and N

Solution: For two dimensional systems density of state $g(E)dE = \pi A\left(\dfrac{2m}{h^2}\right)dE$ and distribution function at temperature $T = 0$ for is given by $f(E) = 1$ if $E < E_F = 0$ if $E > E_F$

a) $N = \int_0^{E_F} g(E)f(E)dE \Rightarrow \dfrac{\pi 2mAE_F}{h^2}$

b) $E = \int_0^{E_F} Eg(E)f(E)dE \quad \Rightarrow E = \dfrac{\pi A2m}{h^2}\int\limits_0^{E_F} EdE, \quad \Rightarrow E = \dfrac{\pi mAE_F^2}{h^2}$

c) $E = \dfrac{NE_F}{2}$

Example: (a) If Fermi gas is at temperature $T > 0$ what will $f(E_F)$.

(b) At $E = E_F + x$, find the fraction of occupied levels.

(c) At $E = E_F - x$, find fraction of unoccupied levels.

Solution: **(**a) It is also interesting to note that at T > 0, when $E = E_F$ we have

$$f(E) = \frac{1}{e^{E-E_F/kT}+1} = \frac{1}{e^{0/kT}+1} = \frac{1}{1+1} = \frac{1}{2}$$

(b) At $T > 0$, fraction of levels above E_F are occupied and a fraction of levels below E_F are vacant. The fraction of occupied levels at the energy E is

$$f(E) = \frac{1}{e^{(E-E_F)/kT}+1}$$

At $E = E_F + x$, , the fraction of occupied levels is $f(E_F + x) = \frac{1}{e^{x/kT}+1}$

(c) The fraction of unoccupied levels at the energy E is

$$1 - f(E) = 1 - \frac{1}{e^{(E-E_F)/kT}+1}$$

At $E = E_F - x$, the fraction of unoccupied levels is $1 - f(E_F - x) = \frac{1}{1+e^{x/kT}}$

Questions

MCQ (Multiple Choice Questions)

Q1. If spin of a fermions is $s = \frac{3}{2}\hbar$ then no of microscope is given

(a) 1 (b) 2 (c) 3 (d) 4

Q2. If 2 classical particle have to to adjusted in 2 different non degenerate quantum state where one particle is in ground state and another is in upper state the no of possible microstate is

(a) 1 (b) 2 (c) 3 (d) 4

Q3. If 3 classical particle have to to adjusted in 2 different quantum state where one particle is in ground state and other is in upper state where ground state is non degenerate and upper state is doubly degenerate

a) 4 (b) 6 (c) 10 (d) 12

Q4. If 2 boson have to to be adjusted in 2 different non degenerate quantum state where one particle is in ground state and another is in upper state the no of possible microstate is

(a) 1 (b) 2 (c) 3 (d) 4

Q5. If 3 boson particle have to adjusted in 2 different quantum state where one particle is in ground state and other is in upper state where ground state is non degenerate and upper state is doubly degenerate

(a) 2 (b) 3 (c) 6 (d) 12

Q6. If 2 fermions have to to be adjusted in 2 different non degenerate quantum state where one particle is in ground state and another is in upper state the no of possible microstate is

(a) 1 (b) 2 (c) 3 (d) 4

Q7. If 3 fermions particle have to adjusted in 2 different quantum state where one particle is in ground state and other is in upper state where ground state is non degenerate and upper state is doubly degenerate

(a) 1 (b) 2 (c) 3 (d) 4

Q8. If ground state of single particle confined in one dimensional potential box of width a have ground state energy is E_0 then if 4 such type of fermions have to adjusted in same box then energy of ground state of configuration is

(a) E_0 (b) $4E_0$ (c) $10E_0$ (d) $12E_0$

Q9. For a two-dimensional free electron gas, the electronic density n, and the Fermi energy E_F at temperature T, are related by

(a) $n = \frac{(2mE_F)^{3/2}}{3\pi^2\hbar^3}$ (b) $n = \frac{mE_F}{\pi\hbar^2}$

(c) $n = \frac{mE_F}{2\pi\hbar^2}$ (d) $n = \frac{2^{3/2}(mE_F)^{3/2}}{\pi\hbar}$

Q10. A system has two energy levels with energies ε and 2ε. The lower level is 4-fold degenerate while the upper level is doubly degenerate. If there are N non-interacting classical particles in the system, which is in thermodynamic equilibrium at a temperature T, the fraction of particles in the upper level is

(a) $\frac{1}{1+e^{\varepsilon/k_BT}}$ (b) $\frac{1}{1+2e^{\varepsilon/k_BT}}$

(c) $\frac{1}{2e^{\varepsilon/k_BT}+4e^{2\varepsilon/k_BT}}$ (d) $\frac{1}{2e^{\varepsilon/k_BT}-4e^{2\varepsilon/k_BT}}$

Q11. For an ideal Fermi gas in three dimensions, the electron velocity V_F at the Fermi surface is related to electron concentration n as,

(a) $V_F \propto n^{2/3}$ (b) $V_F \propto n$ (c) $V_F \propto n^{1/2}$ (d) $V_F \propto n^{1/3}$

Q12. Consider a system whose three energy levels are given by 0, ε and 2ε. The energy level ε is two-fold degenerate and the other two are non-degenerate. The partition function of the system with $\beta = \frac{1}{k_BT}$ is given by

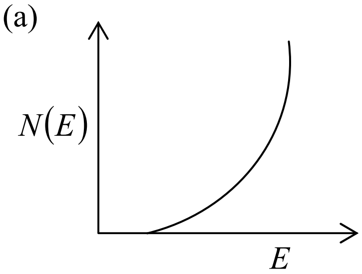
(a) $1+2e^{-\beta\varepsilon}$ (b) $2e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}$ (c) $(1+e^{-\beta\varepsilon})^2$ (d) $1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}$

Q13. Consider a collection of N two-level systems in thermal equilibrium at temperature T. Each system has only two states: a ground state of energy 0 and excited state of energy E. What is probability that system will be found in the excited state.

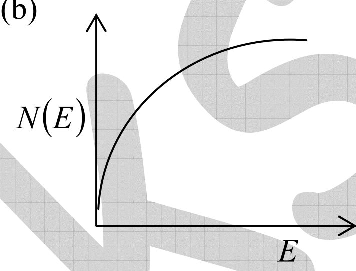
(a) $\frac{1}{1+e^{E/kT}}$ (b) $\frac{e^{-E/kT}}{1+e^{+E/kT}}$ (c) $\frac{1}{1+e^{-E/kT}}$ (d) $\frac{e^{E/kT}}{1+e^{-E/kT}}$

Q14. For a free electron gas in two dimensions the variations of the density of states. $N(E)$ as a function of energy E, is best represented by

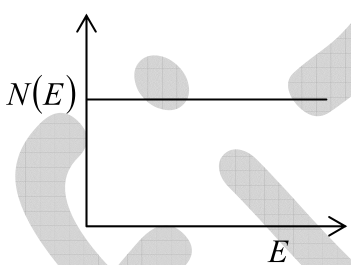
(a)



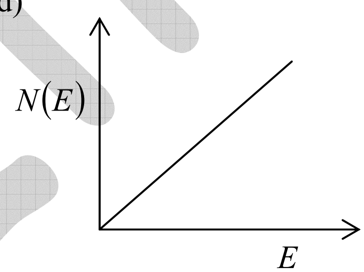
(b)



(c)



(d)



Q15. If 6 fermions of spin half have to be adjusted in two dimensional harmonic isotropic oscillator with angular frequency ω the energy of ground state configuration is given by

(a) $\frac{\hbar\omega}{2}$ (b) $\hbar\omega$ (c) $6\hbar\omega$ (d) $10\hbar\omega$

NAT (Numerical Answer Type)

Q16. 6 spin $\frac{1}{2}$ fermions and 6 boson of spin 1 confined in one dimensional box of width a the ratio of ground state energy configuration of fermions to ground state bosons are ………

Q17. Consider a linear collection of N independent spin $1/2$ particles, each at a fixed location. The entropy of this system is (k_B is the Boltzmann constant) $Nk_B\alpha$ then value α is given by ……..

Q18. Consider a system of 3 fermions which can occupy any of the 4 available energy states with equal probability. The entropy of the system is $\alpha k_B \ln\beta$ where value of α is ………..and β ………..

Q19. For three dimensional system, the energy density is proportional by E^α then value of α is given by …………..

Q20. If fermions are confined in three dimensional energy if energy per particle is αE_F at temperature $T = 0K$ the value of α........ if E_F is Fermi energy at temperature $T = 0^0 K$

Q21. For photon gas in three dimension no density n is proportional to T^α where T is temperature at Kelvin then the value of α is given by ....

Q22. For photon gas in three dimension energy E is proportional to T^α at constant V where T is temperature at Kelvin then the value of α is given by ....

Q23. For photon gas in three dimension energy E per particle is proportional to T^α at constant V where T is temperature at Kelvin then the value of α is given by ....

Q24. For one dimensional classical harmonic oscillator total energy per particle at equilibrium temperature T is given by $\frac{\alpha}{2}k_B T$ then value of α is ………..

MSQ (Multiple Select Questions)

Q25. Which of the following statements is ***correct*** ?

(a) Indistinguishable particles obey Maxwell-Boltzmann statistics

(b) All particles of an ideal Bose gas occupy a single energy state at $T = 0$

(c) The integral spin particles obey Bose-Einstein statistics

(d) Protons obey Fermi-Dirac statistics

Q26. Which of the following statements is/are ***correct*** ?

(a) Distinguishable particles obey Maxwell-Boltzmann statistics

(b) In any quantum state two fermions can be adjusted

(c) The half integral spin particles obey Fermi Dirac statistics

(d) Photon obey Bose Einstein statistics

Q27. Which of the following is correct

(a) fermions and bosons are indistinguishable particle

(b) Electrons are fermions while protons are bosons

(c) Electrons are fermions while photons are bosons

(d) protons are fermions while photons are bosons

Q28. The chemical potential of an ideal Bose gas at any temperature is

(a) necessarily negative (b) zero

(c) necessarily positive (d) negative

Q29. If six fermions of spin $\frac{3}{2}$ have to adjusted in one dimensional quantum mechanical harmonic oscillator then which of the following is correct .

(a) In one particular energy state four fermions can be adjusted

(b) In one particular quantum state only one fermions can be adjusted.

(c) The energy of ground state configuration is $\frac{\hbar\omega}{2}$.

(d) The energy of ground state configuration is $5\hbar\omega$

Q30. Which one of the following is correct

(a) In Micro canonical ensemble the volume and temperature are constant but energy is allow to exchange

(b) In canonical ensemble the volume and temperature and number of particle are constant but energy is allow to exchange

(c) In canonical ensemble the volume and temperature is constant but energy and number of particle are allow to exchange

(d) In Grand canonical ensemble the volume and temperature are constant but energy and number of Particles are allow to exchange

Q31. Which one of the following system has average energy particle per particle is $k_B T$ where T is equilibrium temperature.

(a) Two dimensional classical free particle .

(b) Classical particle confined in one dimensional Harmonic oscillator.

(c) Photon confined in one dimensional Box of length L

(d) Photon confined in two dimensional Box of of area A

Q32. Which of the following is correct if N particle confine in harmonic oscillator potential defined as $V(x) = \frac{m\omega^2 x^2}{2}$ at equilibrium temperature T

(a) for classical particle average energy is $Nk_B T$

(b) For quantum mechanical particles average energy is $U = N\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1}\right]$

(c) For quantum mechanical particles average energy is $U = N\left[\frac{\hbar\omega}{2} - \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1}\right]$

(d) For quantum mechanical particle average energy is $U = N\frac{\hbar\omega}{2}\coth\frac{\hbar\omega}{2}$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (d)

Solution: If spin of fermion then no of microstate is $2s+1$ put $s=\frac{3}{2}$ so no of microstate is 4

Ans. 2: (b)

Solution: $W=\prod_i \frac{\lfloor\underline{N}}{\lfloor\underline{n_i}}g_i^{n_i}$ $N=2, n_1=1, n_2=1$ and $g_1=1$, $g_2=1$ So $W=2$

Ans. 3: (d)

Solution: $W=\prod_i \frac{\lfloor\underline{N}}{\lfloor\underline{n_i}}g_i^{n_i}$ $N=3, n_1=1, n_2=2$ and $g_1=1$, $g_2=2$ So $W=12$

Ans. 4: (a)

Solution: $W=\prod_i \frac{\lfloor\underline{n_i+g_i-1}}{\lfloor\underline{n_i}\lfloor\underline{g_i-1}}$ $N=2, n_1=1, n_2=1$ and $g_1=1$, $g_2=1$ So $W=1$

Ans. 5: (b)

Solution: $W=\prod_i \frac{\lfloor\underline{n_i+g_i-1}}{\lfloor\underline{n_i}\lfloor\underline{g_i-1}}$ $N=3, n_1=1, n_2=2$ and $g_1=1$, $g_2=2$ $W=3$

Ans. 6: (a)

Solution: $W=\prod_i \frac{\lfloor\underline{g_i}}{\lfloor\underline{g_i-n_i}\lfloor\underline{n_i}}$ $N=2, n_1=1, n_2=1$ and $g_1=1$, $g_2=1$ So $W=1$

Ans. 7: (a)

Solution: $W=\prod_i \frac{\lfloor\underline{g_i}}{\lfloor\underline{g_i-n_i}\lfloor\underline{n_i}}$ $N=3, n_1=1, n_2=2$ and $g_1=1$, $g_2=2$ $\Rightarrow W=1$

Ans. 8: (c)

Solution: $E=2\times E_0+2\times 4E_0=10E_0$ where $E_0=\frac{\pi^2\hbar^2}{2ma^2}$

Ans. 9: (c)

Solution: $n=\int_0^{E_F} g(E)f(E)dE$, $g(E)dE=\frac{\pi A 2m}{h^2}dE$, at $T=0$ $\quad f(E)=1 \quad if \quad E<E_F$; $\quad =0 \quad if \quad E>E_F$

$$\Rightarrow n=\frac{2mE_F}{h^2}=\frac{mE_F}{2\pi\hbar^2}$$

Ans. 10: (b)

Solution: Partition function $Z=4e^{-\epsilon/kT}+2e^{-2\epsilon/kT}$

$$P(2\varepsilon)=\frac{2e^{-2\epsilon/kT}}{4e^{-\epsilon/kT}+2e^{-2\epsilon/kT}}=\frac{1}{1+2e^{\epsilon/kT}}$$

Ans. 11: (d)

Solution: $E_F=\frac{1}{2}mV_F^2 \quad \because E_F\propto n^{2/3}\Rightarrow V_F^2\propto n^{2/3}\Rightarrow V_F\propto n^{1/3}$.

Ans. 12: (c)

Solution: $E_1=0,\ E_2=\varepsilon, E_3=2\varepsilon$; $g_1=1,\ g_2=2, g_3=1$ where g_1, g_2 and g_3 are degeneracy.

The partition function $Z=g_1e^{-\beta\cdot E_1}+g_2e^{-\beta\cdot E_2}+g_3e^{-\beta\cdot E_3}=1+2e^{-\beta\varepsilon}+e^{-\beta 2\varepsilon}=\left(1+e^{-\beta\varepsilon}\right)^2$

Ans. 13: (a)

Solution: $\frac{e^{-E/kT}}{1+e^{-E/kT}}=\frac{1}{1+e^{E/kT}}$

Ans. 14: (c)

Ans. 15: (d)

Solution: n^{th} energy state have $n+1$ degeneracy and due to spin s particle the total degeneracy $g=(n+1)\times(2s+1)$ according to Pauli exclusion principle maximum no of fermions are no of degeneracy .

So energy of ground state configuration is $2\times\hbar\omega+4\times 2\hbar\omega=10\hbar\omega$

NAT (Numerical Answer Type)

Ans. 16: 4.66

Solution: Energy of bosons are $E_{boson} = 6\times\frac{\pi^2\hbar^2}{2ma^2} = \frac{6\pi^2\hbar^2}{2ma^2}$ all bosons are in ground state

Fermions obey Pauli exclusion principle if spin of fermions are $\frac{1}{2}$ then maximum 2 fermions will adjusted in one energy state

$$E_{fermoins} = 2\times\frac{\pi^2\hbar^2}{2ma^2} + 2\times\frac{4\pi^2\hbar^2}{2ma^2} + 2\times\frac{9\pi^2\hbar^2}{2ma^2} = \frac{28\pi^2\hbar^2}{2ma^2}$$

$$\frac{E_{fermions}}{E_{boson}} = \frac{28}{6} = 4.66$$

Ans. 17: 0.693

Solution: No of microstate is 2 so entropy is $Nk_B \ln 2$

Ans. 18: $\alpha = 2,\ \beta = 2$

Solution: Number of ways that 3 fermions will adjust in 4 available energy is ${}^4C_3 = 4$ so entropy is $k_B \ln 4 = 2k_B \ln 2$

Ans. 19: 0.5

Solution: For three dimensional system $g(E) \propto E^{1/2}$ so $\alpha = .5$

Ans. 20: $\frac{U}{N} = \frac{3}{5}E_F \quad \alpha = 0.6$

Ans. 21: $\alpha = 3$

Solution: Photon in three dimensional $N \propto VT^3$

Ans. 22: $\alpha = 4$

Solution: Photon in three dimensional $E \propto VT^4$

Ans. 23: $\alpha = 1$

Solution: Photon in three dimensional $N \propto VT^3,\ E \propto VT^4 \quad \Rightarrow \frac{E}{N} \propto T$

Ans. 24: $\alpha = 2$

Solution: The partition function of one dimensional harmonic oscillator at equilibrium temperature T is $Z = \dfrac{2\pi k_B T}{h\omega}$

The internal energy $U = k_B T^2 \dfrac{\partial \ln Z}{\partial T} = k_B T$

MSQ (Multiple Select Questions)

Ans. 25: (b), (c) and (d)

Solution: Distinguishable particles obey Maxwell-Boltzmann statistics

Ans. 26: (a), (c) and (d)

Solution: In any quantum state only one fermions can be adjusted

Ans. 27: (a), (c) and (d)

Solution: Fermions and bosons are indistinguishable ,electrons and protons are fermions but photons are bosons .

Ans. 28: (b) and (d)

Solution: The chemical potential of an ideal Bose gas at any temperature either zero or negative.(zero in case of photon).

Ans. 29: (a), (b) and (d)

Solution: If fermions have spin $\dfrac{3}{2}$ then $s_z = -\dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{3}{2}$.

Due to Pauli exclusion principle only one fermions can be adjusted in one quantum sate $\left| n, s, s_z \right\rangle$

so degeneracy of one particular energy state is 4 ,so 4 out of 6 fermions will adjust in ground state and rest two will in first excited state .

so energy is $4 \times \dfrac{\hbar\omega}{2} + 2 \times \dfrac{3\hbar\omega}{2} = 5\hbar\omega$

Ans. 30: (b) and (d)

Ans. 31: (a), (b) and (c)

Solution: (a)Two dimensional free particle has partition function $Z = \frac{2\pi m k_B T}{h^2}$,

$$U = k_B T^2 \frac{\partial \ln Z}{\partial T} = k_B T$$

(b) The partition function of one dimensional harmonic oscillator at equilibrium temperature T is $Z = \frac{2\pi k_B T}{h\omega}$ The internal energy $U = k_B T^2 \frac{\partial \ln Z}{\partial T} = k_B T$

(c) the partition function for photon in one dimension is $Z = \frac{L k_B T}{hc}$ $U = k_B T^2 \frac{\partial \ln Z}{\partial T} = k_B T$

(d) the partition function for two dimension box $Z = A\left(\frac{k_B T}{hc}\right)^2$ $U = k_B T^2 \frac{\partial \ln Z}{\partial T} = 2k_B T$

Ans. 32: (a), (b) and (d)

6. Basic Nuclear Properties

An ordinary hydrogen atom has as its nucleus a single proton, whose charge is $+e$ and whose mass is 1836 times that of the electron. All other elements have nuclei that contain neutrons as well as protons. As its name suggests, the neutron is uncharged; its mass is slightly greater than that of the proton. Neutrons and protons are jointly called **nucleons**.

The **atomic number** of an element is the number of protons in each of its nuclei, which is the same as the number of electrons in a neutral atom of the element. Thus atomic number of hydrogen is 1, of helium 2, of lithium 3, and of uranium 92. All nuclei of a given element do not necessarily have equal numbers of neutrons. For instance, although over 99.9 percent of hydrogen nuclei are just single protons, a few also contain a neutron, and a very few two neutrons, along with the protons. The varieties of an element that differ in the numbers of neutrons their nuclei contain are called **isotopes**.

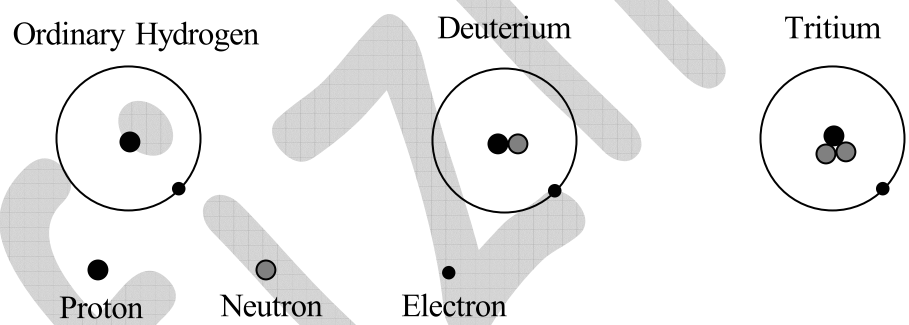


Figure: The isotope of hydrogen

The conventional symbols for nuclear species, or nuclides, follow the pattern ${}_{Z}^{A}X$, where

$X =$ Chemical symbol of the element

$Z =$ Atomic number of the element

$=$ Number of protons in the nucleus

$A =$ Mass number of the nuclide

$=$ Number of nucleons in the nucleus

Nuclear Terminology

- **Isotopes**

If two nuclei have same atomic number Z (proton), then they are called as isotopes.

Example: ${}^{13}_{6}C$ & ${}^{14}_{6}C$, ${}^{16}_{8}O$ & ${}^{17}_{8}O$ and ${}^{1}_{1}H$, ${}^{2}_{1}H$, ${}^{3}_{1}H$

- **Isotones**

If two nuclei have same neutron number N (proton), then they are called as isotones.

Example: ${}^{13}_{6}C$ and ${}^{14}_{7}N$

- **Isobars**

If two nuclei have same mass number A, then they are called as isobars.

Example: ${}^{14}_{6}C$ and ${}^{14}_{7}N$

- **Mirror nuclei**

Nuclei with same mass number A but with proton and neutron number interchanged and their difference is ± 1.

Example: ${}^{11}_{6}C$ & ${}^{11}_{5}B$ and ${}^{13}_{7}N$ & ${}^{13}_{6}C$

Atomic masses: Atomic masses refer to the masses of neutral atoms, not of bare nuclei. Thus an atomic mass always includes the masses of Z electrons. Atomic masses are expressed in **mass units** (u), which are so defined that the mass of a ${}^{12}_{6}C$ atom is exactly $12u$. The value of mass unit is $1u = 1.66054 \times 10^{-27}\, kg \approx 931.4\, MeV$.

Some rest masses in various units are:

Particle	**Mass(kg)**	**Mass(u)**	**Mass(MeV/c^2)**
Proton	1.6726×10^{-27}	1.007276	938.28
Neutron	1.6750×10^{-27}	1.008665	939.57
Electron	9.1095×10^{-31}	5.486×10^{-4}	0.511
${}_{1}H^{1}$	1.6736×10^{-27}	1.007825	938.79

6.1 Size and Density

Majority of atomic nuclei have spherical shape and only very few show departure from spherical symmetry. For spherically symmetrical nuclei, nuclear radius is given by

$$R = R_0 A^{1/3}$$

where A is the mass number and $R_0 = (1.2 \pm 0.1) \times 10^{-15} m \approx 1.2 \ fm$.

R varies slightly from one nucleus to another but is roughly constant for $A > 20$.

The radius of ${}^{12}_{6}\text{C}$ nucleus is

$$R = (1.2)(12)^{1/3} \approx 2.7\, fm$$

Example: The radius of Ge nucleus is measured to be twice the radius of ${}^{9}_{4}\text{Be}$. How many nucleons are there in Ge nucleus?

Solution: $R = R_o (A)^{1/3} \Rightarrow R_{Ge} = 2R_{Be} \Rightarrow R_o (A)^{1/3} = 2R_o (9)^{1/3} \Rightarrow A = 72$

Nuclear Density

Assuming spherical symmetry, volume of nucleus is given by

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A.$$

Mass of one proton = 1.67×10^{-27} kg, Nuclear Mass = $A \times 1.67 \times 10^{-27}$ kg.

$$\text{Nuclear density} = \frac{A \times 1.67 \times 10^{-27}}{\frac{4}{3}\pi R_0^3 \times A} \approx 10^{17}\,\text{kg/m}^3$$

$$\text{Nuclear Particle Density} = \frac{\text{Nuclear Mass Density}}{\text{Nuclear Mass}} = \frac{10^{17}\ \text{Kg/m}^3}{1.67 \times 10^{-27}\ \text{Kg/Nucleon}}$$

$$= 10^{44}\ \text{Nucleons/m}^3$$

6.2 Spin and Magnetic Moment

Proton and neutrons, like electrons, are fermions with spin quantum numbers of $s = \frac{1}{2}$.

This means they have spin angular momenta $\vec{S}$ of magnitude

$$\left|\vec{S}\right| = \sqrt{s(s+1)}\hbar = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\hbar = \frac{\sqrt{3}}{2}\hbar$$

and spin magnetic quantum number of $m_s = \pm\frac{1}{2}$.

As in the case of electrons, magnetic moments are associated with the spins of protons and neutrons. In nuclear physics magnetic moments are expressed in **nuclear magnetons** (μ_N), where

Nuclear magneton $\mu_N = \dfrac{e\hbar}{2m_p} = 5.051\times10^{-27}\,J/T = 3.152\times10^{-8}\,eV/T$ where m_p is the proton mass.

In atomic physics we have defined **Bohr magneton** $\mu_B = \dfrac{e\hbar}{2m_e}$ where m_e is the electron mass.

The nuclear magneton is smaller than the Bohr magneton by the ratio of the proton mass to the electron mass which is 1836.

$$\frac{\mu_B}{\mu_N} = \frac{m_p}{m_e} = 1836.$$

The spin magnetic moments of the proton and neutron have components in any direction of

Proton $\mu_{pz} = \pm 2.793\mu_N$

Neutron $\mu_{nz} = \mp 1.913\mu_N$

There are two possibilities for the signs of μ_{pz} and μ_{nz}, depending on whether m_s is $-\frac{1}{2}$ or $+\frac{1}{2}$. The $\pm$ sign is used for μ_{pz} because $\vec{\mu}_{pz}$ is in the same direction as the spin $\vec{S}$, whereas $\mp$ is used for μ_{nz} because $\vec{\mu}_{nz}$ is opposite to spin $\vec{S}$.

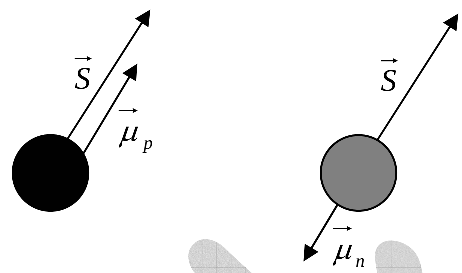


Figure: Spin magnetic moment ($\vec{\mu}$) and spin angular momentum ($\vec{S}$) directions for neutron and protons.

Note: For neutron, magnetic moment is expected to be zero as $e = 0$ but $\vec{\mu}_{nz} = \mp 1.913\mu_N$.

At first glance it seems odd that the neutron, with no net charge, has spin magnetic moment. But if we assume that the neutron contains equal amounts of positive and negative charge, a spin magnetic moment arise if these charges are not uniformly distributed. Thus we can say that neutron has physical significance of negative charges because magnetic moment is opposite to that of its intrinsic spin angular momentum.

6.3 Angular Momentum of Nucleus

The hydrogen nucleus $^1_1\mathrm{H}$ consists of a single proton and its total angular momentum is given by $\left|\vec{S}\right| = \frac{\sqrt{3}}{2}\hbar$. A nucleon in a more complex nucleus may have orbital angular momentum due to motion inside the nucleus as well as spin angular momentum. The total angular momentum of such a nucleus is the vector sum of the spin and orbital angular momenta of its nucleons, as in the analogous case of the electrons of an atom.

When a nucleus whose magnetic moment has z component μ_z is in a constant magnetic field $\vec{B}$, the magnetic potential energy of the nucleus is

Magnetic energy $U_m = -\mu_z B$

The energy is negative when $\vec{\mu}_z$ is in the same direction as $\vec{B}$ and positive when $\vec{\mu}_z$ is opposite to $\vec{B}$. In a magnetic field, each angular momentum state of the nucleus is therefore split into components, just as in the Zeeman Effect in atomic electron states. Figure below shows the splitting when the angular momentum of the nucleus is due to the spin of a single proton.

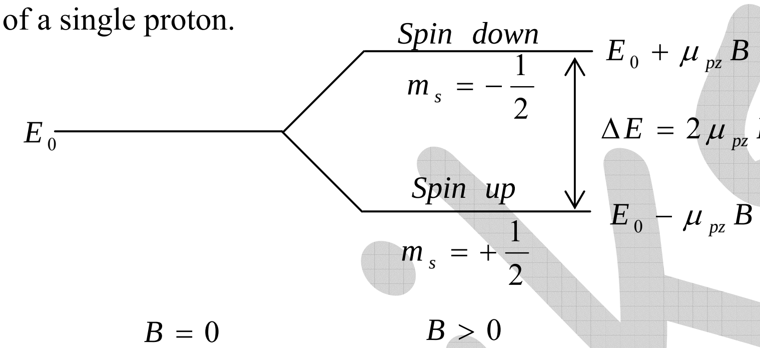


Figure: The energy levels of a proton in a magnetic field are split into spin-up and spin-down sublevels.

The energy difference between the sublevels is $\Delta E = 2\mu_{pz} B$

A photon with this energy will be emitted when a proton in the upper state flips its spin to fall to the lower state. A proton in the lower state can be raised to upper one by absorbing a photon of this energy. The photon frequency ν_L that corresponds to ΔE is

Larmor frequency for photons $$\nu_L = \frac{\Delta E}{h} = \frac{2\mu_{pz} B}{h}$$

This is equal to the frequency with which a magnetic dipole precesses around a magnetic field.

6.4 Stable Nuclei

Not all combination of neutrons and protons form stable nuclei. In general, light nuclei $(A < 20)$ contain equal numbers of neutrons and protons, while in heavier nuclei the proportion of neutrons becomes progressively greater. This is evident in figure as shown below, which is plot of N versus Z for stable nuclides.

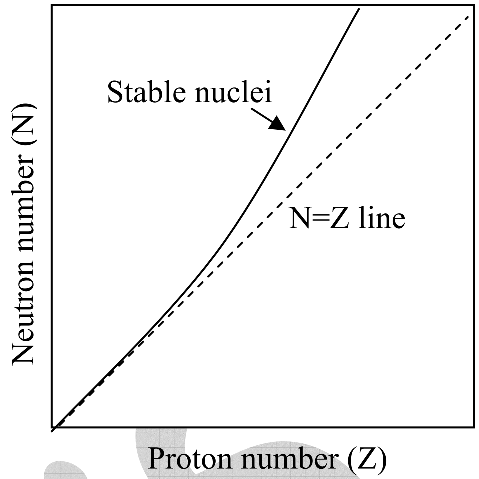


Figure: Neutron-proton diagram for stable nuclides.

The tendency for N to equal Z follows from the existence of nuclear energy levels. Nucleons, which have spin ½, obey exclusion principle. As a result, each energy level can contain two neutrons of opposite spins and two protons of opposite spins. Energy levels in nuclei are filled in sequence, just as energy levels in atoms are, to achieve configurations of minimum energy and therefore maximum stability. Thus the boron isotope ${}^{12}_{5}\mathrm{B}$ has more energy than the carbon isotope ${}^{12}_{6}\mathrm{C}$ because one of its neutrons is in a higher energy level, and ${}^{12}_{5}\mathrm{B}$ is accordingly unstable. If created in a nuclear reaction, a ${}^{12}_{5}\mathrm{B}$ nucleus changes by beta decay into a stable ${}^{12}_{6}\mathrm{C}$ nucleus in a fraction of second.

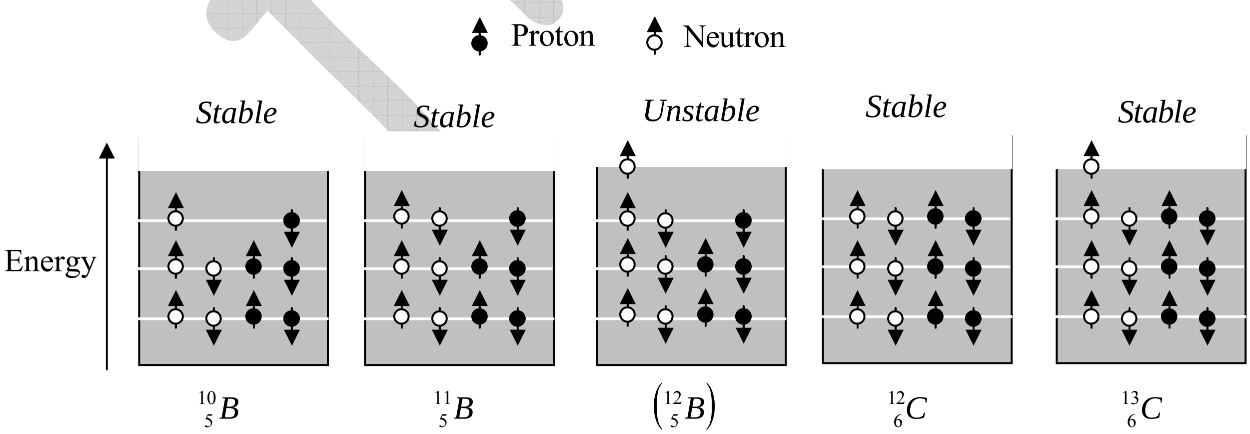


Figure: Simplified energy level diagrams of some boron and carbon isotopes.

The preceding argument is only part of the story. Protons are positively charged and repel one another electrically. This repulsion becomes so great in nuclei with more than 10 protons or so that an excess of neutrons, which produce only attractive forces is required for stability. Thus the curve departs more and more from $N = Z$ line as Z increases.

Sixty percent of stable nuclides have both even Z and even N; these are called **"even-even"** nuclides. Nearly all the others have either even Z and odd N (**"even-odd"** nuclides) or odd Z and even N (**"odd-even"** nuclides) with the numbers of both kinds being about equal. Only five stable **odd-odd** nuclides are known: ${}_1^2\text{H}$, ${}_3^6\text{Li}$, ${}_5^{10}\text{B}$, ${}_7^{14}\text{N}$ and ${}_{73}^{180}\text{Ta}$. Nuclear abundances follow a similar pattern of favoring even numbers for Z and N.

These observations are consistent with the presence of nuclear energy levels that can each contain two particles of opposite spin. Nuclei with filled levels have less tendency to pick up other nucleons than those with partially filled levels and hence were less likely to participate in the nuclear reactions involved in the formation of elements.

Nuclear forces are limited in range, and as a result nucleons interact strongly only with their nearest neighbors. This effect is referred to as the **saturation** of nuclear forces. Because the coulomb repulsion of protons is appreciable throughout the entire nucleus, there is a limit to the ability of neutrons to prevent the disruption of large nucleus. This limit is represented by the bismuth isotope ${}_{83}^{209}\text{Bi}$, which is the **heaviest stable** nuclide.

All nuclei with $Z > 83$ and $A > 209$ spontaneously transform themselves lighter ones through the emission of one or more alpha particles, which are ${}_2^4\text{He}$ nuclei:

Alpha decay $\quad {}_Z^A\text{X} \rightarrow {}_{Z-2}^{A-4}\text{Y} + {}_2^4\text{He}$

Since an alpha particle consists of two protons and two neutrons, an alpha decay reduces the Z and N of the original nucleus by two each. If the resulting daughter nucleus has either too small or too large a neutron/proton ratio for stability, it may beta-decay to a more appropriate configuration.

In negative beta decay, a neutron is transformed into a proton and an electron is emitted:

Negative beta decay $n^0 \rightarrow p^+ + e^-$

In positive beta decay, a proton becomes a neutron and a positron is emitted:

Positron emission $p^+ \rightarrow n^0 + e^+$

Thus negative beta decay decreases the proportion of neutrons and positive beta decay increases it. A process that competes with positron emission is the capture by a nucleus of an electron from its innermost shell. The electron is absorbed by a nuclear proton which is thereby transformed into neutron:

Electron Capture $p^+ + e^- \rightarrow n^0$

Figure below shows how alpha and beta decays enable stability to be achieved.

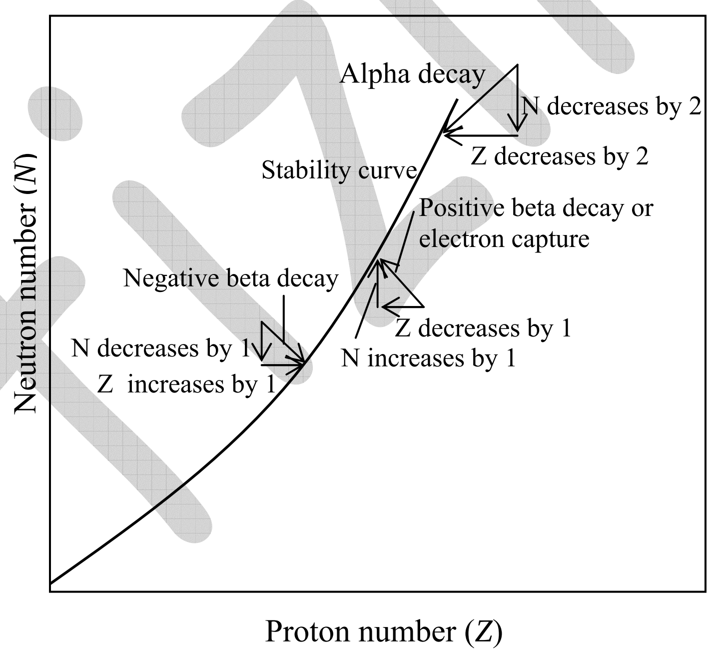


Figure: Alpha and beta decays permit an unstable nucleus to reach a stable configuration.

6.5 Binding Energy

When nuclear masses are measured, it is found that they are less than the sum of the masses of the neutrons and protons of which they are composed. This is in agreement with Einstein's theory of relativity, according to which the mass of a system bound by energy B is less than the mass of its constituents by B/c^2 (where c is the velocity of light).

The Binding energy B of a nucleus is defined as the difference between the energy of the constituent particles and of the whole nucleus. For a nucleus of atom ${}_Z^AX$,

$$B = \left[ZM_p + NM_N - {}_Z^AM\right]c^2 = [ZM_H + NM_N - M({}_Z^AX)]c^2$$

If mass is expressed in atomic mass unit

$$B = \left[ZM_p + NM_N - {}_Z^AM\right]\times 931.5\,MeV = [ZM_H + NM_N - M({}_Z^AX)]\times 931.5\,MeV$$

M_p : Mass of free proton, M_N : M_N: Mass of free neutron,

M_H : mass of hydrogen atom ${}_Z^AM$: mass of the nucleus,

Z : Number of proton, N : Number of neutron,

$M\left({}_Z^AX\right)$: mass of atom.

6.5.1 Binding Energy per Nucleon

The **binding energy per nucleon** for a given nucleus is found by dividing its total binding energy by the number of nucleon it contains. Thus binding energy per nucleon is

$$\frac{B}{A} = \frac{c^2}{A}\left[ZM_p + NM_N - {}_Z^AM\right] = \frac{c^2}{A}[ZM_H + NM_N - M({}_Z^AX)]$$

The binding energy per nucleon for ${}_1^2\text{H}$ is $\frac{2.224}{2} = 1.112\text{ MeV}/\text{nucleon}$ and for ${}_{63}^{209}\text{Bi}$ it is $\frac{1640\text{ MeV}}{209} = 7.8\text{ MeV}/\text{nucleon}$.

Figure below shows the binding energy per nucleon against the number of nucleons in various atomic nuclei.

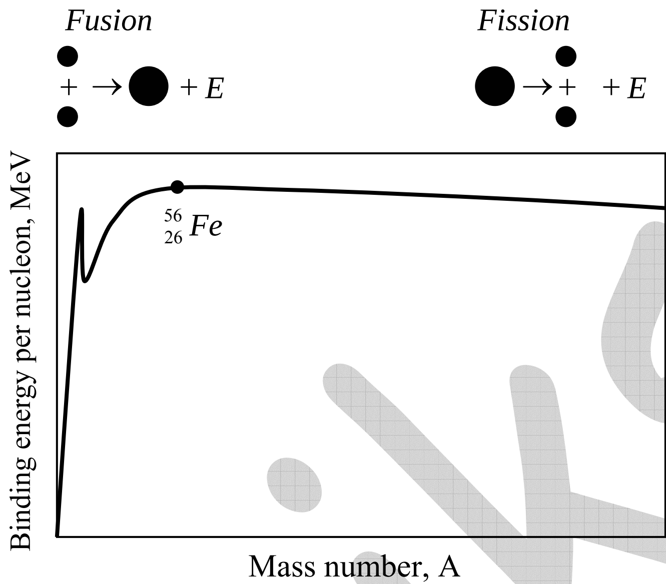


Figure: Binding energy per nucleon as function of mass number.

The greater the binding energy per nucleon, the more stable the nucleus is. The graph has the maximum of 8.8 MeV / nucleon when the number of nucleons is 56. The nucleus that has 56 protons and neutrons is $^{56}_{26}\text{Fe}$ an iron isotope. This is the most stable nucleus of them all, since the most energy is needed to pull a nucleon away from it.

Two remarkable conclusions can be drawn from the above graph.

(i) If we can somehow split a heavy nucleus into two medium sized ones, each of the new nuclei will have more binding energy per nucleon than the original nucleus did. The extra energy will be given off, and it can be a lot. For instance, if the uranium nucleus $^{235}_{92}\text{U}$ is broken into two smaller nuclei, the binding energy difference per nucleon is about 0.8 MeV. The total energy given off is therefore

$$\left(0.8\,\frac{\text{MeV}}{\text{nucleon}}\right)\left(235\ \ \text{nucleon}\right)=188\,\text{MeV}$$

This process is called as **nuclear fission**.

(ii) If we can somehow join two light nuclei together to give a single nucleus of medium size also means more binding energy per nucleon in the new nucleus. For instance, if two $_1^2\text{H}$ deuterium nuclei combine to form a $_2^4\text{He}$ helium nucleus, over 23 MeV is released. Such a process, called **nuclear fusion**, is also very effective way to obtain energy. In fact, nuclear fusion is the main energy source of the sun and other stars.

Example: The measured mass of deuteron atom $\left(_1^2\text{H}\right)$, Hydrogen atom $\left(_1^1\text{H}\right)$, proton and neutron is $2.01649\ u$, $1.00782\ u$, $1.00727\ u$ and $1.00866\ u$. Find the binding energy of the deuteron nucleus (unit $MeV/nucleon$).

Solution: Here $A=2,\ Z=1,\ N=1$

$$B.E.=[ZM_H+NM_N-M(_1^2H)]\times 931.5\,MeV$$
$$=[1\times 1.00782+1\times 1.00866-2.01649]\times 931.5\,MeV$$
$$=[0.00238]\times 931.5\,MeV=2.224\,MeV$$

Example: The binding energy of the neon isotope $_{10}^{20}\text{Ne}$ is 160.647 MeV. Find its atomic mass.

Solution: Here $A=10,\ Z=10,\ N=10$

$$\text{M}\ (_\text{Z}^\text{A}\text{X})=\left[\text{ZM}_\text{H}+\text{NM}_\text{N}\right]-\frac{\text{B}}{931.5\,\text{MeV/u}}$$

$$\text{M}\ (_{10}^{20}\text{Ne})=\left[10\left(1.00782\right)+10\left(1.00866\right)\right]-\frac{160.647}{931.5\,\text{MeV/u}}=19.992\,\text{u}$$

Example:

(a) Find the energy needed to remove a neutron from the nucleus of the calcium isotope $_{20}^{42}Ca$.

(b) Find the energy needed to remove a proton from this nucleus.

(c) Why are these energies different?

Given: atomic masses of $_{20}^{42}Ca=41.958622u$, $_{20}^{41}Ca=40.962278u$, $_{19}^{41}K=40.961825u$, and mass of $_0^1n=1.008665u$, $_1^1p=1.007276u$.

Solution:

(a) ${}^{42}_{20}Ca \to {}^{41}_{20}Ca + {}^{1}_{0}n$ **;**

Total mass of the ${}^{41}_{20}Ca$ *and* ${}^{1}_{0}n = 41.970943\ u$

Mass defect $\Delta m = 41.970943 - 41.958622 = 0.012321\,u$

So, B.E. of missing neutron $= \Delta m \times 931.5 = 11.48\ MeV$

(b) ${}^{42}_{20}Ca \to {}^{41}_{19}K + {}^{1}_{1}p$ **;**

Total mass of the ${}^{41}_{19}K$ *and* ${}^{1}_{1}p = 41.919101\ u$

Mass defect $\Delta m = 41.919101 - 41.958622 = 0.010479\,u$

So, B.E. of missing protron $= \Delta m \times 931.5 = 10.27\ MeV$

(c) The neutron was acted upon only by attractive nuclear forces whereas the proton was also acted upon by repulsive electric forces that decrease its binding energy.

6.6 Salient Features of Nuclear Forces

Nucleus is bounded by nuclear forces. The basic properties of nuclear forces are

(i) It is a short range attractive force.

(ii) It is in general non-central force.

(iii) They have property of saturation i.e. each nucleon interacts only with its nearest neighbors and not with all the constituents in the nucleus. This is apparent from the fact that average $B.E.$ per nuclear remains approximately constant i.e.

$$B.E. \propto A$$

When all interactions are possible then $\frac{A(A-1)}{2}$ interaction may take place;

$$B.E. \propto A^2 \text{ is not valid.}$$

(iv) They are charge independent i.e. $n-n, p-p$ and $n-p$ have same nuclear force.

(v) They are spin dependent force as shown by deuteron.

(vi) They are exchange forces proposed by Yukawa(Meson Theory).

7. Radio Active Decay

Despite the strength of the forces that holds nucleons together to form an atomic nucleus, many nuclides are unstable and spontaneously change into other nuclides by radioactive decay.

Five kinds	**Example**	**Reason for instability**
Alpha decay	$_Z^AX \rightarrow _{Z-2}^{A-4}Y + _2^4He$ Emission of α-particle reduces size of nucleus.	Nucleus is too large.
Beta decay	$_Z^AX \rightarrow _{Z+1}^{A}Y + e^-$ Emission of electron by neutron in nucleus changes the neutron to a proton.	Nucleus has too many neutrons relative to number of protons.
Gamma decay	$_Z^AX^* \rightarrow _Z^AX + \gamma$ Emission of γ-ray reduces energy of the nucleus.	Nucleus has excess energy.
Electron capture	$_Z^AX + e^- \rightarrow _{Z-1}^{A}Y$ Capture of electron by protons changes the proton to a neutron.	Nucleus has too many protons relative to number of neutrons.
Positron emission	$_Z^AX \rightarrow _{Z-1}^{A}Y + e^+$ Emission of positron by proton in nucleus changes the proton to a neutron.	Nucleus has too many protons relative to number of neutrons.

7.1 Alpha Decay

Nuclei which contain 210 or more nucleons are so large that the short range nuclear forces that hold them together are barely able to counterbalance the mutual repulsion of their protons. Alpha decay occurs in such nuclei as a means of increasing their stability by reducing their size

$$^{A}_{Z}X \rightarrow ^{A-4}_{Z-2}Y + ^{4}_{2}He$$

To escape from nucleus, a particle must have K.E., and only the alpha particle mass is sufficiently smaller than that of its constituent nucleons for such energy to be available (α-particle have high B.E. as compared to proton or $^{3}_{2}He$ nuclei).

The energy Q-released when various particles are emitted by a heavy nucleus is,

i.e.

Disintegration energy $Q = \left(m_i - m_f - m_x\right)c^2$ where

$m_i =$ Mass of initial nuclei,

$m_f =$ mass of final nuclei,

$m_x =$ α-particle mass

The KE_α of the emitted α-particle is never quite equal to Q, since momentum must be conserved, the nucleus recoils with a small amount of kinetic energy when the α-particle emerges. Thus

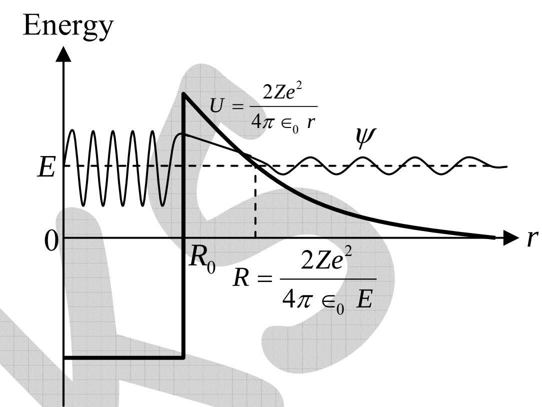
$$KE_\alpha \approx \left(\frac{A-4}{A}\right)Q$$

since $A \geq 210$, most of the disintegration energy appears as the K.E. of the α-particle.

7.1.1 Tunnel Theory of α-decay: (How α-particle can actually escape the nucleus)

The height of the potential barrier is $\approx 25\ MeV$, which is equal to the work that must be done against the repulsive electric force to bring an α-particle from infinity to a position adjacent to the nucleus but just outside the range of its attractive forces.

We may therefore regard an α-particle in such a nucleus as being inside a box whose box requires energy of $25\ MeV$ to be surmounted. However, decay α-particles have energies that range from 4 to $9\ MeV$, depending on the particular nuclide involved, 16 to $21\ MeV$ short of the energy needed for escape.



Although α-decay is inexplicable classically, quantum mechanics provides a straight forward explanation. The basic notions of this theory are:

An α-particle may exist as an entity within a heavy nucleus. Such a particle is in constant motion and is held in the nucleus by potential barrier. There is a small but definite-likelihood that the particle may tunnel through the barrier (despite its height) each time a collision with it occurs.

The decay probability per unit time, i.e decay constant $\lambda = \nu T$

Where ν = number of times per second an α-particle within a nucleus strikes the potential barrier around it and

$T = e^{-2k_2 L}$ = Probability that the particle will be transmitted through the barrier.

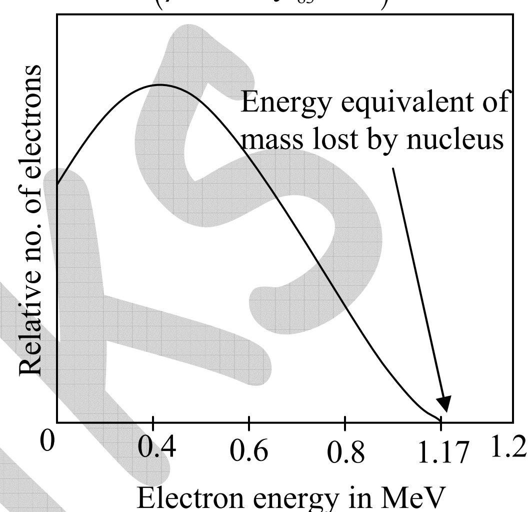
L is the width of the barrier, wave number inside the barrier $k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$, where E is the K.E. , U is height of the barrier and m is the mass of α-particle.

7.2 Beta Decay

It is a means whereby a nucleus can alter its composition to become more stable. The conservation principles of energy, linear momentum and angular momentum are all apparently violated in beta decay: $n \to p + e^-$

(i) The electron energies observed in the β^- decay of a particular nuclide are found to vary continuously from 0 to maximum value $KE_{\max}$ characteristic of the nuclide. The maximum energy $E_{\max} = \left(m_0c^2 + KE_{\max}\right)$ carried by the decay electron is equal to the energy equivalent of the mass difference between the parent and daughter nuclei. However an emitted electron is rarely found with energy of $KE_{\max}$.

$\left(\beta - \text{decay } {}_{83}Bi^{210}\right)$



(ii) When the directions of emitted electron and of the recoiling nuclei are observed, they are almost never exactly opposite as required for linear momentum to be conserved.

(iii) The spins of the neutron, proton and electron are all ½. If beta decay involved just a neutron becoming a proton and an electron, spin (and hence angular momentum) is not conserved.

In 1930 Pauli proposed a "desperate remedy": If an uncharged particle of small or zero rest mass and spin ½ is emitted in $\beta^- -$decay together with the electron, the above discrepancies would not occur. This particle is called **neutrino** which would carry off energy equal to the difference between $KE_{\max}$ and actual $K.E$ of the electron (the recoiling nucleus carry away negligible $K.E$). The neutrino's linear momentum also exactly balances those of the electron and the recoiling daughter nucleus.

Thus in ordinary $\beta^- -$decay $\quad n \to p + e^- + \bar{\nu}$ (also possible outside the nucleus)

The interaction of neutrinos with matter is extremely feeble. The only interaction with matter a neutrino can experience is through a process called **inverse beta decay** with extremely low probability $p + \bar{\nu} \to n + e^+$ and $n + \nu \to p + e^-$.

Note: Parity violates in β^- − decay.

7.2.1 Positron emission

It is the conversion of a nuclear proton into a neutron, a positron and a neutrino:

$$p \to n + e^+ + \nu \qquad \text{(Possible only within a nucleus)}$$

7.2.2 Electron capture

It is closely connected with positron emission. In electron capture a nucleus absorbs one of its inner atomic electron, with the result that a nuclear proton becomes a neutron and neutrino is emitted:

$$p + e^- \to n + \nu .$$

Usually the absorbed electron comes from the *K*-shell, and an *X*-ray photon is emitted when one of the atoms outer electrons falls into the resulting vacant state. The wavelength of the photon will be one of those characteristic of daughter element, not of the original one, and the process can be recognized on that basis.

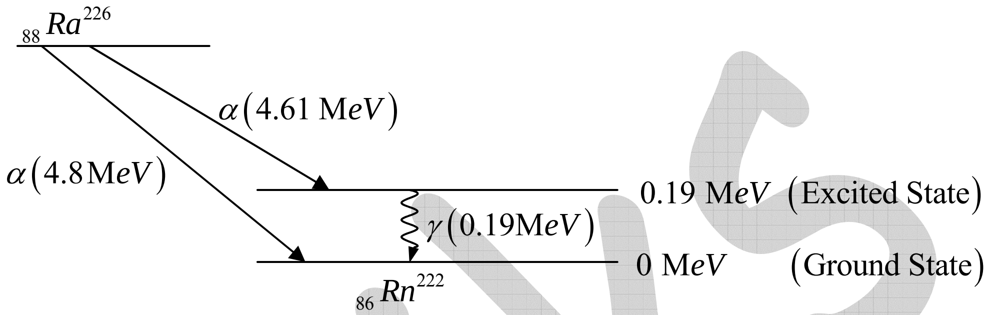
Note:

1. Electron capture is competitive with positron emission since both processes lead to the same nuclear transformation.

2. Electron capture occurs more often than positron emission in heavy nuclides because the electrons in such nuclides are relatively close to the nucleus, which promotes their interaction with it.

7.3 Gamma Decay

Nuclei can exist in definite energy levels just as an atom can. Due to α or β-emission, nuclei get into an excited state. These excited nuclei return to their ground state by emitting photons whose energies correspond to energy difference between the various initial and final states in the transition involved called γ-ray.



γ-rays characteristics

1. It is an electromagnetic wave.

2. Very short wavelength ($\approx 400\text{A}^{\text{o}}$ to 0.4A^{o}).

3. No electric charge and so not detected by magnetic and electric field.

When a beam of γ-rays photons passes through matter, the intensity of beam decreases exponentially i.e. $I = I_0 e^{-\mu x}$ where I_0: Initial Intensity, μ: absorption coefficient of substance, x : thickness of absorber.

7.3.1 Various processes by which γ-rays can lose its energy

Three separate processes responsible for the decrease in intensity of γ-rays.

1. Photoelectric absorption

In this all the energy of γ-ray photon is transferred to a bound electron and γ-ray photon ceases to exist. The ejected electron may either escape from the absorber or may get reabsorbed due to collision. At low photon energies (8 *KeV* for *Al* and 500 *KeV* for *Pb*) the photoelectric effect is chiefly responsible for the γ-ray absorption.

2. Compton Scattering

At energies in neighborhood of 1 MeV, Compton Scattering becomes the chief cause of removal of photons form the γ-ray beam.

3. Pair production

At high enough energies pair production becomes important. In this a γ-ray photon passing close to an atomic nucleus in the absorbing matter disappears and an electron-positron pair is created: $\gamma \rightarrow e^{-} + e^{+}$

The charge is conserved in the reaction. The rest mass m_0 and hence the rest mass energy of e^- and e^+ are same i.e. $0.51\ MeV$. The energy of the γ-ray photon must be at least $2 \times 0.51\ MeV = 1.02\ MeV$ for pair production to be possible. If $h\nu$ greater than $1.02\ MeV$, the balance of the energy appears as $K.E.$ of particles.

7.3.2 Internal Conversion

"Process of Internal Conversion is an alternative to γ-decay". Internal conversion is a process which enables an excited nuclear state to come down to some lower state without the emission of γ-photon. The energy ΔE involved in this nuclear transition gets transferred directly to a bound electron of the atom. Such an electron gets knocked out of the atom. Electrons like this are called "internal conversion" electrons.

This probability is highest for the K-shell electrons which are closest to the nucleus. For such a case, the nucleus may not de-excite by γ-emission but by giving the excitation energy ΔE directly to a K-shell electron. Internal conversion is also possible (though less, as compared to K-shell) for higher atomic shells L, M etc.

The kinetic energy of the converted electron is $K_e = \Delta E - B_e$,

where $\Delta E = E_i - E_f =$ Nuclear excitation energy between initial state i (higher) and final state f (lower) and $B_e =$ atomic binding energy of electron.

We know that the β-spectrum is continuous; usually this continuous $\beta-$ spectra are superimposed by discrete lines due to conversion electrons. These lines are called 'internal conversion' lines.

γ-ray emission and internal conversion are competing process for de-excitation of nucleus.

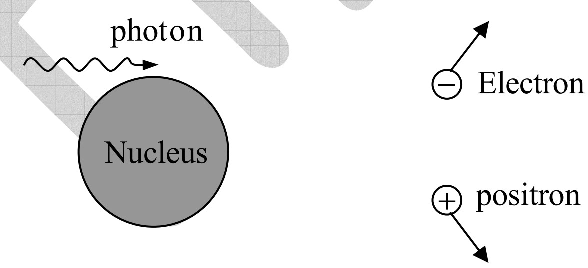
If we neglect the small recoil energy of the γ-emitter nucleus, the energy of the γ-ray is given by $h\nu = \Delta E = E_i - E_f$; where ν is the frequency of the γ-photon.

7.3.3 Pair Production (Energy into matter)

In a collision a photon can give an electron all of its energy (the photoelectric effect) or only part (the Compton Effect). It is also possible for a photon to materialize into an electron and a positron. In this process, electromagnetic energy is converted into matter. This process is called pair production.

No conservation principles are violated when an electron-positron pair is created near an atomic nucleus.

The rest energy m_0c^2 of an electron or positron is $0.51\ MeV$, hence pair production requires photon energy of at least $1.02\ MeV$. Any additional photon energy becomes *K.E.* of the electron and positron.



7.3.4 Pair Annihilation

The inverse of pair production occurs when a positron is near an electron and the two come together under the influence of their opposite electric charges. Both particles vanish simultaneously with the lost mass becoming energy in the form of two gamma ray photon.

$$e^{+} + e^{-} \rightarrow \gamma + \gamma$$

The total mass of the positron and electron is equivalent to $1.02\ MeV$, and each photon has energy $h\nu$ of $0.51\ MeV$ plus half the $K.E.$ of the particles relative to their center of mass.

Note:

1. The directions of the photons are such as to conserve both energy and linear momentum.

2. No nucleus or other particles is needed for this pair annihilation to take place.

7.2.5 Massbauer Effect

"It is the recoilless emission and absorption of photon"

The emission of gamma rays is generally accompanied by the emission of an α or β particle. If after the emission of an α or β particle the product nucleus is left in an excited state, it reaches the ground state by releasing or emitting photons called γ-rays. When a nucleus emits a photon it recoils in the opposite direction. This reduces the energy of the γ-ray from its usual transition energy E_0 to $E_0 - R$, where R is the recoil energy.

Mossbauer Effect almost eliminates the energy of recoil by using solid state properties of a crystal lattice. Also, such recoil-less emission of γ-rays makes it possible to construct a source of essentially mono-energetic and hence monochromatic photons.

The isotope of iron, Fe^{57} is the most often used nucleus to study Mossbauer Effect.

7.4 Activity

The activity of a sample of any radioactive nuclide is the rate at which the nuclei of its constituent atoms decay. If N is the number of nuclei present in the sample at a certain time, its activity R is given by

$$R = -\frac{dN}{dt}, \text{ SI unit is Becquerel.}$$

1 Becquerel = 1 Bq = 1 decay/sec.

The traditional unit of activity is the curie (Ci),

1 Curie = 3.7×10^{10} decay/sec = 37 GBq (1Ci is activity of 1 g of radium $^{226}_{88}Ra$)

Let λ be the probability per unit time for the decay of each nucleus of given nuclide.

Then λdt is the probability that any nucleus will undergo decay in a time interval dt . If a sample contains N undecayed nuclei, the number dN that decay in a time dt is

$$dN = -N\lambda dt$$

$$\Rightarrow R = -\frac{dN}{dt} = \lambda N$$

$$\int_0^{N_0} \frac{dN}{N} = -\int_0^t \lambda dt \Rightarrow N = N_0 e^{-\lambda t}$$

Half Life: $T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} \quad \left(\text{At} \quad t = T_{1/2}, \quad N = \frac{N_0}{2}\right)$

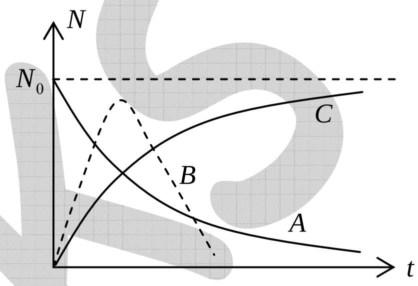
Mean Time: $\bar{T} = \frac{1}{\lambda} = 1.44T_{1/2}$

7.4.1 Successive Growth and Decay Process

In a successive growth/decay process $A \to B \to C$, element C is a stable nucleus. The following parameters are given for the process:

$$\begin{array}{cccc} & A \xrightarrow{\lambda_1} & B \xrightarrow{\lambda_2} & C \\ t=0 & N_0 & 0 & 0 \\ t=t & N_1 & N_2 & N_3 \end{array}$$

where λ_1 and λ_2 are decay constant, N_0 is the concentration of A at $t=0$ and N_1, N_2, N_3 are concentration of A, B, C at any time t.



Thus $$\boxed{N_1 = N_0 e^{-\lambda_1 t}}$$

Rate equation for B: $$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

$$\Rightarrow \frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_0 N_0 e^{-\lambda_1 t}$$

Multiply both side by $e^{\lambda_2 t} dt$ and then integrate $\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1 N_0 e^{(\lambda_2 - \lambda_1)}}{\lambda_2 - \lambda_1} + K$

At $t = 0, N_2 = 0 \Rightarrow K = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$, thus $\boxed{N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]}$

If C is a stable nucleus, the rate of decay of atoms of B into C i.e $\frac{dN_3}{dt}$ is given by,

$$\frac{dN_3}{dt} = -\lambda_2 N_2 \qquad (\text{At } t = 0,\ N_3 = 0)$$

$$\Rightarrow \boxed{N_3 = N_0\left[1 + \frac{\lambda_1 e^{-\lambda_2 t}}{(\lambda_2 - \lambda_1)} - \frac{\lambda_2 e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)}\right]}$$

Then the time at which concentration of intermediate member (B) will reach maxima is:

$$\left(\frac{dN_2}{dt}\right)_{t=t'} = 0 \Rightarrow \boxed{t' = \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}}$$

7.4.2 Branching

A given type of of nuclei will normally decay by one particular mode; say by emission of β-particles. But many cases have been found in which a smaller percentage of nuclei will decay by a different mode such as α-emission.

Let us denote the probability of α-emission by one nucleus, in dime dt emission by $\lambda_\alpha dt$ and that of β-emission by $\lambda_\beta dt$.

Then the probability of decay of a nucleus in time dt by either α or β-emission is: $\left(\lambda_\alpha + \lambda_\beta\right)dt$.

Hence the activity is $\frac{dN}{dt} = -\left(\lambda_\alpha + \lambda_\beta\right)N \Rightarrow N = N_0 e^{-\left(\lambda_\alpha + \lambda_\beta\right)t}$.

Giving mean life $\tau = \frac{1}{\left(\lambda_\alpha + \lambda_\beta\right)}$ and Branching Ratio $= \frac{\lambda_\alpha}{\lambda_\beta}$

7.4.3 Determination of the Age of the Earth

Let us consider successive growth decay process

$$U^{238} \xrightarrow{\lambda_U} X \longrightarrow Y \text{ - - - - - - - - - } \longrightarrow Pb^{206}$$

$\left(\text{Stable Product, } \lambda_{Pb} = 0\right)$

The half life of U^{238} is 4.5×10^9 *Years*. Hence after sufficient time the only element present in any appreciable amount will be uranium and lead.

$$\because N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$

Here $\lambda_1 = \lambda_U$, $\lambda_2 = \lambda_{Pb} = 0$, $N_2 = N_{Pb}$ and $N_0 = N_V$

Thus $\quad N_{Pb} = N_V\left[1 - e^{-\lambda_U t}\right]$

$N_0 = N_V =$ Present no. of Pb atoms+Present no. of U atoms $\Rightarrow N_V = N_{Pb} + N_U$

$$\Rightarrow t = \frac{1}{\lambda_U}\ln\left(\frac{N_{Pb} + N_U}{N_U}\right)$$

Example: Half life of P is 14.3 days. If you have 1.00 g of P today, then what would be the amount remaining in 10 days.

Solution: $t_{1/2} = \frac{0.693}{\lambda} \Rightarrow \lambda = \frac{0.693}{14.3} = 0.04847\ per\,day$

$N = N_o e^{-\lambda t} \Rightarrow N = (1.00)e^{-0.04847\times 10} \Rightarrow N = 0.616\ g$ or $N = 616\ mg$

Example: A radioactive nucleus has a half life of 100 years. If the number of nuclei $t = 0$ is N_0, then find the number of nuclei that have decayed in 300 years.

Solution:

Number of nuclei present after 300 year $\mathrm{N}=\mathrm{N_o}\left(\frac{1}{2}\right)^{T/T_{1/2}} \Rightarrow \mathrm{N'} = \mathrm{N_0}\text{-}\mathrm{N_0}\left(\frac{1}{2}\right)^3 = \frac{7}{8}\mathrm{N_0}$

Example: The atomic ratio between the uranium isotopes ^{238}U and ^{234}U in a mineral sample is found to be 1.8×10^4. The half life of ^{238}U is $4.5\times 10^9\ years$, then find the half life of ^{234}U.

Solution: $N_A\lambda_A = N_B\lambda_B \Rightarrow \frac{N_A}{N_B} = \frac{\lambda_B}{\lambda_A} = \frac{T_{1/2A}}{T_{1/2B}}$

$$\Rightarrow T_{1/2B} = \frac{N_B}{N_A}T_{1/2A} = \frac{1}{1.8\times 10^4}4.5\times 10^9 = 2.5\times 10^5\ years$$

Example: A radioactive sample contains 1.00 mg of radon ^{222}Rn, whose atomic mass is $222\,u$. The half life of the radon is $3.8\ day$. Then find the activity of the radon.

Solution: Decay constant $\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{3.8\times 24\times 60\times 60} = 2.1\times 10^{-6}\ \sec^{-1}$

Number of atoms in 1.00 mg is $N = \frac{1.00\times 10^{-6}\,kg}{(222u)\times 1.66\times 10^{-27}\,kg/u} = 2.7\times 10^{18}$ atoms

$$\text{or } N = \frac{10^{-3}}{222}\times 6.023\times 10^{23} = 2.7\times 10^{18} \text{ atoms}$$

Hence, activity $R = \lambda N = 2.1\times 10^{-6}\times 2.7\times 10^{18} = 5.7\times 10^{12}\ decay/\sec$

8. Nuclear Reaction

All the nuclear reactions are of the form

$$x + X \rightarrow Y + y \quad \text{or} \quad X\left(x, y\right)Y$$

The equation and notation both means that particle x strikes nucleus X to produce nucleus Y and particle y. The particles x and y may be elementary particles or γ-rays or they may themselves be nuclei, eg. α -particles or deuteron. In more general nuclear equations are shown $x + X \rightarrow \begin{cases} X + x \\ X^* + x \\ Y + y \\ Z + z \end{cases}$

The first equation $x + X \rightarrow X + x$ represents elastic scattering in which the total *K. E.* of the system, projectile plus target, is the same before the collision as after.

The second reaction $x + X \rightarrow X^* + x$ represents inelastic scattering, in which the target nucleus X is raised into an excited state X^*, and the total *K.E.* of the system is decreased by the amount of excited energy given to target nucleus.

The last two equations show a general nuclear equation.

8.1 Conservation Laws

In any nuclear reaction certain quantities must be conserved. The following conservation laws hold well during a nuclear reaction.

1. Conservation of Energy
2. Conservation of Linear Momentum
3. Conservation of charges
4. Conservation of Nucleons
5. Conservation of Angular Momentum
6. Conservation of Parity
7. Conservation of spin
8. Conservation of statistics
9. Conservation of Isobaric spins

8.1.1 The Quantities not conserved

These are magnetic dipoles moments and the electrical quadrupole moment of the reacting nuclei. These moments depends upon the internal distribution of mass, charge and current within the nuclei involved and are not subjected to conservation laws.

8.2 Nuclear Reaction Kinematics $(Q-Value)$

Consider the reaction

$$a + A \rightarrow B + b$$

Now according to Law of conservation of energy the total energy of the reactant is equal to the total energy of the product.

$$\left(M_a c^2 + E_a\right) + \left(M_A c^2 + E_A\right) = \left(M_B c^2 + E_B\right) + \left(M_b c^2 + E_b\right)$$

where M_a, M_A, M_b, M_B are the masses of the particles *a*, *A*, *B* and *b* respectively and E_a, E_A, E_b, E_B are their respective kinetic energies.

If we suppose that the target is at rest then $E_A = 0$,

$$Q = E_B + E_b - E_a = \left[\left(M_A + M_a\right) - \left(M_B + M_b\right)\right]c^2$$

Here masses are the nuclear masses. The quantity Q is called the energy balance of the reaction or more commonly Q-value of the reaction. Thus Q is energy appearing due to disappearance of masses or mass defect in a nuclear reaction.

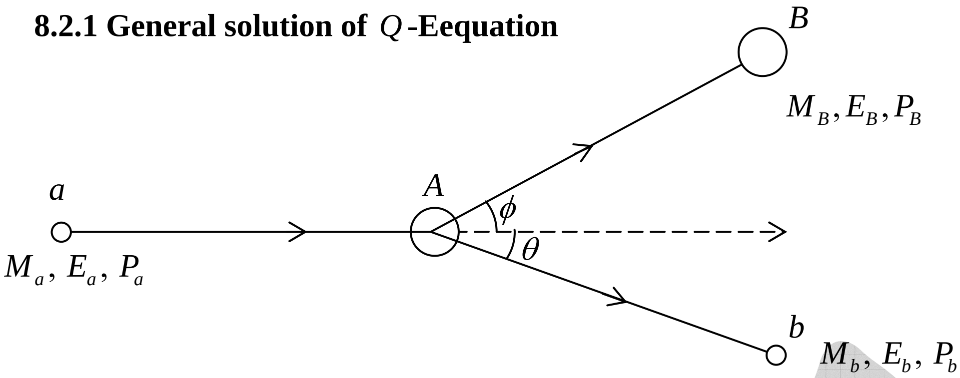
(i) If Q is $+ve$ $\left(Q > 0\right)$

The Kinetic energy of the products is greater than that of the reactants, the reaction is then said to be exothermic or exoergic. The total mass of the reactants is greater than that of the products in this case.

(ii) If Q is $-ve$ $\left(Q < 0\right)$

The reaction is said to be endothermic or endoergic, i.e. energy must be supplied usually as *K. E.* of the incident particles.

8.2.1 General solution of Q-Eequation



Let the particle a strike with target A and the product nuclei B and b are observed at angle ϕ and θ to the direction of incident particle respectively.

Then applying the law of conservation of linear momentum

$\vec{P}_a = \vec{P}_B \cos\phi + \vec{P}_b \cos\theta \Rightarrow \vec{P}_B \cos\phi = \vec{P}_a - \vec{P}_b \cos\theta$ and $\vec{P}_B \sin\phi = \vec{P}_b \sin\theta$

Squaring and adding above two equations $P_B^2 = P_b^2 + P_a^2 - 2P_aP_b\cos\theta$, where $\vec{P}_a, \vec{P}_A, \vec{P}_B, \vec{P}_b$ are the momenta of the particles a, A, B and b respectively.

$\because \left|\vec{P}_a\right| = \sqrt{2M_aE_a}, \ \left|\vec{P}_A\right| = \sqrt{2M_AE_A}$

$\left|\vec{P}_B\right| = \sqrt{2M_BE_B}, \ \left|\vec{P}_b\right| = \sqrt{2M_bE_b}$

Putting these values in the equation $P_B^2 = P_b^2 + P_a^2 - 2P_aP_b\cos\theta$

$$2M_BE_B = 2M_bE_b + 2M_aE_a - 2\sqrt{2M_aE_a}\sqrt{2M_bE_b}\cos\theta$$

or, $$E_B = \frac{M_b}{M_B}E_b + \frac{M_a}{M_B}E_a - \frac{2}{M_B}\left(M_aM_bE_aE_b\right)^{1/2}\cos\theta$$

But $Q = E_B + E_b - E_a$

$$Q = E_a\left(\frac{M_a}{M_B} - 1\right) + E_b\left(1 + \frac{M_b}{M_B}\right) - \frac{2}{M_B}\left(M_aM_bE_aE_b\right)^{1/2}\cos\theta,$$

$$E_b - \frac{2\left(M_aM_bE_a\right)^{1/2}}{\left(M_B + M_b\right)}E_b^{1/2}\cos\theta + \left(\frac{M_a - M_B}{M_b + M_B}\right)E_a - \left(\frac{QM_B}{M_b + M_B}\right) = 0$$

This equation is of the form $x^2 - 2ux - v = 0$

where $x = \sqrt{E_b}$, $u = \dfrac{(M_a M_b E_a)^{1/2}\cos\theta}{(M_B + M_b)}$ and $v = \dfrac{QM_B + E_a(M_B - M_a)}{(M_B + M_b)}$.

The solution of the above equation is given by

$$x = \frac{2u \pm \sqrt{4u^2 + 4v}}{2} \text{ or } x = u \pm \sqrt{4u^2 + 4v}.$$

Thus $\sqrt{E_b} = u \pm \sqrt{u^2 + v}$

The energetically possible reactions are those for which $\sqrt{E_b}$ is real and $+ve$.

8.2.2 Exothermic Reaction $(Q > 0)$

For exothermic reaction $Q > 0$ and these reactions are possible even for $E_a = 0$

$\because Q = E_B + E_b - E_a$ for $E_a = 0$, $u = 0$ and $v = \left(\dfrac{QM_B}{M_B + M_b}\right)$.

Thus, $E_b = \left(\dfrac{QM_B}{M_B + M_b}\right)$.

8.2.3 Exothermic Reaction $(Q < 0)$

All endothermic reactions have negative Q-Values.

When $E_a \to 0$, $u^2 + v = -ve$ and hence $\sqrt{E_b}$ is imaginary.

It means that these reactions are not possible. The smallest value of E_a at which reaction can take place is called threshold energy. The reaction first becomes possible when E_a is large enough to make

$$u^2 + v = 0,$$

$$\Rightarrow \frac{M_a M_b E_a \cos^2\theta}{(M_b + M_B)^2} + \frac{QM_B + E_a(M_B - M_a)}{M_B + M_b} = 0$$

$$\Rightarrow \frac{M_a M_b E_a \cos^2\theta}{(M_b + M_B)} + QM_B + E_a(M_B - M_a) = 0$$

$$\Rightarrow -QM_B(M_b + M_B) = M_a M_b E_a \cos^2\theta + E_a(M_B - M_a)(M_b + M_B)$$

$$\Rightarrow -QM_B\left(M_b+M_B\right)=-M_aM_bE_a\left(1-\cos^2\theta\right)+E_a\left(M_B^2+M_bM_B-M_aM_B\right)$$

$$\Rightarrow E_a=\frac{-Q\left[M_B+M_b\right]}{\left[M_B+M_b-M_a-\frac{M_aM_b}{M_B}\sin^2\theta\right]}$$

At $\theta=0$, E_a is minimum and is the threshold energy E_{th}.

$$\therefore E_{th}=\frac{-Q\left(M_B+M_b\right)}{\left(M_B+M_b-M_a\right)}\quad\Rightarrow\quad E_{th}=-\left(1+\frac{M_a}{M_A}\right)Q$$

NOTE:

Isotopic masses can be used to compute Q except in positive beta decay: (Q in mass units)

(i) ${}^{238}_{92}U\rightarrow{}^{234}_{90}Th+{}^{4}_{2}He$

$$Q=[m(U)-92m_e)]-[m(Th)-90m_e+m(He)-2m_e]$$

$$Q=m(U)-[m(Th)-m(He)]$$

(ii) ${}^{14}_{6}C\rightarrow{}^{14}_{7}N+\bar{e}+\bar{\nu}$

$$Q=m(C)-6m_e-[m(N)-7m_e+m_e]$$

$$Q=m(C)-m(N)$$

(iii) ${}^{63}_{30}Zn\rightarrow{}^{63}_{29}Cu+e^{+}+\nu$

$$Q=m(Zn)-30m_e-[m(Cu)-29m_e+m_e]$$

$$Q=m(Zn)-[m(Cu)+2m_e]$$

Example: The Thermal Neutrons are captured by ${}_5B^{10}$ to form ${}_5B^{11}$ which further decays into α - particle and ${}_3Li^7$. Then find the Q - value.

(Given $m_n = 1.008665\, u$, $m_B = 10.01611\, u$, $m_\alpha = 4.003879\, u$, $m_{Li} = 7.01823\, u$)

Solution:

$${}_5B^{10} + {}_0n^1 \rightarrow {}_5B^{11} \rightarrow {}_2He^4(\alpha) + {}_3Li^7 + Q$$

$$Q = (M_B + M_N - M_\alpha - M_{Li}) \times 931.5\ MeV$$

$$Q = (10.01611 + 1.008665 - 4.003879 - 7.018231) \times 931.5 \approx 2.78\ MeV \approx 2.8\ MeV$$

Example: Consider the nuclear reaction ${}_7N^{14}(\alpha, p)\,{}_8O^{17}$ which occurred in Rutherford's α -range in nitrogen experiment. The mass of $N^{14} = 14.0031\, u$, $He^4 = 4.0026\, u$, $O^{17} = 16.9994\, u$ and $p = 1.0078\, u$ p. Then find the Q -value of the reaction.

Solution: $a + A \rightarrow B + b$; $\alpha + {}_7N^{14} \rightarrow {}_8O^{17} + p$

$$Q = \left[(M_A + M_a) - (M_B + M_b)\right] \times 931.5\, MeV$$

$$Q = \left[(14.003 + 4.0026) - (16.9994 + 1.0078)\right] \times 931.5\, MeV$$

$$Q = -0.0013 \times 931.5\, MeV = -1.49\ MeV$$

Example: A neutron beam is incident on a stationary target of fluorine atoms. The reaction $F^{17}(n, p)O^{19}$ has a Q -value of $-4.0\ MeV$. Then find the lowest neutron energy which will make this reaction possible.

Solution:

$$a + A \rightarrow B + b; \qquad n + F^{17} \rightarrow O^{19} + p$$

$$E_{th} = -Q\left(\frac{M_A + M_a}{M_A}\right) = 4\left(\frac{19+1}{19}\right) \approx 4.2\ MeV$$

8.2.3 Nuclear Fission

Nuclear fission is a process in which a heavy nucleus, after capturing a neutron splits up into two lighter nuclei of comparable masses. The process is accompanied by release of a few fast neutrons and a huge amount of energy in form of K.E. of fission fragments and γ-rays.

$$_{92}^{235}U + _0 n^1 \rightarrow _{92}^{236} U^* \rightarrow _{36}^{92} Kr + _{56}^{141} Ba + 3 _0 n^1 + Q$$

Energy released by 1 kg of Uranium

Mass of $_{92}^{235}U =$ 235.035315 u , Mass of $n =$ 1.008665 u

Total initial mass $= 236.04398\ u$

Mass of $_{56}^{141}Ba = 140.9177\ u$, Mass of $_{36}^{92}Kr = 91.8854\ u$, Mass of $3n = 3.0259\ u$

Total final mass $= 235.8290\ u$

Mass defect $(\Delta m) =$ initial mass- final mass $= 0.21498\ u$

Energy released i.e Q-value $= \Delta m \times 931.5\ MeV \approx 200\ MeV$

Number of uranium nucleus in 1 kg of uranium $= 6.023 \times 10^{23} \times 1000/235$

Energy released by 1 kg of uranium

$$= 6.023 \times 10^{23} \times 1000/235 \times 200\, MeV$$

$$= 5.13 \times 10^{26} MeV$$

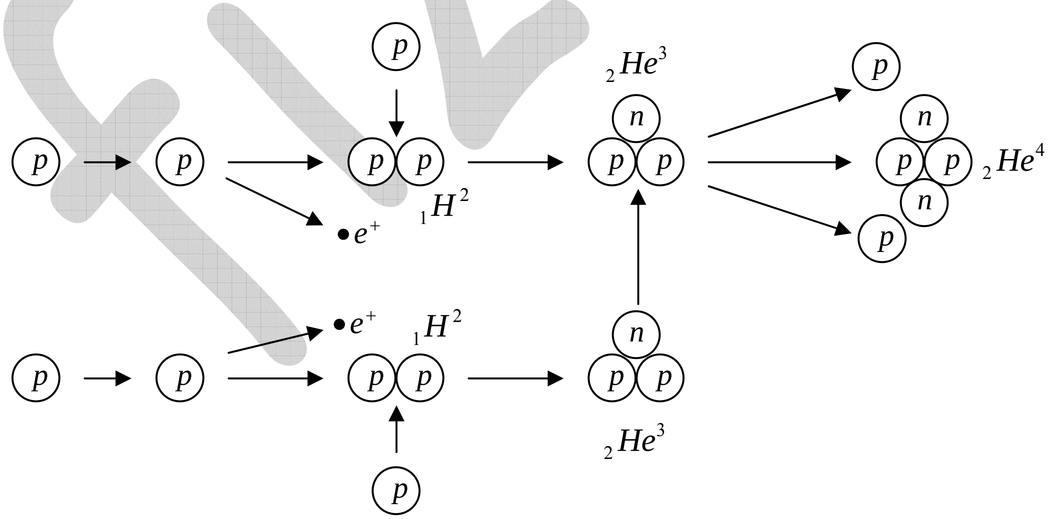
8.3 Nuclear Fusion in Stars

Fusion is a process, in which the lighter nuclei fuse together and produce a heavier nucleus. The sum of the masses of the individual light nuclei is more than would be the mass of the nucleus formed by their fusion, and thus the fusion process result in liberation of energy.

Stellar energy was liberated in the formation of helium from hydrogen because large amounts of hydrogen and helium exist in the sun. Such processes are called thermonuclear reactions because energy is liberated due to very high stellar temperature. In order to interact two nuclei, that must have enough $K.E.$ to permit them to overcome the electrostatic repulsion barrier which tends to keep them apart.

The basic energy producing process in the sun is the fusion of hydrogen nuclei into helium nuclei. This can take place in several different reaction sequences, the most common of which, the **proton-proton cycle**. The total evolved energy is 24.7 MeV per ${}_2^4He$ nucleus formed.

Since 24.7 MeV is 4×10^{-12} $Joules$, the sun's power output of $4\times10^{26}\,Watt$ means each sequence of reactions must occur 10^{38} times per second.



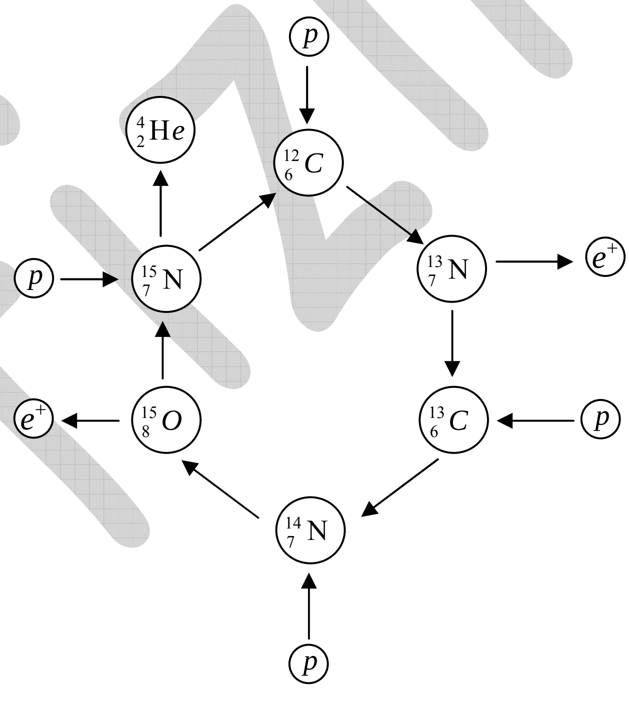
Proton-Proton Cycle

Energy is given off at each step. The net result is the combination of four hydrogen nuclei to form a helium nucleus and two positrons.

Self-sustaining fusion reactions can occur only under conditions of extreme temperature and density. The high temperature ensures that the some nuclei have the energy needed to come close enough together to interact, and the high density ensures that such collisions are frequent. A further condition for the proton-proton cycles is a large reacting mass, such as that of the sun, since much time may elapse between the initial fusion of a particular proton and its eventual incorporation in an α-particle.

Carbon-Cycle

It also involves the combination of four hydrogen nuclei (Stars hotter than sun) to form a helium nucleus with the evolution of energy. The net result again is the formation of an α-particle and two positrons from four protons, with the evolution of $24.7\ MeV$. The initial ${}^{12}_{6}C$ acts as a kind of catalyst for the process, since it reappears at its end.



Carbon Cycle

Multiple Choice Questions (MCQ)

Q1. The radius of Ge nucleus is measured to be twice the radius of ${}^{9}_{4}Be$. How many nucleons are there in Ge nucleus?

(a) 64 (b) 72 (c) 82 (d) 86

Q2. The radius of a ${}^{64}_{29}X$ nucleus is measured to be $4.8\times10^{-13}\,cm$. The radius of a ${}^{27}_{12}Y$ nucleus can be estimated to be

(a) $2\times10^{-13}\,cm$ (b) $4\times10^{-13}\,cm$

(c) $6\times10^{-13}\,cm$ (d) $8\times10^{-13}\,cm$

Q3. According to the empirical observations of charge radii, a ${}^{16}_{8}X$ nucleus is spherical and has charge radius R and a volume $V = \frac{4}{3}\pi R^3$. Then the volume of the ${}^{128}_{54}Y$ nucleus, is

(a) $1.5V$ (b) $2V$ (c) $6.5V$ (d) $8V$

Q4. Assume spherical symmetry of the nucleus ${}^{A}_{Z}X$, where Z is atomic number and A is mass number of the nucleus. Then the nuclear density and nuclear particle density of nucleus is of the order of: $\left(m_p \approx m_n = 1.67\times10^{-27}\,kg \text{ and } R_0 = 1.2 \text{ fermi}\right)$

(a) $10^{15}\,kg/m^3$ and $10^{40}\,nucleons/m^3$ (b) $10^{17}\,kg/m^3$ and $10^{44}\,nucleons/m^3$

(c) $10^{17}\,kg/m^3$ and $10^{40}\,nucleons/m^3$ (d) $10^{15}\,kg/m^3$ and $10^{44}\,nucleons/m^3$

Q5. Consider a nucleus with N neutrons and Z protons. If m_p, m_n and $B.E.$ represents the mass of the proton, the mass of the neutron and binding energy of the nucleus respectively. Then mass of the nucleus is given by (and c is the velocity of light in free space)

(a) $Nm_n + Zm_p$ (b) $Nm_p + Zm_n$

(c) $Nm_n + Zm_p - \frac{B.E.}{c^2}$ (d) $Nm_p + Zm_n + \frac{B.E.}{c^2}$

Q6. If M_e, M_p *and* M_H are the rest masses of electron, proton and hydrogen atom in the ground state (with energy $-13.6\,eV$) respectively. Which of the following is exactly true? (c is the speed of light in free space)

(a) $M_H = M_p + M_e$

(b) $M_H = M_p + M_e - \dfrac{13.6\,eV}{c^2}$

(c) $M_H = M_p + M_e + \dfrac{13.6\,eV}{c^2}$

(d) $M_H = M_p + M_e + K, \text{ where } K \neq \pm\dfrac{13.6\,eV}{c^2}$ or zero

Q7. Let m_p and m_n be the mass of proton and neutron. M_1 is the mass of ${}^{20}_{10}Ne$ nucleus and M_2 is the mass of a ${}^{40}_{20}Ca$ nucleus. Then find the correct relation:

(a) $M_1 = 10(m_p + m_n),\ M_2 = 20(m_p + m_n)$ *and* $M_2 = 2M_1$

(b) $M_1 < 10(m_p + m_n),\ M_2 < 20(m_p + m_n)$ *and* $M_2 = 2M_1$

(c) $M_1 < 10(m_p + m_n),\ M_2 < 20(m_p + m_n)$ *and* $M_2 > 2M_1$

(d) $M_1 < 10(m_p + m_n),\ M_2 < 20(m_p + m_n)$ *and* $M_2 < 2M_1$

Q8. The measured mass of deuteron atom $\left({}^{2}_{1}H\right)$, Hydrogen atom $\left({}^{1}_{1}H\right)$, proton (p) and neutrons (n) are $2.0141\,u$, $1.0078\,u$, $1.0073\,u$ and $1.0087\,u$. Then the binding energy of the deuteron nucleus is:

(a) $1.11\,MeV$ (b) $2.22\,MeV$ (c) $3.33\,MeV$ (d) $4.44\,MeV$

Q9. The masses of a hydrogen atom, neutron and ${}^{238}_{92}U$ atom are given by $1.0078\,u$, $1.0087\,u$ and $238.0508\,u$ respectively. The binding energy per nucleon of ${}^{238}_{92}U$ nucleus is therefore approximately equal to

(a) $6.6\,MeV/nucleons$ (b) $7.6\,MeV/nucleons$

(c) $8.6\,MeV/nucleons$ (d) $9.6\,MeV/nucleons$

Q10. Total binding energy of ${}^{5}_{2}\mathrm{He}$ is approximately equal to:

where $\left(m\left({}^{5}_{2}\mathrm{He}\right)=5.01220u,\ m\left({}^{1}\mathrm{H}\right)=1.007825u,\ \ m_n=1.008665\,u\right)$

(a) 16.5 MeV (b) 40.1 MeV

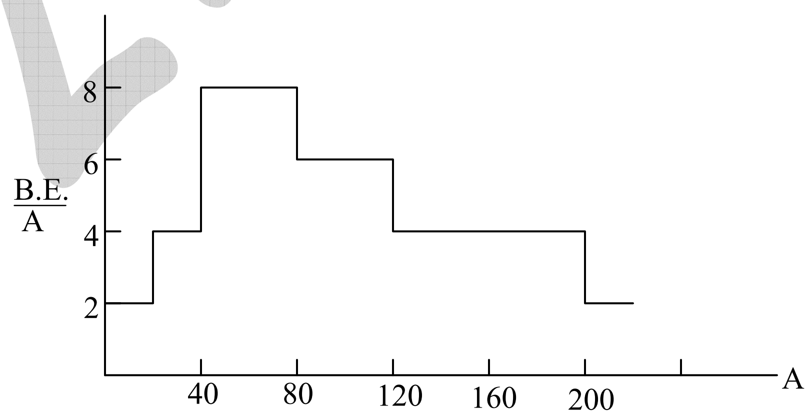
(c) 8.00 MeV (d) 27.4 MeV

Q11. The binding energy of the neon isotope ${}^{20}_{10}Ne$ is $160.647\,MeV$. The atomic mass of hydrogen atom $\left({}^{1}_{1}H\right)$, mass of proton and neutron is $1.0078\,u$, $1.0073\,u$ and $1.0087\,u$. Then the atomic mass of ${}^{20}_{10}Ne$ is

(a) 18.00 u (b) 18.99 u

(c) 19.99 u (d) 20.99 u

Q12. The following histogram represents the binding energy per particle $\left(\frac{B.E.}{A}\right)$ in MeV as a function of the mass number (A) of a nucleus. A nucleus with mass number $A=180$ fission into two nuclei of equal masses. In the process



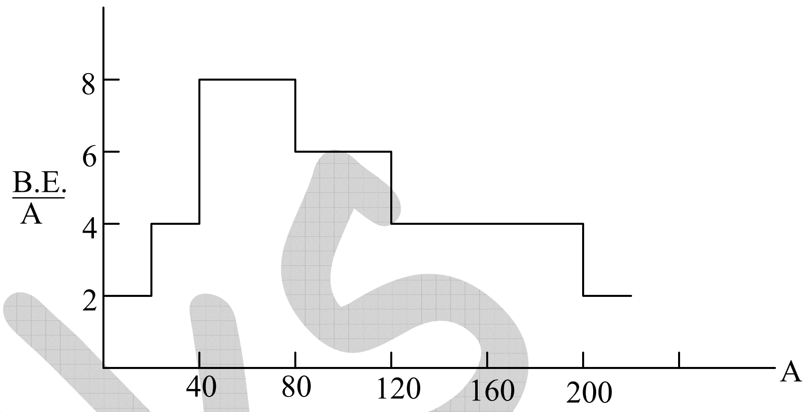
(a) 180 MeV of energy is released

(b) 180 MeV of energy is absorbed

(c) 360 MeV of energy is released

(d) 360 MeV of energy is absorbed

Q13. The following histogram represents the binding energy per particle $\left(\frac{B.E.}{A}\right)$ in MeV as a function of the mass number (A) of a nucleus. A nucleus with mass number $A = 170$ fission into two nuclei of equal masses. In the process



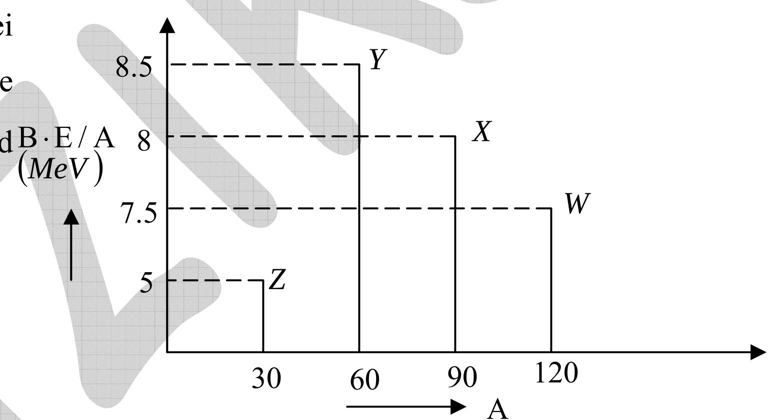
(a) $340\, MeV$ of energy is released

(b) $340\, MeV$ of energy is absorbed

(c) $360\, MeV$ of energy is released

(d) $360\, MeV$ of energy is absorbed

Q14. Binding energy per nucleon Vs mass number curve for nuclei W, X, Y *and* Z is indicated on the curve. The process that would release energy is:



(a) $Y \to 2Z$

(b) $W \to X + Z$

(c) $W \to 2Y$

(d) $X \to Y + Z$

Q15. Six α-decay and four β⁻-decay occurs before ${}^{232}_{90}X$ achieves stability. The final product in the chain is

(a) ${}^{210}_{82}Y$ (b) ${}^{208}_{80}Y$ (c) ${}^{210}_{80}Y$ (d) ${}^{208}_{82}Y$

Q16. A radioactive sample containing N_0 nuclei emits N α-particle per second on decaying. The half life of the sample is $0.693\frac{N}{N_0}$.

(a) $0.693\frac{N}{N_0}$ (b) $0.693\frac{N_0}{N}$ (c) $1.44\frac{N}{N_0}$ (d) $1.44\frac{N_0}{N}$

Q17. According to measurements by Rutherford and Geiger, one gram of radium -226 emits in one second 3.7×10^{10} alpha particles. The half life of radium is

(a) 400 *years* (b) 800 *years*

(c) 1600 *years* (d) 3200 *years*

Q18. A radioactive sample contains 3×10^{-9} kg of active gold ^{200}Au, whose half life is 48 min. Then the activity of the radon sample is

(a) 55 Ci (b) 57 Ci (c) 59 Ci (d) 61 Ci

Q19. The radio isotope ^{14}C maintains a fixed proportion in a living entity by exchanging carbon with the atmosphere. After it dies exchange ceases and proportion of ^{14}C decreases continuously as ^{14}C beta decays with half life of 5500 *years*. Estimate the age of the dead tree whose present activity is $1/3$ of initial activity.

(a) 8717 *years* (b) 6520 *years*

(c) 5500 *years* (d) 4500 *years*

Q20. A radioactive sample emits n β-particles in 2 sec. In next 2 sec it emits $0.75n$ β-particles, then the mean life of the sample is $(\ln 2 = 0.693, \ln 3 = 1.0986)$

(a) 2 sec (b) 5 sec (c) 7 sec (d) 9 sec

Q21. A radioactive substance is initially absent, is formed at constant rate P nuclei per second. If the decay constant of the nuclei formed is λ, then the number of nuclei N present after time t seconds is

(a) $\frac{P}{\lambda}$ (b) $\frac{P}{\lambda}e^{-\lambda t}$

(c) $\frac{P}{\lambda}\left(1-e^{-\lambda t}\right)$ (d) $\frac{P}{\lambda}\left(e^{-\lambda t}-1\right)$

Q22. In a successive growth/decay process $A \to B \to C$, element C is a stable nucleus. The following parameters are given for the process:

$$\begin{array}{cccc} & A \xrightarrow{\lambda_1} & B \xrightarrow{\lambda_2} & C \\ t=0 & N_0 & 0 & 0 \\ t=t & N_1 & N_2 & N_3 \end{array}$$

where λ_1 and λ_2 are decay constant, N_0 is the concentration of A at $t=0$ and N_1, N_2, N_3 are concentration of A, B, C at any time t. Then the concentration of intermediate member (B) will be:

(a) $N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

(b) $N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{+\lambda_1 t} - e^{+\lambda_2 t}\right]$

(c) $N_2 = \frac{\lambda_2 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

(d) $N_2 = \frac{\lambda_2 N_0}{\lambda_2 - \lambda_1}\left[e^{\lambda_1 t} - e^{\lambda_2 t}\right]$

Q23. In a successive growth/decay process $A \to B \to C$, element C is a stable nucleus. The following parameters are given for the process:

$$\begin{array}{cccc} & A \xrightarrow{\lambda_1} & B \xrightarrow{\lambda_2} & C \\ t=0 & N_0 & 0 & 0 \\ t=t & N_1 & N_2 & N_3 \end{array}$$

where λ_1 and λ_2 are decay constant, N_0 is the concentration of A at $t=0$ and N_1, N_2, N_3 are concentration of A, B, C at time t. Then the time at which concentration of intermediate member (B) will reach maxima is:

(a) $t' = \frac{\ln(\lambda_1/\lambda_2)}{(\lambda_1 - \lambda_2)}$

(b) $t' = \frac{\ln(\lambda_2/\lambda_1)}{(\lambda_2 - \lambda_1)}$

(c) $t' = \frac{\ln(\lambda_1\lambda_2)}{(\lambda_1 - \lambda_2)}$

(d) $t' = \frac{\ln(\lambda_1\lambda_2)}{(\lambda_2 - \lambda_1)}$

Q24. The disintegration energy is defined to be the difference in the rest energy between the initial and final states. Consider the following process:

$$ {}^{240}_{94}Pu \rightarrow {}^{236}_{92}U + {}^{4}_{2}He $$

The emitted α-particle has a kinetic energy $5.17\,MeV$. The value of the disintegration energy is

(a) $5.26\,MeV$ (b) $5.17\,MeV$ (c) $5.08\,MeV$ (d) $2.59\,MeV$

Q25. The polonium isotope ${}^{210}_{84}Po$ is unstable and emits a $5.30\,MeV$ α-particle. The atomic mass of ${}^{210}_{84}Po$ is $209.9829\,u$ and that of ${}^{4}_{2}He$ is $4.0026\,u$, then the atomic mass of its daughter nuclei is

(a) $203.9723\,u$ (b) $204.9052\,u$

(c) $205.9754\,u$ (d) $206.1053\,u$

Q26. The Thermal Neutrons are captured by ${}_5B^{10}$ to form ${}_5B^{11}$ which further decays into α - particle and ${}_3Li^7$, the kinetic energy of Li is

(Given $m_n = 1.008665\,u$, $m_B = 10.01611\,u$, $m_\alpha = 4.003879\,u$, $m_{Li} = 7.01823\,u$)

(a) $1.78\,MeV$ (b) $2.5\,MeV$ (c) $1.00\,MeV$ (d) $2\,MeV$

Q27. Neutrons are observed in a nuclear reaction $Li^7(p,n)Be^7$. Then the bombarding energy of proton at which neutrons of zero energy is obtained, will be (Q-value of reaction is $-1.65\,MeV$):

(a) $1.7\,MeV$ (b) $1.9\,MeV$ (c) $2.1\,MeV$ (d) $5.2\,MeV$

Q28. A nuclear decay process is given

$$ {}_Z X^A \rightarrow {}_{Z-1}Y^A + e^+ $$

The atomic masses of X and X are $51.9648\,u$ and $51.9571\,u$. Then the Q-value of the reaction is:

(a) $2.7\,MeV$ (b) $3.7\,MeV$ (c) $4.7\,MeV$ (d) $6.2\,MeV$

Q29. For nuclear fusion reaction to take place, which one of the following is true?

(a) Only very high temperature is required

(b) Normal temperature and comparatively high pressure is required.

(c) Very high temperature and comparatively high pressure is required.

(d) Very high temperature and very low pressure is required.

Q30. Which of the following fusion reaction give more energy? The nuclear mass of the different nuclei is as follows

$M\left({}_1H^1\right)=1.00783\ u$, $M\left({}_1H^2\right)=2.01410\ u$, $M\left({}_1H^3\right)=3.01605\ u$,

$M\left({}_2He^3\right)=3.01603\ u$, $M\left({}_2He^4\right)=4.02603\ u$, $M\left({}_3Li^6\right)=6.01512\ u$,

$M\left(e^-\right)=0.00055\ u$

(a) ${}_1H^1+{}_1H^1\rightarrow {}_1H^2+e^++\nu$ (b) ${}_1H^2+{}_1H^1\rightarrow {}_2He^3$

(c) ${}_1H^3+{}_2He^3\rightarrow {}_3Li^6$ (d) ${}_1H^2+{}_2He^4\rightarrow {}_3Li^6$

Numerical Answer Type Question (NAT)

Q31. The ratio of the sizes of ${}^{208}_{82}Pb$ and ${}^{26}_{12}Mg$ nuclei is approximately……….

Q32. If the nuclear radius of ${}^{27}Al$ is 3.6 Fermi, the approximate nuclear radius of ${}^{64}Cu$ in Fermi is…………

Q33. The atomic masses of ${}^{42}_{20}Ca=41.958622u$, ${}^{41}_{20}Ca=40.962278u$ and mass of ${}^1_0n=1.008665u$, ${}^1_1p=1.007276u$. Then the energy needed to remove a neutron from the nucleus of the calcium isotope ${}^{42}_{20}Ca$ is………… MeV

Q34. The atomic masses of ${}^{42}_{20}Ca=41.958622u$, ${}^{41}_{19}K=40.961825u$ and mass of ${}^1_0n=1.008665u$, ${}^1_1p=1.007276u$. Then the energy needed to remove a proton from the nucleus of the calcium isotope ${}^{42}_{20}Ca$ is………… MeV

Q35. The energy required to break ${}_6C^{12}$ into $3\left({}_2\mathrm{He}^4\right)$ particle is………… MeV

(Given $m\left({}_6C^{12}\right)=12.0\,u,\quad m\left({}_2^4\mathrm{He}\right)=4.0026\ u$)

Q36. A $280\,day$ old radioactive substance shows an activity of $6000\,dps$, $140\,day$ later its activity becomes $3000\ dps$. Then the initial activity of the sample was………… dps

Q37. A radioactive sample contains $1.00\,g$ of radim ${}^{226}Ra$, whose half life is $1622\ years$. Then the activity of the radon sample is ………… $Curie$

Q38. If ${}_{92}U^{235}$ captures a thermal neutron a releases $160\ MeV$ and if the resulting fission fragments have mass numbers 138 and 95, the kinetic energy of the lighter fragment is………… MeV

Q39. In the uranium radioactive series, the initial nucleus is ${}_{92}^{238}\mathrm{U}$ and the final nucleus is ${}_{82}^{206}\mathrm{Pb}$. When the uranium nucleus decays to lead, the number of α-particles emitted are ………. and the number of β-particles emitted are ……….

Q40. The binding energies per nucleon for deuteron $\left({}_1\mathrm{H}^2\right)$ and helium $\left({}_2\mathrm{He}^4\right)$ are 1.1 MeV and 7.0 MeV respectively. The energy released when two deuterons fuse to form a helium nucleus $\left({}_2\mathrm{He}^4\right)$ is ………..

Multiple Select Type Questions (MSQ)

Q41. A radioactive sample emits n β-particles in $2\sec$. In next $2\sec$ it emits $0.75n$ β-particles, then which of the following statements are true $(\ln 2 = 0.693, \ln 3 = 1.0986)$

(a) Decay constant of the sample is $0.14\sec^{-1}$

(b) Decay constant of the sample is $0.28\sec^{-1}$

(c) Mean life of the sample is 3.5 sec

(d) Mean life of the sample is 7.0 sec

Q42. Which of the following statement is true?

(a) Six α-decay and four β^--decay occurs before ${}^{232}_{90}Th$ achieves stability; the final product in the chain being ${}^{208}_{82}Pb$.

(b) A radioactive nucleus has a half life of 100 years. If the number of nuclei $t = 0$ is N_0, then $\frac{7}{8}N_0$ number of nuclei have decayed in 300 years.

(c) The atomic ratio between the uranium isotopes ${}^{238}U$ and ${}^{234}U$ in a mineral sample is found to be 1.8×10^4. Then the half life of ${}^{234}U$ and ${}^{238}U$ is 2.5×10^5 *years* and 4.5×10^9 *years*.

(d) A radioactive sample containing N_0 nuclei emits N α-particle per second on decaying. The half life of the sample is $0.693\frac{N}{N_0}$.

Q43. A Uranium nucleus decays at rest into a Thorium nucleus and a Helium nucleus as shown below $\qquad U^{235} \to Th^{231} + He^4$

Which of the following is not true?

(a) Each decay product has the same kinetic energy

(b) Each decay product has the same speed

(c) The Thorium nucleus has more momentum than the Helium nucleus

(d) The Helium nucleus has more kinetic energy than the Thorium nucleus

Q44. Which of the following statement is true?

(a) Electron capture occurs more often than positron emission in heavy nuclides

(b) Proton outside the nucleus decays into neutron

(c) Neutron outside the nucleus decays into proton

(d) Positron emission leads to daughter nucleus of lower atomic number

Q45. Which of the following statement is not true regarding a negative beta decay

(a) an atomic electron is ejected

(b) an electron which is already present within the nucleus is ejected.

(c) a neutron in the nucleus decays emitting an electron.

(d) a part of the binding energy of the nucleus is converted into an electron.

Q46. From the following equations pick out the possible nuclear fusion reactions:

(a) ${}_6C^{13} + {}_1\mathrm{H}^1 \rightarrow {}_6C^{14} + 4.3\ MeV$

(b) ${}_6C^{12} + {}_1\mathrm{H}^1 \rightarrow {}_7N^{13} + 2\ MeV$

(c) ${}_7N^{14} + {}_1\mathrm{H}^1 \rightarrow {}_8O^{15} + 7.3\ MeV$

(d) ${}_{92}U^{235} + {}_0n^1 \rightarrow {}_{54}Xe^{140} + {}_{38}Sr^{94} + {}_0n^1 + {}_0n^1 + 200\ MeV$

Q47. Which of the following statement is not true regarding a nuclear fusion reaction

(a) a heavy nucleus breaks into two fragments by itself

(b) a light nucleus bombarded by thermal neutrons breaks up

(c) a heavy nucleus bombarded by thermal neutrons breaks up

(d) two light nuclei combine to give a heavier nucleus and possibly other products

Q48. Which of the following statement(s) is (are) correct?

(a) The rest mass of a stable nucleus is less than the sum of the rest masses of its separated nucleons

(b) The rest mass of a stable nucleus is greater than the sum of the rest masses of its separated nucleons.

(c) In nuclear fission, energy is released by fusing two nuclei of medium mass (approximately 100 amu).

(d) In nuclear fission, energy is released by fragmentation of a very heavy nucleus.

Q49. Let m_p be the mass of proton, m_n the mass of neutron, M_1 the mass of a ${}^{20}_{10}\text{Ne}$ nucleus and M_2 the mass of a ${}^{40}_{20}\text{Ca}$ nucleus. Then

(a) $M_2 = 2M_1$ (b) $M_2 > 2M_1$

(c) $M_2 < 2M_1$ (d) $M_1 < 10\left(m_n + m_p\right)$

Q50. Which of the following statement is not true for ${}^{42}_{20}Ca$?

(a) Energy needed to remove a neutron is greater than to remove a proton

(b) Energy needed to remove a proton is greater than to remove a neutron

(c) Energy needed to remove a neutron and proton is same

(d) Energy needed to remove a neutron or proton cannot be predicted

Solution

Multiple Choice Questions (MCQ)

Ans.1: (b) $\because R = R_o(A)^{1/3}$ and given that $R_{Ge} = 2R_{Be} \Rightarrow R_o(A_{Ge})^{1/3} = 2R_o(9)^{1/3} \Rightarrow A_{Ge} = 72$.

Ans.2: (b) Since $R = R_0(A)^{1/3} \Rightarrow \frac{R_Y}{R_X} = \left(\frac{A_Y}{A_X}\right)^{1/3} = \left(\frac{27}{64}\right)^{1/3}$

$$\Rightarrow \frac{R_Y}{R_X} = \frac{3}{4} \Rightarrow R_Y = \frac{3}{4}R_X \Rightarrow R_Y = 3.6\times10^{-13}\,cm \approx 4\times10^{-13}\,cm$$

Ans.3: (d) $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A = \frac{4}{3}\pi R_0^3 \times 16;\ V' = \frac{4}{3}\pi R_0^3 \times 128 = 8V$

Ans.4: (b)

Ans.5: (c)

Ans.6: (b)

Ans.7: (d)

Since nuclear mass is always less than their constituent particles so

$$M_1 < 10(m_p + m_n) \text{ and } M_2 < 20(m_p + m_n)$$

Since B.E. of ${}^{40}_{20}Ca >$ B.E. of ${}^{20}_{10}Ne$

$$\Rightarrow \left[20(m_p + m_n) - \mathrm{M}_2\right]c^2 > \left[10(m_p + m_n) - \mathrm{M}_1\right]c^2 \Rightarrow \mathrm{M}_2 < \left[10(m_p + m_n) + \mathrm{M}_1\right] < 2\mathrm{M}_1$$

Ans.8: (b) $B.E. = \left[Zm_H + Nm_n - m\left({}^2_1H\right)\right]\times 931.5\,MeV$

$$\Rightarrow B.E. = [1\times1.0078 + 1\times1.0087 - 2.0141]\times 931.5\,MeV$$

$$\Rightarrow B.E. = 0.0024\times 931.5\,MeV = 2.2356\,MeV$$

Ans.9: (b)

$$B.E. = \left[Zm_H + Nm_n - m\left({}^{238}_{92}U\right)\right]\times 931.5\,MeV$$

$$\Rightarrow B.E. = [92\times1.0078 + 146\times1.0087 - 238.0508]\times 931.5\,MeV$$

$$\Rightarrow B.E. = 1.937\times 931.5\,MeV = 1804\,MeV$$

$$\Rightarrow \frac{B.E.}{A} = \frac{1804}{238} = 7.6\,MeV/nucleons$$

Ans.10: (d)

$$2\,{}^{1}\text{H}+3\text{n} \rightarrow {}_{2}^{5}\text{He}$$

$$B.E.=\left[2m\left({}^{1}\text{H}\right)+3m(n)-m\left({}_{2}^{5}\text{He}\right)\right]\times 931.5\ MeV$$

$$\Rightarrow B.E.=\left[2\times 1.007825+3\times 1.008665-5.01220\right]\times 931.5\ MeV$$

$$\Rightarrow B.E.=\left[0.02944\right]\times 931.5\ MeV=27.4\ MeV$$

Ans.11: (c)

$$m\left({}_{Z}X^{A}\right)=\left[Zm\left({}_{1}H^{1}\right)+N\,m_{n}\right]-\frac{E_{b}}{931.5\,MeV/u}=\left[10\times 1.0078+10\times 1.0087\right]-\frac{160.647}{931.5}$$

$$=19.99\,u$$

Ans.12: (c)

Since a nucleus with mass number $A=180$ fission into two nuclei of equal masses thus

$$180\rightarrow 90+90.$$

So $B.E.$ of the heavier nucleus is $=180\times 4=720\,MeV$.

Total $B.E.$ of the lighter nuclei is $=90\times 6+90\times 6=1080\,MeV$.

Since product nuclei have higher $B.E.$ so in this process energy is released i.e $=1080-720=360MeV$.

Ans.13: (a)

Since a nucleus with mass number $A=170$ fission into two nuclei of equal masses thus

$$180\rightarrow 85+85.$$

So $B.E.$ of the heavier nucleus is $=170\times 4=680\,MeV$.

Total $B.E.$ of the lighter nuclei is $=85\times 6+85\times 6=1020\,MeV$.

Since product nuclei have higher $B.E.$ so in this process energy is released i.e $=1020-680=340MeV$.

Ans.14: (c)

Let us write $B.E.$ in MeV of both sides.

(a) $Y \rightarrow 2Z$; $8.5\times 60 = 510 \rightarrow 2\times 5\times 30 = 300$

(b) $W \rightarrow X + Z$; $7.5\times 120 = 900 \rightarrow 8\times 90 + 5\times 30 = 870$

(c) $W \rightarrow 2Y$; $7.5\times 120 = 900 \rightarrow 2\times 8.5\times 60 = 1020$

(d) $X \rightarrow Y + Z$; $8\times 90 = 720 \rightarrow 8.5\times 60 + 5\times 30 = 660$

In $W \rightarrow 2Y$, product have higher B.E. than reactant. So energy will release.

Ans.15: (d)

${}_Z^AX \rightarrow {}_{Z-2}^{A-4}Y + {}_2^4He$ and ${}_Z^AX \rightarrow {}_{Z+1}^{A}Y + \beta^-$

Change in mass number $= 232 - 24 = 208$.

Change in atomic number after 6 α-decay $= 90 - 12 = 78$.

Final products mass number $= 82$; 4 β^--decay

Ans.16: (b) $R = \lambda N \Rightarrow N = \lambda N_0 \Rightarrow T_{1/2} = \dfrac{0.693}{\lambda} = 0.693\dfrac{N_0}{N}$

Ans.17: (c)

Number of radium atoms in one gram of radium $= \dfrac{1}{226}\times 6.02\times 10^{23} = 2.7\times 10^{21}$.

Decay constant $\lambda = -\dfrac{dN}{N} - 1 \Rightarrow \lambda = \dfrac{3.7\times 10^{10}}{2.7\times 10^{21}} = 1.37\times 10^{-11}\ \sec^{-1}$

Thus Half Life $T_{1/2} = \dfrac{0.693}{\lambda} = \dfrac{0.693}{1.37\times 10^{-11}} = 5\times 10^{10}\ \sec = \dfrac{5\times 10^{10}}{365\times 24\times 60\times 60} = 1600\ years$

Ans.18: (a)

Decay constant $\lambda = \dfrac{0.693}{T_{1/2}} = \dfrac{0.693}{48\times 60} = 2.406\times 10^{-4}\ \sec^{-1}$

Number of atoms in $3\times 10^{-9}\ kg$ is $N = \dfrac{3\times 10^{-6}\ g}{200}\times 6.023\times 10^{23} = 9.04\times 10^{15}$ atoms

Hence, activity $R = \lambda N = 2.406\times 10^{-4}\times 9.04\times 10^{15} = 2.18\times 10^{12}\ decay/\sec = 59\ Ci$

$\because 1.0\ \text{Ci} = 3.7\times 10^{10}\ decay/\sec$

Ans.19: (a)

$$\because R = R_0 e^{-\lambda t}$$

The age of the dead tree is $t = \frac{1}{\lambda}\ln\left(\frac{R_0}{R}\right) = \frac{T_{1/2}}{0.693}\ln\left(\frac{R_0}{R}\right) = \frac{5500}{0.693}\ln\left(\frac{3}{1}\right) = 8717\, years$

Ans.20: (c)

Let N_0 be the number of initial number of nuclei. Then

$$n = N_0 - N_0 e^{-2\lambda} = N_0(1 - e^{-2\lambda})$$

$$0.75n = N_0 e^{-2\lambda} - N_0 e^{-2\lambda} e^{-2\lambda} = N_0 e^{-2\lambda}(1 - e^{-2\lambda})$$

$$\frac{0.75n}{n} = \frac{\mathrm{N_o e^{-2\lambda}}\left(1 - e^{-2\lambda}\right)}{\mathrm{N_o}\left(1 - e^{-2\lambda}\right)} \Rightarrow e^{-2\lambda} = \frac{3}{4} \Rightarrow 2\lambda = 2\ln 2 - \ln 3 \Rightarrow \lambda = 0.1438\,\mathrm{sec}^{-1}$$

$$\Rightarrow \overline{\mathrm{T}} = \frac{1}{\lambda} = 7\,\mathrm{sec}$$

Ans.21: (c)

$$\frac{dN}{dt} = P - \lambda N \Rightarrow \int_0^N \frac{dN}{P - \lambda N} = \int_0^t dt \Rightarrow -\frac{1}{\lambda}\ln\left[P - \lambda N\right]_0^N = t$$

$$\Rightarrow \ln\left[P - \lambda N\right] - \ln\left[P\right] = -\lambda t \Rightarrow \frac{P - \lambda N}{P} = e^{-\lambda t} \Rightarrow N = \frac{P}{\lambda}\left(1 - e^{-\lambda t}\right)$$

Ans.22: (a)

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \Rightarrow \frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_0 N_0 e^{-\lambda_1 t}$$

Multiply both side by $e^{\lambda_2 t} dt$ and then integrate $\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1 N_o e^{(\lambda_2 - \lambda_1)}}{\lambda_2 - \lambda_1} + K$.

At $t = 0, N_2 = 0 \Rightarrow K = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$, thus $N_2 = \frac{\lambda_1 N_o}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$.

Ans.23: (b)

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \Rightarrow \frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_0 N_0 e^{-\lambda_1 t}$$

Multiply both side by $e^{\lambda_2 t}dt$ and then integrate $\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1 N_o e^{(\lambda_2 - \lambda_1)}}{\lambda_2 - \lambda_1} + K$.

At $t = 0, N_2 = 0 \Rightarrow K = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$, thus $N_2 = \frac{\lambda_1 N_o}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

$$\left(\frac{dN_2}{dt}\right)_{t=t'} = 0 \Rightarrow t' = \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}.$$

Ans.24: (a)

$$K.E_\alpha \approx \left(\frac{A-4}{A}\right)Q \Rightarrow Q = \left(\frac{A}{A-4}\right)KE_\alpha = \frac{240}{236}\times 5.17 = 5.26\,MeV$$

Ans.25: (c)

$$Q = \left(\frac{A}{A-4}\right)KE_\alpha = \frac{210}{206}\times 5.30 = 5.40\,MeV$$

The mass equivalent of this Q-value, $m_Q = \frac{5.40MeV}{931.5MeV/u} = 0.0058\,u$

Hence $m_f = m_i - m_\alpha - m_x = 209.9829 - 4.026 - 0.0058 \Rightarrow m_f = 205.9754\ u$

Ans.26: (c)

$${}_5B^{10} + {}_0n^1 \rightarrow {}_5B^{11} \rightarrow {}_2He^4(\alpha) + {}_3Li^7 + Q$$

$$Q = \left(M_B + M_N - M_\alpha - M_{Li}\right)\times 931.5\ MeV$$

$$Q = (10.01611 + 1.008665 - 4.003879 - 7.018231)\times 931.5 \approx 2.78\ MeV \approx 2.8\ MeV$$

Energy released in the process for α - particle ($Q > 0$ and it is an exothermic reaction).

$$E_\alpha = \frac{Q\cdot M_{Li}}{M_{Li} + M_\alpha} = \frac{2.78\times 7.018221}{4.003879 + 7.018221} \Rightarrow E_\alpha \approx 1.78\,MeV$$

We know that $E_{Li} + E_\alpha = Q \Rightarrow E_{Li} = Q - E_\alpha = 2.78 - 1.78 \approx 1.00\ MeV$

Ans.27: (b)

$a+A\to B+b;\qquad p+Li^7\to Be^7+n$

$$E_{th}=-Q\left(\frac{M_A+M_a}{M_A}\right)=1.65\left(\frac{7+1}{7}\right)\approx 1.9\ MeV$$

Ans.28: (d)

$$Q=X-\left[Y+2m_{e^+}\right]=\left(X-Y\right)-2m_{e^+}=\left(51.9648\ u-51.9571\ u\right)\times 931.5MeV-1.02\ MeV$$

$$Q=7.17\ MeV-1.02\ MeV=6.2\ MeV$$

Ans.29: (c)

Ans.30: (d)

The energy (Q) released in all fusion reaction

(a) ${}_1H^1+{}_1H^1\to{}_1H^2+e^++\nu$

$$Q=\left[M\left({}_1H^1\right)+M\left({}_1H^1\right)-M\left({}_1H^2\right)-M\left(e^+\right)\right]\times 931.5\ MeV$$

$$\Rightarrow Q=\left[1.00783\ u+1.00783\ u-2.01410\ u-0.00055\ u\right]\times 931.5\ MeV$$

$$\Rightarrow Q=\left(1.07\times10^{-3}\right)\times 931.5\ MeV\Rightarrow Q=0.99\ MeV$$

(b) ${}_1H^2+{}_1H^1\to{}_2He^3$

$$Q=\left[M\left({}_1H^2\right)+M\left({}_1H^1\right)-M\left({}_2He^3\right)\right]\times 931.5\ MeV$$

$$\Rightarrow Q=\left[2.01410\ u+1.00783\ u-3.01603\ u\right]\times 931.5\ MeV$$

$$\Rightarrow Q=\left(5.9\times10^{-3}\right)\times 931.5\ MeV\Rightarrow Q=5.5\ MeV$$

(c) ${}_1H^3+{}_2He^3\to{}_3Li^6$

$$Q=\left[M\left({}_1H^3\right)+M\left({}_2He^3\right)-M\left({}_3Li^6\right)\right]\times 931.5\ MeV$$

$$\Rightarrow Q=\left[3.01605\ u+3.01603\ u-6.01512\ u\right]\times 931.5\ MeV$$

$$\Rightarrow Q=\left[.01696\ u\right]\times 931.5\ MeV\Rightarrow Q=15.8\ MeV$$

(d) ${}_1H^2 + {}_2He^4 \rightarrow {}_3Li^6$

$$Q = \left[M\left({}_1H^2\right) + M\left({}_2He^4\right) + M\left({}_3Li^6\right) \right] \times 931.5\ MeV$$

$$\Rightarrow Q = [2.01410\ u + 4.02603\ u - 6.01512\ u] \times 931.5\ MeV$$

$$\Rightarrow Q = [0.02501\ u] \times 931.5\ MeV \ \Rightarrow Q = 23.3\ MeV$$

Numerical Answer Type Question (NAT)

Ans.31: 2

Since $R = R_0 (A)^{1/3} \Rightarrow \dfrac{R_{Pb}}{R_{Mg}} = \left(\dfrac{A_{Pb}}{A_{Mg}}\right)^{1/3} = \left(\dfrac{208}{26}\right)^{1/3} = (8)^{1/3} = 2$

Ans.32: 1.33

Since $R = R_0 (A)^{1/3} \Rightarrow \dfrac{R_{Cu}}{R_{Al}} = \left(\dfrac{A_{Cu}}{A_{Al}}\right)^{1/3} = \left(\dfrac{64}{27}\right)^{1/3} = \dfrac{4}{3} = 1.33$

Ans.33: 11.48

${}^{42}_{20}Ca \rightarrow {}^{41}_{20}Ca + {}^{1}_{0}n$;

Total mass of the ${}^{41}_{20}Ca$ and ${}^{1}_{0}n = 41.970943\ u$.

Mass defect $\Delta m = 41.970943 - 41.958622 = 0.012321\,u$

So, B.E. of missing neutron= $\Delta m \times 931.5 = 11.48\ MeV$

Ans.34: 10.27

${}^{42}_{20}Ca \rightarrow {}^{41}_{19}K + {}^{1}_{1}p$;

Total mass of the ${}^{41}_{19}K$ and ${}^{1}_{1}p = 41.969101\ u$.

Mass defect $\Delta m = 41.969101 - 41.958622 = 0.010479\,u$

So, B.E. of missing proton= $\Delta m \times 931.5 = 10.27\ MeV$.

Ans.35: 7.3

${}_6C^{12} \rightarrow 3\,{}_2He^4$

Mass defect $\Delta m = (3 \times 4.0026 - 12)u = 0.0078u = 0.0078 \times 931.5 = 7.2657\ MeV$

Ans.36: 24000

$$\lambda = \frac{1}{t}\ln\frac{R_0}{R} = \frac{1}{280}\ln\frac{R_0}{6000} = \frac{1}{(280+140)}\ln\frac{R_0}{3000} \Rightarrow R_0 = 24000\ dps.$$

Ans.37: 1.0

Decay constant $\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{1622\times365\times24\times60\times60} = 1.36\times10^{-11}\sec^{-1}$

Number of atoms in 1.00 g is $N = \frac{1}{226}\times6.023\times10^{23}$ $=2.7\times10^{21}$ atoms

Hence, activity $R = \lambda N = 1.36\times10^{-11}\times2.7\times10^{21} = 3.7\times10^{10}$ $decay/\sec = 1.0\,\mathrm{Ci}$

Ans.38: 95

Since the reaction is due to thermal neutron $E_a \approx 0$

$$a + A \rightarrow B + b;$$

$$\Rightarrow E_a = Q\frac{M_B}{M_B+m_a} = 160\times\frac{138}{138+95} \approx 95\text{MeV}$$

Ans.39: 8 and 6

$${}_ZX^A \rightarrow {}_{Z-2}Y^{A-4} + {}_2He^4 \text{ and } {}_ZX^A \rightarrow {}_{Z+1}Y^A + e^-$$

Change in A occurs only due to $\alpha-$particle and change in Z occurs due to α and β both.

Let number of $\alpha-$particles emitted $= n_1$ and number of $\beta-$particles emitted $= n_2$

$$\therefore n_1\times4 = 238-206 = 32 \text{ or } n_1 = 8$$

For Z, $(n_1\times2)-(n_2\times1) = 92-82$ or $(8\times2)-10 = n_2$ or $n_2 = 6$

$\therefore$ $\alpha-$Particle emitted are 8 and $\beta-$particles emitted are 6.

Ans.40: 23.6

$$2\left({}_1H^2\right) \rightarrow {}_2He^4$$

Binding energy of two deuterons $= E_1$ $\quad\therefore E_1 = 2[2\times1.1] = 4.4\,MeV$

Binding energy of helium nucleus $= E_2$ $\quad\therefore E_2 = 4(7.0) = 28.0\,MeV$

$\therefore$ Energy released $\Delta E = E_2 - E_1 = 28 - 4.4 = 23.6MeV$

Multiple Select Type Questions (MSQ)

Ans.41: (a), (d)

Let N_0 be the number of initial number of nuclei. Then

$$n = N_0 - N_0 e^{-2\lambda} = N_0(1-e^{-2\lambda})$$

$$0.75n = N_0 e^{-2\lambda} - N_0 e^{-2\lambda} e^{-2\lambda} = N_0 e^{-2\lambda}(1-e^{-2\lambda})$$

$$\frac{0.75n}{n} = \frac{N_o e^{-2\lambda}\left(1-e^{-2\lambda}\right)}{N_o\left(1-e^{-2\lambda}\right)} \Rightarrow e^{-2\lambda} = \frac{3}{4} \Rightarrow \lambda = 0.1438\,\text{sec}^{-1} \quad \Rightarrow \bar{T} = \frac{1}{\lambda} = 7\,\text{sec}$$

Ans.42: (a), (b) and (c)

(a) ${}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He$ and ${}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + \beta^-$

Change in mass number $= 232 - 208 = 24 \Rightarrow \frac{24}{4} = 6$ α-decay.

Change in atomic number after 6 α-decay $= 90 - 12 = 78$.

Final products mass number $= 82$; 4 β^--decay

(b) Number of nuclei present after 300 year $N = N_o\left(\frac{1}{2}\right)^{T/T_{1/2}}$

$$\Rightarrow N' = N_0 - N_0\left(\frac{1}{2}\right)^3 = \frac{7}{8}N_0$$

(c) $N_A\lambda_A = N_B\lambda_B \Rightarrow \frac{N_A}{N_B} = \frac{\lambda_B}{\lambda_A} = \frac{T_{1/2A}}{T_{1/2B}} = \frac{4.5\times10^9}{2.5\times10^5} = 1.8\times10^4$

(d) $R = \lambda N \Rightarrow N = \lambda N_0 \Rightarrow T_{1/2} = \frac{0.693}{\lambda} = 0.693\frac{N_0}{N}$

Ans.43: (a), (b) and (c)

For momentum conservation momentum of Th and He must be same and opposite.

$$m_{Th}v_{Th} = m_{He}v_{He} \Rightarrow v_{Th} < v_{He} \because m_{Th} > m_{He}$$

$$\text{Also } \frac{1}{2}m_{Th}v_{Th}^2 < \frac{1}{2}m_{He}v_{He}^2$$

Ans.44: (a), (c) and (d)

Mass of the neutron is more than the mass of the proton. Outside the nucleus decay of proton into neutron will led to violation of conservation of mass and energy.

Ans.45: (a), (b) and (d)

For beta decay, $n \to p + e^- + \bar{\nu}$

A neutron disrupts into a proton and a beta particle. The beta particle is ejected from the nucleus. The beta particles are fast moving electrons.

Ans.46: (b), (c)

Nuclear fusion reaction occurs when two or lighter nuclei combine to produce a heavier nucleus.

Ans.47: (a), (b) and (c)

Ans.48: (a), (d)

Ans.49: (c), (d)

M_1 is the mass of ${}_{10}Ne^{20}$ nucleus, M_2 is the mass of ${}_{20}Ca^{40}$ nucleus.

Ne means 10 protons + 10 neutrons, Ca means 20 protons + 20 neutrons.

Due to mass defect, which is necessary for binding the nucleus, mass of nucleus is always less than the sum of masses of protons and neutrons

For ${}_{10}Ne^{20}$ nucleus, $M_1 < 10(m_p + m_n)$

The mass defect is more in case of heavier nucleus where in binding energy needed is more $\therefore 20(m_p + m_n) - M_2 = B.E.$ for ${}_{20}Ca^{40}$

$10(m_p + m_n) - M_1 = B.E.$ for ${}_{10}Ne^{20}$

$\therefore 20(m_p + m_n) - M_2 > 10(m_p + m_n) - M_1$

$\therefore 10(m_p + m_n) > (M_2 - M_1)$ or $M_2 < M_1 + 10(m_p + m_n)$

$\because M_1 < 10(m_p + m_n)$ $\qquad \therefore M_2 < 2M_1$

Ans.50: (b), (c) and (d)