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Modes of Mathematical Modelling

An analysis of how modelling is used and interpreted in and out of school settings

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To Susanne, Max, My and Maya

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Publications

This thesis includes five papers, see below:

[1] Frejd, P, & Bergsten, C. (Submitted). Mathematical modelling as a professional task: Implications for education.

[2] Frejd, P. (2013). An analysis of mathematical modelling in Swedish textbooks in upper secondary school. *Nordic Studies in Mathematics Education, 18*(3), 59-95.

[3] Frejd, P. (2012). Teachers' conceptions of mathematical modelling at upper secondary school in Sweden. *Journal of Mathematical Modelling and Applications, 1*(5), 17-40.

[4] Frejd, P. (2013). Modes of modelling assessment - A literature review. *Educational Studies in Mathematics*, *84*(3), 413-438.

[5] Ärlebäck, J. B., & Frejd, P. (Submitted). The bottleneck problem in modelling revisited.

The publications in the list below are connected to my research about mathematical modelling in mathematics education, but are not presented as 'full version' papers in this thesis except for nr 17, which is related to research within the research field of using history in mathematics education.

[6] Frejd, P. (2010). Revisiting perspectives on mathematical models and modelling. In C. Bergsten, E. Jablonka, & T. Wedege (Eds.)*, Mathematics and mathematics education: Cultural and social dimensions: Proceedings of Madif 7, Stockholm, 26-27, January, 2010* (pp. 80-90). Linköping: SMDF.

[7] Frejd, P., & Ärlebäck, J. B. (2010). On Swedish upper secondary students' description of the notion of mathematical models and modelling. In C. Bergsten, E. Jablonka, & T. Wedege (Eds.)*, Mathematics and mathematics education: Cultural and social dimensions: Proceedings of Madif 7, Stockholm, 26-27, January, 2010* (pp. 91-101). Linköping: SMDF.

[8] Ärlebäck, J. B., & Frejd, P. (2010). First results from a study investigating Swedish upper secondary students' mathematical modelling competencies. In A. Araújo, A. Fernandes, A. Azevedo, & J.F. Rodrigues (Eds.), *Proceedings of EIMI 2010, Educational interfaces between mathematics and industry, Lisbon 19-23 April* (pp. 063-074). Compac, Inc., Bedford MA USA.

[9] Frejd, P. (2011). An investigation of mathematical modelling in the Swedish national course tests in mathematics. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.). *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 947-956). University of Rzeszów, Poland.

[10]Frejd, P. (2011). Mathematical modelling in the Swedish national course tests in mathematics. In M. Mortensen, & C. Winsløw (Eds), *The Anthropological Theory of the Didactical (ATD) - peer reviewed papers from a PhD course at the University of Copenhagen, 2010* (pp. 61-78). IND Skriftserie 20. Copenhagen: Dept. of Science Education, 2011.

[11]Frejd, P. (2011). *Mathematical modelling in upper secondary school in Sweden an exploratory study*. Licentiate thesis. Linköping: Linköpings universitet.

[12]Frejd, P. (2011). *Teachers' conceptions of mathematical modelling at upper secondary school in Sweden* (Rapport nr 2011:04, LiTH-MAT-R--2011--SE). Linköping: Linköpings universitet, Matematiska institutionen.

[13]Frejd, P., & Ärlebäck, J. B. (2011). First results from a study investigating Swedish upper secondary students' mathematical modelling competencies. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 407-416). Springer: New York.

[14]Frejd, P. (2012). Alternative modes of modelling assessment- A literature review. *Pre-proceedings of ICME12, The 12th International congress on mathematics education, Seoul, Korea, July 8-15, 2012* (pp. 3224-3233).

[15]Frejd, P. (2012). Modelling Assessment of Mathematical Modelling – a Literature Review. In C. Bergsten, E. Jablonka, & M. Raman (Eds.), *Evaluation and Comparison of Mathematical Achievement: Dimensions and Perspectives: Proceedings of Madif 8, Umeå, 24-25, January, 2012* (pp. 81-90). Linköping: SMDF.

[16]Frejd, P. (2013). Mathematical modellers' opinions on mathematical modelling in upper secondary education. Paper presented at *ICTMA 16, Blumenau, Brazil, 14-19 July 2013.*

[17]Frejd, P. (2013). Old algebra textbooks: a resource for modern teaching. *BSHM Bulletin Journal of the British Society of the History of Mathematics, 28*(1), 25-36.

[18]Frejd, P., & Geiger, V. (2013). Theoretical approaches to the study of modelling in mathematics education. Paper presented at *ICTMA 16, Blumenau, Brazil, 14-19 July 2013.*

[19]Geiger, V., & Frejd, P. (2013). The direction of applications and modelling as a field of research. Paper presented at *ICTMA 16*, *Blumenau, Brazil, 14-19 July 2013.*

[20]Ärlebäck, J. B., & Frejd, P. (2013). Modelling from the perspective of commognition – an emerging framework, In G. Stillman, G. Kaiser, W. Blum, & J. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 47-56). New York: Springer.

[21]Frejd, P. (accepted). Mathematical modelling discussed by mathematical modellers. CERME 8, *Eighth Conference of European Research in Mathematics Education, Manavgat-Side 6-10 February 2013.*

Content

Abstract

The relevance of using mathematics in and for out-of-school activities is one main argument for teaching mathematics in education. Mathematical modelling is considered as a bridge between the mathematics learned and taught in schools and the mathematics used at the workplace and in society and it is also a central notion in the present Swedish mathematical syllabus for upper secondary school. This doctoral thesis reports on students', teachers' and modelling experts' experiences of, learning, teaching and working with mathematical modelling in and out of school settings and their interpretations of the notion of mathematical modelling.

The thesis includes five papers and a preamble, where the papers are summarised, analysed, and discussed. Different methods are being used in the thesis such as video analysis of students' collaboration working with modelling problem, interview investigations with teachers and expert modellers, content analysis of textbooks and literature review of modelling assessment. Theoretical aspects concerning mathematical modelling and the didactic transposition of modelling are examined.

The results presented in this thesis provide a fragmented picture of the didactic transposition of mathematical modelling in school mathematics in Sweden. There are significant differences in how modellers, teachers and students work with modelling in different practices in terms of the goal with the modelling activity, the risks involved in using the models, the use of technology, division of labour and the construction of mathematical models. However, there are also similarities identified described as important aspects of modelling work in the different practices, such as communication, collaboration, projects, and the use of applying and adapting pre-defined models. Students, teachers and modellers expressed a variety of descriptions of what modelling means. The variety of descriptions in the workplace is not surprising, since their working approaches are quite different, but it makes the notion difficult to transpose into school practise. Questions raised are if it is unrealistic to search for a general definition and if it is really necessary to have a general definition. The consequence, for anyone how uses the notion, is to always be explicit with the meaning.

An implication for teaching is that modelling as it shows in the workplace can never be fully 'mapped' in the mathematical classroom. However, it may be possible to 'simulate' such activity. Working with mathematical modelling in projects is suggested to simulate workplace activities, which include collaboration and communication between different participants. The modelling problems may for example involve economic and environmental decisions, to prepare students to be critically aware of the use of mathematics in private life and in society, where many decisions are based on mathematical models.

Sammanfattning

I skolans läroplaner är ett av huvudargumenten för att lära sig matematik att kunskaper i matematik är användbara utanför skolan, dvs. i vardags-, samhälls- och yrkeslivet. Ett område inom skolmatematiken som tydligt kopplar samman matematik i och utanför skolan är matematisk modellering, vilket även avspeglas i skolans kursplaner där det lyfts fram som ett centralt begrepp. Denna avhandling behandlar elevers, lärares och experters erfarenheter av att arbeta med matematisk modellering i och utanför skolan. Avhandlingen består av en sammanläggning av fem artiklar samt en litteraturöversikt av forskningsområdet och en diskussion av använda metodologiska ansatser (kappa). Den övergripande struktur som binder samman de fem artiklarna bygger på begreppet 'didaktisk transposition', utvecklat av Yves Chevallard. Denna 'transposition' innebär att kunskap som undervisas i skolan har en tidigare existens utanför skolan och kan ses som en produkt av en process där den senare förändras och förvandlas (transponeras) till 'skol-matematik' genom olika etapper, från institutionell etablerad kunskap (kopplat till den vetenskapliga disciplinen matematik), via läroplan och läromedel till vad som undervisas i klassrummet och bedöms (i t.ex. prov), till elevernas egna kunskaper. De fem artiklarna belyser dessa etapper med fokus på matematisk modellering genom att undersöka: 1. Hur arbetar professionella modellerare med matematisk modellering i sitt yrke? 2. Hur och vad presenteras om matematisk modellering i läroböcker? 3. Hur och vad undervisar lärare om matematisk modellering? 4. Vilka metoder används för att bedöma matematisk modellering och vad bedöms? 5. Hur arbetar elever när de formulerar matematiska modeller? I avhandlingen genomförs analyser av intervjuer med professionella modellerare och lärare, innehållsanalyser av läroböcker och av forskningslitteratur kring bedömning, samt diskursanalys av elevsamarbete kring uppgifter i matematisk modellering.

Resultatet ger en osammanhängande bild av den didaktiska transpositionen av matematisk modellering i svensk skolmatematik. Själva begreppet matematisk modellering ges en varierande innebörd både inom och utanför skolan, vilket gör det komplext att diskutera och hantera i undervisningssammanhang. Professionell matematisk modellering i yrkeslivet och lärares och elevers arbete med matematiska modeller i ett klassrum är helt skilda typer av verksamheter, då syfte och konsekvenser av användningen inte är förenliga mellan de två institutionerna. Dessutom bygger professionella modellerare sina modeller utifrån många års erfarenheter, avancerade kunskaper i matematik, samt kunskaper om programmering och tekniska hjälpmedel som saknas i skolan. Läroböcker beskriver matematisk modellering på mycket olika sätt beroende på läroboksserie. Överlag lyfter dock inga av de undersökta läromedlen fram matematisk modellering som en central aktivitet. Inte heller de intervjuade lärarna beskriver matematisk modellering som en viktig del i matematikundervisningen. De kopplar istället ofta matematisk modellering till fysik och ger endast ett fåtal exempel på aktiviteter där de arbetar med modellering. Analysen av forskningslitteratur kring bedömning visar att kunskaper i modellering inte enkelt kan utvärderas med skriftliga prov.

Det som lyfts fram för att kunna bedöma modellering som en helhet är att använda projektarbete. Studien av elevers arbete med en modelleringsuppgift visar hur de formulerar matematiska modeller genom att känna igen en situation och använda sig av redan kända modeller. Dessa modeller anpassas till situationen genom att eleverna förhandlar med varandra om hur de ska tillämpas. Här är det också centralt hur eleverna använder sig av sina kunskaper inom andra kunskapsområden än matematik samt sina tidigare mer personliga erfarenheter.

Resultatens implikationer för undervisning och lärande antyder att läraren bör vara tydlig med att redogöra explicit för sig själv och för sina elever den tolkning av begreppet matematisk modellering som ska bedömas och användas i klassrummet, då det finns många beskrivningar av begreppet. Synen på modellering som används i klassrummet påverkar också i vilken utsträckning läromedel måste kompletteras och vilka metoder som kan används vid bedömning. Lyfter till exempel läraren fram modellering som en helhet och försöker efterlikna professionella modellerares arbetssätt, så visar denna avhandling att undervisningstid kan ges till projektarbeten som innehåller grupparbete, kommunikation och tekniska hjälpmedel. Ett exempel på projekt beskrivs i avhandlingen kopplat till politiska diskussioner angående miljö och ekonomi, då många beslut i dagens samhälle bygger på matematiska modeller. Syftet med projektet är både att skapa förutsättningar för att utveckla en modelleringsförmåga men också att förbereda eleverna att kunna granska beslut, skapa opinion och bli kritiska demokratiska medborgare.

PART I

PREAMBLE

Chapter 1

Introduction

A common practice for Swedish Ph.D. (Doctor of Philosophy) students is to present a licentiate thesis, after half of their research time, which later on is used as a source to produce a doctoral thesis. This thesis is derived in that manner and includes some content presented in Frejd (2011c). In particular, the preamble in Frejd (2011c) is used as a platform for the structure of the preamble in this thesis. The content in the background chapter and the chapter on mathematical modelling are in this thesis updated, adapted and extended compared to Frejd (2011c). There are also some similarities in the chapters of results, discussions and implications, as one of the five articles in this thesis was included in the licentiate thesis.

The title of this thesis is '*Modes of mathematical modelling*'. The word *modes* is used to denote "a way or manner in which something [in this case, mathematical modelling, ...] is experienced, expressed, or done" (mode, 2013). How mathematical modelling is experienced, expressed and done by different actors in and out of school settings is central in this thesis and will be discussed throughout the thesis, starting with the background section.

1.1 Background

Almost all students (99%) in Sweden that have completed compulsory school enter upper secondary school (Skolverket, 2008). The aim of upper secondary school is described in the Educational Act as:

> The upper secondary school should provide a good foundation for work and further studies and also for personal development and active participation in the life of society. The education should be organised so that it promotes a sense of social community and develops students' ability to independently and jointly with others acquire, deepen and apply knowledge. (A translated version of paragraphs 1 and 2, section 2 in chapter 15 in Skollagen (SFS 2010:800) cited in Skolverket, 2012b, p. 8)

The aim above concerns all of the 18 different national programs that presently are found in the upper secondary education. These national programs consist of different subjects and courses (Skolverket, 2012b). One subject which all students in upper secondary school in Sweden study is mathematics. It is one of the nine core subjects[1] (Skolverket, 2012b) and a requirement for further studies at universities. The present official subject syllabus for mathematics, formulated in 2011, states that the aim of teaching mathematics is to foster students' ability to work mathematically (Skolverket, 2012a).

> This involves developing an understanding of mathematical concepts and methods, as well as different strategies for solving mathematical problems and using mathematics in social and professional situations. Teaching should give students the opportunity to challenge, deepen and broaden their creativity and mathematical skills. In addition, it should contribute to students developing the ability to apply mathematics in different contexts, and understand its importance for the individual and society. Teaching should cover a variety of working forms and methods of working, where investigative activities form a part. Where appropriate, teaching should take place in environments that are relevant and closely related to praxis. (Skolverket, 2012a, p. 1)

Both the citation above and the citation in the previous page seem to highlight the importance for students to develop abilities that will help them in their social life, in future occupations and/or in higher education. The teaching methods should, according to the citations, be organised as collaborative and/or individual work in such a way that it promotes investigation activities relevant for out of school situations. One investigation activity in mathematics education related to such situations is mathematical modelling, which is described as one of the seven[2] teaching goals in the subject syllabus mathematics (Skolverket, 2012a). There it is stated that the modelling ability, that students are going to develop with help of teaching, is to "interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations" (ibid., p. 2).

Mathematical models are used for various purposes. For example, economical models and environmental models are used as a source for decision making at different levels in the society and in the workplace (Hunt, 2007). To understand how and why these models have been developed, how they are being used and their limitations is an ability that may create critical and reflecting citizens in a democracy (Skovsmose, 1994, 2005). The descriptions from the Swedish subject syllabus indicate that the use of realistic modelling activities in the mathematics classroom may contribute to develop students' understanding of how and why mathematics is used in the society and in the workplace, at least if the modelling problems are chosen adequately. Using workplace related modelling problems in

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[1] The nine core subjects are E*nglish, history, physical education and health, mathematics, science studies, religion, social studies and Swedish or Swedish as a second language*.

[2] The other teaching goals described as abilities are: concept, procedure, problem solving, reasoning, communication and relevance.

the classroom might also be used to strengthen the connection between school and workplace, which was one aim for the new curriculum reform in 2011, i.e. "[c]oordination between school and working life must be strengthened to ensure high quality of education and strong involvement from industry and the public sphere" (Skolverket, 2012b, p. 12). Arguments are also put forward that *one major* issue described as an interface between mathematics and a workplace is mathematical modelling (Sträßer, Damlamian & Rodrigues, 2012). One suggestion by Drakes (2012), which may strengthen the connection between industry and school as well as strengthen the teaching of mathematical modelling, is to mirror some parts of expert modellers' working practice. She argues that "[s]tudents would ... benefit from seeing real modelling done by experts. Seeing experts deal with being stuck is informative, and helps change the belief that experts simply rely on intuition" (ibid., 2012, p. 207).

In research literature one can find many arguments that promote modelling activity in mathematics education. Blum and Niss (1991) have done a review of arguments from "representative literature" (p. 42) and they set out 'five main arguments' for including mathematical modelling in the curriculum: 1. *The formative argument,* 2. *The 'critical competence' argument,* 3. *The utility argument,* 4. *The 'picture of mathematics' argument,* 5. *The 'promoting mathematics learning' argument* (pp. 42-44). These five arguments are further discussed in section 2.4 in relation to the Swedish curriculum. However, from the five main arguments by Blum and Niss (1991) mathematical modelling can be seen as an activity in school aiming at different goals as for example: teaching mathematics, teaching mathematical modelling, teaching critical evaluation of models, teaching about the relevance of mathematics in society. These goals are also stated in the Swedish official curriculum guidelines either explicitly or implicitly.

Research with focus on teaching and learning mathematical modelling in mathematics education has gained international interest since the middle of the 1960's (Blum, 1995). Today there are researchers around the world engaging in educational research on mathematical modelling and every second year since 1983 there is an international conference with the specific focus on the teaching and learning of mathematical modelling and applications (ICTMA). Also the last five[3] Congresses of the European Society for Research in Mathematics Education (CERME) have had thematic working groups explicitly focusing on issues in mathematics education related to mathematical modelling. Also, the International Congress on Mathematical Education (ICME) has had topic study groups focusing on mathematical modelling. In addition, there have been many articles, papers and books addressing different issues regarding mathematical modelling in mathematics education. An overview of the state and trends of this field of research can

[3] CERME 4 in Sant Feliu de Guíxols, Spain 2005; CERME 5 in Larnaca, Cyprus 2007; CERME 6 in Lyon, France 2009; CERME 7 in Rzeszów, Poland 2011; CERME 8 in Antalya, Turkey 2013.

be found in the ICMI 14 Study: Modelling and applications in mathematics education (Blum, Galbraith, Henn & Niss, 2007).

As described in the introduction, mathematical modelling is one of the seven mathematical abilities to be taught in the present Swedish upper secondary subject syllabus and there is a vast literature of international research concerning modelling in mathematics education. However, mathematical modelling seems not to play a major role in either Swedish research literature in mathematics education nor in the present mathematics classrooms in Swedish upper secondary school, which will be discussed in the next section.

1.1.1 The present situation in Sweden

An analysis of the official subject syllabuses for mathematics between 1965 to the syllabus formulated in the year 2000 showed that the role of mathematical modelling in upper secondary mathematics education in Sweden has been made more and more explicit during the years (Ärlebäck, 2009b). Assuming that the frequency of how often the word model is found in the mathematical syllabuses is one measure of how explicit the notions of models and modelling are. A counting of the word model in the previous syllabus (Skolverket, 2001) gives the result of 33 matches while in the present syllabus (Skolverket, 2012a) 68 matches are found, which may be one indication that the notions are now more explicit in the present syllabus than the ones investigated by Ärlebäck (2009c). In addition, as noted by Ärlebäck (2009c), there were no explicit definitions of either mathematical models or mathematical modelling presented in the syllabuses up to the year of 2000. This lack of descriptions in the older subject syllabuses of mathematical models and mathematical modelling opened up for interpretations of notions from different actors in the Swedish school system. In the present mathematic syllabus the situation is different. A specific and explicit modelling ability to be taught is stated. However, even if the modelling ability is explicitly described it is still open for interpretations. The quote, "interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations" Skolverket, 2012a, p. 2) may for example be interpreted to differentiate between: 1) interpret a realistic situation and design a mathematical model *and* 2) to use and assess a model's properties and limitations. How this statement is turned into practice is an issue discussed in this thesis.

Educational research about mathematical models and modelling has been sparse in Sweden, especially if one compares to what has been done in some other countries, like Germany, Australia, USA, Great Britain, and the Netherlands. A few studies have been carried out concerning mathematical modelling aspects in relation to teacher education and to prospective teachers (Holmquist & Lingefjärd, 2003; Lingefjärd, 2000, 2002a, 2002b, 2006a, 2006b, 2007, 2010, 2112; Lingefjärd & Holmquist, 2001, 2005, 2007). Wikström (1997) investigated students using models in computer programs. Palm's (2002, 2007) research is also related to modelling in the sense of aspects connected to authentic and realistic tasks in school settings. However, resent research by Ärlebäck (2009a, 2009b, 2009c, 2009d, 2010), Frejd (2010, 2011a, 2011b, 2011c, 2011d) and Frejd and Ärlebäck (2011, 2010) has broadened the agenda with an explicit focus on mathematical modelling in upper secondary education with aims to seek for an overall picture of how the notions are being interpreted by different actors in the educational system. The findings from the studies above will be discussed in more detail in section 2.5. According to Ärlebäck (2009c) the general picture about mathematical modelling in upper secondary school has began to emerge, but there is still much research to be done. I agree with Ärlebäck that the picture is far from complete and further exploratory research studies are needed.

To sum up, mathematical modelling is a central notion in the present Swedish mathematical syllabus and described as an ability to be taught in the classroom. The description of modelling is explicit in the syllabus, but it is open for interpretations from different actors in the Swedish school system. The research focused on modelling in mathematics education in Sweden has so far been moderate. Mathematical modelling is also considered as a bridge between the mathematics learned and taught in schools and the mathematics used at the workplace and in society. One category of workers in the workplace that do possess a modelling ability and rely on mathematical modelling in their occupation is 'expert modellers' and there are arguments put forward that experiences from these experts might be useful for teaching modelling.

Pertinent questions to be asked in order to describe the present situation in upper secondary school in Sweden related to mathematical modelling and its relation to mathematical modelling used by modellers at the workplace are: What is the meaning of mathematical modelling in the workplace and in context of mathematics education? How is the notion of mathematical modelling interpreted by those working with mathematical modelling in the workplace and by different actors in the Swedish school system? Why are they interpreted in this manner? Do the interpretations differ? What types of modelling are professional modellers working with? How are these modelling activities related to modelling in the mathematics classroom? How do professional modellers and students solve modelling problems? What conceptions do teachers and professional modellers have about how modelling can be taught and learned in schools? Do teachers at the upper secondary level believe that mathematical modelling is a part of mathematics/ mathematics education? Are students being assessed on mathematical modelling, and if so what methods are there or are used? What interpretations of mathematical modelling do textbook authors make? What type of mathematical modelling items can be found in course literature? Are these modelling items related to those that professional modellers work with? …

In this doctoral thesis I will present, discuss and try to provide answers to some of these pertinent questions.

1.1.2 An orientation about upper secondary school in Sweden

The upper secondary school in Sweden presently consists of 18 national programmes (12 vocational programmes and six higher education preparatory programmes). All programs last for three years and have their own diploma goals. The diploma goals describe the specific orientations and goals with the different programmes including the goals of the programme specific diploma projects. The programmes consist of subjects, which are divided into courses. For example the subject mathematics is divided in six courses. These courses are: *Mathematics 1-Mathematics 5* and *Mathematics-specialization.* However, the courses in mathematics are also divided a second time into three different paths for different programmes (see Figure 1). Path *a* is for all *vocational programmes*, path *b* is for *The Business Management and Economics Programme*, *the Arts Programme*, *the Humanities Programme*, and *the Social Science Programme* and path *c* is for the *Natural Science Programme* and the *Technology Programme*.

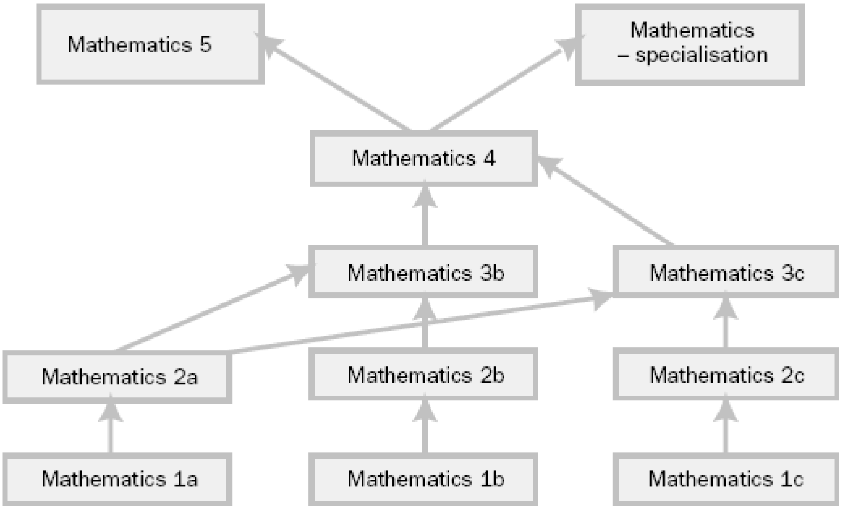

Figure 1. The hierarchy of mathematics courses including the three different paths (Skolverket, 2012b, p. 37).

The mathematics courses *Mathematics 1* to *5* build on each other as displayed in Figure 1. *Mathematics-specialization* is an optional course and builds on *Mathematics 4*. Which of the courses in Figure 1 are optional depends on the programme. *Mathematics 1* is a core course, compulsory for all students in all programmes. Four programmes require more than *Mathematics 1*, they are: *The Technology Programme* (1c, 2c & 3c); *The Natural Science Programme* (1c, 2c & 3c); *The Social Science Programme* (1b & 2b) and *The Business Management and Economics Programme* (1b & 2b).

Those students from the vocational programmes who want to study more mathematics courses may transfer paths from 2a to 3b or 3c. The last two mathematics courses (*Mathematics 4* & *5*) are the same courses for all students.

When a student has completed a course, he or she will get a grade depending on his/her level of proficiency. The grading scale has five pass levels, beginning with the highest grade, A, B, C, D and E and a fail grade F. In case a student has been absent in such an extent that the teacher does not have sufficient assessment information to give a grade, then a dash should be recorded.

All courses are given a set of credits from 50 credits to 200 credits per course. The credits can be seen as a measurement on how extensive a course is. For each mathematics courses shown in Figure 1, 100 credits are distributed. If a student studies all mathematics courses and completes them, he or she will earn 500 credits.

After three years of study in upper secondary school the students can obtain a diploma. The requirement to obtain either a vocational diploma or a diploma for admission to higher education is that the student has gained 2500 credits, of which 2250 credits must be awarded with passing grades. There are also requirements that some specific courses need be passed, for instance *Mathematics 1* is one such course.

1.1.3 My personal entrance into research studies

In the autumn of 2007 the government announced[4] a possibility for in-service teachers to engage in research studies. The offer was that teachers were given a possibility to work 20% of his/her working time at his/her school and the rest of the time do research with full payment during 2,5 years. The goal according to the Swedish minister for higher education and research at the time, Lars Lejonborg, was to create new careers for teachers as well as to increase the competence of teachers in order to improve quality in school. At that time I had been working as a mathematics and physics teacher for almost ten years at an upper secondary school in the south of Sweden. I had always been interested in educational development and I had a wish to do research, so I found the offer from the government very interesting. However, I was not up to date on what issues currently were investigated in mathematics education. I contacted professor Bergsten at Linköping University for further information. He described among other things, that at Linköping University one graduate research student was working with mathematical modelling and a research assistant was working with proofs. Mathematical modelling sounded exciting, but I did not have a clear conception about the notion of mathematical modelling. I contacted the research student, Jonas Bergman Ärlebäck, and he explained about his research and about the notion of mathematical modelling. The way he talked about mathematical modelling sounded familiar to me from the perspective of physics education, but

[4] The announcement can be retrieved from http://www.regeringen.se/sb/d/9985/a/100680

not from mathematics education. I often use modelling activities as a way of teaching in physics and started to question myself on why I had not used it mathematics education. Mathematical modelling at my school played a minor role in mathematics classes and it was not mentioned in our local curriculum. I began to wonder how it is in other schools in Sweden and how other teachers interpret the notion. With these questions in mind I decided to apply for research studies and I was one out of 25 students who was accepted for research studies at The Swedish National Graduate School in Science, Technology and Mathematics Education Research (FontD)[5]. I enjoyed my time as a research student while I was working with my licentiate thesis. After having presented my licentiate thesis I was given the possibility to continue my work and applied for a position as a doctoral student at Linköping University.

1.2 Aim and research questions

In line with Ärlebäck (2009c) and Frejd (2011c) the general aim of this thesis is to investigate the present situation in Sweden upper secondary school regarding mathematical modelling and evaluate the current state and status. However, this thesis also broadens the agenda with an aim to investigate how modelling is used in the workplace. The broadening is done since the use of mathematical models and modelling in the workplace plays a significant role in working decision at different levels in the society and industry (Hunt, 2007) and mathematics education may draw from expert modellers' experiences of how to work with modelling in the mathematical classroom (Drakes, 2012). To complement Ärlebäck's (2009c) and Frejd (2011c) emerging picture the focus in this thesis will be on aspects related to students', teachers' and professional modellers' experiences of working with mathematical modelling:

> *The aim is to present a report of the experiences that students, teachers and modelling experts have of learning, teaching and working with mathematical modelling in and out of school settings and how they interpret the notion of mathematical modelling.*

In a report from Skolinspektionen (2010) about the quality in upper secondary school in Sweden one can read that "[t]he inspection shows that remarkably many teachers pursue teaching which is not in line with all parts of the curriculum guidelines for the subject mathematics. Consequently not all students are educated to provide them with tools to understand mathematics and to use and apply their own full capacity" (p. 6, my translation). Mathematical modelling was not explicitly examined by Skolinspektionen (2010), but it is one part in the curriculum and maybe it does not get much attention in the mathematics classrooms in

[5] see http://www.isv.liu.se/fontd/nationella_fontd/start/1.180370/FolderAllmn_web.pdf.

Sweden. If so, then it may also affect students' knowledge about mathematical modelling and how it is used in and out of school settings.

There are also some evidence that modelling does not play a central part in the present mathematics classroom. In an investigation of 400 students about their modelling competency, only 23% of the students expressed that they had encountered mathematical models or mathematical modelling in their education before participating in the study. (Frejd & Ärlebäck, 2011). Maybe as a consequence of that result, the students in the same investigation seemed not to have a clear view about the meaning of the notion of mathematical modelling (ibid.). An analysis of modelling items in national courses tests indicates that only parts of the modelling process are assessed (Frejd, 2011a). Research has also pointed out teachers' lack of experience of mathematical modelling in Sweden (Ärlebäck, 2010). In a case study with two teachers, Ärlebäck (2010) concluded that the teachers were not able to express coherent formulated descriptions of the notions mathematical models and mathematical modelling. Traditional textbooks are often used as a guide for teaching mathematics in upper secondary school (Jablonka & Johansson, 2010; Skolinspektionen, 2009; Skolverket, 2003; SOU 2004:97) but these textbooks do not treat mathematical modelling explicitly (Jakobsson-Åhl, 2008; Ärlebäck, 2009c).

However, the new implemented curriculum with the mathematics syllabus has a more emphasis on modelling activities related to 'realistic' situations. So, what is the current state and status about students', teachers' and modelling experts' experiences of teaching, learning and working with mathematical modelling in and out of school settings and how do they interpret the notions of mathematical modelling?

How the research questions have been addressed to explore the aim and how the research questions will be operationalised is discussed in section 4.2.

1.3 The structure of the thesis

This thesis is structured in two parts. Part I, the preamble, includes five chapters, while Part II includes the five papers.

Part I will continue with Chapter 2, which discusses theoretical perspectives of mathematical models and modelling in order to explain and clarify the used notions in the thesis. In addition, a section in this chapter is dedicated to research on mathematical modelling related to Swedish upper secondary school, whereas international research is brought up and integrated throughout the text in this thesis. Chapter 3 is devoted to develop 'a red thread' and 'sew the papers together' with use of and inspiration from the notion of didactic transposition (see e.g. Bosch & Gascón, 2006; Hardy, 2009). The overall methodology of the thesis is presented in Chapter 4 where the research questions are specified into more specific research questions. In Chapter 5 the five papers are summarised. The results, conclusions and implications from the papers are finally discussed in Chapter 6.

In Part II the papers are presented in full versions[6].

[6] To fit the format of this thesis the layout of the paper may have been changed compared to the published versions.

Chapter 2

Mathematical modelling

The notion of mathematical modelling is described/ defined in different ways in mathematics education depending on the theoretical perspective adopted (Kaiser & Sriraman, 2006). In research literature in mathematics education one may find several different perspectives and approaches on mathematical modelling (Blum, Galbraith, Henn & Niss, 2007; Frejd & Geiger 2013; Garcia, Gascón, Higueras & Bosch, 2006; Haines, Galbraith, Blum & Khan, 2007; Jablonka & Gellert, 2007; Kaiser & Sriraman, 2006; Sriraman & Kaiser, 2006). Sriraman and Kaiser (2006) write "that there does not exist a homogenous understanding of modelling and its epistemological backgrounds within the international discussion on application and modelling" (p. 45) in a report from the fourth Congress of the European Society for Research in Mathematics Education, CERME4. Section 2.2 will provide a brief overview of a selection of descriptions of theoretical approaches used in research literature about mathematical modelling in mathematics education, but first a short discussion (adapted from Frejd, 2010) on the closely related term mathematical model.

2.1 Different interpretations of the term mathematical model

Blum et al. (2007) attempt to clarify the basic notions and terms related to the term mathematical model and the modelling process (described later in this thesis). They give some examples of standard models (linear, exponential or logistic growth, inverse proportionality, etc.), but it becomes apparent that there is no clear or shared definition of a 'mathematical model'.

Common definitions stress the representational aspects of mathematical models: According to the Encyclopaedia Britannica a "mathematical model is either a physical representation of mathematical concepts or mathematical representation of reality" (mathematical model, 2013a). For example, a physical model is a three-dimensional surface made of wires to visualize some abstract mathematical concept and about mathematical model of reality one reads "anything in the physical or biological world, whether natural or involving technology and human intervention, is subject to analysis by mathematical models if it can be described in terms of mathematical expressions" (ibid.). Wikipedia´s definition is: "A mathematical model is a description of a system using mathematical concepts and language" (mathematical model, 2013b). Representational aspects of models are also found in technology literature, where one can find a definition of a mathematical model as "a representation of essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form" (Eykhoff, 1974, p. 1). A '*purposeful* representation' is another definition from literature about learning how to model (Starfield, Smith & Bleloch, 1990). The definition also highlights that all models can only be discussed and criticized in relation to their specific purposes. In literature from mathematics education

> models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation system, and that are used to construct, describe, or explain the behaviors of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently. A mathematical model focuses on structural characteristics (rather then, for example, physical or musical characteristics) of the relevant systems. (Lesh & Doerr, 2003, p. 10)

According to this description models are situated both in the mind of the learner and in representational media (equations, etc.).

To sum up, mathematical models are described in dictionaries, technology and mathematics education literature to be some kind of representation/ description/ explanation of 'something' in terms of structural characteristics/ expressions from mathematics. These models may be situated in a variety of different places such as the mind of the learner, the discourse where the model is being used or in some form of representational media/ physical object. This 'something' is described in terms of vague expressions, such as conceptual system, existing system, knowledge or reality. The process to create a model of "something" is called modelling.

2.2 Different perspectives on mathematical modelling

Frejd and Geiger (2013) conducted a content analysis of papers from the last five ICTMA proceedings and the 14th ICMI study to explore the expansion of theoretical approaches used in research literature focusing on mathematical applications and modelling. The study identified a set of both local theoretical approaches and more general theoretical approaches that were used.

Examples of local theoretical approaches are *Modelling cycles:* a description of the modelling process as a cyclic activity (e.g. Blomhøj & Højgaard Jensen, 2003; Blum & Niss, 1991); *Modelling competence:* a notion used to define a skill or ability to perform modelling (e.g. Blomhøj & Højgaard Jensen, 2007; Maaβ, 2006); *Emergent modelling:* an instructional approach, with roots from Freudentahl's realistic mathematics education, using how models emerge from contextual problems as means for supporting the emergence of formal mathematics knowledge (e.g. Freudenthal, 1983; Gravemeier, 2007); and *Models and modelling perspective:* a teaching approach with model eliciting activities developed by Lesh and Doerr (2003).

Some examples of general theoretical approaches are *Socio-cultural approaches* that draw on Vygotsky and followers like situated learning and community of practice by Lave and Wenger (1991), and activity theory by Engeström (1987), etc.; *Sociological approaches* where examples are work by Bernstein (2000), Dowling (1996), etc.; *Discursive, linguistics, social linguistics, and semiotics approaches* developed and discussed for example by Sfard (2008), Halliday (1978), Evans and Morgan (2009); *Beliefs, attitudes and affect:* examples of work relate to Leder, Pehkonen and Törner (2002), etc.; *Critical mathematics education* developed by Skovsmose (1994); *ATD/TDS* described by Chevallard(1999)/ Brousseau (1997); *Instrumental approaches* like the work from Rabardel (1995), Artigue (2002) etc.; and *Pragmatic approaches* referring to Dewey.

The content analysis revealed that the local approaches *Modelling cycle* and *Modelling competencies* are used more frequently than all general theoretical approaches together. The notion of mathematical modelling plays a key role in the local theoretical approaches, but not in the general approaches except for *Critical mathematics education* (Skovsmose, 1994, 2005) and *ATD* (Garcia, Gascón, Higueras & Bosch, 2006) which use the notion explicitly.

To illustrate the diversity of meanings associated with mathematical modelling in the context of the teaching and learning of mathematics the next three sections 2.2.1-2.2.3, slightly adapted versions of outlines in Frejd (2011b, 2011d) and Paper 5, will describe three theoretical perspectives of mathematical modelling. The three theoretical approaches are; One local theoretical approach, the *modelling cycle*; One general theoretical approach which explicitly uses the notion of modelling, *ATD*; One general theoretical discursive approach that does not explicitly use mathematical modelling, Anna Sfards' theory of *commognition*. The rationales for illustrating mathematical modelling in this preamble with this selection are that the selection includes both local and general approaches with and without explicit descriptions of modelling. The reason of picking the modelling cycle, sometimes named as the *traditional* description (Williamas & Goos, 2013), as a local approach is beacuse it was the most frequently used approach found in Frejd and Geiger (2013) and is a well established theoretical approach within the research community of mathematical modelling in mathematics education according to Stillman (2012). The modelling cycle is also related to the notion of modelling competence.

Both ATD and Critical mathematics education are general theoretical approaches and explicitly use the notion of mathematical modelling. *ATD* was more frequently used in the study by Frejd and Geiger (2013) than *Critical mathematics education*. Modelling viewed from *ATD* regards all mathematical activity as modelling activity in contrast to the other approaches and are in that sense interesting to discuss. The use of Commognition to interpret modelling is not frequently used, but it has potentials to be used as an analytic tool for analysing students' activities with mathematical modelling (Frejd, 2010; Ärlebäck & Frejd, 2013), which is a reason for the choice.

2.2.1 Modelling cycles and modelling competency

The cyclic process of modelling, known as the 'modelling cycle' in mathematics education related to ICTMA is frequently discussed in the literature (Blomhøj & Højgaard Jensen, 2003; Blum et al., 2007; Blum & Leiβ, 2007; Blum & Niss, 1991; Borromeo Ferri, 2006; Kaiser & Sriraman, 2006; Maaß, 2006, etc.). There are many descriptions of the modelling cycle (Perrenet & Zwaneveld, 2012), depending on the research aim (Borromeo Ferri, 2006; Haines & Crouch, 2010; Jablonka, 1996). According to Kaiser, Blomhøj, and Sriraman (2006) the modelling cycle involves five up to seven sub processes and is split into two domains, one called reality or the extra mathematical world and one called mathematics or the intra mathematical world, see Figure 2 below.

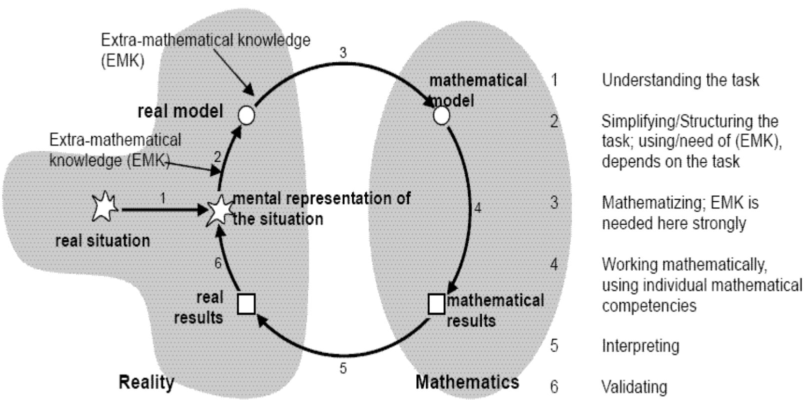

Figure 2. The modelling cycle by Blum and Leiβ (2007) as presented by Borromeo Ferri (2006, p. 92)

A brief explanation of the modelling cycle as described in Figure 2 will be provided here. The modelling problem is situated in the 'real world' called the **real situation** (the problem, which is often formulated in everyday knowledge). The modellers need to *understand the task* to make a **mental representation of the situation** (how the individuals are thinking about the problem situation), then continue to come up with a **real model** (external representations) by *simplify/ structure*, filter and idealize the information from the task. This real model is then translated from the 'real world' with use of extra-mathematical knowledge to 'the mathematics world' by *mathematizing* these criteria and creates a **mathematical model**. The next step is that the modellers work *mathematically* in the 'mathematical world' to produce answers, **mathematical results**. Finally the mathematical results are *interpreted* into **real results** by moving back to the 'real world' and the real results are *validated*. If the validation provides information to the modellers that the real results are not valid and other aspects need to be included, the modeller has to do the modelling cycle over again.

The notion of modelling competence often refers to a modelling cycle. Blomhøj and Højgaard Jensen (2003) illustrate this relation to a modelling cycle with their definition of modelling competence that "[b]y mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context." (p. 126). The 'mathematical modelling process' or modelling cycle they refer to is displayed in Figure 3 below.

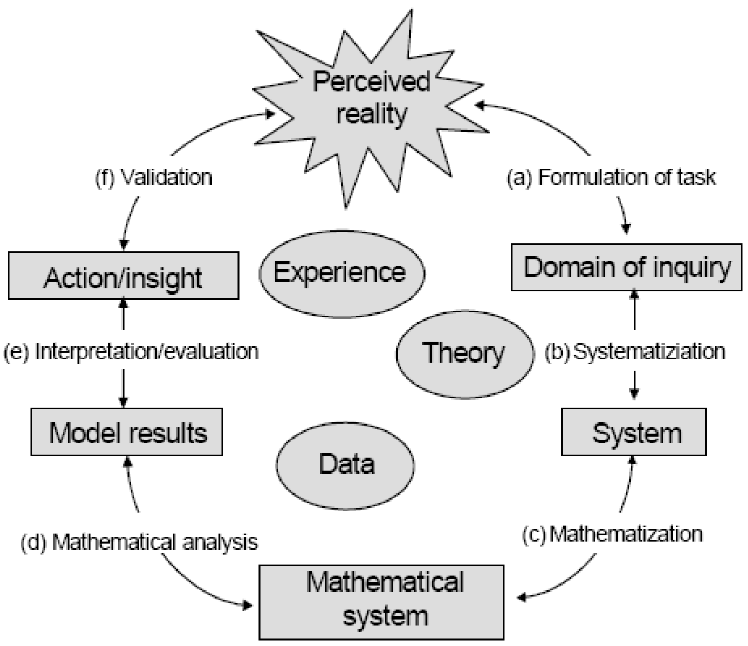

Figure 3. The modelling cycle by Blomhøj and Hoff Kjeldsen (2006, p. 166) adapted from Blomhøj and Højgaard Jensen (2003; 2007).

The modelling cycle in Figure 3 has many similarities to the modelling cycle in Figure 2. The process is composed of six sub-process (a-f), each connected to a modelling sub-competence (Blomhøj & Højgaard Jensen, 2007). In order to investigate a phenomena or a situation from the perceived reality a formulation of a task is done to identify important aspects from the situation, or as Blum and Leiβ's (2007) would describe this, to make a mental representation of the situation. The task is framed in the domain of inquiry and this is not done explicitly in Blum and Leiβ's (2007) description. Other differences are that the arrows goes back and forth in Blomhøj and Højgaard Jensen's (2007) description of the modelling cycle but not in Blum and Leiβ's (2007) description, also the three ellipses in the centre of Figure 3 do not appear in Blum and Leiβ (2007). These ellipses are supposed to show that the epistemological base for the sub-processes is theory, experience or data (Blomhøj & Hoff Kjeldsen, 2006). The action/insight part in Figure 3, or action/realization (Blomhøj & Højgaard Jensen, 2003), is described as new insight gained from the investigated phenomena, which may put into action if it is supported and validated by the empirical data given. The validation process (f) in Figure 3 is a second validation and refers to the questioning of the entire modelling process and for doing that new data is needed (Blomhøj & Højgaard Jensen, 2003). This description from Blomhøj and Højgaard Jensen (2003) of the validation process in two steps is combined in Blum and Leiβ's (2007) description. The rest of the aspects in the modelling cycle in Figure 3 can be found in Blum and Leiβ's (2007) description of modelling, including systematization (identification of essential aspects needed to solve the problem).

According to Blomhøj and Højgaard Jensen (2003) the cyclic model illustrated in Figure 3 can be used in different ways as a tool to investigate mathematical models, modelling processes behind models and analyse and define modelling competence. However, the notion of modelling competence has been used as a 'buzzword' (Blomhøj & Højgaard Jensen, 2007). Buzzwords "are words that add flavour to an analysis, a discussion or the planning of a teaching practice just by being mentioned" (Blomhøj & Højgaard Jensen, 2007, p. 45).

For example Maaβ (2006) has another definition of modelling competencies. She defines:

> Modelling competencies include skills and abilities to perform modelling processes appropriately and goal-oriented as well as the willingness to put these into action. (p. 117)

The definition above is used by Kaiser (2007). Niss, Blum and Galbraith (2007) also have a definition based on the different aspects in a modelling cycle.

> *mathematical modelling competency* means the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given

> situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model etc. (p. 12)

Maaβ's (2006) notion of *modelling competencies*, Blomhøj and Højgaard Jensen's (2003) *modelling competence,* and Niss, Blum and Galbraith's (2007) *modelling competency* all seem to capture the same thing, i.e. the aspects needed to master a modelling process. However, Maaβ (2006) also emphasises students' attitudes to doing modelling in her definition, which is another dimension of students' work with modelling. The relation between willingness and ability, skill or competence is not clear, which seems to makes the notion difficult to operationalise. Nevertheless, a bit confusing still are the words *competencies*, *competence* and *competency*. In this thesis the following definition will be used: competence (plural competences) is used for an individual's skills, while competency (plural competencies) is used as a term for standards to be achieved. For example, a person applying for a job as a modeller may be tested, in a test, on his/her modelling competence (skills) to see is he/she meets the modelling competency (the demanded achievements) required by the company.

There are also descriptions of modelling competencies that differentiate between different levels. Greer and Verschaffel (2007) use the following notions for different levels of competencies: *competencies for implicit*, *explicit*, and *critical modelling.* Implicit modelling is when students are involved in modelling activities without being aware of it and explicit modelling is when students are aware of modelling activities and the aim of the modelling process. Critical modelling is when students are being able to reflect critically on the use and role of mathematical modelling in different subjects and in society. In addition, Henning and Keune (2007) have distinguished between three levels of modelling, *Recognition and understanding of modelling, Independent modelling*, and *Meta-reflection on modelling.* The first level is when the student is aware of the modelling process (*Recognition and understanding of modelling)*, the second level is when the student (himself/herself) can use the modelling process to answer a modelling problem (*Independent modelling*), and the third level is when the student can evaluate, analyse and reflect upon the purpose of the modelling activity in a critical way (*Meta-reflection on modelling*). The *critical modelling* and the *Meta-reflection on modelling* are for instance discussed in Frejd (2010) in relation to the social perspective of modelling (see Jablonka, 1996; Barbosa, 2006; Jablonka & Gellert, 2007; Skovsmose, 1994).

2.2.2 Modelling from the theoretical perspective of ATD

The Anthropological theory of didactics (ATD) founded by Yves Chevallard includes another perspective of mathematical modelling. The theoretical perspective of ATD is broad and some notions will be explained in this section in order to explain the notion of mathematical modelling. For details concerning ATD see for instance Chevallard (1999) and Bosch and Gascón (2006).

One meaning of mathematical modelling from the theoretical perspective of ATD is formulated by Garcia, Gascón, Higueras and Bosch (2006). According to them, mathematical modelling is not another dimension or another aspect of mathematics, instead they propose "that mathematical activity is essentially modelling activity in itself" (Garcia et al., 2006, p. 232). This view of modelling is according to them only meaningful if one defines the notion of mathematical activity and if modelling is considered to include both extra mathematical modelling ('real-world problems') and intra mathematical modelling ('problems related to pure mathematics', such as different representations of algebraic notions). The effect on the statement above will be that the problem situation is not the most important aspect, but the problem itself (a generative question) will be the key-point in order to develop and create new, wider and more complex problems (ibid.). A generative question is "a question with enough `generative power´, in the sense that the work done on it by the group is bound to engender a rich succession of problems that they will have to solve -at least partially- in order to reach a valuable answer to the question studied" (Chevallard, 2007, pp. 7-8). These generative questions, which also are referred to as *crucial questions* or *productive questions* (Garcia et al., 2006), should also be of real interest to the students (Rodriguez, Bosch & Gascón, 2008).

Modelling activity as defined by Barquero, Bosch and Gascón (2007) also stresses the importance of a problematic question (a generative question). They claim that "the modelling activity is a process of reconstruction and articulation of mathematical praxeologies which become progressively broader and more complex. That process starts from the consideration of a (mathematical or extra-mathematical) problematic question that constitutes the rationale of the mathematical models that are being constructed and integrated" (p. 2051).

The notion of praxeology from the quote above is one of the most central notions in ATD (Garcia et al., 2006). A knowledge or a body of knowledge is defined as "a praxeology (or a complex of praxeologies) which has gained epistemic recognition from some culturally dominant institutions, so that mastering that praxeology is equated with mastering a 'true' body of knowledge" (Chevallard, 2007, p. 6). The praxeologies include two main components praxis (or know-how) and logos (or thinking and reasoning about the praxis). These two main components are divided into sub-components as in Table 1.

Table 1. The four sub-components of praxeologies

Praxis (know-how)	Tasks within a specific activity
	Techniques to accomplish the tasks
Logos (know-why)	Technology that justifies the techniques
	Theory that justifies the technology

The praxis part refers to the types of tasks and techniques that are available to solve the tasks and the logos part refers to technology that describes and explains the techniques and the theory that explains the technology. In addition, the praxeologies of mathematics can be analysed as global, regional, local and point praxeologies (Bosch & Gascón, 2006). A point praxeology is characterized by a specific type of problem and an appurtenant specific technique within a technology, a local praxeology is characterized by a set of point praxeologies that are integrated within the same technology, a regional praxeology is characterized by connected local praxeologies within a mathematical theory and a global praxeology is characterized by linked regional praxeologies (ibid.). I will illustrate global, regional, local and point praxeologies by examples inspired from Artigue and Winsløw (2010). A point praxeology is for instance the specific technique to solve the equation x–3=0 by moving the -3 to the right hand side of the equal sign and change minus to plus. A local praxeology may be seen as the discourse relating to solve polynomial equations and a regional praxeology may be an algebraic theory for solving equations. Finally, a global praxeology may be a unified theory of solving equations including number theory, algebraic theory etc.

The process of refining or constructing mathematical praxeologies, which is mentioned in the definition of mathematical modelling by Barquero et al. (2007), is a complex activity called the *process of study* (Rodriguez, Bosch & Gascón, 2008). The process of study is classified into six so called didactic moments: (1) first encounter, (2) exploration, (3) constructing environment for technology and theory, (4) working on the technique, (5) institutionalization and (6) evaluation (ibid.). These six didactical moments do not have to appear in the chronological order stated above. I will describe the process of study with the modelling example used by Ruiz, Bosch and Gascón (2007) about selling and buying T-shirts. The students in the investigation were given a chart with the number of sold T-shirts, the total costs, the total incomes and benefits for three months (May, June and July) and a corresponding question about the possibility to earn 3000 euro in August by selling a reasonable number of T-shirts (first encounter). Based on the given conditions the students started to create a model and did some calculations and estimations in order to develop a technique (exploration) and then continued to improve this technique to set up other models (working on the technique). For instance the students had to find connections between numerical and functional language as well as investigate the roles of parameters and variables (constructing environment for technology and theory) in order to discuss the question. Finally an identification of praxeologies regarding institutional demand is done (institutionalization) and students reflect over the value of those praxeologies (evaluation) (Rodriguez, Bosch & Gascón, 2008).

According to Garcia et al. (2006) the relation to the modelling cycle is that the cyclic perspective does not contradict modelling from an ATD perspective. However, I found no empirical study in the literature that compares and contrasts the different views and I have some problems to see that they do not contradict each other in some respect, especially concerning that there is no clear distinction between intra mathematical modelling and extra mathematical modelling in ATD, which is an important part in the modelling cycle. According to Sriraman and

Kaiser (2006) one consequence of using modelling from the ATD perspective is that "this leads to the fact that every mathematical activity is identified as modelling activity for which modelling is not limited to mathematising of non-mathematical issues" (p. 45). More work need to be done comparing the different perspectives.

2.2.3 Modelling 'realized' in terms of Commognition

Commognition is a discursive theoretical framework developed by Sfard (2008) that combines entities from theories of communication and cognition to describe and explain social and individual aspects of thinking and learning. The framework defines a set of notions and principles that describes thinking as a particular type of interpersonal communication and learning as a change in discourses. The central notion in this framework is *discourse* which is defined as a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions; every discourse defines its own community of discourse; discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives" (p. 297). In other words, a discourse is characterized by the meaning and use of language (in a general sense including written and spoken words, symbols, figures, graphs, etc.). However, words, symbols, expressions, gestures, etc. may have different meanings in different discourses. Metaphorically speaking Sfard (2008) refers to these words, symbols etc. that have a particular meaning for a particular discourse as *discursive objects*. To be a part of a communication in a discourse includes to make consensus on the interpretation of language and to follow the social established conventions for communication and interaction between members of the discourse. The social established conventions for communication by actions paired with re-actions together with the vocabularies, visual mediators, routines, and endorsed narratives are formalized ways on how to determine what is regarded as 'true' within the discourse. A discourse may therefore function as delimiter in that it includes or excludes persons from a given discourse. The notion of discourse is also central in one of the basic principles of commognition, which is that "discourses permeate and shape all human activities, [and] the change in discourse goes hand in hand with the change in all other human doings" (p. 118).

Commonly figuring discourses when a commognitive approach is employed are *colloquial discourses* (or everyday discourses) permeated by personal experiences; the complementary *literate discourses* in which communication often is characterized by the use of specialized symbolic artefacts; and, *classroom discourses* capturing school norms and rules (Sfard, 2008).

All discourses are regulated by rules, *object-level* and *metadiscursive* rules, that govern the processes for communication (Sfard, 2008). The *object-level* rules are narratives about the properties of objects in a discourse whereas the *metadiscursive* rules regulate the activities used for proving and legitimating these narratives. An example of an *object-level* rule is that the sum of the lengths of any two sides in a triangle must be greater than the length of the remaining side (i.e. the triangle inequality) and the *metadiscursive* rules regulate the actions for proving the triangle inequality. A set of metadiscursive rules are called *routines* and they describe *when* (in what situation) and *how* (the course of actions) a repetitive discursive procedure is executed. The equation $2x + 4 = 10$ may be solved with different strategies, (*how of*) routines, such as algebraic procedures, geometric procedures, technology procedures and guessing procedures. However, in a classroom discourse it is equally important and more advanced to chose (*when of*) a routine, since it requires that the discourse participants acknowledge that the chosen routine actually has a positive influence on the solution process and is regarded as an appropriate and socially accepted routine. As described by Sfard (2008), "[w]hereas learning a routine how is often fairly straightforward task, learning when may be a lifelong endeavour" (p. 221). To identify when and how of a routine that addresses a particular problem often includes *recognition*, which is the use of memorized routines (*recalling*) from similar problems. The communicational patterns in educational settings when students discuss problems not recognized and when standard routines not automatically are *recalled* are characterized as ad hoc constructions. However, the underlying recursive sequences in these communicational patterns of ad hoc constructions include a pattern of *conjecture-test-evaluation*. One activity that follows the recursive pattern of *conjecture-test-evaluation* is *negotiations*. Nevertheless, *negotiations* in a teaching-learning situation restrict or facilitate a progress in learning, since the outcome of the negotiation will affect how the communication will proceed. A condition for learning, such as students learning of new *discursive objects*, involves the *learning-teaching agreement*. The *learning-teaching agreement* is an elaborated commognitive view of Brousseau's notion didactical contract (Brousseau, 1997) that frames teachers' and students' classroom communication, due to norms, expectations and obligations.

A discursive object is manifested through its signifier and its realizations. A signifier can for example be a word or an algebraic symbol, and a realization of the signifier is the procedure, or product, of pairing the signifier with another discursive object. Successive meaningful realizations of a signifier can be organized and illustrated in *realization trees*, which are tree-like structures in the sense of graph theory. Sometimes a realization-tree is also productively thought of as a connected graph, possibly containing loops, to stress the "symmetric nature of many signifier-realization relation" (Sfard, 2012, p. 4). Due to the recursive nature inherent in the production of discursive objects, there is often a dual relation between a signifier and its realization making the two notions in different contexts interchangeable. Sfard underlines that this is common in mathematical discourses, and provides the example of a function's values listed in a table realized as a formula and vice versa.

The ability of making successive realizations is a fundamental aspect in problem solving according to Sfard (2008), in particular for problems related to real life. A person's accessibility to be able to effectively take part in an activity to solve practical problem within the mathematical discourse "depends on her ability to decompose signifiers into trees of realizations with branches long enough to reach beyond the discourse, to familiar real-life objects and experiences" (Sfard, 2008, p. 166). These arguments are used to declare how mathematical modelling may be realized in terms of commognition.

Mathematical models and modelling in terms of commognition, a mathematical model may be interpreted to be a discursive object in a subsumed discourse, which means that in a given situation and context a particular mathematical model is a pair of a signifier with its realizations. To be engaged in the activity of modelling means to establish and participate in a modelling discourse which involves singling out the relevant discourses for the problem by finding and making meaningful and productive connections between signifiers and realizations in realization trees belonging to different discourses, subsuming these into the new discourse.

The ideas will briefly be illustrate with help of a modelling problem known from the literature (e.g. an adapted version of the problem from Niss, Blum, and Galbraith's (2007, p. 12) about deciding the best location for speed bumps to calm traffic along a road within college campus); What is the best way to evacuate a school building? This 'evacuation' problem is formulated in a *colloquial discourse* and there are many possibilities for students to decompose the problem into *signifiers*, connect them and *realize* them into *a discursive object*. One possible realization in a classroom discourse in mathematics is the realization of the signifier *best* to the *fastest* way. The following question might be examined: How long time does it take to evacuate a school building? Students may base their solution on their experiences of evacuation situations such as being a part of an evacuation exercises or watching on TV how people evacuate buildings. However, a solution to the problem derived only from experience out of the classroom is usually not regarded as an accepted solution in a mathematics classroom discourse. The communication by the teacher and the textbook set the agenda of the *learning-teaching agreement* and an acceptable solution is verified by *metadiscursive* rules used in the mathematics classroom. Negotiations following the discursive pattern of *conjectures-test-evaluations* may possibly be applied to identify what signifiers/ realization effect the evacuation time and how they are connected.

Examples of signifiers/ realizations that may effect the evacuation time might be (A) number of people, classrooms, exit stairs, and exit doors, etc. (B) The size of the building, (C) walking velocity, (D) initial delay, (E) what time of the day it is. There are several discourses that come into play such as *mathematical discourse* (number of), *colloquial discourse* (time of day), and *physics discourse* (velocity). In a classroom discourse a possible acceptable solution may include an expression for the evacuation time which may include idealisations like assuming everyone walks with constant speed, assigning a noun, such as t, for the evacuation time and an expression for t that may connect the signifiers (A) to (E) with some procedures. The *discursive object* developed may be described as the pair <t, an expression for t> where t is the signifier and the expression is the realization.

Below in Figure 4 signifiers and their realization from the problem described above is presented in a realization tree.

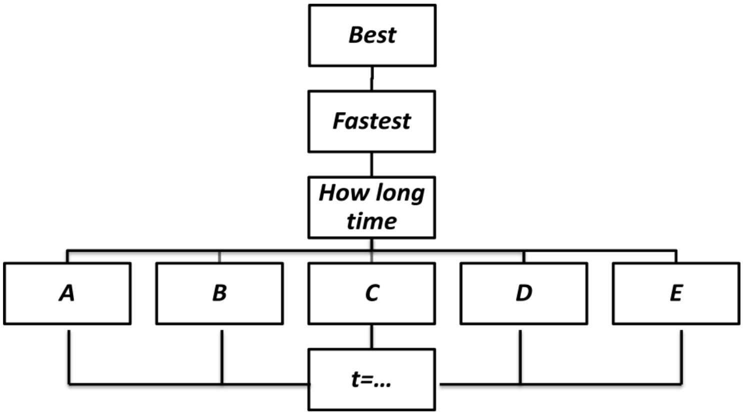

Figure 4. A realization tree from the evacuation problem.

The realization tree in Figure 4 presents an example how one subsumed discourse has been developed with its signifiers. The signifier *Best* in the example was realized to *Fastest* that triggered the question, *how long time* does it take to evacuate a school building? This question was realized into an expression for the evacuation time, t. The example presented was used to illustrate some key notions and other discursive layers, such as: other school subjects, symbolic tools (the use of computers for simulations, etc.), literate discourse (the complement to colloquial discourse), etc. were excluded.

Sriraman and Kaiser (2006) argue that every mathematical activity from the perspective of ATD can be described as a modelling activity. Similar arguments may be put forward for 'modelling' realized in commognition that signifiers/ realization relations take place in all mathematical activity, which makes it necessary to describe what a modelling problem is. The next section will elaborate of differences between modelling, problem solving and application.

2.3 Mathematical modelling in relation to problem solving and applications

What are the differences and similarities between mathematical modelling, problem solving and applications? The question is not easy to answer, because there exist different theoretical perspectives of modelling and the definition of the notion of mathematical modelling depends on perspective and research focus. In addition, the notion of problem solving does not have a single clear definition which is accepted by all researchers in mathematical education (Lesh & Zawojewski, 2007). Already Schoenfeld (1992) expressed that "`problems´ and `problem solving´ have had multiple and often contradictory meanings through the years – a fact that makes interpretation of the literature difficult" (p. 337).

From the theoretical perspective of the modelling cycle with a focus on the entire process (Blum & Niss, 1991; Niss, Blum & Galbraith, 2007), the distinction between modelling and problem solving and application is the problem situation and how the problem is described. According to Niss, Blum and Galbraith (2007) the focus in a *modelling problem* tends to be in the direction from reality to mathematics (what mathematics do I need to solve this problem?) in contrast to *applications of mathematics* where the direction is from mathematics to reality (where can I apply this mathematics?). The notion of applied problem solving is sometimes used in the same manner as modelling and sometimes it is used to solve any extra-mathematical problem (Niss et al., 2007). The consequence is that the problem itself plays the crucial role in defining a modelling problem. Niss et al. (2007) give the following example of common types of problems: *word problems*, *standard applications* and *modelling problems*. The word problem is characterized by an intra mathematical problem dressed up with a given context.

> *How many different menus are possible to serve, if a meal includes a starter, a main course and a dessert and if you have access to three different starters, five main courses and two desserts?*

The stated word problem above is nothing more than an ordinary combinatory problem. Standard applications are characterized by a given implicit model.

> *Can you help a soda company that wants to decrease its costs of aluminium? The company wants you to develop a new cylindrical aluminium can containing 0.5 litre soda.*

The implicit model given in the example above is the function describing the minimizing of the cylindrical volume with constrains. Modelling problems are characterised by the entire modelling process.

> *How long time does it take to evacuate a school building?*

The example above is discussed in section 2.2.3 and it may involve many realizations. In addition, the problem is also discussed in Frejd (2011a) and many parts of a modelling cycle may be present in a solution process.

Ärlebäck (2009b), with the aim to investigate how the notions of mathematical models and modelling have been used in the last six upper secondary curriculums in Sweden, has developed an analytic tool to identify aspects of applications, problem solving and modelling. The aim with the analytic tool is to catch aspects of mathematical modelling described in the curriculum in terms of application and problem solving. Ärlebäck (2009b) takes a broad view of modelling (including only fragments of the modelling process), expressing that mathematical modelling can be differentiated between (Ärlebäck, 2009c, p. 183):

1. *intra-mathematical modelling*; for instance, to solve a geometrical problem using algebra.
2. *direct modelling*; to solve `traditional´ word problems.
3. *complex modelling*; to work with real, open problems.

This view of modelling includes applying mathematics and is associated to problem solving (Ärlebäck, 2009b). Examples of *intra-mathematical, direct* and *complex* -application of mathematics are provided but not of problem solving.

The analytic tool is used in the content analysis of the curriculum in Ärlebäck (2009b) to make a distinction between the notions of applications, problem solving and modelling. The coding category *modelling* includes the explicit notion of modelling, interpret a mathematical expressions like equations and connections, give example on and to formulate problems. The category *apply mathematics* includes to use mathematics in different forms and expressions. *Problem solving* includes tasks, problems and solutions. The notions of applications, problem solving and modelling were also subdivided into sub-categories producing a more nuanced analysis, for more details see Ärlebäck (2009b). Explicit examples of mathematical models and modelling are provided in the study with excerpts from the actual coding procedure to clarify the distinction between the notions of applications, problem solving and modelling. The analytic tool presented by Ärlebäck (2009b) is one possible way to categorise modelling, application and problem solving. However, as Ärlebäck (2009b) states, "is the notion modelling, application and problem solving connected with each other and overlapping in many ways" (my translation, p. 181). There is more literature discussing the issue of differences between the notions, especially between mathematical modelling and problem solving, see for instance section 6 in Lesh, Galbraith, Haines and Hurford (2010).

2.4 The relevance of mathematical modelling in mathematics education

As already mentioned in the introduction, in Blum and Niss' (1991) review of the literature five main arguments for including mathematical modelling in the curriculum were found (pp. 42-44). Those arguments are presented below in more detail.

1. *The formative argument*- to help students to develop general capabilities, attitudes and self confidence by promoting an overall problem solving ability which is explorative, creative and open-mindedness.

2. *The 'critical competence' argument*- to prepare students to be critical of mathematics used in private life and in society, meaning to be able to independently identify, analyse and understand situations and instances where mathematics is being used.

3. *The utility argument*- that by the use of mathematics instructions make students aware of how mathematics can be utilized in different situations especially related to the extra-mathematical domain.

4. *The 'picture of mathematics' argument*- to give the students a broad and colourful picture of mathematic as a science, as an activity in society and in culture.

5. *The 'promoting mathematics learning' argument*- to assists and motivate students to learn mathematical concepts and methods.

It is possible to find similar arguments in the Swedish curriculum for upper secondary school. Below are quotes from the curriculum (Skolverket, 2012a), which are interpreted into the five arguments:

[1. *The formative argument*]

"Teaching should give students the opportunity to challenge, deepen and broaden their creativity and mathematical skills" (p. 1); "Teaching should strengthen students' confidence in their ability to use mathematics in different contexts, and provide scope for problem solving both as a goal and an instrument" (p. 1); "it should provide students with challenges" (p. 1)

[2. *The 'critical competence' argument*]

"understand its importance for the individual and society." (p. 1);

[3. *The utility argument*]

"it should contribute to students developing the ability to apply mathematics in different contexts, and understand its importance for the individual and society." (p. 1); "developing an understanding... different strategies for solving mathematical problems and using mathematics in social and professional situations" (p. 1); "Where appropriate, teaching should take place in environments that are relevant and closely related to praxis" (p. 1)

[4. *The 'picture of mathematics' argument*]

"it should provide students with...experience in the logic, generalisability, creative qualities and multifaceted nature of mathematics." (p. 1)

[5. *The 'promoting mathematics learning' argument*]

"Teaching should cover a variety of working forms and methods of working, where investigative activities form a part. Where appropriate, teaching should take place in environments that are relevant and closely related to praxis." (p. 1)

Comparing the given quotes above with the five arguments given by Blum and Niss (1991) one may conclude that the first, the third and fourth quotes are quite similar to the same three arguments. To turn the second quote into the second argument one needs to make an assumption that to "understand the importance" (Skolverket, 2012a, p. 1) involves being critical how mathematics is being used in the society. This was more elaborated in the previous syllabus Skolverket (2001), or in other words "[t]he subject also aims at pupils being able to analyse, critically assess and solve problems in order to be able to independently determine their views on issues important both for themselves and society, covering areas such as ethics and the environment." (Skolverket, 2001, p. 60).

The last quote may be turned into the last argument by the assumption that one variety of working form is a modelling activity. The quotes above are written in general terms and the connection to mathematical modelling is not explicit. The emphasis of teaching and learning mathematical modelling in upper secondary school is described as "Teaching in mathematics should give students the opportunity to develop their ability to:... interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations" in the subject syllabus mathematics (Skolverket, 2012b, pp. 1-2).

The five arguments are not only possible to identify in the Swedish curriculum, according to Ärlebäck (2009c) these five arguments are analogue to the arguments that are put forward for learning mathematics in general. Jablonka (2009) criticizes such arguments as being too generally formulated, allowing the promotion of different ideological agendas.

The notion of mathematical literacy is put forward as a central aspect in teaching and learning mathematics in for instance the OECD/PISA framework (OECD, 2009). The notion also has connections to mathematical modelling, see the definition below (OECD, 2009): Mathematical literacy is:

> …an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (p. 84)

One connection between mathematical literacy and mathematical modelling is to develop mathematical models and reflect critically on mathematical models used in society (Jablonka, 2003). This critical reflection of mathematical models being used may be related to the critical competence argument given by Blum and Niss (1991).

There are also obstacles for using mathematical modelling in mathematics education. Blum and Niss (1991), drawing on Pollak (1979), Blum (1985), and Niss (1987), identify three types of categories for obstacles from different perspectives. Those three perspectives are from (I) instruction, (II) learners and (III) teachers. From the point of view of instruction, the obstacles described are teachers' limitation of time and a priority for teaching mathematics which by tradition is taught and included in the curriculum, which also is found empirically as the "greatest obstacle" for teachers (Schmidt, 2012, p. 42). In addition, some teachers believe that mathematical modelling does not belong to mathematics instruction at all, because it doesn't belong to the beauty, clarity and purity of mathematics (Blum & Niss, 1991). Obstacles expressed from the learners' point of view are that mathematical modelling is more demanding and requires more of the learner than ordinary teaching. According to Blum and Niss (1991), students do enjoy solving routine tasks by recipes more than being involved in unpredictable modelling activities. Teachers also find it more demanding with instruction that is more open and requires an extra mathematical competence. More obstacles described from the teachers' point of view are that they believe that modelling activities make it difficult to assess students' achievements, that they have not studied modelling themselves, that they do not know enough of what types of modelling activities are suitable and that they do not have the time to prepare or to put them into action (Blum & Niss, 1991; Schmidt, 2012). In addition, some research points at teachers' and students' beliefs of mathematical modelling as a part of mathematics and mathematics education is an obstacle for implementing more modelling activities into school (Kaiser & Maaβ, 2007; Maaβ, 2005).

Instead of just discussing obstacles, Schmidt (2012) also discusses seven empirically found "motivations" for teaching modelling. Those motivations are: *pupils calculate and think more creatively*, *there are long-term, positive effects in mathematics lesson*, *pupils work more independently*, *mathematics gains relevance for pupils' everyday lives*, *there are long-term positive effects beyond mathematics lessons*, *my teaching load is lessened when pupils work on modelling tasks*, *and modelling can be used in the classrooms with large gaps between pupils performance levels* (pp. 50-58). The motivations seem to occur from at least two levels. The first level, as motivation or arguments for teaching modelling, is consistent with the arguments from Blum and Niss (1991), such as it creates creative thinking, or emphasises the relevance of using modelling within and out of school setting. The second level, as motivation for a better teaching and learning environment, such as pupils work more independently, it may be used in mathematical classrooms where students' performance differ and it decreases the working load of the teacher.

Stillman (2010) makes another description of motivations in terms of conditions for success of modelling activities in secondary school. She defines three categories (p. 311) (a) *the structure or nature of the task*, (b) *student conditions* and (c) *teacher conditions*, similar to Blum and Niss (1991). These categories include sub- conditions which are listed on the next page (pp. 311-317):

(a) Allowing students to fly[7]; fostering natural curiosity; allowing freedom in choosing technology; allowing use of multiple representations supporting connection making; requiring the answering of interpretive questions; and scaffolding of recording of key mathematics.

(b) Developing understanding of situation in groups; using physical activities related to the task to develop domain knowledge; and participating in rich dialogue and discussion with peers and the teacher.

(c) Knowing when to intervene; positive expectations of student engagement with modelling, knowing the essence of the task; and tolerating different rates of progress and wrapping up the task.

Reading all conditions required for success in Stillman (2010) gives a picture of the complexity of teaching and learning mathematical modelling. From the condition category a) *the structure or nature of the task,* one can read that one condition is *allowing freedom in choosing technology.* The influence of using different technologies in mathematical modelling is discussed in literature and often technology seems to play an important role in teaching and learning modelling (see e.g. Geiger, Faragher, & Goos, 2010; section 10 in Lesh et al., 2010). Also the condition *scaffolding of recording of key mathematics* is related to students' use of technology and is a condition that students should explicitly record key information such as assumptions and estimations.

2.5 Research related to mathematical modelling in Swedish upper secondary school

Research about mathematical modelling in Sweden has been scarce. This section will briefly examine and provide some information about the research studies that have been done with explicit focus on mathematical modelling in relation to the Swedish upper secondary school.

Looking at Ärlebäck's (2009c) dissertation, it includes and discusses five different papers with different aims. The papers investigate teachers' beliefs and affects (Ärlebäck, 2010), students' modelling competency (Frejd & Ärlebäck, 2011), introducing mathematical modelling by Fermi problems (Ärlebäck, 2009d), curriculum aspects (Ärlebäck, 2009b) and designing and implementing of modelling activities into secondary school (Ärlebäck, 2009a). A brief summary with the main conclusions will be provided below.

Ärlebäck (2010) develops a framework to analyse teachers' beliefs about mathematical models and modelling. He uses his framework in a qualitative study with two Swedish upper secondary teachers to analyse interviews. The main

[7] Metaphor, the task need to be designed to challenge all students at different levels and depths.

conclusion is that the two teachers had some difficulties to express their own beliefs about the notions of mathematical models and modelling (Ärlebäck, 2010).

The notion of Realistic Fermi problems was used in a study in upper secondary school to introduce mathematical modelling (Ärlebäck, 2009d). A so called MAD framework, partly drawn from Schoenfeld's (1985a) problem solving categories, was used to identify modelling sub- activities (*reading, making model, estimating, calculating, validating* and *writing*). The conclusion presented in Ärlebäck (2009d) was that "Realistic Fermi problems may provide a good and potentially fruitful opportunity to introduce mathematical modelling at upper secondary level" (p. 355) and all the sub-activities were represented (but not in a cyclic manner).

The content analysis of the six curriculums between the years of 1965-2000 made by Ärlebäck (2009b) was based on the analytic tool discussed in section 2.3. One result is that the notion of mathematical modelling has gained more explicit emphasis in the curriculum since 1965 (ibid.). "It is also concluded that mathematical modelling as described in Gy2000 [the previous curriculum] can be interpreted both as a goal in itself and as didactical tool, as an instrument for fulfilling other curriculum goals" (Ärlebäck, 2009c, p. 49).

Ärlebäck (2009a) is a design study on how to incorporate modelling activities (*modelling modules*) into the classroom within the present curriculum. The conceptual framework used in the analysis are drawn from; Barab and Squire (2004), and The Design-Based Research Collective (2003) also called *design-based research*; Engeström's (1987) *cultural-historical activity theory*; Wagner's (1997) notion of *co-learning agreement*. The main results were that both teachers expressed that the project was a success and that they wanted to continue. The students also seemed quite happy (58% and 70% of the students in each modelling module expressed a positive experience). However, the students wanted to have more time (schedule time) to execute the modules. This design study shows that it is possible to teach and learn mathematical modelling under prevailing conditions and restrictions in upper secondary school in Sweden.

The licentiate thesis by Frejd's (2011c) consists of five papers (i.e. Frejd, 2010, 2011a, 2011d; Frejd & Ärlebäck, 2010, 2011). All papers provide information about how the notion of mathematical modelling is interpreted and used by different actors in Swedish upper secondary school in mathematics except for one paper, which presents an exploratory and comparative literature review about meanings associated with models and modelling in the context of the teaching and learning of mathematics (Frejd, 2010). The other papers focus on student's modelling competencies (Frejd & Ärlebäck, 2011), student's description of the notions of mathematical models and modelling (Frejd & Ärlebäck, 2010), how and what is assessed in national course tests about mathematical modelling (Frejd, 2011a) and teachers' conceptions of mathematical modelling (Frejd, 2011d) (similar to Paper 3). A summary of the main findings of Frejd and Ärlebäck, (2010, 2011) and Frejd (2011a) are presented below.

Frejd and Ärlebäck (2011) develop and design a research instrument, a questionnaire based on items from the multiple choice test by Hains, Crouch and Davis (2000) together with attitude questions (Likart scale) and questions about background information of the participating students such as gender, last received grade in mathematics and last taken mathematics course to investigate the modelling competency of Swedish upper secondary. About 400 12th grade students participated in the study and it revealed that the students were most proficient in the sub-competencies *to formulate a precise problem* and *to assign variables*, *parameters, and constants in a model on the basis of sound understanding of model and situation*, and least proficient in *clarify the goal of the real model* and *to select a model* (if *to make simplifying assumptions concerning the real world problem* is disregarded). The study also shows that the students' grade, last taken mathematics course, and if they thought the problems in the tests were easy or interesting, were factors positively affecting the students' modelling competency. In addition, only 22.5% of the students stated that they had heard about or used mathematical models or modelling in their education before, and the expressed overall attitudes towards working with mathematical modelling as represented in the test items were negative.

How students describe the notions of mathematical models and modelling was explored in Frejd and Ärlebäck (2010) with an open question in the questionnaire used in Frejd and Ärlebäck (2011). About 2/3 of the 400 students responded and their answers were analysed with a grounded theory inspired approach. The students associate mathematical modelling with problem solving and with using/ applying mathematics as a tool in different situations, and mathematical models with formulas and equations. An indication of a discrepancy between what is prescribed in the upper secondary mathematics curriculum and what the students expressed with respect to the notions of mathematical models and modelling was found. This indication was based on the fact that one fourth of the students expressed that they did not have a clear view on mathematical models and modelling and the descriptions made by the students were short in facts and in words (10 in average). Suggested reasons might be lack of experience of these notions in the classroom, that students have heard the notions but still do not have a clear view about them, or that they find it difficult to describe and express their views in writing.

To investigate how and what is assed about mathematical modelling a content analysis of the last 10 years of national tests in mathematics D[8] (a total of 19 tests) was done, guided by Robson's (2002) guidelines, in Frejd (2011a). An analytic research instrument of 11 coding categories with aim to capture significant aspects of the modelling process, was developed. The primary aspects being assessed are related to the intra-mathematical world, such as the use of an already existing model to calculate a result. Aspects not frequently assessed or left out are related to extra-mathematical parts (the real situation and validation), such as to do

[8] Mathematics D belonging to the syllabus from the year 2000 (see Skolverket, 2001)

simplifying assumptions about the problem, to clarify what facts are most important, to critically assess conditions and interpret the result and relate to the real situation. The *result* (correct answer, correct derived function etc.) was the most frequently category used to explain how to assess. From the holistic view of modelling by Blomhøj and Højgaard Jensen (2003) one conclusion is that there exist no modelling items in the analysed tests, because not all aspects of the process were represented in the data.

In addition, Frejd (2011b) investigated modelling test items in national course tests in mathematics from the theoretical perspective of ATD. Using a reference model of a generative question an analysis of the last four freely available[9] national course tests in the mathematics course C and D was done. It was concluded that no generating questions were found with respect to the reference model. However, suggestions on how to use items from the national course tests and revise them into generative questions to be used under less restricted situations are provided in the study.

Boesen (2006) analysed mathematical creativity in national course tests and in teacher made tests. One aspect analysed was modelling competence as described by Palm et al. (2004). A finding was that the modelling competence is more frequently assessed in the national course test than in teacher made tests.

Wikström (1997) investigated students in upper secondary school about their experience and understanding of dynamic systems, by letting students develop models in a computer environment. These models were examined by the students through changing variables, constants and parameters. One result from the study was that students' conceptions of derivatives and functions had been improved by the experiment. This is one example which shows that mathematical modelling can improve students' conceptions about different concepts.

Lingefjärd has done research about mathematical modelling related to pre-service teachers (also involving upper secondary teachers). His dissertation (Lingefjärd, 2000) includes three studies. The first study is about pre-service teachers' experiences and attitudes related to technology and modelling. One of the findings from the first study was that the pre-service teachers trusted the technology too much and this was followed up and further investigated in the second study. The last study in Lingefjärd (2000) is about pre-service teachers' own responsibility for learning and what authority they use while they are involved in mathematical modelling activities. Lingefjärd (2002a, 2002b, 2006a) and Lingefjärd and Holmquist (2007) study pre-service teachers' strategies and attitudes while they are solving mathematical modelling problems. In Lingefjärd (2002a, 2002b, 2012) and Lingefjärd and Holmquist (2007) the pre-service teachers used strategies that included technology software for finding curves. The pre-service teachers' attitudes were both positive and negative in one study (Lingefjärd, 2002b), while a majority of the teachers were positive in Lingefjärd

[9] freely available meaning there is no secrecy on the test and it is free to download from the internet.

(2002a). However, in Lingefjärd (2006a) it was concluded that ”It simply seems as if students who worked with applied problems became much more involved and engaged in the problem solving process. The context itself seems to be important, especially when the problem offers possibilities to explore at different directions” (pp. 9-10). Another effect of using modelling with technology (VirtualDub and GeoGebra) and letting pre-service teachers act as ‘teachers’ and ‘students’ was that those acting as teachers got a better understanding of theoretical concepts from the teacher course, like concept image (Lingefjärd, 2012). Lingefjärd and Holmquist (2005) also investigate pre-service teachers’ attitudes and experiences of assessment in relation to mathematical modelling. They conclude that peer-to-peer assessment was found to be positive by the pre-service teachers and that peer-to-peer assessment could be used as an introduction to discussions about assessment. Another investigation by Lingefjärd (2007) showed that only four out of 26 mathematics departments in Sweden offered a course in mathematical modelling and that “[t]he underlying arguments [for not offer modelling courses] often showed to be the lack of insight in mathematical modelling among the faculty staff” (p. 336). A project in upper secondary school in both Germany and Sweden about the phenomenon of sunrise/sunset showed, among other things, that the teachers participating adopted a new teacher role as a coach or a manager to help the students (Lingefjärd, 2010) and students used models found on the internet that they had troubles to understand (Lingefjärd, 2011).

Palm’s (2002) dissertation includes four papers which deal with school tasks connected to “real situations” which is one aspect of modelling. Among other things Palm describes a framework for analysing authentic tasks and he analyses Swedish and Finnish national course tests in upper secondary school with use of the framework. From an empirical study with 161 fifth grade students Palm concluded that students can improve their tendency to respond realistically and use extra-mathematical knowledge if the tasks are being more authentic. The dissertation is summarised in Palm (2007).

From the presentation above about research of modelling in upper secondary school in Sweden one may conclude that the research studies conducted have had different aims, goals, and theoretical frameworks. Many of the research results indicate positive attitudes from students and teachers working with modelling activities (Lingefjärd, 2006a, 2002a; Lingefjärd and Holmquist, 2005; Ärlebäck, 2009a), but also some negative attitudes (Frejd & Ärlebäck, 2011). In addition, some results point in a direction that students’ understanding of concepts and realistic responses improves while they are working with aspects of modelling (Wikström, 1996; Palm, 2002). The researchers have investigated teachers, students and tasks, as well as curriculum and assessment, but because of the different aims it is not trivial to find any close connections between the different research (e.g. no follow up studies are made between different researchers). Overall, the use of modelling activities in mathematics education seem to have some positive effect on teaching and learning mathematics, but as Ärlebäck

(2009c) expressed it, there are many issues concerning modelling in upper secondary school in Sweden that still need to be investigated.

Chapter 3

The didactic transposition

I will in this chapter present a literature review in relation to my research, structured by inspiration from the notion of didactic transposition (Chevallard, 1985) to develop a connection between the five papers in the thesis. The notion of didactic transposition is based on the assumption that the knowledge found in curriculums is a reconstruction, an external transposition (Winsløw, 2011), of the knowledge found outside school. Also the knowledge in the curriculum adapts and changes, an internal transposition (Winsløw, 2011), to be a knowledge that is teachable in the mathematical classroom. The word 'transposition' indicates a change in position of the knowledge, from one institution to another, where the fact that the constraints and criteria that operate in the new institution are different from those of the other, by necessity changes the character of the knowledge that is being 'transposed'. Figure 5 depicts the process in more detail.

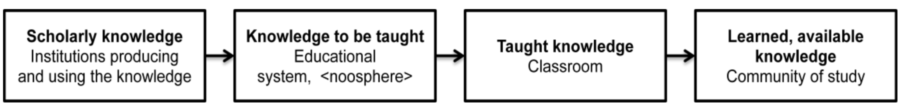

Figure 5. The didactic transposition process (Bosch & Gascón, 2006, p. 56).

Scholarly knowledge, as written in Figure 5, or original mathematical knowledge is knowledge produced by mathematicians at universities and other actors outside school (Bosch & Gascón, 2006). The leftmost arrow indicates the transposition of *scholarly knowledge* to *knowledge to be taught*, taking place within what Chevallard calls the 'noosphere' (an undefined group of policy makers, political stake holders, educators, curriculum developers, etc.). What is formed through this first step is "teaching text" (ibid., p. 56) supposed to guide teachers on what to teach (examples are textbooks, official documents, recommendations to teachers etc.). The middle arrow indicates the move to how and what aspects of the 'teaching text', *knowledge to be taught*, are used and presented by the teacher in the mathematics classroom, *taught knowledge*. The *taught knowledge* may be analysed from observation in the classroom, from teachers' prepared tasks and presentations, etc. (Hardy, 2009). The last step of the transposition of knowledge denotes the step from the *taught knowledge* to the students' *learned or available knowledge*. The *learned or available knowledge* is by Hardy (2009) divided into *knowledge to be learned* and *knowledge actually learned*. *Knowledge to be learned*[10] is the knowledge asked for in assessment and *knowledge actually learned* is the knowledge found in observing students' behaviours and responses to questions, tests, interviews, etc.

The didactic transposition has been used as a research tool (cf. e.g. Bosch & Gascón, 2006, p. 58) for analysing the human practice of mathematics as it unfolds in and between institutions. However, what is not a part of the 'theory' of didactic transposition (Bosch & Gascón, 2006; Hardy, 2009; Winsløw, 2011) are cognitive aspects (e.g. conceptions, beliefs, attitudes) or social aspects (e.g. norms, gender, ethnicity, group compositions, social economic situation of the students etc.), but is a part in this literature review. The following categories from Bosch and Gascón (2006) and Hardy (2009), *Scholarly knowledge*, *knowledge to be taught*, *taught knowledge*, *knowledge to be learned*, and *knowledge actually learned* will be used to structure this chapter. In the end of the chapter a summary of the didactic transposition is presented.

3.1 Scholarly knowledge

The notion of mathematical modelling is not uniquely determined among researchers, as discussed in Chapter 2, so what is regarded as scholarly knowledge outside school is not trivial to determine. According to Bosch and Gascón (2006) scholarly knowledge refers to researchers and others that work with some mathematics, like modelling, and produce new knowledge outside school. The actors that work with mathematical modelling outside school and/or at university level are scientists, researchers, and other workers as well as ordinary people solving everyday problems. To some extent these actors are considered by research conducted about *workplace mathematics* and *authenticity,* which are closely related.

Research about *authenticity* is one aspect of modelling and deals with issues about what is 'real' and what is not 'real' in instructions and pedagogy used in school and what impact it has on teaching and learning (e.g. Palm 2002, 2007, 2009; Vos, 2011). Authenticity refers to the use of mathematics in everyday situations and in workplace situations, which makes the connection to research about *workplace mathematics*.

Workplace mathematics as a research field, may be described as a subset of the research paradigm concerning everyday cognition/ 'cognition in practice' (Lave, 1988; Lave & Wenger, 1991) and 'ethnomathematics' (D'Ambrosio, 1985) analysing the mathematical practices of workers in various workplaces (Naresh & Chahine, 2013). This body of research aims to increase our understanding of how workers conceptualize the role of mathematics in their work. A characterising

[10] May be seen as a subcategory of the knowledge to be taught or of the knowledge actually taught

feature of workplace mathematics as a research field is the use of ethnographic observation, grounded in the view that the mathematical activity is embedded or situated in a social practice influenced by the community or culture at the workplace. In the following two sections I will not try to give an exhaustive description of workplace mathematics or authenticity as a research field, because the aim of the thesis includes workplace in relation to mathematical modelling in particular mathematical modellers. For example, I will not discuss about weather transfer of learning between practices exists or not (Lave, 1988; Saxe 1988), what the meaning of authenticity is (e.g. Palm 2002, 2007, 2009; Vos, 2011) or workplaces literature with focus on arithmetic calculations (e.g. De la Rocha, 1985; Gahamanyi, 2010; Lave, 1988; Millroy, 1992; Nunes, Schliemann & Carraher, 1993; Saxe, 1988). Instead I will present a brief introduction of research concerning workplace mathematics and its relation to technology, to set the scene for discussing modelling and modellers at the workplace.

3.1.1 Workplace mathematics and technology

The relevance of using mathematics in and for out of school activities, in particular in and for waged labour, is one main argument for teaching mathematics in education (Romberg, 1992). However, the synergy between mathematics used in different workplaces and mathematics taught and learned at school is not always straight forward, which seems to be an accepted view among educational researchers in mathematics education. Workplace mathematics is more complex and is situation dependent. It includes specific technologies, social, political and cultural dimensions that are not found in any educational settings (e.g. Harris, 1991; Noss & Hoyles, 1996; Wedege, 2010b). Harris (1991) lists a summary adapted from Hoyles (1991) about the nature of school mathematics and informal mathematics used in out of school activities, illustrated in the Table 2 below.

Table 2. Informal vs. School mathematics (Harris, 1991, p. 129)

Informal Mathematics	*School Mathematics*
Embedded in task	Decontextualized
Motivational is functional	Motivation is intrinsic
Objects of activity are concrete	Objects of activity are abstract
Processes are not explicit	Processes are named and are the object of study
Data is ill-defined and ‘noisy’	Data is well defined and presented tidily
Tasks are particularistic	Tasks are aimed at generalisation
Accuracy is defined by situation	Accuracy is assumed or given
Numbers are messy	Numbers are arranged to work out well
Work is collaborative, social	Work is individualistic
Correctness is negotiable	Answers are right or wrong
Language is imprecise and responsive to settings	Language is précis and carefully differentiated

Table 2 presents a 'stereotype' picture of the characterisation of mathematics used in school and in work (Gainsburg, 2003). On may for example argue that mathematics never is decontextualised in classroom setting, since the classroom in itself is a valid context for doing mathematics, and the motivation for students may not just be intrinsic, because social expectations and the drive for good grades are other examples that may stimulate and motivate students to work with mathematics (ibid.).

There are several goals within the research field of workplace mathematics in mathematics education that explore how and what mathematics is used in specific professions such as, to improve workers' performance (Noss & Hoyles, 1996), to establish information about the role of tools and artefacts (Magajna & Monaghan, 2003; Pozzi, Noss & Hoyles, 1998), to identify similarities and discrepancies between what mathematics is needed in the workplace and what mathematics is taught in school (Triantafillou & Potari, 2010), to analyse communication between employers and visitors (William & Wake, 2007b), and to search for strategies that will improve a general curriculum that better prepares students for future work (Wake, 2012).

Four common trends in relation to mathematical skills needed for the workplace are identified (Hoyles, Wolf, Molyneux-Hodgson & Kent, 2002). Those are: 1. More extended mathematical knowledge is needed within the work force, due to introduction of ICT and business goals. 2. The need to communicate effectively about mathematical data and inferences. 3. Teamwork is commonly used as a way of working. 4. The need of hybrid working skills, like employees that are competent in both management and financial/budgeting (ibid.). Also, Society for Industrial and Applied Mathematics SIAM (2012) specifies a set of skills and experiences needed within industry, which are: *Exposure to a relevant application and real-world problem solving*, *expertise in programming*, *high-performance computing*, and *communication and teamwork* (pp. 34-35). As described in the lists of needs in both SIAM (2012) and Hoyles et al. (2002), communication and teamwork are addressed as well as the use of technology, like programming, computing with computers and other ICT. These skills are also asked for in job advertisements pertaining to mathematical modelling (Sodhi & Son, 2007).

The use of technology may make the mathematics involved invisible. Invisible mathematics, also named implicit mathematics, is mathematics 'embodied', 'crystallised' or 'frozen' into objects (Chevallard, 1989)[11]. The objects can be of mathematical and/or non-mathematical nature and take different material or non-material forms and shapes (ibid.). Most of these objects found at the workplace have particular purposes, which change and manipulate the human environment, from paper and pens to complex space rockets, and may be regarded as some type of technology according to the definitions in Encyclopaedia Britannica.

[11] Reprint in Gellert and Jablonka (2007)

Technology is defined as “scientific knowledge to the practical aims of human life or, as it is sometimes phrased, to the change and manipulation of the human environment” (technology, 2013). This definition seems to capture two different dimensions of technology: the first is that these objects may assist and support human life, and the second is to see technology as something that you can learn (scientific knowledge) about these objects. A metaphor used to describe the invisibility of mathematics hidden in technology is the *black box*. The *black box*, a cybernetic term, is used to depict a complex set of commands or machinery that functions just as an input/ output system (Latour, 1987). These *black boxes* are increasing in the society with the implementation of ‘modern’ information and communication technology (ICT), influencing our lives in different dimensions. As described by Strässer (2007), ICT can be used to speed up the disappearance of mathematics from society by hiding it in these *black boxes*, but at same time ICT can be used to simulate and capture the role of mathematics in applied situations. As said in the beginning of this paragraph a frequent finding within the research filed of workplace mathematics is that the mathematics is hidden in technology at the workplace (e.g. Noss & Hoyles, 1996; Triantafillou & Potari, 2010; Wake, 2007; William & Wake, 2007ab). Technology and mathematics often go hand in hand, in particular at the workplace (SIAM, 2012) and it is difficult (or even impossible) to separate the two notions, at least if one uses a broad description of technology and see mathematics as a *tool* to solve problems. Williams and Goos (2013) argue that the modelling activity is complex and it is not possible to distinguish between technology and mathematics. Instead they describe a fusion of technology and mathematics, “mathematics *is* always mediated by the technology” (p. 554). Modelling can be viewed as three connected dimensions of mathematics, technology, and an activity-or-problem-context, where for instance “[l]anguage and mathematics... could be understood as the supreme modelling tools” (p. 552). These three connected dimensions capture what Noss, Hoyles, and colleagues (e.g. Hoyles et al., 2002; Noss, Bakker, Hoyles & Kent, 2007; Noss & Hoyles, 2011) describe as techno-mathematics or techno-mathematical literacies.

The mathematics hidden in these objects described as technology are often products/ outcomes of mathematical modelling work (Jablonka & Gellert, 2007), which is discussed in the next section.

3.1.2 Modelling and modellers at the workplace

The pre-face to the proceeding of 2009 SIAM Society for Industrial and Applied Mathematics Conference on ‘Mathematics for Industry’ declares that “every paper presents a **mathematical model**...[and]... the term ‘mathematical model[l]ing’ applies to all the articles [in the proceeding]” (Field & Peters, 2009, p. I), which stresses that models and modelling is a central part of the mathematics used in industry. Mathematical models and modelling are frequently used in a variety of contexts and for different reasons in the workplace (e.g. Hoyles et al, 2002; Hunt, 2007; SIAM, 2012; Wake, 2007). Examples of contexts where the use of models and modelling take place are *business analytics*; *mathematical finance*; *systems* *biology*; *oil discovery and extraction*; *manufacturing*; *communications and transportation*; *modelling complex systems*; *computer systems, software, and information technology*; *electronic engineering and optoelectronics*; *food processing*; *health care*; *packaging*; *pharmaceuticals*; *tourism*; etc. (Hoyles et al., 2002; SIAM, 2012). The goals of the modelling activity is workplace specific, but some overall goals may be categorised in *for the drive of effectiveness*, *the companies need to innovate and change constantly*, *improving the quality*, *remaining competitive*, and *the maintenance of apparatus and keeping track of the stock* (Hoyles et al., 2002).

As described above, there are many sectors in the workplace where modelling is used by different actors, which is also displayed in research literature. The EIMI-study (Educational Interfaces between Mathematics and Industry) conference (Araújo, Fernandes, Azevedo & Rodrigues, 2010) includes several papers related to engineering and modelling. Other examples of research literature that include modelling and the use of mathematical models at the workplace are focusing on operators like bankers (Noss & Hoyles, 1996), nurses (Pozzi, Noss & Hoyles, 1998), telecom technicians (Triantafillou & Potari, 2010), operators in a chemical plant (William & Wake, 2007b), finance officers and railway signal engineers (Wake, 2007). A common finding related to the operators that base their decision mainly on computer aided tools with input and output values are that they do not consider the underlying mathematical structure of the models they use. Despite the facts the mathematical models sometimes are hidden in technology and the linguistic conventions of representing mathematical models used in the workplace (formula, graph, table) (Triantafillou & Potari, 2010; William & Wake, 2007b) differ from those in mathematics education, it is argued that mathematical models together with metaphors and gestures offer to facilitate communication of mathematics between workers and clients (William & Wake, 2007b). One of the "principles for strategic curriculum design" that support workplace mathematics (Wake, 2012, p. 1686) emphasises communication about development and validation of mathematical models. Other principles given by Wake (2012) are: to take mathematics in practice into account; facilitate activities that pay attention to technology; and, to let students criticise mathematics used by others.

Mathematical modelling applied by mathematical *modellers* at the workplace also seems to be workplace specific and quite different from school situations (Drakes, 2012; Frejd, accepted, 2013a; Gainsburg, 2003). Gainsburg (2003) observed mathematical behaviour in structural engineering and concluded "that model[l]ing- transforming hypothetical structures into mathematical or symbolic language for the purpose of applying engineering theory- is the heart of the structural engineering profession" (p. 221). She identifies three parts in the modelling process as potentially the most cognitively demanding for the *constructors*. These are: *understanding the phenomena*, *mathematizing* and *keeping track*. The structural engineers in her study struggle with the *understanding of the underlying physical phenomena*, because of the lack of access to data, which is not often the case in classroom situations (ibid.). The struggle is often solved by using past experience with theories about structures and materials as well as general data found in manuals. The *mathematizing* is performed by selecting a pre-defined model, adapt a model or create a model based on the understanding of the phenomena, which may be easy or very complicated. The access to useful technology to mathematize is a factor that influenced how complicated it is. The models used or developed were complex and included several levels, to *keep track* of what has been modelled and justify the different parts of the models was identified as difficult. Another observation was that "communication is a critical part of structural engineering...their work is highly collaborative and they frequently engage in verbal practices" (p. 259) with colleagues inside and outside the firm. This observation is in coherence with Drakes' (2012) finding that experts, i.e. professors of modelling, discuss with others (internal and external) to overcome initial barriers and to assist them when they get stuck. Other skills requested for modelling success such as: *a broad knowledge of mathematics*, *life skills or personal qualities* and *the understanding of the background of the problem*, were also identified. The experts' way of working with modelling problems were to explore the problem and do research and gather data as well as understand the processes involved, simplify and collaborate. To explore the problem, understand the processes, and simplify is referred to as an 'art' by an expert. To identify the relevant information one should try to include the fewest possible factors influencing the phenomena, which is done by data, physical properties or based on previous experiences. To verify the solutions the experts, in Drakes' (2012) study, compared with data, a graph, a simulation or something observed. However, the experts emphasised that there are no correct solutions but only consistent or sensible solutions, which may be negotiable based on experts' experience. This is usually not the case in a mathematical classroom where students are supposed to find an accepted (correct) solution. Frejd (accepted) found that some professional modellers work in groups where the division of labour is specific and predefined (numerical analyst, meteorologist, etc.) in contrast to school settings were the aim is that all students are supposed to learn 'everything'. In addition, there are other aspects that only appear in a limited way in the mathematics classroom, but are large parts of the workplace practice of modelling, such as programming and that the *client's* (the company's) purpose must be taken into consideration or it may be difficult to put the paper product into action (ibid.).

The modellers' conceptions of the notion of mathematical modelling are investigated by Drakes (2012). From the interviews she identifies three common themes of definitions. The first refers to modelling as the activity to set up a model to be used as a "description of a real life situation using a mathematical framework" (p. 39) or "the model would be a simplified or approximate version of the physical system" (p. 39). The second theme emerged was modelling as a process which included setting up the model, but also "solving, analysing and verifying the model are ... parts of the definition... [as well as] refining the model for more accurate results or using the model for prediction" (p. 40). According to Drakes (2012) the first two themes create a dichotomy on the meaning of modelling which may affect teaching of mathematical modelling and gives the example that “one group might mainly focus on setting up the model ‘from scratch’, while another group might focus on continuing past the initial set up to solve the model” (p. 40). The third theme is that mathematical modelling “does not need a definition” (p. 40), because it is just the same as doing problem solving.

There are not much research concerning modellers and their opinions on mathematical modelling in secondary mathematics education. Research has shown that modellers in the Netherlands are sceptical to the use of ′messy′ modelling problems in secondary education (Spandaw, 2011). They argue that modelling is too complex and time consuming for students and that modelling projects are too complicated for teachers to supervise. Instead the aim of secondary education should be to teach basic skills in mathematics (i.e. algebra and analysis). These results contradict to some extent to the results in Frejd (2013). Frejd examines and discusses how nine mathematical modellers have learned mathematical modelling and their opinions on mathematical modelling in upper secondary education. Based on the interviews it was concluded that the modellers mainly learned mathematical modelling during their PhD studies and through their occupation, by working with ‘real modelling’. Their opinions are that mathematical modelling should be a part of mathematics education in upper secondary school, in particular modelling should be more emphasized as a part of general education to develop students’ critical views on how models are used in the society. In addition, they gave suggestions for approaches to teach modelling and suitable modelling problems to work with in mathematics education from their own workplace.

3.2 Knowledge to be taught

Curriculum documents, textbooks, teaching materials, guides to teachers how to teach are connected to *knowledge to be taught*, which is discussed in this section.

3.2.1 Curriculum documents

Blum (1993) writes that “[g]lobally speaking, there is a clear world-wide trend towards including more modelling into mathematics curricula” (p. 7). However, the role of mathematical modelling in school mathematics differs significantly between different countries, as set out in national curriculum frameworks. In some countries it has been strengthened during the last decades, as for example in Swedish upper secondary mathematics (Ärlebäck, 2009), in Denmark and in Germany (Blomhøj & Hoff Kjeldsen, 2006; Schmidt, 2012). In some other countries it has a more weak position, such as for example in Norway (Utdanningsdirektoratet, 2013). In the Norwegian curriculum for the common core subject mathematics (Utdanningsdirektoratet, 2013) the basic skill to aim for is numeracy instead of modelling. The word modelling is not found in relation to numeracy but instead found one time related to the basic skill of using digital tools, because the use of technology “involves learning how to use and assess digital aids for problem solving, simulation and modelling" (p. 5). As a contrast, in the competence oriented new Swedish National syllabus for mathematics for upper secondary school, mathematical modelling is one of the overall seven mathematical "abilities" to aim for: to develop students' ability to "interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations". Mathematical modelling is also emphasised in the knowledge criteria descriptions, though mentioned only at one occasion in the mathematical content descriptions (in connection to differential equations) (Skolverket, 2012a). However, this explicit description of the notion of mathematical modelling is not found in the Swedish syllabus for mathematics in *compulsory school*, where the word modelling only appears in the syllabus for art (Skolverket, 2011).

3.2.2 Textbook analyses of mathematical modelling

The issues of adequate textbooks and their use in mathematics education have been discussed for more than 200 years in Sweden (Frejd, 2013b). Jablonka and Johansson (2010) conclude, based on a literature review of research about Swedish textbooks in mathematics, that the "use of textbooks still is a prevalent practice in Swedish mathematics classrooms" (p. 363) and students are spending much time working independently with exercises from their textbooks (ibid.). However, to what extent these textbooks in Sweden include mathematical modelling is not analysed, but some isolated tasks with a modelling character were found in textbooks from the late 1990s (Jakobsson-Åhl, 2008).

Also from international research there are indications and evidence that the mathematical textbooks play an important role in the mathematics classroom. For instance, the international assessment of mathematics and science knowledge, the Trends in International Mathematics and Science Study (TIMSS) 2007, indicates that a primary source for teaching and learning mathematics is the textbook (see http://timss.bc.edu/timss2007). According to Ikeda (2007), the lack of adequate mathematical textbooks is a common obstacle to teach modelling in lower secondary school, and Cabassut and Wagner (2011) found that modelling was described only implicitly in tasks in primary textbooks in France and Germany. However, there seem to be few research studies of textbooks, with an explicit focus to analyse how mathematical modelling is interpreted and explained in different textbooks. A search in the database ERIC (Educational Resources Information Center, EBSCO) using the terms 'mathematical modelling' together with 'textbooks' gave six references, and the same search in MathEduc gave 17 references, but none of the references focused on textbook analysis with an explicit aim to analyse how mathematical modelling is interpreted and explained in different textbooks.

Nevertheless, there are other research studies with related aims, which refer to numeracy, modelling used in mechanics textbooks, applications and problem solving.

Gatabi, Stacey and Gooya (2012) used the PISA (The Programme for International Student Assessment) definition of mathematical literacy to analyse and compare textbook problems (grade nine) used in two textbooks in Australia and one in Iran. They found that only few textbook problems did require the students to interpret and check their answers and they conclude "[f]inally, even though mathematical modelling is at the heart of the mathematical literacy..., we must acknowledge that neither in this study nor in our other experiences with school textbooks in Australia or Iran, have we seen many examples of problems that really meet the criteria for a genuine modelling problem" (ibid., p. 418).

A-level textbooks used in mechanics was analysed by Rowland (2003). He found that the textbook descriptions of modelling were not specific to mechanics, instead they seemed to stem from a more general approach of mathematical modelling found in mathematics (i.e. a cyclic process as described in Chapter 2), which according to Rowland (2003) is inappropriate in the beginning of A-level since the students need to grasp that "the modelling process is structured according to the scientific model- to which students have to be induced" (p. 103) and thus "[t]he general mathematics modelling procedure is only appropriate after the student has learnt mechanics and is familiar with the translation component" (pp. 103-104).

Research studies that are analysing textbooks related to mathematical applications (Lu & Bao, 2012) as well as word problems and problem solving (e.g. Fan & Zhu, 2007; Hensey, 1996; Jakobsson-Åhl, 2008; Kongelf, 2011; Mayer, Sims & Tajika, 1995; Nibbelink, Stockdale, Hoover & Mangru, 1987; Zhu, 2003; Zhu & Fan, 2006) have some connections to modelling. Some of the results from studies on textbooks analyses with focus on problem solving and applications are: Norwegian lower secondary textbooks do to a large extend only treat problem solving implicitly and in particular heuristic problem solving methods where guidance of when and how to use these methods is lacking in the textbooks (Kongelf, 2011); most of the application examples in upper secondary textbooks used in USA and in China were problems without any connection to the real world, but for those examples that had a relation to the real world the American textbook did emphasise mostly personal contexts compared to the two Chinese textbooks that used more public contexts (Lu & Bao, 2012); there exist gaps between what is written in curriculums regarding problem solving and what is presented in textbooks for lower secondary students in China, USA and Singapore and general problem solving strategies are more explicitly elaborated in Chinese textbooks compared to textbooks in USA and Singapore (Fan & Zhu, 2007); the seventh grade textbooks in Japan presented worked-out examples of problem solving strategies as a guide for students to learn problem solving, but textbooks used in USA did not include much guidance but instead presented many exercises for students to solve on their own (Mayer, Sims & Tajika, 1995).

3.2.3 Other teaching materials, guides to teachers etc.

There exist a plenitude of books about modelling that teachers may read and use. A search in the international database Worldcat with the word ‘mathematical modelling’ gave 322650 books. Some of the books are guides for teachers on what and how to teach (e.g. Swetz & Hartzler, 1991; Treilibls et al., 1982), other modelling books relate to some specific topic like algebra, statistic, optimization etc. (e.g. Kallrath, 2011; Timmons, Johanson & McCook, 2010; Zill, 2009), and some books focus more on modelling as a topic in itself (e.g. Caldwell & Ng, 2004; Giordano, Fox & Horton, 2014; Heinz, 2011; Starfield, Smith & Bleloch, 1990).

There exist online materials such as the LEMA[12]-project or MATHmodels.org that provide examples of modelling activities for the mathematical classroom.

For Swedish upper secondary school teachers there is a guide made by Skolverket (n.d.) with aim to clarify the seven abilities to teach in the present subject syllabus for mathematics (Skolverket, 2012a). The guide described in a section about the mathematical modelling ability by the following text:

> Modelling ability means to be able to formulate a mathematical description- a model- based on a realistic situation. The situation may be, for example, problems or tasks found in the program specific courses and problems or situations relevant for personal finance or with relevance for the individual to take part in the society. It is about to individually develop a connection in terms of a model, rather than to apply already developed models. When the model is developed, then the modelling ability means to be able to use the model’s properties to solve, for example, mathematical problems or standardised tasks. Modelling ability also means to be able to interpret the result in relation to the original problem situation. It also means be able to evaluate the model’s properties and limitations in relation to the initial situation.
>
> For example, in science, technology, social science and economy mathematical models play an important role as a tool for analysing specific questions. (p. 2, my translation)

The description above is an extension (an interpretation) of the curriculum, which gives example of using realistic situations related to private financial and societal issues in the classroom to develop models. Then the students, according to the description, are supposed to apply the developed model to solve a problem or a standardised task. However, it is a bit unclear if it is necessary for the students to develop the model from scratch and if the purpose with the developed models is

[12] LEMA- Learning and Education in and through Modelling and Applications and example of modelling problems may be retrieved from http://www.lema-project.org/web.lemaproject/web/eu/tout.php

mainly for students to solve standardized tasks. No examples of activities or problems are given in the guide that may explain the notion further.

The government has recently, with start in the autumn 2013, introduced a professional development programme (Skolverket, 2012c) for all mathematics teachers to increase the quality of teaching. The municipalities may apply for money so that the teachers can follow the programme during regular working hours. The programme is implemented using 'peer learning', which means that mathematics teachers discuss and evaluate teaching together with a mentor (one or more teachers are appointed by the community to take care of the professional development course and act as mentor). For upper secondary school the programme is based on a module called *Teaching Mathematics by Problem Solving*[13]. This module, as all modules in the programme, consists of eight parts, each including an outline of activities and a sequence for the teachers to follow. The work, the sequence to follow, in a module typically starts with a number of texts to read and discuss, followed by the collaborative development of short teaching activity. The developed activity is then implemented in the participating teachers' mathematics classrooms, and finally evaluated together with their peers. One of the eight parts in the module is "mathematical modelling", which includes two texts about modelling and two (modelling?) problems about geometry and similarity. I will not analyse this module in this thesis, but it may be done in an upcoming investigation.

There also exist research literature about guides to teachers with suggestions how to teach. For example do Leavitt and Ahn (2010) give "recommendations for Group Composition...for Selection of Relevant MEA's [modelling eliciting activities]...Teachers Roles During Group Work...for Group Presentations and Individual Written Work" (pp. 353-358). Geng (2003) provides tables with suggestions how to teach and examples of topics, and Guerra, Hernándes, Kim, Menekse and Middleton, (2010) give another list of suggestions for teachers as displayed in Table 3.

Table 3. Suggestions for teachers (Guerra et al., 2010, p. 306)

(1) supply a classroom environment where students are allowed to question
(2) State the purpose of an activity and present it explicitly
(3) Help students to see pattern from collected data by providing practice with graphic organizers
(4) Make experiment simple and changes should be minimal from one activity to another
(5) Give students enough time to think
(6) Observe and collect students' work to guide them more accurately and for improve of future lessons
(7) Discuss explicitly students' thinking using examples and non-examples.

[13] Available through the web page http://matematiklyftet.skolverket.se/matematik/

To what extent these suggestions for teaching modelling differ from suggestions for other teaching may of course be discussed, since the suggestions in Table 3 is quite generally written. Borromeo Ferri and Blum (2010) focus on modelling in their suggestions. They argue that their version of the modelling cycle may be "indispensible both for teachers (as a basis for their diagnosis and interventions) and for researchers (as a tool for describing actions and cognitive process in learning environments with modelling tasks) for students, a simpler version seems to be appropriate" (p. 431). For further guides, examples and ideas see e.g. the chapter 'Concrete cases' in the ICTMA 14 proceeding (Kaiser, Blum, Borromeo Ferri & Stillman, 2011) or the chapter 'Practice of modelling' in the ICTMA 12 proceeding (Hains, Galbraith, Blum & Kahn, 2007).

3.3 Taught knowledge

From the document analysis of a sample of ICTMA publications over the past decade it was found that a large proportion of the papers are 'professional papers', with an aim to inform teaching and learning practice at any level of education (Frejd & Geiger, 2013; Geiger & Frejd, 2013). Some of these papers write about teaching sequences, teaching activities and professional development courses, where it is described what is actually taught. However, a consensus among researchers as described in the research literature relating mathematical modelling in mathematics education is the low integration of modelling activities in day to day teaching (see e.g. the preface in Kaiser et al., 2011, and the introduction in Kaiser, 2010).

A list that captures all knowledge being taught is obviously not possible, but to describe the diversity of knowledge taught there are some examples given related to elementary arithmetic with base ten blocks (Speiser & Walter, 2010), sound intensity and brightness (Riede, 2010), ranking statistical data (Carmona & Greenstein, 2010; English, 2010; Mousolides et al, 2010), solving linear pattern tasks (Amit & Neria, 2010), solving problems with geometry (Stillman, Brown, & Galbraith, 2010), with technology (Confrey & Maloney, 2007), different representations of functions (Arzarello, Pezzi & Robutti, 2007), calculus (Araújo & Salvador, 2001), non linear situations (De Bock, Van Dooren & Janssens, 2007), multi-variable functions (Nisawa & Moriya, 2011), interdisciplinary projects (Ng, 2011), traffic models (Blomhøj & Hoff Kjeldsen, 2011), etc....

Findings from research about what is taught about modelling in Swedish upper secondary school include modelling introductions with use of Fermi problems (Ärlebäck, 2009d), implementation of modelling modules (Ärlebäck, 2009a), and developing computer models based on dynamic systems (Wikström, 1997).

There are also some research concerning teachers' beliefs and modelling, which impact what is taught in school. However, before discussing the findings of teachers' beliefs and conceptions of modelling a short discussion of the notions belief and conceptions is presented.

3.3.1 Beliefs and conceptions

From a historical point of view, research on beliefs increased during the 1980s. One reason was the interest in problem solving activities. Schoenfeld (1985b), who wanted to explain and characterize problem solving activities, introduced a theoretical framework including a belief system: "[...] the set of (not necessarily conscious) determinants of an individual's behaviour" (Schoenfeld , 1985b, p. 15). The new problem solving activities during the 1980s required teachers to change their classroom behaviour. According to Ernest (1989) it was necessary to understand more about knowledge, beliefs and attitudes of mathematics teachers to be able to change current teaching into the new approach with an increased emphasis on problem solving.

The word beliefs is discussed and defined in articles, books and chapters, such as Thompson (1992), Leder, Pehkonen and Törner (2002), and Philipp (2007), but there does not exist a single definition of belief or any general agreement of the word in the literature (McLeod & McLeod, 2002). Törner (2002) adds "[i]t is clear that only in rare cases can a final precise definition of all components of a belief definition be achieved in a specific context" (p. 91). Pajares (1992) describes the situation as problematic in belief research, using the words "messy construct" (p. 1). The problematic situation has appeared due to "definitional problems, poor conceptualizations, and differing understandings of beliefs and belief structure" (Pajares, 1992, p. 1).

The relation or distinction between knowledge, beliefs and attitudes could be seen as a central question in beliefs research in mathematics education and the question is widely discussed in literature (e.g. Ernest, 1989; Furinghetti & Pehkonen, 2002; Green, 1971; Leder, Pehkonen & Törner, 2002; Liljedahl, 2008; Pajares, 1992; Philipp, 2007; Skott, 2009; Thompson, 1992; Österholm, 2010). Philip (2007) describes the complexity in finding the relations between knowledge, beliefs and attitudes: "[r]esearchers studying teachers' knowledge, beliefs and affect related to mathematics teaching and learning are still trying to tease out the relationships among these constructs and to determine how teachers' knowledge, beliefs and affect relate to their instruction" (p. 257). He defines affect, emotions, attitudes, beliefs and knowledge as quoted below.

> **Affect-** a disposition or tendency or an emotion or feeling attached to an idea or object. Affect is comprised of emotions, attitudes, and beliefs.
>
> **Emotions-** Feelings or states of consciousness, distinguished from cognition. Emotions change more rapidly negative (e.g., the feeling of panic). Emotions are less cognitive than attitudes.
>
> **Attitudes-** manners of acting, feeling, or thinking that show one's disposition or opinion. Attitudes change more slowly than emotions, but they change more quickly than beliefs. Attitudes, like emotions, may involve positive or negative feelings, and they are felt with less intensity

> than emotions. Attitudes are more cognitive than emotions but less cognitive than beliefs.
>
> **Beliefs-** Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (I do not intend this definition under affect because, although beliefs are considered a component of affect by those studying affect, they are not seen in this way by most who study teachers' beliefs.)
>
> ...
>
> **Knowledge-** Beliefs held with certainty or justified true beliefs. What is knowledge for one person may be belief for another, depending upon whether one holds the conception as beyond question." (Philip, 2007, p. 259)

The definition of beliefs above shows that beliefs are quite complex, they can be held with varying degree of uncertainty, they are possible to feel, they have cognitive aspects and they are possible to change. If beliefs are held with certainty then they are called knowledge. The definition of knowledge is actually done in terms of beliefs. This implies that the researcher should consider making a distinction between knowledge and beliefs, e.g. define levels of uncertainty or define justified truth, at least if the research aims to investigate possible changes in teacher behaviour. Beliefs are possible to change, but knowledge is not (Furinghetti & Pehkonen, 2002). The debate about the distinction between beliefs and knowledge as well as if it useful will continue among researchers (Philipp, 2007).

Conceptions are closely related to beliefs, but like beliefs the word conceptions has been used with different meanings both in 'everyday life' (see e.g. http://www.merriam-webster.com/dictionary/conceptions) and in research literature in mathematics education (Furingghetti & Pehkonen, 2002). Lloyd and Wilson (1998) as well as Thompson (1992) argue that it is not useful in research to distinguish between teachers' knowledge and teachers' beliefs and therefore it seems more useful to use the connected concept conceptions. Thompson (1992) defines teachers' conceptions as:

> Teachers' conceptions- mental structures, encompassing beliefs and any aspect of teachers' knowledge that bears on their experience, such as meanings, concepts, propositions, rules mental images, and the like. (Thompson, 1992, p. 141)

To conclude, the notion of conceptions is often used as an umbrella term (Furingghetti & Pehkonen, 2002) and it may therefore be seen better to use the notion of conception in exploratory investigations rather than beliefs in line with Thompson's (1992) argument. That researchers "should search for whether and how teachers' beliefs, or what they take to be knowledge, relate to their experience" (Thompson, 1992, pp. 140-141). However, everything depends on the definitions used.

3.3.2 Teachers' conceptions of mathematical modelling

Teachers' conceptions/beliefs of the use of real-world problem solving or modelling in mathematics education may (or may not) influence how they teach and work with mathematical modelling in their classroom. Chapman (2007) discuss about teachers that stress modelling as an important aspect of mathematics and these conceptions also influence their teaching strategies to support the development of modelling ability. Kaiser (2006) did an empirical investigation and found that the teachers' conceptions did not influence how they taught: "it became clear that although teachers were convinced to considering applications and modelling for daily school practice they still argued for mathematics and mathematics teaching in which application and modelling only played a minor role" (p. 393). In addition, Kaiser and Maass (2007) also stressed that mathematical modelling only played a minor role in teachers' beliefs about mathematics and mathematics education, which may be one reason for the low integration of modelling activity in day to day teaching. To change teachers' beliefs about how to implement modelling is difficult. After an experimental course with teachers in a mathematical modelling project, LEMA, some of the teachers' beliefs (in terms of obstacles for implementing modelling) were still persistent. These beliefs implied that it is enough time for doing mathematical modelling in the mathematical classroom and that assessment of modelling is too complex (Schmidt, 2012). However, some beliefs changed during the course, for example that there exist to little modelling material. Other beliefs (in terms of motivations for implementing modelling) seemed to increase and get stronger due to the professional development course, such as that students think more creatively while working with modelling problems, that there are long-term positive effects in mathematics lesson, that the teachers working load decreased when students work with modelling problems, and that modelling problems are useful when there is a diversity in students' performance levels.

Teachers' conceptions about the modelling process and the purpose of mathematical modelling were investigated by Gould (2013) in a large scale questionnaire study with 260 teachers from grade 7 through 12. It was concluded that teachers describe the modelling process as steps to perform such as checking the result for correctness. The teachers also accept that the result may be either approximate or exact. However, the teachers in Gould's study did not connect modelling to real-world problems but instead argue that the problem may be derived from whimsical and unrealistic situations. The teachers did not emphasise assumptions and choices as a part of the modelling process. The teachers' conceptions of the purpose of modelling addressed the learning goal for students to use modelling in their everyday life, in other school subjects, to think mathematically, and to apply mathematics they have learned (ibid.).

Förster (2011) analysed and categorised secondary teachers' beliefs of goals for teaching modelling in mathematics education into three goals. One group of teachers hold the belief that mathematics should be taught without context and modelling may be used to illustrate mathematical content. The second group of teachers' goal with modelling was to increase the students' motivation to learn the mathematical content. The third group of teachers was categorised to teach modelling as a means for "*empowered citizen*, *problem solving* (inside and outside mathematics), *a positive attitude towards mathematics*, and *learning to ask questions*" (p. 72), which seem to be related to the arguments by Blum and Niss (1991).

The case study by Ärlebäck (2010), discussed in section 2.4, is so far the only research with explicit focus on modelling in Sweden addressing the issue of in-service teachers beliefs about mathematical modelling. However, Paper 3 will provide more results on teachers' conceptions in Sweden.

3.4 Knowledge to be learned

What is asked for in assessment may be categorised as *knowledge to be learned* (Hardy, 2009). The word assessment, rooted from the Latin word assidere, *to sit beside*, meaning metaphorically or literally: an activity where a teacher, peer or parent is *sitting beside* a student to gather and interpret evidence to seek answers about students' learning, about her own teaching, and what actions to take to provide information and feedback to the student, teacher, parents and other institutions about the quality of teaching and learning mathematics (Swaffield, 2011). This description of assessment captures the aspects of assessment as a *process* (Airasian & Russel, 2008) and the focus on *judging* (Niss, 1993a). The sitting beside metaphor also captures the objectives of assessment described by Wiliam (2007) and Niss (1993a). According to Wiliam (2007) assessment has three objectives: assessment as an aid for learning (formative assessment), assessment as guide for certifying students' performance and achievements (summative assessment), and assessment used to monitoring the quality of institutions or educational programs (evaluating assessments). Also, Niss (1993a) categorises the purpose of assessment into three categories: to provide information to students, teachers and others (parents, educational institutions, employers, etc.), as a base for decisions and actions, and as a way to shape social reality by informing about what is regarded as valuable in a society in terms of working habits, attitudes towards social order and competition.

Several assessments modes/ methods exist in mathematics education, such as oral tasks, practical tasks, teacher observation, student journals, peer and self-assessment, and parental assessment (Watt, 2005), but the most common modes of assessment mode is the written test that includes a set of task to be solved (Niss, 1993a).

Teachers argue that it is difficult to assess modelling (Schmidt, 2012), but it is possible to identify several different assessment modes in the research literature. For example written tests (Henning & Keune 2006; Naylor, 1991; Stillman, 1998; Turner, 2007), multiple choice questions (Haines, Crouch & Davis, 2000; Zöttl, Ufer & Reiss, 2011), practical assessment tasks (Vos, 2007), projects (Antonius, 2007; Coxhead, 1997; Kaiser, 2007), observations (Gillespie, Binns, Burkhardt & Swan, 1989), poster presentation (Wake, 2010), students' portfolio (Dunne & Galbraith, 2003; Francis & Hobbs, 1991), and contests (Haan, 2003; Jiang, Xie & Ye, 2007).

In relation to *knowledge to be learned*, the 'highly relevant' question: *What* should be assessed in the applications and modelling sub-space 'content x product x process…?' (Niss 1993b, p. 49) need to be discussed.

The content asked for refers to atomistic parts or sub-competencies of the modelling cycle (e.g. Dunne & Galbraith, 2003; Haines, Crouch & Davis, 2000; Zöttl, Ufer & Reiss, 2011), communication (e.g. Battye & Challis, 1995; Clatworthy, 1989; Edwards & Morton, 1987; Hamson, 1987), the range of contexts in which a student may perform his/her modelling ability and how advanced the mathematics is which the student uses (Højgaard Jensen, 2007), to critically analyze and reflect upon the modelling process (Henning & Keune, 2006). Concerning the mathematical content there is a variety of topics such as fractions and combinatory (e.g. Houston & Breedon, 1998), probability (e.g. Vos, 2007), linear pictorial patterns (e.g. Amit & Neria, 2010), polynomial functions (e.g. Ruiz, Bosch & Gascón, 2007), exponential functions (e.g. Battye & Challis, 1995), geometry (e.g. Wake, 2010; Zöttl et al., 2011), calculus (e.g. Dunne & Galbraith, 2003) etc.

The main focus in written tests and contests seem to be on the product (e.g. Haan, 2003; Haines, Crouch & Davis, 2000; Shouting, Wei & Tonga, 2003; Zöttl, Ufer & Reiss, 2011). It seems easier to achieve the process goal in projects, portfolios and poster presentations (e.g. Clatworthy, 1989; Francis & Hobbs, 1991; Wake, 2010).

A more extended review of assessment modes of modelling and its consequences is found in Paper 4, which draws on and extents the findings from Frejd (2012a, 2012b).

3.5 Knowledge actually learned

The issue about what students learn by working with mathematical modelling concerns *knowledge actually learned*. According to Hardy (2009) knowledge actually learned can be accessed, for example, through students' written or oral responses to tasks, through interviews, by observing students' behaviour in the mathematics classroom or in designed experimental settings. I will present a narrow and small sample of literature review to illustrate the diversity and complexity of knowledge actually learned, instead of attempting to capture 'all' literature that deal with knowledge actually learned in relation to mathematical modelling.

Students in primary school (Kindergarten to grade six) have learned to use different strategies to solve problems related to out-of school situations and include real world knowledge while reasoning mathematically (Bonotto, 2010). In addition do some students in primary school have the knowledge to "interpret the problem information, expressed ideas how to meet the problem goal, tested their ideas against the given criteria and data, revisited the problem information, revised their approach, implemented a new version, tested this, and so on." (English, 2010, p. 292). However, the main finding using modelling problems was that the students learned how to direct their own learning (ibid.). Some difficulties are identified among primary students' working with modelling problems like to understand the problem and verify the solution (Mousolides et al., 2010).

There exists diversity in students' knowledge in secondary school (grade seven to grade nine) how to plan a modelling activity, where some students are well organised and discuss a plan before getting started, others do the planning implicitly and some students need to get more information to get started (Greefrath, 2010). Research findings indicate that secondary students involved in modelling tasks make less overuse of linear models (De Bock, Van Dooren, & Janssens, 2007), know how to find, interpret and recognise the meaning of a mathematical solution (Sol, Giménez & Rosich, 2011), get more motivated to develop a modelling competency (Lakoma, 2007), increase their knowledge and beliefs of the usefulness of mathematics (Maass, 2010), develop a modelling competency if it is integrated in the classroom during a long period (Bracke & Geiger, 2011). Other research studies point to a lack of knowledge, for example that many students do not apply mathematical skills or concepts (Ng, 2011) or use simple mathematics (Biccard & Wessels, 2011) in the solution process. The lack of validating is a common problem among the secondary students (Biccard & Wessels, 2011; Sol et al., 2011) as well as to make generalisations (Amit & Neria, 2010; Lakoma, 2007). However, according to Stillman, Brown, and Galbraith (2010) "the most challenging part of the modelling cycle" is the formulation phase (p. 396).

From an evaluation of a modelling week in upper secondary school (grade 10 to grade 12), it was found that "[t]he students described high learning outcomes reflecting all the goals of modelling in mathematics education, ranging from psychological goals, namely enhancing the understanding of the world around us" (Kaiser, Schwarz & Buchholtz, 2011, p. 600). Ärlebäck and Bergsten (2010) used realistic Fermi problems to introduce mathematical modelling in upper secondary mathematics and concluded that "all the modelling sub-activities proposed by the framework (reading, making model, estimating, calculating, validating and writing) are richly and dynamically represented when students get engaged in solving realistic Fermi problems." (p. 607). Students also seem to have knowledge to use different representations while working with modelling problems (Arzarello, Pezzi & Robutti, 2007; Confrey & Maloney, 2007). However, according to Legé (2007a) "[t]he good news is that students exist who can fuse knowledge about mathematics and science (studied as separate disjoint subjects), apply that synthesis to model a simple situation and critically question the validity of alternative explanations" (p. 284), but the bad news are that there are not so many such students (ibid.). The students have problems with 'formulating the problem' (Kaiser, 2007) and 'select models' for a given situation (Frejd & Ärlebäck, 2011).

University students have gained knowledge in statistical concepts (Makar & Confrey, 2007), in multi-variable functions (Nisawa & Moriya, 2011), calculus (Araújo & Salvador, 2001), environmental modelling (Hamson, 2001), etc. when working with modelling problems. In the working process students construct a diversity of different models (Dominguez, 2010) and they develop internal- (about sub-process) and external- (the role of the model in the actual or potential applications) knowledge about modelling (Blomhøj & Hoff Kjeldsen, 2011). Modelling activities at university seem to motivate students to work with problems (Barbosa, 2001; Hamson, 2001). Some difficulties of students' performance are also detected like "talk about and write out their presuppositions, resolution and validation process, presenting just a generic description of the strategy and the recommendation" (Barbosa, 2001, p. 191). Tavares (2001) draws on (her) earlier research studies and lists some difficulties of students' performance at university level, which are illustrated in Table 4.

Table 4. University students' difficulties (Tavares, 2001, p. 258)

(a) Difficulty to understand the context surrounding to a given situation
(b) Creation of restrictions not mentioned in the work proposal
(c) The non-observation of conditions or important data mentioned in the worksheet
(d) Difficulty to identify the essential aspects of a set situation and translate them into mathematical terms
(e) Difficulty in bearing in mind various aspects of the same situation at the same time
(f) Difficulty in starting to see the situation from different angle/ perspective
(g) Difficulty to identify the meaning of a mathematical operation in terms of the situation at the starting point
(h) Difficulty to identify the mathematical concepts to be used in a given situation
(i) poor mastery of some mathematical concepts

3.6 A summary of the didactic transposition

This section will summarise and discuss the didactic transposition of mathematical modelling based on the brief literature review. Mathematical modelling as a knowledge is transposed through *Scholarly knowledge*, *knowledge to be taught*, *taught knowledge*, *knowledge to be learned*, to *knowledge actually learned*, as illustrated in Figure 6.

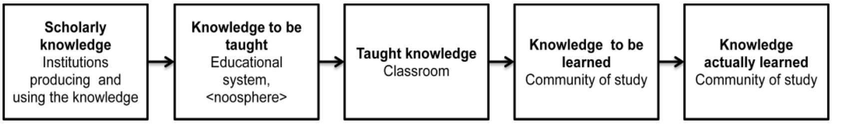

Figure 6. The didactic transposition process (based on Bosch & Gascón, 2006, p. 56 and Hardy, 2009, p. 343)

Scholarly knowledge about mathematical modelling may be categorised as what professional modellers do and know, but it may also be conceived as to adapt and use models that are hidden in black boxes like workers do and/or how textbooks at university level present modelling. The last issue of textbooks may also refer to knowledge to be taught. The short review of scholarly knowledge indicates that 'all' sectors in the workplace frequently include models and modelling, often hidden in technology. *Modellers* have different views about what modelling is. In their work they emphasize aspects such as communication, collaboration, personal experience and technology to identify relevant information and verify solutions. However, other aspects like the unspoken rules at the workplace, the specific community of practice and the division of labour also influence all activities at the workplace (Wake, 2007).

Knowledge to be taught about modelling is complicated to describe. The role of mathematical modelling in school mathematics differs significantly between different countries and between different grades (primary school, secondary school and upper secondary school), as set out in national curriculum frameworks. There are also differences between how textbooks present problem solving and applications in different countries and arguments are put forward to incorporate more explicitly general problem solving strategies. There are indications that modelling is not a central notion in mathematical textbooks. However, it is not always easy to compare different textbook studies, since there is a "lack of common and explicit criteria for textbook comparisons" (Charalambous, Delaney, Hsu & Mesa, 2010, p. 120). There exist an abundance of teaching material and teaching guidance on how to teach about modelling in books, research papers, at the internet. The teaching materials seem to display two objectives of knowledge to be taught, i.e. either is modelling a vehicle to learn mathematics or to learn mathematical modelling. The latter objective is in focus in this thesis.

Taught knowledge about modelling is lacking in everyday teaching. There is a consensus among researchers in mathematics education that mathematical modelling only plays a minor role in the classroom. However, the literature review indicates that there is a diversity of mathematical topics included in the teaching of modelling and there exits many examples of teaching sequences, teaching activities and professional development courses that include different aspects of mathematical modelling. Research results concerning the impact of teachers' conceptions of the role of modelling are contradictory. It may or may not have an impact on what they teach, the main conclusion being that modelling still plays a minor role in regular teaching.

Knowledge to be learned about modelling as displayed in assessments indicates that the type of assessment modes used has an impact on what is assessed. There seems to be a discrepancy in the literature between learning modelling in terms of sub-competencies versus modelling as a holistic activity. Written tests as displayed in the review focus on the product and sub-processes whereas alternative modes may be to assess process goals and modelling as a holistic activity.

Knowledge actually learned: it is difficult to find consistent picture of what the students actually learn. Some researchers argue that students learn a type of problem solving process, while other focus on the mathematics. The findings are to some extent contradictory where some research gives evidence that students do possess some knowledge, like validation, and other researchers argue that the students do not. A common difficulty seems to be to know how to formulate a mathematical model.

To provide a clear outline of the didactic transposition of mathematical modelling based on to this brief literature review is complex. One reason for this complexity is that mathematical modelling outside school has different meanings, which makes it difficult (or even impossible) to adapt in educational settings. The notion of mathematical modelling has no clear description at the level of *scholarly knowledge*, where it lives in a variety of institutions with diverse practices. Modelling at the workplace and in *some* curriculums is however described as an important skill to master. Modelling is closely related to technology at the workplace, which is not so explicitly described in knowledge to be taught. There exist many different textbooks and guides on how to teach modelling, but the common picture is that modelling is not a part in everyday mathematics teaching. Modelling as it exists in assessment, the knowledge to be learned, has a focus on mathematics and sub-competencies and sometimes on a holistic approach. What students actually learn is difficult to grasp, since the research literature is inconsistent.

Chapter 4

Methodology

Regarding the etymology of *methodology*, the word originates from the Latin word *methodologi´a*[14]. *Methodologi´a* includes the Latin word *me´thodus* (systematic or scientific procedure, method of inquiry) and the suffix -*logi´a* (branch of knowledge, science). The word is defined in different ways in dictionaries[15]. The Dictionary.Com[16] defines methodology as "the underlying principles and rules of organization of a philosophical system or inquiry procedure". It appears that the definition of methodology from Dictionary.Com refers to *how* an investigation has been carried out (method, rules of organizations) as well as *why* the chosen methods have been used in an investigation (the underlying principles). This chapter presents the methodology of this doctorial thesis, or in other words different aspects of organization and methods in relation to the research questions and the underlying principles of choice of research questions and of theoretical framework as well as a discussion of ethical considerations.

4.1 Theory and frameworks

One aspect that counts as 'primary quality parameter' in a PhD dissertation is the methodological concerns about justifying answers to the posed research questions (Niss, 2010). In order to justify answers, a theoretical discussion about the used terminology is needed and a theoretical approach that explains the findings is expected. However, the meaning of theory is not clear in mathematics education (Niss, 2007a, 2007b).

The word theory is rooted in the Greek word *theōri´a* meaning 'consideration', 'contemplation' (teori, 2013, my translation) and it has multiple meanings in literature and everyday use. In everyday use it may be discussed in terms of 'it is just a theory' meaning, more or less, that it is a speculation (often not underpinned) of something. In encyclopaedias there are several definitions used, for example:

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[14] http://www.ne.se/lang/metodologi

[15] http://www.wordnik.com/words/methodology

[16] http://dictionary.reference.com/browse/methodology

> scientific theory, systematic ideational structure of broad scope, conceived by the human imagination, that encompasses a family of empirical (experiential) laws regarding regularities existing in objects and events, both observed and posited. A scientific theory is a structure suggested by these laws and is devised to explain them in a scientifically rational manner. (scientific theory, 2013)

and

> theory, a group of assumptions or statements that explain phenomena and systematize our knowledge of them. (teori, 2013, our translation)

Both these encyclopaedia definitions of theory focus on 'a set of postulates or entities' (systematic ideational structure of broad scope, or a group of assumptions or statements) to be used to structure, systematize and interpret observations in order to get a better understanding of them.

Theory as 'a set of postulates or entities' with a specific scientific purpose may also be addressed in research literature within mathematics education. However, according to Niss (2007a),

> is it not clear what 'theory' actually means in mathematics education. Nor is it clear at where the entities referred to as theories invoked in mathematics education come from, how they are developed, what foundations they have, or what roles they play in the field. (p. 97)

Nevertheless, the issue of describing and defining the notion of theory, the use and the role of theory in research in mathematics education is frequently discussed in working and study groups at conferences like ICME and CERME as well as in literature (Cobb, 2007; Bergsten, 2008; Jablonka & Bergsten, 2010; Jablonka, Wagner & Walshaw, 2013; Lester 2005; Niss, 2007a; Radford, 2008; Silver & Herbst, 2007; Sriraman & English, 2010; Sriraman & Nardi, 2013).

The descriptions or definitions used in literature may be characterized as encyclopaedia definitions, like Sriraman and Nardi's (2013) definition

> Theories are cleaned- up bodies of knowledge that are shared by a community. They are the kind of knowledge that gets embodied in textbooks. They emphasize formal/ deductive logic, and they usually try to express ideas elegegantly, using a single language and notion system. (p. 310)

Other attempts to describe theory may be conceptualised in terms of "minimum ingredients" (Jablonka & Bergsten, 2010, p. 27), like in Niss (2007a). He defines five sub-theories as a minimum requirement for a 'grand' theory in mathematics education. These five sub-theories are: *mathematics as a discipline and a subject*; *individuals' affective notions*; *individuals' cognitive notions*; *the teaching of mathematics*; and *teachers of mathematics* (pp. 107-108). In addition, a theory ought to acknowledge an organisation of concepts (structured in a hierarchy) and claims about contexts, situations, normative issues, etc. (ibid.). Theory may also be described in termers of *static* and *dynamic* view (Bikner-Ahsbahs & Prediger, 2010). The *static* view regards theory as 'a set of postulates or entities' to be used to organize and interpret observations, for example the encyclopaedia definitions above. The *dynamic* view regards theory as integrated part of the methodology that need to be developed in order to answer a given research question. The dynamic view is closely related to the role of theory.

The role of theory or the purpose of theory is also discussed, more or less fine-grained, in literature. Silver and Herbst (2007) suggest a quite general description of the role of theory (a *dynamic* view), which is "helping to shape research questions, suggest research methods, and explaining research findings" (p. 41). Related to Silver and Herbst's (2007) *dynamic* view is Radford's (2008) description of three key aspects of theory (a system of principles, the methodology and research questions) and explains that the purpose of theory is "producing understandings and ways of actions" (p. 320). Niss (2007a) includes more roles for theory and identifies six purposes, categorised as: *explanation*, *predictions*, *guidance for action or behaviour*, *a structured set of lenses*, *a safeguard against unscientific approaches* and *protection against attacks* (p. 100). Bikner-Ahsbahs and Prediger, (2010) use similar criteria as Niss (2007a), but use a more *dynamic* view and describe the role of theories to be a guide for researchers for "investigating phenomena in mathematics education or providing the tools for design, the language to observe, understand, describe and even explain or predict phenomena in mathematics education" (p. 488).

Theory and theoretical approach are sometimes used synonymously in literature. Wedege (2010a) does not distinguish between theory and theoretical approach more than describing that the notion of 'theory' is wide when the two words are conceptualized as synonymous. Bikner-Ahsbahs and Prediger (2010) also use the words exchangeably, but suggest that theoretical approach is more preferred with a dynamic view of theory. Theorising refers to development of theory with an aim of making something visible "that cannot be captured without mediating by the theoretical concepts" (Jablonka & Bergsten, 2010, p. 27).

The working definition of theory or theoretical approaches for this thesis is defined by Bikner-Ahsbahs and Prediger (2010) as:

> 'theories' [theoretical approaches] are constructions in a state of flux. They are more or less consistent systems of concepts and relationships, based on assumptions and norms. They consist of a core, of empirical components, and their application area. The core includes basic foundations, assumptions and norms which are taken for granted. The empirical components comprise additional concepts and relationships with paradigmatic examples; it determines the empirical content and usefulness through applicability. (p. 488)

The rationale for using this definition is that it is a dynamic view of theory i.e. a theoretical approach, which may be useful in this thesis. This thesis is based on publications and an overall theoretical approach that combines the different publications needs to be developed in a dynamic process of finding relationships between aims and research questions in the different articles and then establish possible new aims and research questions for this thesis.

A research framework aims, like theorising, to make relationships and abstractions visible in order to understand and justify research findings of some investigated phenomena (Lester, 2005). Theoretical framework and theoretical perspective are closely related notions to theory and theoretical approach and sometimes the notions are distinguished from theory and theoretical approach, since the former notions do not automatically include a methodology (Wedege, 2010a). However, a research framework may also 'guide one's research' to "develop deep understanding by providing a structure for designing research studies, interpreting data resulting from those studies, and drawing conclusions" and include a methodology (Lester, 2005, p. 458). There are different kinds of research frameworks identified by Eisenhart (1991), a *theoretical*, a *practical*, and a *conceptual* framework.

The *theoretical* framework is closely related to theory or in other words "to think of theory as a specific kind of framework" (Lester, 2005, p. 458). For example, Vygotsky's socio-cultural theory may be described as a *theoretical* framework, because it is a theory that may be used for rephrasing research questions, analyse the data and justify the findings.

The *practical* framework does not involve a formal theory, but is instead based on "what works" in practice and accumulated knowledge from practical experiences.

The *conceptual* framework, which is developed from previous research, is based on what the researcher finds relevant for his/her research and may include several theories and knowledge from practitioners (Lester, 2005). The focus in a conceptual framework is justification rather than explanation (ibid.).

Lester (2005) identifies several problems with the theoretical and the practical framework. He lists four problems related to the theoretical framework (p. 459):

1. Theoretical frameworks force the researchers to explain their results who often tend to give 'decree' rather than evidence.
2. Data have to "travel".
3. Standards for theory-based discourses are not helpful in day-to-day practice.
4. No triangulation.

The major problems with the practical framework are that it is dependent on the local practitioners' perspective and the findings cannot be generalized due to the local conditions and constrains (Lester, 2005).

Lester (2005) argues for employing a conceptual framework with the use of the *bricoleur* metaphor. A bricoleur is a person who uses the most appropriate tools that are available in his/her arsenal to solve a problem. Lester (2005) continues that "[i]n this manner, we should appropriate whatever theories and perspectives are available in our pursuit of answers to our research questions" (p. 460). Also Cobb (2007) uses the same metaphor to argue for a conceptual framework and "suggest that rather than adhering to one particular theoretical perspective, we act as bricoleurs by adapting ideas from a range of theoretical sources" (p. 29). As a note or a critic (see also Gellert, 2010), the *bricoleur* metaphor does not really seem to be applicable to the research practice of mathematics education. A bricoleur uses any appropriate tools he/she has access to at the particular time. For example if a bricoleur forgets the hammer he/she might use a shoe or a rock to hit a nail, but a researcher in mathematics education will not use just any tool and he/she would probably use it a longer time and go and get the hammer.

Niss (2007b) has a similar view and defines, even if he argues that it is not well-defined, an *investigational framework*. He proposes that the investigational framework, which is not *a* theory, should include at least the following components: I) Research questions and a suitable perspective of the issues of interest; II) Theoretical constructs (e.g. assumptions, notions, and concepts) and these theoretical constructs should be defined "to capture essential entities of significance to the issues and questions in focus of the framework" (p. 1297); and III) Methods, which are suitable for the research questions and issues investigated, especially, in relation to the theoretical constructs (Niss, 2007b). For using an investigational framework it is important that the components are presented in the research approach either explicitly or implicitly. Despite the fact that an investigational framework is not a theory it has some resemblance with the definition of dynamic theoretical approach suggested by Bikner-Ahsbahs and Prediger (2010). Research questions (I), theoretical constructs (II), and research methods (II) may be argued to be developed and constructed dynamically in a state of flux. The *core* in Bikner-Ahsbahs and Prediger (2010) seems to match (I) and (II) in the investigational framework (Niss, 2007b). The *empirical* component may refer to the method (III) and the application area may relate to how the investigational framework will be used.

The use of single concepts or hybrids from other *less known theories*, which may be a consequence of using a conceptual framework may, according to Jablonka et al. (2013), be productive for the field of research in mathematics education, but it may also be a limitation since it is distorting the field into individual theoretical approaches. This may be captured by Campbell (2006) who cited a colleague that said "[t]heories are like toothbrushes...everyone has their own, and no one wants to use anyone else's" (p. 257). In order not to distort the research field into individual theoretical approaches I have used theoretical approaches well established in research literature in mathematics education. I have followed the arguments from Lester (2005) and used a conceptual framework in terms of an *investigational framework* (Niss, 2007b) as a guide for my research. The reason for choosing an investigational framework is that I have different papers with different aims, research questions and I use different theoretical perspectives. The research questions, discussed in the next section, were developed dynamically with choosing the investigational framework (i.e. choosing theoretical perspectives, research methods, clarifying notions).

4.2 Reasons for choice of theoretical perspectives, research questions and research methods

This section will present a discussion of what aspects of the aim have been investigated, what aspects have been left out, and why. In addition, the research questions and research methods are discussed.

The aim from section 1.2 is:

> *The aim is to present a report of the experiences that students, teachers and modelling experts have of, learning, teaching and working with mathematical modelling in and out of school settings and how they interpret the notion of mathematical modelling.*

This aim includes words that have multiple meanings. Words that need to be clarified are *mathematical modelling, experiences, modelling experts, students, teachers, in and out of school settings,* and *interpret. Experience* is in this thesis, used in a broad sense and refers to "the fact or state of having been affected by or gained knowledge through direct observation or participation" (experience, 2013) with the focus to capture the gained knowledge and/or affect. *Modelling experts* will further on relate to those that have experience from mathematical modelling in their profession and explicitly argue that mathematical modelling has a large role in their work. *Students* are restricted to learners that attend upper secondary education in Sweden and *teachers* are those who are employed to teach at the same level. *In and out of school settings* is the expression used to display a distinction between what takes place within school as a community of practice and what takes place (here restricted to) in the labour market (see Figure 7). The word *interpret* will be used "to clarify or explain the meaning of" (http://www.thefreedictionary.com/interpret). Finally, the notion of *mathematical modelling* has multiple meanings as discussed in Chapter 2. Bosch and Gascón (2006) argue for using a reference model, not necessarily derived from scholarly knowledge, while conducting research studies. The purpose of the reference model is to describe the notion or phenomena the researcher wants to investigate, in order for the researcher to objective. In this thesis I have chosen to describe mathematical modelling by Blomhøj and Højgaard Jensen's (2003, 2007) framework as displayed in Blomhøj and Hoff Kjeldsen (2006). The main reason for the choice is the focus on structural aspects of modelling, which seem to be useful to compare and contrast to issues to be explored.

To present a picture of teachers', students' and modellers' *experiences* in relation to mathematical modelling, it is preferable to present factors or aspects that may influence their experiences. It is of course not possible to take in account all factors that may have such influence, which is also the case in many effect studies exploring and rating factors that influence student achievements, since the factors are complex and multidimensional (Teddlie & Reynolds, 2000). However, one attempt to illustrate and categorise some factors that may influence their experiences is depicted in Figure 6 and is used to describe how the aim of this thesis has turned into research questions. The illustration is inspired by the didactic transposition as described by Hardy (2009).

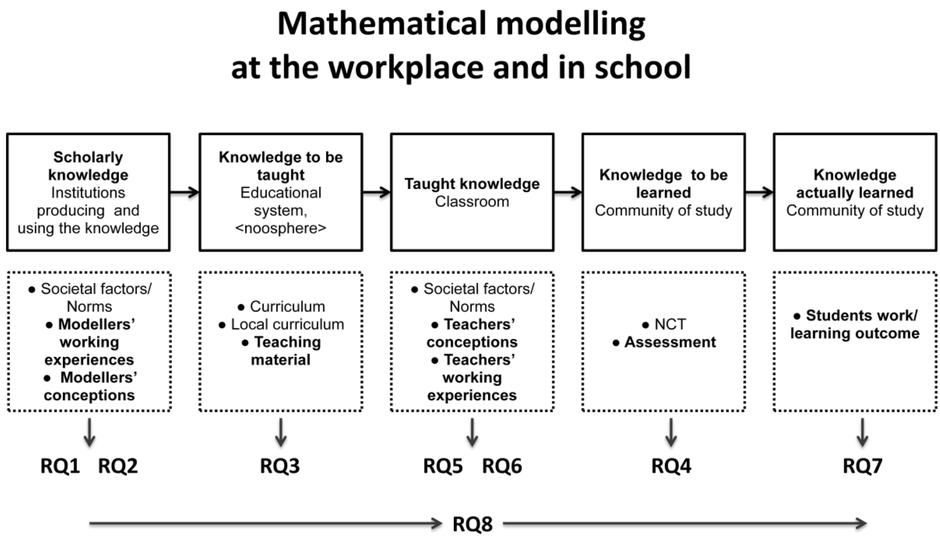

Figure 6. The didactic transposition, factors that may influence students', teachers' and modellers' experience of mathematical modelling and a description of factors that have been investigated (i.e. research questions RQ).

The top left box of Figure 6 relate to *scholarly knowledge* of modellers at the workplace and below are some factors listed that may influence modellers' experience of mathematical modelling. These factors may be workplace specific and include societal factors such as working tasks (modelling problems), to work in groups or individually etc., but also more unspoken factors such as norms and expectations from colleagues and supervisors. Other factors that are more subjective refer to the individual workers' working experiences and their conceptions of mathematical modelling.

The second box from left, *knowledge to be taught*, are texts that are developed by politicians, educators, curriculum developers etc., as guides for teaching. The curriculum, which includes the syllabus for mathematics will influence school policy about how the local curriculum is developed (i.e. a local interpretation of the national curriculum) as well as influence textbooks authors what to include in textbooks. How the textbooks treat and present mathematical modelling are factors that may impact teachers and students experiences of mathematical modelling.

What to teach, *taught knowledge*, and how to teach mathematical modelling is a matter of the school norms that appear in a particular school and is depicted in the centre of Figure 6. The norms may consist of local documents, regulating how much time the students get in the classroom, what equipment is possible to use, etc., but also ‘hidden rules’ (Jablonka, 2011) as a matter of expectations from colleagues, parents, principals, students, etc. of what to teach, learn and behave in a school situation. Teachers’ conceptions of mathematical modelling will influence what will be taught and learned in the mathematics classroom, which may give impact on students’ experiences of mathematical modelling.

What knowledge is valued, *knowledge to be learned*, in assessment and national course tests (NCT) affect teachers and students experiences of mathematical modelling.

In addition, how students work, and their learning outcomes, *knowledge actually learned*, may also influence teachers’ and students’ experiences of teaching and working with mathematical modelling, which is depicted to the right in Figure 6.

Other factors not described in Figure 6, but maybe having impact on teaching and learning about modelling, are factors that influence educational effectiveness such as the social economic situation of the students, gender, ethnicity, personality traits, subject motivation etc. (Creemers & Kyriakides, 2009).

However, the different ‘transitions’ in the didactic transposition are not totally separated from each other, but may be connected in different ways (not illustrated in Figure 6). For example, the *noosphere* determines both how the curriculum is described and is in charge of the development of national course tests. It is also possible that what is used by teachers and students in the classroom may influence national policy, for example teachers in upper secondary education have the possibilities to give comments on the national course test, which later is analysed and discussed by those who construct the test.

The aim to present a report of different actors’ experiences and interpretations of mathematical modelling is very broad and the following restrictions have therefore been made. Following the structure of the didactic transposition in Figure 6, I will in the ‘workplace institution’, *scholarly knowledge,* investigate how expert modellers work with mathematical modelling in their profession and how they interpret the notion of modelling. In the ‘school institution’ I will explore how textbooks present mathematical modelling, *knowledge to be taught*, teachers’ conceptions about mathematical modelling, *knowledge taught*, assessment approaches for modelling, *knowledge to be learned*, and how students actually work and formulate mathematical models, *knowledge actually learned*.

Students' interpretation of the notion of mathematical modelling (Frejd & Ärlebäck, 2010) has been investigated. In addition, the development of the notion of modelling in the Swedish curriculum over time (Ärlebäck, 2009c) and modelling in national course tests (Frejd, 2011a) have been investigated.

In order to examine the area of research as defined above, I will use the following research questions (RQ in Figure 6):

RQ1. What conceptions do expert modellers express about the notion of mathematical modelling?

RQ2. How do expert modellers work with mathematical modelling in their workplace?

RQ3. How is the notion of mathematical modelling presented and treated in textbooks in the first mathematical course in upper secondary school in Sweden?

RQ4. What conceptions do mathematics teachers in upper secondary school express about the notion of mathematical modelling?

RQ5. To what extent do teachers describe mathematical modelling activities as part of mathematics education?

RQ6. What modes of assessment are being used in the context of research in mathematics education to assess mathematical modelling and what is actually being assessed?

RQ7. How do students formulate mathematical models?

RQ8. What similarities and differences can be found between 'School' and 'Workplace' concerning how to work with mathematical modelling and concerning the notion of mathematical modelling?

The reasons for using these research questions will be discussed below as well as the methods used.

The connection between workplace and school needs to be strengthened according to Skolverket (2012b), and mathematical modelling can be a bridge between the two practices (discussed in section 4). As argued by Drakes (2012), expert modellers' experiences are an asset for teaching mathematical modelling. To present a report of how modellers interpret the notion of mathematical modelling may help developing teaching, in terms of what modelling is and to ease communication between school and the workplace. Presenting information about how modellers work may help teachers to teach as well as students to learn. Teachers may use the information to, for example, develop realistic modelling problems or identify teaching strategies closely related to working practice, and students may identify solution strategies that experts use such as getting information about how to validate, make assumptions etc. RQ1 and RQ2 are discussed in Paper 1.

Teaching of mathematics in Sweden does to a large extent depend on textbooks, both as a guide for teachers on what to teach and for students to work individually with exercises (Jablonka & Johansson, 2010; Skolinspektionen, 2009; Skolverket, 2003; SOU 2004:97). Therefore, it is highly relevant how the mathematical topics are presented and treated in the textbooks to support students' development of the abilities described in the seven goals in the syllabus for mathematics. How the textbook authors describe mathematical modelling in textbooks, addressed in RQ3, will be one factor that influences how students and teachers experience mathematical modelling, which is discussed in Paper 2.

Teachers' conceptions (beliefs) of mathematics and mathematics education is an essential factor for how they teach (Thompson, 1992), which will influence students' experiences of mathematics. The fifth and the six research questions (RQ4 and RQ5), discussed in Paper 3, explore teachers' experiences of teaching mathematical models and modelling in upper secondary school and what conceptions they have about the notions of mathematical models and modelling.

Assessment is a part of mathematics education and it informs students, teachers and institutions how and what is regarded as valuable knowledge to teach and learn (Niss, 1993a), which may influence their experience of mathematical modelling. Frejd (2011a) analysed test items in the national course test and concluded that there did not exist one single item that assessed all aspects of modelling (i.e. a holistic view). Using this result and following the principle by Blomhøj and Hoff Kjeldsen (2006) that "the pedagogical idea behind identifying mathematical modelling competency as a specific competency is exactly to highlight the holistic aspect of modelling" (p. 166), raise some questions related to national assessment in Sweden regarding mathematical modelling. Is it possible to assess all aspects of modelling in the NCT? If yes, how? If it is not possible to assess modelling in written tests in a holistic way, what other assessment modes are being used or suggested? These questions were used to formulate RQ6, which is discussed in Paper 4.

A part of students' (as well as teachers') experiences with mathematical modelling is how they work with modelling problems in the classroom. One of the most demanding and difficult part for students engaged in a modelling activity is to formulate a mathematical model (Hickman, 1986; Stillman 2012), which empirically has been substantiated in research (Frejd & Ärlebäck, 2011; Haines, Crouch & Davis, 2000; Houston & Neill, 2003; Kaiser 2007). Consequently, how students develop mathematical models is considered as one priority area for research in mathematics education regarding mathematical modelling (Stillman, 2012). According to Hickman (1986), "in order to teach modelling effectively we must understand the processes that underly the formulation stage of the modelling process. This stage is well recognised by professional modellers and practising teachers to be the 'bottleneck' stage of the process as a whole. The unplugging of this blockage must be a primary step towards a theory of instruction for mathematical modelling" (p. 284). The issue of unplugging the 'bottleneck' stage gave raise to RQ7, discussed in Paper 5.

The last research question (RQ8) is an extension of the five papers, in the sense that answers to the other research questions in this thesis may contribute to the research literature, by shedding some light on similarities and differences between the two practices.

There are several methods used for investigating the research questions. The first and second research questions were explored based on a qualitative study rather than a quantitative. There are several reasons for that. First of all it is difficult to identify expert modellers in the workplace, which makes it difficult to conduct a large scale survey. Second, the intention of the research is not to give any transferable description to be representative for the entire population of modellers' work practice, rather the intentions are to generalize to theory. The intention was to use a sample of expert modellers from different kinds of workplaces and in more detail investigate how they work with and conceptualise mathematical modelling. One method to capture the complexity of workplace mathematics is to use observations in a workplace together with interviews (Wedege, 2010b). For these research questions (RQ1 and RQ2), I have developed and used semi-structured interview questions that pay attention to how modellers work and their conceptions of modelling. Later the interviews were transcribed and analysed with grounded theory. Grounded theory is frequently used in research studies to analyse qualitative data in form of transcripts or field notes from interviews or observations (Bryman, 2004), in particular in exploratory studies, to develop insights that are representative of the data, rather than test already existing ones. It handles much data in a systematic, transparent and concept oriented approach (Strauss & Corbin, 1998). Drakes (2012) and Gainsburg (2003) both used grounded theory for analysing professional modellers at the workplace. The main source used in the construction of the interview questions is the set of critical questions developed by Jablonka (1996) for analysing a mathematical model. According to Jablonka, the key aspect when someone is working with mathematical modelling is to judge the quality of the mathematical model.

There are many alternatives for how to analyse textbooks, such as using a historical epistemological perspective (Jakobsson-Åhl, 2008), a sociological approach with concepts and expressions from researchers like Bernstein and Foucault (Dowling, 1996), or a broad analytic scheme with pre-defined questions (Haggarty & Pepin, 2002). However, a common approach suggested by Johansson (2005) and Robson (2002) is the method of content analysis to examine school textbooks, frequently used for analysing textbooks (Fan & Zhu, 2007; Gatabi et al, 2012; Kongelf, 2011; Lu and Bao, 2012) focusing on structural organisation of content (counting frequency of words, expressions, strategies, etc.) and it is a transparent method (Robson, 2002). Research question four (RQ 3) concerns characteristic features, *the structural organisation*, on of how modelling is presented and treated in Swedish textbooks. That is the reason I have chosen to follow Robson's (2002) guidelines for a content analysis together with Blomhøj and Højgaard Jensens' (2003) framework.

The two research questions RQ4 and RQ5, about how teacher work and their conceptions of modelling, are answered through a qualitative interview study with teachers. The reasons for developing and using a semi structured interview study and analyse the written transcripts with grounded theory are similar to the reasons explained for the research questions RQ1 and RQ2. Both studies are exploratory with an aim toward 'quality' in contrast to 'quantity' and grounded theory is useful to systematize much transcribed data to be able to develop, relate, and identify different meanings of concepts (Strauss & Corbin, 1998).

To get a broad picture of what modes of modelling assessment are used in mathematics education (RQ6), I turned to research literature. Another suggestion could have been to ask teachers about assessment modes, which also is done in Paper 3, but according to Niss (1993) the most prevailing assessment mode in mathematics education is the written test, which is not suitable for testing modelling. The reason for using research literature was therefore to identify not just written tests but also other less common approaches that might be more suitable to assess mathematical modelling. Choosing a sample that span over a long time period, which seems to have potential to be a representative sample of a large part of the literature focusing on modelling within the research community of modelling, the analysis was done inspired by grounded theory (Strauss & Corbin, 1998). Grounded theory is not just used to analyse qualitative data in form of transcripts and field notes (Bryman, 2004) but may also be applied on published articles "as a method for rigorously reviewing literature" (Wolfswinkel et al., 2013, p. 45). The rationale for adopting a grounded theory approach for analysing papers in a literature review is, according to Wolfswinkel et al. (2013), that it captures phenomena through an accurate, transparent, systematic, concept oriented approach that provides insights that are representative of the data. These insights may be used to challenge existing theoretical approaches and identify gaps in knowledge that need further research (ibid.).

How students collaborate and develop mathematical models is explored in research question seven, RQ7. Three groups of students took part in a study aiming to investigate the potential use of open *realistic Fermi problems* for introducing the notion of mathematical modelling at the upper secondary level (Ärlebäck, 2009c). These types of problems are open and discussion promoting with clear real-world connections and the problems can be understood both by individuals and groups on different levels of complexity. The reason for using the data from one group was partly based on convenience since the data already were available, and partly on the fact that the chosen group, consisting of two male students, was a known well functioning, dynamic and highly verbal duo when engaged in joint collaborative work. Working collaborative with problems requires both communication between members and individual cognition, which was the reason for using Sfards' (2008) theory of commognition for the analysis. The five methodological principles for doing commognitive research (Sfard, 2012) were followed during the analysis of the transcribed video sequence, with focus on how the students developed signifier-realization relation.

The last research question, RQ8, about identifying similarities and differences between the two practices is investigated with a template approach, as described by Robson (2002), because it is a systematic method for identifying and comparing text segments. The key categories, 'similarities' and 'differences', are derived from the research question and serves as a template as suggested by Robson. The text segments identified from the other research questions are used for empirical evidence in the comparison.

As said before, there are restrictions made and many aspects are not investigated related to my aim and research questions. Aspects related to students' experience in the classroom not investigated are for example; What modelling activities/ modelling items are presented to the students in the textbooks? What actually happens in the classroom -do the teachers do as the say they do? Do the students experience mathematical modelling in other situations than in the mathematics classroom? However, the main focus is to find some initial understanding, so choices had to be done due to time limitation. For instance, instead of actually going into the classroom, which may be very time consuming, I decided to interview teachers and hopefully receive some answers to what extent mathematical modelling activities form part of mathematics education.

4.3 Ethical considerations

In this section I will describe what ethical considerations were made.

Paper 1 investigates how modellers work and their conceptions of mathematical modelling. The modellers were invited to participate in the interview study by e-mail. In the invitation (see Appendix) I informed about my name, my position at the university and about my research interest in how modelling is used in the workplace. In addition I explained about the structure of the interviews in terms of *general questions* (e.g. academic background and working experiences), *the notion of mathematical modelling* (e.g. what does modelling mean to you?), *mathematical modelling in work/ research* (e.g. how do you work with modelling in your profession?), and *mathematical modelling in education* (e.g. How have you learned mathematical modelling and what are your opinions on modelling in school mathematics?). The invitation also included information on the ethical principles: "This project follows the Swedish Research Council´s ethical principles[17] with the requirements of information, approval, confidentiality and how the research material will be used. Confidentiality will be guaranteed and the information that you leave will not be tracked to any specific person." No names that connect to the participants have been used and the information about the workplace is sometimes, when necessary, described in fewer details than the data permitted to assure confidentiality. In addition, as a part of ethical consideration

[17] See for instance CODEX for rules and guidelines in research about humans may be retrieved from http://codex.vr.se/forskningmanniska.shtml

(and reliability) the participants were given the opportunity to give comments on a draft version of paper.

Paper 2 and Paper 4 is a textbook analysis respectively a critical literature review and they did not involve ethical considerations more than to pay attention to that the content from a specific author agreed with the references.

Paper 3 examines teachers' conceptions about mathematical modelling. One ethical problem was to get in contact with the teachers when they have answered a teacher questionnaire anonymously. The teacher questionnaire included questions about gender, years of working experience as teachers, the name of the school, if they had heard the notion of mathematical modelling before, if they could describe the meaning of the notion as well as if they were working with modelling activities and how. The purpose of the teacher questionnaire was to provide some initial information about the teachers' views of mathematical modelling and to identify the schools that were participating in the study, because one aim was to investigate differences between how modelling activities are used in different geographical regions. The teacher questionnaire did not provide with enough information, and I wanted to contact the teachers for more information. It may be seen as an ethical dilemma, on the one hand the teachers had answered the questionnaire anonymously and on the other hand I as a researcher had access to the name of their school and their working experience. That meant that I might have had the possibility to identify them. To overcome the dilemma I asked all research students from FontD who delivered the questionnaires to help me. The research students asked the participating teachers if I was allowed to contact them. All teachers agreed to be contacted and their names were given to me by the research students, and an interview investigation could take place.

Paper 5 is based on data from Ärlebäck (2009c). I was informed by him that the data could be used for our analysis. The participants had been informed in Ärlebäck (2009c) that the data could be used for his 2009-study as well as in future research, with reference to the ethical principles described above in relation to Paper 1.

Chapter 5

Summary of papers

To facilitate a discussion about the papers a brief summary of each of the five papers enclosed in this doctoral thesis are provided in this chapter.

5.1 Paper 1

[1] Frejd, P, & Bergsten, C. (Submitted). Mathematical modelling as a professional task: Implications for education.

Scholarly knowledge is in focus in Paper 1. The aim is here to give a description of how professional mathematical modellers in the workplace construct their models.

Nine persons working professionally with mathematical modelling were interviewed with the use of semi-structured interview questions which were designed, based on a set of critical questions developed by Jablonka (1996) for analysing mathematical models, to pay attention to the research aim. Subsequently, the interviews were transcribed and analysed in accordance with the principles of grounded theory (Strauss & Corbin, 1998). As this paper takes a focus on the constructors of mathematical models three phases (pre-construction, construction and post-construction) were set up to explore how they work in their professional practice.

The results are drawn from three themes generated from the data, which relate to model construction as an *empirical*, *theoretical* and *applicational modelling*.

Empirical modelling is the work of gathering, interpreting, synthesizing, and transforming data as the underlying base for identifying variables, relationships, and constrains about a phenomena used in the modelling process. The aims of the problems set up are to describe, simulate, predict, design, and construct a reality. The models will be used for several purposes, eventually (to different extents) hidden in technologies as black boxes of different kinds, such as measurement instruments, algorithms for investments, traffic routes etc. Communication and collaboration between the constructor and consumer is described as a central part in the process of clarifying, adapting and reformulating the problem. The use of technology and the quality of data play a major role in the construction, which is also the case for the evaluation/ simulation process when in- and outputs are tested, validated and compared with given data, outcomes of experiments or expert opinions. According to the constructors there are risks attached to the use of their models, for example that people will lose money, time, or inexperienced people may use models they do not understand well enough.

Theoretical modelling is the work of developing and setting up new equations based on already 'theorised' and established physical equations, followed by the activation of computer resources for computational purposes to solve the new equations with aim to get information about the 'theorised' equations. The aims of the problems are similar to the empirical activity, i.e. to describe, simulate, predict, design, and construct a reality. Examples are to predict the climate change, including descriptions of the current situation and simulating the future, and design and construct a new material with some specific properties. However, an overall aim is to get a better understanding of physical phenomena (i.e. develop a theory). The models will be realized in computer programs. Briefly, the modellers work is to translate the physical equation to a computer model, use the computer model to get information about the physical equations by interpreting and evaluating the result of the computer model, and finally evaluate the validity of the computer model. Communication between different experts are also emphasised as a central role in the theoretical activity. The modellers describe that there are risks with using their models without having an understanding of the weaknesses of the models.

Applicational modelling is the construction of models by identifying situations where some mathematics or some established mathematical models can be applied. This activity is a part of all modellers' work. One of the modellers explicitly emphasises the application of mathematics as a way to work, which means that he identifies applications to be useful in different contexts, based on experience. For example the same type of mathematics may be used in chemical concentrations as well as in population distribution of oak trees. To apply already developed models is also a part of all the constructors' work, either due to limitation of time or because there exist models that take care of standardized problems. All constructors also emphasise that they use computers and software that include some types of established models. The use of the already defined models stems from the working experience or is as an outcome of communication with other experts.

Implications for education are that the *applicational modelling* to some extent is a part of textbook descriptions of modelling as well as assessment of modelling in national course tests regarding the use of, and sometimes adaption of already defined models (Frejd, 2011, 2013). However, *empirical modelling* and *theoretical* modelling present a challenge or an "inaccessible phenomena" (Gainsburg, 2003, p. 263) for teachers and students to 'imitate' in a mathematics classroom, because the professional modellers' work is complex for upper secondary education. It involves years of modelling experience, knowledge of advanced mathematics, specialised knowledge of other fields, the use of technology and programming, and collaboration with other experts. This empirical study suggests, in line with other research (e.g. Hoyles et al, 2002; SIAM, 2012), that teamwork, mathematical communication and the use of technology should be widely employed in the mathematics classroom in modelling work.

5.2 Paper 2

[2] Frejd, P. (2013). An analysis of mathematical modelling in Swedish textbooks in upper secondary school. *Nordic Studies in Mathematics Education, 18*(3), 59-95.

Paper 2 addresses the issue of knowledge to be taught, or more specifically, how the notion of mathematical modelling is presented and treated in mathematical textbooks. There seem to be few research studies that systematically have analysed how textbooks treat mathematical modelling in upper secondary school. The aim of this paper is to investigate how Swedish mathematical textbooks for upper secondary school interpret and explain the notion of mathematical modelling. By following Robson's (2002) guidelines for a content analysis, 14 mathematical textbooks recently published for the new Swedish National Curriculum 2011 in five textbook series (*Exponent*, *Matematik 1a*, *Matematik 5000*, *Matematik M*, and *Matematik Origo*) were analysed with use of an analytic coding scheme developed to include three parts. The *first* part is to identify texts that treat modelling in the textbooks and how explicitly the textbooks use the words modelling and models. The *second* part of the analytic scheme examines how textbook instructions (instructional texts and worked examples) describe different sub-processes of modelling related to the framework by Blomhøj and Højgaard Jensen (2003), as well as examines the aim of the instruction. The *third* part concerns how the modelling tasks are presented.

The reliability of the coding was assessed by letting an independent researcher do the same analysis of the second and the third part of the analytic scheme. The first part (counting the words and identifying where in the text they are placed) was controlled by myself by doing the analysis a second time at a later occasion. Overall the result produced by the application of the analytic scheme can be described as consistent between coders.

The outcomes from the *first* part of the analysis show that the word *model* (model and mathematical model) is mainly found in items for students to solve and is frequently appearing in three of the text book series, but is less frequent or absent in the other two textbook series. The word *modelling* (modelling and modelling ability) is only stated in the introduction in the *Exponent* series and it occurs a few times in the *Matematik 5000* series.

The *second* part of the analysis shows that three of the textbook series only use the term model as *implicit* descriptions in instructional texts or worked examples, which means that it is up to the students to interpret the notion. The other two textbook series offer *explicit* descriptions of models as "formula or an equation describing reality in a simplified way" (Origo, p. 74 in 1c; my translation) or as "a simplification of reality "(5000, p. 224 in 1b; my translation). The notion of modelling is not discussed at all in two of the textbooks series. In the *Exponent* series modelling is described *implicitly* in some items as an ability and in the *Origo* series the word modelling is not used (*implicit* description), but the notion of modelling is described *explicitly* as a problem solving activity with three phases. The *Matematik 5000* series describes modelling *explicitly* as a group activity to develop mathematical models, including making assumptions, estimations, calculations and evaluation. Relating the instructional texts or worked example to Blomhøj and Højgaard Jensen's (2003) sub-processes, it is found that focus is towards *translation into a mathematical representation, using mathematics to solve the corresponding mathematical problem* and *making an interpretation of the results in the initial domain of inquiry.* Only the *Matematik 5000* series is considering *selecting the relevant objects, relations and idealisations* and *evaluating the validity of the model.* No series is discussing *formulating a task in the domain of inquiry.*

In the *third* part of the analysis it was found that two textbook series have items with an aim at modelling, described as Fermi problems and are categorised as estimation models. Four of the textbook series have quite few items dealing with models and these mathematical models are mainly *exponential* and/or *linear* functions (an item may include both an exponential and a linear function). The Exponent series on the other hand uses a variety of mathematical models.

It is concluded that mathematical modelling is not treated as a central notion in the analysed textbooks. The variety of descriptions of modelling in the different textbook series indicate that all students do not get the same opportunities to develop modelling ability as stated in the subject syllabus mathematics (2012a). This implies that the teachers need to complement their teaching of modelling with other teaching material.

5.3 Paper 3

[3] Frejd, P. (2012). Teachers' conceptions of mathematical modelling at upper secondary school in Sweden. *Journal of Mathematical Modelling and Applications, 1*(5), 17-40.

Paper 3 presents a qualitative interview investigation (telephone interviews) of 18 upper secondary school teachers in order to search for a deeper and more extended insight into teachers' conceptions about mathematical modelling and their experiences of working with modelling activities. The investigation addresses the issue of knowledge actually taught.

The use of the notion of *conception* in Paper 3 refers to Lloyd and Wilson's (1998) definition "[w]e use the word conceptions to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, preferences, and views" (p. 249). The reason for this decision was mainly that this study is an exploratory investigation with an aim to find some first indication of teachers' conceptions in a *broad* sense about mathematical modeling and how teachers work with modeling in their classrooms.

An interview guide was developed based on suggestions from Kvale (1997) to make the transition from research question to interview question explicit. The two research questions were divided into seven auxiliary questions, which were used to develop the interview questions. In addition, test items from Haines et al. (2000) were attached to the interview questions to be discussed during the interviews. The interview questions were sent in advance to the teachers after a pilot interview had been carried out. The interviews were transcribed (189 transcribed pages) and coded with a coding strategy inspired from grounded theory.

The results in Paper 3 indicate that teachers have limited experience of the notion of mathematical modelling in mathematics education. Only half of the teachers had heard the notion before taking part in study which included these interviews and no descriptions or definitions were found in any of the teachers' local curricula. The teachers' conceptions relate mainly to designing a mathematical model based on a situation (i.e. simplify and describe something with mathematics). Only five teachers intentionally used modelling activities in mathematics class, and the overall lack of assessment in relation to modelling items in teachers' own tests may indicate that mathematical modelling is not a frequently occurring activity in many teachers' mathematics classrooms. However, mathematical modelling was described as a part of physics or chemistry education. 15 out of 17 teachers explicitly expressed that they use mathematical modelling more in physics or chemistry than in mathematics. From analyzing teachers' expressions about the test items in the student questionnaire, one may conclude that the teachers' conceptions about mathematical modelling in relation to mathematics (as measured by the test items used in the 'student questionnaire') emphasise mainly the mathematical aspects of mathematical modelling and leave other aspects out or use them less frequently. The fact that 10 out of 18 teachers expressed that you should not work with all test items in the 'student questionnaire' in the mathematics classroom, because they considered it not to be mathematics, may be an obstacle for implanting modelling into more everyday teaching in mathematics.

5.4 Paper 4

[4] Frejd, P. (2013). Modes of modelling assessment - A literature review. *Educational Studies in Mathematics*, *84*(3), 413-438.

The knowledge to be learned about modelling is investigated in Paper 4, with a focus on assessment. The aim of this paper is to review a selection of literature focusing on mathematical modelling in mathematics education in order to analyse approaches used or suggested to assess students' mathematical modelling competence. Is it possible to assess a holistic approach to modelling by using written tests? If yes, how? If no, what other assessment modes are being used or suggested?

Guided by Bryman's (2004) central features of any qualitative literature review, I first identified relevant and adequate research literature related to the aim, and secondly I related, organised, and connected the literature (constructing intertextual coherence) as well as pointed at research literature that seems to be incomplete (problematizing the situation).

The sample reviewed was the ICMI14 Study, all 15 ICTMA proceedings up to date, all proceedings from the 'modelling working group' at CERME, and the special issues of ZDM 2006 focusing on mathematical modelling (issues 38(2) and 38(3)). The main reason for choosing this sample is that it includes a variety of cited resources (a state of art book, proceedings, discussions and papers from thematic working groups, and a highly cited journal with two thematic issues about modelling) that span over a long time period, which seem to have potential to be a representative sample of a large part of the literature focusing on modelling within the research community of modelling in mathematics education.

To identify the relevant articles in the literature sample, the method used is based on key words in the titles (assessment, assessing, evaluating, etc.), an examination of all abstracts in book sections relating to assessment, and a search for the word assessment in the index. The analyses of the articles are based on grounded theory, because it captures phenomena through a systematic, accurate, transparent concept oriented approach that provides insights that are representative of the data (Wolfswinkel et al., 2013).

The results describe several different modes of modelling assessment, i.e. *written tests*, *projects*, *hands-on tests*, *portfolio* and *contests*.

Written tests like Haines, Crouch and Davis' (2000) multiple-choice test assessing six aspects of a modelling process (with similarities to the sub-processes described in Blomhøj and Højgaard Jensen, 2003) has been used in a variety of settings with different aims such as to investigate the levels of students' modelling competencies (Frejd & Ärlebäck, 2011). However, some critique has been given to these types of written tests, such as the lack of ICT and collaborative work (which are other important aspects of modelling) but the main critical point concerns their atomistic view of modelling. An atomistic view seems to be refelcted in most of the written tests discussed in the sample of literature. However, written tests that include hands-on tasks (Vos, 2007) seem to have potentials for including a more holistic approach of modelling if the tasks are chosen adequately.

Projects are described as an ideal method for assessing modelling (Berry & Le Masurier, 1984; Niss, 1993b). For example Antonius (2007) argues that ”the different competences seem to be more visible in project examination” (p. 414) than in traditional written examination because it is more extensive (includes both written reports and oral examinations). Oral examinations for projects seem to have at least two objectives. The first is to assess students’ ability to communicate and present the result of a project (Battye & Challis, 1995; Clatworthy, 1989; Edwards & Morton, 1987; Hamson, 1987). The second objective is to certify that the students can argue for their solutions in order to show that they have done the projects ‘themselves’ and that they have acquired insights in relation to the problem (Antonius, 2007; Clatworthy, 1989; Hamson, 1987; Niss, 1993b). However, a problematic issue regarding projects as an assessment mode is the reliability of the assessment (e.g. Antonius, 2007; Hall, 1984; Houston & Breedon, 1998; Niss, 1993b)

Students’ portfolio is identified and discussed in only two papers in this sample (Dunne & Galbraith, 2003; Francis & Hobbs, 1991). A students’ portfolio seems to be a collection of students’ work completed in the classroom or at home and other material that exhibit the student’s progress, efforts and performances and can be used both as formative and summative assessment (Francis & Hobbs, 1991). According to Simon and Forgette-Giroux (2000, p. 83) ”portfolio assessment has shown the greatest promise in terms of its use in evaluating higher-order, cross-curricular competencies” and thus seems to be of interest for assessing modelling.

Contests refer to upper secondary tests A-lympiad (Netherlands), HiMCM (USA), and CAMK (China), and to undergraduate tests CUMCM (China) and MCM (USA). Teams of students get one day (Haan, 2003) or 72 hours (Jiang, Xie & Ye, 2007) to develop a complete solution to a ‘realistic’ problem and the students are allowed to use computers or textbooks (Asgaard Andersen, 1998). The assessing process of the contests is to rank students’ solutions (from the best to the worst) based on a number of general and problem specific criteria provided by the test committee. However, these criteria are not included in any of the other reviewed papers dealing with contest and the assessing process is therefore not very transparent.

5.5 Paper 5

[5] Ärlebäck, J. B., & Frejd, P. (Submitted). The bottleneck problem in modelling revisited.

The final paper in this thesis deals with the last part of the didactic transposition, the knowledge students actually have learnt. The aim explored in Paper 5 is how students who work collaboratively in small groups construct and develop the mathematical model they need to solve a modelling problem.

The data used comes from two upper secondary students that took part in a study aiming to investigate the potential use of open realistic Fermi problems for introducing the notion of mathematical modelling at the upper secondary level (Ärlebäck, 2009). The problem that students were presented and engaged in was about snow clearance of a soccer field, a problem motivated in connection to the opening of the 2010 soccer game in Sweden, when a local soccer club had to ask their supporters for help to clear away all the snow from the soccer field in order for the opening game to take place. The work of two students was videotaped and transcribed. In addition, the students wrote down their solution explaining their assumptions, estimates and reasoning as part of the task. The analysis drew on both, the transcribed video recording and the written solution.

We used the commognitive perspective (Sfard, 2008), guided by the five methodological principles by Sfard (2012), to analyse the transcribed data, with a focus on the vocabulary (signifiers) used and how the relationships between different signifiers evolves as these enfold and connect (successively constituting a realization trees) during model formulation stage. One of the methodological principles is that of *alternative perspectives*, that is that we as observers alter between being an insider and an outsider to the discourse studied. Taking an insider perspective in the analysis refers to be well acquainted with the contextual situation and to consider the students' expectations, norms and values. An outsider perspective focuses on what is visible in the conversation when the context is disregarded.

The result of the analysis of the transcription and the written solution indicates that the two students applied pre-defined models that silently are agreed upon and are a part of their discourse, by *recognizing* the situation and adapting the model by *negotiating* the values and justifications in order to formulate a mathematical model. It is difficult to see the process of recognition and negotiation from just a written solution. For example, in the transcribed data the two students express doubts about the size of the soccer field, but from the written solution the size of the field seem to be well defined. In addition the transcription includes negotiations about the relation between the density of snow and water which can not be found in the solution. Instead, in the written solution it appears to be a well established rule that the density of water is four times as much as the density of snow. When the two students are involved in negotiations to realize the signifiers, they mainly use several different colloquial discourses and often with an explicit reference to their personal experiences. However, also literate discourses are visible in the data arising from mathematics, physics and chemistry.

The results of what type of signifiers the students used, in what order they were evoked, whether they were negotiated and how those signifiers have been realized are presented through what Sfard (2008) calls *realization trees*. Overall, the discerned signifiers in this study are manifested by a big variety of numbers and issues negotiated. The links between the signifiers constructed during the modelling activity, the realization tree, is a result of the interplay between what the two students brought with them as previous knowledge and experience, the communication between the two, and, the social setting in which the modelling activity took place.

Chapter 6

Discussion and conclusions

This final chapter discusses the main results presented in the five papers with a special attention to the specific research questions addressed in this thesis. In addition, the results in relation to other research results are discussed as well as limitations and possible extensions of the five papers. The chapter will end with a conclusion of the findings and some possible implications of my research.

6.1 Discussion

The main results from Paper 1 – 5 are discussed below.

6.1.1 Paper 1

Research question one (RQ1), *what conceptions do expert modellers express about the notion of mathematical modelling?*, and research question two (RQ2), *how do expert modellers work with mathematical modelling in their workplace?* are explored in Paper 1.

In Table 5 the modellers' answers to the interview question, what does mathematical modelling mean to you, are summarised.

Table 5. Modellers' expressions about the notion of mathematical modelling.

	The modeller's answer to the question *What does mathematical modelling mean to you?*
1	To build a mathematical model one must understand the dynamics of reality and then try to simplify it and catch the most characteristic elements; here a mathematical model is always an approximation, one tries to find the most critical components
2	To mathematically describe a reality; assumptions and simplifications; are just models
3	A mathematical abstraction of some kind of reality, also a computer program is a model, no clear separation between model construction and solving method
4	Describe some kind of system by way of mathematics, can describe only parts of it, adjusting a number of parameters, what is missing

5	One must have the whole problem clear, not just the mathematical part, what variables and quantitative data there are, validation. You don't solve problems by mathematical models, but they can be used to analyse problems
6	To translate physics to mathematics (e.g. the pendulum movement into differential equations)
7	In real world modelling, by recreating reality mathematically you construct a relatively good description of reality that can be used for predictions, extrapolations and also for analysing the present state. Numerical mathematical modelling means that by physical and mathematical models one should construct different kinds of simulation tools and to analyse problems in industry
8	Tools to describe something, create images for observing and better understanding the world
9	Break down observations and data to a model for how reality works, in mathematical terms and notions; solving equations with approximations; theoretical physics builds on established equations;

From Table 5 it is clear that most of the modellers responded, to the meaning of modelling, in terms of a mathematical description, or abstraction, of reality. The modellers are dealing with different kinds of "reality", but that by necessity there are constraints and limitations of the models were generally pointed out, and especially in the financial sector it was emphasised that it is "just models" and in this case of "behaviour" and not of "nature". Two of the three themes identified by Drakes (2012) may also be found in this study. The first theme, modelling as the activity to set up a model to be used as a "description of a real life situation using a mathematical framework" (p. 39) or "the model would be a simplified or approximate version of the physical system" (p. 39) may be found in Table 3. For example number 2 'to mathematically describe a reality' or number 4 'describe some kind of system by way of mathematics'. The second theme by Drakes (2012) was modelling as a process which included setting up the model, but also "solving, analysing and verifying the model are ... parts of the definition... [as well as] refining the model for more accurate results or using the model for prediction" (p. 40). This theme is also addressed in Table 3. For example number 5 'one must have the whole problem clear, not just the mathematical part, what variables and quantitative data there are, validation' include aspects related to validation. However, the third theme that mathematical modelling "does not need a definition" (Drakes, 2012, p. 40), because it is just the same as doing problem solving is not explicitly expressed in Table 5. A comparison of these descriptions of meanings of modelling with Blomhøj and Højgaard Jensen's (2003) framework indicates similarities like that modelling may be seen as a process including sub-processes like validation etc., but none of the interviewed professional modellers describes modelling as a cyclic process.

Research question two (RQ2), how mathematical modellers as constructors work when they develop models, is answered through a description of three different types of activities. In the *empirical modelling* the main focus lies within the data and the quality of data. The data frame the problem formulation and is used for the identification of parameters, variables, constants, process etc., making up the base for the construction of the models. Validation is also data dependent and the results are seen as acceptable solutions rather than correct solutions. *Theoretical modelling* is a process of re-formulating already established physical equations to computer based models, using the computer models to get information about the physical equations by interpreting and evaluating the result of the computer models, and finally evaluating the validity of the computer models. The *applicational* activity, a part of all constructors' work, consists mainly of the use of working experience and the application of mathematics or already defined models to particular situations. Model construction is usually team work, in particular for larger projects, and the use of technology is a very central part of all three activities. Communication between different actors (clients, operators and other experts) are by all constructors pointed at as a vital part of the work, as stated by Finance (insurance): "communication, getting feedback and discuss with others is an *incredible* qualification that one needs working with applied mathematics". The communication with clients often involved discussions about the usefulness of the models, while communication with other experts often addressed the effectiveness of the models. In the post constructing phase all constructors pointed out that there are always risks involved in the use of their models and that all actors should be aware that it is "just" models and critically reflect upon that fact.

The three ellipses, theory, experience or data, in the illustration in Figure 3 (Chapter 2), indicate the epistemological base for the sub-processes (Blomhøj & Hoff Kjeldsen, 2006), correlating to some extent to the findings in this paper. Empirical modelling sets the data in focus, to improve theory is one of the goals of theoretical modelling, and the modeller needs working experience to apply either mathematics or already defined models in order to solve the problem (applicational modelling). The sub-process in Blomhøj and Højgaard Jensen's (2003) framework may be found in the empirical activity, but the sub-process in the framework are less visible in the theoretical activity and quite invisible in the applicational activity. Communication, collaboration and technology are emphasised by the professional modellers but is not an explicit part of the reference framework. To summarise, the scholarly knowledge of mathematical modelling, as described in this interview study, is quite different from the reference model (Blomhøj & Højgaard Jensen, 2003).

Several of the outcomes from this study, for example that communication, collaboration, and division of labour play an important role for modelling work, reflect the findings reported by Gainsburg (2003) and Drakes (2012). The empirical based schemes of the three activities depicted in Paper 1 attempt to differentiate the communication and collaboration between the actors involved in the process. The observed aspects of mathematizing during the modelling work is also found by Gainsburg (2003) and to some extent by Drakes (2012), both of which, however, focus mainly on cognitive aspects such as on challenges, skills and understandings. These studies also concur with the present study in the identification of the key roles of communication, collaboration, and division of labour for modelling work.

The term empirical modelling has been used for one of the identified activities, which is quite similar to how it is commonly used. For example:

> Empirical modelling refers to any kind of (computer) modelling based on empirical observations rather than on mathematically describable relationships of the system modelled (empirical modelling, 2013)

The description above is partly similar to the employment of the term empirical modelling in Paper 1, though space has been left also for the use of pre-defined mathematics or models (applicational modelling) during the modelling process. The activity of empirical modelling as identified in the professional practice seems most similar (but much more complex) to mathematical modelling as set out in some curriculum.

Theoretical modelling as described in Paper 1 has similarities to what is called computational modelling, which is employed in weather forecasts, earthquakes simulations, molecular protein folding, etc. (computational model, 2013). The following definition illustrate the use of computers to simulate and solve complex systems that do not have simple analytical solutions available, which is similar to theoretical modelling:

> Computational model[l]ing is the use of mathematics, physics and computer science to study the behavior of complex systems by computer simulation. A computational model contains numerous variables that characterize the system being studied. Simulation is done by adjusting these variables and observing how the changes affect the outcomes predicted by the model.[18]

The words modelling and application are, since the call by Pollak (1969), often seen together in the expression *modelling and application*, as for example in the international conference with the specific focus on the teaching and learning of mathematical *modelling and applications* (ICTMA). In section 2.4 the notions were discussed and examples to differentiate the notions were given based on Niss et al. (2007). However, the activity from the data in this study identified as applicational modelling illustrates that they are difficult to separate, and that 'applications' includes not only some specific mathematical apparatus fitting

[18] Retrieved from http://www.nibib.nih.gov/science-education/science-topics/computational-modeling

different but similarly structured problems, but also the application of already developed models.

6.1.2 Paper 2

Paper 2 presents research question three (RQ3): *How is the notion of mathematical modelling presented and treated in textbooks in the first mathematical course in upper secondary school in Sweden?*

The conclusion in Paper 2 is that mathematical modelling is not treated as a central notion in the analysed textbooks, even though modelling is one of the seven main abilities to be taught in the national curriculum (Skolverket, 2012). The descriptions of both mathematical models and modelling vary between the analysed textbooks both in terms of how frequently the words models and modelling have been used and how the notions have been described. For example the description of the notion of modelling ranges from more *explicit* descriptions such as a cyclic problem solving method (Szabo et al., 2011) and an activity to solve Fermi problems (Alfredsson et al., 2011), to more *implicit* descriptions in form of tasks for the students to solve (Gennow et al., 2011) as well as tasks that include the word model without further explanations (Viklund et al., 2011; Sjunnesson et al., 2011). The aim of the analysed models and modelling items is towards intra-mathematical aspects of Blomhøj and Højgaard Jensen's (2003) sub-processes except for a few Fermi problems. That result seems to be consistent with the results in Frejd (2011a) where national course test items also focused mainly on intra-mathematical aspects. There seems to be a large potential for Swedish textbook authors to further develop the presentation of the notion of mathematical modelling and mathematical models. The five textbook series, assumed to be representative of the textbooks used in Swedish mathematics education at upper secondary level, do not strongly, and to very different extent, support the current curriculum in terms of modelling ability and the consequences will be that students who want to meet the standards in the curriculum to develop a holistic modelling ability need to get support from a teacher with complementing material.

However, there may be many reasons why the descriptions of modelling found in the textbooks varied. Some suggestions are that the modelling ability in the curriculum is not interpreted as a holistic ability by the textbook authors, that the authors emphasise other aspects of mathematics as more important, or that the authors did not have enough time to reflect about the new curriculum and relied on older textbooks. A reason may also be found in the fact that the textbooks are commercially produced, where the most important drive might be economic interest to gain a large proportion of the market share rather than pedagogical influences (Cháves, 2003).

It may be argued that qualitative teaching may compensate for possible inadequacy of mathematical textbooks. However, according to Schoenfeld (1988) there is no evidence that it does, which brings forward the question whether any textbook series more adequately than another is presenting models and modelling. Some evidence indicate that students are left to themselves to work in the textbooks in Swedish upper secondary classroom (see Jablonka & Johansson, 2010), and if that is the case, the content of modelling should be presented explicitly (Niss et al., 2007). Only two of the textbook series present *explicit* descriptions of modelling (Szabo et al., 2011; Alfredsson et al., 2011). Nevertheless, modelling is still only presented in a marginalized way, which means that the textbooks only do not enough support students to develop a holistic modeling ability as described by Blomhøj and Højgaard Jensen (2003). In order for a modelling ability to be developed, modelling needs to be explicitly taught with support from a teacher (Niss et al., 2007).

This study also contributes to the ongoing research from a methodological point of view, with an emerging framework for analysing textbooks with focus on modelling. The analytic scheme provided information about similarities and discrepancies of how models and modelling are described in different textbooks, but at the same time it also revealed that in terms of analysing items for students to solve, the method needs to be further developed.

A comparison of the knowledge to be taught, as found in Paper 2, with the reference model (Blomhøj & Højgaard Jensen, 2003), indicates that textbooks do not consider modelling as a cyclic process, except for Origo. Only the Matematik 5000 series includes the sub-processes *selecting the relevant objects, relations and idealisations* and *evaluating the validity of the model* and no series is discussing *formulating a task in the domain of inquiry.* The epistemological base for the sub-processes in terms of theory, experience or data (Blomhøj & Hoff Kjeldsen, 2006) is not in focus in the analysed textbooks, except maybe for students' own experience in related to Fermi problems.

6.1.3 Paper 3

Paper 3 examines research questions four (RQ4), *What conceptions do mathematics teachers in upper secondary school express about the notion of mathematical modelling?,* and five (RQ5), *To what extent do teachers describe mathematical modelling activities as part of mathematics education?*

The 18 teachers in this study seemed to have limited knowledge about the notion of mathematical modelling in mathematics education. Only 50% of the teachers had heard about the notion before taking part in the study which included these interviews. Additionally, there were no descriptions or definitions found in any of the teachers' local curricula. The teachers' conceptions in this study mainly related the notion of mathematical modelling to designing a mathematical model based on a situation (i.e. to simplify and describe something with mathematics). However, a few teachers also brought up problem solving and validation of a model in relation to the meaning of mathematical modelling. Compared to the professional modellers in Paper 1, the teachers give fewer details and are less confident, but both groups of teachers and modellers express two of the themes identified by Drakes (2012), i.e. modelling as the formulation of models and modelling as a holistic process. Maybe there is a connection with teachers' conceptions about the notion of mathematical modelling, as designing or formulating models, and what is assessed in national course tests, because *to set up a mathematically formulated statement (a mathematical model) describing the problem addressed* is a phase frequently assessed in the national course tests (Frejd, 2011a). In addition, textbooks frequently include the phase *translation into a mathematical representation* in instructions (see Paper 2), which may influence the teachers' conception of the meaning of modelling. The following statement from the curriculum was given to the teachers in advance: "The school in its teaching of mathematics should aim to ensure that pupils:… develop their ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models" (Skolverket, 2001, pp. 60-61). This statement may also have influenced the teachers' conception about modelling as a "design" activity. However, the other aspects mentioned in the statement, to fine-tune mathematical models and critically assess the conditions, opportunities and limitations of different mathematical models were expressed only in rare cases. As these aspects were not brought up spontaneously by the teachers and are essential aspects for teaching and learning modelling, they might be more emphasized in students' textbooks.

A predominant conception expressed by the teachers in this study is the emphasis of mathematical modelling as a part of physics or chemistry. Almost all teachers (15 out of 17 teachers[19]) explicitly explained that mathematical modelling is more used in physics or in chemistry than in mathematics. A follow up study investigating teachers with other subject combinations, like mathematics and English, and examining their conceptions about mathematical modelling in relation to mathematics and other subjects, might provide interesting information on this issue.

That only five teachers used modelling activities intentionally in their mathematics class might suggest that mathematical modelling is not a frequently occurring activity in many teachers' mathematics classrooms. This is also indicated by the general lack of assessment of modelling items in teachers' own tests. The teachers described their experiences in working with mathematical modelling in relation to projects, laboratory activities in mathematics and regular classroom activities. Three of the teachers expressed that they were using modelling examples from the book during a 'project' in mathematics course D[20]. One of the examples they described is related to the storage of petrol in the desert, an item used in an investigation by Ärlebäck (2009b). The teachers gave other modelling examples related to laboratory activities in mathematics and in regular

[19] One teacher was not asked, because the teacher expressed that you do not work with modelling in upper secondary school it belongs to further education at University.

[20] Mathematic course D is found in former Swedish upper secondary curriculum formulated in the year 2000, see, http://www3.skolverket.se/ki/eng/nv_eng.pdf

classrooms activities. Some example originated from physics, such as density in a linear model [teacher C], and working with a pendulum and regression [teacher D]. Others provided examples related to percentage like car price change [teacher M], some related to rental cars or buying mobile phones [teacher O]. One person gave examples of Fermi problems that they work with, like how much sand there is on a (specific named local) beach and how much water runs under a bridge over a (specific named local) river [teacher B]. Other ideas of modelling activities were why there are no giants [teacher F] and how many knots it is possible to tie on a rope [teacher K].

Mathematical laboratory activities were often used by four of the teachers, sometimes used by five of the teachers, seldom used by eight of the teachers and never used by one teacher. The main focus of the laboratory activities was related to specific parts of mathematics like geometry, functions, probability and statistics.

Overall, the integration of mathematical modelling, the knowledge to be taught, into everyday teaching in mathematics education was not the prioritized by the teachers. From a holistic view of modelling in line with Blomhøj and Højgaard Jensen's (2003) definition of modelling competence, this study concluded that the teachers' conceptions (as measured by the test items used in the "student questionnaire") emphasised mainly the mathematical aspects of mathematical modelling and left other aspects out, or use them less frequently. The teachers' arguments that some of the test items did not include any mathematics, may be an obstacle for implementing more modelling activities in mathematics education in upper secondary school.

6.1.4 Paper 4

Paper 4 focuses on research question six (RQ6): *What modes of assessment are being used in the context of research in mathematics education to assess mathematical modelling and what is actually being assessed?*

The following modes for modelling assessment are identified in this study: *written tests*, *projects*, *hands-on tests*, *portfolio* and *contests*. The *written tests* as described in the reviewed papers draw on an atomistic view of assessment focusing more on the product than on the whole process (i.e. parts of the sub-process discussed in the reference model by Blomhøj and Højgaard Jensen, 2003), whereas *projects* are described to assess a more holistic modelling competence. Arguments are put forward that the use of projects is an ideal method for assessment, but obstacles regarding reliability of assessing projects are identified. A method described as reliable for assessing students' modelling competence is used in *contests*, which is for referees to rank students' solutions based on criteria. *Hands-on tests* and *portfolio* assessment modes seem to have potential to be developed as valid and reliable assessment modes for modelling, but there are in this sample too few research studies focusing on these modes to provide evidence for any further claims.

No general answers were found in the literature review to the question if it is possible to construct justifiable holistic approaches to assess modelling in written national tests. The result in Frejd (2011a) showing a lack of a holistic assessment approach in national course tests was also found in other countries (Naylor, 1991; Stillman, 1998; Vos, 2013). On the other hand, since the written tests as described in the reviewed papers draw on an atomistic view of assessment, one may discuss whether this process also can be assessed atomistically (i.e. focusing on how students cope with a part of the process, a sub-process) in a written test. However, the phases in the modelling process described by Blomhøj and Højgaard Jensen (2003) are not really step-by-step but related. That is perhaps why it is also problematic to assess one single phase, except perhaps the validation of results of a given model.

The assessment modes found in this study are also used in mathematics education to assess other mathematical abilities. For example, Watt (2005) described, based on syllabi and curriculum literature, six alternative assessment methods (*oral tasks, practical tasks, teacher observation, student journals, peer and self- assessment*, and *parental assessment*) which have similarities to the modes found in this study, except from parental assessment.

For the purpose of assessing a holistic approach, in light of Blomhøj and Højgaard Jensen's (2003) sub-process, the use of projects is suggested (e.g. Niss, 1993b). Two advantages of projects pointed at are that the time constraints are less restricted than in ordinary written tests, hands-on tests or contests and that the students have more possibilities in projects to show their modelling ability (Antonius, 2007). A complement to the national course tests could be to use projects in order to assess a holistic modelling competence in the Swedish education, either as summative (Antonius, 2007) or formative (Wake, 2010) assessment. However, two issues need to be discussed if the use of projects is going to have a more prominent role in mathematics education regarding modelling, i.e. how projects are going to be presented and how they are going to be assessed.

Suggestions of possible ways to present a modelling project are several. "It may be a written report of some kind - a textbook or a popular book, an article for a scientific journal, a newspaper or a magazine. It may also less frequently be (the design of) an exhibition, a film, a lecture, a photo slide show, a radio or a TV program" (Niss, 1993b, p. 47). These suggestions were not in focus in the reviewed papers, where most often the students were supposed to hand in an 'ordinary' written report, which opens up for more research concerning Niss' (1993b) suggestions above. For example, one suggestion may be to arrange discussions in relation to workplace 'meetings' as done by Edwards and Morton (1987) that include communications between those who develop the models (constructors)[21], those who use models (operators) and those who listen and/or read about expert opinions about some models in order to establish an own

[21] The words in brackets in the sentence are my interpretation of Edwards and Morton's (1987) intentions in terms of Skovsmose (2005).

opinion and act (consumers). Communication between constructors, operators and consumers is an essential part of mathematics education, which prepares students for a critical citizenship (Skovsmose, 2005).

The most intriguing problem of projects is to develop a common ground for how to assess modelling projects and to agree on what assessment criteria are to be used. According to Jablonka (1997), the most crucial aspect to assess in students' work with modelling is to judge the quality of a mathematical model. The critical questions about evaluation of efficiency ("To what extent does the model fulfill its main goal?") and assessment of usefulness ("What is the contribution to the solution of the main problem and how can the goals and consequences be evaluated?") suggested by Jablonka (1996) could be one way to deal with assessment criteria.

To summarise, the knowledge to be learned related to Blomhøj and Højgaard Jensen's (2003) framework, will be seen to some extent as a consequence of the assessment mode used as well as regarding the explicitness of meaning and the goal of modelling and its relation to assessment criteria.

6.1.5 Paper 5

Paper 5 is examining research question seven (RQ7), *How do students formulate mathematical models?*, by analysing *how the realization trees evolve when the students are formulating mathematical models working with the snow clearance problem and what signifier/realizations can be discerned during this process.*

The analysis illustrated that the students formulated mathematical models based on using already known mathematical models. These pre-defined models used were not explicitly mentioned or discussed by the two students during the solution process. They were identified from analysing the data material from an insider perspective. The realization trees in this study evolved through activities of recognition and negotiations based on recursive patterns of conjecture-test-evaluation. The signifiers and realizations discerned though the analysis and displayed in two realizations trees, mainly reflect the negotiations of numeric estimated values of the quantities needed as input to these silently used and pre-defined models.

An issue need to be discussed is to what extent these findings are specific for this particular problem. The students had little experiences of modelling before they were introduced to this Fermi problem, because it was intentionally used as an introduction to mathematical modelling. The lack of experience made the students rely on their understandings of the learning-teaching agreement from traditional mathematics classrooms, which is displayed in their written solution. The students may have recognised the problem as an ordinary word problem, excluding the context and focusing solemnly on calculating the volume of a cuboid and determining its weight, which may be one reason for their use of the 'implicit' models. However, the problem required the students to collaboratively participate in a mathematical discourse lasting about half an hour, which would not have been the case for a regular classroom task. Gainsburg (2003) and Paper 1 also indicate that the use of pre-defined models constitutes a large part of mathematical modelling in the workplace practice, thus implying that it could also be a part of mathematics education.

Research literature (Blum et al., 2007; Stillman, 2012) sometimes describes the distinction between mathematical modelling and applications as a dichotomy, to some extent contradicting the results in Paper 5 that modelling and applications are intertwined and non-separable. This is consistent with Jablonka and Gellert (2007) who argue that "[m]athematics is not only the sphere where formalised problems find their solutions; mathematics is from the outset the vantage point from which the problem is constructed" (p. 6). The outsets for these students were that the problem was constructed by means of mathematics and they silently identified known models (recognition) that were adapted (negotiated) to the situation. The interaction between the two students seemed to explore how they most efficiently could use the pre-defined models. Also evident in the analysis is that the model formulation stage, transitioning from the phenomenon or situation to the mathematical model, is not linear, which also has been discussed in literature (e.g. Borromeo Ferri, 2006; Oke & Bajpai, 1986; Ärlebäck, 2009c). The experienced based models being part of someone's discourse will effect the assumptions, parameters etc. he or she will identify in formulating a new model within this discourse. Based on this study it is difficult to separate the two notions modelling and application. The experience of applying and recognising mathematical models will effect (i.e. frame) the modelling work, but modelling work is a creative problem solving process that will effect how and in what ways pre-defined models are connected and applied in developing new models that are useful and effective in a particular discourse.

To summarise, the experience of recognising and applying mathematical models will frame the modelling work. The result indicates that modelling work is a creative problem solving process that includes to connect, adapt and apply pre-defined models as means for developing new models that are useful and effective for a given purpose. A comparison of the result with Blomhøj and Højgaard Jensen's (2003) framework suggests that the *mathematical system*, the pre-defined models, is the outset for the modelling process, which means that the sub-process of *systematization* and the *mathematization* are done implicitly. The *mathematical analysis* done by the students is based on *experience* (recognition) and negotiation of quantities. However, the act of negotiation and evaluation of quantities is not visible in Blomhøj and Højgaard Jensen's (2003) framework as a part of the mathematical analysis, but seems to fit more into the sub-process *systematization* (i.e. to select the relevant objects, relations and idealisations). It appears that the model formulation done by the two students working collaboratively as described in Paper 5 is difficult to compare with Blomhøj and Højgaard Jensen's (2003) framework, since the sub-processes are linked in many different ways.

Paper 5 has also given some interesting insights, from a methodological perspective, about using commognition for analysing students' communication while collaborative working with mathematical modelling problems. Sfard's (2012) guidelines, the five methodological principles, guided our research to "develop deep understanding by providing a structure for designing research studies, interpreting data resulting from those studies, and drawing conclusions" (Lester, 2005, p. 458). The reconstruction of students' emerging realization trees and the use of complementary perspectives (insider/outsider) made it possible to make the relationships and abstractions (signifiers and realizations) visible for the justification of the research findings. The method is time consuming, but the outcome from using the method with the realization trees within the analysis presents fine grain results about *when*, *how* and *why* students formulated the mathematical models as they did. These aspects are difficult to discern from the written solution only, as discussed above, and further research studies of group collaborations, in particularly with more complex modelling problems, using the same methodology, may have potentials to give further insights into how to go about in unblocking the bottleneck problem.

6.1.6 Research question 8

The final research question, *What similarities and differences can be found between 'School' and 'Workplace' concerning how to work with mathematical modelling and concerning the notion of mathematical modelling?*, is summarised around the didactic transposition of mathematical modelling and then discussed.

The description and the meaning of the notion of mathematical modelling expressed by the modellers varied (scholarly knowledge, Paper 1), which was also the case for descriptions in textbooks (knowledge to be taught, Paper 2), expressions by the teachers (knowledge actually taught, Paper 3), descriptions in assessment (knowledge to be learned, Paper 4) and students descriptions (knowledge actually learned; Frejd and Ärlebäck, 2010). In addition, to apply and adapt models as a part of how to work with modelling is another similarity identified in all the parts of the didactic transposition related to Paper 1-5. Communication and collaboration is emphasised by the professional modellers (Paper 5), which to some extent is explicitly described to be a part in some tasks/ items/ problems/ activities in textbooks (Paper 2), in teacher activities in the classroom (Paper 3), in assessment i.e. projects (Paper 4) and by students formulating models by collaborating (Paper 5).

Differences identified concerning how to work include for example: *the goal of mathematical modelling*, *the risks involved in using the models*, *the use of technology*, *division of labour* and *the construction of mathematical models*. While in the workplace *the goal of mathematical modelling* is to develop a model that is going to be used as a tool for decisions about something, as for example building roundabouts, how to invest, how to use the staff, etc. (Paper 1), the textbook's (Paper 2) and the teachers' (Paper 3) goal is mainly to teach mathematics. In addition, the goal of assessment of modelling is assessing students' modelling competence (Paper 4), and students are restricted by the teaching and learning agreement (Sfard, 2008) with an aim to learn modelling (Paper 5) even if the students may have different goals with their activities in the classroom. The modellers in Paper 5 all point to *risks involved in using the models* that have been developed, for example that people get injured or lose money, while in the classroom mathematical models worked on are seldom put to use in a context of practice, or in other ways involving risks. For the student, though, one “risk” of producing or using a less accurate model is of course that it might lead to a low evaluation mark. *Technology* as displayed in Paper 5 plays a fundamental role in the modelling work of the professional modellers, but seems to play a very limited role in textbooks, teacher conceptions, assessments and students (Paper 2- 5). In school, as displayed in Paper 2-5, the *division of labour* focuses that ‘all’ students are to learn everything, while in the workplace (Paper 1) the division of labour is based on individual skills or the distribution of the amount of work. *The construction of mathematical models* differs significantly between school and workplace, not just in terms of use of technology and programming. Much of the modellers’ work (see Paper 1) is based on years of modelling experience, collaboration with other experts, knowledge of advanced mathematics, and specialised knowledge of other fields, making up an experience and knowledge base reaching far beyond what can be found in school (see Paper 2-5). In addition, the construction of mathematical models that includes for students to consider and evaluate the quality of data involved (empirical modelling, Paper 1), as well as the construction of models not possible to solve without technology (theoretical modelling, Paper 1), are not identified as a part of school practice (Paper 2-5).

However, there is no consensus among researchers in mathematics education of what modelling ‘is’ (Sriraman & Kaiser). Based on the results in Paper 1, there is also no agreement among the professional mathematical modellers on the meaning of mathematical modelling, which is not surprising since their working approaches are quite different. This makes it a challenge for textbook authors (Paper 2), teachers (Paper 3), assessment authors (Paper 4) and students (Frejd & Ärlebäck, 2010) to discuss a general definition of the notion of mathematical modelling. Similar to the case of ‘mathematics’, for ‘mathematical modelling’ it may be intrinsically difficult or even unrealistic to search for a general definition or description, which may also not be necessary as suggested by the mathematical modeller in the excerpt below, found in Drakes (2012, p. 40).

> I don’t think it needs a definition really. People just pretend it’s something which is different. I don’t really think it’s any different to anybody works in any particular subject you know? I mean you just do it. Everyone does it if they have a problem.

Argued by Drakes (2012), the modeller in this excerpt suggests that mathematical modelling is no different from problem solving. Nevertheless, in the curriculum documents for Swedish upper secondary school problem solving ability *and* modelling ability are included. The problem solving ability is for students to "formulate, analyse and solve mathematical problems, and assess selected strategies, methods and results" (Skolverket, 2011, p. 1), while the modelling ability is to "interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations" (ibid., p. 2). Assuming that the mathematical problem to solve is set within a realistic situation, the problem solver need to analyse the problem, i.e. interpret the realistic situation, choose a method, for example to design a mathematical model, and assess the result in terms of the model's properties and limitations. A common feature in Paper 1 is that modellers analyse problems with the help of mathematics. So the main distinction between modelling and problem solving, as displayed in the curriculum document, refers to the question whether the problem is regarded as a mathematical problem or not (i.e. a realistic situation). From a school perspective the answer seems quite obvious, in a mathematics class all problems are going to be solved with mathematics (at least in the current tradition of school mathematics), i.e. the problems in a mathematics class are mathematical problems by definition, also the problems that include contexts beyond school situations. In a workplace practice the situation is different: a problem does not require to be solved with mathematics, sometimes it may be resolved by expert opinions as discussed in Paper 1, which may be a reason for a distinction between the notions. A follow up study could be to explore professional modellers' conceptions of modelling vs. problem solving and the benefits of separating the two notions. The other similarities identified, the emphasis of applications and communication as parts of modelling, should get an increased attention in mathematics education. How this may be done and how the differences may be overcome or not will be discussed in the last section on implications and future research.

6.1.7 Validity and reliability of the findings

Validity and reliability are issues that impact on the quality of the research findings (Bryman, 2004).

Validity concerns the degree to which a study accurately assesses the aim and the research questions the researchers set out to investigate and may be discussed in terms of internal and external validity (ibid.). Internal validity refers to the accuracy of the match between the research observations and the conclusions developed while external validity refers to how transferable the results are to other situations or to other people beyond the investigated setting. Bryman (2004) defines reliability as "[t]he degree to which a measure of a concept is stable" (p. 543) and distinguish between external and internal reliability. Internal reliability refers to the accuracy of the instrument or procedure used for the investigation, such as how consistent two coders have coded the material whereas external reliability concerns the replicability of the study.

Regarding *Paper 1*, the use of observations together with interviews might have been more accurate to establish how the modellers worked, as argued by Wedege (2010), but also more time consuming. The convenience sample displays a set of different workplaces, which is not possible to generalize to other types of workplaces. However, there are indications that there are some aspects that seem to be common regarding how modellers work with modelling. Model construction as an empirical and applicational activity frequently appearing in the data set is also found in other research literature in mathematics education (e.g. Bissell & Dillon, 2000; Gainsburg, 2003). A limitation of the study is that the research literature reviewed did not include science literature. Theoretical modelling is less frequent and may be similar to computational modelling (computational model, 2013). Other descriptions of the modelling processes are possible, as they were based on a grounded theory approach. The process of coding the data was a collaboration between the authors and the categories were discussed and criticised to increase internal reliability. The modellers were given the opportunity to examine and give comments on a draft version of the paper to increase validity, which Bryman (2004) calls respondent validation. Four participants gave some comments, like 'when it comes to predict the future' should be e.g. 'when it comes to gain knowledge about possible outcomes', and their overall comments were that the result seemed acceptable and no major critic was raised. To facilitate replication of the study details about participants and their occupation, interview questions and excerpts of how coding was done are given in the text.

Paper 2. There is a "lack of common and explicit criteria for textbook comparisons" (Charalambous, Delaney, Hsu & Mesa, 2010, p. 120), in particular for mathematical modelling, and the development of an analytic scheme with focus on word counting and explicit/ implicit descriptions of models and modelling makes the research process possible to replicate. The internal reliability of the coding was assessed by letting an independent researcher do the same analysis of some parts of the analytic scheme. The independent researcher was given the analytic scheme and the two analysed examples as an instruction for how to proceed and the result can be described as consistent between coders. The parts not assessed by the other independent researcher were controlled by myself a second time at a later occasion. The sample chosen was at the time of the study all publishers that had developed a mathematics textbook for the new course and only one textbook from the published series was missing in the analysis (see list of 'Swedish publishers selling mathematical textbooks' at www.ncm.gu.se). Thus the sample seems to be comprehensive. However, the sample only refers to the first mathematics course of upper secondary school and it is possible that mathematical modelling plays a larger role in the other mathematics courses.

Paper 3 does not claim any generalized conclusion about teachers' conceptions about mathematical modelling in Sweden, since in line with Paper 1 the sample is chosen based on convenience and the grounded theory inspired approach used gives one interpretation out of many possible interpretations of the data. An additional observation of the mathematics classrooms might have provided a more solid base for the conclusions than only interviews, but it would have been much more time consuming. The internal reliability of the coding was controlled with Holsti's method (Holsti, 1969) (i.e. a formula for computing reliability for two coders) and I found no indications that the teachers participating in the study should be biased. However, there might be a problem with this study as it took place during the previous curriculum from 2000 where modelling had a less explicit description than the present curriculum from 2011. Research literature claim that curriculum reforms are challenging, in particular if new promoted modes of teaching are different in nature to 'traditional' ways of teaching as teachers tend to preserve the teaching methods they are accustomed to (Geiger, 2013). To change teaching due to a reform teachers need support (ibid.) and the present syllabus for mathematics (Skolverket, 2012a) was enforced with new textbooks where some include modelling items (Paper 2), complementary comments to the new syllabus in mathematics (Skolverket, n.d.), new NCTs that include assessment of modelling competency and a professional development course for in-service teachers that includes modelling (Skolverket, 2012c), which may have had impact on teachers' ways of teaching. Therefore, more research about the knowledge actually taught is necessary to provide a more up-to-date description.

Paper 4. To get hold of information of assessment modes, assessment guidelines etc. from teachers around the world is difficult and a literature review seems accurate as method. The main concern about validity and reliability is twofold, the choice of the sample and the analysis of the sample. The chosen sample is considered as a representative sample within the research community of mathematical modelling and application, because it includes all the ICTMA proceedings etc. However, it is not necessarily a valid sample outside this community. Many of the researchers attending ICTMA conferences are also attending CERME conferences and have written papers to the ICMI 14 Study and the ZDM journal, which may narrow the scope. On the other hand, many contributions in high ranked scientific journals and in international books have been written by members of the ICTMA community. Therefore it is not obvious that the outcome of the review would have been much different if the sample had been chosen to include other scientific journals and international books. With hindsight, the use of grounded theory as the method for analysis to establish the different modes of modelling assessment could have been replaced by use of the pre-defined categories by Watt (2005), but it may also be seen as a measure of reliability to find similar modes as described in other research studies. The study had no aim to generalize in a broad sense, as the focus was to bring up on the table critical issues about assessing mathematical modelling for further exploration.

Paper 5. The method used, Sfard's (2012) guidelines, is valid and reliable in the sense that is quite transparent and possible to replicate, because what is said and done when the two students collaborated is illustrated in excerpts and how the analysis is performed is explicitly explained. The findings about applying already known models relate to, for example, how professional modellers work (Paper 1). However, the outcome of Paper 5 is much depending on the problem given to the students, and to some extent it is questionable to generalize the findings too much. How would the students have acted, worked and formulated mathematical models if they had been exposed to a problem where they had no personal experience to rely on? That could be project for further research.

6.2 Conclusions

This doctoral thesis is a product of explorative research and contributes with information about the experiences that students, teachers and modelling experts have of learning, teaching and working with mathematical modelling in and out of school settings, and how they interpret the notions of mathematical modelling. In this section, I will try to extract and summarize some of the major findings from the discussion (6.1.1-6.1.8), set within the didactic transposition frame, starting with the interpretation of the notion of mathematical modelling and then continue with how the actors work with mathematical modelling.

6.2.1 The notion of mathematical modelling

Scholarly knowledge. The modellers' descriptions of the notion of mathematical modelling varied. Briefly, the modellers expressed modelling as mathematical tools to describe, recreate, abstract, characterize and simplify different kinds of reality. In addition, modelling was expressed as an activity to set up a mathematical model and an activity that also included validation (i.e. a holistic process like Blomhøj and Højgaard Jensen's (2003) description of modelling)

Knowledge to be taught. The textbooks analysed display a variety of descriptions. Two textbook series did not discuss the notion at all. One series described modelling *implicitly* as an ability related to some items to solve. Finally, two textbook series *explicitly* described modelling, in a few pages, as a problem solving activity with three phases or as a group activity to develop mathematical models, including making assumptions, estimations, calculations and evaluation .

Knowledge actually taught. Only 50% of the teachers had heard the notion before participating in the interview study. The teachers, like the modellers, expressed different descriptions of modelling. Briefly, they described modelling as an activity to design a mathematical model based on a situation (i.e. simplify and describe something with mathematics).

Knowledge to be learned. The literature review in Paper 4 does not focus on the descriptions of the notion of mathematical modelling, but it is evident from the review that the choice of assessment mode will impact on the possibilities to assess mathematical modelling in terms of an atomistic or a holistic approach.

Knowledge actually learned. Paper 5 does not include questions to the students about the notion of modelling. However, the students in Paper 5 developed mathematical models by applying pre-defined models, which may to some extent mirror the students' description of modelling as problem solving and a way of using/applying mathematics as a tool in different situations as found also in Frejd and Ärlebäck (2010).

Conclusion related to the notion of mathematical modelling: The variety of description in the workplace is not surprising, since their working approaches are quite different. This variety makes the notion difficult to transpose into the school practise, which is described in this study. The questions raised are if it is unrealistic to search for a general definition and if it is really necessary to have a general definition. A consequence for anyone how uses the notion, is to always be explicit with the meaning.

6.2.2 How modellers, teachers and students work with mathematical modelling

Scholarly knowledge. Three different types of activities of how professional modellers construct their models are identified as *empirical*, *theoretical* and *applicational modelling*. The empirical activity focuses on the data and the quality of data: the data is the central aspect to frame the problem formulation, to identify parameters, variables, constants, and processes as well as to do the validation. The theoretical activity is a process of re-formulating already established physical equations to computer based models, using the computer models to get information about the physical equations by interpreting and evaluating the result of the computer models, and finally evaluating the validity of the computer models. The applicational activity is the process of using working experience and applying mathematics or already defined models to a particular situation, which is a part of all constructors' work. Overall, mathematical modelling at the workplace is often an activity performed by teams, in particular for larger projects, and the use of technology is a central part in all three activities. The modellers search for 'acceptable' solutions rather than 'correct' solutions. Another vital part of the work is communication between different actors (consumers, operators and other experts). In addition, all modellers pointed out that there are, for several reasons, risks with using their models and that all actors should be aware that it is just models and critically reflect upon that.

Knowledge to be taught. Mathematical modelling is in the mathematical syllabus (Skolverket, 2012) described as an ability to be taught, but it is not treated as a central notion in the analysed textbooks. The descriptions of both mathematical models and modelling vary between the analysed textbooks both in terms of how frequently the words models and modelling have been used and how the notions have been described. Only two of the textbook series present *explicit* descriptions of modelling (Szabo et al., 2011; Alfredsson et al., 2011). Nevertheless, modelling is still only presented in a marginalized way, which means that the textbooks do not strongly support students to develop a holistic modeling ability, as described by Blomhøj and Højgaard Jensen (2003). This means that in order to develop modelling ability, modelling needs to be explicitly taught with support from a teacher (Niss et al., 2007).

Knowledge actually taught. The mathematics teachers interviewed (still working with the curriculum from the year 2000) did not prioritize the integration of mathematical modelling into their everyday teaching. Almost all of them explicitly explained that mathematical modelling is more used in physics or in chemistry than in mathematics. Only five teachers out of 18 used modelling activities intentionally in their mathematics class in relation to projects, laboratory activities in mathematics and regular classroom work. The teachers discussed the items shown from Haines et al. (2000) and emphasised mainly the mathematical aspects of mathematical modelling, leaving other aspects out or intended to use them less frequently, as they argued that those items did not include mathematics.

Knowledge to be learned. The modes of modelling assessment will impact on what to be learned. It clearly relates to the complexity of any assessment endeavour (Niss, 1993a), illustrated by Izard's (1997) statement that "[n]o single assessment method is capable of providing evidence about the full range of achievement" (p. 109). The *written tests* as described in the papers that were analysed all draw on an atomistic view of assessment focusing on sub-process, whereas *projects* are described to assess a more holistic modelling competence. Projects are identified as an ideal method for assessment, but obstacles regarding reliability of assessing projects are found. A method used in *contests* for assessing students' modelling competence is for referees to rank students' solutions based on given criteria. There are indications, based on a few research studies, that *hands-on tests* and *portfolio* assessment modes have potential to be developed as valid and reliable assessment modes for modelling, though further research on this issue is needed.

Knowledge actually learned. What students have learned about formulating mathematical models, based on the snow clearance problem used in the study, depends on their experiences of recognising and applying mathematical models and their ability to collaboratively negotiate the meaning of the model. The students' modelling work can be characterised as a creative problem solving process, including the connection, adaption and application of pre-defined models as means for developing new models. Negotiation aims at making the models useful and effective for a given purpose.

To summarise, the results presented in this thesis provide a fragmented picture of the didactic transposition of mathematical modelling in school mathematics in Sweden. There are significant differences in how modellers, teachers and students work with modelling in different practices in terms of the goal with the modelling activity, the risks involved in using the models, the use of technology, division of labour and the construction of mathematical models. However, there are also similarities identified described as important aspects of modelling work in the different practices, such as communication, collaboration, projects, and the use of applying and adapting pre-defined models.

Considering Niss, Blum, and Galbraith's (2007) concern that "[i]n spite of a variety of existing materials, innovative programmes, and sustained arguments for the inclusion of modelling in mathematics education, it is necessary to ask why its presence in everyday teaching practice remains limited in so many places" (pp.

22-23), this thesis suggest some possible answers: there is no consensus across levels what modelling ‘means’ (Paper 1 and Paper 3); as described by professional modellers, modelling is a very complex endeavour (Paper 1); textbooks do not emphasise modelling as a central notion (Paper 2); teachers consider modelling as a part of science rather than mathematics (Paper 3); modelling is difficult to assess (Paper 4).

6.3 Implications for teaching

There are several foundations or principles that teachers may rely on in their mathematics teaching practice, depending on what they see as the goals of mathematic teaching and what use of mathematics they consider important for students to learn. For example, the teachers in Paper 3 express that is important for students to learn mathematics to prepare for further studies, for use other subjects, and as a scientific language, whereas other arguments emphasise the use of mathematics outside school such as in everyday life, in society, and in the workplace. To prepare students for higher education in mathematics requires that students are well acquainted with algebra, arithmetic and calculus. Teaching will then be based on mathematical principles like to handle procedures to solve equations, to know about mathematical concepts, establish proofs, etc. To prepare students for the use of mathematics outside school requires other approaches, such as teaching mathematics in relation to how people employ mathematics in their work. This thesis has contributed to the latter, with a focus on mathematical modelling.

Assuming that a teacher in upper secondary mathematics education wants to follow the argument by Niss, Blum and Galbraith (2007, pp. 6-7) that “if we want students to develop applications and modelling competency as one outcome of their mathematical education, applications and modelling have to be explicitly put on the agenda of the teaching and learning of mathematics” the teacher needs to clarify what applications and modelling competency could mean to be able to make the meaning explicit to students. Based on this thesis that is a puzzling and difficult task for the teacher, as there is no consensus on this issue among the different actors investigated. One reason for this is that it is not possible to set up a unique definition due to incommensurable working approaches in the different workplaces, as discussed in Paper 1. Thus, the teacher may need to turn to what is supposed to guide his/her teaching, i.e. the syllabus for mathematics (Skolverket, 2012b). The formulation found there, “interpret a realistic situation and design a mathematical model, as well as use and assess a model’s properties and limitations” (Skolverket, 2012b, p. 2), then needs to be interpreted. One interpretation (i.e. atomistic) is to differentiate between, for example: 1) interpret a realistic situation and design a mathematical model *and* 2) to use and assess a model’s properties and limitations. Another interpretation is to view the quote as describing a complete process (i.e. holistic), including all parts.

The choice of interpretation entails consequences for teaching. The first interpretation (atomistic) suggests that the teacher may use the analysed textbooks (Paper 2) and written tests (Paper 4) to justify their teaching, since both the textbooks and the written test focus on sub-processes. The teacher must in addition consider to be explicit with what sub-process is found in the textbooks and in written tests, and not just focus on the purely mathematical aspects, which are most frequently found in textbooks and written tests (Paper 2 and Paper 4). However, the common argument in research literature in mathematics education is that an atomistic view of mathematical modelling is not enough (e.g. Blomhøj & Hoff Kjeldsen, 2006; Legé, 2007b; Niss et al, 2007), a student will not develop the full amount of modelling competence by training the sub-competences one at a time. Metaphorically, it is like learning to swim: one cannot expect to know how to swim after first exercising the arm strokes and then the leg kicks on the beach, one has to go into the water and do it all at the same time; the sub-processes in mathematical modelling are intertwined and connected. Thus, following Blomhøj and Hoff Kjeldsen's (2006) principle that "the pedagogical idea behind identifying mathematical modelling competency as a specific competency is exactly to highlight the holistic aspect of modelling" (p. 166) will, based on results from this thesis, suggest that the textbooks need to be complemented with other teaching material (Paper 2) and that alternative assessment modes, such as projects, may be used for assessing the holistic modelling competency (Paper 4).

To find arguments why it is important to teach modelling one may review the five arguments put forward by Blum and Niss (1991) for including modelling in mathematics education. From a critical analysis of the five arguments and a holistic view of mathematical modelling one may argue that some of the arguments also relate to other parts of the mathematics curriculum, while some may be more strongly linked specifically to modelling activities. Both *The formative argument* and *The 'promoting mathematics learning' argument* may, for example, also be supported by general problem solving activities, and *The utility argument* by different kinds of applications of mathematics. *The 'picture of mathematics' argument* could be developed in many different ways of which one is modelling. However, for *The 'critical competence' argument* it seems necessary to employ a holistic view of mathematical models and modelling in order to prepare students to be critical to the use of mathematics in private life and in society, when for example many economic and environmental decisions are based on mathematical models (e.g. Jablonka, 2003; Skovsmose, 1994, 2005). This was also highlighted by the interviewed modellers (Paper 1; Frejd, 2013a; accepted) as an important aspect of modelling.

Another principle related to the holistic interpretation of modelling is that "[m]odelling competency is developed through the practice of modelling" (Blomhøj & Hoff Kjeldsen, 2006, p. 166), meaning the students should be exposed to 'realistic' modelling problems in their education. The question is how realistic, in terms of similarities to the workplace, it is possible to be in a classroom situation at upper secondary school, due to the differences presented in section 6.1.6. Research literature describes the difficulties in trying to map and integrate the way of work outside school into school, for example:

> A problem with including “authentic” examples of these practices in the school curriculum is caused by the fact that the technological transformation of academic mathematical knowledge is a process embedded in a highly specialised division of labour. The transformation is mediated by several disciplines (resulting in software development) and the mathematics involved is in general too sophisticated (Jablonka, 2007, p. 198).

Paper 1 indicates, in line with the quote above, that two descriptions of how to work with mathematical modelling, empirical modelling and theoretical modelling, present a challenge or “inaccessible phenomena” (Gainsburg, 2003, p. 263) for students and teachers. The knowledge required is not accessible in upper secondary school, because the modellers’ work is based collaboration with other experts, knowledge of advanced mathematics, specialised knowledge of other fields, the use of technology and programming, and on years of modelling experience. Other differences as discussed in section 6.1.6. are: *the goal with mathematical modelling*, *the risks involved in using the models*, *the use of technology*, and *division of labour*. A consequence of these differences is that modelling as it shows in the workplace can never be fully ‘mapped’ in the mathematical classroom. However, it may be possible to simulate such activity.

In organised educational settings a frequently appearing activity is learning by simulation, for example in kindergarten it is common that the environment is organised into different pedagogical ‘rooms’ (SOU 2006:75). A ‘room’ may be called ‘doll-room’ and be furnished with a small stove and a sink, household items, small tables and chairs, dolls with accessories, etc. to prepare the children to become adults (ibid.). The goals and the risks with the pedagogic activity in these ‘rooms’ is very different from real life. Similar arrangements may be found in teaching activities for helicopter pilots or train drivers that may include controlling helicopters or trains in simulators.

According to professional modellers (Frejd, 2013a) it is possible to teach modelling in upper secondary school and a teacher may be guided to follow the principle to teach ‘holistic’ modelling as close to workplace practice as possible, even if this was not a teacher conception in Paper 3. Based on this thesis, the following example is constructed, to exemplify one possibility to realize a ‘simulated’ modeling project:

"The long-term goal of the predator policy in Sweden is to achieve and conserve a healthy population of wolf, bear, wolverine, lynx and golden eagles. A credible predator policy also requires an active management to strengthen the confidence in the predator policy and contributes to a better coexistence between humans and large predators "(p.14)... "the aim is to create a good balance between the predator population and the impact it causes on business, public and individual interests" (p. 16)... (SOU 2012/13:191, my translation)

How many wolves should we have in Sweden?

Figure 7. Example of a modelling problem derived from SOU 2012/13:191

The example above in Figure 7, inspired by the work of the biology modeller in Paper 1, is picked up from a government proposal (SOU 2012/13:191) about sustainable predator policy. The problem would be regarded by Vos (2007) as a problem "clearly not created for educational purposes" (p. 721). It means that the problem originally is developed for out of school purposes, it is binary (*yes*, it is authentic or not, *no*), separate task aspects can be authentic and be applied in the classroom, and the authenticity can be certified by different stakeholders and modellers and make that clear to anyone (ibid.). According to Vos (2010) authenticity "is a social construct on which a community agrees on its qualification" (p. 720). In Frejd (2013a) modellers suggested that mathematics teachers could invite people from the workplace (or other organizations) to present how they work to increase the motivation for the study of mathematics. In addition, the modellers described that modelling is important for students to become a democratic citizen, because modelling enhances the ability to understand results, critically examine statistics and be able to form opinions (ibid.). To invite a politician to the mathematical classroom to introduce the predator question could serve three purposes. First, the politician may act as the client (the one that gives the problem as discussed in Paper 1), second, it may certify the authenticity of the predator problem, and third, the politician can explain how mathematical models are used in political discussions and decisions. According to Skovsmose (2005), the use of mathematical models in decision-making often has the role of "*dehumanization*" (p. 94), "*authorization*" (p. 95) and "*kept at a convenient distance"* (p. 95), meaning to eliminate the human factor, such as feelings, to justify for decisions and the responsibility for the actions based on the decisions are distributed between the politicians and the modeller. To follow up the introduction the teacher may invite experts in the area to whom students may ask questions, since the collaboration between modellers and experts is a part of the workplace practice (Paper 1). For example, experts from some of the consultation bodies may be invited, like ‘Svenska jägarförbundet’ (The Swedish Association for Hunting) and ‘Sveriges lantbruksuniversitet’ (The Swedish University of Agricultural Sciences).

To work within a project is a common activity in the workplace (Paper, 1) and project reports together with oral presentations is an adequate way to assess a holistic modelling competence (Paper 4), implying that projects should be given more time in the classroom. One idea could be to split the teaching group into smaller teams and let them choose to listen and collaborate with the experts from The Swedish Association for Hunting or from The Swedish University of Agricultural Sciences, with an aim to develop a model to be used for decision of how many wolves there should be in Sweden. The two consultation bodies have clearly different views on how many wolves there should be in Sweden as described in the government proposal (SOU 2012/13:191). The Swedish Association for Hunting argues for 200 wolves whereas The Swedish University of Agricultural Sciences argues for 1250-2000 (ibid.), which suggests that there are no non-political models as one professional modeller described it (Frejd, accepted).

The students in the teams could, to some extent, divide the workload between them, i.e. make a division of labour. One student may search for historical data on the wolf population and aspects that effect the population like inbreeding, poaching, the amount of prey, etc. Another student may search for data on the impact that population growth has on business, public and individual interests and what it costs. A third student may focus on the mathematical relations and a fourth student prepares the report and the presentation. However, what is significantly important is the collaboration and communication with the aim that all students learn about all parts of the issue discussed. The modelling work of the students may include aspects of an *empirical*, *theoretical*, and *applicational modelling activity* (Paper 1). The students are involved in the empirical activity when they identify the data and parameters, variables, constants, and processes to develop models and maybe they also consider the quality of data. The involvement of statistic data may be displayed and analysed with the use of technology, which is an essential part of modelling in the workplace practice (Paper 1). Aspects of the theoretical activity could also be a part of the students’ work if the students read something about predator-prey relations and about Lotka-Volterra equations. These equations could for example be visualised (and solved) with technology. To apply already defined models, the applicational activity, is a part of modelling work (Paper 1 and Paper 5) and may include some economic models, statistic models, Lotka-Volterra equations, etc. The students’ collaborative team work in relation to applicational activity will imply that the activities of recognition and negotiation will be given more time in mathematics classrooms as requested in Paper 5 to facilitate to overcome the ‘bottleneck problem’.

The end of the project may be organised as a political debate inspired by Edwards and Morton’s (1987) idea discussed in Paper 4, about to “simulate a boardroom meeting between a management panel (a mixture of technical and non technical mangers) and a modelling team" (p. 53). The goal of the political debate is for the consumer, the politician, to make a decision on the number of wolves, based on the modelling teams' (students') oral and visual (e.g. powerpoint) presentations as well as based on their written project report. The political panel may include the politician, experts and teachers. This will imply that the students' need to communicate and adapt their language, which was an important part of the modellers' workplace (Paper 1), as well as explain the mathematical models they have used. One risk with using mathematical models as described by the professional modellers is that the users of the model do not understand the model or its limitations (Paper 1). It has been argued that it is better to use simple mathematical models in decision-making based on complicated problems that one understands than use complex models that one does not understand (Kaijser, 1993), which address the issue of assessment. The teachers in the panel also have another aim, which is to assess the student work. Based on Paper 4 the criteria need to be explicit and may for example include the critical questions suggested by Jablonka (1996) or be derived from Blomhøj and Højgaard Jensen's (2003) framework. In addition, what seems to be approved in the investigated research literature about assessing projects in Paper 4 (Antonius, 2007; Niss, 1993b; etc.) is that more than one marker should be involved to increase reliability. One aspect that is described as an important ability in modelling in all papers, but difficult to perform, is the justification of the model (i.e. validation). Also in this wolf example this may be problematic, since it is a predictive question. However, it still should be a part of the assessment criteria, because students need to convince others *why* their solution is (more) appropriate.

Finally, the main implication for teaching regarding modelling assessment is for the teacher to be explicit regarding the meaning and the goal of modelling and its relation to assessment criteria (Paper 4).

6.4 Future research

This thesis displays a broad analysis focusing on different aspects of mathematical modelling in practices at different levels of the didactic transposition, which suggests many opportunities for future research. Here some examples are given.

In Paper 1, on mathematical modellers, modelling is discussed from an expert point of view, but there are other persons at the workplace that use mathematical modelling in their workplace (i.e. consumers and operators). A follow up study could be to follow up how the modellers' models in Paper 1 are actually used.

The sample of textbooks in Paper 2 refers only to the first mathematics course at Swedish upper secondary school, and it is possible that mathematical modelling plays a larger role in other mathematics courses, which may be of research interest. In addition, Paper 2 and Paper 5 both include 'new' research methods for analysing how mathematical modelling is treated and presented in textbooks as well as how students in small groups work with modelling, that need further research.

The result in Paper 4 is depending on the ICTMA sample and a follow up study could include a broader sample and focus on books and research journals.

The fact that teachers in Paper 3 did not express clear conceptions about the notion of mathematical modelling, suggests that further research into teacher education is needed. Internationally there is much research about modelling courses and other projects for in-service teachers and pre-service teachers (see section 9 in Lesh, Galbraith, Haines & Hurford, 2010; Blomhøj & Hoff Kjeldsen, 2006). There are examples in Sweden of successful introductions of modelling activities to pre-service teachers as well as to in-service teachers. Lingefjärd's (2006a, 2002a) and Lingefjärd and Holmquist's (2005) research include examples of modelling activities for pre-service teachers and one example for further education of in-service teachers is the design project by Ärlebäck (2009a). It would be interesting to do follow up studies of those teachers participating in Ärlebäck (2009a) and Lingefjärd's (2006a, 2002a) and Lingefjärd and Holmquist's (2005) teacher students (e.g. those who work at upper secondary school today) to see to what extent they use modelling activities in their present teaching.

One may also ask several exploratory questions about the present state of modelling in mathematics education, such as: What aspects about mathematical modelling are presented to pre-service teachers in Sweden? How is it possible to develop pre-service teachers' conceptions about mathematical modelling during their education? What type of teaching programmes have proven to be effective to teach and learn mathematical modelling? Other issues regarding support to teachers' in-service and questions related to this could be: What kind of modelling activities with instructions are available to teachers? What are the outcomes of government introduced professional development course regarding mathematical modelling? How are the statements in the curriculum practised and taught by teachers in the mathematics classrooms?

One way to deal with the last question above may be to include mathematical modelling items that have holistic aspects of modelling in the national course tests (NCT) or in projects. One could begin to include small modelling items in the NCT like "realistic Fermi problems" (Ärlebäck, 2009d) or other introduction problems (see for instance the LEMA[22]-project), and in the future use more complex modelling problems. A project as the one described in the section implications for teaching could serve as a research project, with aim to explore teachers' and students' experiences of working with modelling projects.

[22] LEMA- Learning and Education in and through Modelling and Applications and example of modelling problems may be retrieved from http://www.lema-project.org/web.lemaproject/web/eu/tout.php

The teachers' conceptions in Paper 3 related mathematical modelling to physics or chemistry. However, there are many students that do not study physics in upper secondary school in Sweden and physics or chemistry problems are only a minor part of possible modelling problems. Maybe knowledge from physics or chemistry education about teaching and learning modelling activities can be applied also in mathematics education. Questions that need more research are for instance, what type of modelling activities are done in physics or chemistry? What aspects of mathematical modelling from physics or chemistry education are useful to incorporate in mathematics classrooms? What aspects are missing?

Finally, this study has indicated that (holistic) modelling activities are quite rare in the participating classrooms in upper secondary school in Sweden and one may wonder how students experience mathematics in general without mathematical modelling. The use of (holistic) modelling activities in Swedish classrooms will need to increase in order to develop in students a competence called *modelling competency* in upper secondary school in Sweden.

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Appendix

Till .... (titel och för och efternamn)

Hej,

Mitt namn är Peter Frejd och jag är doktorand i matematikdidaktik vid matematiska institutionen (MAI) vid Linköpings universitet. Mitt avhandlingsarbete handlar om matematisk modellering och dess roll i matematikundervisningen. Ett pågående delprojekt fokuserar på hur matematisk modellering används i yrkeslivet/ forskning och hur ”professionella modellerare” ser på begreppet modellering och dess koppling till matematikundervisning.

Jag vore mycket tacksam om jag fick möjlighet att genomföra en intervju med dig, då du har en stor erfarenhet och stor kompetens av matematisk modellering i ditt yrke - fokus skulle då vara att få en bild av din syn på begreppet matematisk modellering och hur du arbetar med matematisk modellering professionellt. Intervjun beräknas ta ca 30 minuter. Frågorna är kopplade till följande teman:

- bakgrund (t.ex. Vad är din akademiska bakgrund och yrkeslivserfarenhet?)

- begreppet matematisk modellering (t.ex. Vad innebär matematisk modellering för dig?)

- matematisk modellering i arbetet/forskningen (t.ex. Hur arbetar du med matematisk modellering i ditt arbete?)

- matematisk modellering i undervisningen (t.ex. Hur har du "lärt dig" matematisk modellering och hur ser du på modellering i skolmatematiken /främst gymnasiet/?)

Projektet följer Vetenskapsrådets etiska forskningsprinciper med dess krav på information, samtycke, konfidentialitet samt hur forskningsmaterialet får användas. Anonymitet kommer att garanteras och i forskningsrapporter som publiceras kommer den information du lämnar inte att kunna knytas till dig. Ingen annan än jag och mina handledare (professor Christer Bergsten, lektor Jonas Bergman Ärlebäck) kommer att ta del av den ljudinspelade/transkriberade intervjun och diskutera den enbart internt i samband med mitt avhandlingsarbete.

Jag vore mycket tacksam om du kan tänka dig att ställa upp på en intervju och bidra till matematikdidaktisk forskning kring matematisk modellering.

Meddela mig om du kan avvara lite av din tid för en intervju och när det i sådana fall skulle kunna passa att jag kommer. Har du tid någon gång de närmaste veckorna vore jag tacksam.

M.v.h.

Peter Frejd

PART II

PAPERS

Part II

Papers