Module 04
Linear Time-Varying Systems

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Introduction to State Transition Matrix (STM)

- For the linear autonomous system

$$\dot{x}(t) = Ax(t), x(t_0) = x_0, t \geq 0$$

the state solution is

$$x(t) = e^{A(t-t_0)}x_0$$

- **Define the state transition matrix (STM):**

$$\phi(t, t_0) = e^{A(t-t_0)}$$

  - STM ($\phi(t, t_0)$) propagates an initial state along the LTI solution t time forward. Note that:

$$\phi(t_1 + t_2, t_0) = \phi(t_1, t_0)\phi(t_2, t_0) = \phi(t_2, t_0)\phi(t_1, t_0), \forall t_1, t_2 \geq 0$$

- In general, for an linear time varying system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), x(t_0) = x_0,$$

the state solution is given in terms of the STM:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Properties of the STM

For the linear autonomous system

$$\dot{x}(t) = Ax(t), x(t_0) = x_0, t \geq 0$$

the STM is:

$$\phi(t, t_0) = e^{A(t-t_0)}$$

1. $\phi(t_0, t_0) = \phi(t, t) = I$
2. $\phi^{-1}(t_1, t_2) = \phi(t_2, t_1)$
3. $\phi(t_1, t_2) = \phi(t_1, t_0)\phi(t_0, t_2)$
4. $\dfrac{d}{dt}(\phi(t, t_0)) = A\phi(t, t_0)$

Proofs:

Solution Space and System Modes

- **Solution space** $\mathcal{X}$ of the LTI system $\dot{x}(t) = Ax(t)$ is the set of all its solutions:

$$\mathcal{X} := \{x(t), t \geq 0 \mid \dot{x} = Ax\}$$

- $\mathcal{X}$ is a vector space
- Dimension of $\mathcal{X}$ is n
- **System modes:** A **mode** of the LTI system $\dot{x} = Ax$ is its solution from an eigenvector of A:

$$x(t) = e^{At} v_i = e^{\lambda_i t} v_i$$

- This is one property of the matrix exponential (see Module 3)
- The n (possibly repeated) **modes** form a basis of the solution space $\mathcal{X}$

Decomposition of State Solution

- Any state solution for an autonomous system can be written as a linear combination of **system modes**, assuming that A is diagonalizable
- This means that the solution space $\mathcal{X}$ can be formed by these linear combinations
- $A = TDT^{-1}$ is assumed to be diagonalizable
- Assume that we start from $x_0 = TT^{-1}x_0 = x_0$
- This means that we start from a linear combinations of $v_i, i = 1, \ldots, n$ since

$$x_0 = TT^{-1}x_0 = \sum_{i=1}^{n}(x_0^\top w_i)v_i = \sum_{i=1}^{n}\alpha_i v_i$$

  where w_i's are the rows of the T^{-1} matrix (or the left evectors)
- Given that construction, we can see that the solution $x(t)$ is a LC of the modes $e^{\lambda_i t}v_i$

Changing Coordinates

- Changing of coordinates of an LTI system: basically means we're scaling the coordinates in a different way
- Assume that $T \in \mathbb{R}^{n\times n}$ is a nonsingular transformation matrix
- Define $\tilde{x} = T^{-1}x$. Recall that $\dot{x} = Ax + Bu$, then:

$$\dot{\tilde{x}} = (T^{-1}AT)\tilde{x} + T^{-1}Bu = \tilde{A}\tilde{x} + \tilde{B}u$$

  with initial conditions $\tilde{x}(0) = T^{-1}x(0)$
- Remember the diagonal canonical form? We can get to it if the transformation T is the matrix containing the eigenvectors of A
- What if the matrix is not diagonlizable? Well, we can still write $A = TJT^{-1}$, which means that $\tilde{A} = J$ is the new state-space matrix via the eigenvector transformation
- In fact, you can show that if $A = TJT^{-1}$ with j Jordan blocks (i.e., $J = \text{diag}(J_1, J_2, \ldots, J_j)$), then after the transformation $\tilde{x} = T^{-1}x$, the LTI system becomes decoupled:

$$\dot{\tilde{x}}_1 = J_1\tilde{x}_1,\ \dot{\tilde{x}}_2 = J_2\tilde{x}_2,\ \ldots,\ \dot{\tilde{x}}_j = J_j\tilde{x}_j.$$

STM of LTV Systems

- In the previous module, we learned how to compute the state and output solution
- We assumed that the system is time invariant, i.e.,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- What if the system is time varying:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t) \quad (*)$$

- How can we compute $x(t)$ and $y(t)$?
- That relies on finding the STM of the LTV system $(*)$
- To do so, we have to find the exponential of a time-varying matrix

STM of LTV Systems — 2

Theorem — STM of $\dot{x}(t) = A(t)x(t)$

The STM of $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is given by

$$\phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(q)dq\right)$$

if the following conditions are satisfied:

1. $A(t)$ has piecewise continuous entries for all t, t_0[a]
2. $A(t)$ commutes with its integral $M(t, t_0) = \int_{t_0}^{t} A(q)dq$, i.e., $A(t)M(t, t_0) = M(t, t_0)A(t)$

[a] A function is piecewise continuous if: (a) it is defined throughout that interval, (b) its functions are continuous on that interval, and (c) there is no discontinuity at the endpoints of the defined interval.

- This theorem is very important, but can be very difficult to assess
- Consider a large system with TV $A(t)$. Then, numerical integration needs to be performed the check the conditions

STM of LTV Systems — 3

- Given this analytical challenge, a natural question arises
- **What are easily testable** conditions that are sufficient for $A(t)$ to commute with $M(t, t_0)$?
- The following theorem investigates this question

Theorem — STM Testing Conditions

$A(t)$ and $M(t, t_0)$ commute if any of the following conditions hold:

1. $A(t) = A$ is a constant matrix
2. $A(t) = \beta(t)A$ where $\beta(\cdot) : \mathbb{R} \to \mathbb{R}$ is a scalar function and A is a constant matrix
3. $A(t) = \sum_{i=1}^{m} \beta_i(t)A_i$ where $\beta_i(\cdot) : \mathbb{R} \to \mathbb{R}$ are all scalar functions and A_i's are all constant matrices that commute with each other, i.e., $A_iA_j = A_jA_i, \ \forall i, j \in \{1, 2, \dots, m\}$
4. There exists a factorization $A(t) = TD(t)T^{-1}$ where $D(t) = \text{diag}(\lambda_1(t), \dots, \lambda_n(t))$

Example 1

- $A(t) = \dot{\alpha}(t) \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}$
- What is the state transition matrix?
- **Solution**: notice that $A(t)$ fits with the second characterization, hence $A(t)$ and $M(t, t_0)$ commute (assume that $\alpha(t)$ is continuous differentiable function)
- Note that A is nilpotent of order 2
- Solution:

$$\phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(q)dq\right) = I + \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} (\alpha(t) - \alpha(t_0))$$

Example 2

- $A(t) = \begin{bmatrix} \dot{a}(t) & \dot{b}(t) \\ \dot{b}(t) & \dot{a}(t) \end{bmatrix}$, find the STM
- Note that $A(t) = \dot{a}(t)I + \dot{b}(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- We can apply the third case to obtain:

$$\phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(q)dq\right) = e^{a(t)-a(t_0)} exp\left(\begin{bmatrix} 0 & b(t) - b(t_0) \\ b(t) - b(t_0) & 0 \end{bmatrix}\right)$$

- Recall that if

$$A_2 = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \Rightarrow e^{A_2 t} = \begin{bmatrix} \cosh(bt) & \sinh(bt) \\ \sinh(bt) & \cosh(bt) \end{bmatrix}$$

- Hence,

$$\phi(t, t_0) = e^{a(t)-a(t_0)} \begin{bmatrix} \cosh(b(t) - b(t_0)) & \sinh(b(t) - b(t_0)) \\ \sinh(b(t) - b(t_0)) & \cosh(b(t) - b(t_0)) \end{bmatrix}$$

Overall Solution

- So, given that we have the state transition matrix, how can we find the overall solution of the LTV system?
- The answer is simple:

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau$$

More Examples on STM Computations

Find the state transition matrix of

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) & \beta \\ 0 & \sin(t) & \cos(t) \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

Intro to Discretization

We want to discretize and transform this dynamical system

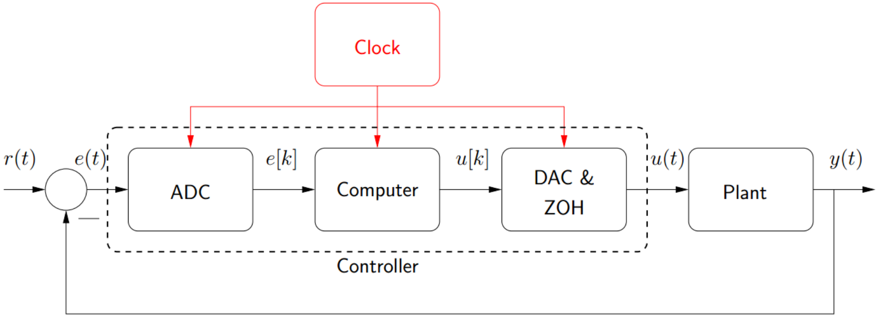
$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

to

$$\begin{aligned}x(k+1) &= \tilde{A}x(k) + \tilde{B}u(k) \\ y(k) &= \tilde{C}x(k) + \tilde{D}u(k)\end{aligned}$$

- Why do we need that?
- Because if you want to use a computer to compute numerical solutions to the ODE, you'll have to give the computer a language it understands
- Also, many dynamical systems are naturally discrete, not continuous, i.e., sampling doesn't happen continuously

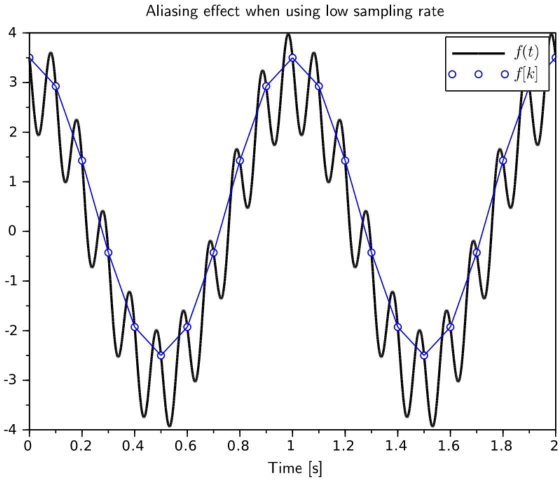
What Computers Understand



What is zero order hold? It's basically the model of the signal reconstruction of the digital to analog converter (DAC):

$$u_{\mathrm{ZOH}}(t) = \sum_{k=-\infty}^{\infty} u(k) \cdot \mathrm{rect}\left(\frac{t - T/2 - kT}{T}\right)$$

Discretization Errors



Discretization — 1

1. Use the derivative rule:

$$\dot{x}(t) = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T}$$

2. You can use this approximation:

$$\frac{x(t+T) - x(t)}{T} = Ax(t) + Bu(t) \Rightarrow x(t+T) = x(t) + ATx(t) + BTu(t)$$

3. Hence,

$$x(t+T) = (I + AT)x(t) + BTu(t)$$

4. Now, if we compute $x(t)$ and $y(t)$ only at $t = kT$ for $k = 0, 1, \ldots,$ then the dynamical system equation for the discretized, approximate system is:

$$\begin{aligned} x((k+1)T) &= \underbrace{(I + AT)}_{\tilde{A}} x(kT) + \underbrace{BT}_{\tilde{B}} u(kt) \\ y(kT) &= \tilde{C}x(kT) + \tilde{D}u(kT) \end{aligned}$$

Discretization — 2

- The aforementioned discretization is a valid discretization for a continuous time system
- This method is based on forward Euler differentiation method
- Easily computed by the computer, i.e., no need for matrix exponentials—just simple computations
- While this discretization is the easiest, it's the least accurate
- Solution: a different discretization method

Another Discretization Method — 1

- Recall that if the input $u(t)$ is generated by a computer then followed by DAC, then $u(t)$ will be **piecewise constant**:

$$u(t) = u(kT) =: u(k) \qquad \text{for } \; kT \leq t \leq (k+1)T, \quad k = 0, 1, \ldots, k_f$$

- Note that this input only changes values at discrete time instants
- Recall the solution to the state-equation:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Setting $t = KT$ in the previous equation, then we can write:

$$x(k) := x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

$$x(k+1) := x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

Another Discretization Method — 2

$$x(k+1) := x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

- Note that the above equation can be written as:

$$x(k+1) = e^{AT}\left(e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right) + \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau$$

- Recall that we're assuming that:

$$u(t) = u(kT) =: u(k) \qquad \text{for } kT \le t \le (k+1)T, \quad k = 0, 1, \ldots, k_f$$

  i.e., the input is constant between two sampling instances
- Look at $x(k)$ and let $\alpha = kT + T - \tau$, then:

$$x(k+1) = e^{AT}x(k) + \left(\int_0^{T} e^{A\alpha}d\alpha\right)Bu(k)$$

Another Discretization Method — 3

$$x(k+1) = e^{AT}x(k) + \left(\int_0^T e^{A\alpha}d\alpha\right)Bu(k)$$

- Hence, the discretized system with sampling time-period T can be written as:

$$\begin{aligned} x(k+1) &= \tilde{A}x(k) + \tilde{B}u(k) \\ y(k) &= \tilde{C}x(k) + \tilde{D}u(k) \end{aligned}$$

where

$$\tilde{A} = e^{AT}, \tilde{B} = \left(\int_0^T e^{A\alpha}d\alpha\right)B, \tilde{C} = C, \tilde{D} = D$$

- Note that there is no approximation in this solution
- We only assumed that $u(k)$ is piecewise constant between the two sampling instances
- It's easy to compute the new discretized SS matrices (besides $\tilde{B}$)

Another Discretization Method — 4

- To compute $\tilde{B}$, you can simply evaluate the formula:

$$\tilde{B} = \left(\int_0^T e^{A\alpha} d\alpha\right) B = \left(\int_0^T I + A\alpha + \frac{1}{2}A^2\alpha^2 + \ldots d\alpha\right) B$$

- Which can be evaluated:

$$\tilde{B} = \left(TI + \frac{1}{2}T^2A + \frac{1}{3}T^3A^2 + \ldots\right) B$$

$$= A^{-1}\left(TA + \frac{1}{2}T^2A^2 + \frac{1}{3}T^3A^3 + \ldots\right) B$$

$$= A^{-1}\left(I + TA + \frac{1}{2}T^2A^2 + \frac{1}{3}T^3A^3 + \ldots - I\right) B$$

$$\Rightarrow \tilde{B} = A^{-1}(\tilde{A} - I)B$$

- This result is only valid for nonsingular A
- This formula helps in avoiding infinite series
- You can also use MATLAB's `c2d(A,B,...)` command

Examples

Discretize the following CT-LTI system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} u(t)$$

where the controller's sampling time is $T = 0.1\,\text{sec}$.
We can try the three approaches we learned:

1. Approach 1:

$$\tilde{A} = I + AT = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 1.2 \end{bmatrix}, \quad \tilde{B} = BT = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

2. Approach 2:

$$\tilde{A} = e^{AT} = \begin{bmatrix} e^{-T} & \frac{1}{3}(e^{2T} - e^{-T}) \\ 0 & e^{2T} \end{bmatrix} = \begin{bmatrix} 0.9048 & 0.1055 \\ 0 & 1.2214 \end{bmatrix}$$

$$\tilde{B} = \left(\int_0^T e^{A\alpha} d\alpha \right) B = \texttt{int(expm(A*h),h,0,T)*B} = \begin{bmatrix} 0.1903 & 0.0207 \\ 0 & 0.4428 \end{bmatrix}$$

Examples (Cont'd)

Approach 3:

$$\tilde{A} = e^{AT} = \begin{bmatrix} e^{-T} & \frac{1}{3}(e^{2T} - e^{-T}) \\ 0 & e^{2T} \end{bmatrix} = \begin{bmatrix} 0.9048 & 0.1055 \\ 0 & 1.2214 \end{bmatrix}$$

$$\tilde{B} = A^{-1}(\tilde{A} - I)B = \begin{bmatrix} 0.1903 & 0.0207 \\ 0 & 0.4428 \end{bmatrix}$$

What do we notice? What are some preliminary conclusions?

Remarks

- There's plenty of other discretization methods in the literature
- This question has no specific golden answer
- It often depends on the properties of the system
- Basically the sampling time period (how often your control is fixed or changing)
- The singularity of the A matrix also plays an important role

Questions And Suggestions?

Thank You!

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IFF you want to know more ☺