MODULE 1
Mechanical Measurements
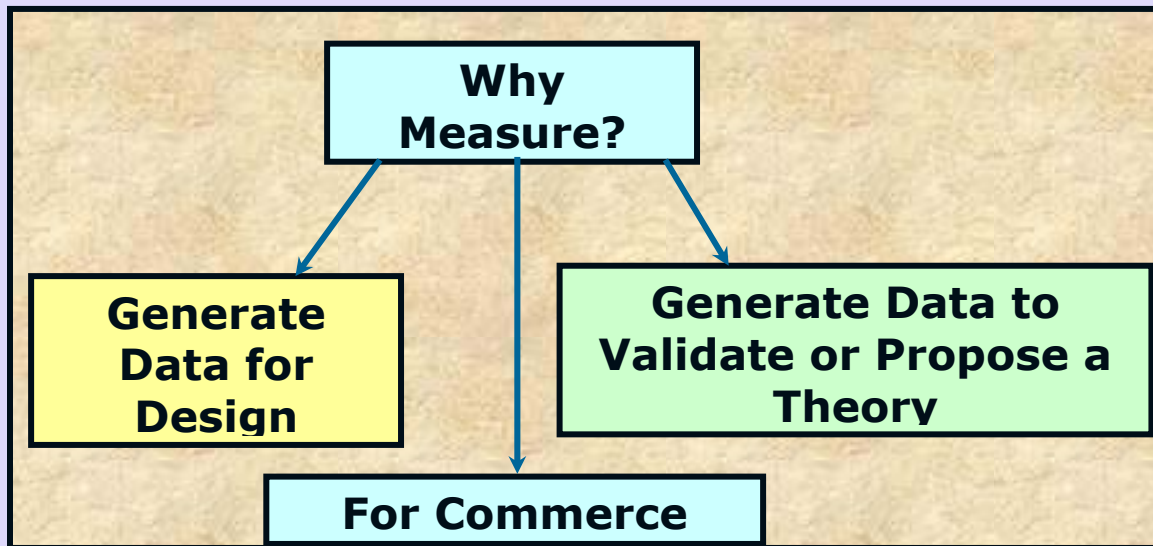
1. Introduction to Mechanical Measurements

Figure 1 Why make measurements?

We recognize three reasons for making measurements as indicated in Figure 1. From the point of view of the course measurements for commerce is outside its scope.

Engineers design physical systems in the form of machines to serve some specified functions. The behavior of the parts of the machine during the operation of the machine needs to be examined or analyzed or designed such that it functions reliably. Such an activity needs data regarding the machine parts in terms of material properties. These are obtained by performing measurements in the laboratory.

The **scientific method** consists in the study of nature to understand the way it works. Science proposes hypotheses or theories based on observations and need to be validated with carefully performed experiments that use many measurements. When once a theory has been established it may be used to make predictions which may themselves be confirmed by further experiments.

Measurement categories

1. *Primary quantity*
2. *Derived quantity*
3. *Intrusive – Probe method*
4. *Non-intrusive*

Measurement categories are described in some detail now.

1. Primary quantity:

It is possible that a **single quantity** that is **directly measurable** is of interest. An example is the measurement of the diameter of a cylindrical specimen. It is directly measured using an instrument such as vernier calipers. We shall refer to such a quantity as a **primary quantity**.

2. Derived quantity:

There are occasions when a quantity of interest is not **directly measurable** by a single measurement process**.** The quantity of interest needs to be estimated by using an appropriate relation involving several measured primary quantities. The measured quantity is thus a **derived quantity**. An example of a derived quantity is the determination of acceleration due to gravity (g) by finding the period (T) of a simple pendulum of length (L). T and L are the measured primary quantities while g is the derived quantity.

3. Probe or intrusive method:

Most of the time, the measurement of a physical quantity uses a **probe** that is placed inside the system. Since a probe invariably affects the measured quantity the measurement process is referred to as an **intrusive** type of measurement.

4. Non-intrusive method:

When the measurement process does not involve insertion of a probe into the system the method is referred to as being **non-intrusive**. Methods that use some naturally occurring process like radiation emitted by a body is used to measure a desired quantity relating to the system the method may be considered as non-intrusive.   The measurement process may be **assumed to be non-intrusive** when the probe has negligibly small interaction with the system. A typical example for such a process is the use of laser Doppler velocimeter (LDV) to measure the velocity of a flowing fluid.
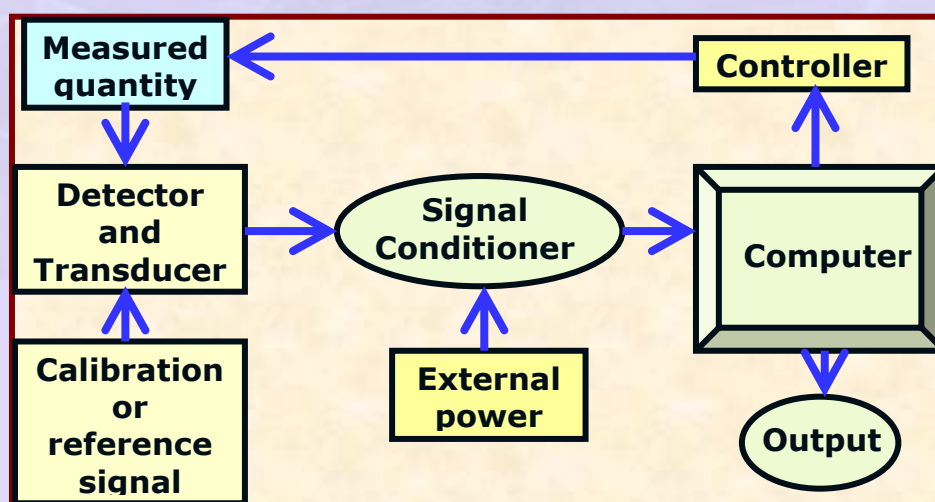
General measurement scheme:

Figure 2 Schematic of a general measurement system

Figure 2 shows the schematic of a general measurement scheme. Not all the elements shown in the Figure may be present in a particular case. The measurement process requires invariably a **detector** that responds to the measured quantity by producing a measurable change in some property of the **detector**. The change in the property of the detector is converted to a measurable output that may be either mechanical movement of a pointer over a scale or an electrical output that may be measured using an appropriate electrical circuit. This action of converting the measured quantity to a different form of output is done by a **transducer.** The output may be manipulated by a signal conditioner before it is recorded or stored in a computer. If the measurement process is part of a control application the computer can use a controller to control the measured quantity. The relationship that exists between the measured quantity and the output of the transducer may be obtained by **calibration** or by **comparison** with a **reference value**. The measurement system requires external power for its operation.

Some issues:

1. *Errors – <u>Systematic</u> or <u>Random</u>*
2. *Repeatability*
3. *Calibration and Standards*
4. *Linearity or Linearization*

Any measurement, however carefully it is conducted, is subject to measurement **errors**. These errors make it difficult to ascertain the **true value** of the measured quantity. The nature of the error may be ascertained by **repeating** the measurement a number of times and looking at the spread of the values. If the spread in the data is small the measurement is **repeatable** and may be

termed as being good. If we compare the measured quantity obtained by the use of **any instrument** and compare it with that obtained by a **standardized instrument** the two may show different performance as far as the **repeatability** is concerned.   If we add or subtract a certain correction to make the two instruments give data with similar spread the correction is said to constitute a **systematic error**. The spread of each of the instruments will constitute **random** error.

The process of ascertaining the systematic error is **calibration**. The response of a detector to the variation in the in the measured quantity may be **linear** or **non-linear**. In the past the tendency was to look for a linear response as the desired response. Even when the response of the detector was non-linear the practice was to make the response linear by some manipulation. With the advent of automatic recording of data using computers this practice is not necessary since software can take care of this aspect.

Sub Module 1.2

2. Errors in measurements

Errors accompany any measurement, however well it is conducted. The error may be inherent in the measurement process or it may be induced due to variations in the way the experiment is conducted. The errors may be classified as:

1. Systematic errors (Bias):

Systematic errors due to faulty or improperly calibrated instruments. These may be reduced or eliminated by careful choice and calibration of instruments. Sometimes bias may be linked to a specific cause and estimated by analysis. In such a case a correction may be applied to eliminate or reduce bias. Bias is an indication of the accuracy of the measurement. Smaller the bias more accurate the data

2. Random errors:

Random errors are due to non-specific causes like natural disturbances that may occur during the measurement process. These cannot be eliminated. The magnitude of the spread in the data due to the presence of random errors is a measure of the precision of the data. Smaller the random error more precise is the data. Random errors are statistical in nature. These may be characterized by statistical analysis.

We shall explain these through the familiar example shown in Figure 3. Three different individuals with different skill levels are allowed to complete a round of target practice. The outcome of the event is shown in the figure.
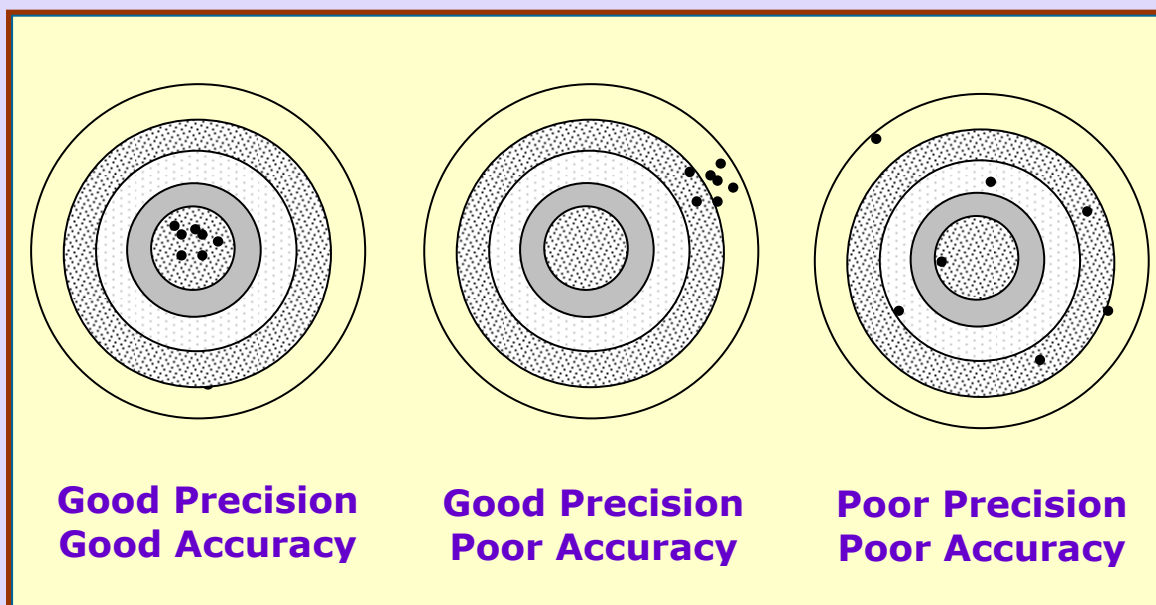

Good Precision Good Accuracy **Good Precision Poor Accuracy** **Poor Precision Poor Accuracy**

Figure 3 Precision and accuracy explained through a familiar example

It is evident that the target at the left belongs to a highly skilled shooter. This is characterized by all the shots in the inner most circle. The result indicates good accuracy as well as good precision. A measurement made well must be like this case! The individual in the middle is precise but not accurate. Maybe it is due to a faulty bore of the gun. The individual at the right is an unskilled person who is behind on both counts. Most beginners will fall into this category. The analogy is quite realistic since most students performing a measurement in the laboratory may be put into one of the three categories. A good experimentalist has to work hard to excel in it!
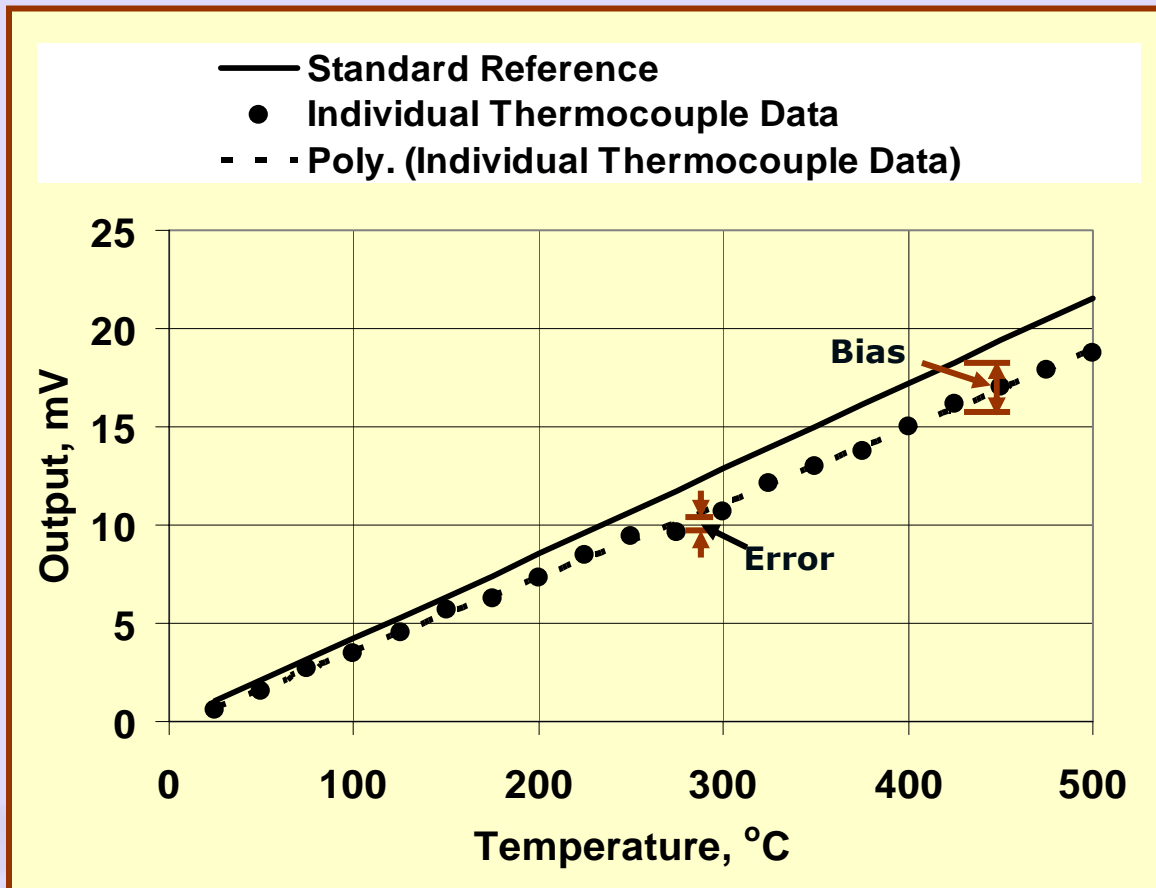
Another example:

Figure 4 Example showing the presence of systematic and random errors in data.

The results shown in Figure 4 compare the response of a particular thermocouple (that measures temperature) and a standard thermocouple. The measurements are reported between room temperature (close to 20°C and 500°C. That there is a systematic variation between the two is clear from the figure that shows the trend of the measured temperatures indicated by the particular thermocouple. The systematic error appears to vary with the

temperature. The data points indicated by the full symbols appear also to hug the trend line. However the data points do not lie on it. This is due to random errors that are always present in any measurement. Actually the standard thermocouple would also have the random errors that are not indicated in the figure. We have deliberately shown only the trend line for the standard thermocouple.

Sub Module 1.3

3. Statistical analysis of experimental data

Statistical analysis and best estimate from replicate data:

- Let a certain quantity X be measured repeatedly to get

$$X_i, \ i=1,n \qquad\qquad (1)$$

- Because of random errors these are all *different*.
- How do we find the best estimate X_b for the true value of X?
- It is reasonable to assume that the best value be such that the measurements are as precise as they can be!
- In other words, the experimenter is confident that he has conducted the measurements with the best care and he is like the skilled shooter in the target practice example presented earlier!
- Thus, we minimize the variance with respect to the best estimate X_b of X.
- Thus we minimize:

$$S = \sum_{i=1}^{n} \left[X_i - X_b \right]^2 \qquad\qquad (2)$$

❏ This requires that:

$$\frac{\partial S}{\partial X_b} \; 2\sum_{i=1}^{n}\left[X_i - X_b\right]\,(-1) = 0$$

(3)

$$\text{or } X_b = \frac{\sum_{i=1}^{n}X_i}{n}$$

❏ The best estimate is thus nothing but the mean of all the individual measurements!

Error distribution:

When a quantity is measured **repeatedly** it is expected that it will be distributed around the best value according to some **distribution**. Many times the random errors may be distributed as a **normal distribution**. If μ and σ are the mean and the standard deviation, then, the probability density is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

(4)

The probability that the error around the mean is $(x-\mu)$ is the area under the probability density function between $(x-\mu)+dx$ and $(x-\mu)$ represented by the product of the probability density and dx. The probability that the error is anywhere between $-\infty$ and x is thus given by the following integral:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left[\frac{v-\mu}{\sigma}\right]^2}dv$$

(5)

This is referred to as the **cumulative probability**. It is noted that if $x\rightarrow\infty$ the integral tends to 1. Thus the probability that the error is of all possible magnitudes (between $-\infty$ and $+\infty$) is unity! The integral is **symmetrical** with

respect to x=μ as may be easily verified. The above integral is in fact the **error integral** that is a tabulated function. A plot of f(x) and F(x) is given in Figure 5.
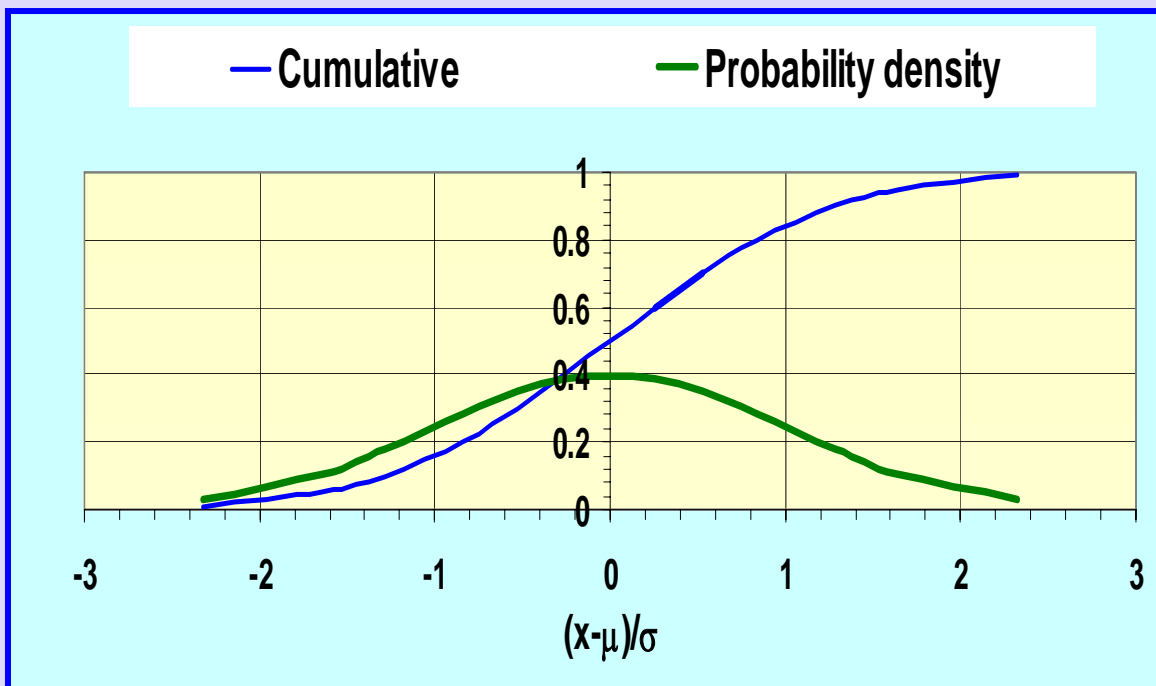

Figure 5 Normal distribution and its integral

Many times we are interested in finding out the chances of error lying between two values in the form ±**p**σ. This is referred to as the "confidence interval" and the corresponding cumulative probability specifies the chances of the error occurring within the confidence interval. Table 1 gives the confidence intervals that are useful in practice:

Table 1

Confidence intervals according to normal distribution

Cumulative Probability	0	0.95	0.99	0.999
Interval p	0	± 1.96	± 2.58	± 3.29

The table indicates that error of magnitude greater than $\pm 3.29\sigma$ is very unlikely to occur. In most applications we specify **$\pm 1.96\sigma$** as the error bounds based on **95%** confidence.

Example 1

⊙ Resistance of a certain resistor is measured repeatedly to obtain the following data.

No.	1	2	3	4	5	6	7	8	9
R, kΩ	1.22	1.23	1.26	1.21	1.22	1.22	1.22	1.24	1.19

⊙ What is the best estimate for the resistance? What is the error with 95% confidence?

⊙ Best estimate is the mean of the data.

$$\overline{R} = \frac{1.22 \times 4 + 1.23 + 1.26 + 1.21 + 1.24 + 1.19}{9}$$

$$= 1.223 \approx 1.22 \text{ k}\Omega$$

⊙ Standard deviation of the error σ:

$$\text{Variance} = \frac{1}{9}\sum_{1}^{9}\left[Ri - \overline{R}\right]^2$$

$$= 3.33 \times 10^{-4}$$

$$\text{Hence}:$$

$$\sigma = \sqrt{3.33 \times 10^{-4}}$$

$$= 0.183 \approx 0.02 \text{ k}\Omega$$

⊙ Error with 95% confidence :

$$\text{Error}_{95\%} = 1.96\sigma = 1.96 \times 0.0183$$

$$= 0.036 \approx 0.04 \text{ k}\Omega$$

Thickness of a metal sheet (in mm) is measured repeatedly to obtain the following replicate data. What is the best estimate for the sheet thickness? What is the variance of the distribution of errors with respect to the best value? Specify an error estimate to the mean value based on 99% confidence.

Experiment No.	1	2	3	4	5	6
t, mm	0.202	0.198	0.197	0.215	0.199	0.194
Experiment No.	7	8	9	10	11	12
t, mm	0.204	0.198	0.194	0.195	0.201	0.202

⊙ The best estimate for the metal sheet thickness is the mean of the 12 measured values. This is given by

$$t_b = \overline{t} = \frac{\sum_1^{12} t_i}{12} = \frac{\begin{bmatrix} 0.202 + 0.198 + 0.197 + 0.215 + 0.199 + 0.194 + 0.204 \\ +0.198 + 0.194 + 0.195 + 0.201 + 0.202 \end{bmatrix}}{12} = 0.2 \text{ mm}$$

⊙ The variance with respect to the mean or the best value is given by (on substituting $\overline{t}$ for t_b) as

$$\sigma^2_b = \frac{\sum_1^{12}[t_i - \overline{t}]^2}{12} = \frac{\sum_1^{12} t_i}{12} - \overline{t}^2$$

⊙
$$= \frac{\begin{bmatrix} 0.202^2 + 0.198^2 + 0.197^2 + 0.215^2 + 0.199^2 + 0.194^2 + 0.204^2 \\ +0.198^2 + 0.194^2 + 0.195^2 + 0.201^2 + 0.202^2 \end{bmatrix}}{12} - 0.2 \text{ mm}$$

$$= 3.04 \times 10^{-5} \text{mm}^2$$

⊙ The corresponding standard deviation is given by

$$\sigma_b = \sqrt{3.04 \times 10^{-5}} = 0.0055 \approx 0.006 \text{ mm}$$

⊙ The corresponding error estimate based on 99% confidence is

$$\text{Error} = \pm 2.58\sigma_b = \pm 2.58 \times 0.0055 \approx \pm 0.014 \text{ mm}$$

Principle of Least Squares

Earlier we have dealt with the method of obtaining the best estimate from replicate data based on **minimization of variance**. No mathematical proof was given as a basis for this. We shall now look at the above afresh, in the light of the error distribution that has been presented above.

Consider a set of replicate data x_i. Let the best estimate for the measured quantity be x_b. The probability for a certain value x_i within the interval $x_i, x_i + dx_i$ to occur in the measured data is given by the relation

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_b - x_i)^2}{2\sigma^2}} dx_i \qquad \textbf{(6)}$$

The probability that the particular values of measured data are obtained in replicate measurements must be given by the compound probability given by

$$p = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_{i=1}^{n} e^{-\frac{(x_b - x_i)^2}{2\sigma^2}} dx_i = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} e^{-\sum_{i=1}^{n}\frac{(x_b - x_i)^2}{2\sigma^2}} \prod_{i=1}^{n} dx_i \qquad \textbf{(7)}$$

The reason the set of data was obtained as replicate data is that it was the **most probable**! Since the intervals dx_i are arbitrary, the above will have to be maximized by the proper choice of x_b and σ such that the exponential factor is a maximum. Thus we have to choose x_b and σ such that

$$p' = \frac{1}{\sigma^n} e^{-\sum_{i=1}^{n}\frac{(x_b - x_i)^2}{2\sigma^2}} \qquad \textbf{(8)}$$

has the largest possible value. As usual we set the derivatives $\dfrac{\partial p'}{\partial x_b} = \dfrac{\partial p'}{\partial \sigma} = 0$ to

get the values of the two parameters x_b and σ. We have:

$$\frac{\partial p'}{\partial x_b} = -\frac{1}{2\sigma^{n+2}} e^{\sum\limits_{i=1}^{n} \frac{(x_i - x_b)^2}{2\sigma^2}} \underbrace{\sum_{i=1}^{n} 2(x_i - x_b)(-1)}_{\text{This part should go to zero}} = 0 \qquad (9)$$

Or

$$\sum_{i=1}^{n}(x_i - x_b) = 0 \text{ or } x_b = \sum_{i=n}^{n} x_i = \overline{x} \qquad (10)$$

It is clear thus that the best value is nothing but the mean of the values! We also

have:

$$\frac{\partial p'}{\partial \sigma} = \underbrace{\left[-\frac{n}{\sigma^{n+1}} + \frac{1}{\sigma^{n+3}} \sum_{i=0}^{n}(x_i - x_b)^2 \right]}_{\text{This part should go to Zero}} e^{\sum\limits_{i=1}^{n} \frac{(x_i - x_b)^2}{2\sigma^2}} = 0 \qquad (11)$$

Or

$$\sigma^2 = \frac{\sum\limits_{i=1}^{n}(x_i - x_b)^2}{n} \qquad (12)$$

This last expression indicates that the parameter σ^2 is nothing but the variance of the data with respect to the mean! Thus the best values of the measured quantity and its spread is based on the minimization of the squares of errors with respect to the mean. This embodies what is referred to as the **"Principle of Least Squares"**.

Propagation of errors:

Replicate data collected by measuring a single quantity enables us to calculate the best value and characterize the spread by the variance with respect to the best value using the principle of least squares. Now we look at the case of a **derived quantity** that is estimated from the measurement of several **primary** quantities. The question that needs to be answered is the following:

"A derived quantity Q is estimated using a formula that involves the primary quantities. $a_1, a_2, ...a_n$ Each one of these is available in terms of the respective best values $\bar{a}_1, \bar{a}_2, ...\bar{a}_n$ and the respective variances $\sigma_1, \sigma_2 ... \sigma_n$. What is the best estimate for Q and what is the corresponding variance σ_Q σ_Q ?"

We have, by definition

$$Q = Q(a_1, a_2, ...a_n) \tag{13}$$

It is obvious that the best value of Q should correspond to that obtained by using the **best values** for the a's. Thus, the best estimate for Q given by $\bar{Q}$ as

$$\bar{Q} = Q(\bar{a}_1, \bar{a}_2, ...\bar{a}_n) \tag{14}$$

Again, by definition, we should have:

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left(Q_i - \bar{Q}\right)^2 \tag{15}$$

The subscript i indicates the experiment number and the ith estimate of Q is given by

$$Q_i = Q\left(a_{1i}, a_{2i}, ...a_{ni}\right) \tag{16}$$

If we assume that the spread in values are small compared to the mean or the best values (this is what one would expect from a well conducted experiment), the difference between the ith estimate and the best value may be written using a Taylor expansion around the best value as

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=2}^{N}\left(\frac{\partial Q}{\partial a_1}\Delta a_{1i} + \frac{\partial Q}{\partial a_2}\Delta a_{2i} + ... + \frac{\partial Q}{\partial a_2}\Delta a_{ni}\right)^2 \qquad \textbf{(17)}$$

where the partial derivatives are all evaluated at the best values for the a's. If the

a's are all **independent** of one another then the errors in these are unrelated to

one another and hence the cross terms. $\sum_{i=1}^{N}\Delta a_{mi}\Delta a_{ki} = 0$ for $m \neq k$ Thus equation

(17) may be rewritten as

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{\partial Q}{\partial a_1}\Delta a_{1i}\right)^2 + \left(\frac{\partial Q}{\partial a_2}\Delta a_{2i}\right)^2 + ... + \left(\frac{\partial Q}{\partial a_n}\Delta a_{ni}\right)^2\right] \qquad \textbf{(18)}$$

Noting that $\sum_{i=1}^{N}\left(\Delta a_{ji}\right)^2 = N\sigma_j^2$ we may recast the above equation in the form

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial a_1}\right)^2\sigma_1^2 + \left(\frac{\partial Q}{\partial a_2}\right)^2\sigma_2^2 + ... + \left(\frac{\partial Q}{\partial a_n}\right)^2\sigma_n^2 \qquad \textbf{(19)}$$

Equation (19) is the error propagation formula. It may also be recast in the form

$$\sigma_Q = \sqrt{\left(\frac{\partial Q}{\partial a_1}\right)^2\sigma_1^2 + \left(\frac{\partial Q}{\partial a_2}\right)^2\sigma_2^2 + ... + \left(\frac{\partial Q}{\partial a_n}\right)^2\sigma_n^2} \qquad \textbf{(20)}$$

Example 3

The volume of a sphere is estimated by measuring its diameter by vernier calipers. In a certain case the diameter has been measured as D = 0.0502 $\pm$ 0.00005 m. Determine the volume and specify a suitable uncertainty for the same.

Nominal volume of sphere:

$$V = \pi \frac{D^3}{6} = 3.14159 \times \frac{0.0502^3}{6} = 6.624 \times 10^{-5} \, m^3$$

⊙ The error in the measured diameter is specified as:

$$\Delta D = \pm 0.00005 m$$

⊙ The influence coefficient is defined as

$$I_D = \frac{\partial V}{\partial D} = \pi \frac{D^2}{2} = 3.14159 \times \frac{0.0502^2}{2} = 3.958 \times 10^{-3} \, m^2$$

⊙ Using the error propagation formula, we have

$$\Delta V = I_D \Delta D = 3.958 \times 10^{-3} \times 0.00005 = 1.979 \times 10^{-7} \, m^3$$

⊙ Thus

$$V = 6.624 \times 10^{-5} \pm 1.979 \times 10^{-7} \, m^3$$

Alternate solution to the problem

⊙ By logarithmic differentiation we have

$$\frac{dV}{V} = 3 \frac{dD}{D}$$

⊙ This may be recast as

$$\Delta V = \pm 3V \frac{\Delta D}{D} = \pm 3 \times 6.624 \times 10^{-5} \times \frac{0.00005}{0.0502} = \pm 0.0198 \times 10^{-5} m^3$$

This is the same as the result obtained earlier.

Example 4

Two resistances R_1 and R_2 are given as 1000 ± 25 and 500 ± 10. Determine the equivalent resistance when these two are connected in a) series and b) parallel. Also determine the uncertainties in these two cases.

- Given Data:

$$R_1 = 1000, \sigma_1 = 25; R_2 = 500 \; \sigma_2 = 10 \rightarrow \text{All Values are in } \Omega$$

  Case a) Resistances connected in series:

- Equivalent resistance is

$$R_s = R_1 + R_2 = 1000+500=1500\Omega$$

- Influence coefficients are:

$$I_1 = \frac{\partial R_s}{\partial R_1} = 1 \; ; I_2 = \frac{\partial R_s}{\partial R_2} = 1$$

- Hence the uncertainty in the equivalent resistance is

$$\sigma_s = \pm \sqrt{(I_1\sigma_1)^2 + (I_2\sigma_2)^2} = \pm \sqrt{(25)^2 + (10)^2} = \pm 26.93 \; \Omega$$

Case b) Resistances connected in parallel:

- Equivalent resistance is given by

$$R_p = \frac{R_1 R_2}{R_1 + R_2} = \frac{1000 \times 500}{1000 + 500} = 333.3 \; \Omega$$

$$R_p = \frac{R_1 R_2}{(R_1 + R_2)}$$

⊙ Influence coefficients are:

$$I_1 = \frac{\partial R_p}{\partial R_1} = \frac{R_2}{(R_1 + R_2)} - \frac{R_1 R_2}{(R_1 + R_2)^2} = \frac{500}{1500} - \frac{1000 \times 500}{1500} = 0.111$$

$$I_1 = \frac{\partial R_p}{\partial R_2} = \frac{R_1}{(R_1 + R_2)} - \frac{R_1 R_2}{(R_1 + R_2)^2} = \frac{1000}{1500} - \frac{1000 \times 500}{1500^2} = 0.444$$

⊙ Hence the uncertainty in the equivalent resistance is

$$\sigma_s = \pm \sqrt{(I_1\sigma_1)^2 + (I_2\sigma_2)^2} = \pm \sqrt{(0.111 \times 25)^2 + (0.444 \times 10)^2} = \pm 5.24 \; \Omega$$

Thus the equivalent resistance is 1500 ± 26.9 in the series arrangement

and 333.6 ± 5.24 in the parallel arrangement.

Error estimation – some results without proof

Standard deviation of the means

The problem occurs as indicated below:

- **Replicate** *data is collected with **n** measurements in a set*
- *Several such sets of data are collected*
- *Each one of them has a **mean** and a **variance** (precision)*
- *What is the mean and standard deviation of the means of all sets?*

Population mean

Let **N** be the total number of data in the entire population. Mean of all the sets **m** will be nothing but the population mean (i.e. the mean of all the collected data taken as a whole).

Population variance

Let the population variance be

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - m)^2}{N} \tag{21}$$

Variance of the means

Let the variance of the means be σ_m^2. Then we can show that:

$$\sigma_m^2 = \frac{(N-n)}{n(N-1)}\sigma^2 \tag{22}$$

If n<<N the above relation will be approximated as

$$\sigma_m^2 = \frac{(N-n)}{n(N-1)}\sigma^2$$

$$= \frac{(1\text{-}n/N)}{n(1\text{-}1/N)}\sigma^2 \approx \frac{\sigma^2}{n}$$

(23)

Estimate of variance

- *Sample and its variance*

    - *How is it related to the population variance?*

- *Let the sample variance from its own mean ms be se2.*

- *Then we can show that:*

$$\sigma_e^2 = \frac{(N-n)}{n(N-1)}\sigma^2 \approx \sigma^2\left(1-\frac{1}{n}\right)$$

(24)

Error estimator

The last expression may be written down in the more explicit form:

$$\sigma_e^2 = \frac{\sum\limits_{1}^{n}(x_i\text{-}m_s)^2}{(n\text{-}1)}$$

(25)

Physical interpretation

Equation (25) may be interpreted using physical arguments. Since the **mean** (the best value) is obtained by one use of **all** the available data, the **degrees of freedom** available (**units of information available**) is **one** less than before. Hence the error estimator uses the factor **(n-1)** rather than **n** in the denominator!

Example 5
(Example 1 revisited)

⊙ Resistance of a certain resistor is measured repeatedly to obtain the following data.

#	1	2	3	4	5	6	7	8	9
R, k	1.22	1.23	1.26	1.21	1.22	1.22	1.22	1.24	1.19

⊙ What is the best estimate for the resistance? What is the error with 95% confidence?

⊙ Best estimate is the mean of the data.

$$\bar{R} = \frac{1.22 \times 4 + 1.23 + 1.26 + 1.21 + 1.24 + 1.19}{9}$$
$$= 1.223 \approx 1.22 \text{ k}\Omega$$

⊙ Standard deviation of the error se:

$$\sigma_e^2 = \frac{1}{8} \sum_1^9 \left[R_i - \bar{R} \right] = 3.75 \times 10^{-4}$$

⊙ Hence

$$\sigma_e^2 = \sqrt{3.75 \times 10^{-4}} = 0.019 \approx 0.02 \text{k}\Omega$$

⊙ Error with 95% confiden

Heat flux

1. Measurement of heat flux

Heat flux is defined as the amount of heat transferred per unit area per unit time from or to a surface. In a basic sense it is a derived quantity since it involves, in principle, two quantities viz. the amount of heat transfer per unit time and the area from/to which this heat transfer takes place. In practice, the heat flux is measured by the change in temperature brought about by its effect on a sensor of known area. The incident heat flux may set up either a steady state temperature field or a transient temperature field within the sensor. The temperature field set up may either be perpendicular to the direction of heat flux or parallel to the direction of heat flux. We study the various types of heat flux gages in what follows.

2. Foil type heat flux gage:

The foil type heat flux gage (also known as the Gardon gage after its inventor) consists of a thin circular foil of constantan stretched tightly over a cooled copper annulus as shown in Figure 1. One surface of the foil is exposed to the heat flux that is to be measured while the other surface may be taken as insulated. A copper wire is attached at the geometric center of the foil as indicated in the figure. A second copper wire is attached to the cooled copper annulus. The constantan foil forms two junctions with copper, the first one at its center and the second one at its periphery. Under steady state, the thermoelectric voltage across the two copper leads is a direct measure of the temperature difference set up between the center and the periphery of the

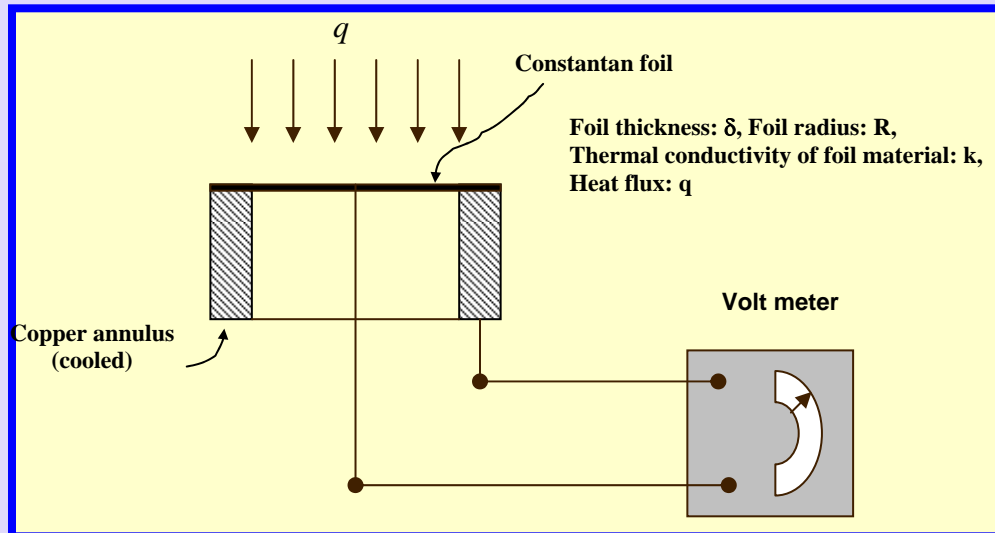constantan disk. The temperature difference is obtained by performing the following analysis.

Figure 1 Schematic of a foil type heat flux gage

Heat balance for an annular element of the foil shown in Figure 2 is made as follows:

Heat gained by the foil element is $(q)(2\pi r dr)$

Net heat conducted in to the foil element $(2\pi k\delta)\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right)dr$

Sum of these should be zero. We cancel the common factor $2\pi\,dr$ to get

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q}{k\delta}r=0 \qquad\qquad \textbf{(1)}$$

The boundary conditions are

$$T \text{ is finite at } r=0;\ T=T_R \text{ at } r=R \qquad\qquad \textbf{(2)}$$

Integrate equation 1 once with respect to *r* to get $r\dfrac{dT}{dr}+\dfrac{q}{k\delta}\dfrac{r^2}{2}=A$, where A is a

constant of integration. This may be rearranged to get $\dfrac{dT}{dr}+\dfrac{q}{2k\delta}r=\dfrac{A}{r}$. Integrate

this once more with respect to r to get $T + \dfrac{qr^2}{4k\delta} = A \ln(r) + B$ where *B* is a second

constant of integration. The constant *A* has to be chosen equal to zero in order

that the solution does not diverge at. $r = 0$. The constant *B* is obtained from the

boundary condition at r = R as $B = T_R + \dfrac{qR^2}{4k\delta}$. With this the solution for the
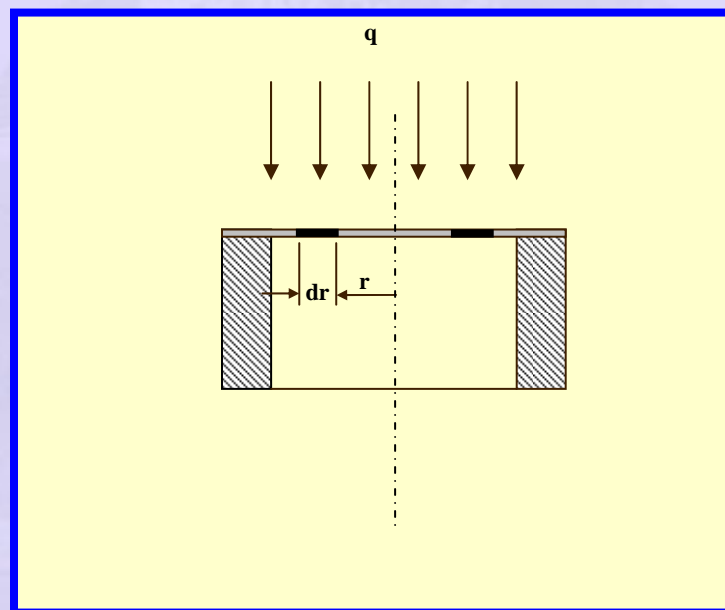
temperature is obtained as

Figure2 Energy balance over a foil element in the form of an annular ring

$$T = T_R + \frac{q}{4k\delta}\left(R^2 - r^2\right)$$ (3)

It may be noted that the constant *B* is nothing but the temperature at the center

of the constantan disk. In view of this, equation 3 may be recast as

$$q = \frac{T_0 - T_R}{\left(R^2 / 4k\delta\right)} = K(T_0 - T_R) = K\Delta T$$ (4)

In the above T_0 is the temperature at the center of the disk and the coefficient K is the gage constant given by $K = \dfrac{4k\delta}{R^2}$ W/m^2K. The temperature difference between the center of the disk and the periphery is the output that appears as a proportional voltage ΔV across the terminals of the differential thermocouple. There is thus a linear relationship between the heat flux and the output of the heat flux gage.

Example 1

A typical gage may be constructed using a 6 mm diameter foil of $50\,\mu m$ thickness. The thermal conductivity of the foil material is typically $k = 20$ $W/m°C$. Copper Constantan thermocouple pair gives an output of about 40 $\mu V/°C$. The given data corresponds to.

$$\delta = 50\,\mu m = 50 \times 10^{-6}\ m,\ R = 3\,mm = 0.003\ m,\ k = 20\ W/m°C$$

The gage constant then works out to $K = \dfrac{4 \times 20 \times 5 \times 10^{-5}}{0.003^2} = 444.4$ $W/m^2°C$. This may be rewritten in terms of the thermocouple output

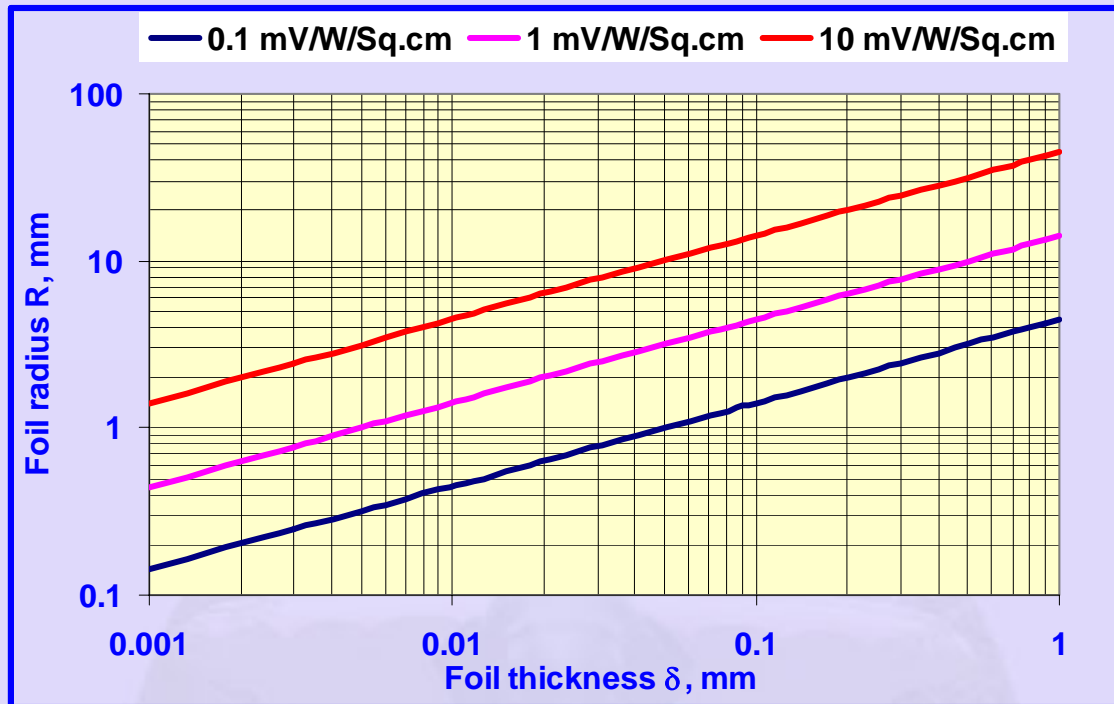as $K = \dfrac{444.44}{40} = 11.1$ $W/m^2\mu V$.

Figure 3 Proportions of foil gage

Figure 3 gives a plot that is useful in finding the R,   combinations that will have

a given sensitivity. The sensitivity (1/K) is specified in units of $\dfrac{mV}{\left(\dfrac{W}{cm^2}\right)}$ with both

R and   being in mm.

Transient Analysis of the Foil Gage:

Under steady state we have seen that the temperature distribution in the

foil is given by a quadratic variation with respect to r.

In fact we have:

$$T - T_R = \frac{q}{4k\delta}\left(R^2 - r^2\right) \tag{5}$$

With $$T_0 - T_R = \frac{qR^2}{4k\delta} \tag{6}$$

From these, by division we have

$$\frac{(T - T_R)}{(To - T_R)} = \left[1 - \left(\frac{r}{R}\right)^2\right].$$

(7)

In the steady sate the energy stored in the foil is given by

$$E = \int_0^R \left[\rho C_p \delta \, 2\pi r dr\right]\left[T - T_R\right]$$

Using equation (5) this is recast as

$$E = 2\rho C_p \delta \, \pi\left[T_o - T_R\right]\int_0^R r\left[1 - \left(\frac{r}{R}\right)^2\right]dr$$

(8)

But $\quad \int_0^R r\left[1 - \left(\frac{r}{R}\right)^2\right]dr = \left(\frac{R^2}{2} - \frac{R^2}{4}\right) = \frac{R^2}{4}.$

$$\therefore E = \frac{2\rho C_p \delta R^2 \left[T_o - T_R\right]}{2} = \rho C_p \delta R^2 \left[T_o - T_R\right]$$

(9)

Consider now the unsteady state heat transfer in the foil. The input heat flux is partially stored in the foil and partially removed by the coolant at the foil periphery. The stored energy is the change in E with respect to time given by the above expression.

Thus, $\dfrac{dE}{dt} = \rho C_p \delta\pi R^2 \dfrac{d(T_o - T_R)}{dt}$ the heat loss at r = R, is in fact given by the instantaneous conductive heat transfer at the periphery,

$$-2\pi R k\delta\frac{dT}{dr}\bigg|_{r = R} = \frac{q}{4k\delta}(-2R)(-2\pi R k\delta) = \pi R^2 q$$

$$= \pi R^2 4k\delta\frac{\left(T_o - T_R\right)}{R^2} = \pi 4k\delta\left(T_o - T_R\right)$$

Thus we have

$$\rho C_p \delta\pi R^2 \cdot \frac{d(T_o - T_R)}{dt} + 4\pi\delta k\left(T_o - T_R\right) - \pi R^2 q = 0.$$

Or

$$\underbrace{\frac{d\left(T_o - T_R\right)}{dt} + \frac{4k\left(T_o - T_R\right)}{\rho C_p R^2}} - \frac{2q}{\rho C_p f} = 0 \qquad\qquad \textbf{(10)}$$

These two terms determine
first order system behaviour

The time constant is identified as

$$\tau = \frac{\rho C_p . R^2}{4k} = \frac{R^2}{4\alpha} \qquad\qquad \textbf{(11)}$$

Where α is the thermal diffusivity of the foil material.

Figure 4 shows the relationship (11) between the foil radius and the first order time constant of the sensor in graphical form. This figure may be used in tandem with Figure 3 for determining the proportions and the time constant for the chosen dimensions of the gage. Figure 5 shows the actual view of a commercially available heat flux gage which has arrangement for cooling of the cylinder by water. The figure also gives the typical ranges of gage available and their characteristics, the sensitivity and time constant. The illustration has been taken from the manufacturer's web site indicated therein.
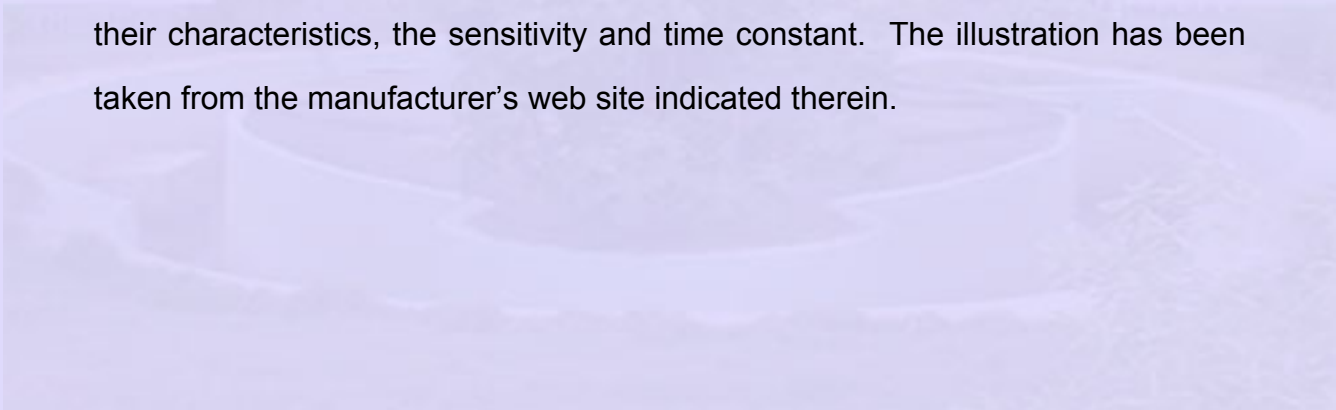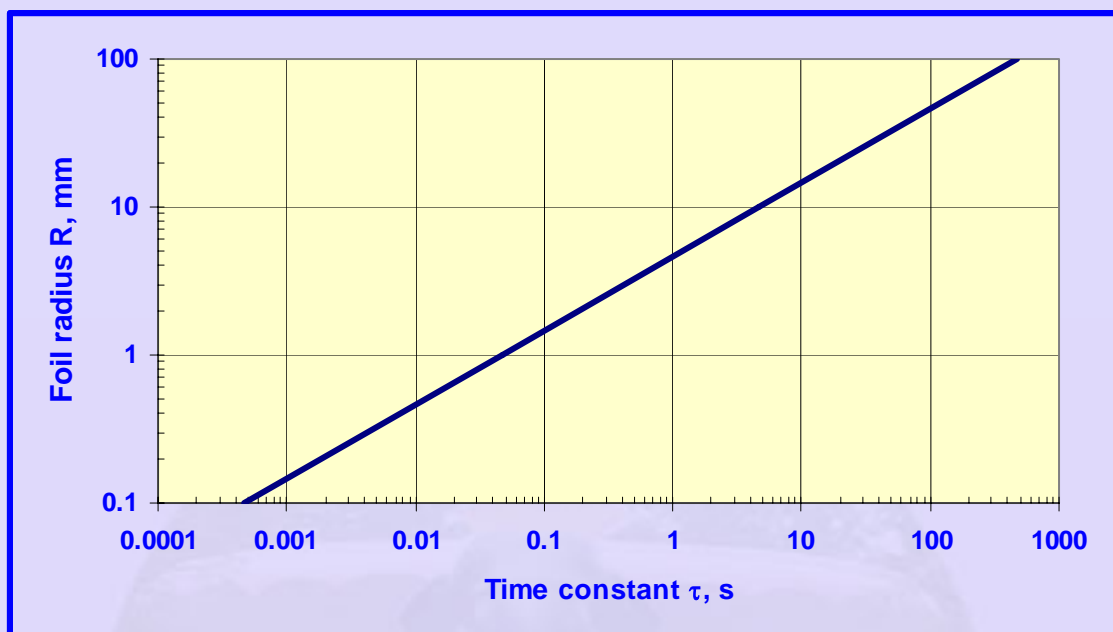
Figure 4 Time constant for a foil type heat flux gage

Figure 5 Commercial foil type heat flux gag

Thin film sensors:

q

**Thermal barrier
of known k:
Thickness = 0.25 mm,
k = 0.2 W/m °C
Kapton® (polyimide)**

ΔT

**Differential temperature measured using
40 junctions (thin foil thermocouple)
Maximum temperature is limited to 200°C
for this material.**

**Bonded on to the
surface over which
heat flux is incident.**
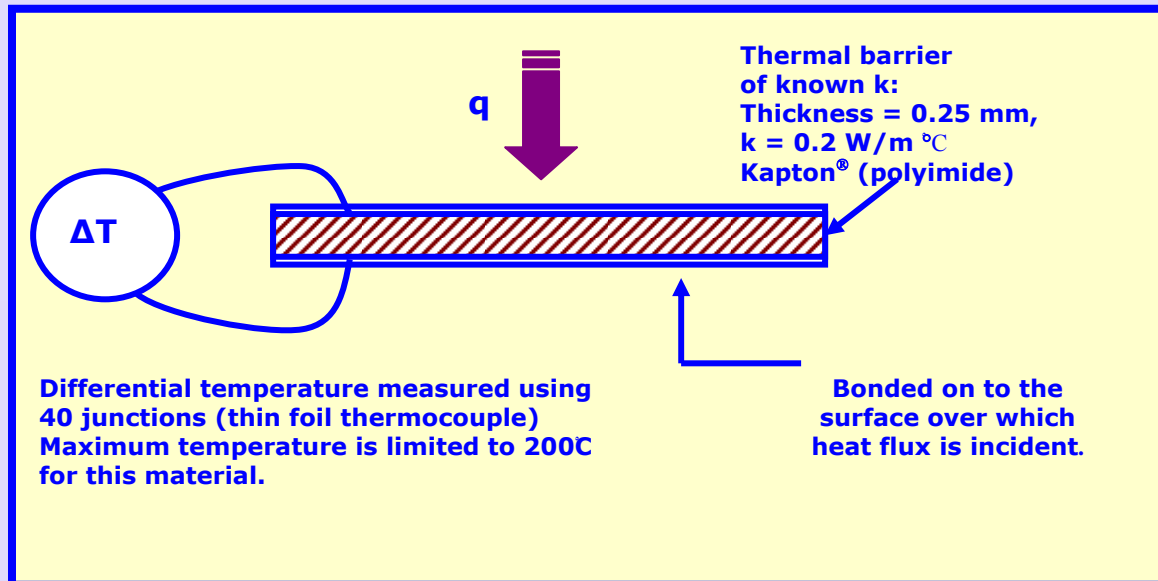
Figure 6 Schematic of a thin film heat flux sensor

The operation of a thin film heat flux sensor, shown schematically in Figure 6, is very simple. A thin barrier of known thermal conductivity is attached to a surface that is receiving the heat flux to be measured. The barrier imposes a thermal resistance parallel to the direction of the heat flux and the heat conduction in the barrier is one-dimensional. The temperature difference across the barrier is measured using a thermopile arrangement wherein several hot and cold junctions are connected in opposition. The output is proportional to the heat flux. Example 2 below brings out the typical characteristics of such a heat flux gage.

Example 2

The geometrical description of a thin film heat flux sensor is given in Figure 5. The thermal conductivity of the barrier material is also given therein. Output of the gage is known to be V = 0.1 V (100 mV) at the maximum rated heat flux. Determine the maximum rated heat flux from this data.

Since there are 40 junctions, the total output corresponds to 0.1 / 40 = 0.0025 V per junction. Assuming that 40 μV corresponds to 1°C this translates to a temperature difference of

$$\Delta T = \frac{0.0025}{40 \times 10^{-6}} = 62.5°C$$

Using known k and δ values we then have

$$q = k\frac{\Delta T}{\delta} = \frac{(0.2)(62.5)}{(0.25 \times 10^{-3})}\frac{W}{m^2} = 50000\frac{W}{m^2}$$

Typically the solar heat flux is 1000 W / m^2. The above heat flux is some 50 times larger!

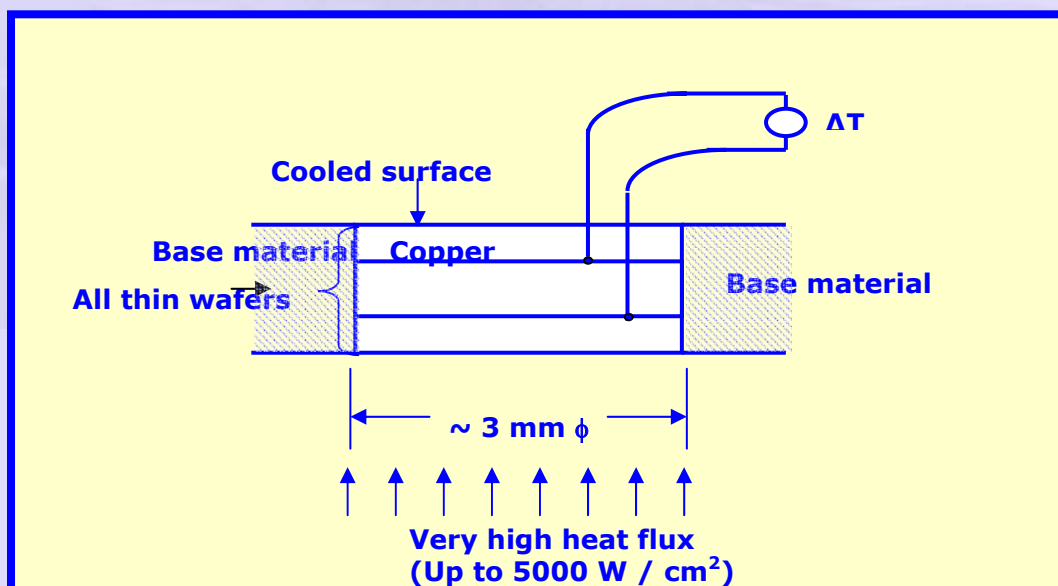**Cooled thin wafer heat flux gage:**

Figure 7 Thin wafer type cooled heat flux gage

The operational principle of the thin wafer type cooled heat flux gage is the same as the thin film gage above. Temperature drop across the constantan wafer is measured by the differential thermocouple arrangement shown in Figure 7. There are two T type junctions formed by the constantan wafer sandwiched between the two copper wafers

Axial conduction guarded probe:

This probe (see Figure 8) is based on conduction through the probe in a direction parallel to the heat flux that is being measured. The gage consists of a cylinder of known thermal conductivity with an annular guard. The guard consists of outer annular cylinder made of the same material as that of the gage. It is exposed to the same heat flux and cooled at the back by the same coolant that is also used to cool the probe itself. Since the outer annulus experiences roughly the same axial temperature gradient as the probe, one dimensional conduction is achieved in the probe. The temperatures are measured by two embedded thermocouples as indicated in the figure. Fourier law is used to derive the heat flux from the measured temperature difference, the distance between the thermocouples and the known thermal conductivity of the probe material.
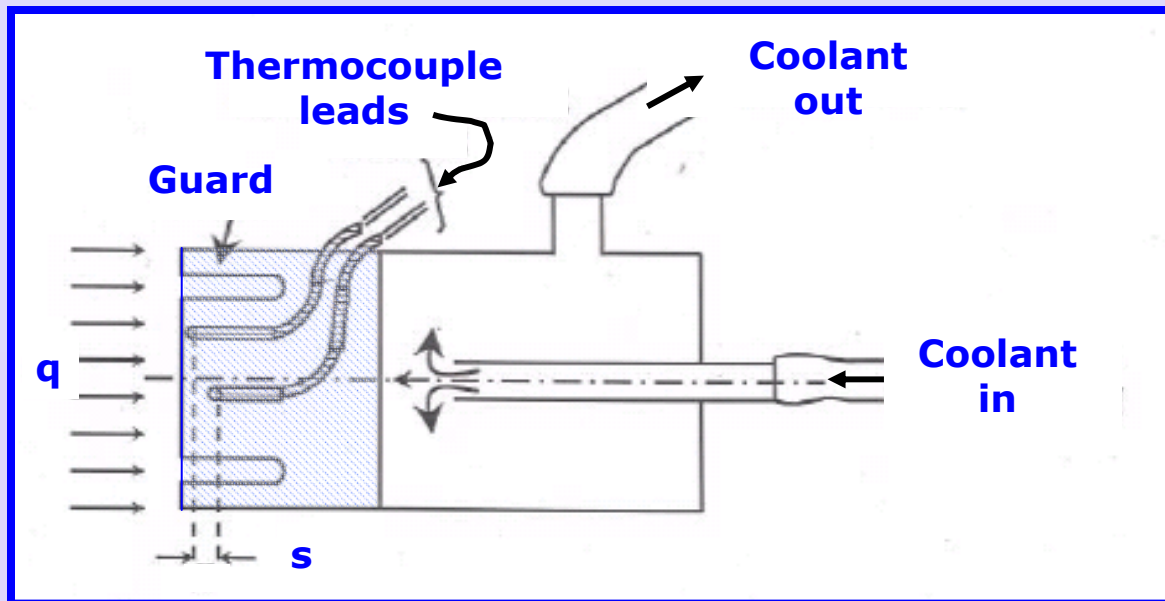
Figure 8 Axial conduction guarded heat flux probe

Example 3

Consider an axial conductivity guarded heat flux probe made of an alloy material of thermal conductivity equal to 45 W/m°C. The two thermocouples are placed 1 cm apart. The incident heat flux is known to be 10^5 W/m². The probe has a diameter of 25 mm. Determine the indicated temperature difference ΔT and the heat Q that needs to be removed from the back surface of the probe.

$$q = 100000 \ \frac{W}{m^2} = k \frac{\Delta T}{s} \ (by \ Fourier \ law)$$

$$\therefore \Delta T = \frac{100000 \times 0.01}{45} = 22.2°C$$

$$Heat \ to \ be \ removed = Q = qA = 100000 \times \pi \times \frac{0.025^2}{4} = 49.1W$$

This amount of heat is removable by an air stream.

Slug type sensor:

Schematic of a slug type heat flux sensor is shown in Figure 9. A mass M of a material of specific heat c is embedded in the substrate as shown. Frontal area A of the slug is exposed to the heat flux to be measured while all the other surfaces of the slug are thermally insulated as indicated. When the incident flux is absorbed at the surface of the slug, it heats the slug, and uniformly so if it is made of a material of high thermal conductivity.
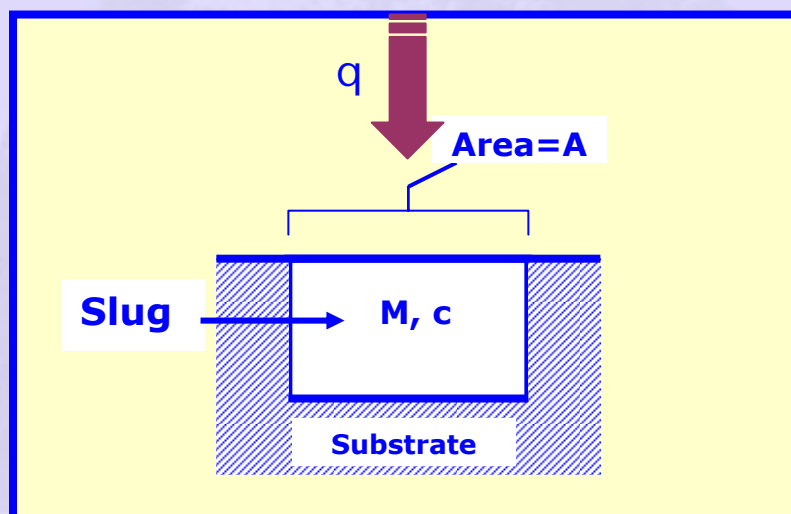

Figure 9 Slug type heat flux sensor

In the ideal case with no heat loss the equation for the temperature of the slug is given by

$$Mc \frac{dT}{dr} = qA \qquad \text{(12)}$$

On integration this will yield the slug temperature as a linear function of time given by

$$T = T_0 + \frac{qA}{Mc} t \qquad \text{(13)}$$

Obviously we have to allow the process of heating to terminate by stopping the exposure of the slug to the incident heat flux when the slug temperature reaches its maximum allowable temperature.

Response of a slug type sensor with a small heat loss:

Consider now the case when there is a small heat leak from the slug sensor. Let the loss be proportional to the temperature excess of the slug with respect to the casing or the substrate. The loss coefficient is given as K_L W/°C. Equation (12) will now be replaced by

$$Mc\frac{dT}{dr} = qA - K_L(T - T_c) \qquad (14)$$

Let $T - T_c = \theta$ and $\frac{K_L}{Mc} = \in$. With these equation (14) will be recast as

$$\frac{d\theta}{dt} - \frac{qA}{Mc} = - \in \theta \qquad (15)$$

Since K_L is expected to be small, the parameter $\in$ is small. The solution may be sought by expanding it in the form $\theta = \theta^{(1)} + \in \theta^{(2)} + ...$ Substituting this in equation (15) we have

$$\frac{d\theta^{(1)}}{dt} - \frac{qA}{Mc} + \in \frac{d\theta^{(2)}}{dt} + ... = -\left[\in \theta^{(1)} + \in^2 \theta^{(2)} + ...\right] \qquad (16)$$

Collecting terms of same order, the above is replaced by

$$\frac{d\theta^{(1)}}{dt} = \frac{qA}{Mc} \qquad (17a)$$

$$\frac{d\theta^{(2)}}{dt} = -\frac{qA}{Mc}\theta^{(1)} \qquad (17b)$$

These equations are solved to get

$$\theta^{(1)} = \frac{qAt}{Mc}; \ \theta^{(2)} = -\frac{qAt^2}{2Mc} \qquad (17c)$$

Thus the response of the slug follows the relation

$$\theta \approx \frac{qAt}{Mc} - \in \frac{qA}{2Mc}.t^2 = \frac{qA}{Mc}\left[t - \frac{K_L}{Mc}.\frac{t^2}{2}....\right] \qquad \textbf{(18)}$$

Thus a slight amount of nonlinearity is seen in the solution.

Example 4

A slug type of sensor is made of a copper slug of 3 mm thickness. The specification is that the temperature of the slug should not increase by more than 40°C.

a) What is the time for which the slug can be exposed to an incident heat flux of 10000 W/m^2 and there is negligible heat loss from the slug?

b) What is the time for which the slug can be exposed to an incident heat flux of 10000 W/m^2 and there is small heat loss from the slug specified by a loss coefficient of 50 W/°C?

Note: We make all calculations based on unit area of slug exposed to the incident heat flux. Properties of Copper are taken from hand book

Density ρ = 8890 kg / m^3, specific heat C = 398 J/kg Č

The maximum temperature rise during operation is ΔT_{max} = 40°C

a) *If there is negligible heat loss the temperature increases linearly and the*

maximum exposure time is given by equation (13).

$$t_{max} = \frac{3 \times 10^{-3} \times 8890 \times 398 \times 40}{10000} = 42.46 \text{ s}$$

b)   *When the heat loss is taken into account, what happens is worked out*

below.We   have,   from   the   given   data

$$\frac{K_L}{Mc} = \frac{50}{8890 \times 3 \times 10^{-3} \times 398} = 0.00047$$

and $\frac{qA}{Mc} = \frac{q}{\rho\delta c} = \frac{10000}{8890 \times 3 \times 10^{-3} \times 398} = 0.942$. *With these, we get, using*

equation xx $\theta \approx 0.942\left[t - 0.00047\frac{t^2}{2} \right]$.

With $\Delta T_{max} = 40°C$ *we have* $\Delta T_{max} = 0.942\, t_{max}\left[1 - 0.000235\, t_{max}\right]$

or $t_{max} = \frac{\Delta T_{max}}{0.942\left[1 - 0.000235 t_{max}\right]}$. *This gives the maximum exposure time*

approximately as $t_{max} \approx \frac{40}{0.942\left[1 - 0.000235 \times 42.46\right]} = 47.2s$