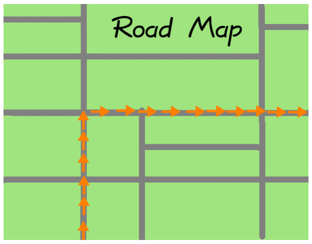
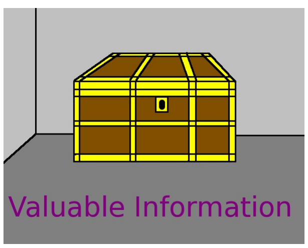
Module 3.9: What is a Logarithm?



The logarithm is an excellent mathematical tool that has several distinct uses. First, some measurements, like the Richter scale for earthquakes or decibels for music, are fundamentally logarithmic. Second, plotting on logarithmic scales can reveal exponential relationships like those you learned about in connection with inflation, compound interest, depreciation, radiation, and population growth. Third, logarithms are the basis of slide-rules and tables of logarithms, which were useful methods of calculation before the invention of the hand-held calculator—but the reason we like logarithms in this text is because they allow us to solve equations of the form

$$(\text{ some number }) = (\text{ another number })^x$$

This will prove vital in topics that you are already familiar with, such as compound interest, radiation, depreciation, population growth, and so on.

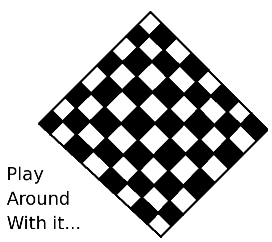


If $f(x)$ is a function such that

$$f(xy) = f(x) + f(y)$$

then we say that $f(x)$ is a *logarithm*.

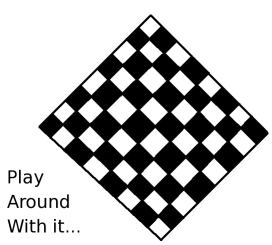
There are three logarithms in common use: the binary logarithm, the common logarithm, and the natural logarithm. For simplicity, we will study the common logarithm here.



3-9-1

Using the logarithm button on your calculator, find the logarithms of the following values:

- What is $\log 100$? [Answer: 2.]
- What is $\log 1000$? [Answer: 3.]
- What is $\log 10,000$? [Answer: 4.]
- What is $\log 100,000$? [Answer: 5.]



3-9-2

Using your calculator, find the following:

- What is $10^{0.301029995}$? [Answer: 2.]
- What is $\log 2$? [Answer: 0.301029995.]
- What is $10^{0.477121254}$? [Answer: 3.]
- What is $\log 3$? [Answer: 0.477121254.]

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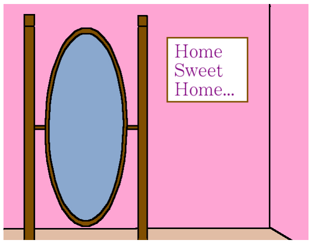
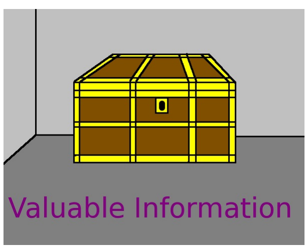


A Pause for Reflection...
Describe in your own words the pattern you've discovered in the previous two boxes. Concentrate, and make sure you understand it. If you do, then you will become very skilled in the use of the common logarithm.



Mathematics has many maneuvers that can be considered move & counter-move. For example, addition and subtraction are opposites, squaring and square-rooting are opposites, multiplying and dividing are opposites, and we are now exploring that exponentiating is the opposite of the logarithm. This can be summarized by the following list, an expansion of what was found on Page 28 and Page 349. We will expand it for the last time on Page 523. This important topic is called "the theory of inverse functions."

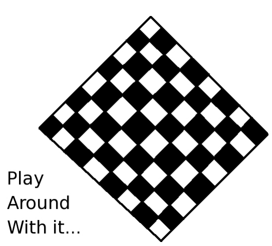
- If $4x = 64$ and you want to "undo" the "times 4," you do $64/4$ to learn $x = 16$.
- If $x/2 = 64$ and you want to "undo" the "divide by 2," you do 64×2 to learn $x = 128$.
- If $x + 13 = 64$ and you want to "undo" the "plus 13," you do $64 - 13$ to learn $x = 51$.
- If $x - 12 = 64$ and you want to "undo" the "minus 12," you do $64 + 12$ to learn $x = 76$.
- If $x^2 = 64$ and you want to "undo" the "square," you do $\sqrt{64}$ to learn $x = 8$.
- If $x^3 = 64$ and you want to "undo" the "cube," you do $\sqrt[3]{64}$ to learn $x = 4$.
- If $x^6 = 64$ and you want to "undo" the "sixth power," you do $\sqrt[6]{64}$ to learn $x = 2$.

To which we now add:

- If $10^x = 64$ and you want to "undo" the "ten to the," you do $\log 64$ to learn $x = 1.80617\cdots$.
- If $\log x = 64$ and you want to "undo" the "logarithm," you do 10^{64} to learn $x = 10,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000$.

Which can be explained alternatively:

- If $10^x = 100$ and you want to "undo" the "ten to the," you do $\log 100$ to learn $x = 2$.
- If $\log x = 3$ and you want to "undo" the "logarithm," you do 10^3 to learn $x = 1000$.



3-9-3

Try finding the common logarithm of the following numbers on your calculator. Do not write the answer in scientific notation, but rather, in regular notation.

- What is $\log(3.14159 \times 10^0)$? [Answer: $0.49714\cdots$.]
- What is $\log(6.65702 \times 10^9)$? [Answer: $9.82327\cdots$.]
- What is $\log(7.188173236 \times 10^9)$? [Answer: $9.85661\cdots$.]
- What is $\log(5.9620 \times 10^4)$? [Answer: $4.77539\cdots$.]
- What is $\log(6.02 \times 10^{23})$? [Answer: $23.7795\cdots$.]

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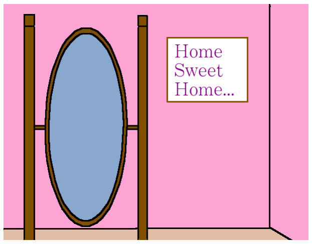


A Pause for Reflection...
In each case in the previous box, I want you to look at the exponent in scientific notation, and the number that you find to the left of the decimal point in the logarithm. Do you see a pattern? What is the relationship?

As you can see from the previous chessboard box, there is a pattern. If x is some number in scientific notation, with the exponent on the ten being y, then $\log x$ will have y to the left of the decimal point.

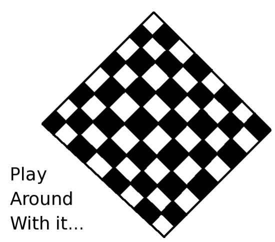
This is a great way to check your work. One could write

$$\log(j.unk \times 10^{x}) = x.crud$$

but that isn't very mathematical. In any case, notice the positions of x on the left, and the x on the right.

Still, when you see a common logarithm, you always know at least the exponent of the number in scientific notation that it represents. It works the other way too. If you have a number in scientific notation, and you take the common logarithm, you know what to expect to the left of the decimal point. If you are alert, this will help prevent many accidental errors. This number, which is simultaneously the exponent in scientific notation, and the left-of-the-decimal value of the logarithm, is called the *mantissa*.

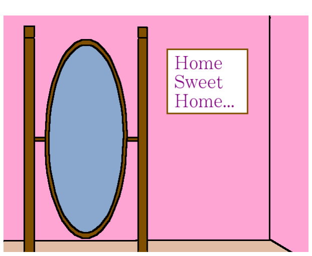
So what about numbers less than one?



3-9-4

Try finding the common logarithm of the following numbers on your calculator. Do not write the answer in scientific notation, but rather, in regular notation.

- What is $\log 2.5 \times 10^{-1}$? [Answer: $-0.602059\cdots$.]
- What is $\log 4 \times 10^{-2}$? [Answer: $-1.39794\cdots$.]
- What is $\log 5 \times 10^{-3}$? [Answer: $-2.30102\cdots$.]
- What is $\log 9.10938 \times 10^{-31}$? [Answer: $-30.0405\cdots$.]



A Pause for Reflection...
Again, I want you to look at the exponent in scientific notation, and the number that you find to the left of the decimal point in the logarithm. How has the relationship changed?

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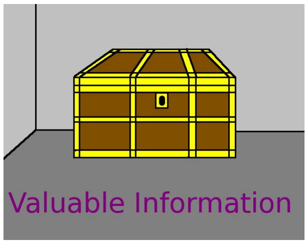


The laws of logarithms, like the laws of exponents, are numbered differently in essentially every text book. I number them as follows. First, the major laws:

1. For any positive real numbers a and c: $\log ac = \log a + \log c$.
2. For any real numbers a and c, with $a > 0$: $\log a^c = c \log a$.
3. For any positive real numbers a and c: $\log \frac{a}{c} = \log a - \log c$.
4. For the common logarithm, and any real a: $\log(10^a) = a$.

Note: For a logarithm that is not base 10 but is base b, then # 4 becomes $\log_b b^a = a$.

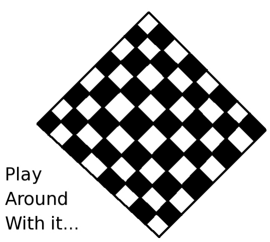
5. For the common logarithm, and any positive real a: $10^{\log a} = a$.

Note: For a logarithm that is not base 10 but is base b, then # 5 becomes $b^{\log_b a} = a$.

Second, the minor laws:

1. $\log 1 = 0$
2. For any positive real number a: $\log(1/a) = -\log a$
3. For any positive real number a, and positive integer n: $\log(\sqrt[n]{a}) = \frac{\log a}{n}$
4. If log(junk) = log(stuff) then (junk) = (stuff).

It would be really awesome if you knew all of those laws perfectly. However, it is really only the major laws that actually appear to come up in solving problems, especially within this textbook. Even then, only the common-logarithm cases of the fourth and fifth are frequently used. The $b \neq 10$ situations are rather rare.

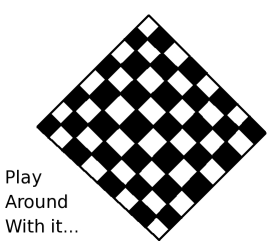


3-9-5

Now, let's test these laws, to make sure we have them right! We will learn, over the course of this module and the next, that logarithms are extremely sensitive to rounding error. Therefore, for this box and the next few, we will use 9 digits instead of 6 digits.

- What is $\log 2$? [Answer: $0.301029995 \cdots$.]
- What is $\log 3$? [Answer: $0.477121254 \cdots$.]
- What is $\log 6$? [Answer: $0.778151250 \cdots$.]

Note: $0.301029995 \cdots + 0.477121254 \cdots = 0.778151249 \cdots$, off only by one part per billion. Thus we've shown $\log 2 + \log 3 = \log(2 \times 3) = \log 6$.



3-9-6

- What is $\log 12$? [Answer: $1.07918124 \cdots$.]
- What is $\log 4$? [Answer: $0.602059991 \cdots$.]
- You found $\log 3$ in the previous box, what was it? [Answer: $0.477121254 \cdots$.]

Note: $1.07918124 \cdots - 0.602059991 \cdots = 0.477121249 \cdots$, again, only off by one part in a billion. Thus we've shown $\log 12 - \log 4 = \log(12/4) = \log 3$.

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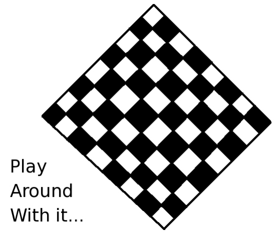
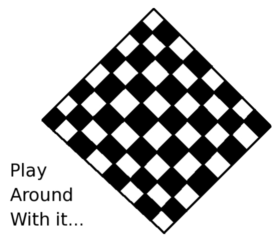


3-9-7

- What is $\log 9$? [Answer: $\log 9 = 0.954242509\cdots$.]
- You found $\log 3$ in the previous box, what was it? [Answer: $0.477121254\cdots$.]

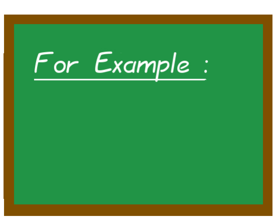
Note: $2\log 3 = 2 \times 0.477121254\cdots = 0.954242508\cdots$, again, only off by one part in a billion. Thus we've shown $2\log 3 = \log 3^2 = \log 9$.



3-9-8

Okay, we've verified the first three major laws now. The first minor law is easy to verify, because you can ask your calculator if $\log 1 = 0$ if you like. Meanwhile, consider:

- First, we start with 64. What is $\log 64$? [Answer: $\log 64 = 1.80617997$.]
- Because $\sqrt{64} = 8$, we anticipate $\log 8 = (1/2)\log 64$. Is it true? [Answer: $\log 8 = 0.903089986\cdots$, and $(1/2)\log 64 = 0.903089986\cdots$, so yes.]
- Because $\sqrt[3]{64} = 4$, we anticipate $\log 4 = (1/3)\log 64$. We found $\log 4$ above. Is our anticipation true? [Answer: $(1/3)\log 64 = 0.602059991\cdots$, exact to the digits shown.]
- Lastly, because $\sqrt[6]{64} = 2$, we anticipate $\log 2 = (1/6)\log 64$. Is this true? [Answer: $(1/6)\log 64 = 0.301029995\cdots$, and $\log 2 = 0.301029995\cdots$, exact to the digits shown.]
- Just for completeness, we could ask what $\log(1/2)$ is, and compare it to $\log 2$. [Answer: $\log(1/2) = -0.301029995\cdots$, exactly the negative of $\log 2$, as anticipated.]



3-9-9

The next step is to tear apart some expressions with a logarithm. Suppose we have, in a long computation $\log[(1+x)(2+y)] = 8$. Suppose we know, from earlier in the computation, that $y = 3$, and we plug that in to get $\log(1+x)(5) = 8$. We will proceed as follows.

$$\begin{aligned}
\log[(1+x)(5)] &= 8 \\
\log(1+x) + \log 5 &= 8 \\
\log(1+x) &= 8 - \log 5 \\
\log(1+x) &= 8 - 0.698970004\cdots \\
\log(1+x) &= 7.30102999\cdots \\
1+x &= 10^{7.30102999\cdots} \\
1+x &= 20,000,000 \\
x &= 19,999,999
\end{aligned}$$

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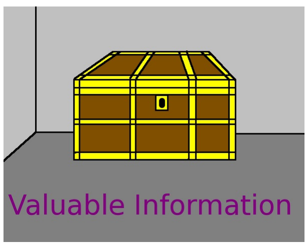
The last example was a bit complicated, so let's check our work.

$$\begin{aligned} \log\left[(1+x)(5)\right] &= \log\left[(1+19,999,999)(5)\right] \\ &= \log\left[(20,000,000)(5)\right] \\ &= \log 100,000,000 \\ &= 8 \qquad\qquad \text{Hooray!} \end{aligned}$$



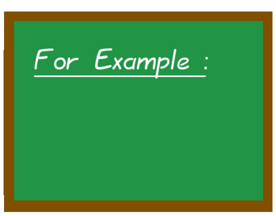
Remember the move-countermove strategies? The previous box worked because

$$10^{\log x} = x$$

and other problems (such as the first checkerboard on Page 504) are solvable because

$$\log 10^x = x$$

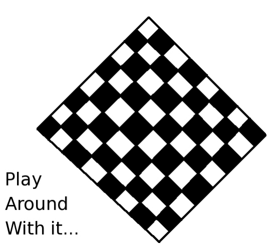
As you can see, these are the fourth and fifth major laws of logarithms.



3-9-10

Suppose we have $5\log\frac{x+25}{100} = 1$, and we want to find x.

$$\begin{aligned} 5\log\frac{x+25}{100} &= 1 \\ \log\frac{x+25}{100} &= 1/5 = 0.2 \\ \frac{x+25}{100} &= 10^{0.2} \\ \frac{x+25}{100} &= 1.58489\cdots \\ x+25 &= 158.489\cdots \\ x &= 158.489\cdots - 25 = 133.489\cdots \end{aligned}$$



3-9-11

How about these equations? Can you solve them for x?

- $5 + 3(\log x^2) = 11$ [Answer: $x = 10$.]
- $(\log x^2) + (\log x^3) = 2.38560627\cdots$ [Answer: $x = 3$.]

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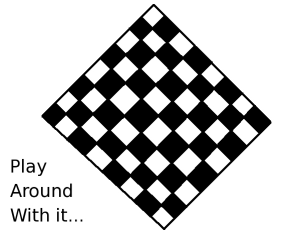
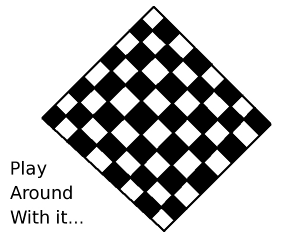


3-9-12

Now, let's solve these equations for x.

- $\log \frac{x+3}{10} = 2$ [Answer: 997.]
- $\log(xy^2) = 3$ when $y = 5$ [Answer: $x = 40$.]



3-9-13

These equations will be fun to solve!

- $\frac{2}{3} - \log(2x+1) = \frac{5}{3}$ [Answer: $x = \frac{-9}{20} = -0.45$.]
- $4\log(5x+678) = 12$ [Answer: $x = \frac{322}{5} = 64.4$.]

There is a common point of confusion that traps many students. Sometimes students will try to do something with logarithms resembling the distributive law. Just to be clear...

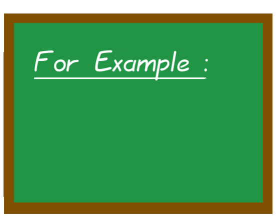
$$\begin{array}{lrcll} \text{Wrong!} \rightarrow & \log(a+c) & = & (\log a) + (\log c) & \leftarrow \text{Wrong!} \\ \text{Wrong!} \rightarrow & \log(a-c) & = & (\log a) - (\log c) & \leftarrow \text{Wrong!} \end{array}$$

...because instead it is actually the case that ...

$$\begin{array}{lrcll} \text{Correct!} \rightarrow & \log(ac) & = & (\log a) + (\log c) & \leftarrow \text{Correct!} \\ \text{Correct!} \rightarrow & \log(a/c) & = & (\log a) - (\log c) & \leftarrow \text{Correct!} \end{array}$$

Accordingly, if you were to attempt either or both of these forbidden steps in any sort of homework or quiz, to say nothing of an examination, the entire problem would probably be ruined beyond repair.

In my years of teaching, I have seen many students fall victim to this trap! Remember, we do not "distribute" the logarithm. Instead we carefully apply proven laws of logarithms, which we have already studied, one step at a time.



3-9-14

Here are some formulae from our work with finance, and now we shall explore if it is possible to use logarithms to convert the formulas to a new form.

- Since we know $A = P(1+rt)$, for simple interest, what is $\log A$? Well, $\log A = \log P + \log(1+rt)$, using the first major law. There is no law of logarithms that deals with the logarithm of a sum (see the above box), and so we cannot break up that last logarithm any further.
- Since we know $A = P(1+i)^n$, what is $\log A$? Well, $\log A = \log P + \log(1+i)^n$, using the first major law. Then, we can apply the second major law to get $\log A = \log P + n\log(1+i)$. This is as far as we can go.

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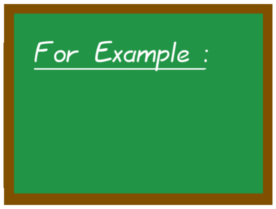


3-9-15

At first glance, the previous box looks disappointing. We were able to crack open the simple interest and compound interest formulas partially, but it did not seem to serve a useful purpose. However, imagine that in a compound interest problem, you wanted to know what n, would turn a given P into a given A.
Then you could take our last line and simplify to

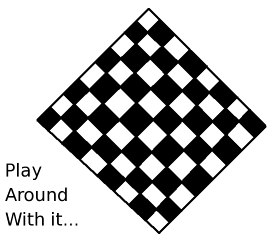
$$\log A - \log P = n \log(1+i)$$

or equivalently

$$\frac{\log A - \log P}{\log(1+i)} = n$$

or if you prefer

$$\frac{\log(A/P)}{\log(1+i)} = n$$



3-9-16

This problem might be extremely challenging, but using the methods of the previous box, can you show that

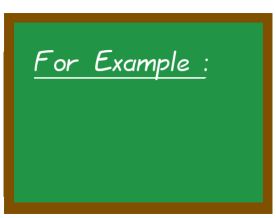
$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

can be made to become

$$t = \frac{\log A/P}{m \log(1 + r/m)}$$

by giving a series of valid mathematical steps that leads from one to the other? [Answer given at the end of the module.]

This one is very hard, so please don't feel bad if you cannot get it to work without peeking at the answers.



3-9-17

Suppose there is a tax-free municipal-bond fund that is giving a 5% rate of return (let us assume it compounds annually). Jimmy has \$ 20,000 as a graduation gift from his grandparents, and he wants to save it for the down payment on a house. He thinks he needs \$ 50,000 for that. How long is it going to take him?

We start with $A = P(1+i)^n$. Recall, $i = r/m$ and we have $m = 1$ and so $r = 0.05 = i$. Then $A = 50,000$ and $P = 20,000$. Also, because $m = 1$ then $n = mt = t$. Thus we have

$$\begin{aligned}
A &= P(1+i)^n \\
50,000 &= 20,000(1+0.05)^n \\
\frac{50,000}{20,000} &= (1.05)^n \\
2.5 &= (1.05)^n \\
\log 2.5 &= \log(1.05)^n \\
\log 2.5 &= n \log(1.05) \\
\frac{\log 2.5}{\log 1.05} &= n \\
\frac{0.397940\cdots}{0.0211892\cdots} &= n \\
18.7802\cdots &= n
\end{aligned}$$

and so we learn that Jimmy will need almost 19 years to reach that amount. It sounds like Jimmy should either seek a more aggressive investment or contribute some cash of his own.

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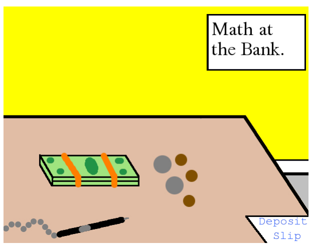
Let's check our work from the previous box. What would happen if someone were to invest \$ 20,000 for 18.7802 years at 5% compounded annually? We would have

$$A = P(1+i)^n = (20,000)(1+0.05)^{18.7802} = (20,000)(2.49999\cdots) = 49,999.9\cdots$$

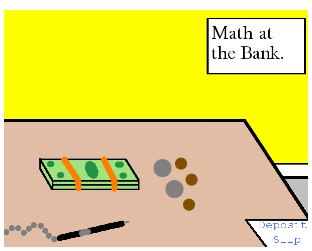
which is extremely close. If we had used nine digits instead of six, it would be exact.



I neglected to say what municipal-bond funds are, and why they are tax free, at least in the USA.

Towns, cities, and counties will offer municipal bonds to fund large projects, like state-highways, bridges, and airports. Since those items cost many millions of dollars to build, it isn't a small purchase and there would be no way to "scrape together" the cash from ordinary city and county budgets.

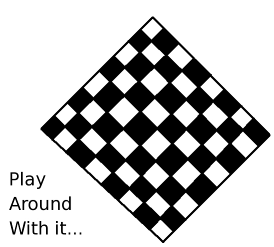
On the other hand, since city and county budgets are often not very large, a high interest rate might be difficult for the city or county to pay. Therefore, the interest rates on municipal bonds tend to be rather low. This is a problem, because citizens might not find bonds with a low rate of interest to be very attractive investments, as compared to perhaps corporate bonds of large and famous "blue chip" companies.



The federal government, of course, wants citizens to buy these municipal bonds. Without the municipal bond system, the federal government would have to build those bridges and airports itself!

With that in mind, in order to make the municipal bonds more attractive, the federal government wisely decided to waive its right to collect income taxes on the interest that the bonds pay. Since the interest rates on municipal bonds are low, the federal government is giving up only a microscopic amount (compared to income taxes in general), yet it keeps the system alive, which is very convenient.

A municipal-bond fund is merely a mutual fund that buys and sells tax-free municipal bonds. Their dividends are (almost always) tax-free, as a result.



3-9-18

The stock market has historically returned a rate of roughly 9%. (We'll get more precise about that in later chapters.) Referring back to the previous example, if Jimmy used some stocks rather than a tax-free investment fund, and had earned 9% compounded annually, then how long would it take for him to have the \$ 50,000 before taxes?
[Answer: $10.6325\cdots$ years.]

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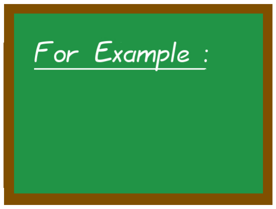


3-9-19

Consider that Bob has \$ 5000 and wants to know how long it will take to become \$ 6000, if invested at 6% compounded monthly. Of course, if $r = 0.06$ and we work monthly, then $i = 0.06/12 = 0.005$. Then we have

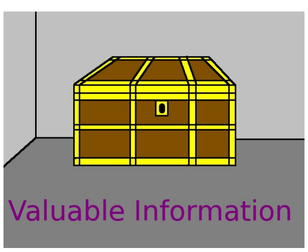
$$\begin{aligned} A &= P(1+i)^n \\ 6000 &= 5000(1+0.005)^n \\ \frac{6000}{5000} &= 1.005^n \\ 1.2 &= 1.005^n \\ \log 1.2 &= \log 1.005^n \\ \log 1.2 &= n \log 1.005 \\ \frac{\log 1.2}{\log 1.005} &= n \\ \frac{0.0791812\cdots}{0.00216606\cdots} &= n \\ 36.5553\cdots &= n \end{aligned}$$

As you can see, it will require 36.5553 months, (which really means 37 months), for the \$ 5000 to turn into \$ 6000.

Once more, let's check our work from the previous example. What would happen if someone were to invest \$ 20,000 for 36.5553 months at 6% compounded monthly? We would have

$$A = P(1+i)^n = (5,000)(1+0.005)^{36.5553} = (5,000)(1.199999\cdots) = 5,999.99\cdots$$

which is excellent.

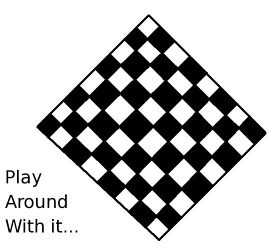


Whenever you are presented with a compound interest problem, and are asked to find how long it will take (or how many compounding periods are required) to reach some financial goal, you have two choices.

You can either use the shortcut formula,

$$\frac{\log(A/P)}{\log(1+i)} = n$$

or you can solve the problem in the manner of the previous box. Personally, I think that memorizing formulas is unwise, so I recommend you try to follow the pattern of the previous two examples.



3-9-20

Alice has \$ 1000 and she wants to know how long it will take to become \$ 1500. The investment she is considering will yield 7% compounded monthly.

- How many months are required? [Answer: 69.7108$\cdots$ months.]
- What is that in years, as a decimal? [Answer: 5.80923$\cdots$ years.]
- How many years and months? [Answer: 5 years and 9.71 months.]

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Play Around With it...

3-9-21

Continuing with the previous box...

- If the yield became 7.25%, then how many months would be required?
  [Answer: 67.3139··· months, or 5 years and 7.31 months.]
- If the yield became 7.75%, then how many months would be required?
  [Answer: 62.9842··· months, or 5 years and 2.98 months.]

Play Around With it...

3-9-22

Suppose that I lose my job and cannot make payments on my credit card, where perhaps I have a $ 5,000 balance. The rate will then switch to a 29.95% interest rate in that case. It is compounded monthly. How much time will it take for that debt to balloon to $ 20,000? [Answer: 56.2346 compounding periods, which is 4 years and 8.23 months.]

Play Around With it...

3-9-23

- Repeat the above problem for $ 25,000.
  [Answer: 65.2864··· months or 5 years 5.28 months.]
- Repeat the above problem for $ 30,000.
  [Answer: 72.6822··· months or 6 years and 0.68 months.]

Play Around With it...

3-9-24

Continuing with the previous two boxes, suppose instead that the rate were 24.95%, but everything else is the same—I start with a principal of $ 5000.

- How long would it take for my debt to reach $ 20,000?
  [Answer: 67.3662··· months which means 5 years and 7.36 months.]
- How long would it take for my debt to reach $ 25,000?
  [Answer: 78.2097··· months which means 6 years and 6.20 months.]
- How long would it take for my debt to reach $ 30,000?
  [Answer: 87.0696··· months which means 7 years and 3.06 months.]

Let's summarize the answers to the last few checkerboards.

	29.95% interest	24.95% interest
$ 5000 growing to $ 20,000:	4 years and 8.23 months	5 years and 7.36 months.
$ 5000 growing to $ 25,000:	5 years and 5.28 months	6 years and 6.20 months.
$ 5000 growing to $ 30,000:	6 years and 0.68 months	7 years and 3.06 months.

Is it not shocking how rapidly debts can spiral out of control?

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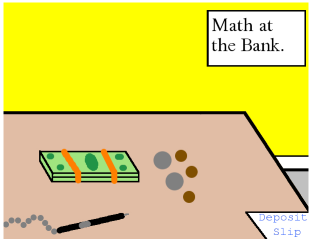
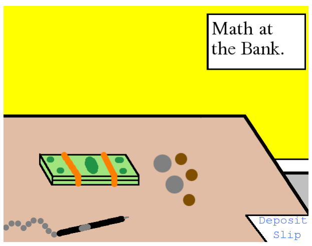
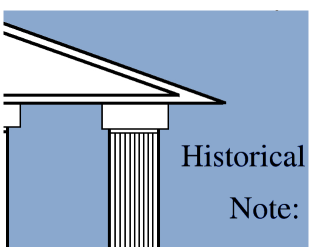


You might think that the above rates are harsh, and you'd be right, they are harsh—but they are also typical. When you have a credit card, if you *go into default*, which means violating the terms and conditions of the credit card's contract, then an extremely high rate will kick in. This is because the credit card company believes that you are slowly sliding into bankruptcy, and they would like to get the maximum quantity of money out of you as rapidly as possible. This is why it is absolutely vital that you *never* go into default.

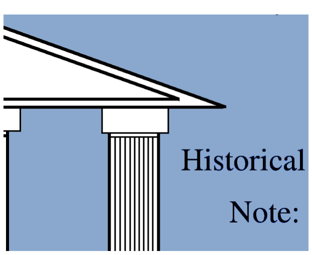


Often missing two payments or sometimes even one payment for a credit card is sufficient for a bank to declare you in default. Then your interest rate would skyrocket.

I hope that you never have long-term credit card debt, but if you do, it is a good idea to keep two or three minimum payment's worth in your savings account, to ensure that if you suddenly lose your job, that you do not default on your obligations.



When we are in high school, we often have to memorize lists like "Alexander Graham Bell invented the telephone" and "Thomas Edison invented the light bulb" or "Sir Thomas Crapper popularized the flushing toilet." Not only are such memorizations intellectually sterile, they are highly misleading. Each of those inventions is known to have been a collaboration. Like them, logarithms were a collaborative work as well.



The two people who contributed the most, in the case of the logarithm, are Henry Briggs and John Napier.

When doing the research for these boxes, I was shocked to learn that the natural logarithm, which is based on the special number e, and which we will study on Page 520, came first. The common logarithm came second. Moreover, the concept of the logarithm predated and inspired the idea of non-integer exponents. Yet, today, because we learn in school about exponents several years before we learn about logarithms, many people including myself assume that exponents came first, and logarithms came later.

We'll discuss Henry Briggs now, and John Napier on Page 534.

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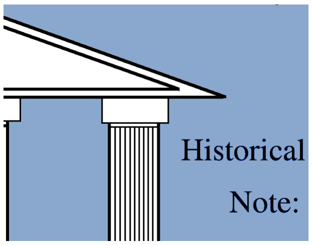


Henry Briggs (1561–1630) was an English mathematician, and is best known for publishing tables of common logarithms and popularizing their use. In fact, he wrote two books on this topic: *Logarithmorum Chilias Prima*, first published in 1617, and later *Arithmetica Logarithmetica*, first published in 1624. However, he did not invent the logarithm, but rather John Napier did. A very practical man, Briggs not only saw the utility of logarithms, but also published *Tables for the Improvement of Navigation*, in 1610.

Briggs was born in Yorkshire, and it is known that he came from a rather poor family. He studied at St. John's College of Cambridge University starting in 1577. Furthermore, he took the Bachelor of Arts degree in 1581 and Master of the Arts in 1585. Like most scholars of his era, he was active in theology as well as mathematics, being a noted Presbyterian and Puritan. It is said that "From the flow of rivers in Virginia and in the Hudson Bay area, he also deduced the existence of the mid-continental range of mountains" which would not be located until more than a century and a half later.

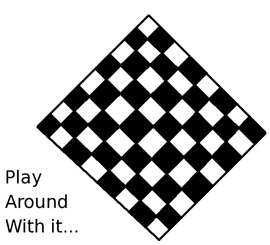
Briggs served as Professor of Geometry in Gresham College of Cambridge University, and finally as Savillian Professor of Geometry in Merton College of Oxford University until his death. That last Oxford posting has also been filled by Edmond Halley, (the discoverer of Halley's comet's orbit), and Baden Powell, the father of Robert Baden-Powell who founded the Boy Scout Movement.

You can read more about Henry Briggs at
`http://www.thocp.net/biographies/briggs_henry.html`

Logarithms are extremely sensitive to rounding error. Let's explore that. Suppose a calculation is being performed by Alice, Bob, and Charlie, and we arrive at $\log a = 5.92185\cdots$. All three students are happy to be near the end of a problem, and they know they have one step left. The 10^x button on their calculator will undo the common logarithm. Alice will be diligent and enter all six given digits into the calculator—but on the other hand, Bob and Charlie will be lazy. Bob will enter 5.92 and Charlie will enter 5.93. Let's see what they get

- Alice gets $10^{5.92185} = 835,314.\cdots$.
- Bob gets $10^{5.92} = 831,763.\cdots$.
- Charlie gets $10^{5.93} = 851,138.\cdots$.

Is it not shocking that there is a huge difference in the answers, given only a slight rounding on the part of the students? (I find the difference between Bob and Charlie particularly shocking.) As you can see, logarithms, like interest rates (see Page 281) are very sensitive numbers. That's why, at times, I've used 9 digits of accuracy.



3-9-25

Find the relative errors for Bob and Charlie in the previous box, assuming Alice is correct. [Answer: Bob has error of -0.43% and Charlie has error of +1.89%, pretty abysmal considering that the data given was to six decimal places.]

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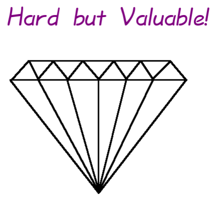
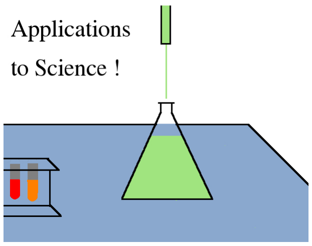


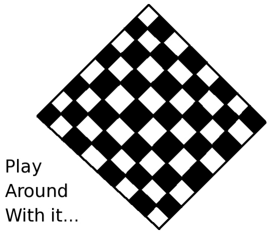
We're now going to explore how scientists use logarithms to explore forms of physical laws that are more convenient for repeated calculations. This brief foray might be too difficult for some readers, and not all instructors will include it.



Newton's Law of Gravitation states that the force of gravity between a planet and the sun is

$$F = \frac{GM_sM_p}{r^2}$$

where $G = 6.67428 \times 10^{-11}$ is the gravitational constant, and M_s is the mass of the sun (approximately 1.98844×10^{30} kg), while M_p is the mass of the planet (e.g. for earth, approximately 5.97220×10^{24} kg). Finally, r is the distance, from the center of the sun to the center of the planet. For planet-sized objects, you can just think of this as the distance between them (e.g. from the earth to the sun the distance is typically on the order of 1.49598×10^{11} m).



\# 3-9-26

Using Newton's Law of Gravity from the previous box, and using 9 digits of accuracy:

First, can you write a series of valid steps to transform the formula

$$F = \frac{GM_sM_p}{r^2}$$

into the formula

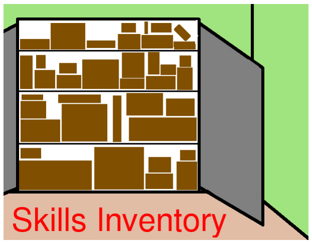
$$\log F = (\log G) + (\log M_s) + (\log M_p) - 2(\log r)$$

and second, show for the specific case of the earth and the sun

$$\log F = 44.8990512 \cdots - 2(\log r)$$

[Answer given at the end of the module.]

Note: This sort of short cut formula can come up when computing orbits numerically, in that you need a quick and easy formula to give you the force on the object for any given r at any given moment.



We have learned the following skills in this module:

- What logarithms are.
- The five major and four minor laws of logarithms.
- How to use logarithms to solve simple equations involving exponents.
- How to use scientific notation to estimate the common logarithm.
- How to manipulate general exponential formulas using logarithms.
- How to find n in a compound interest problem.
- As well as the vocabulary terms: "go into default," logarithm, and mantissa.

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What follows is the answer to an earlier chessboard box, that was found on Page 511:

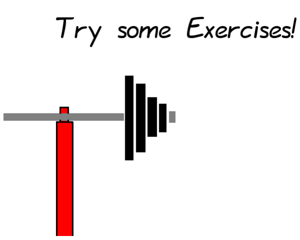
$$\begin{aligned}
A &= P(1+r/m)^{mt} \\
A/P &= (1+r/m)^{mt} \\
\log(A/P) &= \log(1+r/m)^{mt} \\
\log(A/P) &= mt\log(1+r/m) \\
\frac{\log(A/P)}{m\log(1+r/m)} &= t
\end{aligned}$$

What follows is the answer to an earlier chessboard box, that was found on Page 517:

$$\begin{aligned}
F &= \frac{GM_sM_p}{r^2} \\
\log F &= \log\frac{GM_sM_p}{r^2} \\
\log F &= \log\left(GM_sM_p\right) - \log\left(r^2\right) \\
\log F &= (\log G) + (\log M_s) + (\log M_p) - \log\left(r^2\right) \\
\log F &= (\log G) + (\log M_s) + (\log M_p) - 2(\log r)
\end{aligned}$$

For the particular case of the earth and the sun, we plug in $M_s = 1.98844 \times 10^{30}$ as well as $M_p = 5.97220 \times 10^{24}$ and $G = 6.67428 \times 10^{-11}$ to get

$$\begin{aligned}
\log F &= (\log G) + (\log M_s) + (\log M_p) - 2(\log r) \\
\log F &= (\log 6.67428 \times 10^{-11}) + (\log 1.98844 \times 10^{30}) + (\log 5.97220 \times 10^{24}) - 2(\log r) \\
\log F &= (-10.1756137\cdots) + (30.2985124\cdots) + (24.7761343\cdots) - 2(\log r) \\
\log F &= (44.8990330\cdots) - 2\log r
\end{aligned}$$



Coming Soon!

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