Module 7

Simple Beam Theory

Learning Objectives

- Review simple beam theory
- Generalize simple beam theory to three dimensions and general cross sections
- Consider combined effects of bending, shear and torsion
- Study the case of shell beams

7.1 Review of simple beam theory

Readings: BC 5 Intro, 5.1

A beam is a structure which has one of its dimensions much larger than the other two. The importance of beam theory in structural mechanics stems from its widespread success in practical applications.

7.1.1 Kinematic assumptions

Readings: BC 5.2

Beam theory is founded on the following two key assumptions known as the Euler-Bernoulli assumptions:

- Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid

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Concept Question 7.1.1. With reference to Figure 7.1,

1. what is the main implication of this assumption on the kinematic description (overall displacement field) of the cross section?

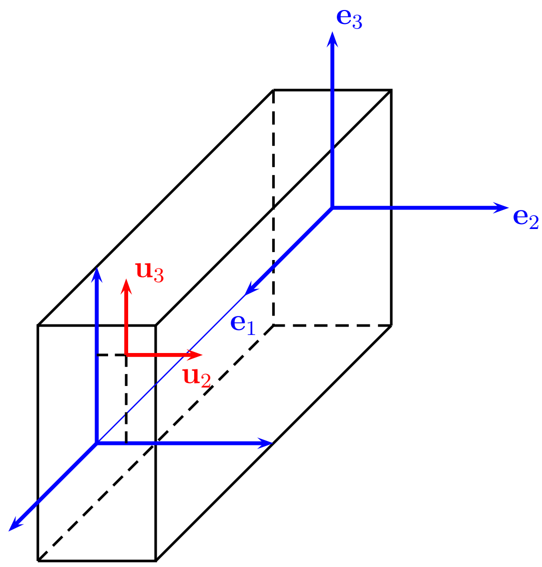

Figure 7.1: First kinematic assumption in Euler-Bernoulli beam theory: rigid in-plane deformation of cross sections.

2. To simplify further the discussion, assume for now that there is no rotation of the cross section around the $\mathbf{e}_3$ axis. Write the most general form of the cross-section in-plane displacement components:

- During deformation, the cross section of the beam is assumed to remain planar and normal to the deformed axis of the beam.

Concept Question 7.1.2. With reference to Figure 7.3,

1. what is the main implication of this assumption on the kinematic description (overall displacement field) of the cross section?
2. Based on these kinematic assumptions, write the most general form of the cross-section out-of-plane displacement component:
3. If you noticed, we have not applied the constraint that the sections must remain normal to the deformed axis of the beam. Use this part of the assumption to establish a relation between the displacements fields $\bar{u}_2(x_1), \bar{u}_3(x_1)$ and the angle fields $\theta_2(x_1), \theta_3(x_1)$, respectively, see Figure **??**.

Concept Question 7.1.3. Combine the results from all the kinematic assumptions to obtain a final assumed form of the general 3D displacement field in terms of the three unknowns $\bar{u}_1(x_1), \bar{u}_2(x_1), \bar{u}_3(x_1)$:

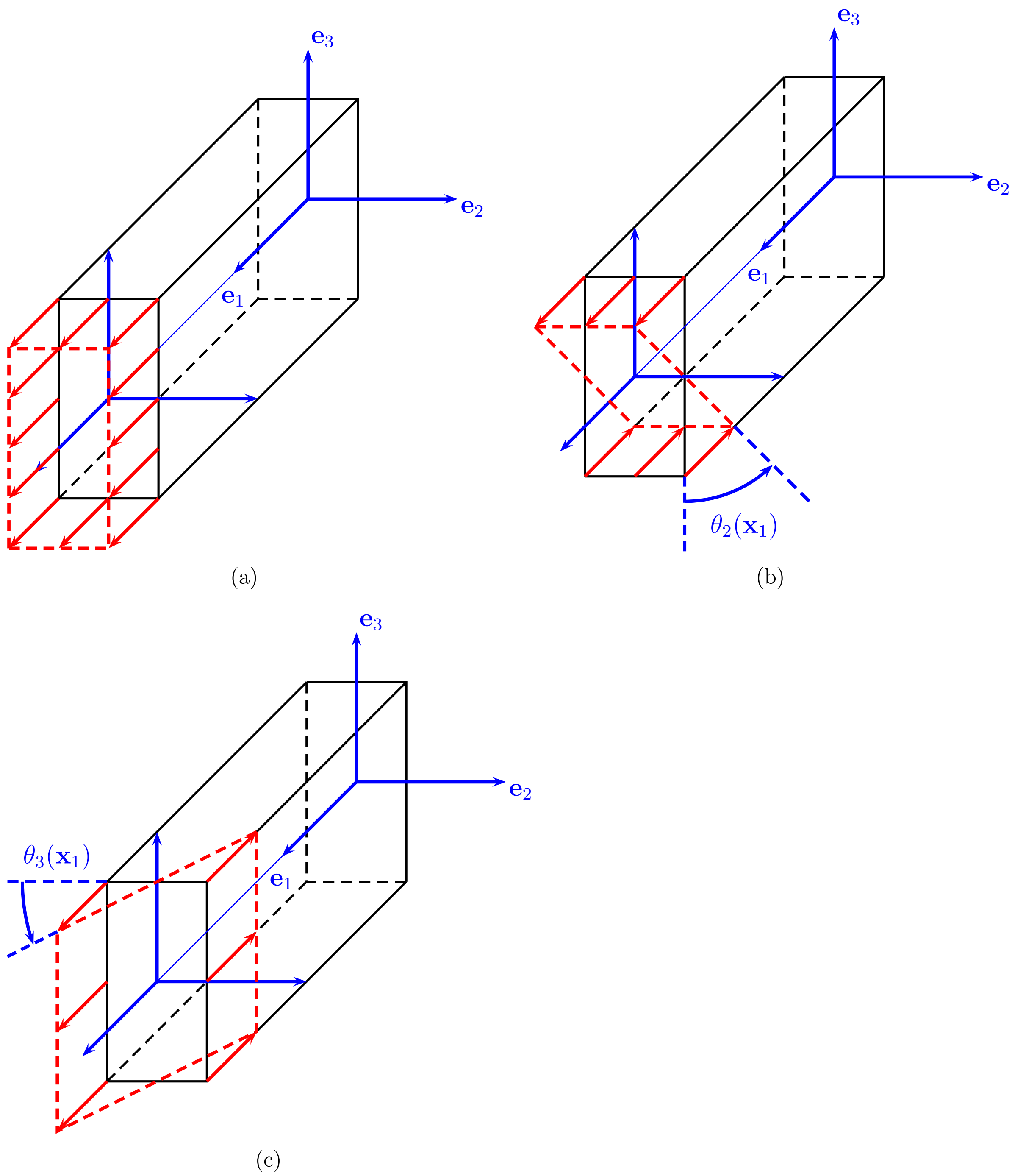

Figure 7.2: Second kinematic assumption in Euler-Bernoulli beam theory: cross sections remain planar after deformation.

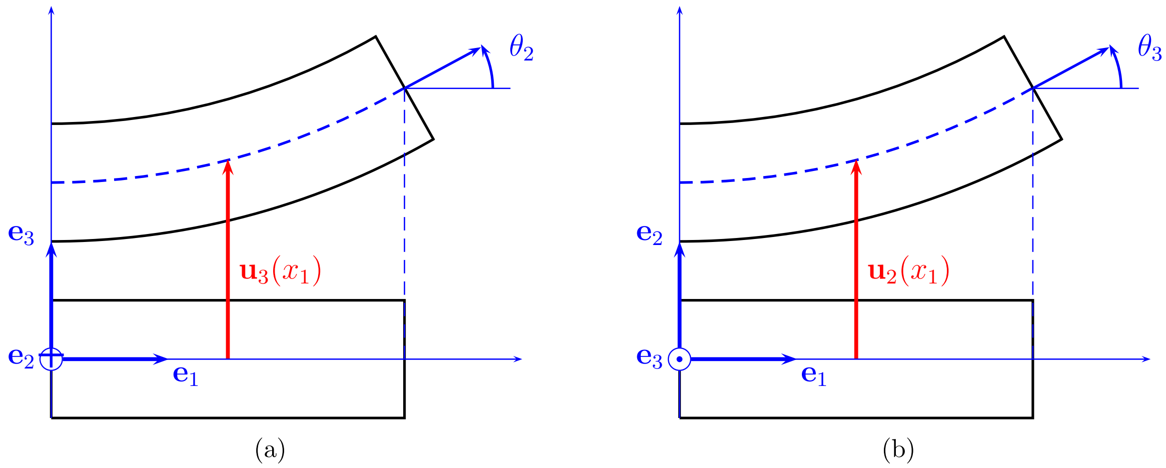

Figure 7.3: Implications of the assumption that cross sections remain normal to the axis of the beam upon deformation.

These assumptions have been extensively confirmed for slender beams made of isotropic materials with solid cross-sections.

Concept Question 7.1.4. Use the kinematic assumptions of Euler-Bernoulli beam theory to derive the general form of the strain field:

Concept Question 7.1.5. It is important to reflect on the nature of the strains due to bending. Interpret the components of the axial strain ϵ_{11} in Euler-Bernoulli beam theory

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One of the main conclusions of the Euler-Bernoulli assumptions is that in this particular beam theory the primary unknown variables are the three displacement functions $\bar{u}_1(x_1), \bar{u}_2(x_1), \bar{u}_3(x_1)$ which are only functions of x_1. The full displacement, strain and therefore stress fields do depend on the other independent variables but in a prescribed way that follows directly from the kinematic assumptions and from the equations of elasticity. The purpose of formulating a beam theory is to obtain a description of the problem expressed entirely on variables that depend on a single independent spatial variable x_1 which is the coordinate along the axis of the beam.

7.1.2 Definition of stress resultants

Readings: BC 5.3

Stress resultants are equivalent force systems that represent the integral effect of the internal stresses acting on the cross section. Thus, they eliminate the need to carry over the dependency of the stresses on the spatial coordinates of the cross section x_2, x_3.

The *axial or normal force* is defined by the expression:

$$N_1(x_1) = \int_A \sigma_{11}(x_1, x_2, x_3) dA \tag{7.1}$$

The *transverse shearing forces* are defined by the expressions:

$$V_2(x_1) = \int_A \sigma_{12}(x_1, x_2, x_3) dA \tag{7.2}$$

$$V_3(x_1) = \int_A \sigma_{13}(x_1, x_2, x_3) dA \tag{7.3}$$

(7.4)

The *bending moments* are defined by the expressions:

$$M_2(x_1) = \int_A x_3 \sigma_{11}(x_1, x_2, x_3) dA \tag{7.5}$$

$$M_3(x_1) = -\int_A x_2 \sigma_{11}(x_1, x_2, x_3) dA \tag{7.6}$$

(7.7)

The negative sign is needed to generate a positive bending moment with respect to axis $\mathbf{e}_3$, see Figure 7.4

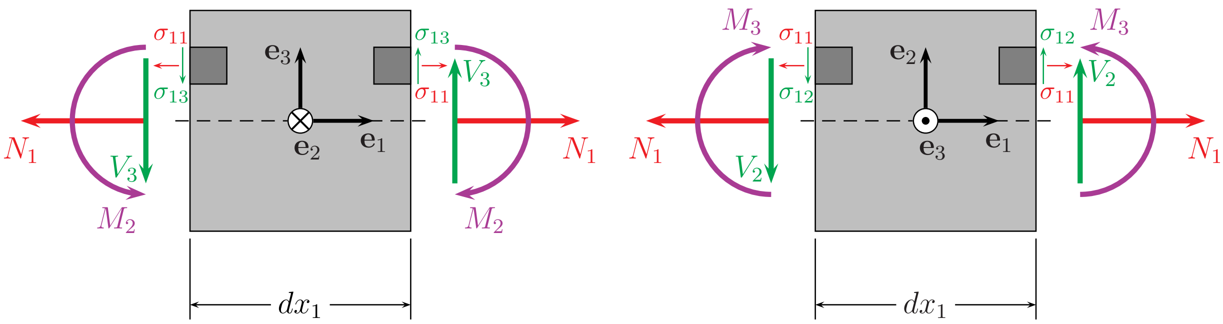

Figure 7.4: Sign conventions for the sectional stress resultants

7.2 Axial loading of beams

Readings: BC 5.4

Consider the case where there are no transverse loading on the beam, Figure 7.5. The only external loads possible in this case are either concentrated forces such as the load P_1, or distributed forces per unit length $p_1(x_1)$.

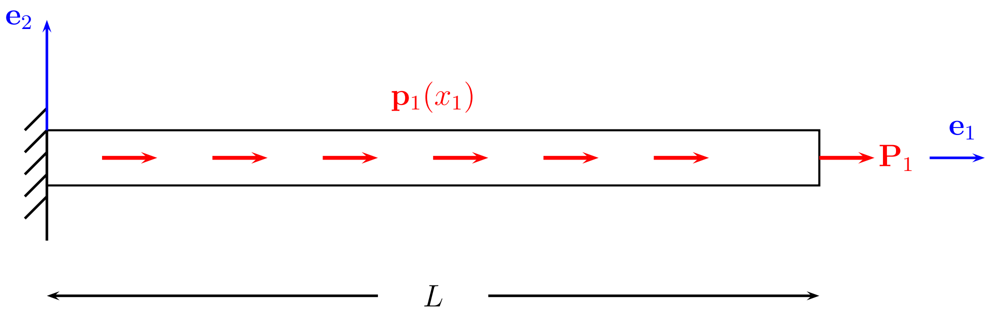

Figure 7.5: Beams subjected to axial loads.

7.2.1 Kinematics

In this case, we will assume that the cross sections will not rotate upon deformation.

Concept Question 7.2.1. Based on this assumption, specialize the general displacement (**??**) and strain field (**??**) to this class of problems and comment on the implications of the possible deformations obtained in this theory

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7.2.2 Constitutive law for the cross section

We will assume a linear-elastic isotropic material and that the transverse stresses $\sigma_{22}, \sigma_{33} \sim 0$. By Hooke's law, the axial stress σ_{11} is given by:

$$\sigma_{11}(x_1, x_2, x_3) = E\epsilon_{11}(x_1, x_2, x_3)$$

Replacing the strain field for this case:

$$\sigma_{11}(x_1, x_2, x_3) = E\bar{u}_1'(x_1) \tag{7.8}$$

In other words, we are assuming a state of uni-axial stress.

Concept Question 7.2.2. This exposes an inconsistency in Euler-Bernoulli beam theory. What is it and how can we justify it?

The axial force can be obtained by replacing (7.8) in (7.1):

$$N_1(x_1) = \int_{A(x_1)} E\bar{u}_1'(x_1)dA = \underbrace{\int_{A(x_1)} EdA}_{S(x_1)}\,\bar{u}_1'(x_1)$$

We will define:

$$S(x_1) = \int_{A(x_1)} E(x_1, x_2, x_3)dA \tag{7.9}$$

as the *axial stiffness* of the beam, where we allow the Young's modulus to vary freely both in the cross section and along the axis of the beam, and we allow for non-uniform cross section geometries. In the case that the section is homogeneous in the cross section ($E = E(x_1, \cancel{x_2}, \cancel{x_3})$), we obtain: $S(x_1) = E(x_1)A(x_1)$ (This may happen for example in functionally-graded materials). Further, if the section is uniform along x_1 and the material is homogeneous ($E = const$), we obtain: $S = EA$.

We can then write a constitutive relation between the axial force and the appropriate measure of strain for the beam:

$$N_1(x_1) = S(x_1)\bar{u}_1'(x_1) \tag{7.10}$$

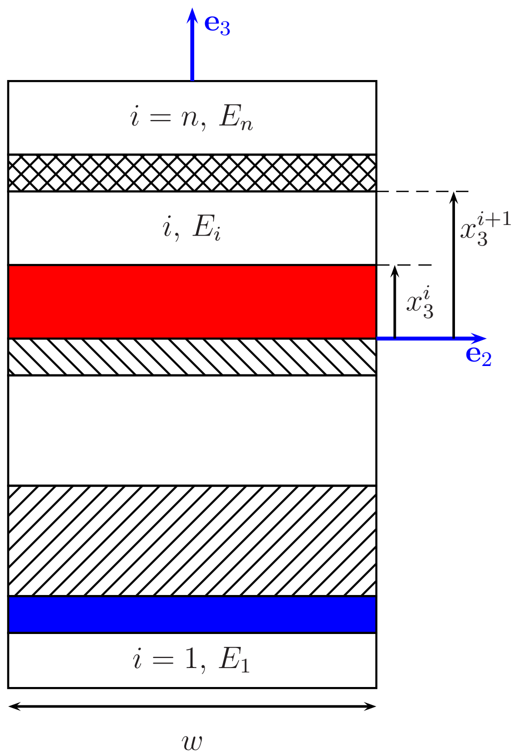

Figure 7.6: Cross section of a composite layered beam.

Concept Question 7.2.3. Axial loading of a composite beam

1. Compute the axial stiffness of a composite beam of width w, which has a uniform cross section with n different layers in direction $\mathbf{e}_3$, where the elastic modulus of layer i is E^i and its thickness is $t^i = x_3^{i+1} - x_3^i$, as shown in the figure.

2. Compute the stress distribution in the cross section assuming the axial force distribution $N_1(x_1)$ is known:

3. Interpret the stress distribution obtained.

Having completed a kinematic and constitutive description, it remains to formulate an appropriate way to enforce equilibrium of beams loaded axially.

7.2.3 Equilibrium equations

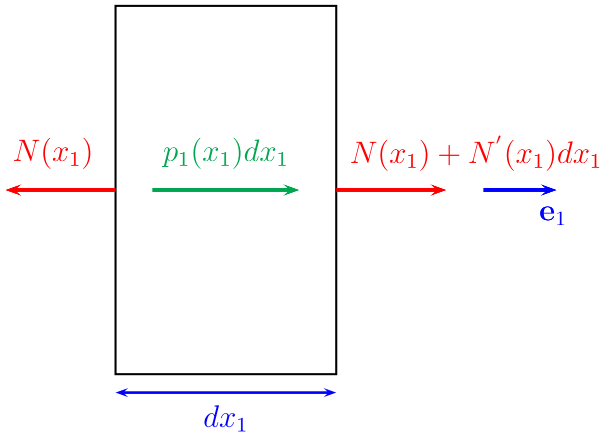

Figure 7.7: Axial forces acting on an infinitesimal beam slice.

For structural elements, we seek to impose equilibrium in terms of resultant forces (rather than at the material point as we did when we derived the equations of stress equilibrium). To this end, we consider the free body diagram of a slice of the beam as shown in Figure 7.7. At x_1 the axial force is $N_1(x_1)$, at $x_1 + dx_1$, $N_1(x_1 + dx_1) = N_1(x_1) + N'(x_1)dx_1$. The distributed force per unit length $p_1(x_1)$ produces a force in the positive x_1 direction equal to $p_1(x_1)dx_1$. Equilibrium of forces in the $\mathbf{e}_1$ direction requires:

$$-N_1(x_1) + p_1(x_1)dx_1 + N_1(x_1) + N'(x_1)dx_1 = 0$$

which implies:

$$\boxed{\frac{dN_1}{dx_1} + p_1 = 0} \tag{7.11}$$

7.2.4 Governing equations

Concept Question 7.2.4. 1. Derive a governing differential equation for the axially-loaded beam problem by combining Equations (7.10) and (7.11).

2. What type of elasticity formulation does this equation correspond to?

3. What principles does it enforce?

Concept Question 7.2.5. The derived equation requires boundary conditions.

1. How many boundary conditions are required?
2. What type of physical boundary conditions make sense for this problem and how are they expressed mathematically?

This completes the formulation for axially-loaded beams.

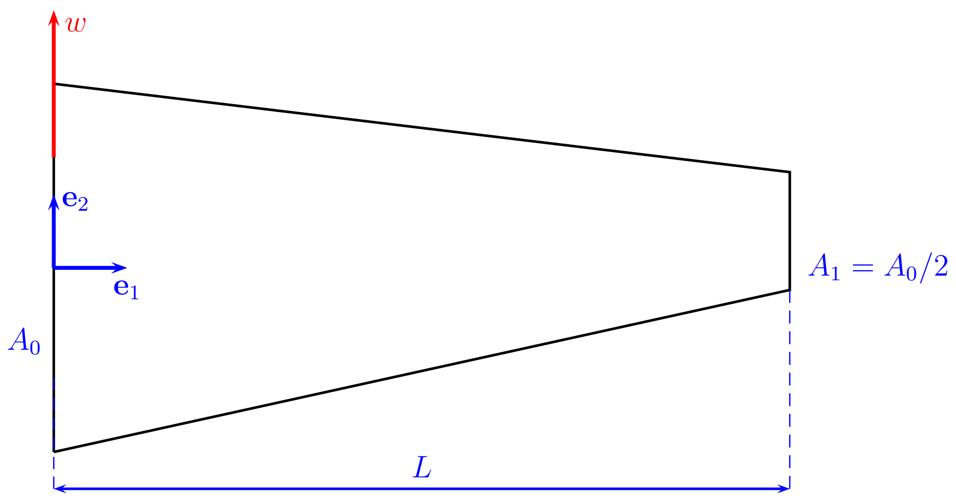

Figure 7.8: Schematic of a helicopter blade rotating at an angular speed ω

Concept Question 7.2.6. *Helicopter blade under centrifugal load* A helicopter blade of length $L = 5m$ is rotating at an angular velocity $\omega = 600$rpm about the $\mathbf{e}_2$ axis. The blade is made of a carbon-fiber reinforced polymer (CFRP) composite with mass density $\rho = 1500\text{kg} \cdot \text{m}^{-3}$, a Young's modulus $E = 80GPa$ and a yield stress $\sigma_y = 50$MPa. The area of the cross-section of the blade decreases linearly from a value $A_0 = 100\text{cm}^2$ at the root to $A_1 = A_0/2 = 50\text{cm}^2$ at the tip.

1. give the expression of the distributed axial load corresponding to the centrifugal force
2. Integrate the equilibrium equation (7.11) and apply appropriate boundary conditions to obtain the axial force distribution $N_1(x_1)$ in the blade.
3. What is the maximum axial force and where does it happen?
4. Provide an expression for the axial stress distribution $\sigma_{11}(x_1)$
5. What is the maximum stress, where does it happen, does the material yield?
6. The displacement can be obtained by integrating the strain:

$$\epsilon_{11} = \bar{u}_1'(x_1) = \frac{\sigma_{11}(x_1)}{E}$$

and applying the boundary condition $\bar{u}_1(0) = 0$. The solution can be readily found to be:

$$\bar{u}_1(x_1) = \frac{\rho\omega^2 L^3}{3E}\left[2\eta + \frac{\eta^2}{2} - \frac{\eta^3}{3} + 2\log\left(1 - \frac{\eta}{2}\right)\right]$$

Verify that the correct strain is obtained and that the boundary condition is satisfied:

7.3 Beam bending

Readings: BC 5.5

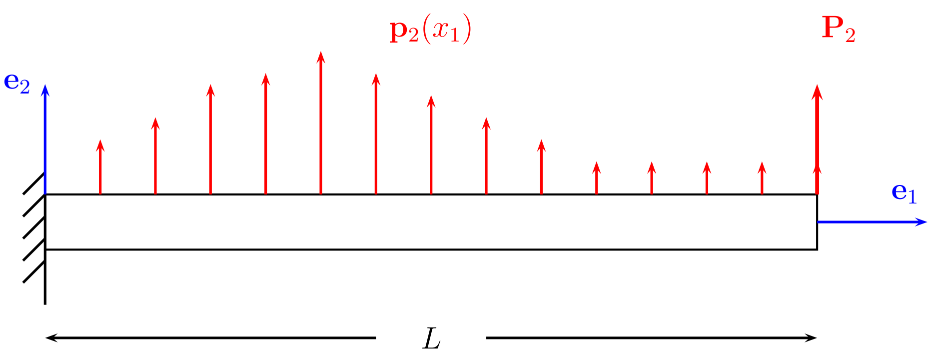

Figure 7.9: Beam subjected to tranverse loads

Beams have the defining characteristic that they can resist loads acting transversely to its axis, Figure 7.9 by bending or deflecting outside of their axis. This bending deformation causes internal axial and shear stresses which can be described by equipolent stress resultant moments and shearing forces.

We will start by analyzing beam bending in the plane $\mathbf{e}_1, \mathbf{e}_2$. Combined bending in different planes can be treated later by using the principle of superposition.

The Euler-Bernoulli kinematic hypothesis (**??**) reduces in this case to

$$u_1(x_1, x_2, x_3) = -x_2\bar{u}_2'(x_1),\ u_2(x_1, x_2, x_3) = \bar{u}_2(x_1),\ u_3(x_1, x_2, x_3) = 0$$

The strains to:

$$\epsilon_{11}(x_1, x_2, x_3) = u_{1,1} = -x_2\bar{u}_2''(x_1)$$

7.3.1 Constitutive law for the cross section

Hooke's law reduces one more time to:

$$\sigma_{11}(x_1, x_2, x_3) = E\epsilon_{11}(x_1, x_2, x_3) = -Ex_2\bar{u}_2''(x_1) \tag{7.12}$$

Concept Question 7.3.1. Assuming E is constant in the cross section, comment on the form of the stress distribution

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We now proceed to compute the stress resultants of section 7.1.2.

Concept Question 7.3.2. *Location of the neutral axis:* We will see in this question that the x_2 location of the fibers that do not stretch in the $\mathbf{e}_1$ direction, which is where we are going to place our origin of the x_2 coordinates is determined by the requirement of axial equilibrium of internal stresses.

1. The only applied external forces are in the $\mathbf{e}_2$ direction. Based on this, what can you say about the axial force $N_1(x_1)$?
2. Write the expression for the axial force
3. From here, obtain an expression that enforces equilibrium in the $\mathbf{e}_1$ direction. Interpret its meaning
4. Define the *modulus-weighted centroid* of the cross section by the condition:
$$S(x_1)x_2^c(x_1) = \int_{A(x_1)} Ex_2 dA, \Rightarrow \boxed{x_2^c(x_1) = \frac{1}{S(x_1)} \int_{A(x_1)} Ex_2 dA}$$
and interpret the meaning of using x_2^c as the origin of the coordinate system
5. Compare the location of the modulus-weighted centroid, the center of mass and the center of area for a general cross section and for a homogeneous one

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We now consider the internal bending moment.

Concept Question 7.3.3. Specialize the definition of the M_3 stress resultant (7.5)

$$M_3(x_1) = -\int_{A(x_1)} x_2 \sigma_{11}(x_1, x_2, x_3) dA$$

to the case under consideration by using the stress distribution resulting from the Euler-Bernoulli hypothesis, $\sigma_{11}(x_1, x_2, x_3) = -Ex_2\bar{u}_2''(x_1)$ to obtain a relation between the bending moment and the local curvature $\bar{u}_2''(x_1)$.

We can see that we obtain a linear relation between the bending moment and the local curvature (*moment-curvature relationship*):

$$\boxed{M_3(x_1) = H_{33}^c(x_1)\bar{u}_2''(x_1)} \tag{7.13}$$

The constant of proportionality will be referred to as the *centroidal bending stiffness* (also sometimes known as the *flexural rigidity*):

$$\boxed{H^c_{33}(x_1) = \int_{A(x_1)} E x_2^2 dA} \tag{7.14}$$

In the case of a homogeneous cross section of Young's modulus $E(x_1)$:

$$H^c_{33}(x_1) = E(x_1) \underbrace{\int_{A(x_1)} x_2^2 dA}_{I_{33}}$$

we obtain the familiar:

$$M = EI\bar{u}_2''(x_1)$$

Concept Question 7.3.4. *Modulus-weighted centroid, Bending stiffness and bending stress distribution in a layered composite beam* A composite beam of width b, which has a uniform

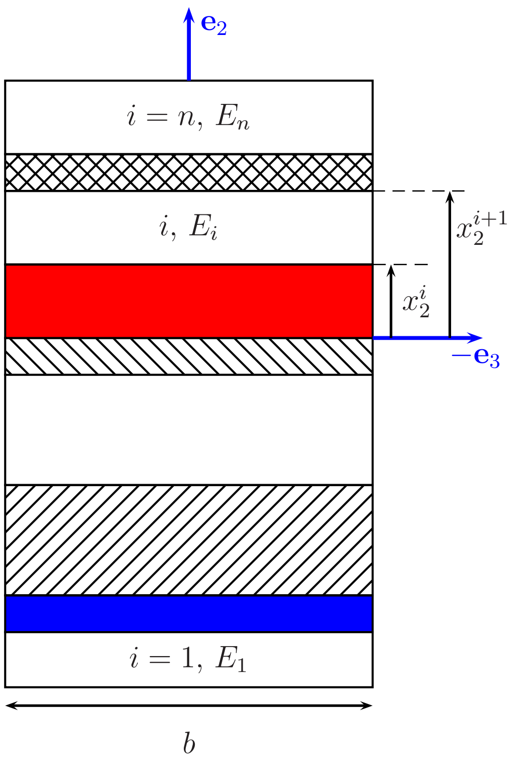

Figure 7.10: Cross section of a composite layered beam.

cross section with n different layers in direction $\mathbf{e}_2$, where the elastic modulus of layer i is E^i and its thickness is $t^i = x_2^{i+1} - x_2^i$, as shown in Figure 7.10.

1. Compute the position of the modulus-weighted centroid

2. Compute the bending stiffness
3. Compute the σ_{11} stress distribution in the cross section assuming the bending moment M_3 is known:
4. Interpret the stress distribution obtained.
5. Specialize to the case that the section is homogeneous with Young's modulus E:

Concept Question 7.3.5. *Bi-material cross section properties*

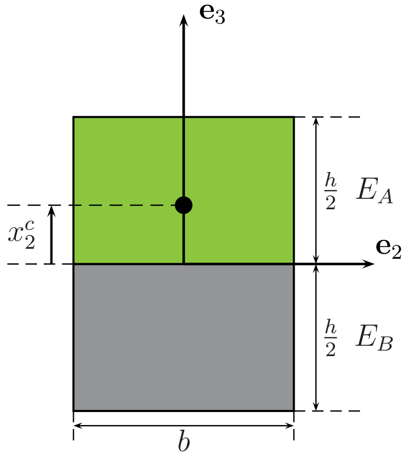

Figure 7.11: Bi-material beam

For the cantilevered beam shown in Figure 7.11,

1. Compute the location of the modulus-weighted centroid:

7.3.2 Equilibrium equations

Concept Question 7.3.6. Consider the equilibrium of a slice of a beam subjected to transverse loads. Using the figure,

1. write the equation of equilibrium of forces in the $\mathbf{e}_2$ direction and then derive a differential equation relating the shear force $V_2(x_1)$ and the distributed external force $p_2(x_1)$.
2. do the same for equilibrium of moments in the $\mathbf{e}_3$ axis about point $\mathbf{O}$.
3. Eliminate the Shear force from the previous two equations to obtain a single equilibrium equation relating the bending moment and the applied distributed load:

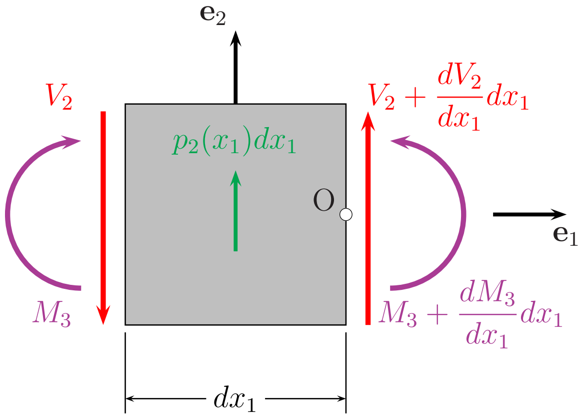

Figure 7.12: Equilibrium of a beam slice subjected to transverse loads

7.3.3 Governing equations

Concept Question 7.3.7.
1. Derive a governing differential equation for the transversely-loaded beam problem by combining Equations (7.13) and (**??**).
2. What type of elasticity formulation does this equation correspond to?
3. What principles does it enforce?

Concept Question 7.3.8. The equation requires four boundary conditions since it is a fourth-order differential equation. Express the following typical boundary conditions mathematically

1. clamped at one end
2. simply supported or pinned
3. free (unloaded)
4. subjected to a concentrated transverse load P_2

Concept Question 7.3.9. *Cantilevered beam under uniformly distributed transverse load*
A cantilevered beam (clamped at $x_1 = 0$ and free at $x_1 = L$) is subjected to a uniform load per unit length gp_0.

1. Specialize the general beam equation to this problem
2. Write down the boundary conditions for this problem:
3. Integrate the governing equation and apply the boundary conditions to obtain the deflection $\bar{u}_2(x_1)$, bending moment $M_3(x_1)$ and shear force $V_2(x_1)$.

4. Compute the maximum deflection, maximum moment and maximum σ_{11} stress (for the case of a solid rectangular wing spar of length $L = 1$m, width $b = 5$mm, height $h = 3$cm, and Young's modulus $E = 100GPa$ when the load is $p_0 = 10N/m$.
