Module B

# Transportation and Assignment Solution Methods

## Solution of the Transportation Model

The following example was used in Chapter 6 of the text to demonstrate the formulation of the transportation model. Wheat is harvested in the Midwest and stored in grain elevators in three different cities—Kansas City, Omaha, and Des Moines. These grain elevators supply three flour mills, located in Chicago, St. Louis, and Cincinnati. Grain is shipped to the mills in railroad cars, each of which is capable of holding one ton of wheat. Each grain elevator is able to supply the following number of tons (i.e., railroad cars) of wheat to the mills on a monthly basis:

| *Grain Elevator* | *Supply* |
|---|---|
| 1. Kansas City | 150 |
| 2. Omaha | 175 |
| 3. Des Moines | 275 |
| Total | 600 tons |

Each mill demands the following number of tons of wheat per month.

| *Mill* | *Demand* |
|---|---|
| A. Chicago | 200 |
| B. St. Louis | 100 |
| C. Cincinnati | 300 |
| Total | 600 tons |

The cost of transporting one ton of wheat from each grain elevator (source) to each mill (destination) differs according to the distance and rail system. These costs are shown in the following table. For example, the cost of shipping one ton of wheat from the grain elevator at Omaha to the mill at Chicago is $7.

| | **Mill** | | |
|---|---|---|---|
| **Grain Elevator** | *A. Chicago* | *B. St. Louis* | *C. Cincinnati* |
| 1. Kansas City | $6 | $ 8 | $10 |
| 2. Omaha | 7 | 11 | 11 |
| 3. Des Moines | 4 | 5 | 12 |

The problem is to determine how many tons of wheat to transport from each grain elevator to each mill on a monthly basis in order to minimize the total cost of transportation.

The linear programming model for this problem is formulated in the equations that follow:

$$\text{minimize } Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$$

subject to

$$\begin{aligned}
x_{1A} + x_{1B} + x_{1C} &= 150 \\
x_{2A} + x_{2B} + x_{2C} &= 175 \\
x_{3A} + x_{3B} + x_{3C} &= 275 \\
x_{1A} + x_{2A} + x_{3A} &= 200 \\
x_{1B} + x_{2B} + x_{3B} &= 100 \\
x_{1C} + x_{2C} + x_{3C} &= 300 \\
x_{ij} &\geq 0
\end{aligned}$$

In this model the decision variables, $x_{ij}$, represent the number of tons of wheat transported from each grain elevator, $i$ (where $i = 1, 2, 3$), to each mill, $j$ (where $j = $ A, B, C). The objective function represents the total transportation cost for each route. Each term in the objective function reflects the cost of the tonnage transported for one route. For example, if 20 tons are transported from elevator 1 to mill A, the cost of \$6 is multiplied by $x_{1A}(=20)$, which equals \$120.

The first three constraints in the linear programming model represent the supply at each elevator; the last three constraints represent the demand at each mill. As an example, consider the first supply constraint, $x_{1A} + x_{1B} + x_{1C} = 150$. This constraint represents the tons of wheat transported from Kansas City to all three mills: Chicago ($x_{1A}$), St. Louis ($x_{1B}$), and Cincinnati ($x_{1C}$). The amount transported from Kansas City is limited to the 150 tons available. Note that this constraint (as well as all others) is an equation (=) rather than a $\leq$ inequality because all the tons of wheat available will be needed to meet the *total demand* of 600 tons. In other words, the three mills demand 600 total tons, which is the exact amount that can be supplied by the three grain elevators. Thus, all that *can* be supplied *will be,* in order to meet demand. This type of model, in which supply exactly equals demand, is referred to as a *balanced transportation* model. The balanced model will be used to demonstrate the solution of a transportation problem.

*Transportation problems are solved manually within a tableau format.*

Transportation models are solved manually within the context of a *tableau,* as in the simplex method. The tableau for our wheat transportation model is shown in Table B-1.

**Table B-1**
The Transportation Tableau

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| ***1*** | 6 | 8 | 10 | 150 |
| ***2*** | 7 | 11 | 11 | 175 |
| ***3*** | 4 | 5 | 12 | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

*Each cell in a transportation tableau is analogous to a decision variable that indicates the amount allocated from a source to a destination.*

*The supply and demand values along the outside rim of a tableau are called **rim requirements**.*

Each cell in the tableau represents the amount transported from one source to one destination. Thus, the amount placed in each cell is the value of a decision variable for that cell. For example, the cell at the intersection of row 1 and column A represents the decision variable $x_{1A}$. The smaller box within each cell contains the unit transportation cost for that route. For example, in cell 1A the value, \$6, is the cost of transporting one ton of wheat from Kansas City to Chicago. Along the outer rim of the tableau are the supply and demand constraint quantity values, which are referred to as **rim requirements**.

*Transportation models do not start at the origin where all decision variables equal zero; they must be given an initial feasible solution.*

The two methods for solving a transportation model are the *stepping-stone method* and the *modified distribution method* (also known as *MODI*). In applying the simplex method, an initial solution had to be established in the initial simplex tableau. This same condition must be met in solving a transportation model. In a transportation model, an initial feasible solution can be found by several alternative methods, including the *northwest corner method,* the *minimum cell cost method,* and *Vogel's approximation model.*

## The Northwest Corner Method

*In the* ***northwest corner method*** *the largest possible allocation is made to the cell in the upper left-hand corner of the tableau, followed by allocations to adjacent feasible cells.*

With the **northwest corner method**, an initial allocation is made to the cell in the upper left-hand corner of the tableau (i.e., the "northwest corner"). The amount allocated is the most possible, *subject to* the supply and demand constraints for that cell. In our example, we first allocate as much as possible to cell 1A (the northwest corner). This amount is 150 tons, since that is the maximum that can be supplied by grain elevator 1 at Kansas City, even though 200 tons are demanded by mill A at Chicago. This initial allocation is shown in Table B-2.

We next allocate to a cell *adjacent* to cell 1A, in this case either cell 2A or cell 1B. However, cell 1B no longer represents a feasible allocation, because the total tonnage of wheat available at source 1 (i.e., 150 tons) has already been allocated. Thus, cell 2A represents the only feasible alternative, and as much as possible is allocated to this cell. The amount allocated at 2A can be either 175 tons, the supply available from source 2 (Omaha), or 50 tons, the amount *now* demanded at destination A. (Recall that 150 of the 200 tons demanded at A have already been supplied.) Because 50 tons is the most constrained amount, it is allocated to cell 2A, as shown in Table B-2.

**Table B-2**
The Initial NW Corner Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | [6] 150 | [8] | [10] | 150 |
| **2** | [7] 50 | [11] 100 | [11] 25 | 175 |
| **3** | [4] | [5] | [12] 275 | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

The third allocation is made in the same way as the second allocation. The only feasible cell adjacent to cell 2A is cell 2B. The most that can be allocated is either 100 tons (the amount demanded at mill B) or 125 tons (175 tons minus the 50 tons allocated to cell 2A). The smaller (most constrained) amount, 100 tons, is allocated to cell 2B, as shown in Table B-2.

*The initial solution is complete when all rim requirements are satisfied.*

The fourth allocation is 25 tons to cell 2C, and the fifth allocation is 275 tons to cell 3C, both of which are shown in Table B-2. Notice that all of the row and column allocations add up to the appropriate rim requirements.

The transportation cost of this solution is computed by substituting the cell allocations (i.e., the amounts transported),

$$\begin{aligned} x_{1A} &= 150 \\ x_{2A} &= 50 \\ x_{2B} &= 100 \\ x_{2C} &= 25 \\ x_{3C} &= 275 \end{aligned}$$

into the objective function:

$$\begin{aligned} Z &= \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C} \\ &= 6(150) + 8(0) + 10(0) + 7(50) + 11(100) + 11(25) + 4(0) + 5(0) + 12(275) \\ &= \$5{,}925 \end{aligned}$$

The steps of the northwest corner method are summarized here:

1. Allocate as much as possible to the cell in the upper left-hand corner, subject to the supply and demand constraints.
2. Allocate as much as possible to the next adjacent feasible cell.
3. Repeat step 2 until all rim requirements have been met.

## The Minimum Cell Cost Method

*In the* ***minimum cell cost method*** *as much as possible is allocated to the cell with the minimum cost.*

With the **minimum cell cost method,** the basic logic is to allocate to the cells with the lowest costs. The initial allocation is made to the cell in the tableau having the lowest cost. In the transportation tableau for our example problem, cell 3A has the minimum cost of $4. As much as possible is allocated to this cell; the choice is either 200 tons or 275 tons. Even though 275 tons could be supplied to cell 3A, the most we can allocate is 200 tons, since only 200 tons are demanded. This allocation is shown in Table B-3.

**Table B-3**
The Initial Minimum Cell Cost Allocation

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | [6] | [8] | [10] | 150 |
| *2* | [7] | [11] | [11] | 175 |
| *3* | [4] 200 | [5] | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

Notice that all the remaining cells in column A have now been eliminated, because all the wheat demanded at destination A, Chicago, has now been supplied by source 3, Des Moines.

The next allocation is made to the cell that has the minimum cost and also is feasible. This is cell 3B, which has a cost of $5. The most that can be allocated is 75 tons (275 tons minus the 200 tons already supplied). This allocation is shown in Table B-4.

**Table B-4**
The Second Minimum Cell Cost Allocation

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | [6] | [8] | [10] | 150 |
| *2* | [7] | [11] | [11] | 175 |
| *3* | [4] 200 | [5] 75 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

The third allocation is made to cell 1B, which has the minimum cost of $8. (Notice that cells with lower costs, such as 1A and 2A, are not considered because they were previously ruled out as infeasible.) The amount allocated is 25 tons. The fourth allocation of 125 tons is made to cell 1C, and the last allocation of 175 tons is made to cell 2C. These allocations, which complete the initial minimum cell cost solution, are shown in Table B-5.

**Table B-5**
The Initial Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | 6 | 8<br>25 | 10<br>125 | 150 |
| **2** | 7 | 11 | 11<br>175 | 175 |
| **3** | 4<br>200 | 5<br>75 | 12 | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

*The minimum cell cost method will provide a solution with a lower cost than the northwest corner solution because it considers cost in the allocation process.*

The total cost of this initial solution is $4,550, as compared with a total cost of $5,925 for the initial northwest corner solution. It is not a coincidence that a lower total cost is derived using the minimum cell cost method; it is a logical occurrence. The northwest corner method does not consider cost at all in making allocations—the minimum cell cost method does. It is therefore quite natural that a lower initial cost will be attained using the latter method. Thus, the initial solution achieved by using the minimum cell cost method is usually better in that, because it has a lower cost, it is closer to the optimal solution; fewer subsequent iterations will be required to achieve the optimal solution.

The specific steps of the minimum cell cost method are summarized next:

1. Allocate as much as possible to the feasible cell with the minimum transportation cost, and adjust the rim requirements.
2. Repeat step 1 until all rim requirements have been met.

## Vogel's Approximation Model

*A **penalty cost** is the difference between the largest and next largest cell cost in a row (or column).*

The third method for determining an initial solution, Vogel's approximation model (also called VAM), is based on the concept of **penalty cost** or *regret*. If a decision maker incorrectly chooses from several alternative courses of action, a penalty may be suffered (and the decision maker may regret the decision that was made). In a transportation problem, the courses of action are the alternative routes, and a wrong decision is allocating to a cell that does not contain the lowest cost.

*VAM allocates as much as possible to the minimum cost cell in the row or column with the largest penalty cost.*

In the VAM method, the first step is to develop a penalty cost for each source and destination. For example, consider column A in Table B-6. Destination A, Chicago, can be supplied by Kansas City, Omaha, and Des Moines. The best decision would be to supply Chicago from source 3 because cell 3A has the minimum cost of $4. If a wrong decision was made and the next higher cost of $6 was selected at cell 1A, a "penalty" of $2 per ton would result (i.e., $6 − 4 = $2). This demonstrates how the penalty cost is determined for each row and column of the tableau. The general rule for computing a penalty cost is to subtract the minimum cell cost from the next higher cell cost in each row and column. The penalty costs for our example are shown at the right and at the bottom of Table B-6.

**Table B-6**
The VAM Penalty Costs

| From \ To | A | B | C | Supply | |
|---|---|---|---|---|---|
| **1** | 6 | 8 | 10 | 150 | 2 |
| **2** | 7 | 11 | 11 | 175 | 4 |
| **3** | 4 | 5 | 12 | 275 | 1 |
| **Demand** | 200 | 100 | 300 | 600 | |
| | 2 | 3 | 1 | | |

The initial allocation in the VAM method is made in the row or column that has the highest penalty cost. In Table B-6, row 2 has the highest penalty cost of $4. We allocate as much as possible to the feasible cell in this row with the minimum cost. In row 2, cell 2A has the lowest cost of $7, and the most that can be allocated to cell 2A is 175 tons. With this allocation the greatest penalty cost of $4 has been avoided because the best course of action has been selected. The allocation is shown in Table B-7.

**Table B-7**
The Initial VAM Allocation

| From \ To | A | B | C | Supply | |
|---|---|---|---|---|---|
| **1** | 6 | 8 | 10 | 150 | 2 |
| **2** | 7<br>175 | 11 | 11 | 175 | |
| **3** | 4 | 5 | 12 | 275 | 1 |
| **Demand** | 200 | 100 | 300 | 600 | |
| | 2 | 3 | 2 | | |

*After each VAM cell allocation, all row and column penalty costs are recomputed.*

After the initial allocation is made, *all* the penalty costs must be recomputed. In some cases the penalty costs will change; in other cases they will not change. For example, the penalty cost for column C in Table B-7 changed from $1 to $2 (because cell 2C is no longer considered in computing penalty cost), and the penalty cost in row 2 was eliminated altogether (because no more allocations are possible for that row).

Next, we repeat the previous step and allocate to the row or column with the highest penalty cost, which is now column B with a penalty cost of $3 (see Table B-7). The cell in column B with the lowest cost is 3B, and we allocate as much as possible to this cell, 100 tons. This allocation is shown in Table B-8.

Note that all penalty costs have been recomputed in Table B-8. Since the highest penalty cost is now $8 for row 3 and since cell 3A has the minimum cost of $4, we allocate 25 tons to this cell, as shown in Table B-9.

**Table B-8**
The Second VAM Allocation

| From \ To | A | B | C | Supply | |
|---|---|---|---|---|---|
| **1** | (6) | (8) | (10) | 150 | 4 |
| **2** | (7) 175 | (11) | (11) | 175 | |
| **3** | (4) | (5) 100 | (12) | 275 | 8 |
| **Demand** | 200 | 100 | 300 | 600 | |
| | 2 | | 2 | | |

**Table B-9**
The Third VAM Allocation

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | (6) | (8) | (10) | 150 |
| **2** | (7) 175 | (11) | (11) | 175 |
| **3** | (4) 25 | (5) 100 | (12) | 275 |
| **Demand** | 200 | 100 | 300 | 600 |
| | | | 2 | |

Table B-9 also shows the recomputed penalty costs after the third allocation. Notice that by now only column C has a penalty cost. Rows 1 and 3 have only one feasible cell, so a penalty does not exist for these rows. Thus, the last two allocations are made to column C. First, 150 tons are allocated to cell 1C because it has the lowest cell cost. This leaves only cell 3C as a feasible possibility, so 150 tons are allocated to this cell. Both of these allocations are shown in Table B-10.

**Table B-10**
The Initial VAM Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | (6) | (8) | (10) 150 | 150 |
| **2** | (7) 175 | (11) | (11) | 175 |
| **3** | (4) 25 | (5) 100 | (12) 150 | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

*VAM and minimum cell cost both provide better initial solutions than the northwest corner method.*

The total cost of this initial Vogel's approximation model solution is $5,125, which is not as high as the northwest corner initial solution of $5,925. It is also not as low as the minimum cell cost solution of $4,550. Like the minimum cell cost method, VAM typically results in a lower cost for the initial solution than does the northwest corner method.

The steps of Vogel's approximation model can be summarized in the following list:

1. Determine the penalty cost for each row and column by subtracting the lowest cell cost in the row or column from the next lowest cell cost in the same row or column.
2. Select the row or column with the highest penalty cost (breaking ties arbitrarily or choosing the lowest-cost cell).
3. Allocate as much as possible to the feasible cell with the lowest transportation cost in the row or column with the highest penalty cost.
4. Repeat steps 1, 2, and 3 until all rim requirements have been met.

## The Stepping-Stone Solution Method

*Once an initial solution is derived, the problem must be solved using either the* ***stepping-stone method*** *or the* ***modified distribution method (MODI)****.*

Once an initial *basic feasible solution* has been determined by any of the previous three methods, the next step is to solve the model for the optimal (i.e., minimum total cost) solution. There are two basic solution methods: the **stepping-stone solution method** and the **modified distribution method (MODI)**. The stepping-stone solution method will be demonstrated first. Because the initial solution obtained by the minimum cell cost method had the lowest total cost of the three initial solutions, we will use it as the starting solution. Table B-11 repeats the initial solution that was developed from the minimum cell cost method.

**Table B-11**
The Minimum Cell Cost Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | [6] | [8] 25 | [10] 125 | 150 |
| **2** | [7] | [11] | [11] 175 | 175 |
| **3** | [4] 200 | [5] 75 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

*The stepping-stone method determines whether there is a cell with no allocation that would reduce cost if used.*

The basic solution principle in a transportation problem is to determine whether a transportation route not at present being used (i.e., an empty cell) would result in a lower total cost if it were used. For example, Table B-11 shows four empty cells (1A, 2A, 2B, and 3C) representing unused routes. Our first step in the stepping-stone method is to evaluate these empty cells to see whether the use of any of them would reduce total cost. If we find such a route, then we will allocate as much as possible to it.

First, let us consider allocating one ton of wheat to cell 1A. If one ton is allocated to cell 1A, cost will be increased by $6—the transportation cost for cell 1A. However, by allocating one ton to cell 1A, we increase the supply in row 1 to 151 tons, as shown in Table B-12.

**Table B-12**
The Allocation of One Ton to Cell 1A

| From \ To | A | B | C | Supply | |
|---|---|---|---|---|---|
| *1* | +1 $\boxed{6}$ | $\boxed{8}$ 25 | $\boxed{10}$ 125 | 150 | 151 |
| *2* | $\boxed{7}$ | $\boxed{11}$ | $\boxed{11}$ 175 | 175 | |
| *3* | $\boxed{4}$ 200 | $\boxed{5}$ 75 | $\boxed{12}$ | 275 | |
| **Demand** | 200 | 100 | 300 | 600 | |

The constraints of the problem cannot be violated, and feasibility must be maintained. If we add one ton to cell 1A, we must subtract one ton from another allocation along that row. Cell 1B is a logical candidate because it contains 25 tons. By subtracting one ton from cell 1B, we now have 150 tons in row 1, and we have satisfied the supply constraint again. At the same time, subtracting one ton from cell 1B has reduced total cost by $8.

However, by subtracting one ton from cell 1B, we now have only 99 tons allocated to column B, where 100 tons are demanded, as shown in Table B-13. To compensate for this constraint violation, one ton must be added *to a cell that already has an allocation*. Since cell 3B has 75 tons, we will add one ton to this cell, which again satisfies the demand constraint of 100 tons.

**Table B-13**
The Subtraction of One Ton from Cell 1B

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | +1 $\boxed{6}$ | –1 $\boxed{8}$ 25 | $\boxed{10}$ 125 | 150 |
| *2* | $\boxed{7}$ | $\boxed{11}$ | $\boxed{11}$ 175 | 175 |
| *3* | $\boxed{4}$ 200 | $\boxed{5}$ 75 | $\boxed{12}$ | 275 |
| **Demand** | 200 | 100 | 300 | 600 |
| | | 99 | | |

A requirement of this solution method is that units can be added to and subtracted from only those cells that already have allocations. That is why one ton was added to cell 3B and not to cell 2B. It is from this requirement that the method derives its name. The process of adding and subtracting units from allocated cells is analogous to crossing a pond by stepping on stones (i.e., only allocated-to cells).

By allocating one extra ton to cell 3B we have increased cost by $5, the transportation cost for that cell. However, we have also increased the supply in row 3 to 276 tons, a violation of the supply constraint for this source. As before, this violation can be remedied by subtracting one ton from cell 3A, which contains an allocation of 200 tons. This satisfies the supply constraint again for row 3, and it also reduces the total cost by $4, the transportation cost for cell 3A. These allocations and deletions are shown in Table B-14.

**Table B-14**

The Addition of One Ton to Cell 3B and the Subtraction of One Ton from Cell 3A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | +1 $\boxed{6}$ | −1 $\boxed{8}$ 25 | $\boxed{10}$ 125 | 150 |
| 2 | $\boxed{7}$ | $\boxed{11}$ | $\boxed{11}$ 175 | 175 |
| 3 | −1 $\boxed{4}$ 200 | +1 $\boxed{5}$ 75 | $\boxed{12}$ | 275 |
| Demand | 200 | 100 | 300 | 600 |

Notice in Table B-14 that by subtracting one ton from cell 3A, we did not violate the demand constraint for column A, since we previously added one ton to cell 1A.

Now let us review the increases and reductions in costs resulting from this process. We initially increased cost by \$6 at cell 1A, then reduced cost by \$8 at cell 1B, then increased cost by \$5 at cell 3B, and, finally, reduced cost by \$4 at cell 3A:

$$1A \rightarrow 1B \rightarrow 3B \rightarrow 3A$$
$$+\$6 - 8 + 5 - 4 = -\$1$$

*An empty cell that will reduce cost is a potential entering variable.*

In other words, for each ton allocated to cell 1A (a route not at present used), total cost will be reduced by \$1. This indicates that the initial solution is not optimal because a lower cost can be achieved by allocating additional tons of wheat to cell 1A (i.e., cell 1A is analogous to a pivot column in the simplex method). Our goal is to determine the cell or entering "variable" that will reduce cost the most. Another variable (empty cell) may result in an even greater decrease in cost than cell 1A. If such a cell exists, it will be selected as the entering variable; if not, cell 1A will be selected. To identify the appropriate entering variable, the remaining empty cells must be tested as cell 1A was.

*To evaluate the cost reduction potential of an empty cell, a closed path connecting used cells to the empty cell is identified.*

Before testing the remaining empty cells, let us identify a few of the general characteristics of the stepping-stone process. First, we always start with an empty cell and form a *closed path* of cells that now have allocations. In developing the path, it is possible to skip over both unused and used cells. In any row or column there can be only *one* addition and *one* subtraction. (For example, in row 1, wheat is added at cell 1A and is subtracted at cell 1B.)

Let us test cell 2A to see if it results in a cost reduction. The stepping-stone closed path for cell 2A is shown in Table B-15. Notice that the path for cell 2A is slightly more complex

**Table B-15**

The Stepping-Stone Path for Cell 2A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | $\boxed{6}$ | − $\boxed{8}$ 25 | + $\boxed{10}$ 125 | 150 |
| 2 | + $\boxed{7}$ | $\boxed{11}$ | − $\boxed{11}$ 175 | 175 |
| 3 | − $\boxed{4}$ 200 | + $\boxed{5}$ 75 | $\boxed{12}$ | 275 |
| Demand | 200 | 100 | 300 | 600 |

$2A \rightarrow 2C \rightarrow 1C \rightarrow 1B \rightarrow 3B \rightarrow 3A$
$+\$7 - 11 + 10 - 8 + 5 - 4 = -\$1$

than the path for cell 1A. Notice also that the path crosses itself at one point, which is perfectly acceptable. An allocation to cell 2A will reduce cost by $1, as shown in the computation in Table B-15. Thus, we have located another possible entering variable, although it is no better than cell 1A.

The remaining stepping-stone paths and the resulting computations for cells 2B and 3C are shown in Tables B-16 and B-17, respectively.

**Table B-16**
The Stepping-Stone Path for Cell 2B

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | 6 | + 8<br>25 | − 10<br>125 | 150 |
| **2** | 7 | − 11 | + 11<br>175 | 175 |
| **3** | 4<br>200 | 5<br>75 | 12 | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

2B → 2C → 1C → 1B
+$11 − 11 + 10 − 8 = +$2

**Table B-17**
The Stepping-Stone Path for Cell 3C

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | 6 | + 8<br>25 | − 10<br>125 | 150 |
| **2** | 7 | 11 | 11<br>175 | 175 |
| **3** | 4<br>200 | − 5<br>75 | + 12 | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

3C → 1C → 1B → 3B
+$12 − 10 + 8 − 5 = +$5

*After all empty cells are evaluated, the one with the greatest cost reduction potential is the entering variable.*

Notice that after all four unused routes are evaluated, there is a tie for the entering variable between cells 1A and 2A. Both show a reduction in cost of $1 per ton allocated to that route. The tie can be broken arbitrarily. We will select cell 1A (i.e., $x_{1A}$) to enter the solution.

Because the total cost of the model will be reduced by $1 for each ton we can reallocate to cell 1A, we naturally want to reallocate as much as possible. To determine how much to allocate, we need to look at the path for cell 1A again, as shown in Table B-18.

*When reallocating units to the entering variable (cell), the amount is the minimum amount subtracted on the stepping-stone path.*

The stepping-stone path in Table B-18 shows that tons of wheat must be subtracted at cells 1B and 3A to meet the rim requirements and thus satisfy the model constraints. Because we cannot subtract more than is available in a cell, we are limited by the 25 tons in cell 1B. In other words, if we allocate more than 25 tons to cell 1A, then we must subtract more than 25 tons from 1B, which is impossible because only 25 tons are available. Therefore, 25 tons is the amount we reallocate to cell 1A according to our path. That is, 25 tons are added to 1A, subtracted from 1B, added to 3B, and subtracted from 3A. This reallocation is shown in Table B-19.

**Table B-18**
The Stepping-Stone Path for Cell 1A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | + [6] | − [8] 25 | [10] 125 | 150 |
| *2* | [7] | [11] | [11] 175 | 175 |
| *3* | − [4] 200 | + [5] 75 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

**Table B-19**
The Second Iteration of the Stepping-Stone Method

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | [6] 25 | [8] | [10] 125 | 150 |
| *2* | [7] | [11] | [11] 175 | 175 |
| *3* | [4] 175 | [5] 100 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

The process culminating in Table B-19 represents one *iteration* of the stepping-stone method. We selected $x_{1A}$ as the entering variable, and it turned out that $x_{1B}$ was the leaving variable (because it now has a value of zero in Table B-19). Thus, at each iteration one variable enters and one leaves (just as in the simplex method).

Now we must check to see whether the solution shown in Table B-19 is, in fact, optimal. We do this by plotting the paths for the unused routes (i.e., empty cells 2A, 1B, 2B, and 3C) that are shown in Table B-19. These paths are shown in Tables B-20 through B-23.

**Table B-20**
The Stepping-Stone Path for Cell 2A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | − [6] 25 | [8] | + [10] 125 | 150 |
| *2* | + [7] | [11] | − [11] 175 | 175 |
| *3* | [4] 175 | [5] 100 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

2A → 2C → 1C → 1A
+\$7 − 11 + 10 − 6 = \$0

**Table B-21**
The Stepping-Stone Path for Cell 1B

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | − [6] 25 | + [8] | [10] 125 | 150 |
| **2** | [7] | [11] | [11] 175 | 175 |
| **3** | + [4] 175 | − [5] 100 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

1B ⟶ 3B ⟶ 3A ⟶ 1A
+\$8 − 5 + 4 − 6 = +\$1

**Table B-22**
The Stepping-Stone Path for Cell 2B

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | − [6] 25 | [8] | + [10] 125 | 150 |
| **2** | [7] | + [11] | − [11] 175 | 175 |
| **3** | + [4] 175 | − [5] 100 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

2B ⟶ 3B ⟶ 3A ⟶ 1A ⟶ 1C ⟶ 2C
+\$11 − 5 + 4 − 6 + 10 − 11 = +\$3

**Table B-23**
The Stepping-Stone Path for Cell 3C

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | + [6] 25 | [8] | − [10] 125 | 150 |
| **2** | [7] | [11] | [11] 175 | 175 |
| **3** | − [4] 175 | [5] 100 | + [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

3C ⟶ 3A ⟶ 1A ⟶ 1C
+\$12 − 4 + 6 − 10 = +\$4

*The stepping-stone process is repeated until none of the empty cells will reduce cost (i.e., an optimal solution).*

Our evaluation of the four paths indicates no cost reductions; therefore, the solution shown in Table B-19 is optimal. The solution and total minimum cost are

$$x_{1A} = 25 \text{ tons} \qquad x_{2C} = 175 \text{ tons} \qquad x_{3A} = 175 \text{ tons}$$
$$x_{1C} = 125 \text{ tons} \qquad x_{3B} = 100 \text{ tons}$$
$$\begin{aligned} Z &= \$6(25) + 8(0) + 10(125) + 7(0) + 11(0) + 11(175) + 4(175) + 5(100) + 12(0) \\ &= \$4{,}525 \end{aligned}$$

*Multiple optimal solutions occur when an empty cell has a cost change of zero and all other empty cells are positive.*

*An alternative optimal solution is determined by allocating to the empty cell a zero cost change.*

However, notice in Table B-20 that the path for cell 2A resulted in a cost change of $0. In other words, allocating to this cell would neither increase nor decrease total cost. This situation indicates that the problem has **multiple optimal solutions**. Thus, $x_{2A}$ could be entered into the solution and there would not be a change in the total minimum cost of $4,525. To identify the alternative solution, we would allocate as much as possible to cell 2A, which in this case is 25 tons of wheat. The alternative solution is shown in Table B-24.

**Table B-24**
The Alternative Optimal Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| **1** | [6] | [8] | [10] 150 | 150 |
| **2** | [7] 25 | [11] | [11] 150 | 175 |
| **3** | [4] 175 | [5] 100 | [12] | 275 |
| **Demand** | 200 | 100 | 300 | 600 |

The solution in Table B-24 also results in a total minimum cost of $4,525. The steps of the stepping-stone method are summarized here:

1. Determine the stepping-stone paths and cost changes for each empty cell in the tableau.
2. Allocate as much as possible to the empty cell with the greatest net decrease in cost.
3. Repeat steps 1 and 2 until all empty cells have positive cost changes that indicate an optimal solution.

## The Modified Distribution Method

The *modified distribution method (MODI)* is basically a modified version of the stepping-stone method. However, in the MODI method the individual cell cost changes are determined mathematically, without identifying all the stepping-stone paths for the empty cells.

*MODI is a modified version of the stepping-stone method in which math equations replace the stepping-stone paths.*

To demonstrate MODI, we will again use the initial solution obtained by the minimum cell cost method. The tableau for the initial solution with the modifications required by MODI is shown in Table B-25.

**Table B-25**
The Minimum Cell Cost Initial Solution

| | $v_j$ | $v_A =$ | $v_B =$ | $v_C =$ | |
|---|---|---|---|---|---|
| $u_i$ | **From \ To** | **A** | **B** | **C** | **Supply** |
| $u_1 =$ | **1** | [6] | [8] 25 | [10] 125 | 150 |
| $u_2 =$ | **2** | [7] | [11] | [11] 175 | 175 |
| $u_3 =$ | **3** | [4] 200 | [5] 75 | [12] | 275 |
| | **Demand** | 200 | 100 | 300 | 600 |

The extra left-hand column with the $u_i$ symbols and the extra top row with the $v_j$ symbols represent column and row values that must be computed in MODI. These values are computed for all *cells with allocations* by using the following formula:

$$u_i + v_j = c_{ij}$$

The value $c_{ij}$ is the unit transportation cost for cell *ij*. For example, the formula for cell 1B is

$$u_1 + v_B = c_{1B}$$

and, since $c_{1B} = 8$,

$$u_1 + v_B = 8$$

The formulas for the remaining cells that presently contain allocations are

$$\begin{aligned} x_{1C}&: \quad u_1 + v_C = 10 \\ x_{2C}&: \quad u_2 + v_C = 11 \\ x_{3A}&: \quad u_3 + v_A = 4 \\ x_{3B}&: \quad u_3 + v_B = 5 \end{aligned}$$

Now there are five equations with six unknowns. To solve these equations, it is necessary to assign only one of the unknowns a value of zero. Thus, if we let $u_1 = 0$, we can solve for all remaining $u_i$ and $v_j$ values.

$$\begin{aligned} x_{1B}&: \quad u_1 + v_B = 8 \\ &\quad\ \ 0 + v_B = 8 \\ &\qquad\quad v_B = 8 \\ x_{1C}&: \quad u_1 + v_C = 10 \\ &\quad\ \ 0 + v_C = 10 \\ &\qquad\quad v_C = 10 \\ x_{2C}&: \quad u_2 + v_C = 11 \\ &\quad\ \ u_2 + 10 = 11 \\ &\qquad\quad u_2 = 1 \\ x_{3B}&: \quad u_3 + v_B = 5 \\ &\quad\ \ u_3 + 8 = 5 \\ &\qquad\quad u_3 = -3 \\ x_{3A}&: \quad u_3 + v_A = 4 \\ &\quad\ -3 + v_A = 4 \\ &\qquad\quad v_A = 7 \end{aligned}$$

Notice that the equation for cell 3B had to be solved before the cell 3A equation could be solved. Now all the $u_i$ and $v_j$ values can be substituted into the tableau, as shown in Table B-26.

**Table B-26**
The Initial Solution with All $u_i$ and $v_j$ Values

| | $v_j$ | $v_A = 7$ | $v_B = 8$ | $v_C = 10$ | |
|---|---|---|---|---|---|
| $u_i$ | From \ To | A | B | C | Supply |
| $u_1 = 0$ | 1 | [6] | [8] 25 | [10] 125 | 150 |
| $u_2 = 1$ | 2 | [7] | [11] | [11] 175 | 175 |
| $u_3 = -3$ | 3 | [4] 200 | [5] 75 | [12] | 275 |
| | Demand | 200 | 100 | 300 | 600 |

Next, we use the following formula to evaluate all *empty cells:*

$$c_{ij} - u_i - v_j = k_{ij}$$

where $k_{ij}$ equals the cost increase or decrease that would occur by allocating to a cell.

For the *empty cells* in Table B-26, the formula yields the following values:

$$\begin{aligned}
x_{1A}&: \quad k_{1A} = c_{1A} - u_1 - v_A = 6 - 0 - 7 = -1\\
x_{2A}&: \quad k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 7 = -1\\
x_{2B}&: \quad k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 8 = +2\\
x_{3C}&: \quad k_{3C} = c_{3C} - u_3 - v_C = 12 - (-3) - 10 = +5
\end{aligned}$$

*Each MODI allocation replicates the stepping-stone allocation.*

These calculations indicate that either cell 1A or cell 2A will decrease cost by \$1 per allocated ton. Notice that those are exactly the same cost changes for all four empty cells as were computed in the stepping-stone method. That is, the same information is obtained by evaluating the paths in the stepping-stone method and by using the mathematical formulas of the MODI.

*After each allocation to an empty cell, the $u_i$ and $v_j$ values must be recomputed.*

We can select either cell 1A or 2A to allocate to because they are tied at −1. If cell 1A is selected as the entering nonbasic variable, then the stepping-stone path for that cell must be determined so that we know how much to reallocate. This is the same path previously identified in Table B-18. Reallocating along this path results in the tableau shown in Table B-27 (and previously shown in Table B-19).

**Table B-27**
The Second Iteration of the MODI Solution Method

| | $v_j$ | $v_A =$ | $v_B =$ | $v_C =$ | |
|---|---|---|---|---|---|
| $u_i$ | From \ To | A | B | C | Supply |
| $u_1 =$ | 1 | [6] 25 | [8] | [10] 125 | 150 |
| $u_2 =$ | 2 | [7] | [11] | [11] 175 | 175 |
| $u_3 =$ | 3 | [4] 175 | [5] 100 | [12] | 275 |
| | Demand | 200 | 100 | 300 | 600 |

The $u_i$ and $v_j$ values for Table B-27 must now be recomputed using our formula for the allocated-to cells:

$$\begin{aligned}
x_{1A}&: \quad u_1 + v_A = 6\\
&\quad\; 0 + v_A = 6\\
&\quad\; v_A = 6\\
x_{1C}&: \quad u_1 + v_C = 10\\
&\quad\; 0 + v_C = 10\\
&\quad\; v_C = 10\\
x_{2C}&: \quad u_2 + v_C = 11\\
&\quad\; u_2 + 10 = 11\\
&\quad\; u_2 = 1\\
x_{3A}&: \quad u_3 + v_A = 4\\
&\quad\; u_3 + 6 = 4\\
&\quad\; u_3 = -2\\
x_{3B}&: \quad u_3 + v_B = 5\\
&\quad\; -2 + v_B = 5\\
&\quad\; v_B = 7
\end{aligned}$$

These new $u_i$ and $v_j$ values are shown in Table B-28.

**Table B-28**
The New $u_i$ and $v_j$ Values for the Second Iteration

| | $v_j$ | $v_A = 6$ | $v_B = 7$ | $v_C = 10$ | |
|---|---|---|---|---|---|
| $u_i$ | From \ To | A | B | C | Supply |
| $u_1 = 0$ | 1 | [6] 25 | [8] | [10] 125 | 150 |
| $u_2 = 1$ | 2 | [7] | [11] | [11] 175 | 175 |
| $u_3 = -2$ | 3 | [4] 175 | [5] 100 | [12] | 275 |
| | Demand | 200 | 100 | 300 | 600 |

The cost changes for the empty cells are now computed using the formula $c_{ij} - u_i - v_j = k_{ij}$:

$$
\begin{aligned}
x_{1B}&: \quad k_{1B} = c_{1B} - u_1 - v_B = 8 - 0 - 7 = +1 \\
x_{2A}&: \quad k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 6 = 0 \\
x_{2B}&: \quad k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 7 = +3 \\
x_{3C}&: \quad k_{3C} = c_{3C} - u_3 - v_C = 12 - (-2) - 10 = +4
\end{aligned}
$$

Because none of these values is negative, the solution shown in Table B-28 is optimal. However, as in the stepping-stone method, cell 2A with a zero cost change indicates multiple optimal solutions.

The steps of the modified distribution method can be summarized as follows:

1. Develop an initial solution using one of the three methods available.
2. Compute $u_i$ and $v_j$ values for each row and column by applying the formula $u_i + v_j = c_{ij}$ to each cell that has an allocation.
3. Compute the cost change, $k_{ij}$, for each empty cell using $c_{ij} - u_i - v_j = k_{ij}$.
4. Allocate as much as possible to the empty cell that will result in the greatest net decrease in cost (most negative $k_{ij}$). Allocate according to the stepping-stone path for the selected cell.
5. Repeat steps 2 through 4 until all $k_{ij}$ values are positive or zero.

## The Unbalanced Transportation Model

Thus far, the methods for determining an initial solution and an optimal solution have been demonstrated within the context of a balanced transportation model. Realistically, however, an *unbalanced problem* is a more likely occurrence. Consider our example of transporting wheat. By changing the demand at Cincinnati to 350 tons, we create a situation in which total demand is 650 tons and total supply is 600 tons.

*When demand exceeds supply, a dummy row is added to the tableau.*

To compensate for this difference in the transportation tableau, a "dummy" row is added to the tableau, as shown in Table B-29. The dummy row is assigned a supply of 50 tons to balance the model. The additional 50 tons demanded, which cannot be supplied, will be allocated to a cell in the dummy row. The transportation costs for the cells in the dummy row are zero because the tons allocated to these cells are not amounts really transported but the amounts by which demand was not met. These dummy cells are, in effect, *slack variables.*

**Table B-29**
An Unbalanced Model (Demand > Supply)

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 150 |
| 2 | 7 | 11 | 11 | 175 |
| 3 | 4 | 5 | 12 | 275 |
| Dummy | 0 | 0 | 0 | 50 |
| Demand | 200 | 100 | 350 | 650 |

*When a supply exceeds demand, a dummy column is added to the tableau.*

Now consider our example with the supply at Des Moines increased to 375 tons. This increases total supply to 700 tons, while total demand remains at 600 tons. To compensate for this imbalance, we add a dummy column instead of a dummy row, as shown in Table B-30.

**Table B-30**
An Unbalanced Model (Supply > Demand)

| From \ To | A | B | C | Dummy | Supply |
|---|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 0 | 150 |
| 2 | 7 | 11 | 11 | 0 | 175 |
| 3 | 4 | 5 | 12 | 0 | 375 |
| Demand | 200 | 100 | 300 | 100 | 700 |

The addition of a dummy row or a dummy column has no effect on the initial solution methods or on the methods for determining an optimal solution. The dummy row or column cells are treated the same as any other tableau cell. For example, in the minimum cell cost method, three cells would be tied for the minimum cost cell, each with a cost of zero. In this case (or any time there is a tie between cells) the tie would be broken arbitrarily.

## Degeneracy

In all the tableaus showing a solution to the wheat transportation problem, the following condition was met:

$$m \text{ rows} + n \text{ columns} - 1 = \text{the number of cells with allocations}$$

For example, in any of the balanced tableaus for wheat transportation, the number of rows was three (i.e., $m = 3$) and the number of columns was three (i.e., $n = 3$); thus, $3 + 3 - 1 = 5$ cells with allocations.

*In a transportation tableau with m rows and n columns, there must be $m + n - 1$ cells with allocations; if not, it is **degenerate**.*

These tableaus always had five cells with allocations; thus, our condition for normal solution was met. When this condition is not met and fewer than $m + n - 1$ cells have allocations, the tableau is said to be **degenerate**.

Consider the wheat transportation example with the supply values changed to the amounts shown in Table B-31. The initial solution shown in this tableau was developed using the minimum cell cost method.

**Table B-31**
The Minimum Cell Cost Initial Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | [6] | [8] 100 | [10] 50 | 150 |
| 2 | [7] | [11] | [11] 250 | 250 |
| 3 | [4] 200 | [5] | [12] | 200 |
| Demand | 200 | 100 | 300 | 600 |

The tableau shown in Table B-31 does not meet the condition

$$m + n - 1 = \text{the number of cells with allocations}$$
$$3 + 3 - 1 = 5 \text{ cells}$$

*In a degenerate tableau, not all of the stepping-stone paths or MODI equations can be developed.*

because there are only four cells with allocations. The difficulty resulting from a degenerate solution is that neither the stepping-stone method nor MODI will work unless the preceding condition is met (there is an appropriate number of cells with allocations). When the tableau is degenerate, a closed path cannot be completed for all cells in the stepping-stone method, *and* not all the $u_i + v_j = c_{ij}$ computations can be completed in MODI. For example, a closed path cannot be determined for cell 1A in Table B-31.

*To rectify a degenerate tableau, an empty cell must artificially be treated as an occupied cell.*

To create a closed path, one of the empty cells must be artificially designated as a cell with an allocation. Cell 1A in Table B-32 is designated arbitrarily as a cell with artificial allocation of zero. (However, any symbol, such as ∅, could be used to signify the artificial allocation.) This indicates that this cell will be treated as a cell with an allocation in determining stepping-stone paths or MODI formulas, although there is no real allocation in this cell. Notice that the location of 0 was *arbitrary* because there is no general rule for allocating the artificial cell. Allocating zero to a cell does not guarantee that all the stepping-stone paths can be determined.

**Table B-32**
The Initial Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | [6] 0 | [8] 100 | [10] 50 | 150 |
| 2 | [7] | [11] | [11] 250 | 250 |
| 3 | [4] 200 | [5] | [12] | 200 |
| Demand | 200 | 100 | 300 | 600 |

For example, if zero had been allocated to cell 2B instead of to cell 1A, none of the stepping-stone paths could have been determined, even though technically the tableau would no longer be degenerate. In such a case, the zero must be reallocated to another cell and all paths determined again. This process must be repeated until an artificial allocation has been made that will enable the determination of all paths. In most cases, however, there is more than one possible cell to which such an allocation can be made.

The stepping-stone paths and cost changes for this tableau follow:

$$\begin{array}{ll} & \text{2A} \quad \text{2C} \quad \text{1C} \quad \text{1A} \\ x_{2A}: & 7 - 11 + 10 - 6 = 0 \\ & \text{2B} \quad \text{2C} \quad \text{1C} \quad \text{1B} \\ x_{2B}: & 11 - 11 + 10 - 8 = +2 \\ & \text{3B} \quad \text{1B} \quad \text{1A} \quad \text{3A} \\ x_{3B}: & 5 - 8 + 6 - 4 = -1 \\ & \text{3C} \quad \text{1C} \quad \text{1A} \quad \text{3A} \\ x_{3C}: & 12 - 10 + 6 - 4 = +4 \end{array}$$

Because cell 3B shows a $1 decrease in cost for every ton of wheat allocated to it, we will allocate 100 tons to cell 3B. This results in the tableau shown in Table B-33.

**Table B-33**
The Second Stepping-Stone Iteration

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | 6<br>100 | 8 | 10<br>50 | 150 |
| *2* | 7 | 11 | 11<br>250 | 250 |
| *3* | 4<br>100 | 5<br>100 | 12 | 200 |
| **Demand** | 200 | 100 | 300 | 600 |

*A normal problem can become degenerate at any iteration and vice versa.*

Notice that the solution in Table B-33 now meets the condition $m + n - 1 = 5$. Thus, in applying the stepping-stone method (or MODI) to this tableau, it is not necessary to make an artificial allocation to an empty cell. It is quite possible to begin the solution process with a normal tableau and have it become degenerate or begin with a degenerate tableau and have it become normal. If it had been indicated that the cell with the zero should have units subtracted from it, no actual units could have been subtracted. In that case the zero would have been moved to the cell that represents the entering variable. (The solution shown in Table B-33 is optimal; however, multiple optimal solutions exist at cell 2A.)

## Prohibited Routes

*A **prohibited route** is assigned a large cost such as M so that it will never receive an allocation.*

Sometimes one or more of the routes in the transportation model are *prohibited*. That is, units cannot be transported from a particular source to a particular destination. When this situation occurs, we must make sure that no units in the optimal solution are allocated to the cell representing this route. In our study of the simplex tableau, we learned that assigning a large relative cost or a coefficient of $M$ to a variable would keep it out of the final solution. This same principle can be used in a transportation model for a **prohibited route**. A value of $M$ is assigned as the transportation cost for a cell that represents a prohibited route. Thus, when the prohibited cell is evaluated, it will always contain a large positive cost change of $M$, which will keep it from being selected as an entering variable.

## Solution of the Assignment Model

*An **assignment model** is a special form of the transportation model in which all supply and demand values equal one.*

The **assignment model** is a special form of a linear programming model that is similar to the transportation model. There are differences, however. In the assignment model, the supply at each source and the demand at each destination are limited to one unit each.

The following example from the text will be used to demonstrate the assignment model and its special solution method. The Atlantic Coast Conference has four basketball games on a particular night. The conference office wants to assign four teams of officials to the four games in a way that will minimize the total distance traveled by the officials. The distances in miles for each team of officials to each game location are shown in Table B-34.

**Table B-34**
The Travel Distances to Each Game for Each Team of Officials

| | Game Sites | | | |
|---|---|---|---|---|
| **Officials** | Raleigh | Atlanta | Durham | Clemson |
| A | 210 | 90 | 180 | 160 |
| B | 100 | 70 | 130 | 200 |
| C | 175 | 105 | 140 | 170 |
| D | 80 | 65 | 105 | 120 |

The supply is always one team of officials, and the demand is for only one team of officials at each game. Table B-34 is already in the proper form for the assignment.

*An **opportunity cost table** is developed by first substracting the minimum value in each row from all other row values and then repeating this process for each column.*

The first step in the assignment method of solution is to develop an **opportunity cost table**. We accomplish this by first subtracting the minimum value in each row from every value in the row. These computations are referred to as *row reductions*. We applied a similar principle in the VAM method when we determined penalty costs. In other words, the best course of action is determined for each row, and the penalty or "lost opportunity" is developed for all other row values. The row reductions for this example are shown in Table B-35.

**Table B-35**
The Assignment Tableau with Row Reductions

| | Game Sites | | | |
|---|---|---|---|---|
| **Officials** | Raleigh | Atlanta | Durham | Clemson |
| A | 120 | 0 | 90 | 70 |
| B | 30 | 0 | 60 | 130 |
| C | 70 | 0 | 35 | 65 |
| D | 15 | 0 | 40 | 55 |

Next, the minimum value in each column is subtracted from all column values. These computations are called *column reductions* and are shown in Table B-36, which represents the completed opportunity cost table for our example. Assignments can be made in this table wherever a zero is present. For example, team A can be assigned to Atlanta. An *optimal solution* results when each of the four teams can be uniquely assigned to a different game.

**Table B-36**
The Tableau with Column Reductions

| | Game Sites | | | |
|---|---|---|---|---|
| **Officials** | Raleigh | Atlanta | Durham | Clemson |
| A | 105 | 0 | 55 | 15 |
| B | 15 | 0 | 25 | 75 |
| C | 55 | 0 | 0 | 10 |
| D | 0 | 0 | 5 | 0 |

*Assignments are made to locations with zeros in the opportunity cost table.*

Notice in Table B-36 that the assignment of team A to Atlanta means that no other team can be assigned to that game. Once this assignment is made, the zero in row B is infeasible, which indicates that there is not a unique optimal assignment for team B. Therefore, Table B-36 does not contain an optimal solution.

*An optimal solution occurs when the number of independent unique assignments equals the number of rows or columns.*

A test to determine whether four unique assignments exist in Table B-36 is to draw the minimum number of horizontal or vertical lines necessary to cross out all zeros through the rows and columns of the table. For example, Table B-37 shows that three lines are required to cross out all zeros.

**Table B-37**
The Opportunity Cost Table with the Line Test

| | Game Sites | | | |
|---|---|---|---|---|
| **Officials** | Raleigh | Atlanta | Durham | Clemson |
| A | 105 | 0 | 55 | 15 |
| B | 15 | 0 | 25 | 75 |
| C | 55 | 0 | 0 | 10 |
| D | 0 | 0 | 5 | 0 |

*If the number of unique assignments is less than the number of rows (or columns), a line test must be used.*

The three lines indicate that there are only three unique assignments, whereas four are required for an optimal solution. (Note that even if the three lines could have been drawn differently, the subsequent solution method would not be affected.) Next, subtract the minimum value that is not crossed out from all values not crossed out. Then, add this minimum value to those cells where two lines intersect. The minimum value not crossed out in Table B-37 is 15. The second iteration for this model with the appropriate changes is shown in Table B-38.

**Table B-38**
The Second Iteration

| | Game Sites | | | |
|---|---|---|---|---|
| **Officials** | Raleigh | Atlanta | Durham | Clemson |
| A | 90 | 0 | 40 | 0 |
| B | 0 | 0 | 10 | 60 |
| C | 55 | 15 | 0 | 10 |
| D | 0 | 15 | 5 | 0 |

*In a line test, all zeros are crossed out by horizontal and vertical lines; the minimum uncrossed value is subtracted from all uncrossed values and added to cell values where two lines cross.*

No matter how the lines are drawn in Table B-38, at least four are required to cross out all the zeros. This indicates that four unique assignments can be made and that an optimal solution has been reached. Now let us make the assignments from Table B-38.

First, team A can be assigned to either the Atlanta game or the Clemson game. We will assign team A to Atlanta first. This means that team A cannot be assigned to any other game, and no other team can be assigned to Atlanta. Therefore, row A and the Atlanta column can be eliminated. Next, team B is assigned to Raleigh. (Team B cannot be assigned to Atlanta, which has already been eliminated.) The third assignment is of team C to the Durham game. This leaves team D for the Clemson game. These assignments and their respective distances (from Table B-34) are summarized as follows:

| ***Assignment*** | ***Distance*** |
|---|---|
| Team A → Atlanta | 90 |
| Team B → Raleigh | 100 |
| Team C → Durham | 140 |
| Team D → Clemson | 120 |
| | 450 miles |

Now let us go back and make the initial assignment of team A to Clemson (the alternative assignment we did not initially make). This will result in the following set of assignments:

| ***Assignment*** | ***Distance*** |
|---|---|
| Team A $\rightarrow$ Clemson | 160 |
| Team B $\rightarrow$ Atlanta | 70 |
| Team C $\rightarrow$ Durham | 140 |
| Team D $\rightarrow$ Raleigh | 80 |
| | 450 miles |

These two assignments represent *multiple optimal solutions* for our example problem. Both assignments will result in the officials traveling a minimum total distance of 450 miles.

*When supply exceeds demands, a dummy column is added to the assignment tableau.*

Like a transportation problem, an assignment model can be unbalanced when supply exceeds demand or demand exceeds supply. For example, assume that, instead of four teams of officials, there are five teams to be assigned to the four games. In this case a *dummy column* is added to the assignment tableau to balance the model, as shown in Table B-39.

**Table B-39**
An Unbalanced Assignment Tableau with a Dummy Column

| **Officials** | **Game Sites** | | | | |
|---|---|---|---|---|---|
| | RALEIGH | ATLANTA | DURHAM | CLEMSON | DUMMY |
| A | 210 | 90 | 180 | 160 | 0 |
| B | 100 | 70 | 130 | 200 | 0 |
| C | 175 | 105 | 140 | 170 | 0 |
| D | 80 | 65 | 105 | 120 | 0 |
| E | 95 | 115 | 120 | 100 | 0 |

*When demand exceeds supply, a dummy row is added to the assignment tableau.*

In solving this model, one team of officials would be assigned to the dummy column. If there were five games and only four teams of officials, a *dummy row* would be added instead of a dummy column. The addition of a dummy row or column does not affect the solution method.

*A **prohibited assignment** is given a large relative cost of M so that it will never be selected.*

**Prohibited assignments** are also possible in an assignment problem, just as prohibited routes can occur in a transportation model. In the transportation model, an $M$ value was assigned as a large cost for the cell representing the prohibited route. This same method is used for a prohibited assignment. A value of $M$ is placed in the cell that represents the prohibited assignment.

The steps of the assignment solution method are summarized here:

1. Perform row reductions by subtracting the minimum value in each row from all row values.
2. Perform column reductions by subtracting the minimum value in each column from all column values.
3. In the completed opportunity cost table, cross out all zeros, using the minimum number of horizontal or vertical lines.
4. If fewer than $m$ lines are required (where $m$ = the number of rows or columns), subtract the minimum uncrossed value from all uncrossed values, and add this same minimum value to all cells where two lines intersect. Leave all other values unchanged, and repeat step 3.
5. If $m$ lines are required, the tableau contains the optimal solution and $m$ unique assignments can be made. If fewer than $m$ lines are required, repeat step 4.

## Problems

1. Green Valley Mills produces carpet at plants in St. Louis and Richmond. The carpet is then shipped to two outlets located in Chicago and Atlanta. The cost per ton of shipping carpet from each of the two plants to the two warehouses is as follows:

| | To | |
|---|---|---|
| **From** | *Chicago* | *Atlanta* |
| St. Louis | $40 | $65 |
| Richmond | 70 | 30 |

The plant at St. Louis can supply 250 tons of carpet per week; the plant at Richmond can supply 400 tons per week. The Chicago outlet has a demand of 300 tons per week, and the outlet at Atlanta demands 350 tons per week.The company wants to know the number of tons of carpet to ship from each plant to each outlet in order to minimize the total shipping cost. Solve this transportation problem.

2. A transportation problem involves the following costs, supply, and demand:

| | To | | | | |
|---|---|---|---|---|---|
| **From** | *1* | *2* | *3* | *4* | **Supply** |
| 1 | $500 | $750 | $300 | $450 | 12 |
| 2 | 650 | 800 | 400 | 600 | 17 |
| 3 | 400 | 700 | 500 | 550 | 11 |
| **Demand** | 10 | 10 | 10 | 10 | |

a. Find the initial solution using the northwest corner method, the minimum cell cost method, and Vogel's approximation model. Compute total cost for each.
b. Using the VAM initial solution, find the optimal solution using the modified distribution method (MODI).

3. Consider the following transportation tableau and solution:

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| *1* | [12] | [10] | [6] 600 | 600 |
| *2* | [4] 400 | [15] | [3] | 400 |
| *3* | [9] 300 | [7] | [*M*] | 300 |
| *4* | [11] | [8] 500 | [6] 300 | 800 |
| **Dummy** | [0] 200 | [0] | [0] | 200 |
| **Demand** | 900 | 500 | 900 | 2,300 |

a. Is this a balanced or an unbalanced transportation problem? Explain.
b. Is this solution degenerate? Explain. If it is degenerate, show how it would be put into proper form.
c. Is there a prohibited route in this problem?
d. Compute the total cost of this solution.
e. What is the value of $x_{2B}$ in this solution?

4. Solve the following transportation problem:

| | To | | | |
|---|---|---|---|---|
| **From** | *1* | *2* | *3* | **Supply** |
| 1 | $ 40 | $ 10 | $ 20 | 800 |
| 2 | 15 | 20 | 10 | 500 |
| 3 | 20 | 25 | 30 | 600 |
| **Demand** | 1,050 | 500 | 650 | |

5. Given a transportation problem with the following costs, supply, and demand, find the initial solution using the minimum cell cost method and Vogel's approximation model. Is the VAM solution optimal?

| | To | | | |
|---|---|---|---|---|
| **From** | *1* | *2* | *3* | **Supply** |
| A | $ 6 | $ 7 | $ 4 | 100 |
| B | 5 | 3 | 6 | 180 |
| C | 8 | 5 | 7 | 200 |
| **Demand** | 135 | 175 | 170 | |

6. Consider the following transportation problem:

| | To | | | |
|---|---|---|---|---|
| **From** | *1* | *2* | *3* | **Supply** |
| A | $ 6 | 9 | *M* | 130 |
| B | 12 | 3 | 5 | 70 |
| C | 4 | 8 | 11 | 100 |
| **Demand** | 80 | 110 | 60 | |

a. Find the initial solution by using VAM and then solve it using the stepping-stone method.
b. Formulate this problem as a general linear programming model.

7. Solve the following linear programming problem:

$$\text{minimize } Z = 3x_{11} + 12x_{12} + 8x_{13} + 10x_{21} + 5x_{22} + 6x_{23} + 6x_{31} + 7x_{32} + 10x_{33}$$

subject to

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 90 & \quad x_{11} + x_{21} + x_{31} &\leq 70 \\ x_{21} + x_{22} + x_{23} &= 30 & x_{12} + x_{22} + x_{32} &\leq 110 \\ x_{31} + x_{32} + x_{33} &= 100 & x_{13} + x_{23} + x_{33} &\leq 80 \\ & & x_{ij} &\geq 0 \end{aligned}$$

8. Consider the following transportation problem:

| | To | | | |
|---|---|---|---|---|
| **From** | *1* | *2* | *3* | **Supply** |
| A | $ 6 | $ 9 | $ 7 | 130 |
| B | 12 | 3 | 5 | 70 |
| C | 4 | 11 | 11 | 100 |
| **Demand** | 80 | 110 | 60 | |

a. Find the initial solution using the minimum cell cost method.
b. Solve using the stepping-stone method.

9. Steel mills in three cities produce the following amounts of steel:

| ***Location*** | ***Weekly Production (tons)*** |
|---|---|
| A. Bethlehem | 150 |
| B. Birmingham | 210 |
| C. Gary | 320 |
| | 680 |

These mills supply steel to four cities where manufacturing plants have the following demand:

| ***Location*** | ***Weekly Demand (tons)*** |
|---|---|
| 1. Detroit | 130 |
| 2. St. Louis | 70 |
| 3. Chicago | 180 |
| 4. Norfolk | 240 |
| | 620 |

Shipping costs per ton of steel are as follows:

| | To | | | |
|---|---|---|---|---|
| **From** | *1* | *2* | *3* | *4* |
| A | $14 | 9 | 16 | 18 |
| B | 11 | 8 | 7 | 16 |
| C | 16 | 12 | 10 | 22 |

Because of a truckers' strike, shipments are at present prohibited from Birmingham to Chicago.

a. Set up a transportation tableau for this problem and determine the initial solution. Identify the method used to find the initial solution.
b. Solve this problem using MODI.
c. Are there multiple optimal solutions? Explain. If so, identify them.
d. Formulate this problem as a general linear programming model.

10. In Problem 9, what would be the effect on the optimal solution of a reduction in production capacity at the Gary mill from 320 tons to 290 tons per week?

11. Coal is mined and processed at the following four mines in Kentucky, West Virginia, and Virginia:

| *Location* | *Capacity (tons)* |
|---|---|
| A. Cabin Creek | 90 |
| B. Surry | 50 |
| C. Old Fort | 80 |
| D. McCoy | 60 |
| | 280 |

These mines supply the following amount of coal to utility power plants in three cities:

| *Plant* | *Demand (tons)* |
|---|---|
| 1. Richmond | 120 |
| 2. Winston-Salem | 100 |
| 3. Durham | 110 |
| | 330 |

The railroad shipping costs ($1,000s) per ton of coal are shown in the following table. Because of railroad construction, shipments are now prohibited from Cabin Creek to Richmond:

| | **To** | | |
|---|---|---|---|
| **From** | *1* | *2* | *3* |
| A | $ 7 | $10 | $ 5 |
| B | 12 | 9 | 4 |
| C | 7 | 3 | 11 |
| D | 9 | 5 | 7 |

a. Set up the transportation tableau for this problem, determine the initial solution using VAM, and compute total cost.
b. Solve using MODI.
c. Are there multiple optimal solutions? Explain. If there are alternative solutions, identify them.
d. Formulate this problem as a linear programming model.

12. Oranges are grown, picked, and then stored in warehouses in Tampa, Miami, and Fresno. These warehouses supply oranges to markets in New York, Philadelphia, Chicago, and Boston. The following table shows the shipping costs per truckload ($100s), supply, and demand. Because of an agreement between distributors, shipments are prohibited from Miami to Chicago.

| | **To** | | | | |
|---|---|---|---|---|---|
| **From** | *New York* | *Philadelphia* | *Chicago* | *Boston* | **Supply** |
| Tampa | $ 9 | $ 14 | $ 12 | $ 17 | 200 |
| Miami | 11 | 10 | 6 | 10 | 200 |
| Fresno | 12 | 8 | 15 | 7 | 200 |
| **Demand** | 130 | 170 | 100 | 150 | |

a. Set up the transportation tableau for this problem and determine the initial solution using the minimum cell cost method.
b. Solve using MODI.
c. Are there multiple optimal solutions? Explain. If so, identify them.
d. Formulate this problem as a linear programming model.

13. A manufacturing firm produces diesel engines in four cities—Phoenix, Seattle, St. Louis, and Detroit. The company is able to produce the following numbers of engines per month:

| *Plant* | *Production* |
|---|---|
| 1. Phoenix | 5 |
| 2. Seattle | 25 |
| 3. St. Louis | 20 |
| 4. Detroit | 25 |

Three trucking firms purchase the following numbers of engines for their plants in three cities:

| *Firm* | *Demand* |
|---|---|
| A. Greensboro | 10 |
| B. Charlotte | 20 |
| C. Louisville | 15 |

The transportation costs per engine ($100s) from sources to destinations are shown in the following table. However, the Charlotte firm will not accept engines made in Seattle, and the Louisville firm will not accept engines from Detroit; therefore, these routes are prohibited.

| | **To** | | |
|---|---|---|---|
| **From** | *A* | *B* | *C* |
| 1 | $ 7 | $ 8 | $ 5 |
| 2 | 6 | 10 | 6 |
| 3 | 10 | 4 | 5 |
| 4 | 3 | 9 | 11 |

a. Set up the transportation tableau for this problem. Find the initial solution using VAM.
b. Solve for the optimal solution using the stepping-stone method. Compute the total minimum cost.
c. Formulate this problem as a linear programming model.

14. The Interstate Truck Rental firm has accumulated extra trucks at three of its truck leasing outlets, as shown in the following table:

| **Leasing Outlet** | **Extra Trucks** |
|---|---|
| 1. Atlanta | 70 |
| 2. St. Louis | 115 |
| 3. Greensboro | 60 |
| Total | 245 |

The firm also has four outlets with shortages of rental trucks, as follows:

| **Leasing Outlet** | **Truck Shortage** |
|---|---|
| A. New Orleans | 80 |
| B. Cincinnati | 50 |
| C. Louisville | 90 |
| D. Pittsburgh | 25 |
| Total | 245 |

The firm wants to transfer trucks from those outlets with extras to those with shortages at the minimum total cost. The following costs of transporting these trucks from city to city have been determined:

| | To | | | |
|---|---|---|---|---|
| **From** | *A* | *B* | *C* | *D* |
| 1 | $ 70 | 80 | 45 | 90 |
| 2 | 120 | 40 | 30 | 75 |
| 3 | 110 | 60 | 70 | 80 |

a. Find the initial solution using the minimum cell cost method.
b. Solve using the stepping-stone method.

15. The Shotz Beer Company has breweries in two cities; the breweries can supply the following numbers of barrels of draft beer to the company's distributors each month:

| Brewery | Monthly Supply (bbl) |
|---|---|
| A. Tampa | 3,500 |
| B. St. Louis | 5,000 |
| Total | 8,500 |

The distributors, which are spread throughout six states, have the following total monthly demand:

| Distributor | Monthly Demand (bbl) |
|---|---|
| 1. Tennessee | 1,600 |
| 2. Georgia | 1,800 |
| 3. North Carolina | 1,500 |
| 4. South Carolina | 950 |
| 5. Kentucky | 1,250 |
| 6. Virginia | 1,400 |
| Total | 8,500 |

The company must pay the following shipping costs per barrel:

| | To | | | | | |
|---|---|---|---|---|---|---|
| **From** | *1* | *2* | *3* | *4* | *5* | *6* |
| A | $0.50 | 0.35 | 0.60 | 0.45 | 0.80 | 0.75 |
| B | 0.25 | 0.65 | 0.40 | 0.55 | 0.20 | 0.65 |

a. Find the initial solution using VAM.
b. Solve using the stepping-stone method.

16. In Problem 15, the Shotz Beer Company management has negotiated a new shipping contract with a trucking firm between its Tampa brewery and its distributor in Kentucky that reduces the shipping cost per barrel from $0.80 per barrel to $0.65 per barrel. How will this cost change affect the optimal solution?

17. Computers Unlimited sells microcomputers to universities and colleges on the East Coast and ships them from three distribution warehouses. The firm is able to supply the following numbers of microcomputers to the universities by the beginning of the academic year:

| Distribution Warehouse | Supply (microcomputers) |
|---|---|
| 1. Richmond | 420 |
| 2. Atlanta | 610 |
| 3. Washington, D.C. | 340 |
| Total | 1,370 |

Four universities have ordered microcomputers that must be delivered and installed by the beginning of the academic year:

| University | Demand (microcomputers) |
|---|---|
| A. Tech | 520 |
| B. A and M | 250 |
| C. State | 400 |
| D. Central | 380 |
| Total | 1,550 |

The shipping and installation costs per microcomputer from each distributor to each university are as follows:

| | To | | | |
|---|---|---|---|---|
| From | *A* | *B* | *C* | *D* |
| 1 | $22 | 17 | 30 | 18 |
| 2 | 15 | 35 | 20 | 25 |
| 3 | 28 | 21 | 16 | 14 |

a. Find the initial solution using VAM.
b. Solve using MODI.

18. In Problem 17, Computers Unlimited wants to better meet demand at the four universities it supplies. It is considering two alternatives: (1) expand its warehouse at Richmond to a capacity of 600 at a cost equivalent to an additional $6 in handling and shipping per unit or (2) purchase a new warehouse in Charlotte that can supply 300 units with shipping costs of $19 to Tech, $26 to A and M, $22 to State, and $16 to Central. Which alternative should management select based solely on transportation costs (i.e., no capital costs)?

19. Computers Unlimited in Problem 17 has determined that when it is unable to meet the demand for microcomputers at the universities it supplies, the universities tend to purchase microcomputers elsewhere in the future. Thus, the firm has estimated a shortage cost for each microcomputer demanded but not supplied that reflects the loss of future sales and goodwill. These costs for each university are as follows:

| University | Cost/Microcomputer |
|---|---|
| A. Tech | $40 |
| B. A and M | 65 |
| C. State | 25 |
| D. Central | 50 |

Solve Problem 17 with these shortage costs included. Compute the total transportation cost and the total shortage cost.

20. A severe winter ice storm has swept across North Carolina and Virginia, followed by over a foot of snow and frigid, single-digit temperatures. These weather conditions have resulted in numerous downed power lines and power outages in the region causing dangerous conditions for much of the population. Local utility companies have been overwhelmed and have requested assistance from unaffected utility companies across the Southeast. The following table shows the number of utility trucks with crews available from five different companies in Georgia, South Carolina, and Florida; the demand for crews in seven different areas that local companies cannot get to; and the weekly cost ($1,000s) of a crew going to a specific area (based on the visiting company's normal charges, the distance the crew has to come, and living expenses in an area):

| | Area (Cost = $1,000s) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Crew** | *NC-E* | *NC-SW* | *NC-P* | *NC-W* | *VA-SW* | *VA-C* | *VA-T* | **Crews Available** |
| GA-1 | 15.2 | 14.3 | 13.9 | 13.5 | 14.7 | 16.5 | 18.7 | 12 |
| GA-2 | 12.8 | 11.3 | 10.6 | 12.0 | 12.7 | 13.2 | 15.6 | 10 |
| SC-1 | 12.4 | 10.8 | 9.4 | 11.3 | 13.1 | 12.8 | 14.5 | 14 |
| FL-1 | 18.2 | 19.4 | 18.2 | 17.9 | 20.5 | 20.7 | 22.7 | 15 |
| FL-2 | 19.3 | 20.2 | 19.5 | 20.2 | 21.2 | 21.3 | 23.5 | 12 |
| Crews Needed | 9 | 7 | 6 | 8 | 10 | 9 | 7 | |

Determine the number of crews that should be sent from each utility to each affected area that will minimize total costs.

21. A large manufacturing company is closing three of its existing plants and intends to transfer some of its more skilled employees to three plants that will remain open. The number of employees available for transfer from each closing plant is as follows:

| Closing Plant | Transferable Employees |
|---|---|
| 1 | 60 |
| 2 | 105 |
| 3 | 70 |
| Total | 235 |

The following number of employees can be accommodated at the three plants remaining open:

| Open Plants | Employees Demanded |
|---|---|
| A | 45 |
| B | 90 |
| C | 35 |
| Total | 170 |

Each transferred employee will increase product output per day at each plant as shown in the following table. The company wants to transfer employees so as to ensure the maximum increase in product output.

| From | To A | B | C |
|---|---|---|---|
| 1 | 5 | 8 | 6 |
| 2 | 10 | 9 | 12 |
| 3 | 7 | 6 | 8 |

a. Find the initial solution using VAM.
b. Solve using MODI.

22. The Sav-Us Rental Car Agency has six lots in Nashville, and it wants to have a certain number of cars available at each lot at the beginning of each day for local rental. The agency would like a model it could quickly solve at the end of each day that would tell it how to redistribute the cars among the six lots in the minimum total time. The times required to travel between the six lots are as follows:

| | To (minutes) | | | | | |
|---|---|---|---|---|---|---|
| **From** | *1* | *2* | *3* | *4* | *5* | *6* |
| 1 | — | 12 | 17 | 18 | 10 | 20 |
| 2 | 14 | — | 10 | 19 | 16 | 15 |
| 3 | 14 | 10 | — | 12 | 8 | 9 |
| 4 | 8 | 16 | 14 | — | 12 | 15 |
| 5 | 11 | 21 | 16 | 18 | — | 10 |
| 6 | 24 | 12 | 9 | 17 | 15 | — |

The agency would like the following numbers of cars at each lot at the end of the day. Also shown are the numbers of available cars at each lot at the end of a particular day. Determine the optimal reallocation of rental cars using any initial solution approach and any solution method.

| | Lot | | | | | |
|---|---|---|---|---|---|---|
| **Cars** | *1* | *2* | *3* | *4* | *5* | *6* |
| Available | 37 | 20 | 14 | 26 | 40 | 28 |
| Desired | 30 | 25 | 20 | 40 | 30 | 20 |

23. Bayville has built a new elementary school so that the town now has a total of four schools—Addison, Beeks, Canfield, and Daley. Each has a capacity of 400 students. The school wants to assign children to schools so that their travel time by bus is as short as possible. The school has partitioned the town into five districts conforming to population density—north, south, east, west, and central. The average bus travel time from each district to each school is shown as follows:

| District | Travel Time (min.) *Addison* | *Beeks* | *Canfield* | *Daley* | Student Population |
|---|---|---|---|---|---|
| North | 12 | 23 | 35 | 17 | 250 |
| South | 26 | 15 | 21 | 27 | 340 |
| East | 18 | 20 | 22 | 31 | 310 |
| West | 29 | 24 | 35 | 10 | 210 |
| Central | 15 | 10 | 23 | 16 | 290 |

Determine the number of children that should be assigned from each district to each school to minimize total student travel time.

24. In Problem 23, the school board has determined that it does not want any one school to be more crowded than any other school. It would like to assign students from each district to each school so that enrollments are evenly balanced among the four schools. However, the school board is concerned that this might significantly increase travel time. Determine the number of students to be assigned from each district to each school so that school enrollments are evenly balanced. Does this new solution appear to result in a significant increase in travel time per student?

25. The Easy Time Grocery chain operates in major metropolitan areas on the eastern seaboard. The stores have a "no-frills" approach, with low overhead and high volume. They generally buy their stock in volume at low prices. However, in some cases they actually buy stock at stores in other areas and ship it in. They can do this because of high prices in the cities they operate in compared with costs in other locations. One example is baby food. Easy Time purchases baby food at stores in Albany, Binghamton, Claremont, Dover, and Edison and then trucks it to six stores in and around New York City. The stores in the outlying areas know what Easy Time is up to, so they limit the number of cases of baby food Easy Time can purchase. The following table shows the profit Easy Time makes per case of baby food based on where the chain purchases it and at which store it's sold, plus the available baby food per week at purchase locations and the shelf space available at each Easy Time store per week:

| Purchase Location | Easy Time Store *1* | *2* | *3* | *4* | *5* | *6* | Supply |
|---|---|---|---|---|---|---|---|
| Albany | 9 | 8 | 11 | 12 | 7 | 8 | 26 |
| Binghamton | 10 | 10 | 8 | 6 | 9 | 7 | 40 |
| Claremont | 8 | 6 | 6 | 5 | 7 | 4 | 20 |
| Dover | 4 | 6 | 9 | 5 | 8 | 10 | 40 |
| Edison | 12 | 10 | 8 | 9 | 6 | 7 | 45 |
| **Demand** | 25 | 15 | 30 | 18 | 27 | 35 | |

Determine where Easy Time should purchase baby food and how the food should be distributed in order to maximize profit. Use any initial solution approach and any solution method.

26. Suppose that in Problem 25 Easy Time can purchase all the baby food it needs from a New York City distributor at a price that will result in a profit of $9 per case at stores 1, 3, and 4, $8 per case at stores 2 and 6, and $7 per case at store 5. Should Easy Time purchase all, none, or some of its baby food from the distributor rather than purchasing it at other stores and trucking it in?

27. In Problem 25, if Easy Time could arrange to purchase more baby food from one of the outlying locations, which should it be, how many additional cases could be purchased, and how much would this increase profit?

28. The Roadnet Transport Company has expanded its shipping capacity by purchasing 90 trailer trucks from a competitor that went bankrupt. The company subsequently located 30 of the purchased trucks at each of its shipping warehouses in Charlotte, Memphis, and Louisville. The company makes shipments from each of these warehouses to terminals in St. Louis, Atlanta, and New York. Each truck is capable of making one shipment per week. The terminal managers have indicated their capacity of extra shipments. The manager at St. Louis can accommodate 40 additional trucks per week, the manager at Atlanta can accommodate 60 additional trucks, and the manager at New York can accommodate 50 additional trucks. The company makes the following profit per truckload shipment from each warehouse to each terminal. The profits differ as a result of differences in products shipped, shipping costs, and transport rates:

| | Terminal | | |
|---|---|---|---|
| **Warehouse** | *St. Louis* | *Atlanta* | *New York* |
| Charlotte | $1,800 | $2,100 | $1,600 |
| Memphis | 1,000 | 700 | 900 |
| Louisville | 1,400 | 800 | 2,200 |

Determine how many trucks to assign to each route (i.e., warehouse to terminal) in order to maximize profit.

29. During the Gulf War, Operation Desert Storm required large amounts of military matériel and supplies to be shipped daily from supply depots in the United States to bases in the Middle East. The critical factor in the movement of these supplies was speed. The following table shows the number of planeloads of supplies available each day from each of six supply depots and the number of daily loads demanded at each of five bases. (Each planeload is approximately equal in tonnage.) Also included are the transport hours per plane, including loading and fueling, actual flight time, and unloading and refueling:

| | Military Base | | | | | |
|---|---|---|---|---|---|---|
| **Supply Depot** | *A* | *B* | *C* | *D* | *E* | **Supply** |
| 1 | 36 | 40 | 32 | 43 | 29 | 7 |
| 2 | 28 | 27 | 29 | 40 | 38 | 10 |
| 3 | 34 | 35 | 41 | 29 | 31 | 8 |
| 4 | 41 | 42 | 35 | 27 | 36 | 8 |
| 5 | 25 | 28 | 40 | 34 | 38 | 9 |
| 6 | 31 | 30 | 43 | 38 | 40 | 6 |
| **Demand** | 9 | 6 | 12 | 8 | 10 | |

Determine the optimal daily flight schedule that will minimize total transport time.

30. PM Computer Services produces personal computers from component parts it buys on the open market. The company can produce a maximum of 300 personal computers per month. PM wants to determine its production schedule for the first six months of the new year. The cost to produce a personal computer in January will be $1,200. However, PM knows the cost of component parts will decline each month such that the overall cost to produce a PC will be 5% less each month. The cost of holding a computer in inventory is $15 per unit per month. Following is the demand for the company's computers each month:

| **Month** | **Demand** | **Month** | **Demand** |
|---|---|---|---|
| January | 180 | April | 210 |
| February | 260 | May | 400 |
| March | 340 | June | 320 |

Determine a production schedule for PM that will minimize total cost.

31. In Problem 30, suppose the demand for personal computers increased each month as follows:

| **Month** | **Demand** |
|---|---|
| January | 410 |
| February | 320 |
| March | 500 |
| April | 620 |
| May | 430 |
| June | 380 |

In addition to the regular production capacity of 300 units per month, PM Computer Services can also produce an additional 200 computers per month using overtime. Overtime production adds 20% to the cost of a personal computer.

Determine a production schedule for PM that will minimize total cost.

32. National Foods Company has five plants where it processes and packages fruits and vegetables. It has suppliers in six cities in California, Texas, Alabama, and Florida. The company has owned and operated its own trucking system in the past for transporting fruits and vegetables from its suppliers to its plants. However, it is now considering outsourcing all its shipping to outside trucking firms and getting rid of its own trucks. It currently spends $245,000 per month to operate its own trucking system. It has determined monthly shipping costs (in $1,000s per ton) using outside shippers from each of its suppliers to each of its plants as shown in the following table:

| | **Processing Plants ($1,000s per ton)** | | | | | |
|---|---|---|---|---|---|---|
| **Suppliers** | *Denver* | *St. Paul* | *Louisville* | *Akron* | *Topeka* | **Supply (tons)** |
| Sacramento | 3.7 | 4.6 | 4.9 | 5.5 | 4.3 | 18 |
| Bakersfield | 3.4 | 5.1 | 4.4 | 5.9 | 5.2 | 15 |
| San Antonio | 3.3 | 4.1 | 3.7 | 2.9 | 2.6 | 10 |
| Montgomery | 1.9 | 4.2 | 2.7 | 5.4 | 3.9 | 12 |
| Jacksonville | 6.1 | 5.1 | 3.8 | 2.5 | 4.1 | 20 |
| Ocala | 6.6 | 4.8 | 3.5 | 3.6 | 4.5 | 15 |
| **Demand (tons)** | 20 | 15 | 15 | 15 | 20 | 90 |

Should National Foods continue to operate its own shipping network or sell its trucks and outsource its shipping to independent trucking firms?

33. In Problem 32, National Foods would like to know what the effect would be on the optimal solution and the company's decision regarding its shipping if it negotiates with its suppliers in Sacramento, Jacksonville, and Ocala to increase their capacity to 25 tons per month? What would be the effect of negotiating instead with its suppliers at San Antonio and Montgomery to increase their capacity to 25 tons each?

34. Orient Express is a global distribution company that transports its clients' products to customers in Hong Kong, Singapore, and Taipei. All the products Orient Express ships are stored at three distribution centers, one in Los Angeles, one in Savannah, and one in Galveston. For the coming month the company has 450 containers of computer components available at the Los Angeles center, 600 containers available at Savannah, and 350 containers available in Galveston. The company has orders for 600 containers from Hong Kong, 500 containers from Singapore, and 500 containers from Taipei. The shipping costs per container from each U.S. port to each of the overseas ports are shown in the following table:

| **U.S. Distribution Center** | **Overseas Port** | | |
|---|---|---|---|
| | *Hong Kong* | *Singapore* | *Taipei* |
| Los Angeles | $300 | $210 | $340 |
| Savannah | 490 | 520 | 610 |
| Galveston | 360 | 320 | 500 |

The Orient Express, as the overseas broker for its U.S. customers, is responsible for unfulfilled orders, and it incurs stiff penalty costs from overseas customers if it does not meet an order. The Hong Kong customers charge a penalty cost of $800 per container for unfulfilled demand, Singapore customers charge a penalty cost of $920 per container, and Taipei customers charge $1,100 per container. Formulate and solve a transportation model to determine the shipments from each U.S. distribution center to each overseas port that will minimize shipping costs. Indicate what portion of the total cost is a result of penalties.

35. Binford Tools manufactures garden tools. It uses inventory, overtime, and subcontracting to absorb demand fluctuations. Expected demand, regular and overtime production capacity, and subcontracting capacity are provided in the following table for the next four quarters for its basic line of steel garden tools:

| **Quarter** | **Demand** | **Regular Capacity** | **Overtime Capacity** | **Subcontracting Capacity** |
|---|---|---|---|---|
| 1 | 9,000 | 9,000 | 1,000 | 3,000 |
| 2 | 12,000 | 10,000 | 1,500 | 3,000 |
| 3 | 16,000 | 12,000 | 2,000 | 3,000 |
| 4 | 19,000 | 12,000 | 2,000 | 3,000 |

The regular production cost per unit is $20, the overtime cost per unit is $25, the cost to subcontract a unit is $27, and the inventory carrying cost is $2 per unit. The company has 300 units in inventory at the beginning of the year.

Determine the optimal production schedule for the four quarters that will minimize total costs.

36. Solve the following linear programming problem:

$$\begin{aligned}\text{minimize } Z = {} & 18x_{11} + 30x_{12} + 20x_{13} + 18x_{14} + 25x_{21} + 27x_{22} + 22x_{23} \\ & + 16x_{24} + 30x_{31} + 26x_{32} + 19x_{33} + 32x_{34} + 40x_{41} + 36x_{42} \\ & + 27x_{43} + 29x_{44} + 30x_{51} + 26x_{52} + 18x_{53} + 24x_{54}\end{aligned}$$

subject to

$$\begin{aligned}x_{11} + x_{12} + x_{13} + x_{14} &\le 1 \\ x_{21} + x_{22} + x_{23} + x_{24} &\le 1 \\ x_{31} + x_{32} + x_{33} + x_{34} &\le 1 \\ x_{41} + x_{42} + x_{43} + x_{44} &\le 1 \\ x_{51} + x_{52} + x_{53} + x_{54} &\le 1 \\ x_{11} + x_{21} + x_{31} + x_{41} + x_{51} &= 1 \\ x_{12} + x_{22} + x_{32} + x_{42} + x_{52} &= 1 \\ x_{13} + x_{23} + x_{33} + x_{43} + x_{53} &= 1 \\ x_{14} + x_{24} + x_{34} + x_{44} + x_{54} &= 1 \\ x_{ij} &\ge 0\end{aligned}$$

37. A plant has four operators to be assigned to four machines. The time (minutes) required by each worker to produce a product on each machine is shown in the following table. Determine the optimal assignment and compute total minimum time.

| | Machine | | | |
|---|---|---|---|---|
| **Operator** | *A* | *B* | *C* | *D* |
| 1 | 10 | 12 | 9 | 11 |
| 2 | 5 | 10 | 7 | 8 |
| 3 | 12 | 14 | 13 | 11 |
| 4 | 8 | 15 | 11 | 9 |

38. A shop has four machinists to be assigned to four machines. The hourly cost of having each machine operated by each machinist is as follows:

| | Machine | | | |
|---|---|---|---|---|
| **Machinist** | *A* | *B* | *C* | *D* |
| 1 | $12 | 11 | 8 | 14 |
| 2 | 10 | 9 | 10 | 8 |
| 3 | 14 | 8 | 7 | 11 |
| 4 | 6 | 8 | 10 | 9 |

However, because he does not have enough experience, machinist 3 cannot operate machine B.

a. Determine the optimal assignment and compute total minimum cost.
b. Formulate this problem as a general linear programming model.

39. The Omega pharmaceutical firm has five salespersons, whom the firm wants to assign to five sales regions. Given their various previous contacts, the salespersons are able to cover the

regions in different amounts of time. The amount of time (days) required by each salesperson to cover each city is shown in the following table. Which salesperson should be assigned to each region to minimize total time? Identify the optimal assignments and compute total minimum time.

| | **Region** | | | | |
|---|---|---|---|---|---|
| **Salesperson** | *A* | *B* | *C* | *D* | *E* |
| 1 | 17 | 10 | 15 | 16 | 20 |
| 2 | 12 | 9 | 16 | 9 | 14 |
| 3 | 11 | 16 | 14 | 15 | 12 |
| 4 | 14 | 10 | 10 | 18 | 17 |
| 5 | 13 | 12 | 9 | 15 | 11 |

40. The Bunker Manufacturing firm has five employees and six machines and wants to assign the employees to the machines to minimize cost. A cost table showing the cost incurred by each employee on each machine follows. Because of union rules regarding departmental transfers, employee 3 cannot be assigned to machine E and employee 4 cannot be assigned to machine B. Solve this problem, indicate the optimal assignment, and compute total minimum cost.

| | **Machine** | | | | | |
|---|---|---|---|---|---|---|
| **Employee** | *A* | *B* | *C* | *D* | *E* | *F* |
| 1 | $12 | $ 7 | $20 | $14 | $ 8 | $10 |
| 2 | 10 | 14 | 13 | 20 | 9 | 11 |
| 3 | 5 | 3 | 6 | 9 | 7 | 10 |
| 4 | 9 | 11 | 7 | 16 | 9 | 10 |
| 5 | 10 | 6 | 14 | 8 | 10 | 12 |

41. Given the following cost table for an assignment problem, determine the optimal assignment and compute total minimum cost. Identify all alternative solutions if there are multiple optimal solutions.

| | **Machine** | | | |
|---|---|---|---|---|
| **Operator** | *A* | *B* | *C* | *D* |
| 1 | $10 | $2 | $ 8 | $ 6 |
| 2 | 9 | 5 | 11 | 9 |
| 3 | 12 | 7 | 14 | 14 |
| 4 | 3 | 1 | 4 | 2 |

42. An electronics firm produces electronic components, which it supplies to various electrical manufacturers. Quality control records indicate that different employees produce different numbers of defective items. The average number of defects produced by each employee for each of six components

is given in the following table. Determine the optimal assignment that will minimize the total average number of defects produced by the firm per month.

| | **Component** | | | | | |
|---|---|---|---|---|---|---|
| **Employee** | *A* | *B* | *C* | *D* | *E* | *F* |
| 1 | 30 | 24 | 16 | 26 | 30 | 22 |
| 2 | 22 | 28 | 14 | 30 | 20 | 13 |
| 3 | 18 | 16 | 25 | 14 | 12 | 22 |
| 4 | 14 | 22 | 18 | 23 | 21 | 30 |
| 5 | 25 | 18 | 14 | 16 | 16 | 28 |
| 6 | 32 | 14 | 10 | 14 | 18 | 20 |

43. A dispatcher for the Citywide Taxi Company has six taxicabs at different locations and five customers who have called for service. The mileage from each taxi's present location to each customer is shown in the following table. Determine the optimal assignment(s) that will minimize the total mileage traveled.

| | **Customer** | | | | |
|---|---|---|---|---|---|
| **Cab** | *1* | *2* | *3* | *4* | *5* |
| A | 7 | 2 | 4 | 10 | 7 |
| B | 5 | 1 | 5 | 6 | 6 |
| C | 8 | 7 | 6 | 5 | 5 |
| D | 2 | 5 | 2 | 4 | 5 |
| E | 3 | 3 | 5 | 8 | 4 |
| F | 6 | 2 | 4 | 3 | 4 |

44. The Southeastern Conference has nine basketball officials who must be assigned to three conference games, three to each game. The conference office wants to assign the officials so that the total distance they travel will be minimized. The distance (in miles) each official would travel to each game is given in the following table:

| | **Game** | | |
|---|---|---|---|
| **Official** | *Athens* | *Columbia* | *Nashville* |
| 1 | 165 | 90 | 130 |
| 2 | 75 | 210 | 320 |
| 3 | 180 | 170 | 140 |
| 4 | 220 | 80 | 60 |
| 5 | 410 | 140 | 80 |
| 6 | 150 | 170 | 190 |
| 7 | 170 | 110 | 150 |
| 8 | 105 | 125 | 160 |
| 9 | 240 | 200 | 155 |

a. Should this problem be solved by the transportation method or by the assignment method? Explain.
b. Determine the optimal assignment(s) that will minimize the total distance traveled by the officials.

45. In Problem 44, officials 2 and 8 had a recent confrontation with one of the coaches in the game in Athens. They were forced to eject the coach after several technical fouls. The conference office decided that it would not be a good idea to have these two officials work the Athens game so soon after this confrontation, so they decided that officials 2 and 8 will not be assigned to the Athens game. How will this affect the optimal solution to this problem?

46. State University has planned six special catered events for the November Saturday of its homecoming football game. The events include an alumni brunch, a parents' brunch, a booster club luncheon, a postgame party for season ticket holders, a lettermen's dinner, and a fund-raising dinner for major contributors. The university wants to use local catering firms as well as the university catering service to cater these events and it has asked the caterers to bid on each event. The bids (in $1,000s) based on menu guidelines for the events prepared by the university are shown in the following table:

| | Event | | | | | |
|---|---|---|---|---|---|---|
| **Caterer** | *Alumni Brunch* | *Parents' Brunch* | *Booster Club Lunch* | *Postgame Party* | *Lettermen's Dinner* | *Contributor's Dinner* |
| Al's | $12.6 | $10.3 | $14.0 | $19.5 | $25.0 | $30.0 |
| Bon Apetít | 14.5 | 13.0 | 16.5 | 17.0 | 22.5 | 32.0 |
| Custom | 13.0 | 14.0 | 17.6 | 21.5 | 23.0 | 35.0 |
| Divine | 11.5 | 12.6 | 13.0 | 18.7 | 26.2 | 33.5 |
| Epicurean | 10.8 | 11.9 | 12.9 | 17.5 | 21.9 | 28.5 |
| Fouchés | 13.5 | 13.5 | 15.5 | 22.3 | 24.5 | 36.0 |
| University | 12.5 | 14.3 | 16.0 | 22.0 | 26.7 | 34.0 |

The Bon Apetít, Custom, and University caterers can handle two events, whereas the other four caterers can handle only one. The university is confident all the caterers will do a high-quality job, so it wants to select the caterers for the events that will result in the lowest total cost.

Determine the optimal selection of caterers that will minimize total cost.

47. A university department head has five instructors to be assigned to four different courses. All the instructors have taught the courses in the past and have been evaluated by the students. The rating for each instructor for each course is given in the following table (a perfect score is 100). The department head wants to know the optimal assignment of instructors to courses that will maximize the overall average evaluation. The instructor who is not assigned to teach a course will be assigned to grade exams. Solve this problem using the assignment method.

| | Course | | | |
|---|---|---|---|---|
| **Instructor** | *A* | *B* | *C* | *D* |
| 1 | 80 | 75 | 90 | 85 |
| 2 | 95 | 90 | 90 | 97 |
| 3 | 85 | 95 | 88 | 91 |
| 4 | 93 | 91 | 80 | 84 |
| 5 | 91 | 92 | 93 | 88 |

48. The coach of the women's swim team at State University is preparing for the conference swim meet and must choose the four swimmers she will assign to the 800-meter medley relay team. The medley relay consists of four strokes—the backstroke, breaststroke, butterfly, and freestyle. The coach has computed the average times (in minutes) each of her top six swimmers has achieved in each of the four strokes for 200 meters in previous swim meets during the season as follows:

| | **Stroke (min)** | | | |
|---|---|---|---|---|
| **Swimmer** | *Backstroke* | *Breaststroke* | *Butterfly* | *Freestyle* |
| Annie | 2.56 | 3.07 | 2.90 | 2.26 |
| Beth | 2.63 | 3.01 | 3.12 | 2.35 |
| Carla | 2.71 | 2.95 | 2.96 | 2.29 |
| Debbie | 2.60 | 2.87 | 3.08 | 2.41 |
| Erin | 2.68 | 2.97 | 3.16 | 2.25 |
| Fay | 2.75 | 3.10 | 2.93 | 2.38 |

Determine the medley relay team and its total expected relay time for the coach.

49. Biggio's Department Store has six employees available to assign to four departments in the store—home furnishings, china, appliances, and jewelry. Most of the six employees have worked in each of the four departments on several occasions in the past and have demonstrated that they perform better in some departments than in others. The average daily sales for each of the six employees in each of the four departments is shown in the following table.

| | **Department Sales ($)** | | | |
|---|---|---|---|---|
| **Employee** | *Home Furnishings* | *China* | *Appliances* | *Jewelry* |
| 1 | 340 | 160 | 610 | 290 |
| 2 | 560 | 370 | 520 | 450 |
| 3 | 270 | — | 350 | 420 |
| 4 | 360 | 220 | 630 | 150 |
| 5 | 450 | 190 | 570 | 310 |
| 6 | 280 | 320 | 490 | 360 |

Employee 3 has not worked in the china department before, so the manager does not want to assign this employee to china.

Determine which employee to assign to each department and indicate the total expected daily sales.

50. The Vanguard Publishing Company has eight college students it hires as salespeople to sell encyclopedias during the summer. The company desires to allocate them to three sales territories. Territory 1 requires three salespeople, and territories 2 and 3 require two salespeople each. It is estimated that each salesperson will be able to generate the amounts of dollar sales per day in each of the three territories as given in the following table. The company desires to allocate the salespeople to the three territories so that sales will be maximized. Solve this problem using any method to determine the initial solution and any solution method. Compute the maximum total sales per day.

| Salesperson | Territory 1 | 2 | 3 |
|---|---|---|---|
| A | $110 | $150 | $130 |
| B | 90 | 120 | 80 |
| C | 205 | 160 | 175 |
| D | 125 | 100 | 115 |
| E | 140 | 105 | 150 |
| F | 100 | 140 | 120 |
| G | 180 | 210 | 160 |
| H | 110 | 120 | 90 |

51. Carolina Airlines, a small commuter airline in North Carolina, has six flight attendants whom it wants to assign to six monthly flight schedules in a way that will minimize the number of nights they will be away from their homes. The numbers of nights each attendant must be away from home with each schedule are given in the following table. Identify the optimal assignments that will minimize the total number of nights the attendants will be away from home.

| Attendant | Schedule A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | 7 | 4 | 5 | 10 | 5 | 7 |
| 2 | 4 | 5 | 4 | 12 | 7 | 5 |
| 3 | 9 | 9 | 10 | 7 | 10 | 7 |
| 4 | 11 | 6 | 7 | 5 | 9 | 9 |
| 5 | 5 | 8 | 5 | 10 | 7 | 5 |
| 6 | 10 | 12 | 10 | 9 | 9 | 9 |

52. The football coaching staff at Tech focuses its recruiting on several key states, including Georgia, Florida, Virginia, Pennsylvania, New York, and New Jersey. The staff includes seven assistant coaches, two of whom are responsible for Florida, a high school talent–rich state, whereas one coach is assigned to each of the other five states. The staff has been together for a long time and at one time or another all the coaches have recruited all of the states. The head coach has accumulated some data on the past success rate (i.e., percentage of targeted recruits signed) for each coach in each state as shown in the following table:

| Coach | State GA | FL | VA | PA | NY | NJ |
|---|---|---|---|---|---|---|
| Allen | 62 | 56 | 65 | 71 | 55 | 63 |
| Bush | 65 | 70 | 63 | 81 | 75 | 72 |
| Crumb | 46 | 53 | 62 | 55 | 64 | 50 |
| Doyle | 58 | 66 | 70 | 67 | 71 | 49 |
| Evans | 77 | 73 | 69 | 80 | 80 | 74 |
| Fouch | 68 | 73 | 72 | 80 | 78 | 57 |
| Goins | 72 | 60 | 74 | 72 | 62 | 61 |

Determine the optimal assignment of coaches to recruiting regions that will maximize the overall success rate and indicate the average percentage success rate for the staff with this assignment.