Moments in 3D

Shortest distance between two jokes?

A straight line.

Objectives

- Understand what a moment represents in mechanics
- Understand the vector formulation of a moment

Tools

- Basic Trigonometry
- Pythagorean Theorem
- Algebra
- Visualization
- Position Vectors
- Unit Vectors

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Three Dimensions

- Clockwise and counter-clockwise really don't have any meaning in three dimensional problems
- Vectors make life much easier in three dimensions

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Three Dimensions

○ Once again, we construct a moment arm from the center of rotation to the line of action of the force causing the rotation

○ The moment arm is nothing more than a position vector from the moment center to the line of action of the force

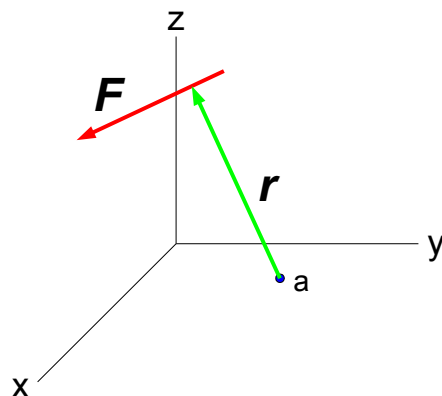Moments in 3D   Wednesday ,September 19, 2012

Three Dimensions

○ *F* is the force vector and *r* is the moment arm vector
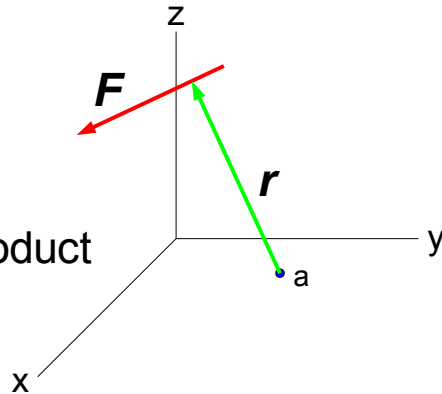

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Three Dimensions

- The moment generated about point a by the force F is given by the expression

$$\vec{M} = \vec{r} \otimes \vec{F}$$

where $\otimes$ is the cross product
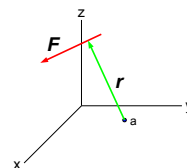
z

F

r

y

a

x

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Cross Product

- The cross product is the second type of vector multiplication
- Unlike the dot product which produced a scalar, the cross product produces a vector
- Unlike the dot product, the order in which we write the terms is important

$$\vec{M} = \vec{r} \otimes \vec{F} \neq \vec{F} \otimes \vec{r}$$
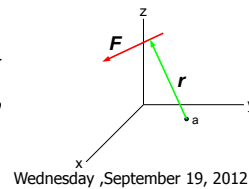
z

F

r

y

a

x

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Cross Product

○ One of the most commonly made mistakes when dealing with moments in three dimensions is to put the order of the cross product in the incorrect order

$$\vec{M} = \vec{r} \otimes \vec{F} \neq \vec{F} \otimes \vec{r}$$

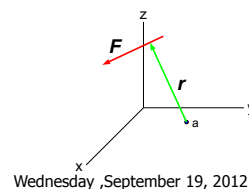Moments in 3D   Wednesday ,September 19, 2012

Cross Product

○ For the dot product, the product of two like unit vectors was 1 and any other product equals 0

$$\vec{i} \odot \vec{i} = 1$$

$$\vec{j} \odot \vec{j} = 1$$

$$\vec{k} \odot \vec{k} = 1$$

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Cross Product

o For the cross product, things are a bit more complicated

$$\vec{i} \otimes \vec{i} = 0 \qquad \vec{i} \otimes \vec{j} = \vec{k} \qquad \vec{i} \otimes \vec{k} = -\vec{j}$$

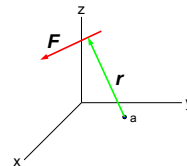$$\vec{j} \otimes \vec{i} = -\vec{k} \qquad \vec{j} \otimes \vec{j} = 0 \qquad \vec{j} \otimes \vec{k} = \vec{i}$$

$$\vec{k} \otimes \vec{i} = \vec{j} \qquad \vec{k} \otimes \vec{j} = -\vec{i} \qquad \vec{k} \otimes \vec{k} = 0$$

Cross Product

o A simple way to remember

i

k ——————— *j*

$$\vec{i} \otimes \vec{i} = 0 \qquad \vec{i} \otimes \vec{j} = \vec{k} \qquad \vec{i} \otimes \vec{k} = -\vec{j}$$
$$\vec{j} \otimes \vec{i} = -\vec{k} \qquad \vec{j} \otimes \vec{j} = 0 \qquad \vec{j} \otimes \vec{k} = \vec{i}$$
$$\vec{k} \otimes \vec{i} = \vec{j} \qquad \vec{k} \otimes \vec{j} = -\vec{i} \qquad \vec{k} \otimes \vec{k} = 0$$
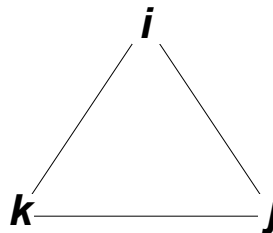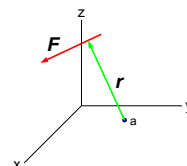
6

Cross Product

○ Positive results

$$\vec{i} \otimes \vec{i} = 0 \qquad \vec{i} \otimes \vec{j} = \vec{k} \qquad \vec{i} \otimes \vec{k} = -\vec{j}$$
$$\vec{j} \otimes \vec{i} = -\vec{k} \qquad \vec{j} \otimes \vec{j} = 0 \qquad \vec{j} \otimes \vec{k} = \vec{i}$$
$$\vec{k} \otimes \vec{i} = \vec{j} \qquad \vec{k} \otimes \vec{j} = -\vec{i} \qquad \vec{k} \otimes \vec{k} = 0$$
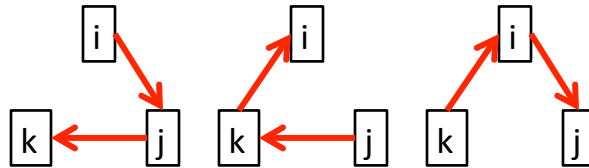
13   Moments in 3D   Wednesday ,September 19, 2012
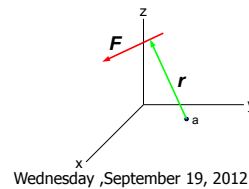

Cross Product

○ Negative results

$$\vec{i} \otimes \vec{i} = 0 \qquad \vec{i} \otimes \vec{j} = \vec{k} \qquad \vec{i} \otimes \vec{k} = -\vec{j}$$
$$\vec{j} \otimes \vec{i} = -\vec{k} \qquad \vec{j} \otimes \vec{j} = 0 \qquad \vec{j} \otimes \vec{k} = \vec{i}$$
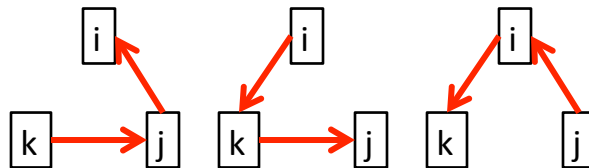$$\vec{k} \otimes \vec{i} = \vec{j} \qquad \vec{k} \otimes \vec{j} = -\vec{i} \qquad \vec{k} \otimes \vec{k} = 0$$

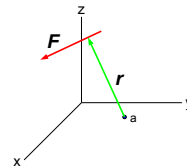14   Moments in 3D   Wednesday ,September 19, 2012
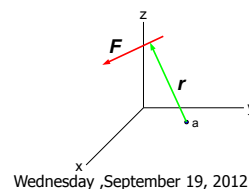
Cross Product

○ If we have a position vector *r* and a force vector *F* defined as

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

$$\vec{r} = r_x \vec{i} + r_y \vec{j} + r_z \vec{k}$$

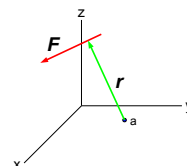Moments in 3D   Wednesday ,September 19, 2012

Cross Product

○ We can calculate the moment of the force about the point by taking the cross product

$$\overrightarrow{M} = \vec{r} \otimes \vec{F}$$

$$\overrightarrow{M} = \left(r_x \vec{i} + r_y \vec{j} + r_z \vec{k} \right) \otimes \left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \right)$$

$$\overrightarrow{M} = r_x \vec{i} \otimes \left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \right)$$

$$+ r_y \vec{j} \otimes \left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \right)$$

$$+ r_z \vec{k} \otimes \left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \right)$$
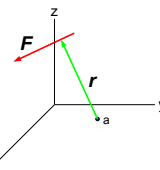
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Cross Product

○ Expanding

$$\overrightarrow{M} = r_x\vec{i} \otimes F_x\vec{i} + r_x\vec{i} \otimes F_y\vec{j} + r_x\vec{i} \otimes F_z\vec{k}$$
$$+ r_y\vec{j} \otimes F_x\vec{i} + r_y\vec{j} \otimes F_y\vec{j} + r_y\vec{j} \otimes F_z\vec{k}$$
$$+ r_z\vec{k} \otimes F_x\vec{i} + r_z\vec{k} \otimes F_y\vec{j} + r_z\vec{k} \otimes F_z\vec{k}$$
$$\overrightarrow{M} = \left(r_x F_x\right)\left(\vec{i} \otimes \vec{i}\right) + \left(r_x F_y\right)\left(\vec{i} \otimes \vec{j}\right) + \left(r_x F_z\right)\left(\vec{i} \otimes \vec{k}\right)$$
$$+ \left(r_y F_x\right)\left(\vec{j} \otimes \vec{i}\right) + \left(r_y F_y\right)\left(\vec{j} \otimes \vec{j}\right) + \left(r_y F_z\right)\left(\vec{j} \otimes \vec{k}\right)$$
$$+ \left(r_z F_x\right)\left(\vec{k} \otimes \vec{i}\right) + \left(r_z F_y\right)\left(\vec{k} \otimes \vec{j}\right) + \left(r_z F_z\right)\left(\vec{k} \otimes \vec{k}\right)$$

Moments in 3D

Cross Product

○ Using our cross product rules for unit vectors

$$\overrightarrow{M} = \left(r_x F_x\right)(0) + \left(r_x F_y\right)\left(\vec{k}\right) + \left(r_x F_z\right)\left(-\vec{j}\right)$$
$$+ \left(r_y F_x\right)\left(-\vec{k}\right) + \left(r_y F_y\right)(0) + \left(r_y F_z\right)\left(\vec{i}\right)$$
$$+ \left(r_z F_x\right)\left(\vec{j}\right) + \left(r_z F_y\right)\left(-\vec{i}\right) + \left(r_z F_z\right)(0)$$
$$\overrightarrow{M} = \left(r_x F_y\right)\left(\vec{k}\right) - \left(r_x F_z\right)\left(\vec{j}\right) - \left(r_y F_x\right)\left(\vec{k}\right)$$
$$+ \left(r_y F_z\right)\left(\vec{i}\right) + \left(r_z F_x\right)\left(\vec{j}\right) - \left(r_z F_y\right)\left(\vec{i}\right)$$
$$\overrightarrow{M} = \left(r_y F_z - r_z F_y\right)\vec{i} + \left(r_z F_x - r_x F_z\right)\vec{j} + \left(r_x F_y - r_y F_x\right)\vec{k}$$
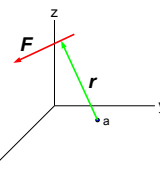
Moments in 3D

Cross Product

○ One thing to notice here

$$\overrightarrow{M} = \left(r_y F_z - r_z F_y \right)\vec{i} + \left(r_z F_x - r_x F_z \right)\vec{j} + \left(r_x F_y - r_y F_x \right)\vec{k}$$

If we are in two dimensions (x and y) there will be no i and j components to the resulting moment. The moment will be either into the page or out of the page. Since we follow the right hand rule for all our axes, into the page would be negative and out of the page would be positive. This corresponds to CW and CCW.
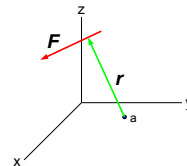
Cross Product
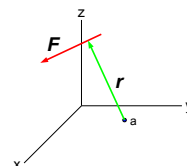
○ We can also set up the cross product as a matrix

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$
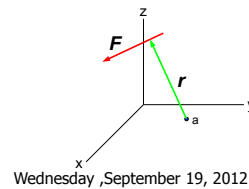
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Cross Product

o There are a number of ways to expand this matrix to find the solution, use whatever way you are comfortable with

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

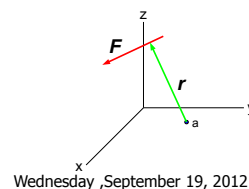21   Moments in 3D   Wednesday ,September 19, 2012

Cross Product

o Since I could never keep the signs straight, I always use what appeared to me to be a very simple technique

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

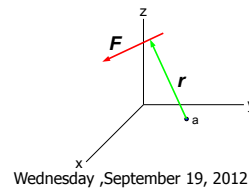22   Moments in 3D   Wednesday ,September 19, 2012

Cross Product

○ Copy the first two columns to the end of the matrix

$$\vec{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{matrix}$$

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Cross Product

○ Start with the *i* and move down and right

$$\vec{i}\left(r_y F_z\right)$$

$$\vec{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{matrix}$$

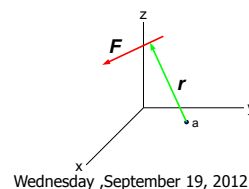24   Moments in 3D   Wednesday ,September 19, 2012

Cross Product

o Then insert a negative sign and move down
and left.

$$\vec{i}\left(r_y F_z - r_z F_y\right)$$

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{matrix}$$

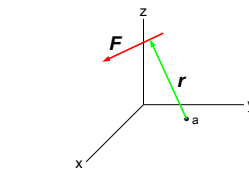25   Moments in 3D   Wednesday ,September 19, 2012

Cross Product

o Move everything to the right and do the
process again.

$$\vec{j}\left(r_z F_x\right)$$

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{matrix}$$

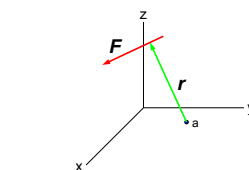26   Moments in 3D   Wednesday ,September 19, 2012

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Cross Product

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○ Move everything to the right and do the process again.

$$\vec{j}\left(r_z F_x - r_x F_z\right)$$

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{matrix}$$

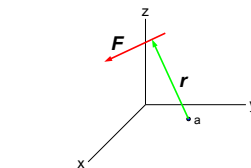27   Moments in 3D   Wednesday ,September 19, 2012

Cross Product

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○ Move everything to the right and do the process again.

$$\vec{k}\left(r_x F_y\right)$$

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{matrix}$$

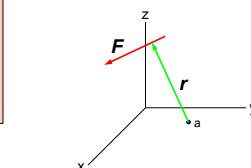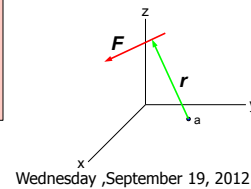28   Moments in 3D   Wednesday ,September 19, 2012

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Cross Product

○ Move everything to the right and do the process again.

$$\vec{k}\left(r_x F_y - r_y F_x\right)$$

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \begin{array}{cc} \vec{i} & \vec{j} \\ r_x & r_y \\ F_x & F_y \end{array}$$
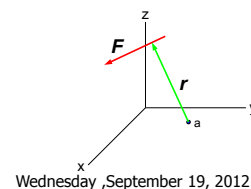

Cross Product

○ Add the individual results.

$$\overrightarrow{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{M} = \vec{i}\left(r_y F_z - r_z F_y\right) + \vec{j}\left(r_z F_x - r_x F_z\right) + \vec{k}\left(r_x F_y - r_y F_x\right)$$

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Problem F4-11

Determine the moment of force F about point O. Express the result as a Cartesian vector.

z

$F = 120$ lb
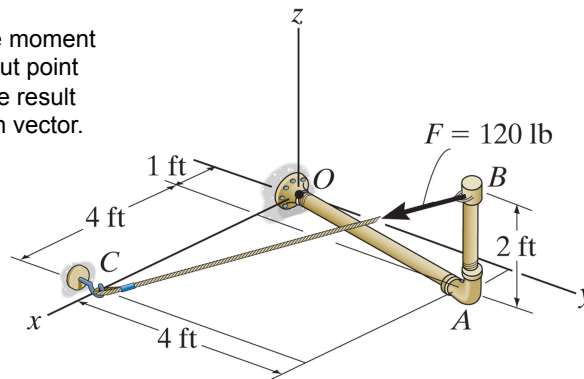
1 ft

4 ft

O

B

C

2 ft

x

A

y

4 ft

Figure: 04_FP011

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Problem F4-12

If F_1 = {100i – 120j +75k} lb and F_2 = {-200i – 250j +100k} lb, determine the resultant moment produced by these forces about point O. Express the result as a Cartesian vector.

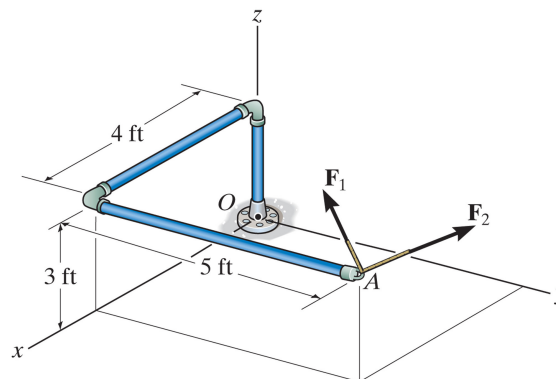z

4 ft

O

F_1

F_2

3 ft

5 ft

A

y

x

Figure: 04_FP012

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Homework

o Problem 4-30

o Problem 4-33

o Problem 4-34

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