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SHORT COMMUNICATION

Moore-Penrose's inverse and solutions of linear systems

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ABSTRACT

We employ the generalized inverse matrix of Moore-Penrose to study the existence and uniqueness of the solutions for over- and under-determined linear systems, in harmony with the least squares method.

***Keywords*:** Linear systems, SVD, Least squares technique, Pseudoinverse of Moore-Penrose

1. INTRODUCTION

For any real matrix A_{nxm}, Lanczos [1, 2] introduces the matrix:

$$S_{(n+m)x(n+m)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \tag{1}$$

with A^T denoting the transpose matrix, and studies the eigenvalue problem:

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$$S\vec{\omega} = \lambda\vec{\omega}, \tag{2}$$

where the proper values are real because S is a real symmetric matrix. Besides:

$$\text{rank}\, A \ \equiv p = \text{Number of positive eigenvalues of}\, S, \tag{3}$$

such that $1 \le p \le \min(n, m)$. Then the singular values or canonical multipliers follow the scheme:

$$\lambda_1, \lambda_2, \dots, \lambda_p, -\lambda_1, -\lambda_2, \dots, -\lambda_p, 0, 0, \dots, 0, \tag{4}$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of S, named 'essential axes' by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m)x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \begin{matrix} n \\ m \end{matrix}, \tag{5}$$

then (1) and (2) imply the Modified Eigenvalue Problem:

$$A_{nxm}\vec{v}_{mx1} = \lambda\, \vec{u}_{nx1}, \qquad A^T{}_{mxn}\vec{u}_{nx1} = \lambda\, \vec{v}_{mx1}, \tag{6}$$

hence:

$$A^T A\vec{v} = \lambda^2\vec{v}, \qquad A\, A^T\vec{u} = \lambda^2\vec{u}, \tag{7}$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{nxp} = (\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p), \qquad V_{mxp} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p), \tag{8}$$

verifying $U^T U = V^T V = I_{pxp}$ because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk}, \tag{9}$$

therefore $\vec{\omega}_j \cdot \vec{\omega}_k = 2\delta_{jk}$, $j, k = 1,2, \dots, p$. Thus, the Singular Value Decomposition (SVD) express [1-5] that A is the product of three matrices:

$$A_{nxm} = U_{nxp}\,\Lambda_{pxp}\,V^T{}_{pxm}, \qquad \Lambda = \text{Diag}\,(\lambda_1, \lambda_2, \dots, \lambda_p). \tag{10}$$

This relation tells that in the construction of A we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to A. Golub [6] mentions that the SVD has played a very important role in computations, in solving least squares problems [7], in signal processing problems, and so on; it is just a very simple decomposition, yet it is of fundamental importance in many problems arising in technology.

It is important to observe that the symmetric matrices $(UU^T)_{nxn}$ and $(VV^T)_{mxm}$ are identity matrices for arbitrary vectors into their respective spaces of activation [5], that is:

$$UU^T\vec{u} = \vec{u}, \quad \forall\ \vec{u}\ \epsilon\ Col\,U, \qquad VV^T\vec{v} = \vec{v}, \quad \forall\ \vec{v}\ \epsilon\ Col\,V; \tag{11}$$

besides, (10) allows obtain the SVD of the Gram matrices:

$$(A\,A^T)_{nxn} = U\Lambda^2U^T\,, \qquad (A^TA)_{mxm} = V\Lambda^2V^T, \tag{12}$$

such that $p = \text{rank}\,A = \text{rank}\,(AA^T) = \text{rank}\,(A^TA)$. From (10) and (12) we observe that:

$$Col\,A = Col\,(A\,A^T) = Col\,U, \qquad Col\,A^T = Col\,(A^TA) = Col\,V. \tag{13}$$

The eigenvectors associated with $\lambda = 0$ verify the equations:

$$\begin{aligned} &A\vec{v}_j^{\circ} = \vec{0}, \quad j = 1,\dots,m-p, \qquad A^T\vec{u}_k^{\circ} = \vec{0}, \quad k = 1,\dots,n-p, \\ &\vec{v}_r\cdot\vec{v}_j^{\circ} = 0, \quad \forall\ r,j\,, \qquad \vec{u}_t\cdot\vec{u}_k^{\circ} = 0, \quad \forall\ t,k \end{aligned} \tag{14}$$

therefore:

$$\begin{aligned} &V^T\vec{v}_j^{\circ} = \vec{0}, \quad \forall\ j, \qquad U^T\vec{u}_k^{\circ} = \vec{0}, \quad \forall\ k\,, \\ &A\vec{x}\ \epsilon\ Col\,U \text{ and } A^TA\,\vec{x}\ \epsilon\ Col\,V, \quad \forall\ \vec{x}\ \epsilon\ E^m, \\ &A^T\vec{y}\ \epsilon\ Col\,V \text{ and } AA^T\vec{y}\ \epsilon\ Col\,U, \quad \forall\ \vec{y}\ \epsilon\ E^n. \end{aligned} \tag{15}$$

In Sec. 2 we exhibit the Moore-Penrose's pseudoinverse of A [8-13] via the corresponding SVD [14-16], which is useful in Sec. 3 to study the solutions of over- and under-determined linear systems [2, 5] in the spirit of the least squares method [7, 17].

2. GENERALIZED INVERSE

The Moore-Penrose's inverse [2, 8-13] is given by:

$$A^+{}_{mxn} = V_{mxp}\,\Lambda_{pxp}^{-1}\,U^T{}_{pxn}\,, \tag{16}$$

which coincides with the natural inverse obtained by Lanczos [2, 5]. The matrix (16) satisfies the relations [10, 11, 13]:

$$A\,A^+A = A, \qquad A^+A\,A^+ = A^+, \qquad (A\,A^+)^T = A\,A^+, \qquad (A^+A)^T = A^+A, \tag{17}$$

that characterize the pseudoinverse of Moore-Penrose. In particular, from (10), (11) and (16):

$$A\,A^{+} = U\,U^{T} \quad \therefore \quad A\,A^{+}\,\vec{u} = \vec{u}, \quad \forall\ \vec{u}\ \epsilon\ Col\,U,$$
$$A^{+}A = V\,V^{T} \quad \therefore \quad A^{+}A\,\vec{v} = \vec{v}, \quad \forall\ \vec{v}\ \epsilon\ Col\,V. \tag{18}$$

The use of (8) and (10) into (16) implies the following expression for the Lanczos generalized inverse:

$$A^{+} = (\vec{t}_1\ \ \vec{t}_2\ \cdots\ \vec{t}_n)\,, \qquad \vec{t}_j = \frac{u_1^{(j)}}{\lambda_1}\,\vec{v}_1 + \frac{u_2^{(j)}}{\lambda_2}\,\vec{v}_2 + \cdots + \frac{u_p^{(j)}}{\lambda_p}\,\vec{v}_p\,, \qquad j = 1, \dots, n, \tag{19}$$

where $u_k^{(j)}$ means the *j th*- component of $\vec{u}_k$; similarly:

$$(A^{+})^{T} = (\vec{r}_1\ \ \vec{r}_2\ \cdots\ \vec{r}_m), \quad \vec{r}_k = \frac{v_1^{(k)}}{\lambda_1}\,\vec{u}_1 + \frac{v_2^{(k)}}{\lambda_2}\,\vec{u}_2 + \cdots + \frac{v_p^{(k)}}{\lambda_p}\,\vec{u}_p\,, \qquad k = 1, \dots, m, \tag{20}$$

therefore:

$$Col\,A^{+} = Col\,V, \qquad Col\,(A^{+})^{T} \equiv Col\,(U\Lambda^{-1}V^{T}) = Col\,U. \tag{21}$$

We can use (16) to construct the pseudoinverse of each Gram matrix, in fact [13]:

$$(A^{T}A)^{+}{}_{mxm} = V\,\Lambda^{-2}\,V^{T}, \qquad (A\,A^{T})^{+}{}_{nxn} = U\,\Lambda^{-2}\,U^{T}, \tag{22}$$

with the interesting properties:

$$(A^{T}A)^{+}A^{T} = A^{+}\,, \quad (A\,A^{T})^{+}A = (A^{+})^{T}\,, \quad (A^{T}A)^{+}(A^{T}A) = A^{+}A = V\,V^{T}. \tag{23}$$

Each matrix has a unique inverse because every matrix is complete within its own spaces of activation. The activated *p*-dimensional subspaces (eigenspaces / operational spaces) are uniquely associated with the given matrix [5].

3. LINEAR SYSTEMS

We want to find $\vec{x}\ \epsilon\ E^{m}$ verifying the linear system:

$$A\,\vec{x} = \vec{b}\,, \tag{24}$$

for the data A_{nxm} and $\vec{b}\ \epsilon\ E^{n}$. It is convenient to consider two situations:

a). *Over-determined linear system* [2,5]: In this case we have more equations than unknowns, that is, $m < n$.

Lanczos [18] comments that the ingenious method of least squares makes it possible to adjust an arbitrarily over-determined and incompatible set of equations. The problem of minimizing $(A\vec{x}-\vec{b})^2$ has always a definite solution, no matter how compatible or incompatible the given system is. The least square solution of (24) satisfies [5, 17]:

$$A^T A\,\vec{x} = A^T\vec{b}, \qquad \vec{x}\ \epsilon\ Col\,V, \qquad p = m, \tag{25}$$

and the remarkable fact about (25) is that it always gives an even-determined (balanced) system, no matter how strongly over-determined the original system has been.

The system (25) is compatible because from (13) and (15) we have that $A^T\vec{b}$ is into $Col\ (A^T A) = Col\,V$. Now we multiply (25) by $(A^T A)^+$ and we use (11) and (23) to obtain the solution:

$$\vec{x} = A^+\vec{b}, \tag{26}$$

which is unique because $p = m$, that is, $Col\ V = E^m$, then in (14) the system $A\vec{v}_j^{\circ}$ only has the trivial solution; hence the Moore-Penrose's inverse gives the least square solution of (24). The expression (26) is in harmony with the results in [19-22].

We have eliminated over-determination (and possibly incompatibility) by the method of multiplying both sides of (24) by A^T. The unique solution thus obtained coincides with the solution generated with the help of A^+ [5].

b). *Under-determined linear system* [2, 5]: There are more unknowns than equations, that is, $n < m$.

In this case we may try the least square formulation of (24), that is, to accept (26), however, now the solution is not unique because $p < m$ and the system $A\vec{v}_j^{\circ}$ has $m-p$ non-trivial independent solutions; an under-determined system remains thus under-determined, even in the least square approach.

An alternative process is to transform the original $\vec{x}$ into the new unknown $\vec{z}$ via the relation [5]:

$$\vec{x} = A^T\vec{z}, \tag{27}$$

then (24) acquires the structure $AA^T\vec{z} = \vec{b}$ whose least square solution is given by the pseudoinverse of Moore-Penrose:

$$\vec{z} = (AA^T)^+\,\vec{b} + \sum_{j=1}^{n-p} c_j\,\vec{z}_j^{\circ}, \tag{28}$$

where the quantities c_j are arbitrary and the $\vec{z}_j^{\circ}$ are $n-p$ independent vectors generating the $Kernel\ (AA^T) = Kernel\ (A^T)$ [13], that is:

$$A^T\vec{z}_j^{\circ} = \vec{0}, \qquad j = 1, \dots, n-p. \tag{29}$$

Thus, from (16), (22), (23), (28) and (29) we have that the solution of (27) is given by:

$$\vec{x} = A^T(AA^T)^+\vec{b} = V\Lambda^{-1}U^T\vec{b} = A^+\vec{b},$$

in agreement with (26).

Although that (26) is not unique for the under-determined case, we can say that it is the 'natural solution' for the linear system (24).

4. CONCLUSIONS

Our study shows the importance of the SVD [1-6, 14-16] of a matrix and of the corresponding Moore-Penrose's inverse [8-13], to elucidate the least square solution [7, 17-22] for over- and under-determined linear systems [2, 5].

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