Moyal and Clifford Algebras in the Bohm Approach.

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The Bohm Story Unfolded.

Classical physics.

Things go on in space-time. dynamics ⟹ phase space

t | Particles or fields — x

symplectic symmetry

p | Particles — x or $\pi(x,t)$ | Fields — $\psi(x,t)$

Quantum physics.

t | Fields — x

abandon phase space ⟹ Operators in Hilbert space

symplectic encoded in Heisenberg group

Primitive Bohm approach.

Looks like a return to particles in phase space.

$p = \nabla_x S$ | Particles — x

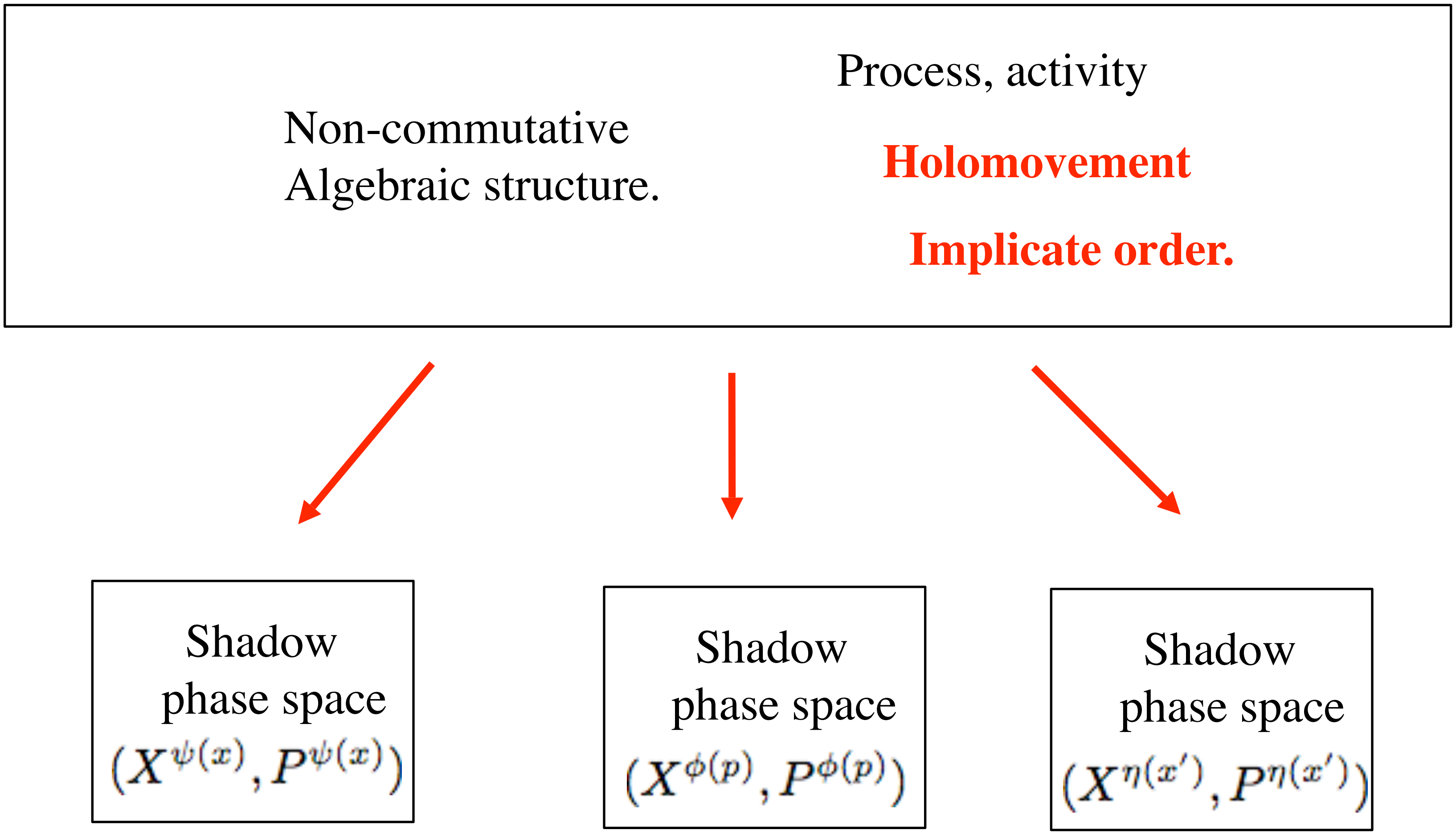
Where has the symplectic symmetry gone? Its there still!

Can also construct p | Particles — $x = -\nabla_p S$ and more!

[Brown, PhD thesis 2000]

[Brown and Hiley quant-ph/0005026]

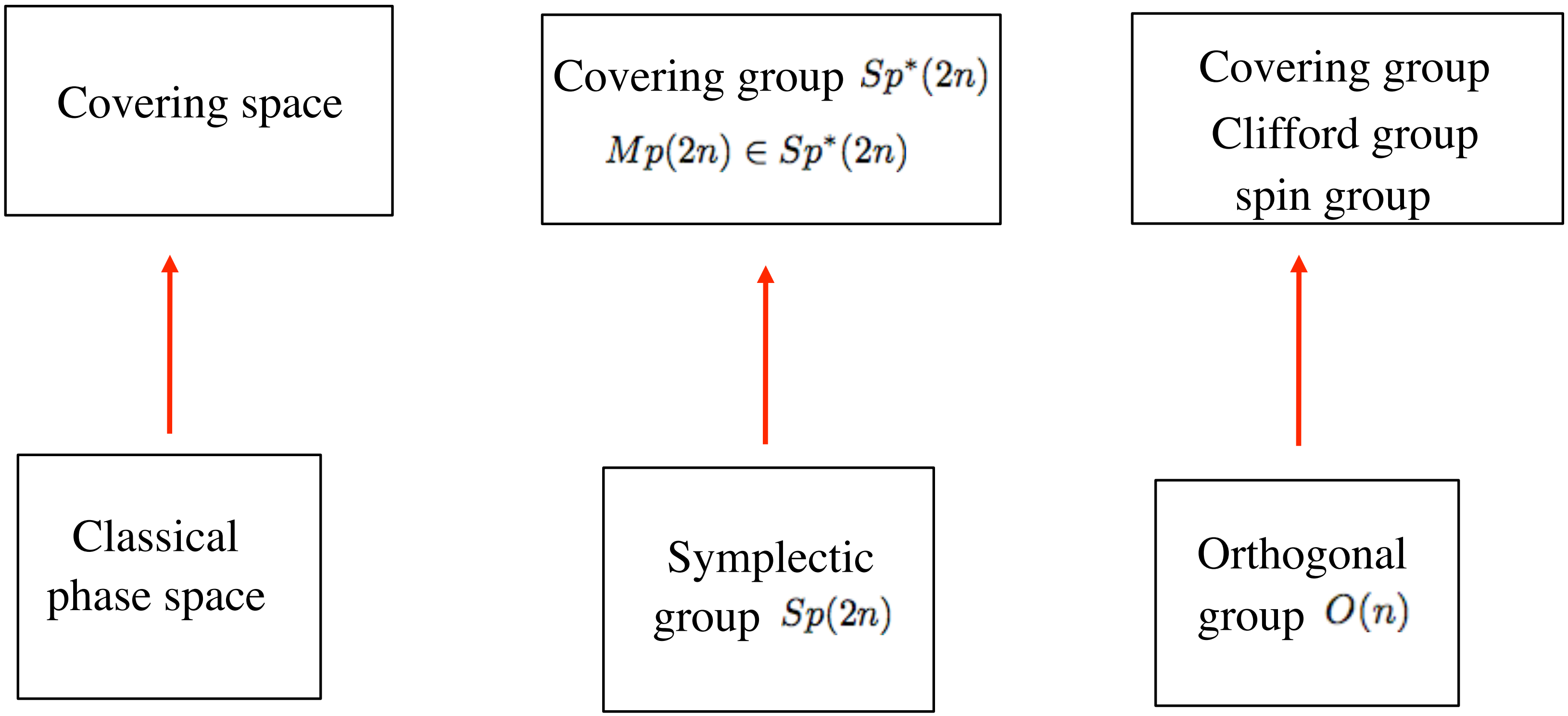
The Overarching Structure



Possible explicate orders.

[D. Bohm Wholeness and the Implicate Order (1980])

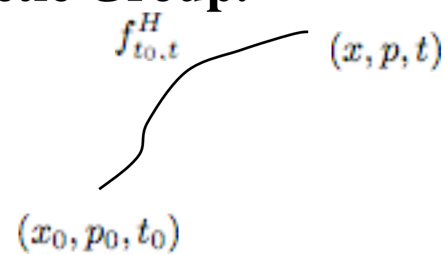
Some Mathematical Facts.



Can we lift the classical properties on to the covering spaces?

[Souriau, Fond. Phys. 13 (1983) 113-151]

Dynamics, Symplectic Group.



Start with classical mechanics.

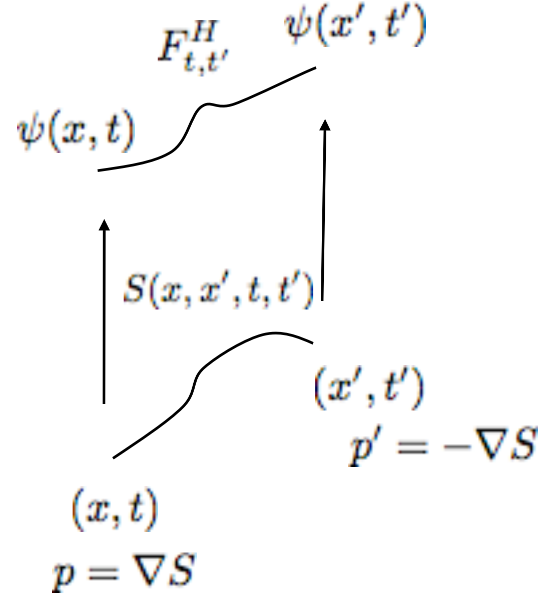
If H is a function of t,

$$f^H_{t,t'} \circ f^H_{t',t''} = f^H_{t,t''}$$

Hamiltonian groupoid

Motion is generated by Hamilton-Jacobi function $S(x, x', t, t')$

Lift this onto a covering space.



de Gosson has shown that $F^H_{t,t'}$ is Schrödinger for all Hamiltonians.

Key object:- $\rho(x, x', t) = \psi^*(x', t)\psi(x, t)$

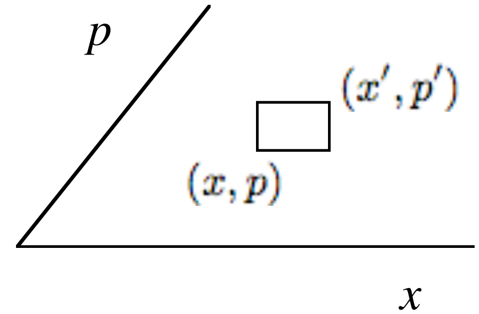
Non-local object

What is this object and how does it develop in time?

[de Gosson, *The Principles of Newtonian and Quantum Mechanics*, 2001].

[de Gosson and Hiley, pre-print 2010]

The Two-point Density Matrix.



Start with $\rho(x, x', t) = \psi^*(x', t)\psi(x, t)$

Go to *p*-space $\psi(x, t) = (2\pi)^{-1} \int \phi(p, t) e^{ipx} dp$

$$\rho(x, x', t) = (2\pi)^{-2} \iint \phi^*(p', t) e^{ixp} \phi(p, t) e^{ix'p'} dp dp'$$

Use $X = (x' + x)/2 \quad \eta = x' - x \quad P = (p' + p)/2 \quad \pi = p' - p$

Then $\rho(X, \eta, t) = (2\pi)^{-2} \iint \phi^*(P - \pi/2, t)\phi(P + \pi/2, t) e^{iX\pi} d\pi \, e^{i\eta P} dP$

Write as $\rho(X, \eta, t) = (2\pi)^{-1} \int F(X, P, t) e^{i\eta P} dP$

$\rho(X, \eta, t) \Leftrightarrow F(X, P, t)$

So that $F(X, P, t) = (2\pi)^{-1} \int \phi^*(P - \pi/2, t) e^{iX\pi} \phi(P + \pi/2, t) d\pi$

$$= (2\pi)^{-1} \int \psi^*(X - \eta/2, t) e^{-iP\eta} \psi(X + \eta/2, t) d\eta$$

Wigner distribution

Try to use $F(X, P, t)$ as a classical distribution function $\Rightarrow$ negative probabilities

It is essentially a ‘density matrix’ in the (*X, P*) representation.

NB. It describes a ‘quantum blob’, not a classical particle.

Symplectic capacity

Quantum Phase space.

1. Change of representation ⇒ return to phase space of functions?

   NB *X* and *P* are not coordinates of a simple particle. $[\hat{X}, \hat{P}] = 0$

   [Bohm and Hiley, Found. Phys. **11**, (1981) 179-203]

2. Treat $F(X,P)$ as a quasi-probability density? **Don't!!**

3. We can generate a new non-commutative algebra of functions with a new product.

$$F(X,P) * G(X,P) = F(X,P)e^{i\hbar/2(\overleftarrow{\partial}_X \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_X)}G(X,P)$$

Moyal product

This product is non-commutative $F * G \neq G * F$

But it is associative $F * (G * H) = (F * G) * F$

We find that we can do quantum mechanics in the phase space without operators.

No operators in Hilbert space!

[Moyal, Proc. Camb. Phil. Soc. 45, 99-123, 1949.]

[Dubin,Hennings & Smith, Math. Aspects of Weyl Quantization, 2000]

Moyal * Multiplication is Matrix Multiplication.

Write in general

$$A(X,P,t) = (2\pi)^{-1} \int \rho_a(X-\eta/2, X+\eta/2) e^{-i\eta P} d\eta$$

write as

$$A(X,P) = (2\pi)^{-1} \int \hat{A}(X,\eta) e^{-i\eta P} d\eta$$

Then

$$A(X,P) * B(X,P) = C(X,P)$$

is equivalent to

$$\hat{C}(X-\eta/2, X+\eta/2) = \int \hat{A}(X-\eta/2, y)\hat{B}(y, X+\eta/2) dy$$

Essentially matrix multiplication. **NB Non-local.**

For proof write

$$A * B = \iint d\eta d\eta' \hat{A}(X-\eta/2, X+\eta/2) e^{-i\eta P} e^{i(\overleftarrow{\partial}_X \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_X)} e^{-i\eta' P} \hat{B}(X-\eta'/2, X+\eta'/2)$$

[Cnockaert, Moyal's Proc. Modave Summer School, 2005]

Properties of the Moyal *-Product?

Moyal bracket (commutator)

$$\{A,B\}_{MB} = \frac{A*B - B*A}{i\hbar} = 2A(X,P)\sin\frac{\hbar}{2}\left[\overleftarrow{\partial}_X\overrightarrow{\partial}_P - \overrightarrow{\partial}_X\overleftarrow{\partial}_P\right]B(X,P)$$

Baker bracket (Jordan product or anti-commutator).

$$\{A,B\}_{BB} = \frac{A*B + B*A}{2} = 2A(X,P)\cos\frac{\hbar}{2}\left[\overleftarrow{\partial}_X\overrightarrow{\partial}_P - \overrightarrow{\partial}_X\overleftarrow{\partial}_P\right]B(X,P)$$

Deform to obtain classical limit.

Sine bracket becomes **Poisson** bracket. $\{A,B\}_{MB} = \{A,B\}_{PB} + O(\hbar^2)$

Cosine bracket becomes ordinary product $\{A,B\}_{BB} = AB + O(\hbar^2)$

[Baker, Jn., Phys. Rev. 109, (1958) 2198-2206.]

Quantum Dynamics.

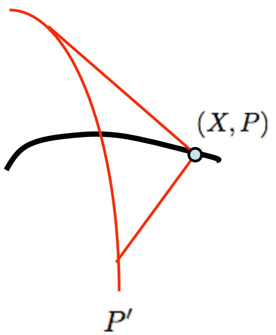
Equation of motion for $\rho(x, x', t)$

$$-i\frac{\partial \rho}{\partial t} = H\rho - \rho H = \left[\frac{1}{2m}\left(\frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x^2}\right) - V(x') + V(x)\right]\rho(x, x', t)$$

Changing variables $(x, x') \to (X, \eta)$ we find

$$-i\frac{\partial \rho}{\partial t} = \left[\frac{1}{m}\frac{\partial}{\partial X}\frac{\partial}{\partial \eta} + V(X - \eta/2) - V(X + \eta/2)\right]\rho(X, \eta)$$

Write $V(X) = \int V_k e^{ikX} dk$ so that



$$V(X + \eta/2) - V(X_\eta/2) = \int V_k e^{ikX}(e^{ik\eta/2} - e^{-ik\eta/2})dk$$

Use the Wigner-Moyal transformation $\rho(X, \eta, t) \to F(X, P, t)$

Finally

$$\frac{\partial F(X, P, t)}{\partial t} + \frac{P}{m}\frac{\partial F(X, P, t)}{\partial t} + i\int L(P, P')F(X, P')dP' = 0$$

Non-local transformation.

The Two Wigner-Moyal Equations.

Define two equations of motion

$$H * F = i(2\pi)^{-1} \int e^{-i\eta p} \psi^*(x - \eta/2, t) \partial_t \psi(x + \eta/2, t) d\eta$$

$$F * H = -i(2\pi)^{-1} \int e^{-i\eta p} \partial_t \psi^*(x - \eta/2, t) \psi(x + \eta/2, t) d\eta$$

[I have written for simplicity η for $\hbar\eta$]

Subtracting gives **Moyal bracket** equation

$$i\hbar \partial_t F = \{H, F\}_{MB}$$

Classical Louville equation to $O(\hbar)$

Adding gives **Baker bracket** equation

$$2\{H, F\}_{BB} = i(2\pi)^{-1} \int e^{-i\eta p} \left[\psi^*(x - \eta/2, t) \partial_t \psi(x + \eta/2, t) - \partial_t \psi^*(x - \eta/2, t) \psi(x + \eta/2, t) \right] d\eta$$

$$= i(2\pi)^{-1} \int e^{-\eta p} \psi^*(x - \eta/2, t) \overleftrightarrow{\partial}_t \psi(x + \eta/2) d\eta$$

Writing $\psi = Re^{iS/\hbar}$ we obtain

Classical Limit.

We find

$$\psi^\star \overleftrightarrow{\partial}_t \psi = \left[\frac{\partial_t R(x+\hbar\eta/2)}{R(x+\hbar\eta/2)} - \frac{\partial_t R(x-\hbar\eta/2)}{R(x-\hbar\eta/2)}\right] + i\left[\frac{\partial_t S(x+\hbar\eta/2)}{S(x+\hbar\eta/2)} - \frac{\partial_t S(x-\hbar\eta/2)}{S(x-\hbar\eta/2)}\right]\psi^\star\psi$$

Expanding in powers of $\hbar$

$$\{H,F\}_{BB} = -\frac{\partial S}{\partial t}F + O(\hbar^2)$$

which becomes $$\frac{\partial S}{\partial t} + H = 0$$ Classical Hamilton-Jacobi

Deformation again gives classical mechanics.

Two key equations

$$i\hbar\partial_t F = \{H,F\}_{MB}$$ **Quantum Liouville**

$$-\frac{\partial S}{\partial t} + O(\hbar^2) = \{H,F\}_{BB}$$ **Hamilton-Jacobi**

Summary so far

1. We have constructed a non-commuting algebra in the covering structure of classical phase space.

2. This reproduces all the standard results of quantum mechanics

3. We do not need operators in a Hilbert space.

4. This algebraic structure contains classical mechanics as a natural limit.

No **fundamental** role for decoherence

5. The structure is intrinsically non-local.

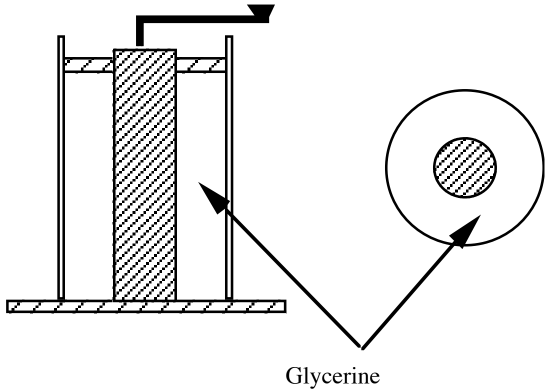
CM uses point to point transformations in phase space.

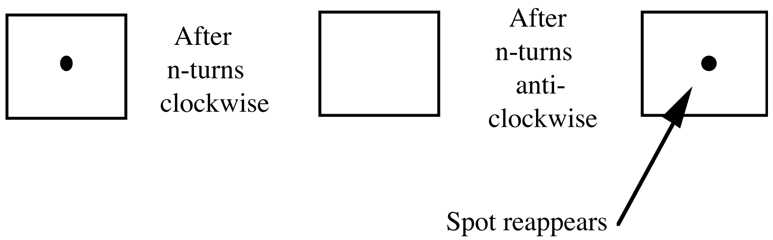
QM involve non-local transformations expressed through matrices

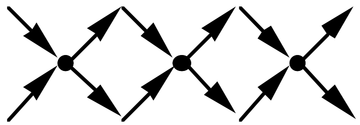
Basic unfolding and enfolding movements

A New Order:the Implicate Order.

Enfolding-unfolding movement







Approximates Bohm trajectories?

Continuity of form not substance.

There are two types of order:-

Implicate order.

Explicate order.

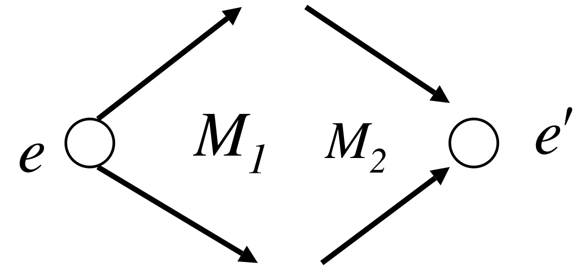
[Bohm, Wholeness and the Implicate Order, 1980]

Evolution of Process in the Implicate Order.

Continuity of form

$$eM_1 = M_2 e'$$

$$e' = M_2^{-1} e M_1$$



$e, e', M_1, M_2 \in \mathcal{A}$

Evolution is an algebraic automorphism.

Assume:-

$$M_1 = M_2 = M \qquad M = \exp[iH\tau]$$

τ is the UNFOLDING PARAMETER.

What about *h*?

Just a scaling parameter

For small τ

$$e' = (1 - iH\tau)e(1 + iH\tau)$$

$$i\frac{(e' - e)}{\tau} = He - eH$$

$$i\frac{\partial e}{\partial \tau} = [H, e]$$

Think of *e* as the density operator ρ. For pure states ρ is idempotent.

QUANTUM LIOUVILLE EQUATION OF MOTION.

Schrödinger Equation ?

If we write formally $e = \psi\phi$ and place in

$$i\frac{de}{d\tau} = [H, e]$$

We find

$$i\frac{d\psi}{d\tau}\phi + i\psi\frac{d\phi}{d\tau} = (H\psi)\phi - \psi(\phi H)$$

Now split into two equations

$$i\frac{d\psi}{d\tau} = H\psi$$ Schrödinger-like equation.

$$-i\frac{d\phi}{d\tau} = \phi H$$ Conjugate equation.

Since $e \in \mathcal{A}$, what are ψ and ϕ ?

[Baker, Phys. Rev. 6 (1958) 2198-2206.]

Minimal Ideals of the enfolding Algebra.

$$\rho = |\psi\rangle\langle\phi| \Rightarrow \psi\rangle\langle\phi \Rightarrow \psi\epsilon\phi = \Psi_L\Psi_R$$

NB we use Dirac's **standard** ket. $\psi\rangle \in \mathcal{A}$

Here ε is an idempotent $\varepsilon^2 = \varepsilon$

Symplectic spinors

$\Psi_L = \psi\epsilon \in \mathcal{A}$ Algebraic equivalent of a wave function — Left ideal

$\Psi_R = \epsilon\phi \in \mathcal{A}$ Algebraic equivalent of conjugate wave function. — Right ideal

Need two Schrödinger-like algebraic equations

$$i\frac{\partial\Psi_L}{\partial t} = \overrightarrow{H}\Psi_L \qquad -i\frac{\partial\Psi_R}{\partial t} = \Psi_R\overleftarrow{H}$$

The Two More Algebraic Equations.

Sum the two algebraic Schrödinger equations

$$i\left[(\overrightarrow{\partial}_t\Psi_L)\Psi_R+\Psi_L(\Psi_R\overleftarrow{\partial}_t)\right]=\left(\overrightarrow{H}\Psi_L\right)\Psi_R-\Psi_L\left(\Psi\overleftarrow{H}\right)$$

Write $\rho=\Psi_L\Psi_R$ so that

$$i\frac{\partial\rho}{\partial t}=[H,\rho]_-$$

Liouville equation

Conservation of Probability

Subtract the two algebraic Schrödinger equations

$$i\left[(\overrightarrow{\partial}_t\Psi_L)\Psi_R-\Psi_L(\Psi_R\overleftarrow{\partial}_t)\right]=\left(\overrightarrow{H}\Psi_L\right)\Psi_R+\Psi_L\left(\Psi\overleftarrow{H}\right)$$

Polar decompose $\Psi_L=Re^{iS}\epsilon$ and $\Psi_R=\epsilon Re^{-iS}$

$$\rho\frac{\partial S}{\partial t}+\tfrac{1}{2}[H,\rho]_+=0$$

New equation

Conservation of Energy.

Moyal and Quantum Algebraic Equations.

$$\frac{\partial F}{\partial t} + [F, H]_{MB} = 0$$

$$2\frac{\partial S}{\partial t}F + [F, H]_{BB} = 0$$

Moyal algebra

$$i\frac{\partial \rho}{\partial t} + [\rho, H]_{-} = 0$$

$$2\frac{\partial S}{\partial t}\rho + [\rho, H]_{+} = 0$$

Quantum algebra

Where is the quantum potential?

Project Quantum Algebraic Equations into a Representation.

Project into representation using $P_a = |a\rangle\langle a|$

$$i\frac{\partial P(a)}{\partial t} + \langle[\rho, H]_-\rangle_a = 0$$

$$2P(a)\frac{\partial S}{\partial t} + \langle[\rho, H]_+\rangle_a = 0$$

Still no quantum potential

Choose $P_x = |x\rangle\langle x|$

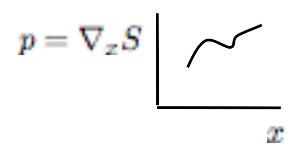
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K\hat{x}^2}{2}$$

$$\frac{\partial P}{\partial t} + \nabla.\left(P\frac{\nabla S_x}{m}\right) = 0$$

Conservation of probability

Quantum H-J equation.

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x}\left(\frac{\partial^2 R_x}{\partial x^2}\right) = 0$$

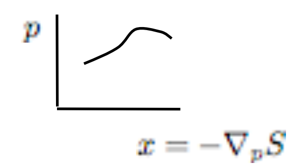


quantum potential

Choose $P_p = |p\rangle\langle p|$

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}x_r^2 - \frac{K}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0$$

$$x_r = -\nabla_p S_p$$



[M. R. Brown & B. J. Hiley, quant-ph/0005026]

The Overarching Structure

Non-commutative
Algebraic structure.

Process, activity

Holomovement

Implicate order.

Shadow
phase space
$(X^{\psi(x)}, P^{\psi(x)})$

Shadow
phase space
$(X^{\phi(p)}, P^{\phi(p)})$

Shadow
phase space
$(X^{\eta(x')}, P^{\eta(x')})$

Possible explicate orders.

[D. Bohm Wholeness and the Implicate Order (1980)]

Four Roads to Quantum Mechanics.

Standard.

Operators in Hilbert space.

Generalized phase space.

Uses ordinary functions in phase space with a non-commutative product.

Moyal star product. Deformed Poisson algebra.

Advantage: Nice classical limit.

Algebraic approach.

Everything is done in the algebra.

Wave functions replaced by elements in the algebra.

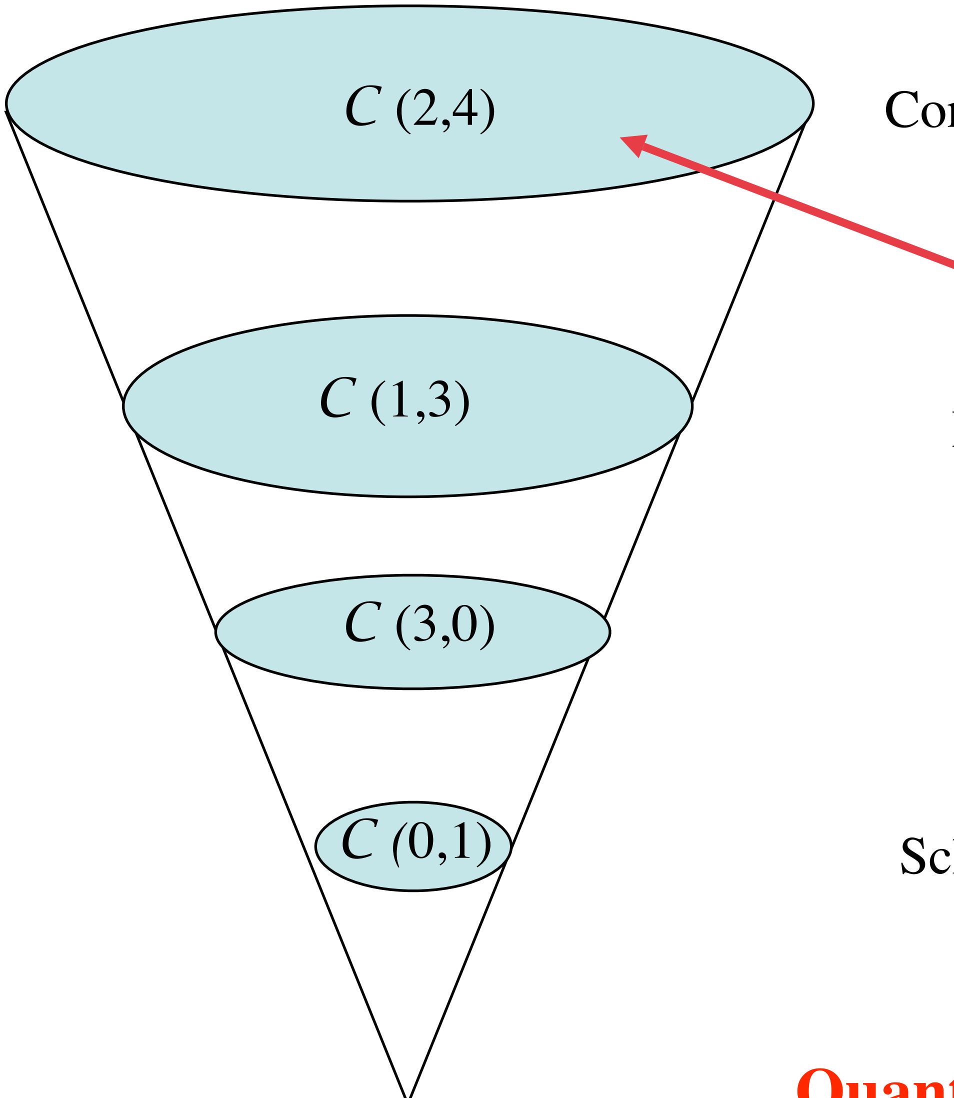
Advantage: Uses Clifford algebra therefore includes Pauli and Dirac.

Can also Schrödinger exploiting $\mathbb{C} \approxeq C_{0,1}$

de Broglie-Bohm.

Contained in all of the above three.

Hierarchy of Clifford Algebras



	Generating elements.
Conformal	$\{1, e_0, e_1, e_2, e_3, e_4, e_5\}$
Twistors	$\{\omega, \pi\}$
Dirac	$\{1, e_0, e_1, e_2, e_3\}$
	$\{\mathbf{1}, \gamma_0, \gamma_1, \gamma_2, \gamma_3\}$
Pauli	$\{1, e_1, e_2, e_3\}$
	$\{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3\}$
Schrödinger	$\{1, e_1\}$
	$\{\mathbf{1}, i\}$

Quantum?

How does it work?

How do we specify the state of the system?

$$\hat{\rho}(x,t) = \Phi_L(x,t)\Phi_R(x,t) = \phi_L(x,t)\epsilon\phi(x,t) = \phi_L(x,t)\epsilon\widetilde{\phi}(x,t)$$

Clifford density element

How do we choose the idempotent?

Decided by the physics.

For Dirac	$\epsilon = (1+\gamma_0)/2$	Picks a time frame
For Pauli	$\epsilon = (1+\sigma_3)/2$	Picks direction of space
For Schrödinger	$\epsilon = 1$	

NB we use Clifford algebras over the reals!

Physical Content of Schrödinger.

$$\phi_L = g_0 + eg_1 \quad \text{and} \quad \phi_R = \widetilde{\phi}_L = g_0 - eg_1 \qquad e \in C_{0,1}$$

$$\rho = \Phi_L \widetilde{\Phi}_L = \phi_L \widetilde{\phi}_L = g_0^2 + g_1^2.$$

Relation to wave function: Cliff $\rightarrow$ Hilbert space.

$$\phi_L \Rightarrow \psi_i$$

Then

$$g_0 = (\psi^* + \psi)/2 \qquad g_1 = i(\psi^* - \psi)/2$$

If we write $\psi = Re^{iS}$ then

$$g_0 = R\cos(S) \qquad g_1 = R\sin(S)$$

Then

$$\rho = g_0^2 + g_1^2 = R^2.$$

satisfies

$$i\frac{\partial \rho}{\partial t} + [\rho, H]_- = 0$$

MISSING information about the phase!

Pauli Particle continued.

$$\phi_L = g_0 + g_1 e_{23} + g_2 e_{13} + g_3 e_{12} \qquad e_{23}, e_{13}, e_{12} \in C_{3,0}$$

$$\phi_L = RU$$

$$\hat{\rho} = \Phi_L \widetilde{\bar{\Phi}}_L = \phi_L \epsilon \widetilde{\phi}_L = R^2 U \epsilon \widetilde{U} = R^2(1 + U\sigma_3\widetilde{U})/2$$

probability spin

$$\rho \boldsymbol{s} = \phi_L \sigma_3 \widetilde{\phi}_L/2$$

$$\hat{\rho} = R^2(1 + \boldsymbol{s} \cdot \boldsymbol{\sigma})/2 \qquad R^2 = \rho$$

It looks as if we have 4 real parameters to specify the state, $\{\rho, s_1, s_2, s_3\}$

But $\quad s^2 = 1/4$

Something missing again!

Dirac Particle.

$$\phi_L = a + b\gamma_{12} + c\gamma_{23} + d\gamma_{13} + f\gamma_{01} + g\gamma_{02} + h\gamma_{03} + n\gamma_5. \qquad \gamma \in C_{1,3}$$

$$\hat{\rho} = \Phi_L \widetilde{\Phi}_L = \phi_L \epsilon \widetilde{\phi}_L = \phi_L (1 + \gamma_0) \widetilde{\phi}_L / 2$$

This will give 8-real dimension spinor. We need 4 complex spinor.

We need a different but related idempotent.

$$\hat{\rho} = \phi_L (1 + \gamma_0 + i\gamma_{12} + i\gamma_{012}) \widetilde{\phi}_L / 2$$

probability

velocity

spin

axial vector

Proca current.

Bi-linear invariants.

Only 7 independent. Need 8 $\therefore$ **Still one missing!**

NB we describe physical processes by physical properties.

[Takabayasi, Prog. Theor. Phys., Supplement No.4 (1957) pp. 2-80]

Dirac Current.

$$\mathbf{J} = \phi_L \gamma_0 \widetilde{\phi}_L$$

With $\phi_L = a + b\gamma_{12} + c\gamma_{23} + d\gamma_{13} + f\gamma_{01} + g\gamma_{02} + h\gamma_{03} + n\gamma_5.$

To show it is the usual current we need Cliff → Hilbert space.

$$\phi_L \Rightarrow \psi_i$$

$$\psi_1 = a - ib; \quad \psi_2 = -d - ic; \quad \psi_3 = h - in \quad \psi_4 = f + ig$$

After some work

$$J^0 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

$$J^1 = \psi_1\psi_4^* + \psi_2\psi_3^* + \psi_3\psi_2^* + \psi_4\psi_1^*$$

$$J^2 = i[\psi_1\psi_4^* - \psi_2\psi_3^* + \psi_3\psi_2^* - \psi_4\psi_1^*]$$

$$J^3 = \psi_1\psi_3^* - \psi_2\psi_4^* + \psi_3\psi_1^* - \psi_4\psi_2^*$$

Dirac current in the standard representation.

What is Missing?

Phase information? Energy-momentum?

In conventional terms $2iT^{\mu\nu} = \bar{\psi}\gamma^\mu(\partial^\nu\psi) - (\partial^\nu\bar{\psi})\gamma^\mu\psi = \bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi$

In Clifford terms $2iT^{\mu\nu} = tr[\gamma^\mu\phi_L\gamma_{012}\overleftrightarrow{\partial}^\nu\widetilde{\phi}_L]$

Only non-vanishing term in trace is when $\phi_L\gamma_{012}\overleftrightarrow{\partial}^\nu\widetilde{\phi}_L$ is a vector

After some work we find

$$\phi_L\gamma_{012}\overleftrightarrow{\partial}^\nu\widetilde{\phi}_L = A^\nu_\sigma(x^\mu)\gamma_\sigma$$

where

$$A^\nu_0 = -(a\overleftrightarrow{\partial}^\nu b + c\overleftrightarrow{\partial}^\nu d + f\overleftrightarrow{\partial}^\nu g + h\overleftrightarrow{\partial}^\nu n)$$

$$A^\nu_1 = -(a\overleftrightarrow{\partial}^\nu g + b\overleftrightarrow{\partial}^\nu f + c\overleftrightarrow{\partial}^\nu h + d\overleftrightarrow{\partial}^\nu n)$$

$$A^\nu_2 = \ \ (a\overleftrightarrow{\partial}^\nu f - b\overleftrightarrow{\partial}^\nu g - c\overleftrightarrow{\partial}^\nu n + d\overleftrightarrow{\partial}^\nu h)$$

$$A^\nu_3 = \ \ (a\overleftrightarrow{\partial}^\nu n - b\overleftrightarrow{\partial}^\nu h + c\overleftrightarrow{\partial}^\nu f - d\overleftrightarrow{\partial}^\nu g)$$

Bohm Energy-Momentum Density Dirac.

Using $\psi_1 = a - ib; \quad \psi_2 = -d - ic; \quad \psi_3 = h - in \quad \psi_4 = f + ig$

$$T^{00} = i\sum_{j=1}^{4}(\psi_j^*\partial^0\psi_j - \psi_j\partial^0\psi_j^*) = -\sum R_j^2\partial^0 S_j$$

$$\psi_i = R_i e^{iS_i}$$

This is just the Bohm energy density, ρE_B

$$T^{0k} = -i\sum_{j=1}^{4}(\psi_j^*\partial^k\psi_j - \psi_j\partial^k\psi_j^*) = \sum R_j^2\nabla S_j$$

This is just the Bohm momentum density, ρP_B^k

Why do we call these Bohm energy-momentum?

Bohm Energy-Momentum for Pauli and Schrödinger

Pauli.

$$2\rho P^{\mu} = -i(\phi_L \sigma_3 \overleftrightarrow{\partial}^{\mu} \widetilde{\phi}_L) = 2\rho D^{\mu} \sigma_{123}$$

Where $D^{\mu} = -(\partial^{\mu} g_0) g_2 + (\partial^{\mu} g_1) g_2 - (\partial^{\mu} g_2) g_1 + (\partial^{\mu} g_3) g_0$

$$E_B = -\sum_{i=1}^{2} R_j^2 \partial_t S_j \qquad P_B = \sum_{i=1}^{2} R_j^2 \nabla S_j$$

Schrödinger

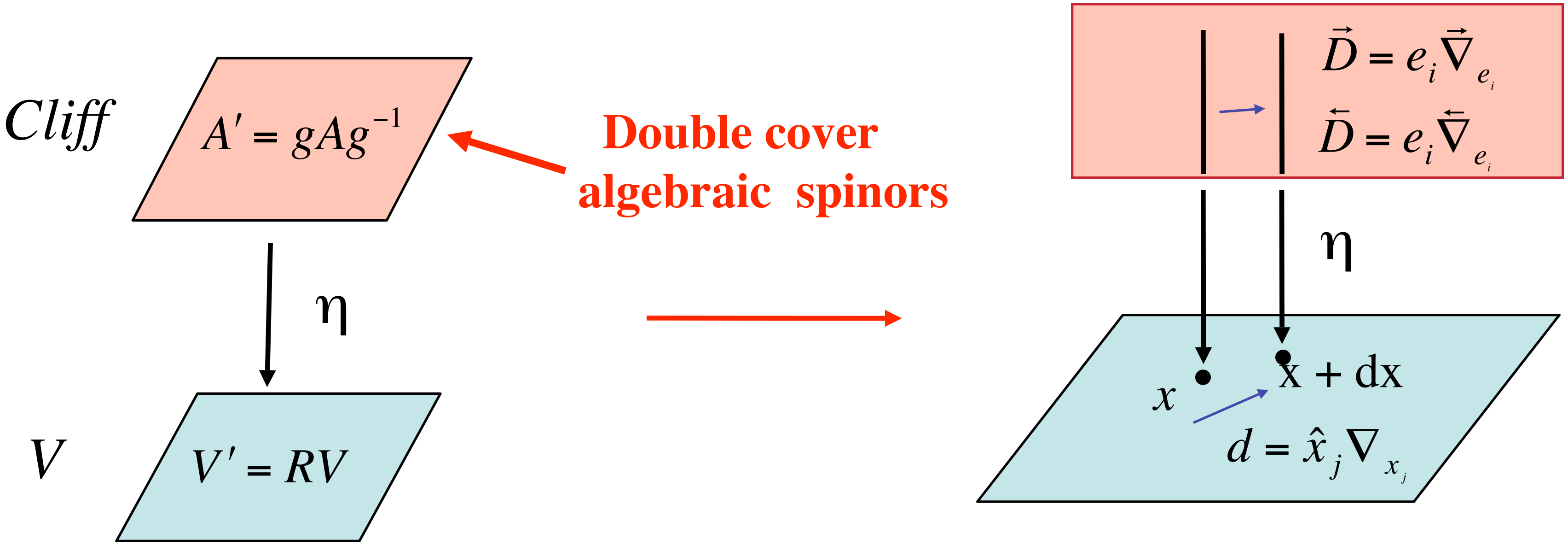
$$\rho E_B = -e(\phi_L \overleftrightarrow{\partial}_t \widetilde{\phi}_L) = -R^2 \partial_t S \qquad E_B = -\partial_t S$$

$$\rho P_B = -e(\phi_L \overleftrightarrow{\nabla} \widetilde{\phi}_L) = R^2 \nabla S \qquad P_B = \nabla S$$

[Bohm and Hiley The Undvided Universe, 1993]

Translations and Time Derivatives.

Construct a Clifford bundle.



Spin bundle with connection.

N.B. We need TWO derivatives in the bundle space $\overrightarrow{D}$ and $\overleftarrow{D}$.

$$\overrightarrow{D} = e_\mu \overrightarrow{\partial}_\mu \qquad \overleftarrow{D} = \overleftarrow{\partial}_\mu e_\mu$$

Therefore we need to use two time development equations.

Time Evolutions: Differences and Sums.

Two equations for time evolution

$$i\partial_t \Phi_L = \overrightarrow{H}\Phi_L \quad \text{and} \quad -i\partial_t \Phi_R = \Phi_R \overleftarrow{H}$$

$$\overrightarrow{H} = H(\overrightarrow{D}, V, m) \qquad \overleftarrow{H} = H(\overleftarrow{D}, V, m)$$

Difference:-

$$i[(\partial_t \Phi_L)\widetilde{\Phi}_L + \Phi_L(\partial_t \widetilde{\Phi}_L)] = (\overrightarrow{H}\Phi_L)\widetilde{\Phi}_L - \Phi_L(\widetilde{\Phi}_L \overleftarrow{H})$$

We can rewrite this as $i\partial_t \hat{\rho} = [H, \hat{\rho}]_-$ **Liouville equation.**

Conservation of probability

Sum:-

$$i[(\partial_t \Phi_L)\widetilde{\Phi}_L - \Phi_L(\partial_t \widetilde{\Phi}_L)] = (\overrightarrow{H}\Phi_L)\widetilde{\Phi}_L + \Phi_L(\widetilde{\Phi}_L \overleftarrow{H})$$

Conservation of energy

Schrödinger Quantum Hamilton-Jacobi Equation

The LHS is

$$i[(\partial_t\Phi_L)\widetilde{\Phi}_L - \Phi_L(\partial_t\widetilde{\Phi}_L)] = i\phi_L \overleftrightarrow{\partial}_t \widetilde{\phi}_L = 2\rho E_B = -2\rho\partial_t S$$

$$2\rho\partial_t S = (\overrightarrow{H}\Phi_L)\widetilde{\Phi}_L + \Phi_L(\widetilde{\Phi}_L\overleftarrow{H})$$

Quantum Hamilton-Jacobi

Since we have written $\Phi_L = R\exp(eS)$ with $\varepsilon = 1$,

using $H = P^2/2m + V(x)$

$$\partial_t S + (\nabla S)^2/2m - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0$$

Conservation of energy.

Quantum Potential

Back to the Two Key Equations.

$$i\partial_t(\Phi_L\widetilde{\Phi}_L) = (\overrightarrow{H}\Phi_L)\widetilde{\Phi}_L - \Phi_L(\widetilde{\Phi}_L\overleftarrow{H})$$ **Quantum Liouville**

$$i\Phi_L\overleftrightarrow{\partial}_t\widetilde{\Phi}_L = (\overrightarrow{H}\Phi_L)\widetilde{\Phi}_L + \Phi_L(\widetilde{\Phi}_L\overleftarrow{H})$$ **Quantum H-J**

Shortened forms.

$$i\partial_t\hat{\rho} = [H,\hat{\rho}]_- \qquad i\Phi_L\overleftrightarrow{\partial}_t\widetilde{\Phi}_L = [H,\hat{\rho}]_+$$

Conservation equations

Probability

Spin

Energy

Always produces a quantum potential

The Pauli Quantum Liouville Equation.

$$i\partial_t \hat{\rho} = [H, \hat{\rho}]_-$$

LHS becomes

$$i\partial_t \hat{\rho} = i\partial_t[\phi_L \epsilon \widetilde{\phi}_L] = i\partial_t[\rho + \phi_L \sigma_3 \widetilde{\phi}_L] = i\partial_t \rho + 2\partial_t(\rho S)$$

PseudoSalar **Bivector**

$$2\rho S = i\phi_L \sigma_3 \widetilde{\phi}_L$$

Look at Pseudoscalar part.

$$[H, \hat{\rho}]_{-\text{pseudo}} = (H\phi_L)\sigma_3 \widetilde{\phi}_L - \phi_L \sigma_3 (H\widetilde{\phi}_L)$$

$$2m[H, \hat{\rho}]_{-\text{pseudo}} = 2i\rho[4S \cdot (P \cdot W) - \nabla P] = -2i\rho[(\nabla \ln \rho)P + \nabla P] = -2i\nabla.(\rho P)$$

$$\boxed{\partial_t \rho + \nabla.(\rho P / m) = 0}$$

Conservation of probability equation

The Bivector part of the QLE.

$$[H, \hat{\rho}]_{-\text{bivector}} = (\overrightarrow{H}\phi_L)\widetilde{\phi}_L - \phi_L(\widetilde{\phi}_L\overleftarrow{H})$$

Then

$$m\partial_t(\rho \boldsymbol{S}) = -[\nabla P \cdot \boldsymbol{S} + \boldsymbol{S} \wedge \nabla W + P \cdot W]$$

Again after some tedious work we find

$$\left(\partial_t + \frac{P.\nabla}{m}\right)\boldsymbol{S} = \frac{1}{m}\left[\nabla^2 \boldsymbol{S} + (\nabla \ln \rho)\nabla \boldsymbol{S}\right] \wedge \boldsymbol{S}$$

Remembering $\boldsymbol{S} = i\boldsymbol{s}$ and $A \wedge B = i(A \times B)$

$$\frac{d\boldsymbol{s}}{dt} = \left(\partial_t + \frac{P.\nabla}{m}\right)\boldsymbol{S} = \frac{1}{m}\boldsymbol{s} \times \nabla(\rho \nabla \boldsymbol{s})$$

Equation for spin time evolution.

The Quantum Torque

[Dewdney et al Nature 336 (1988) 536-44]

Spin trajectories and orientations.

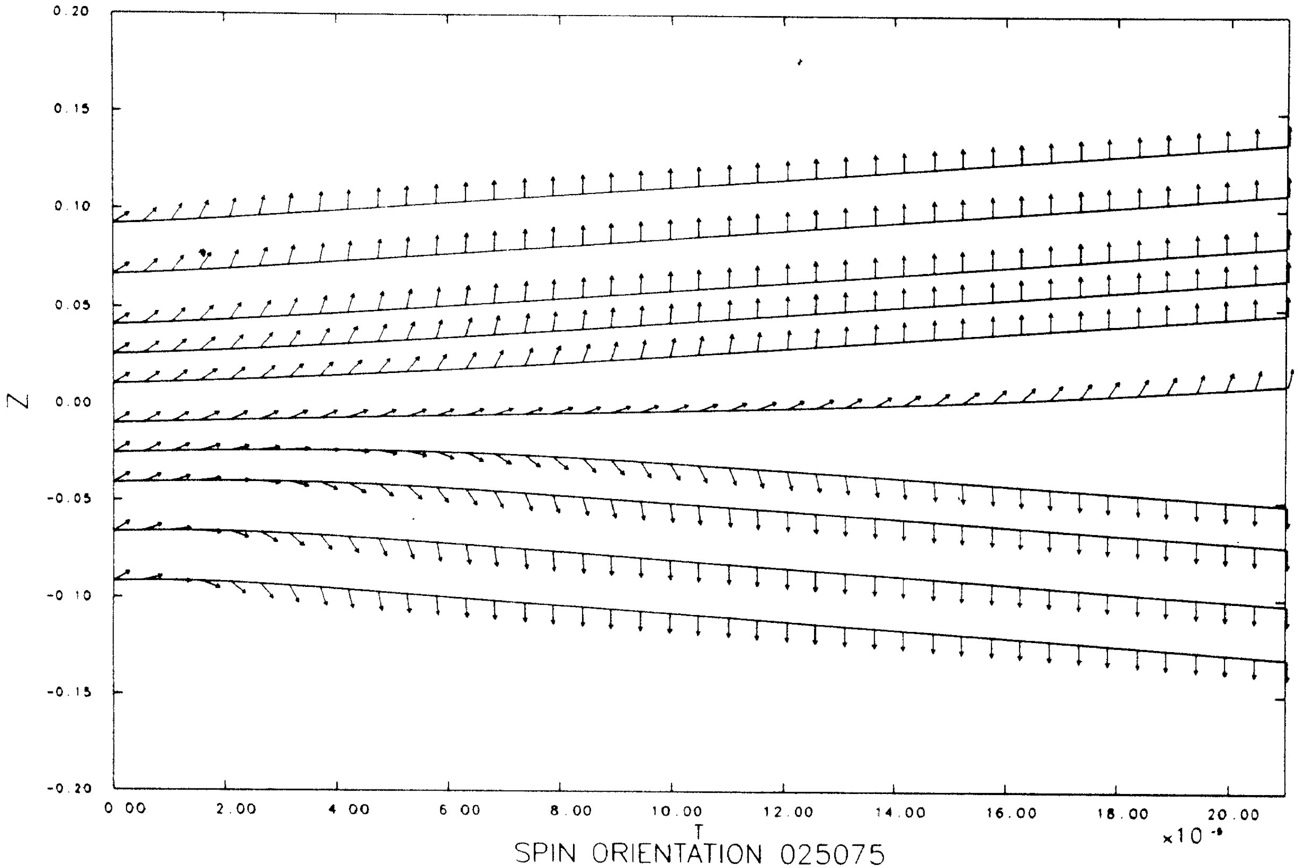


Fig. 4. Trajectories and orientations θ associated with figs. 1 and 2.

The Quantum Hamilton-Jacobi Equation.

$$i\Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L = [H, \hat{\rho}]_+$$

Working the LHS.

$$i\Phi_L \overleftrightarrow{\partial}_t \widetilde{\Phi}_L = i[(\partial_t \phi_L)\epsilon\widetilde{\phi}_L - \phi_L \epsilon(\partial_t \widetilde{\phi}_L)]$$

Writing

$$i\epsilon = \sigma_{123}(1+\sigma_3)/2 = (\sigma_{12}+\sigma_{123})/2$$

$$\rho\Omega_t \cdot S + i\rho\Omega_t = [H, \hat{\rho}]_+$$

$$\Omega_t = 2(\partial_t U)\widetilde{U}$$

$$\phi_L = RU$$

S B

$$\Omega_t \cdot S + 2i\Omega_t \Rightarrow \frac{\partial S_{phase}}{\partial t}$$

The scalar part using Euler angles gives the same as BST, namely

$$E(t) = \Omega_t \cdot S = \partial_t \psi + \cos\theta(\partial_t \phi)$$

[Bohm, Schiller, Tiomno, Nuovo Cim., 1, (1955) 48-66]

The Quantum Hamilton-Jacobi Equation.

Working the RHS of $\rho\Omega_t \cdot \boldsymbol{S} + i\rho\Omega_t = [H,\hat{\rho}]_+$

Scalar part of $[H,\tilde{\rho}]_+$ is $(\overrightarrow{H}\Phi_L)\widetilde{\Phi}_L + \Phi_L(\widetilde{\Phi}_L\overleftarrow{H})$

Bivector part of $[H,\tilde{\rho}]_+$ is $(\overrightarrow{H}\Phi_L)\widetilde{\sigma}_3\Phi_L + \Phi_L\sigma_3(\widetilde{\Phi}_L\overleftarrow{H})$

After tedious but straight forward working

$$2m[H,\hat{\rho}]_{+\text{scalar}} = 2\rho\left[2(\boldsymbol{S}\cdot\nabla W) + P^2 + W^2\right]$$

where $W = \rho^{-1}\nabla(\rho\boldsymbol{S})$

This becomes

$$\Omega_t \cdot \boldsymbol{S} = \frac{P^2}{2m} + \underbrace{\frac{1}{2m}\left[2(\nabla W\cdot\boldsymbol{S}) + W^2\right]}_{\text{Quantum Potential}}$$

Quantum Hamilton-Jacobi

The Quantum Hamilton-Jacobi Equation.

$$2mQ = \left[2(\nabla W \cdot \boldsymbol{S}) + W^2\right] = \left[\boldsymbol{S}^2\left(2\nabla \ln \rho + (\nabla \ln \rho)^2\right)\right] + \boldsymbol{S} \cdot \nabla^2 \boldsymbol{S}$$

$$-\frac{\nabla^2 R}{R}$$

Again using Euler angles

$$\tfrac{1}{4}\left[(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2\right]$$

Putting this all together we get the QHL equation

$$\tfrac{1}{2}\left[\partial_t \psi + \cos\theta(\partial_t \phi)\right] + \frac{P^2}{2m} + Q = 0$$

Quantum HJ equation.

where

$$Q = -\frac{\nabla^2 R}{2mR} + \frac{1}{8m}\left[(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2\right]$$

Quantum Potential

This is exactly the equation obtained in the BST theory.

[Dewdney et al Nature 336 (1988) 536-44]

Dirac Energy-Momentum Conservation Equation.

Slight difference.

$$(\partial_\mu \partial^\mu \Phi_L)\widetilde{\Phi}_L + \Phi_L(\partial_\mu \partial^\mu \widetilde{\Phi}_L) + 2m^2 \Phi_L \widetilde{\Phi}_L = 0$$ Energy-momentum

$$\Phi_L(\partial_\mu \partial^\mu \widetilde{\Phi}_L) - (\partial_\mu \partial^\mu \Phi_L)\widetilde{\Phi}_L = 0$$ Spin torque

In order to proceed we need to start with

$$2\rho P^\mu = [(\partial^\mu \phi_L)\gamma_{012}\widetilde{\phi}_L - \phi_L \gamma_{012}(\partial^\mu \widetilde{\phi}_L)]$$

and use

$$2\rho J = \phi_L \gamma_{012}\widetilde{\phi}_L$$ and $$2\rho W^\mu = -\partial^\mu(\phi_L \gamma_{012}\widetilde{\phi}_L)$$

we find

$$P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] + [J\partial_\mu P^\mu - \partial_\mu P^\mu J] - m^2 = 0$$

Separate Clifford scalar and pseudo-scalar parts, we find

$$P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0$$

c.f.

$$p_\mu p^\mu - m^2 = 0$$

Dirac Continued.

We have

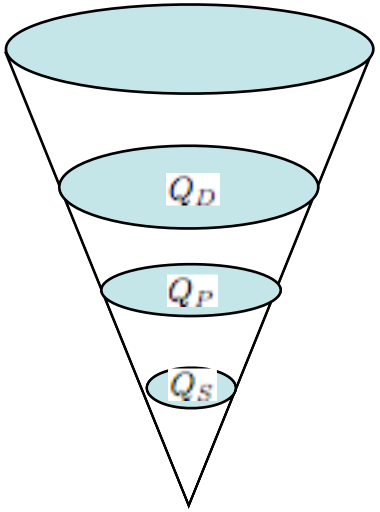
$$P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0$$

but

$$4\rho^2 P^2 = 4\rho^2 P_B^2 + \sum_{i=1}^{3} A_{i\nu} A_i^\nu = 4\rho^2 P_B^2 + 4\rho^2 \Pi^2$$

Thus

$$P_B^2 + \Pi^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0$$



Compare with

$$p_\mu p^\mu - m^2 = 0$$

Find the quantum potential is

$$Q_D = \Pi^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J]$$

Compare with quantum potential of Pauli

$$Q_P = W_P^2 + [S(\nabla W_P) + (\nabla W_P)S]$$

$$[2\rho S = \phi_L \sigma_{12} \widetilde{\phi}_L]$$

Quantum potential of Schrödinger

$$Q_S = -\frac{1}{2m}\frac{\nabla^2 R}{R}$$

Dirac Spin Torque.

Go back to $$\Phi_L(\partial_\mu\partial^\mu\widetilde{\Phi}_L) - (\partial_\mu\partial^\mu\Phi_L)\widetilde{\Phi}_L = 0$$

and get $$J\cdot\partial_\mu P^\mu - P\cdot W + J\wedge\partial_\mu W^\mu = 0$$

with

$$\left.\begin{aligned} 2J\cdot\partial_\mu P^\mu &= J\partial_\mu P^\mu + \partial_\mu P^\mu J \\ 2P\cdot W &= PW + WP \\ 2J\wedge\partial_\mu W^\mu &= J\partial_\mu W^\mu - \partial_\mu W^\mu J \end{aligned}\right\}$$

All Clifford bivectors

since $$\rho(P\cdot W) = -(\partial^\mu\rho)(P_\mu\cdot J) - \rho(P_\mu\cdot\partial^\mu J)$$

we find $$\partial_\mu(\rho P^\mu)\cdot J + \rho(P_\mu\cdot\partial^\mu J) + \rho(J\wedge\partial_\mu W^\mu) = 0$$

Since $$2\partial_\mu(\rho P^\mu) = \partial_\mu(T^{\mu 0}) = 0$$

$$\boxed{P_\mu\cdot\partial^\mu J + J\wedge\partial_\mu W^\mu = 0}$$

Quantum torque equation for Pauli is

$$\left(\partial_t + \frac{P\cdot\nabla}{m}\right)S = \frac{2}{m}(\nabla W\wedge S)$$

Conclusions.

1. Do quantum mechanics entirely within the Clifford algebra.

   No need for wave functions! **[von Neumann algebra]**

2. All terms used are bilinear invariants, i.e. observable quantities.

   No wave functions

3. Use local energy-momentum density $T^{\mu 0}(x^\mu)$

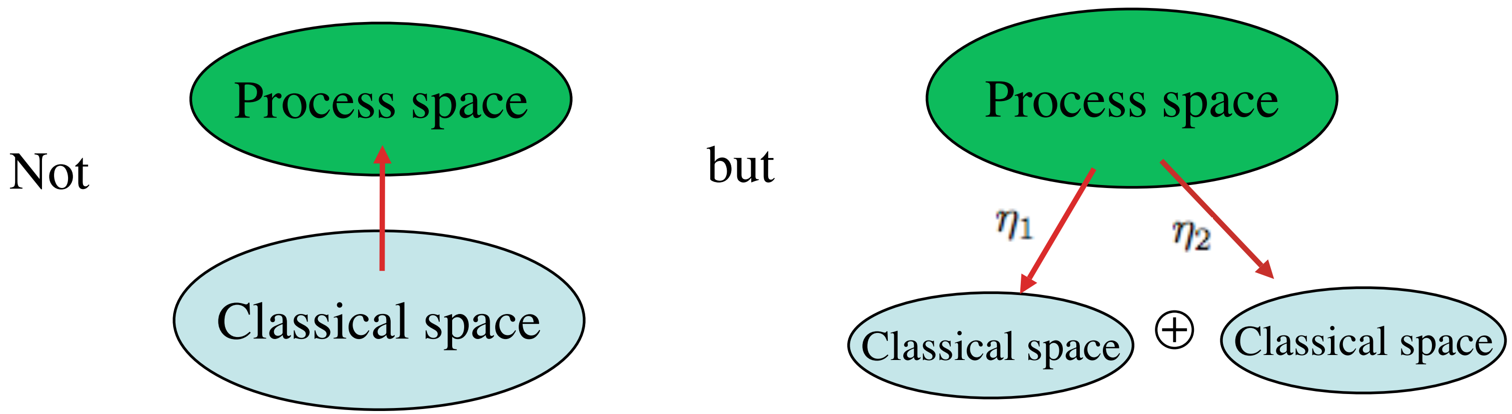
4. The Bohm model follows immediately.

$$2\rho P_B^\mu(x^\mu) = T^{\mu 0}(x^\mu)$$

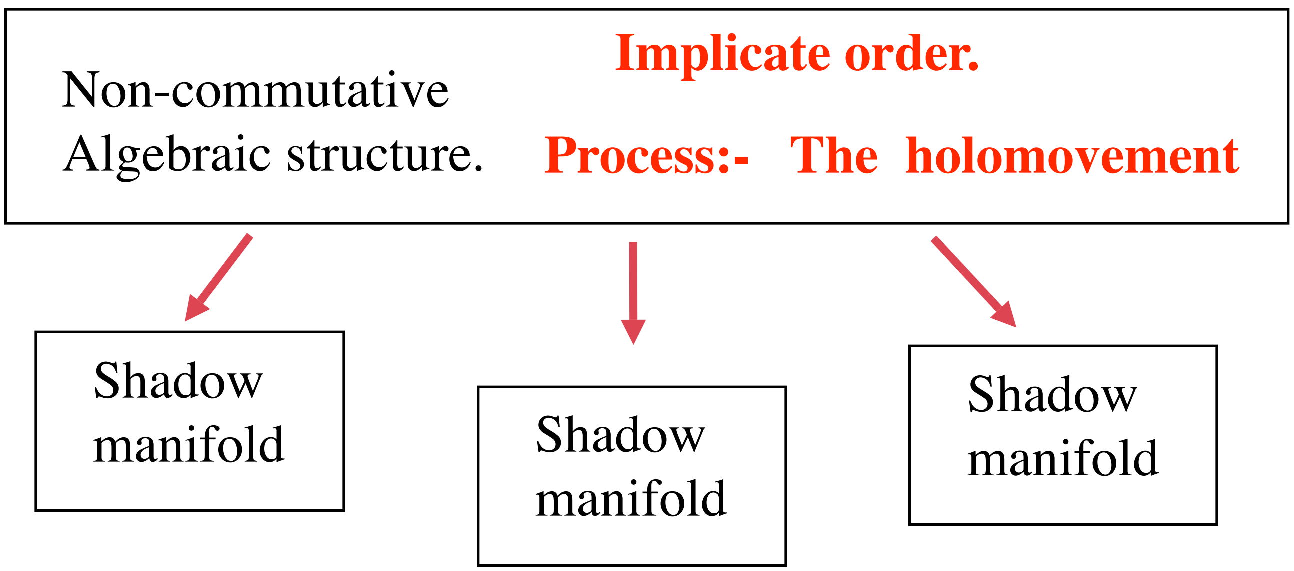
No appeal to classical mechanics at all.

Yet the Clifford is about classical space-time

What does it all mean Physically?



[Hiley, Lecture Notes in Physics, vol 813, ed B. Coecke, 2010.]



Possible explicate orders.

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