# MSC Lecture: The geometry and topology of Coxeter groups

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**Abstract**

These notes are intended as an introduction to the theory of Coxeter groups. They closely follow my talk in the Lectures on Modern Mathematics Series at the Mathematical Sciences Center in Tsinghua University on May 10, 2013. They were prepared from the beamer presentation which I used during my talk.

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## 1 Geometric reflection groups

By a *geometric reflection group* we mean a discrete group generated by reflections across the faces of a convex polytope in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ (the $n$-sphere, Euclidean $n$-space or hyperbolic $n$-space, respectively). The simplest reflection group is the cyclic group of order two acting as a reflection on the real line. The next simplest are the dihedral groups.

**Dihedral groups** A *dihedral group* is any group which is generated by two involutions, call them $s$ and $t$. Such a group is determined up to isomorphism

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by the order $m$ of $st$ ($m$ is an integer $\geq 2$ or $\infty$). Let $\mathbf{D}_m$ denote the dihedral group corresponding to $m$.

For $m \neq \infty$, $\mathbf{D}_m$ can be represented as the subgroup of $O(2)$ which is generated by reflections across lines $L$ and $L'$, making an angle of $\pi/m$. (See Figure 1.)

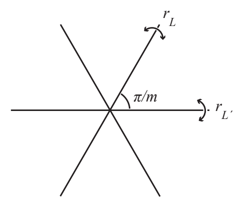


Figure 1: A finite dihedral group

## 1.1 Some history

In 1852 Möbius determined the finite subgroups of $O(3)$ that are generated by isometric reflections on the 2-sphere. The fundamental domain for such a group on the 2-sphere is a spherical triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, with $p$, $q$, $r$ integers $\geq 2$. Since the sum of the angles is $> \pi$, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. For $p \geq q \geq r$, the only possibilities are: $(p, 2, 2)$ for any $p \geq 2$ and $(p, 3, 2)$ with $p = 3$, 4 or 5. The last three cases are the symmetry groups of the Platonic solids. (See Figure 2.)

Later work by Riemann and Schwarz showed there were discrete groups of isometries of $\mathbb{E}^2$ or $\mathbb{H}^2$ generated by reflections across the edges of triangles with angles integral submultiples of $\pi$. (See Figure 3 for a hyperbolic triangle group.) Poincaré and Klein showed there were similar groups for polygons in $\mathbb{H}^2$. (See Figure 4 for a right-angled pentagon group.)

In the second half of the nineteenth century work began on finite reflection groups on $\mathbb{S}^n$, $n > 2$, generalizing Möbius' results for $n = 2$. It developed along two lines. First, around 1850, Schläfli classified regular polytopes in $\mathbb{R}^{n+1}$, $n > 2$. The symmetry group of such a polytope was a finite group generated by reflections and, as in Möbius' case, the projection of a fundamental domain to $\mathbb{S}^n$ was a spherical simplex with dihedral angles integral submultiples of $\pi$. Second, around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925,

Figure 2: $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ triangle groups

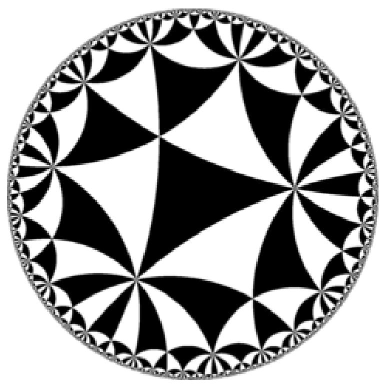

Figure 3: $(3,6,6)$ triangle group

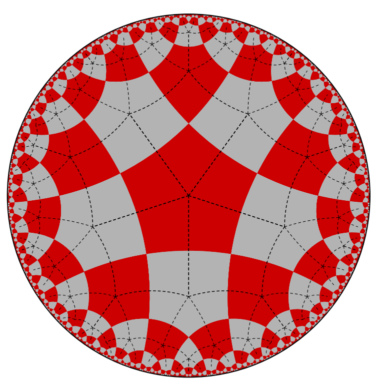

Figure 4: Right-angled pentagon group

Weyl showed the symmetry group of such a root system was a finite reflection group. These two lines were united by Coxeter [7] in the 1930's. He classified discrete reflection groups on $\mathbb{S}^n$ or $\mathbb{E}^n$.[1] (See Figure 5.)

## 1.2 Properties

Let $K$ be a fundamental polytope for a geometric reflection group. For $\mathbb{S}^n$, $K$ is a simplex (i.e., the generalization of a triangle to a higher dimension). For $\mathbb{E}^n$, $K$ is a product of simplices, for example, $K$ could be a product of intervals. For $\mathbb{H}^n$ there are other possibilities, e.g., a right-angled pentagon in $\mathbb{H}^2$ (see Figure 4) or a right-angled dodecahedron in $\mathbb{H}^3$. For a survey of the known examples of geometric reflection groups on $\mathbb{H}^n$, see Vinberg's article [18].

Conversely, given a convex polytope $K$ in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ so that all dihedral angles have form $\pi/\text{integer}$, there is a discrete group $W$ generated by isometric reflections across the codimension one faces of $K$ (cf. [8, Thm. 6.4.3]).

Let $S$ be the set of reflections across the codimension one faces of $K$. Then $S$ generates $W$. For $s, t \in S$, let $m(s,t)$ be the order of $st$. The faces corresponding to $s$ and $t$ intersect in a codimension two face if and only if $m(s,t) \neq \infty$, and for $s \neq t$, the dihedral angle along that face is $\pi/m(s,t)$. (The $S \times S$ symmetric matrix $(m(s,t))$ is the *Coxeter matrix.*) Moreover, $W$ has a presentation of the form

$$\langle S \mid (st)^{m(s,t)}, \quad \text{where } (s,t) \in S \times S \rangle \tag{1.1}$$

**Coxeter diagrams** Associated to $(W,S)$, there is a labeled graph $\Gamma$ called its *Coxeter diagram.* The vertex set of $\Gamma$ is $S$. Connect distinct vertices $s$ and $t$ by an edge if and only if $m(s,t) \neq 2$. Label the edge by $m(s,t)$ if $m(s,t)$ is $> 3$ or $= \infty$ and leave it unlabeled if it is $= 3$. $(W,S)$ is *irreducible* if $\Gamma$ is connected. (The components of $\Gamma$ give the irreducible factors of $W$.)

Figure 5 shows Coxeter's classification in [7] of irreducible spherical and irreducible cocompact Euclidean reflection groups. Figure 6 shows Lannér's classification in [13] of the hyperbolic reflection groups with fundamental polytope equal to a simplex in $\mathbb{H}^n$.

[1] This subsection is based on the Historical Notes in [3].

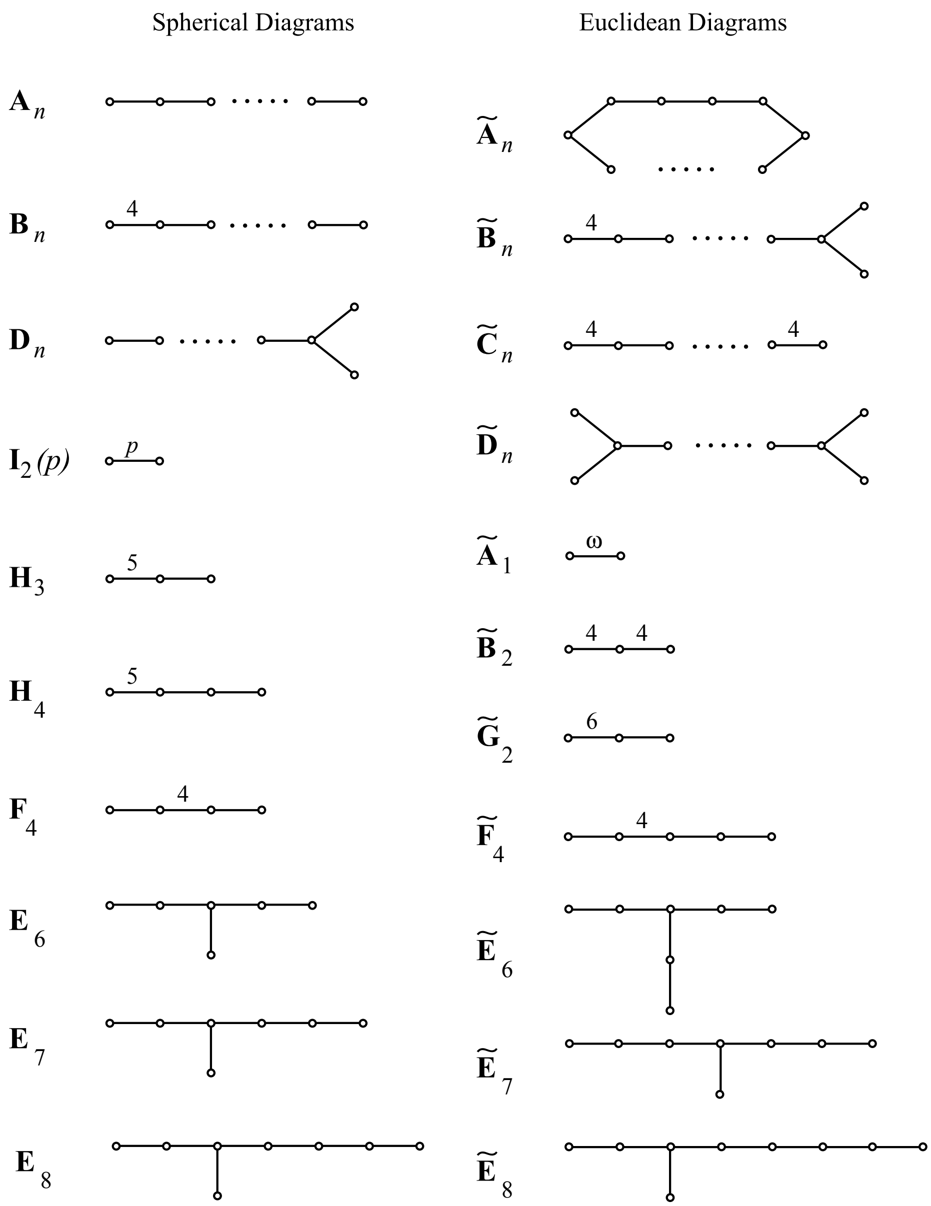


Figure 5: Coxeter diagrams

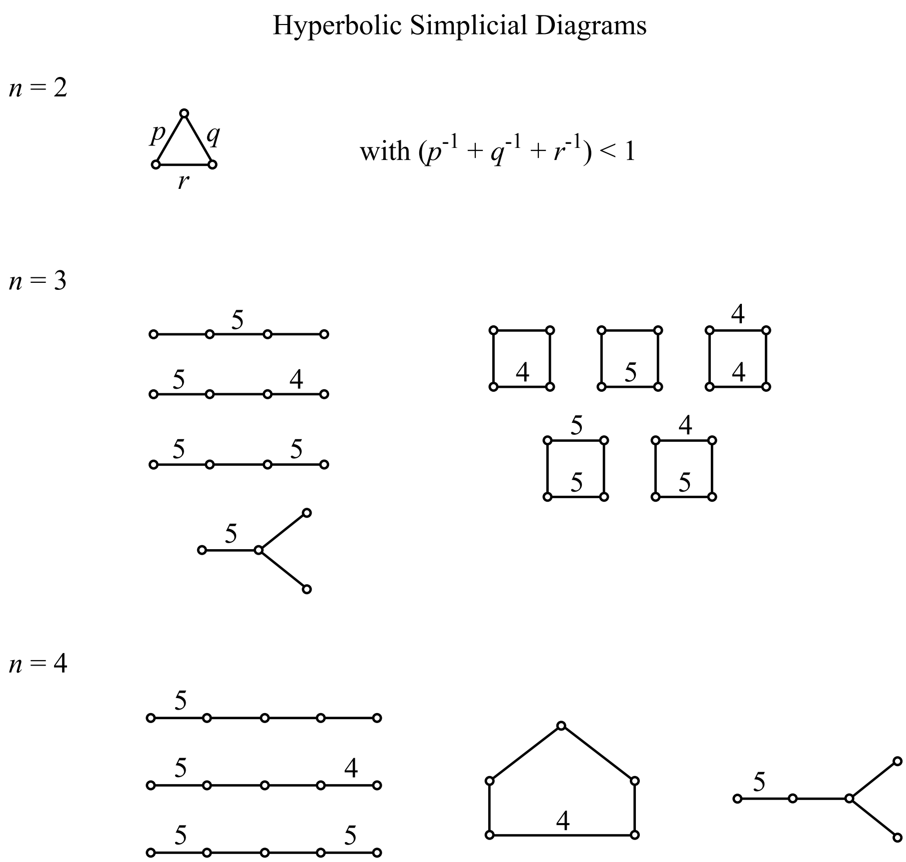


Figure 6: Hyperbolic simplicial diagrams

## 1.3 The relationship with Lie theory

The class of finite Coxeter groups coincides with the class of geometric reflection groups on spheres. These groups are intimately tied to Lie theory.

Suppose $G$ is a simple Lie group with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{h}$ of a maximal torus is a "Cartan subalgebra." The derivative of the adjoint action of $G$ on $\mathfrak{g}$ leads to the adjoint action of $\mathfrak{g}$ on itself by Lie brackets, defined by $\mathrm{ad}(x)(y) :== [x, y]$. Restricting this action to $\mathfrak{h}$, we get a decomposition of $\mathfrak{g}$ into simultaneous eigenspaces:

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha,$$

where the $\alpha$ are certain linear functionals on $\mathfrak{h}$, called the *roots*. The set of such $\alpha \in \mathfrak{h}^*$ is a *root system*. The classification of simple Lie algebras over $\mathbb{C}$ reduces to the classification of finite root systems. As mentioned earlier, Weyl showed that the automorphism group of a root system is a finite reflection group. (Each root defines a hyperplane and the group generated by reflections across these hyperplanes is a reflection group on $\mathfrak{h}$. For this reason, automorphism groups of root systems are usually called "Weyl groups.") The spherical diagrams in Figure 5 are automorphism groups of root systems except in the cases $\mathrm{H}_3$, $\mathrm{H}_4$, and the dihedral groups $\mathrm{I}_2(5)$ and $\mathrm{I}_2(p)$, $p > 6$. The classification of root systems is almost the same as the remaining list of spherical diagrams except that there are two types of root systems for $\mathrm{B}_n$. These are usually denoted by $\mathrm{B}_n$ and $\mathrm{C}_n$.

The connection between spherical Coxeter groups and root systems has many ramifications. For one, the field need not be the field of complex numbers. For example, algebraic groups over finite fields (i.e., finite groups of "Lie type") have corresponding spherical Coxeter diagrams. "Spherical buildings" are associated to these groups. Euclidean reflection groups arise in a number of related ways. One is as the automorphism group of the weight lattice in $\mathfrak{h}^*$. (This automorphism group is an extension of the finite Coxeter group by a lattice in $\mathfrak{h}^*$.) Euclidean reflection groups also occur in the theory of algebraic groups over local fields. The corresponding building is called "Euclidean" or "affine."

Finally, there are the theories of Kac-Moody algebras, groups and buildings. These are related to the more general Coxeter groups that are considered in the next section. In particular, the Tits representations of 2.2 are related to root systems for Kac-Moody Lie algebras.

## 2 Abstract reflection groups

The theory of abstract reflection groups was invented by Tits [16] around 1961. His term for an abstract reflection group was a "Coxeter group." This theory of Coxeter groups is explained in the beautiful book of Bourbaki [3]; Ken Brown's book [4] is also an excellent source.

**Question 2.1.** *Given a group $W$ and a set $S$ of involutions which generate $W$, when does $(W, S)$ deserve to be called an "abstract reflection group"?*

There are two answers to Question 2.1. Suppose $\mathrm{Cay}(W, S)$ is the Cayley graph of $(W, S)$ (i.e., its vertex set is $W$ and $\{w, v\}$ spans an edge if and only if $v = ws$ for some $s \in S$)).

***First answer*:** For each $s \in S$, the fixed point set of $s$ on $\mathrm{Cay}(W, S)$ separates $\mathrm{Cay}(W, S)$.

***Second answer*:** $W$ has a presentation of the same form as (1.1):

$$\langle S \mid (st)^{m(s,t)}, \quad \text{where } (s, t) \in S \times S\rangle. \tag{2.1}$$

Here $(m(s, t))$ is a symmetric $(S \times S)$-matrix with $1$s on the diagonal and off-diagonal elements integers $\geq 2$ or the symbol $\infty$. Such a matrix is called a *Coxeter matrix*.

*It turns out that these two answers are equivalent!* (This is implicit in [3]. It also is stated explicitly and proved in [8, Chapter 3].)

Some of the terms used above should be defined.

**Definition 2.2.** (*Cayley graphs*). Given a group $G$ and a set of generators $S$, let $\mathrm{Cay}(G, S)$ denote the graph with vertex set $G$ which has a (directed) edge from $g$ to $gs$, for all $g \in G$ and for all $s \in S$. The group $G$ acts on $\mathrm{Cay}(G, S)$ (written $G \curvearrowright \mathrm{Cay}(G, S)$), the action is simply transitive on the vertex set, and the edges starting at a given vertex can be labeled by the elements of $S$.

**Definition 2.3.** (*Presentations*). Suppose $S$ is a set of letters and $\mathcal{R}$ is a set of words in $S$. Let $F_S$ be the free group on $S$ and let $N_{\mathcal{R}}$ be the smallest normal subgroup containing $\mathcal{R}$. Set $G := F_S/N_{\mathcal{R}}$ and write

$$G = \langle S \mid \mathcal{R}\rangle.$$

It is a *presentation* for $G$.

## 2.1 Coxeter systems

**Definition 2.4.** If either of the previous two answers holds, $(W, S)$ is a *Coxeter system* and $W$ is a *Coxeter group*.

The terms "Coxeter group" and "Coxeter system" were coined by Tits [16]. The second answer is usually taken to be the official definition: $W$ has a presentation of the form given by (2.1).

**Definition 2.5.** Given a subset $T \subset S$, put $W_T := \langle T \rangle$ and call it a *special subgroup*. $T$ is *spherical* if $W_T$ is finite. Let $\mathcal{S}$ denote the poset of spherical subsets of $S$. (N.B. $\emptyset \in \mathcal{S}$.)

**Definition 2.6.** The *nerve* of $(W, S)$ is the simplicial complex $L$ $(= L(W, S))$ with vertex set $S$ and with $T \subset S$ a simplex if and only if $T$ is spherical. (So, the poset of simplices in $L$, including the empty simplex, is $\mathcal{S}$.)

**Definition 2.7.** (*Right-angled Coxeter systems or* RACS*s*). A Coxeter system $(W, S)$ is *right-angled* (a RACS for short) if all off-diagonal $m(s, t)$ are $= 2$ or $\infty$. $W$ is a *right-angled Coxeter group* (a RACG).

Suppose $\Gamma$ is a graph with vertex set $S$. Define an $(S \times S)$ Coxeter matrix $(m(s, t))$ with off-diagonal elements given by

$$m(s,t) = \begin{cases} 2, & \text{if } \{s,t\} \in \operatorname{Edge}\Gamma \\ \infty, & \text{otherwise.} \end{cases}$$

This defines a RACS, $(W, S)$. A subset $T \subset S$ is spherical if and only if it spans a complete subgraph of $\Gamma$. So, the nerve $L$ is the associated flag complex of $\Gamma$ (called the "clique complex" of $\Gamma$ by combinatorialists).

**Question 2.8.** *Does every Coxeter system have a geometric realization?*

**Answer 2.9.** *Yes. In fact, there are two different realizations:*

(1) *the Tits representation,*

(2) *the cell complex $\Sigma$.*

Both realizations use the following basic construction.

**The basic construction** A *mirror structure* on a space $X$ is a family of closed subspaces $\{X_s\}_{s \in S}$. For $x \in X$, put $S(x) = \{s \in S \mid x \in X_s\}$. Define

$$\mathcal{U}(W, X) := (W \times X)/\sim ,$$

where $\sim$ is the equivalence relation: $(w, x) \sim (w', x') \iff x = x'$ and $w^{-1}w' \in W_{S(x)}$ (the special subgroup generated by $S(x)$). In other words, $\mathcal{U}(W, X)$ is formed by gluing together copies of $X$ (the *chambers*), one for each element of $W$. The group $W$ acts on $\mathcal{U}(W, X)$ via $w \cdot [v, x] = [wv, x]$, where $[v, x]$ denotes the equivalence class of $(v, x)$. (Think of $X$ as the fundamental polytope and the $X_s$ as its codimension one faces.)

**Definition 2.10.** Suppose $(W, S)$ is a Coxeter system. The action of $W$ on a space $\mathcal{U}$ is a *reflection group* if there is a mirror structure $\{X_s\}_{s \in S}$ on a space $X$ such that $\mathcal{U}$ is equivariantly homeomorphic to $\mathcal{U}(W, X)$. (In the past I considered various possible definitions of a "reflection group" - always with an eye to proving that the action was equivalent to the action of $W$ on $\mathcal{U}(W, X)$. Finally, I decided this should be taken as the definition.)

**The properties a geometric realization should have:** It should be an action of $W$ on a space $\mathcal{U}$ so that

(i) $W$ is a reflection group on $\mathcal{U}$.

(ii) The stabilizer of each $x \in \mathcal{U}$ is finite.

(iii) $\mathcal{U}$ is contractible.

(iv) $\mathcal{U}/W$ $(= X)$ is compact.

## 2.2 The first realization: the Tits representation

**Definition 2.11.** (*Linear reflections*). Two data determine a (not necessarily orthogonal) linear reflection on $\mathbb{R}^n$:

(1) linear form $\alpha \in (\mathbb{R}^n)^*$ (the fixed hyperplane is $\alpha^{-1}(0)$).

(2) a $(-1)$-eigenvector $h \in \mathbb{R}^n$ (normalized so that $\alpha(h) = 2$

The formula for the reflection is $v \mapsto v - \alpha(v)h$.

**A symmetric bilinear form** Let $(e_s)_{s\in S}$ be the standard basis for $\mathbb{R}^S$. Given a Coxeter matrix $m(s,t)$ define a symmetric bilinear form $B$ on $R^S$ by

$$B(e_s, e_t) = -2\cos(\pi/m(s,t)).$$

(In the case of geometric reflection group, if the $e_s$ denote the unit normal vectors to the codimension one faces of a fundamental polytope, then the matrix on the right hand side of this formula is the matrix of inner products, $(e_s \cdot e_t)$.) For each $s \in S$, we have a linear reflection $r_s : v \mapsto v - B(e_s, v)e_s$. Tits showed this defines a linear action $W \curvearrowright \mathbb{R}^S$. We are interested in the dual representation $\rho : W \to GL(\mathbb{R}^S)$ defined by $s \mapsto \rho_s := (r_s)^*$.

**Theorem 2.12.** (Properties of Tits representation $W \to GL(\mathbb{R}^S)$, cf. [3]).

(i) *The $\rho_s$ are reflections across the faces of the standard simplicial cone $C \subset (\mathbb{R}^S)^*$.*

(ii) $\rho : W \hookrightarrow GL(\mathbb{R}^S)$*, that is, $\rho$ is a faithful representation.*

(iii) $WC$ *($:= \bigcup_{w\in W} wC$) is a convex cone. Let $\mathcal{I}$ denote the interior of this cone.*

(iv) $\mathcal{I} = \mathcal{U}(W, C^f)$*, where $C^f$ denotes the complement of the nonspherical faces of $C$ (a face is* spherical *if its stabilizer is finite). So, $W$ is a "discrete reflection group" on $\mathcal{I}$.*

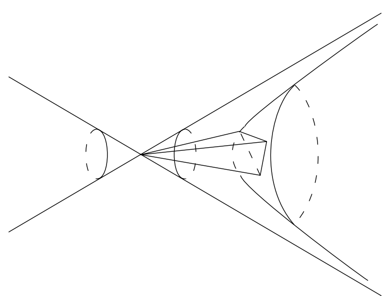

Figure 7: A hyperbolic triangle group

Selberg's Lemma asserts that a finitely generated linear group is residually finite and virtually torsion-free. So, a consequence of Theorem 2.12 is the following.

**Corollary 2.13.** *$W$ is residually finite and virtually torsion-free.*

**Advantages of the Tits representation:** $\mathcal{I}$ is contractible (since it is convex) and $W$ acts properly (i.e., with finite stabilizers) on it.

**Disadvantage:** $\mathcal{I}/W$ is not compact (since $C^f$ is not compact).

**Remarks 2.14.** (1) By dividing by scalar matrices, we get a representation $W \hookrightarrow PGL(\mathbb{R}^S)$. So, $W \curvearrowright P\mathcal{I}$, the image of $\mathcal{I}$ in projective space. When $W$ is infinite and irreducible, this is a proper convex subset of $\mathbb{R}P^{n-1}$, $n = \#S$.

(2) In [17] Vinberg showed one can get linear representations across the faces of more general polyhedral cones. As before, $W \curvearrowright \mathcal{I}$, where $\mathcal{I}$ is a convex cone; $P\mathcal{I}$ is a open convex subset of $\mathbb{R}P^{n-1}$. The fundamental chamber is a convex polytope with some faces deleted. Sometimes it can be a compact polytope, for example, a pentagon. (See Figure 4 again).

Yves Benoist [2] has written a series of papers (see [2]) about these projective representations $W \hookrightarrow PGL(n, \mathbb{R})$. In particular, there are interesting examples with fundamental chamber a compact polytope which are not equivalent to a Euclidean or hyperbolic reflection group. In particular, Moussong [14, 15] had given examples of word hyperbolic Coxeter groups whose nerves were boundary complexes of simplicial polytopes yet could not be realized as reflection groups on any hyperbolic space. Benoist showed that certain of Moussong's examples admit cocompact projective representations as reflection groups.

**Question 2.15.** *For $(W, S)$ to be a reflection group in $PGL(n+1, \mathbb{R})$ with fundamental chamber a compact convex polytope $P^n \subset \mathbb{R}P^n$ there is an obvious necessary condition: its nerve $L(W, S)$ (defined by the spherical subsets of $S$) must be dual to $\partial P^n$. In other words, the simplicial complex $L(W, S)$ must be the boundary complex of a simplicial polytope. Is this condition sufficient?*

Almost certainly the answer is no. In the opposite direction, one could ask the following.

**Question 2.16.** *Are there irreducible, non-affine examples of such $W \subset PGL(n+1, \mathbb{R})$ with $P^n \subset \mathbb{R}P^n$ a compact convex polytope, for $n$ arbitrarily large?*

## 2.3 The second realization: the cell complex $\Sigma$

The second answer is to construct a contractible cell complex $\Sigma$ on which $W$ acts properly and cocompactly as a group generated by reflections. Its advantage is that $\Sigma/W$ will be compact.

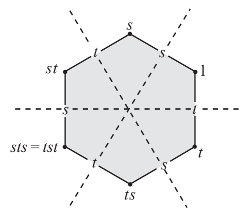


Figure 8: The Cayley 2-complex of a finite dihedral group

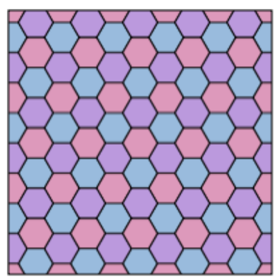

Figure 9: Cayley 2-complex of $(3, 3, 3)$-triangle group

There are two dual constructions of $\Sigma$:

(1) Build the correct fundamental chamber $K$ with mirror structure $\{K_s\}_{s \in S}$, then apply the basic construction to get $\Sigma = \mathcal{U}(W, K)$.

(2) Fill in the Cayley graph of $(W, S)$.

**Filling in the Cayley graph** Let $W_{\{s,t\}}$ be the dihedral subgroup $\langle s, t \rangle$. Whenever $m(s,t) < \infty$ each coset of $W_{\{s,t\}}$ spans a polygon in $\mathrm{Cay}(W, S)$. If we fill in these polygons, we get a simply connected 2-dimensional complex, which will be the 2-skeleton of $\Sigma$. If we want to obtain a contractible space, then we need to fill in higher dimensional polytopes (”cells”). Corresponding to a spherical subset $T$ with $\#T = k$, there is a $k$-dimensional convex polytope called a *Coxeter zonotope*. (A *zonotope* means a convex polytope which is dual to the face poset of an arrangement of hyperplanes in $\mathbb{R}^n$.) When $k = 2$, a zonotope is the polygon with an even number of edges.

**Coxeter zonotopes** Suppose $W_T$ is finite reflection group on $\mathbb{R}^T$. Choose a point $x$ in the interior of fundamental simplicial cone and let $P_T$ be convex hull of $W_T x$. The 1-skeleton of $P_T$ is $\mathrm{Cay}(W_T, T)$ (See Figures 8 and 10.)

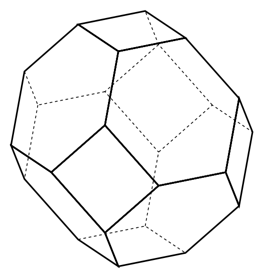

Figure 10: The permutohedron

**Example 2.17.** When $W_T = (\mathbb{Z}/2)^n$, $P_T$ is an $n$-cube.

Let $W\mathcal{S}$ be the *poset of spherical cosets*, i.e., it is the disjoint union of all cosets of spherical special subgroups (partially ordered by inclusion):

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T. \tag{2.2}$$

The idea is to build a cell complex with one cell (a Coxeter zonotope) for each spherical coset.

**Definition 2.18.** (*Geometric realization of a poset*). Associated to any poset $\mathcal{P}$ there is a simplicial complex $|\mathcal{P}|$ called its *geometric realization*. The vertex set of $|\mathcal{P}|$ is $\mathcal{P}$. A set of $k+1$ distinct vertices $\{p_0, p_1, \ldots, p_k\}$ spans a $k$-simplex if and only if it is totally ordered, i.e., if, possibly after renumbering, $p_0 < \cdots < p_k$.

Put $\Sigma := |W\mathcal{S}|$. There is a cell structure on $\Sigma$ with

$$\{\text{cells in } \Sigma\} = W\mathcal{S}.$$

This follows from fact that poset of cells in $P_T$ is $\cong W_T\mathcal{S}_{\leq T}$. The cells of $\Sigma$ are defined as follows: the geometric realization of subposet of cosets $\leq wW_T$ is $\cong$ barycentric subdivision of $P_T$.

**Theorem 2.19.** (Properties of this cell structure on $\Sigma$, cf. [8]).

(i) *$W$ acts cellularly on $\Sigma$.*

(ii) *$\Sigma$ has one $W$-orbit of cells for each spherical subset $T \in \mathcal{S}$; the dimension of the cell is* $\mathrm{Card}(T)$*.*

(iii) *The* $0$*-skeleton of $\Sigma$ is $W$*

(iv) *The* $1$*-skeleton of $\Sigma$ is* $\mathrm{Cay}(W, S)$*.*

(v) *The* $2$*-skeleton of $\Sigma$ is the Cayley* $2$ *complex of the presentation.*

(vi) *If $W$ is right-angled (i.e., if each $m(s,t)$ is* $1, 2$ *or* $\infty$*), then each Coxeter cell is a cube.*

(vii) (Gromov [11] and Moussong [14]): *the induced piecewise Euclidean metric on $\Sigma$ is* $\mathrm{CAT}(0)$ *(meaning that it is non positively curved).*

(viii) *$\Sigma$ is contractible. (This follows from the fact it is* $\mathrm{CAT}(0)$*).*

(ix) *The $W$-action is proper (by construction each isotropy subgroup is conjugate to some spherical $W_T$).*

(x) *$\Sigma/W$ is compact.*

(xi) *If $W$ is finite, then $\Sigma$ is a Coxeter zonotope.*

**Theorem 2.20.** (A typical application of CAT(0)-ness, cf. [9, Remark(5b.2), p. 384]). *There is a closed manifold $M$ with a nonpositively curved (polyhedral) metric so that $M$ is not homotopy equivalent to a nonpositivley curved Riemannian manifold (the reason being that the universal cover of $M$ is not homeomorphic to Euclidean space.)*

**The dual construction of $\Sigma$** Recall $\mathcal{S}$ is the poset of spherical subsets of $S$. The *fundamental chamber* $K$ is defined by $K := |\mathcal{S}|$. ($K$ is the cone on the barycentric subdivision of $L$.) The mirror structure is given by $K_s := |\mathcal{S}_{\geq \{s\}}|$. The proof of the following proposition is an easy exercise - it is essentially just a matter of unraveling the definitions.

**Proposition 2.21.** (cf. [8]). $\Sigma := \mathcal{U}(W, K)$*. So, $K$ is homeomorphic to $\Sigma/W$.*

The construction of $\Sigma$ is very useful for constructing examples. The basic reason is that the chamber $K$ is the cone over a fairly arbitrary simplicial complex (for example, any barycentric subdivision can occur). This means we can construct $\Sigma$ with whatever local topology we like.

**Relationship with geometric reflection groups** If $W$ is a geometric reflection group on $\mathbb{X}^n = \mathbb{E}^n$ or $\mathbb{H}^n$, then $K$ can be identified with the fundamental polytope, $\Sigma$ with $\mathbb{X}^n$ and the cell structure of $\Sigma$ is dual to the tessellation of $\Sigma$ by translates of $K$. (See Figure 4; the dotted black lines form the 1-skeleton of $\Sigma$.)

**Relationship with Tits representation** Suppose $W$ is infinite. Then $K$ is subcomplex of $b\Delta$, the barycentric subdivision of the simplex $\Delta \subset C$. Consider the vertices which are barycenters of spherical faces. They span a subcomplex of $b\Delta$ (the barycentric subdivision of the simplex $\Delta$). This subcomplex is $K$. It is a subspace of $\Delta^f$. So, $\Sigma = \mathcal{U}(W, K) \subset \mathcal{U}(W, \Delta^f) \subset \mathcal{U}(W, C^f) = \mathcal{I}$. Thus, $\Sigma$ is the "cocompact core" of $\mathcal{I}$.

## 2.4 The relationship between Coxeter groups and Artin groups

Given a Coxeter matrix $(m(s,t))$ there is an associated *Artin group* $A$ with set of generators $\{x_s\}_{s \in S}$ and with relations defined by

$$x_s x_t \cdots = x_t x_s \cdots$$

for all $s, t$ with $s \neq t$ and $m(s,t) \neq \infty$. Here the words on each side on the equation are alternating words of length $m(s,t)$ in $x_s$ and $x_t$. There is a natural epimorphism from $A$ to the associated Coxeter group $W$ sending $x_s$ to $s$. For example, if $W$ is the symmetric group $A_n$ on $n+1$ letters than $A$ is the braid group on $n+1$ strands. (Artin was the first to study these braid groups.)

The Tits representation $W \hookrightarrow GL(N; \mathbb{R})$, where $N = \#S$, has complexification $W \hookrightarrow GL(N; \mathbb{C})$. The subset $\mathbb{R}^N + i\mathcal{I}$ is $W$-stable. For each $r \in R$ (where $R$ is the set of conjugates of $S$), we have the complexified hyperplane $H_r$ that is fixed by $r$. Denote the complement of the union of these reflecting hyperplanes by

$$M := (\mathbb{R}^N + i\mathcal{I}) - \bigcup_{r \in R} H_r.$$

The group $W$ acts freely on $M$ and $\pi_1(M/W) = A$. The major unknown question in the area is the $K(\pi, 1)$-Problem for Artin groups. This asks whether $M$ is a model for $K(A, 1)$ (see [5, 6]). The answer is affirmative for Artin groups of spherical type, for right-angled Artin groups (as defined below) and for many other cases (see [5]).

In analogy with Definition 2.7, an Artin group is *right-angled* if all the off-diagonal $m(s,t)$ are 2 or $\infty$. The term "right-angled Artin group" is commonly abbreviated RAAG.

There is a beautiful model for the classifying space $K(A, 1)$ of a RAAG $A$. Let $T^S$ denote the product of $S$ copies of $S^1$. For each simplex $\sigma \in L$ $(= L(W,S))$, let $T^\sigma$ denote the subproduct over the vertices of $\sigma$ of copies of $S^1$. Then $\bigcup_{\sigma \in L} T^\sigma$ is a $K(A, 1)$. Its universal cover is a CAT(0) cube complex (see [6]).

It is proved in [10] that every RAAG is commensurable with a RACG. (In fact, any RAAG has a RACG as a subgroup of finite index as well as a RACG as finite index super group.) Using this, one can show that RAAGs have very strong separability properties. For example, since RACGs are linear groups via the Tits representation, so are RAAGs. In particular, RAAGs are residually finite. Furthermore, Haglund proved that every quasiconvex subgroup of a RACG is a virtual retract of it; so, the same is true for any RAAG.

Some recent spectacular advances in geometric group theory and in 3-manifold theory involve RACGs and RAAGs. First, Haglund and Wise proved that if a group $\Gamma$ acts on a sufficiently nice CAT(0) cube complex, then $\Gamma$ virtually embeds in a RAAG or RACG. These nice cube complexes are the "special" cube complexes of [12]. An overview of this theory can be found in [19]. In [1] Ian Agol proved a conjecture of Wise asserting that if a word hyperbolic groups acts on a CAT(0) cube complex, then it can also act on a special CAT(0) cube complex. Using this, he proved Thurston's remaining conjectures on 3-manifolds and their fundamental groups. In particular, he proved Thurston's Virtual Fibering Conjecture and Virtual Haken Conjecture. For this work Agol and Wise were awarded the Veblen Prize in 2013.

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