

Imperial College London
Department of Materials

MSE 203: Mechanical Behaviour
Continuum Mechanics (Stress Tensors)

Dr David Dye
2013–14

Comments and corrections to david.dye@imperial.ac.uk

Lecture notes may be found on Blackboard (<http://learn.imperial.ac.uk>) and at <http://dyedavid.com/mse203>

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 1 |
| 1.1 | States of Stress: Definitions | 2 |
| 2 | Rotating the Axes that Describe a Stress State in 2D | 6 |
| 2.1 | Stresses on an Inclined Plane (in 2D) | 6 |
| 2.2 | Principal Stresses and Principal Axes (in 2D) | 7 |
| 2.3 | Mohr's Circle | 10 |
| 3 | 3D Stress Tensors | 12 |
| 3.1 | 3D Stress Tensors, Eigenvalues and Rotations | 12 |
| 3.2 | Aside: Calculating Shear Stresses in Sections | 13 |
| 3.3 | Diagonalising matrices | 14 |
| 3.4 | Principal Stresses in 3 Dimensions | 14 |
| 3.5 | Finding stresses for a general rotation | 16 |
| 3.6 | Example: Finding principal stresses | 17 |
| 3.7 | Example: Stresses on crystal axes | 17 |
| 3.8 | Stresses on a Single Plane | 18 |
| 4 | Strain and Elasticity | 19 |
| 4.1 | Strains | 19 |
| 4.2 | Isotropic Elasticity | 20 |
| 4.3 | Anisotropic Elasticity | 22 |
| 4.4 | Strain Measurement | 24 |
| 4.5 | Example: Stress near a weld | 25 |
| 5 | Yield Surfaces and Yield Criteria | 26 |
| 5.1 | Tresca Criterion | 26 |
| 5.2 | Von Mises Criterion | 27 |
| 5.3 | Hydrostatic and Deviatoric Stresses | 27 |
| 5.4 | Graphical Representation of Yield Criteria | 28 |

Chapter 1

Introduction

Aim: To develop the ability to set up and manipulate descriptions of the state of stress in a material, to understand and apply yield criteria.

Objectives

- To derive equations for the normal and shear stresses in multiaxial stress situations.
- To be able to write these as a stress matrix.
- To introduce the concepts of principal stress and strain and maximum shear stress.
- To recognize the principal stresses / strains as the eigenvalues of the stress / strain matrix.
- To be able to rotate a stress or strain matrix and find the orientation of the principal axes.
- To be able to analyse the stress and strain state for the cases of a rotating shaft, a pressure vessel and a bending beam.
- To be able to use Nye's convention for calculating stresses from the stiffness constants and strain tensor.
- To be able to determine the stress tensor from diffraction data.
- To be able to use the Von Mises and Tresca Yield Criteria, and understand the π -plane convention.

The course is composed of 9 lectures, 2 tutorials and a 1-hr test. The test will be the week after the end of the course. The course composes one quarter of MSE203, and is therefore worth around $100/4 = 25$ marks in the summer exams. We will go through problem sheets and previous tests in the tutorials.

Reading List

- 1 JF Nye, Physical Properties of Crystals, OUP, 1957.
- 2 GE Dieter, Mechanical Metallurgy, McGraw-Hill, 1988.
- 3 Cottrell, Mechanical Properties of Matter.
- 4 Le May, Principles of Physical Metallurgy.
- 5 A Mostofi, MSE201 notes on Tensors.

1.1 States of Stress: Definitions

In order to analyse stresses, let us first consider an infinitesimal 2D element in a material. On each face of the element, there must be a maximum of four (net) forces acting, P , Q , R and S , one on each plane of the element.

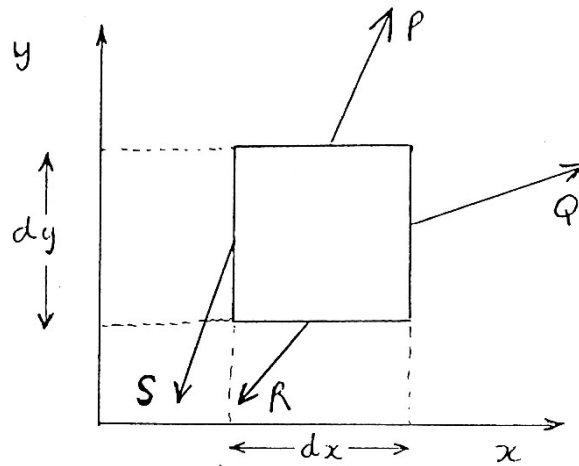


Figure 1: Forces on a 2D element.

Immediately we can resolve these forces into their x and y components, denoted P_x , P_y , Q_x , Q_y , R_x , R_y , S_x , and S_y .

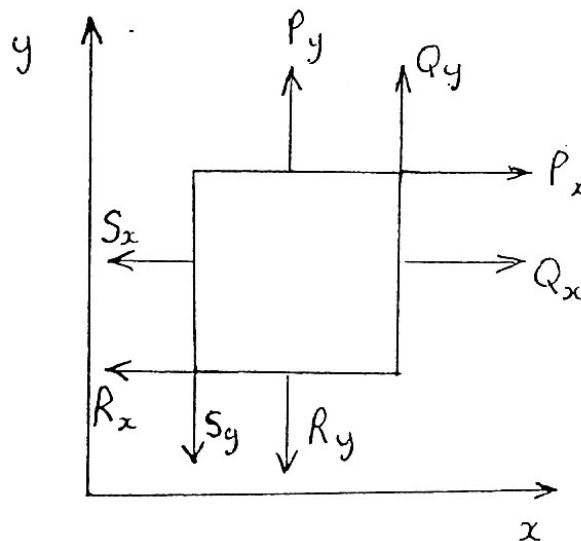


Figure 2: Forces on a 2D element, resolved into components.

We can then convert these into stresses (stress = force / area) if we consider the element to be of unit thickness and divide the force by the element lengths, 1 and dx or dy . If we require that the element is in equilibrium we find that the net moments and forces must be zero, otherwise the element would be in motion. Hence;

By convention, the first subscript denotes the direction of the normal to the plane and the second subscript the direction of the stress. The requirement for equilibrium means that $\tau_{xy} = \tau_{yx}$, otherwise the element would spin. Therefore to fully specify the state of stress in a 2D body

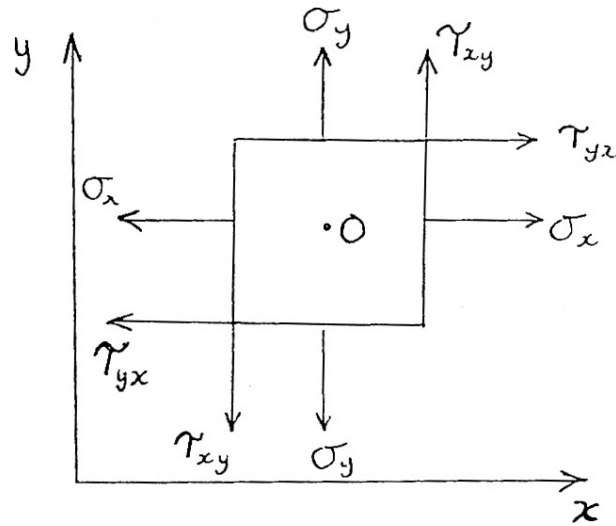


Figure 3: Stresses on a 2D element.

requires only 3 stresses, σ_{xx} , σ_{yy} and τ_{xy} . *Note: there are other sign and notation conventions in use in different textbooks.*

Stress Matrices

Stress can be written as a *second rank tensor*, or matrix. Hence in the 3D case it can be written as a 3x3 matrix σ_{ij} , with $i, j = x, y, z$, where i is the direction of the stress and j is the direction of the plane normal to the face of the volume element. So the normal stresses are σ_{ii} (summation convention) and the shear stresses are the σ_{ij} , $i \neq j$. We can visualise these components as acting on the cube as follows.

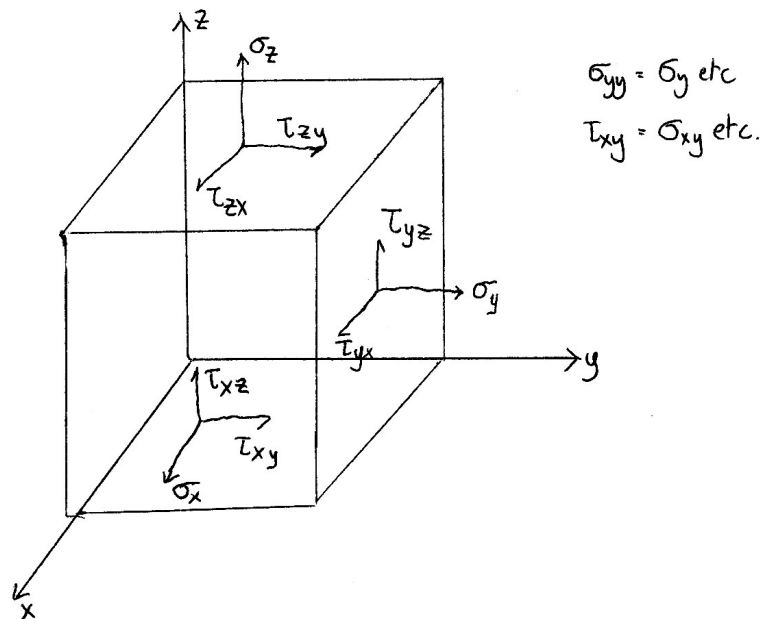


Figure 4: Stresses on a 3D element (obscured matching stresses to produce equilibrium not shown).

So we write the general 3D stress matrix as

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \quad (1)$$

Immediately we see that the requirement for equilibrium, which gives $\sigma_{ij} = \sigma_{ji}$, reduces the 3D stress matrix to 6 independent components and therefore the stress tensor is symmetrical about the leading diagonal; it is a *SYMMETRIC TENSOR*.

Example: Uniaxial Tension

This is the situation for a simple tensile test, Figure 5.

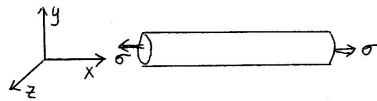


Figure 5: Uniaxial tension.

So the stress matrix, for this arrangement of the axes, is given by

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2)$$

There are two points to note: (i) that the σ_{ij} are dependent on the axes used and (ii) that $\sigma > 0$ for a tensile stress and $\sigma < 0$ for a compressive stress.

Example: Hydrostatic Compression

For example, a body immersed in a liquid.

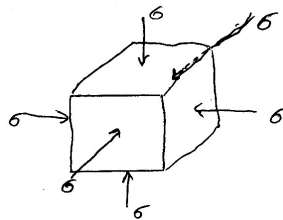


Figure 6: Hydrostatic compression applied to a unit cube (e.g. a fish under the sea).

Here $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$, where $\sigma < 0$. Then the stress matrix is written as

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} \quad (3)$$

Whereas a uniaxial stress is *not* the same for all axis orientations, and some rotations may give shear stresses, a purely hydrostatic stress state is invariant to axis rotation and so it is an *ISOTROPIC TENSOR*.

Example: Pure Shear

Here $\sigma_{xy} = \sigma_{yx} = \tau$, as follows.

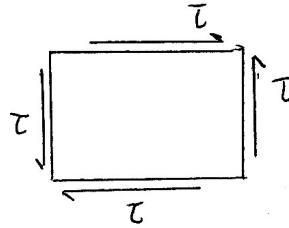


Figure 7: Pure Shear applied to a unit cube.

So the stress matrix is given by

$$\sigma_{ij} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4)$$

This appears simple because of the choice of axes to coincide with the symmetry of the stress state. It would appear more complex for arbitrary axes. Also notice that by convention the system of axes is right handed and so σ_{yx} is in the anticlockwise direction. If the direction of the arrows on the shear stresses in the diagram were reversed, then the shear stress in the stress tensor would be negative.

Note on Notation

The co-ordinate system used must be a right handed, locally orthogonal set of axes, that is, with all the axes at right angles to each other, but can otherwise be chosen arbitrarily for convenience. We will see later that the effect of this is to change the values in the stress matrix. In fact, it is always possible to find a set of axes where the shear stresses are zero. This suggests that the normal stresses σ and shear stresses τ are not fundamentally different, hence the use of the notation σ_{ij} .

Chapter 2

Rotating the Axes that Describe a Stress State in 2D

2.1 Stresses on an Inclined Plane (in 2D)

In 2D, consider the stresses acting on an inclined plane. If the block in the figure below is sliced in two, a (right angled) triangular block remains, Figure 8. We aim to find the values of the stress σ and shear τ on the inclined plane required to balance the applied σ_x , σ_y and τ_{xy} by applying force equilibrium.

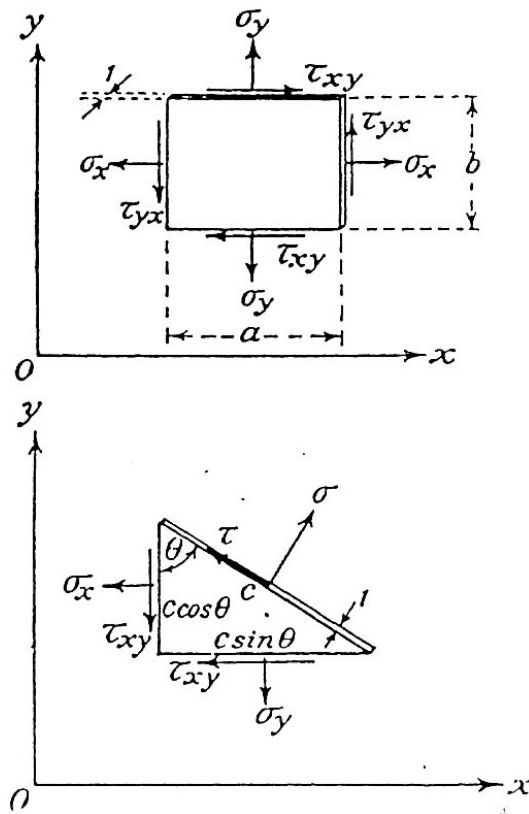


Figure 8: General 2D state of stress (top). If the 2D element is cut to form an inclined plane (bottom), then stresses σ and τ must be placed for it to remain in equilibrium.

Resolving forces parallel to σ , we find

$$\sigma c = \sigma_x(c \cos \theta \cos \theta) + \sigma_y(c \sin \theta \sin \theta) + 2\tau_{xy}(c \sin \theta \cos \theta) \quad (5)$$

Which gives

$$\sigma = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta \quad (6)$$

Resolving forces parallel to τ gives

$$\tau c = -\sigma_x(c \cos \theta \sin \theta) + \sigma_y(c \sin \theta \cos \theta) + \tau_{xy}(c \cos \theta \cos \theta) - \tau_{xy}(c \sin \theta \sin \theta) \quad (7)$$

So

$$\tau = (\sigma_y - \sigma_x) \cos \theta \sin \theta + \tau_{xy}(\cos^2 \theta - \sin^2 \theta) \quad (8)$$

We can then use the identities $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ and $\sin 2\theta = 2 \sin \theta \cos \theta$ to write these expressions for σ and τ as

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \quad (9)$$

$$\tau = -\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta \quad (10)$$

If we plot Equations 9 and 10, we obtain Figure 9.

2.2 Principal Stresses and Principal Axes (in 2D)

From Equation 10 we can deduce that the shear stress τ vanishes when $\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta = \tau_{xy} \cos 2\theta$, so when

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (11)$$

Since \tan is periodic with a period of π , this may be written as

$$\theta = \frac{1}{2} \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{or} \quad \frac{1}{2} \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} + \frac{\pi}{2} \quad (12)$$

We can also deduce that the normal stress σ is max/minimised by differentiating Equation 9 and setting it to zero to find the turning point,

$$\frac{d\sigma}{d\theta} = -(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0 \quad (13)$$

that is, when

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (14)$$

Which is the same as Equation 11. Therefore there always exist two perpendicular directions where the shear stresses are zero and the normal stresses are max/minimised, called the *PRINCIPAL AXES*, the corresponding normal stresses are called the *PRINCIPAL STRESSES*. These form the subject of the next section. It turns out that these properties are a natural consequence of the fact the the stress tensor is a square, symmetric tensor.

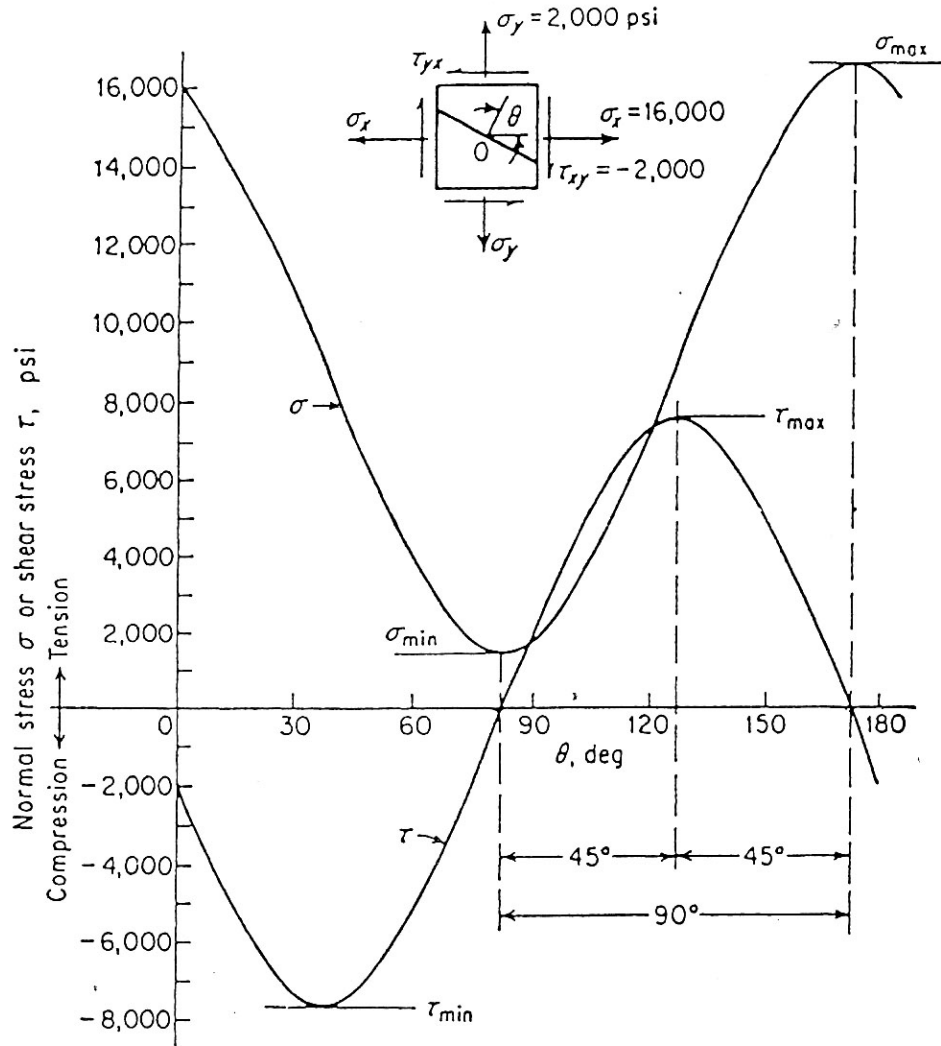


Figure 9: Variation of normal and shear stress on the inclined plane with angle θ .

Values of the Principal Stresses (in 2D)

To find the values of the principal stresses, we could find the directions of the principal axes, Equations 11 and 14 and calculate the two values of θ , and then substitute into Equation 9. However, there is an easier solution to the problem, as follows. If the plane in Figure 10 is a principal plane then there is no shearing stress. We can then say that, since the block is in equilibrium in the x -direction,

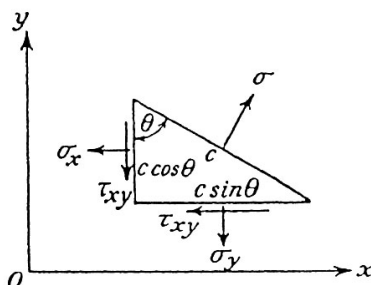


Figure 10: If the inclined plane is orientated such that σ is a principal stress, then no τ acts.

$$\sigma c \cos \theta - \sigma_x c \cos \theta = \tau_{xy} c \sin \theta \quad (15)$$

and so

$$\sigma - \sigma_x = \tau_{xy} \tan \theta \quad (16)$$

Similarly in the y -direction,

$$\sigma c \sin \theta - \sigma_y c \sin \theta = \tau_{xy} c \cos \theta \quad (17)$$

and so

$$\sigma - \sigma_y = \tau_{xy} \cot \theta \quad (18)$$

We can then multiply the LHS and RHS of equations 16 and 18 to find

$$(\sigma - \sigma_x)(\sigma - \sigma_y) = \tau_{xy}^2 \quad (19)$$

This is a quadratic in σ ,

$$\sigma^2 - \sigma(\sigma_x + \sigma_y) + \sigma_x \sigma_y - \tau_{xy}^2 = 0 \quad (20)$$

The solutions of this quadratic are then found to be

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad (21)$$

By convention, we denote the maximum value of σ to be σ_1 and the minimum σ_2 .

Values of the Maximum Shear Stress (in 2D)

Finding the maximum shear stress is slightly more complicated. If we differentiate Equation 10 and set it to zero to find the turning points, we find

$$\frac{d\tau}{d\theta} = -(\sigma_x - \sigma_y) \cos 2\theta - 2\tau_{xy} \sin 2\theta = 0 \quad (22)$$

which gives

$$\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad (23)$$

So in this case, $\tan 2\theta$ is the negative reciprocal of the value when the principal stresses are found, Equation 14, which implies that the 2θ are at 90° to each other in the two cases. So the directions of maximum shear and principal stress are at 45° to each other.

We can substitute the value of $\tan 2\theta$ found in Equation 23 into Equation 10, and find the maxima for τ ,

$$\tau_{\max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad (24)$$

However, we can simplify this, because from the equation for the principal stresses, Equation 21,

$$\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} = \sigma_1 - \frac{1}{2}(\sigma_x + \sigma_y) = -\sigma_2 + \frac{1}{2}(\sigma_x + \sigma_y) \quad (25)$$

So therefore

$$2\tau_{\max} = 2\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} = \sigma_1 - \frac{1}{2}(\sigma_x + \sigma_y) - \sigma_2 + \frac{1}{2}(\sigma_x + \sigma_y) \quad (26)$$

Which gives us the result that

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad (27)$$

2.3 Mohr's Circle

Mohr's circle is a graphical method for finding the principal and maximum shear stresses. If we rearrange Equations 9 and 10, we find that

$$\sigma - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \quad (28)$$

$$\tau = -\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta \quad (29)$$

If we square these and add them, we find that (using $\cos^2 2\theta + \sin^2 2\theta = 1$ and noticing that the $\cos 2\theta \sin 2\theta$ terms disappear),

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \quad (30)$$

This is the equation of a circle, of the form $(x - h)^2 + y^2 = r^2$, that is with a radius of τ_{\max} , Equation 24. We call this circle 'Mohr's Circle', which is a circle plotted on axes of σ and τ , with a radius of τ_{\max} and displaced $\frac{1}{2}(\sigma_x + \sigma_y)$ to the right of the origin.

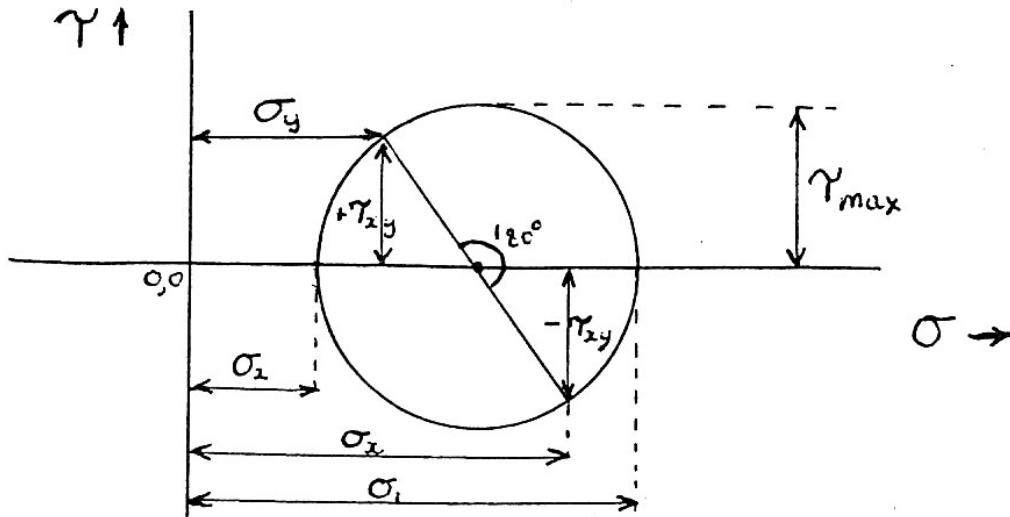


Figure 11: Mohr's circle of stress.

Using Mohr's Circle: Example

Question. A thin-walled pipe of diameter 10 mm and wall thickness 1 mm is pressurised with an internal pressure of 50 MPa. In addition, a shear stress of 50 MPa is applied. Find the axial and hoop stress in the pipe and hence the principal stresses and maximum shear stress.

Solution. First, consider the axial stress component.

The load on the ends is simply the pressure (P) multiplied by the area it is acting over (πr^2), and this is borne by an area of pipe $2\pi r t$. So the axial stress σ_{ax} is given by

$$\sigma_{ax} = \frac{P\pi r^2}{2\pi r t} = \frac{Pr}{2t} = \frac{50 * 5}{2 * 1} = 125 \text{ MPa} \quad (31)$$

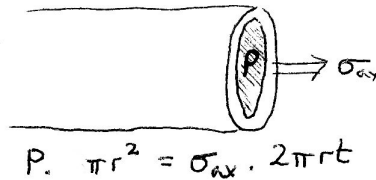


Figure 12: Determination of the axial stress component.

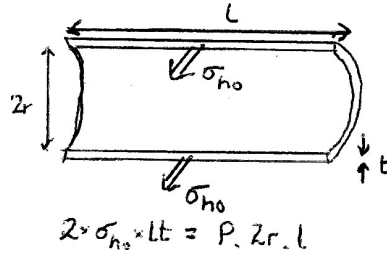


Figure 13: Determination of the hoop stress component.

Similarly, the hoop component is given in Figure 13. The load on the walls in the hoop direction is again the pressure multiplied by an area $2rl$, where l is the length of the pipe. This is borne by an area of wall of $2tl$, so the hoop stress σ_{ho} is

$$\sigma_{ho} = \frac{P2rl}{2tl} = \frac{Pr}{t} = \frac{50 * 5}{1} = 250 \text{ MPa} \tag{32}$$

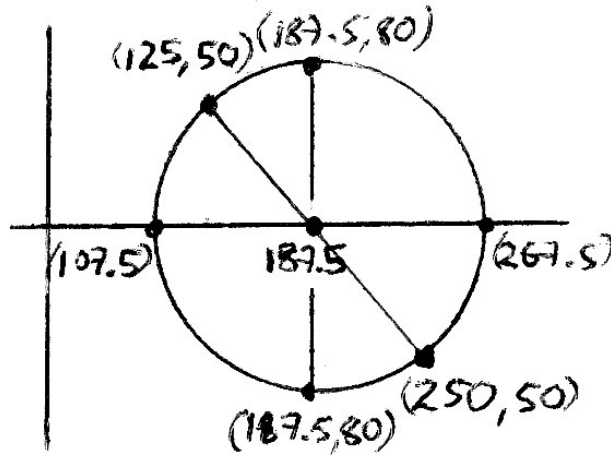


Figure 14: Mohr's circle for the pipe with an additional shear.

So we can now draw Mohr's circle as shown; it is a circle of radius τ_{max} , given by (Equations 27 and 21)

$$\tau_{max} = \frac{1}{2} \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2} = \frac{1}{2} \sqrt{(250 - 125)^2 + 4 * 50^2} = 80 \text{ MPa} \tag{33}$$

centred at a point $\frac{1}{2}(125 + 250) = 187.5$ MPa from the origin. So the principal stresses are $187.5 \pm 80 = 267.5$ and 107.5 MPa.

Chapter 3

3D Stress Tensors

3.1 3D Stress Tensors, Eigenvalues and Rotations

Recall that we can think of an $n \times n$ matrix M_{ij} as a transformation matrix that transforms a vector x_i to give a new vector y_j (first index = row, second index = column), *e.g.* the equation $Mx = y$. We define x to be an *eigenvector* of M if there exists a scalar λ such that

$$Mx = \lambda x \quad (34)$$

The value(s) λ are called the *eigenvalues* of M . We can find the eigenvalues simply, as follows. First we can infer that

$$Mx - \lambda x = 0 \quad (35)$$

where 0 is a zero (null) vector. Then we can simplify this by introducing the Identity matrix I of the same size as M ,

$$(M - \lambda I)x = 0 \quad (36)$$

This implies that $\det(M - \lambda I) = 0$, since the only solution is found where the size of $M - \lambda I$ is zero, where \det signifies the determinant. We are interested in solving this problem for 3x3 matrices, that is for 3D states of stress (or, as we shall see later, strain). You will recall the checkerboard pattern for finding a determinant of a matrix A ;

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad (37)$$

We find the determinant of A by multiplying each of the terms in any row or column by (i) the corresponding sign from the checkerboard pattern and (ii) the determinant of the 2x2 matrix left out once that term's row and column have been removed. *e.g.*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \quad (38)$$

So if we have a square symmetric stress matrix σ_{ij} then its eigenvalues will be given by

$$\begin{aligned} |\sigma - \lambda I| = & +(\sigma_{11} - \lambda) \{(\sigma_{22} - \lambda)(\sigma_{33} - \lambda) - \sigma_{23}^2\} \\ & - \sigma_{12} \{\sigma_{12}(\sigma_{33} - \lambda) - \sigma_{23}\sigma_{13}\} \\ & + \sigma_{13} \{\sigma_{12}\sigma_{23} - (\sigma_{22} - \lambda)\sigma_{13}\} \quad (39) \end{aligned}$$

This is a cubic in λ . For many cases, many of the indices will be zero and the cubic will be easy to solve. Typically, the first root is found by inspection (*e.g.* $\lambda = 2$ is a root), at which point the problem can be reduced to a quadratic by substitution and the remaining roots found trivially.

To find the eigenvectors we then solve the equation $(\sigma - \lambda I)x = 0$ for each of the n eigenvalues in turn. For a 3x3 (square) symmetric (stress) matrix, this will produce three linearly independent eigenvectors. Each eigenvector will be scale-independent, since if x is an eigenvector, it is trivial to show that αx is also an eigenvector.

It turns out to be possible to show that in this case the eigenvalues are the principal stresses, and the eigenvectors are the equations of the axes along which the principal stresses act.

3.2 Aside: Calculating Shear Stresses in Sections

Question. A bar of radius 50 mm transmits 500 kW and 6000 rpm. What is the shear stress in the bar?

Solution. We remember that Power = Torque x Angular Velocity,

$$P = T\omega \quad (40)$$

and that the shear stress τ is related to the torque through the polar moment of inertia J and the outer radius R by

$$T/J = \tau/R \quad (41)$$

J is given, for any section, by $J = \int r^2 dA$, so for a shaft of inner radius r and outer radius R ,

$$J = \frac{1}{2}\pi(R^4 - r^4) \quad (42)$$

Therefore, in this case,

$$J = \frac{1}{2}\pi(0.05)^4 = 9.817 \times 10^{-6} \text{ m}^4 \quad (43)$$

$$\begin{aligned} T = P/\omega &= 500,000 / (6000 * 2\pi/60) \\ &= 500,000 / 628.3 = 795.78 \text{ Nm} \end{aligned} \quad (44)$$

So the result is given by

$$\tau = TR/J = 795.78 * 0.05 / 9.817 \times 10^{-6} = 4.05 \text{ MPa} \quad (45)$$

3.3 Diagonalising matrices

In the previous section, we found the eigenvectors and eigenvalues of a matrix M . Consider the matrix of the eigenvectors X composed of each of the (column) eigenvectors x in turn, e.g. $X_{ij} = x_i, j$, and the matrix D with the corresponding eigenvalues on the leading diagonal and zeroes as the off-axis terms, e.g. $D_{ii} = \lambda_i$ and $D_{ij} = 0 \quad i \neq j$. So for a 3x3 matrix M ,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \text{ By inspection, we can see that}$$

$$MX = XD \quad (46)$$

because for each column, $Mx = \lambda x$. Hence,

$$D = X^{-1}MX \quad (47)$$

A neat example of this is finding large powers of a matrix. For example,

$$M^2 = (XDX^{-1})(XDX^{-1}) = (XD^2X^{-1}) \quad (48)$$

and so on for higher powers.

It turns out that the matrix of eigenvectors X is highly significant. Later, we will look at how to rotate a stress matrix in the general case. However, you will already be able to see that it is always possible to rotate the stress matrix using X , the rotation matrix composed of the unit eigenvectors, to produce a matrix of eigenvalues D . It turns out that this matrix is the matrix of principal stresses, *i.e.* that the eigenvalues of the stress matrix are the principal stresses.

3.4 Principal Stresses in 3 Dimensions

Generalising the 2D treatment of the inclined plane to 3D, we consider an inclined plane. We take a cube with a stress state referred to the 1, 2, 3 axes, and then cut it with an inclined plane with unit normal $x = (l, m, n)$ and area A .

The components of x along the 1, 2, 3 axes are its direction cosines, that is, the cosines of the angles between x and the axes. We require that the stress σ normal to the inclined plane is a principal stress, that is that there are no shears on the inclined plane, Figure 15

First we notice that the components of σ in the 1, 2 and 3 directions are σl , σm and σn , respectively. The areas of the triangles forming the walls of the original cube are also

$$KOL = Al \quad JOK = Am \quad JOL = An \quad (49)$$

If we resolve all the forces in the 1 direction, we find that

$$\sigma Al - \sigma_{11}Al - \sigma_{21}Am - \sigma_{31}An = 0 \quad (50)$$

or

$$(\sigma - \sigma_{11})l - \sigma_{21}m - \sigma_{31}n = 0 \quad (51)$$

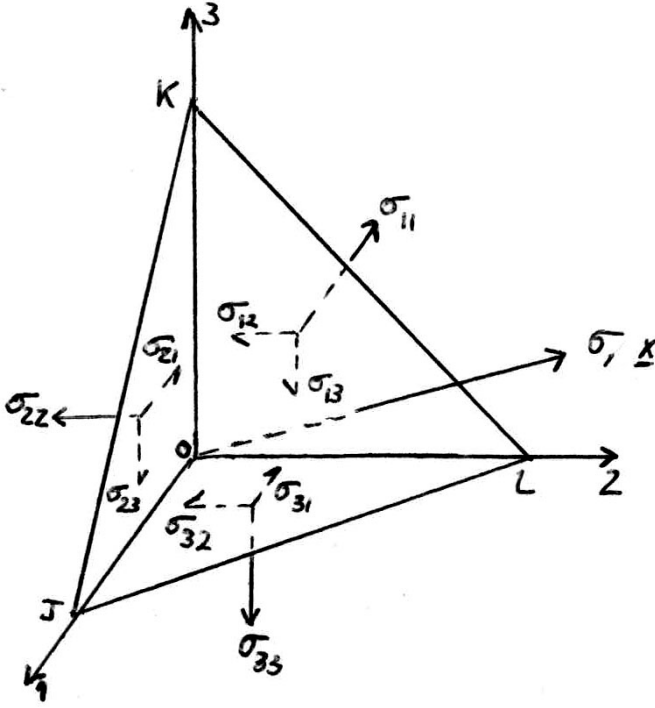


Figure 15: Inclined plane cut through the unit stress cube to give a principal stress σ along the x vector.

Similarly for the other two axes,

$$-\sigma_{12}l + (\sigma - \sigma_{22})m - \sigma_{32}n = 0 \quad (52)$$

$$-\sigma_{31}l - \sigma_{32}m + (\sigma - \sigma_{33})n = 0 \quad (53)$$

So we can write this set of simultaneous equations as (multiplying through by -1 for convenience and requiring that $\sigma_{ij} = \sigma_{ji}$)

$$\begin{pmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \quad (54)$$

The only nontrivial solution is where the determinant of the left-hand matrix is zero, so by comparison with Equation 36 we find that the solutions for σ are the eigenvalues of the stress matrix. This is given by

$$\begin{aligned} \sigma^3 - \sigma^2(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \sigma(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2) \\ - (\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2) = 0 \end{aligned} \quad (55)$$

The three roots of this equation are the principal stresses. Note that the three coefficients of this equation determine the principal stresses. Therefore these coefficients cannot change under a rotation of the coordinate axes and are *invariant*. These invariants are

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (56)$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 \quad (57)$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 \quad (58)$$

The first invariant we identify as 3 times the hydrostatic stress σ_{hyd} , which is the average of the σ_{ii} . When we come to consider yielding, the hydrostatic stress will assume a new significance. This also implies that this hydrostatic stress is the same, in any coordinate system.

3.5 Finding stresses for a general rotation

Having found the principal stresses, we now come to determine the stresses acting for some rotation of axes, where the relation between the *new* axes $i = 1, 2$ and 3 and the *old* axes are determined by the direction cosines a_{ij} between the new axes and the original $j = 1, 2$ and 3 axes, respectively, Figure 16.

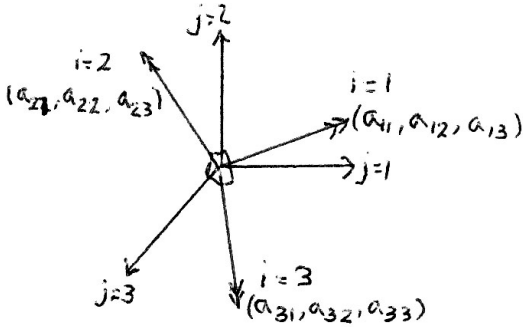


Figure 16: Rotation of axes between an old basis set j and new basis set i .

We find that for each new axis i , we can draw an inclined plane and set the equilibrium condition as before in order to find the new stresses σ'_{kl} , Figure 17.

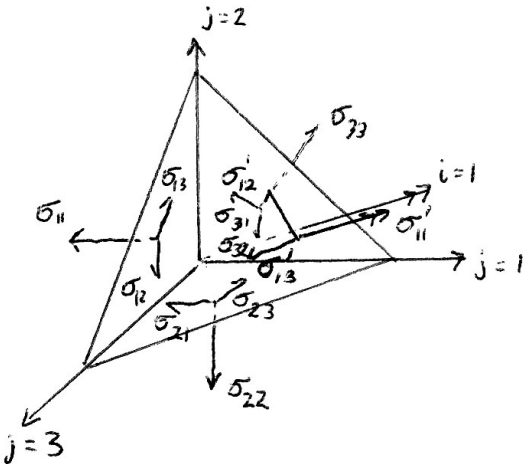


Figure 17: Inclined plane cut through the unit stress cube to give a stresses σ'_{1l} along the 1st new vector.

So now we can resolve the applied stresses along the normal to the plane and each of the perpendicular shears in the plane, as follows,

$$\begin{aligned} \sigma'_{kl} = & \sigma_{k1}a_{k1}a_{l1} + \sigma_{12}(a_{k1}a_{l2} + a_{k2}a_{l1}) \\ & + \sigma_{13}(a_{k1}a_{l3} + a_{k3}a_{l1}) + \sigma_{22}a_{k2}a_{l2} \\ & + \sigma_{23}(a_{k2}a_{l3} + a_{k3}a_{l2}) + \sigma_{33}a_{k3}a_{l3} \end{aligned} \quad (59)$$

which is the same as

$$\sigma' = a\sigma a^T \quad (60)$$

where T denotes the transpose, which for this *orthonormal* matrix is the same as the inverse. If you prefer suffix notation, this can be written

$$\sigma_{kl} = a_{ki}a_{lj}\sigma_{ij} \quad (61)$$

3.6 Example: Finding principal stresses

Question. A material is subject to the following stress state. What are the principal stresses in the material?

$$\sigma = \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \text{MPa} \quad (62)$$

Solution. We need to find the eigenvalues, so;

$$\begin{vmatrix} 100 - \lambda & 20 & 0 \\ 20 & 0 - \lambda & 20 \\ 0 & 20 & 100 - \lambda \end{vmatrix} = 0 \quad (63)$$

Which gives the cubic equation

$$(100 - \lambda) \{-\lambda(100 - \lambda) - 20^2\} - 20 \{20(100 - \lambda) - 0(20)\} + 0 \{20^2 - 0(-\lambda)\} = 0 \quad (64)$$

so

$$(100 - \lambda)(\lambda^2 - 100\lambda - 2 * 20^2) = 0 \quad (65)$$

which has solutions $\lambda = 100, 50 \pm 10\sqrt{17}$ MPa.

3.7 Example: Stresses on crystal axes

Question. A single crystal is loaded as follows on its [100], [010] and [001] axes. What are the normal stresses on the (orthonormal) $\frac{1}{\sqrt{3}}[111]$, $\frac{1}{\sqrt{2}}[1\bar{1}0]$ and $\frac{1}{\sqrt{6}}[11\bar{2}]$ axes?

$$\sigma = \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \text{MPa} \quad (66)$$

Solution. First we write down the rotation matrix a , which is the matrix with the vectors entered in the rows;

$$a = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \quad (67)$$

Then using Equation 59, we find that

$$\sigma' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}^T \quad (68)$$

So, performing the first matrix multiplication,

$$\sigma' = 20 \begin{pmatrix} \frac{6}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{6}{\sqrt{3}} \\ \frac{\sqrt{2}}{4} & \frac{1}{\sqrt{2}} & \frac{-9}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{4} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \quad (69)$$

Now we perform the second matrix multiplication to obtain;

$$\sigma' = 20 \begin{pmatrix} \frac{14}{\sqrt{9}} & \frac{4}{\sqrt{6}} & \frac{-4}{\sqrt{18}} \\ \frac{4}{\sqrt{6}} & \frac{3}{\sqrt{4}} & \frac{7}{\sqrt{12}} \\ \frac{-4}{\sqrt{18}} & \frac{7}{\sqrt{12}} & \frac{23}{\sqrt{36}} \end{pmatrix} = \begin{pmatrix} 93.3 & 32.7 & -18.9 \\ 32.7 & 30 & 40.4 \\ -18.9 & 40.4 & 76.7 \end{pmatrix} \text{ MPa} \quad (70)$$

The rotated matrix is still symmetric (this must be true, by theorem). When choosing the set of new basis vectors to use care must be taken to ensure that they are orthogonal to each other and of unit length (orthonormal).

Notice how it is possible, if computationally intensive, to find the resolved shear stress on any slip system, e.g. on the (111)[110] slip system (32.7 MPa). For plasticity, however, our consideration of yielding later in the course will force us to use, rather than the total stress tensor we have so far considered, the deviatoric stress tensor which has the hydrostatic stress subtracted. This is because the hydrostatic stress cannot cause yielding.

3.8 Stresses on a Single Plane

Fortunately, the case for a general rotation of axes can be simplified if we only want to find the stresses on a single plane. Here we simply find that the normal stress on a plane with unit normal vector $a_i = (a_1, a_2, a_3)$ is given by

$$\sigma_a = \sigma_{ij} a_i a_j \quad (71)$$

Example Lets re-do the previous example and find the normal stress on the {111} and $\{1\bar{1}0\}$ planes. The {111} will be given by

$$\sigma = \frac{20}{3} \{5 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + 5\} = 93.3 \text{ MPa} \quad (72)$$

and for the $\{1\bar{1}0\}$;

$$\sigma = \frac{20}{2} \{5 - 1 - 1\} = 30 \text{ MPa} \quad (73)$$

as before.

Chapter 4

Strain and Elasticity

4.1 Strains

Strain is defined as the distortion of the elemental cube, as opposed to a rigid body translation or rotation of the cube. Defining the displacement field $u = (u_1, u_2, u_3)$, we first assume that u is a linear function of position x , and so define the deformation tensor e as

$$e_{ij} = \frac{\partial u_i}{\partial x_j} \quad (74)$$

e.g. $e_{11} = \frac{\partial u_1}{\partial x_1}$, $e_{21} = \frac{\partial u_2}{\partial x_1}$, $e_{12} = \frac{\partial u_1}{\partial x_2}$, etc. However, consider the following three cases.

The first case in Figure 18, (a), $e_{12} = e_{21}$, shows a situation called *pure shear*. In (b), $e_{12} = -e_{21}$, the block is simply rotating and doesn't change shape at all. In (c) $e_{21} = 0$ and $e_{12} \neq 0$, which is called *simple shear*.

Therefore we define the strain tensor ε as the symmetric part of the deformation tensor e , and a separate rotation tensor ω which is the antisymmetric part of e . Thus

$$\varepsilon_{ij} = \frac{1}{2}(e_{ij} + e_{ji}), \quad \omega_{ij} = \frac{1}{2}(e_{ij} - e_{ji}) \quad (75)$$

Notice that $e = \varepsilon + \omega$. Thus, like stress, strain is by definition a symmetric tensor and has only 6 independent components.

There is a problem however! Conventionally, a shear strain is defined by the shear angle γ produced in simple shear, below.

So in this case the tensor shear strain $\varepsilon_{12} = \frac{1}{2}(e_{12} + e_{21}) = \frac{1}{2}(\gamma + 0) = \gamma/2$. This problem is simply one of definition. The notation used in each case is quite standard so this is easily overcome.

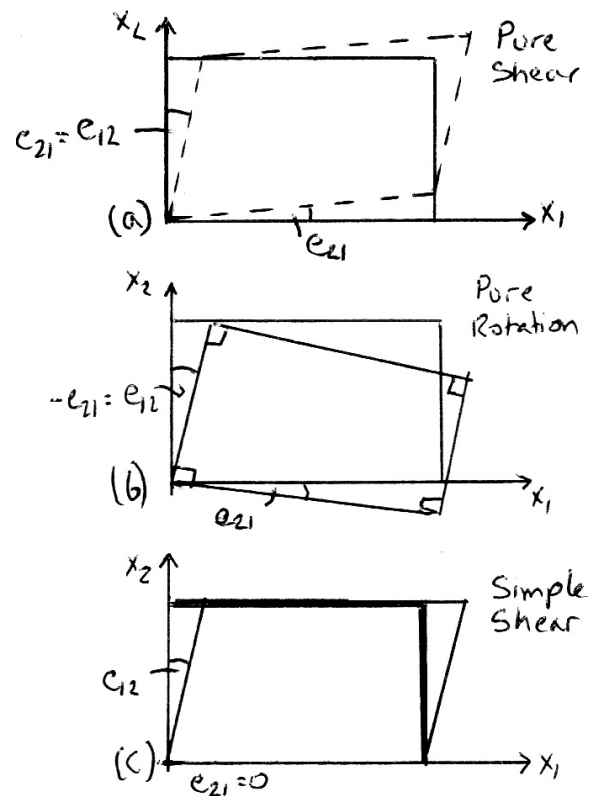


Figure 18: Different deformation states of an elemental cube.

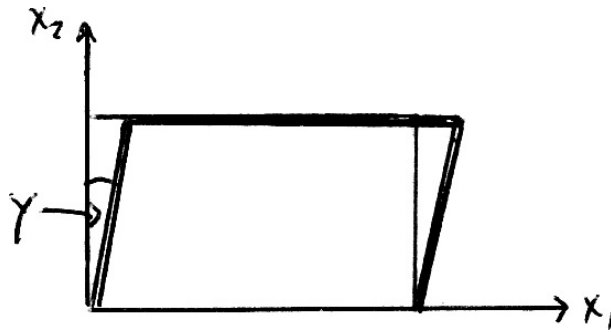


Figure 19: Definition of the simple shear strain γ .

4.2 Isotropic Elasticity

In general, very few materials are actually isotropic, even elastically. This is because single crystals are usually elastically and plastically anisotropic, and because most manufacturing processes produce some ‘uneven-ness’ in the orientation distribution, or texture (c.f. 203a). Therefore we need to consider, in the general case, how to convert from stress to strain and vice-versa.

Starting with the case of simple isotropic elasticity, the basic equations are based on Hooke’s Law, which in the general case gives

$$E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3) \quad (76)$$

where E is Young’s Modulus and ν is Poisson’s ratio. Stresses and strains are superposable, so we can combine stresses along different axes. In shear, a similar equation can be written, $\tau = G\gamma$, where G is the Shear Modulus. In tensor notation, because $\varepsilon_{12} = \gamma/2$, this gives

$$\sigma_{ij} = 2G\varepsilon_{ij} \quad (77)$$

However, it is easy to see that E , ν and G must be inter-related for an isotropic material, as follows. consider a system in a state of simple shear stress,

$$\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (78)$$

Using Mohr’s Circle, we can rotate this to find the Principal Stresses, which are that $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$,

So we can find the strains along the principal axes by using Hookes Law, so $E\varepsilon_1 = \sigma - \nu(-\sigma + 0) = \sigma(1 + \nu)$, and $E\varepsilon_2 = -\sigma - \nu(\sigma + 0) = -\sigma(1 + \nu)$. We can then use Mohr’s Circle for strain, in exactly the same way as for stress, Figure 21.

Since the rotation in Mohr’s circle is the same in the two cases, the strains must be equivalent and so ε_{12} in the original axes is given by

$$\varepsilon_{12} = \frac{\sigma}{E}(1 + \nu) \quad (79)$$

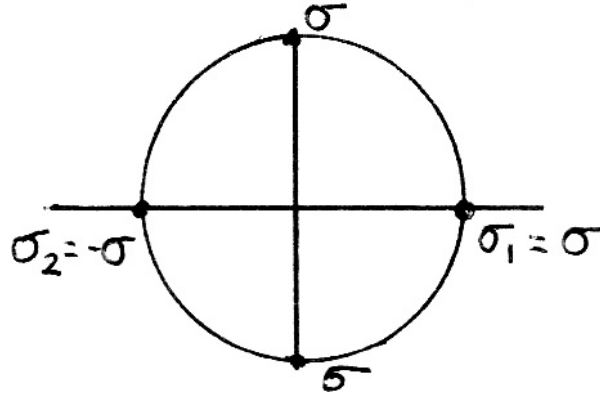


Figure 20: Principal stresses for applied shear σ .

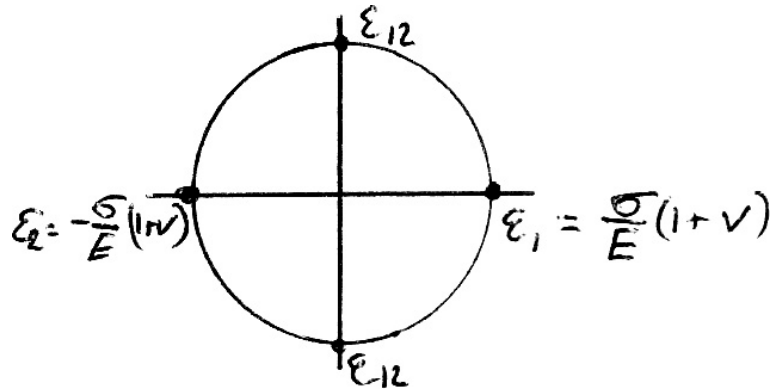


Figure 21: Determination of applied shear strains from the principal strains $\pm \frac{\sigma}{E}(1 + \nu)$.

By comparison with Equation 76 we can therefore say that

$$G = \frac{E}{2(1 + \nu)} \quad (80)$$

Hence for an isotropic material there are only two independent elastic constants.

We can also define other related Elastic constants that are useful in different stress states. The *Bulk Modulus* or dilatational modulus K is useful for hydrostatic stress problems, and is defined by

$$\sigma_H = K \frac{\Delta V}{V} = K \Delta \quad (81)$$

where σ_H is the hydrostatic stress defined by $\sigma_H = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$, and the hydrostatic strain or dilatation Δ is the change in volume $\Delta V/V$ which is given by $\Delta = \Delta V/V = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$. If we add three equations like $E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3)$ we get

$$E(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) = (1 - 2\nu)(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (82)$$

$$E\Delta = 3(1 - 2\nu)\sigma_H \quad (83)$$

It should be remembered that the values of the hydrostatic strain and stress are invariant. By comparison with Equation 74, we find

$$K = \frac{E}{3(1 - 2\nu)} \quad (84)$$

The Generalized Hooke's Law, Equation 69, gives strain in terms of stress. If we want to find stress in terms of strain, we find that

$$\sigma_{11} = \frac{E}{1+\nu}\varepsilon_{11} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \quad (85)$$

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda\Delta \quad (86)$$

where λ is *Lame's Constant*. The proof of Equation 78 is left to the student, and can easily be shown by solving the 3 simultaneous equations from the permutation of Hooke's Law.

4.3 Anisotropic Elasticity

In general, we define the relationship between the 2nd rank tensors for stress and strain using fourth rank tensors, as follows

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{and} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} \quad (87)$$

where the summation convention applies. C is called the Stiffness tensor and S the Compliance tensor (yes, they are the 'wrong way' around). We can therefore say that $C = S^{-1}$. This equation relays that a single stress can, in general, produce strains in all 9 strain components. Since ε and σ are symmetric, then C and S must also be symmetric;

$$C_{ijkl} = C_{jikl} \quad \text{and} \quad C_{ijkl} = C_{ijlk} \quad (88)$$

which means that instead of there being $3^4 = 81$ independent components, C and S have only 36 independent components. We can see this as follows

$$\begin{aligned} \sigma_{11} = & C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} + \\ & C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23} + \\ & C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33} \end{aligned} \quad (89)$$

So

$$\begin{aligned} \sigma_{11} = & C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} \\ & + 2C_{1112}\varepsilon_{12} + 2C_{1113}\varepsilon_{13} + 2C_{1123}\varepsilon_{23} \end{aligned} \quad (90)$$

However, we can go further than this. First, however, we need to reduce the magnitude of the imagination problem caused by the Fourth Rank tensors C and S . Since the stress components and strain components are independent, we can define a 6x1 stress vector, as follows,

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{pmatrix} \quad (91)$$

It is not possible to rotate this vector, but it does contain all the information of the stress matrix. Similarly for strain, we can define

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \varepsilon_1 & \frac{1}{2}\varepsilon_6 & \frac{1}{2}\varepsilon_5 \\ \frac{1}{2}\varepsilon_6 & \varepsilon_2 & \frac{1}{2}\varepsilon_4 \\ \frac{1}{2}\varepsilon_5 & \frac{1}{2}\varepsilon_4 & \varepsilon_3 \end{pmatrix} \quad (92)$$

Note that for strain, we have defined ε_4 , ε_5 and ε_6 such that they are equal to the simple shear strains γ . We can then relate stress and strain as follows

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \quad (93)$$

or, in matrix notation,

$$\sigma_i = C_{ij}\varepsilon_j \quad (94)$$

In this scheme, there is a direct correspondence between the 4-tensor and 2-tensor stiffnesses, e.g. $C_{mn} = C_{ijkl}$. For S , then it is a bit more complicated;

$$S_{mn} \begin{cases} S_{ijkl} & \text{both } m, n = 1, 2, 3, \\ 2S_{ijkl} & m \text{ or } n = 4, 5, 6, \\ 4S_{ijkl} & \text{both } m, n = 4, 5, 6. \end{cases} \quad (95)$$

Effectively, the 6-vector notation uses a set of basis vectors in stress-space to define a new set of co-ordinate axes. Other choices of the co-ordinate axes are possible, and even desirable in some cases (see Kock's, Tome and Wenk; Texture and Anisotropy, CUP).

It is possible to make an energy argument that shows that the 6x6 matrix C must be symmetric, and hence instead of 36 independent components the stiffness and compliance tensors can have only 21 independent components (c.f. Nye, Physical Properties of Crystals, OUP, p136), in a general anisotropic elastic crystal.

In a cubic crystal, the potential exists to reduce this considerably. Shear on any face of the crystal must produce the same effect, as must normal stress. Similarly, the Poisson contraction must be the same in each direction. Finally, shear on one axis cannot produce shear strain on another, and shears cannot produce normal stresses and vice versa (see Nye). It does not matter which axes one considers, this must still be correct. Therefore we can reduce C to a matrix given by

$$C_{ij} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix} \quad (96)$$

Thus, in general, cubic crystals have only 3 independent stiffness constants (and 3 compliances). Conventionally, these are referred to the $[x00]$ crystal axes. Similar arguments can be used to show that a hexagonal crystal has only 5 independent constants.

4.4 Strain Measurement

In principle, it is difficult to measure stresses. The stress is, effectively, the strain energy per unit volume, and as such it is only possible to infer it from the forces required when the stress state is changed. In practice, it is easier to measure strains, and then to convert these to stresses.

There are many methods available for measuring strains, either directly by measuring the surface movement produced when material is removed, or by using some change in the properties associated with strain, such as resistance or the speed of sound. The most direct method, however, lies in measuring the interatomic spacing, since it is expansion of the lattice that gives rise to the restoring force that is Stiffness, and since the lattice spacing is directly the elastic strain. Diffraction provides a convenient tool for measuring lattice spacings, as given by Bragg's Law

$$\lambda = 2d \sin \theta \quad (97)$$

where λ is the wavelength of the radiation used, d is the interatomic spacing of the plane used and θ is the diffraction angle. The experimental setup is shown in Figure 22.

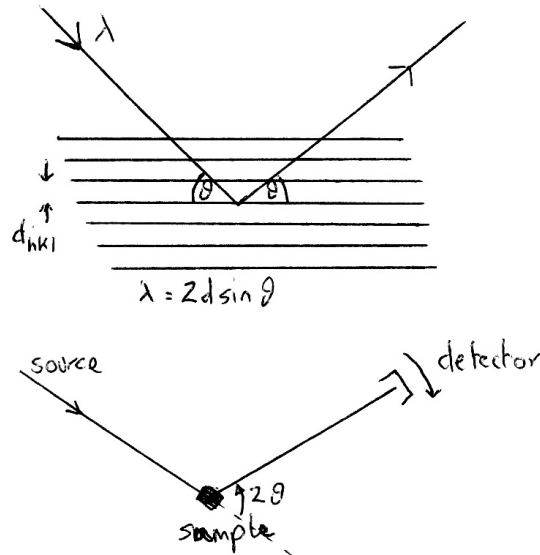


Figure 22: Diffraction geometry - (top) constructive interference condition to produce a diffraction beam and hence derive Bragg's Law, (bottom) real-space diffractometer layout.

If we define the elastic strain ε as the change in lattice spacing, then

$$\varepsilon = \frac{\partial d}{d} \quad (98)$$

we can therefore see that, by differentiating the Bragg equation,

$$\partial \lambda_n = 0 = 2d \sin \theta + 2d \partial \theta \cos \theta \quad (99)$$

where θ is half the diffraction angle, d is the interplanar spacing of the diffracting $\{hkl\}$ plane and λ_n is the neutron wavelength. So

$$\varepsilon = \frac{\partial d}{d} = -\partial \theta \cot \theta \quad (100)$$

This provides a convenient basis for finding states of stress in a material.

4.5 Example: Stress near a weld

Question The state of stress near the weld in a welded thin sheet of aluminium is measured using a neutron diffractometer. At a nominal diffraction angle of 90° , a change in diffraction angle compared to an unstrained reference of 0.12° is measured in the welding direction and a change of 0.06° is measured in the transverse direction. Assuming that these two axes are principal axes and that the stiffness of the material is $E = 70$ GPa and that $\nu = 0.3$, find the stress state.

[Note that with a good diffractometer diffraction peak position uncertainties on the order of 0.003° are routinely obtainable. This translates into a strain uncertainty of around 50×10^{-6}].

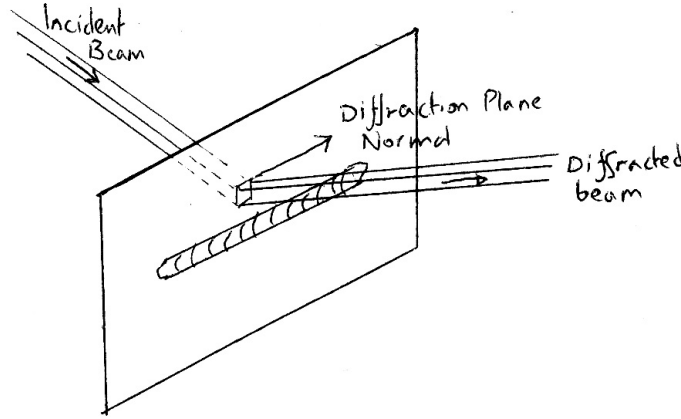


Figure 23: Arrangement of a weld sample in the diffractometer for the measurement of longitudinal strain in the welding direction.

Answer First, we remember that the diffraction angle measured is 2θ , not θ , and that we should always calculate in radians. Then, we can find the strain in the longitudinal (l) and transverse directions (t), remembering that $\tan 90 = 1$;

$$\varepsilon_l = \frac{1}{2} 0.12 * \pi / 180 = 0.001 \quad (101)$$

$$\varepsilon_t = \frac{1}{2} 0.006 * \pi / 180 = 0.0005 \quad (102)$$

We know that $2G = E/(1+\nu) = 53.8$ GPa and that Lamé's constant $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 40.4$ GPa. Also,

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \quad (103)$$

So then

$$\begin{aligned} \sigma_l &= 53.8 * 0.001 + 40.4(0.001 + 0.0005 + \varepsilon_n) \\ \sigma_t &= 53.8 * 0.0005 + 40.4(0.001 + 0.0005 + \varepsilon_n) \\ \sigma_n &= 53.8 * \varepsilon_n + 40.4(0.001 + 0.0005 + \varepsilon_n) \end{aligned} \quad (104)$$

(all in GPa). However, since the sheet is thin, then the stress in the normal (n) direction must be zero, so we can rearrange the last equation to find $\varepsilon_n = \frac{-40.4 * 0.0015}{53.8 + 40.4} = -0.00064$. Then $\Delta = \sum \varepsilon_{ii} = 0.00085$, and the other two stresses are given by $\sigma_l = 0.0538 + 40.4 * 0.00085 = 88$ MPa and $\sigma_t = 61$ MPa.

Chapter 5

Yield Surfaces and Yield Criteria

Generally, we would like to be able to predict whether a material will yield when it is subjected to some biaxial or triaxial stress state. Most engineering components are subjected to complex stress states in practice and so this subject is of key importance. We know that plastic flow occurs by dislocation movement in most cases and that this happens when a critical resolved shear stress is exceeded in the single crystal. Therefore the most obvious criterion to use for yielding is when the maximum shear stress (radius of Mohr's circle) exceeds a critical value; this is called the *Tresca* yield criterion.

5.1 Tresca Criterion

If we consider the *Principal Stresses* σ_1 , σ_2 and σ_3 , where these are quoted in descending order ($\sigma_1 > \sigma_2 > \sigma_3$), then by taking Mohr's circle in any of the three dimensions, the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \quad (105)$$

Let us define σ_y to be the yield stress in uniaxial tension. Then considering the case of uniaxial tension, $\sigma_1 = \sigma_y$, and $\sigma_2 = \sigma_3 = 0$. Therefore from Equation 104 we can say that

$$\tau_{\max} = \sigma_y/2 \quad (106)$$

By comparing Equations 104 and 105, we can therefore define the Tresca criterion as

$$\sigma_1 - \sigma_3 = \sigma_y \quad (107)$$

For the case of pure torsion or shear, then $\sigma_{11} = \sigma_{22} = 0$ and the stress matrix looks like $\begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}$. Using Mohr's circle we can see that $\sigma_1 = -\sigma_3 = \tau$ and $\sigma_2 = 0$. Therefore at yield, using the Tresca criterion, $\sigma_1 - \sigma_3 = \tau - -\tau = 2\tau = \sigma_y$, so the yield stress in pure shear is equal to half the tensile yield stress.

5.2 Von Mises Criterion

Another yield criterion is the *Von Mises* criterion, which is based on a stress invariant (for examples, see *e.g.* Equation 55-57). This defines the Effective, Equivalent or Von Mises stress σ_{VM} to be

$$2\sigma_{VM}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \quad (108)$$

Note that it is based on the Principal Stresses only. Thus, the Von Mises yield stress is equal to the yield stress in uniaxial tension where $\sigma_y = \sigma_1$ and $\sigma_2 = \sigma_3 = 0$. In this case, the equation is

$$2\sigma_{VM}^2 = (\sigma_1 - 0)^2 + (0 - 0)^2 + (0 - \sigma_1)^2 = 2\sigma_y^2 \quad (109)$$

so at yield $\sigma_{VM} = \sigma_y$.

For pure torsion, we can again define $\tau = \sigma_1 = -\sigma_3$ and $\sigma_2 = 0$ so

$$2\sigma_{VM}^2 = (\tau - 0)^2 + (0 + \tau)^2 + (-\tau - \tau)^2 = 6\tau^2 \quad (110)$$

so at yield $\sigma_y = \sigma_{VM} = \sqrt{3}\tau$. Therefore according to the Von Mises yield criterion, the yield stress in torsion is $1/\sqrt{3} = 0.577$ times the yield stress in tension.

5.3 Hydrostatic and Deviatoric Stresses

Since slip does not change the volume of a material, it does not relieve any stress; hence it is not possible for a *hydrostatic*, or pressure, stress to cause yielding. One can see this quite simply in both the Tresca and Von Mises criteria - there if the stress state is simply composed of a pressure that is the same in all directions so $\sigma_1 = \sigma_2 = \sigma_3$, then both the maximum shear stress (Tresca) and Von Mises stress are zero. We define the hydrostatic stress σ_H as the average of the three normal stresses;

$$\sigma_H = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (111)$$

This quantity is invariant to rotations, Equation 55. We can always divide a stress matrix into a *deviatoric* component, σ' , and a hydrostatic one;

$$\sigma = \sigma' + \sigma_H I \quad (112)$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} - \sigma_H & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_H & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_H \end{pmatrix} + \begin{pmatrix} \sigma_H & 0 & 0 \\ 0 & \sigma_H & 0 \\ 0 & 0 & \sigma_H \end{pmatrix} \quad (113)$$

The deviatoric stress is the component of the normal stress state that causes shape change rather than just expansion / contraction of the unit cube. The components of the deviatoric stress state we denote σ'_{ij} and the principal deviatoric stresses σ'_1 , σ'_2 and σ'_3 .

The deviatoric stress is the stress that is capable of producing plastic deformation, and the corresponding deviatoric strains are the strains produced by plastic deformation. You can now calculate the applied deviatoric shear stress on any slip system in a single crystal for any applied stress state, and hence if you know the CRSS you can determine whether or not that crystal will yield.

Finally, note that the Tresca and Von Mises criteria can be applied to either the full or deviatoric stress tensor, because the subtraction in each makes the hydrostatic stress subtraction irrelevant. Also note that the sum of the normal deviatoric stresses (the *trace* of the matrix) is zero, by definition.

5.4 Graphical Representation of Yield Criteria

The Von Mises criterion we can visualise for the case where $\sigma_3 = 0$, which reduces the Von Mises criterion to

$$\sigma_1^2 + \sigma_2^2 + (\sigma_1 - \sigma_2)^2 = 2\sigma_y^2 \quad (114)$$

Therefore

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2 \quad (115)$$

This is the equation of an ellipse. Therefore, in 2D, the Von Mises criterion yield locus is an ellipse. Similarly, the Tresca yield criterion is a distorted hexagon in 2D, Figure 24.

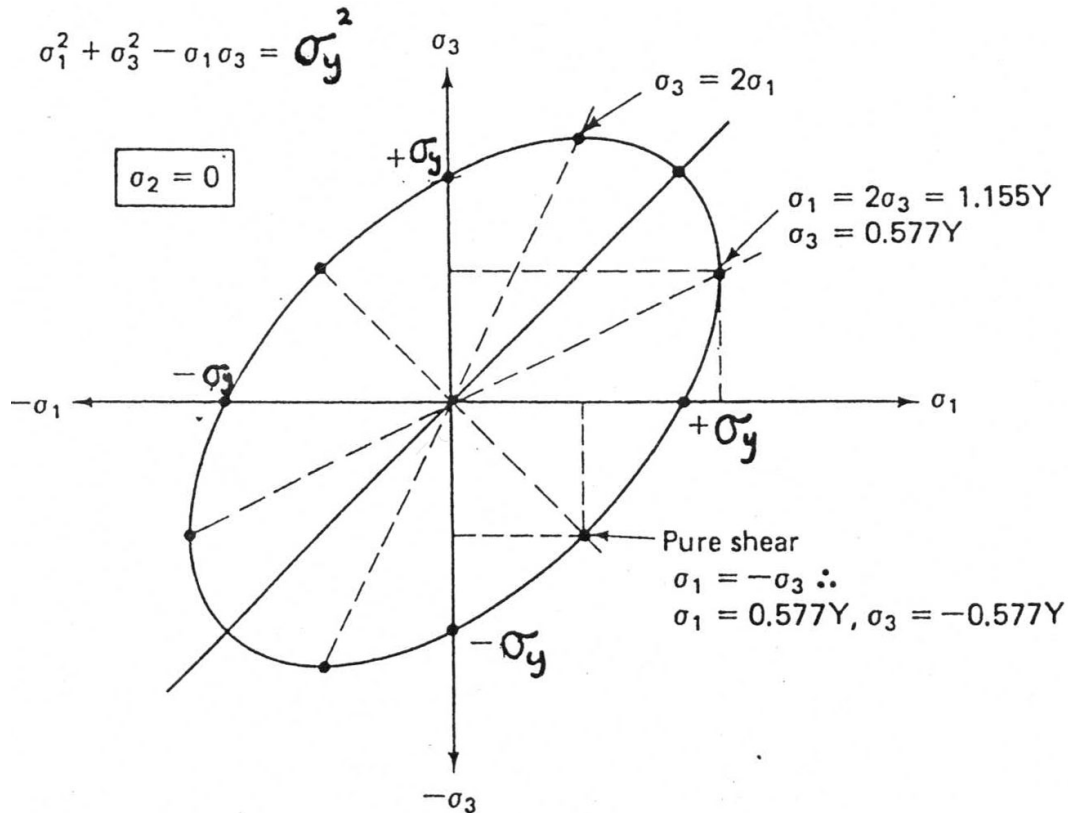


Figure 24: Von Mises yield locus, in 2D.

In 3D, the Tresca and Von Mises yield loci describe a hexagonal prism and a cylinder, with the long axes along the hydrostatic stress axis, Figure 26.

Notice how this is actually a 2D locus wrapped around the hydrostatic stress axis. This makes sense, because once one degree of freedom is removed (the hydrostatic stress), then in principal stress space only 2 remain. The loci can therefore be visualised very simply by looking at the plane with its normal parallel to the hydrostatic stress axis, which is called the π -plane projection. In this project, the Von Mises criterion appears as a circle and Tresca as a hexagon, Figure 25.

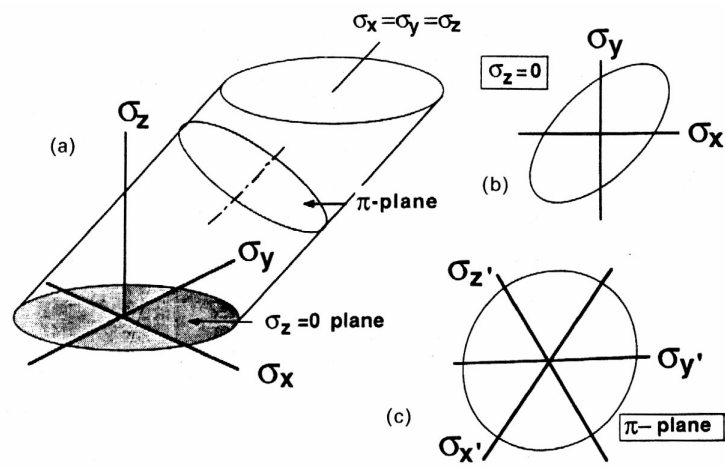


Figure 25: The Von Mises yield surface and π -plane projection.

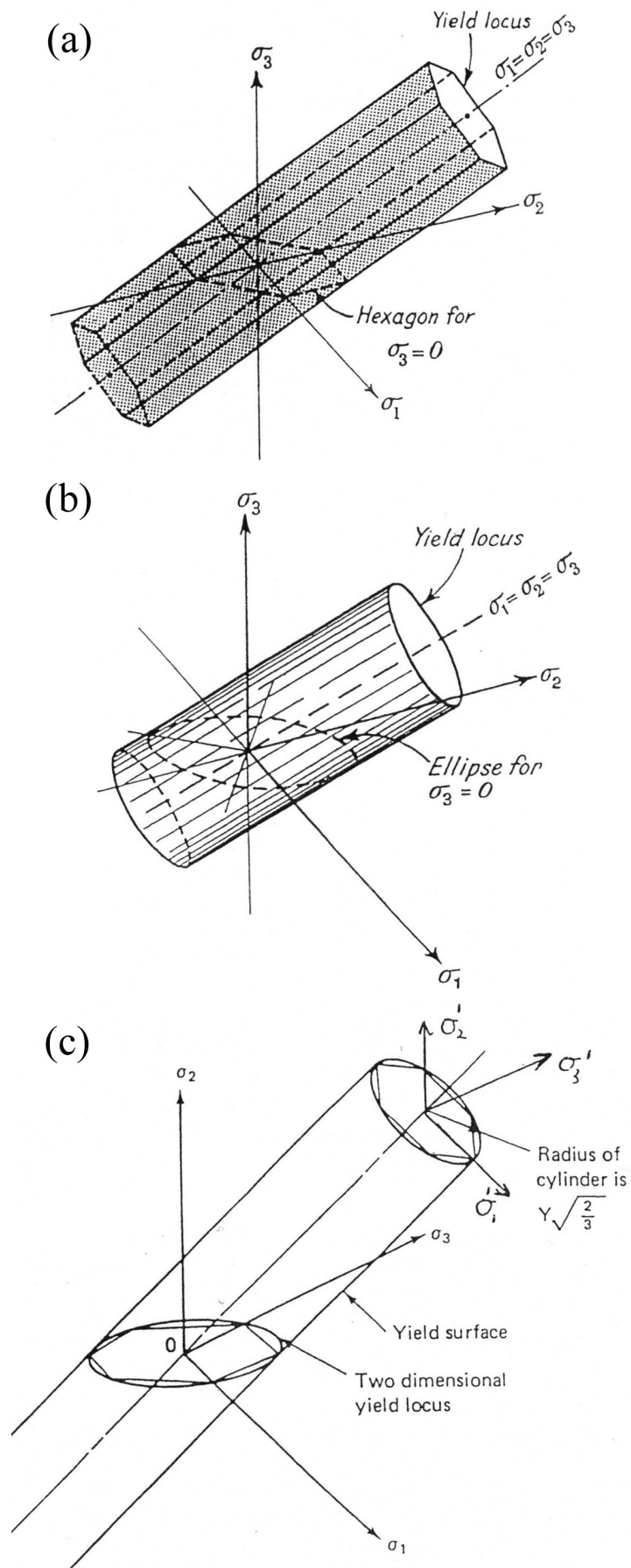


Figure 26: Arrangement of a weld sample in the diffractometer for the measurement of longitudinal strain in the welding direction.