Name: ____________________

Section: __________ **Recitation Instructor**: ____________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ____________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Evaluate the following limits:

   (a) (3 points) $\lim_{x\to 2} \frac{x^2-4}{x^2-2x}$

   Solution:

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{x^2-2x} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x(x-2)} \\ &= \lim_{x\to 2} \frac{(x+2)}{x} = \frac{4}{2} = 2\end{aligned}$$

   (b) (3 points) $\lim_{x\to 3} \frac{(x-4)(x+2)}{x-3}$

   Solution: The limit DNE since

$$\lim_{x\to 3^+} \frac{(x-4)(x+2)}{x-3} = -\infty \qquad \lim_{x\to 3^-} \frac{(x-4)(x+2)}{x-3} = \infty$$

2. (6 points) Let $f(x) = \frac{1}{x+3}$. Calculate $f'(x)$ using the **definition** of the derivative.

   Solution:

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+3-(x+h+3)}{(x+h+3)(x+3)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(x+h+3)(x+3)}}{h} \\ &= \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} = \frac{-1}{(x+3)^2}\end{aligned}$$

3. Find the derivative for each of the following functions:

   (a) (3 points) $f(x) = 3x^2 \sec x$

Solution:

$$f'(x) = (6x)(\sec x) + (3x^2)(\sec x \tan x)$$

   (b) (3 points) $g(x) = (x + \sin x)^3$

Solution:

$$g'(x) = 3(x + \sin x)^2 \cdot (1 + \cos x)$$

4. (6 points) Find an equation of the tangent line to the curve $y^2 = 3xy + x^3$ at the point $(4, -4)$.

Solution:

$$\begin{aligned} 2y \cdot y' &= 3y + 3x \cdot y' + 3x^2 && \text{(plug in } x = 4,\ y = -4\text{)} \\ 2(-4) \cdot y' &= 3(-4) + 3(4) \cdot y' + 3(4)^2 \\ -8y' &= -12 + 12y' + 48 \\ -20y' &= 36 \\ y' &= -9/5 \end{aligned}$$

so the tangent line can be given by the equation $y + 4 = -\frac{9}{5}(x - 4)$.

5. Evaluate the following integrals:

   (a) (3 points) $\displaystyle\int \left(\sec^2 x - 5\sqrt[3]{x^2} + \frac{1}{x^2}\right)\,dx$

   Solution:

$$\begin{aligned}\int \left(\sec^2 x - 5\sqrt[3]{x^2} + \frac{1}{x^2}\right)\,dx &= \int \left(\sec^2 x - 5x^{2/3} + x^{-2}\right)\,dx \\ &= \tan x - 3x^{5/3} - x^{-1} + C\end{aligned}$$

   (b) (4 points) $\displaystyle\int \frac{x^4}{\sqrt{3+2x^5}}\,dx$

   Solution: Set $u = 3 + 2x^5$ so then $du = 10x^4\,dx$

$$\begin{aligned}\int \frac{x^4}{\sqrt{3+2x^5}}\,dx &= \int \frac{\frac{1}{10}\,du}{\sqrt{u}} \\ &= \frac{1}{10}\int u^{-1/2}\,du \\ &= \frac{1}{5}u^{1/2} + C \\ &= \frac{1}{5}\sqrt{3+2x^5} + C\end{aligned}$$

6. (5 points) Solve the following initial value problem:

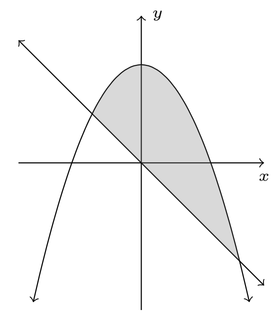
$$f'(x) = \sin x + 3x^2, \qquad f(0) = 2$$

Solution:

$$\begin{aligned}f(x) &= -\cos x + x^3 + C \\ f(0) &= -\cos(0) + 0 + C \\ 2 &= -1 + C \\ 3 &= C\end{aligned}$$

so therefore $f(x) = -\cos x + x^3 + 3$

7. (6 points) Find the area of the region bounded by the curves $y = 2 - x^2$ and $y = -x$.



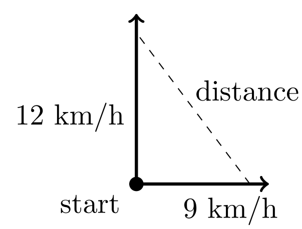
Solution: The curves intersect at

$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 \\ 0 &= (x-2)(x+1) \\ x &= -1,\ 2 \end{aligned}$$

and $2 - x^2 \geq -x$ on $[-1, 2]$ so the area is given by

$$\begin{aligned} A &= \int_{-1}^{2} (2 - x^2 + x)dx \\ &= \left[2x - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_{-1}^{2} \\ &= \left[4 - \tfrac{8}{3} + 2\right] - \left[-2 + \tfrac{1}{3} + \tfrac{1}{2}\right] = \frac{9}{2} \end{aligned}$$

8. (6 points) Two boats start sailing from the same point. One boat travels north at 12 km/h and the other travels east at 9 km/h. At what rate is the distance between the boats increasing two hours later?



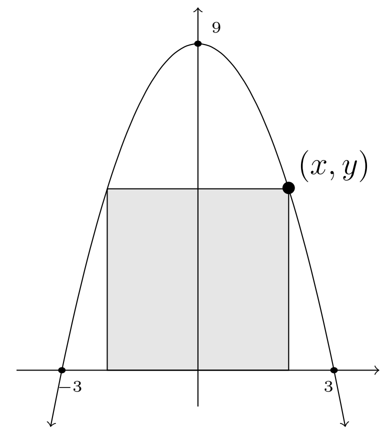
Solution: If we take $y(t)$ to be the distance (in km) from the starting point of the north sailing boat, $x(t)$ the distance (in km) from the starting point of the east sailing boat, and $d(t)$ the distance between the boats where t is in hours then we have:

$$\begin{aligned} [x(t)]^2 + [y(t)]^2 &= [d(t)]^2 \qquad y'(t) = 12 \qquad x'(t) = 9 \\ 2x(t)x'(t) + 2y(t)y'(t) &= 2d(t)d'(t) \\ 2(18)(9) + 2(24)(12) &= 2(30)d'(2) \end{aligned}$$

since $x(2) = 18$, $y(2) = 24$, and $d(2) = 30$.

$$\begin{aligned} 225 &= 15 \cdot d'(2) \\ 15 \text{ km/h} &= d'(2) \end{aligned}$$

9. (6 points) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 9 - x^2$. Find the maximum area of such a rectangle.



Solution:

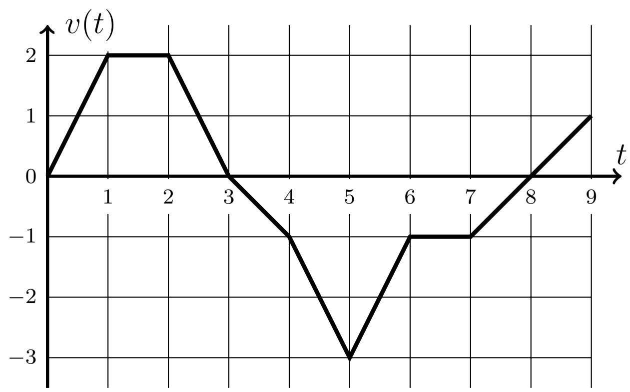
$$\begin{aligned} A &= 2xy \\ A(x) &= 2x(9-x^2), \qquad x \in [0,3] \\ A(x) &= 18x - 2x^3 \\ A'(x) &= 18 - 6x^2 \end{aligned}$$

So there is a critical point in the domain at $x = \sqrt{3}$.
Using the closed interval method we have

$$A(0) = 0 \qquad A(3) = 0 \qquad A(\sqrt{3}) = 12\sqrt{3}$$

so the maximum area of such a rectangle is $A = 12\sqrt{3}$.

10. The graph to the right shows the **velocity** (in m/s) of a particle moving on a horizontal line, for t (in sec) in the closed interval $[0, 9]$. Use this graph to answer the following questions. *Your answers should be in interval notation.*



(a) (1 point) What is the particle's maximum speed?

Solution: $|v(5)| = |-3| = 3$ m/s

(b) (1 point) When is the particle slowing down?

Solution: When speed is decreasing. That is $t \in (2,3) \cup (5,6) \cup (7,8)$ s.

(c) (1 point) When is the particle's acceleration positive?

Solution: When $v'(t) > 0$. That is $t \in (0,1) \cup (5,6) \cup (7,9)$ s.

(d) (1 point) When is the particle moving in the negative direction?

Solution: When $v(t) < 0$. That is $t \in (3,8)$ s.

(e) (2 points) What is the total distance traveled by the particle from $t = 0$ to $t = 4$?

Solution:

$$\int_0^4 |v(t)|\, dt = 4.5 \text{ m.} \qquad \text{(think: add up area from 0 to 4, count all positively)}$$

Multiple Choice. Circle the best answer. No work needed.
No partial credit available. No credit will be given for choices not clearly marked.

11. (3 points) Use Newton's method to approximate a solution of the equation $x^3 = 20 - x$ starting with $x_1 = 2$. Then $x_2 =$?

A. $2 - \frac{2}{3}$

B. $2 - \frac{5}{2}$

C. 20

D. $2 + \frac{10}{13}$

E. None of the above

12. (3 points) What is the average value f_{ave} of $f(x) = x^2$ on the interval [0,4]?

A. $\frac{64}{3}$

B. $\frac{16}{3}$

C. $\frac{16}{2}$

D. $\frac{15}{2}$

E. 16

13. (3 points) The height (in meters) of a ball thrown straight into the air is given by the function $h(t) = -t^2 + 4t + 5$ for $t \geq 0$. What is the speed of the ball when it hits the ground?

A. 0 m/s

B. 2 m/s

C. -2 m/s

D. 5 m/s

E. 6 m/s

Use the information following information to answer questions 14, 15, and 16:

$$f(x) = \frac{2x^2}{x-4}, \qquad f'(x) = \frac{2x(x-8)}{(x-4)^2}, \qquad f''(x) = \frac{64}{(x-4)^3}$$

14. (3 points) Identify the critical points of f and determine whether each is a local maximum, local minimum, or neither.

   A. Critical Points: $x = 0$ (local max), $x = 4$ (local min)

   B. Critical Points: $x = 0$ (local max), $x = 4$ (neither)

   C. Critical Points: $x = 0$ (local max), $x = 8$ (local min)

   D. Critical Points: $x = 0$ (local max), $x = 4$ (neither), $x = 8$ (local min)

   E. Critical Points: $x = 0$ (local max), $x = 4$ (local min), $x = 8$ (local min)

15. (3 points) Identify the intervals over which f is increasing and/or decreasing.

   A. Increasing: $(-\infty, 0] \cup [8, \infty)$; Decreasing: $[0, 8]$

   B. Increasing: $(-\infty, 0] \cup [8, \infty)$; Decreasing: $[0, 4) \cup (4, 8]$

   C. Increasing: $[0, 4) \cup (4, 8]$; Decreasing: $(-\infty, 0] \cup [8, \infty)$

   D. Increasing: $(-\infty, 4]$; Decreasing: $[4, \infty)$

   E. Increasing: $[4, \infty)$; Decreasing: $(-\infty, 4]$

16. (3 points) Identify the intervals over which f is concave up and/or concave down.

   A. Concave up: $(-\infty, 4) \cup (4, \infty)$; Concave down: NONE

   B. Concave up: NONE; Concave down: $(-\infty, 4) \cup (4, \infty)$

   C. Concave up: $(-\infty, 0) \cup (8, \infty)$; Concave down: $(0, 4) \cup (4, 8)$

   D. Concave up: $(0, 4) \cup (4, 8)$; Concave down: $(-\infty, 0) \cup (8, \infty)$

   E. Concave up: $(4, \infty)$; Concave down: $(-\infty, 4)$

17. (3 points) Calculate: $\displaystyle\sum_{i=1}^{100} 2i + 3$

A. 203

B. 10103

C. 10400

D. 20203

E. 20500

18. (3 points) What is the derivative of $f(x) = \dfrac{\sin x}{x^2}$

A. $\dfrac{\cos x}{2x}$

B. $2x \sin x + x^2 \cos x$

C. $2x \sin x - x^2 \cos x$

D. $\dfrac{x \cos x - 2 \sin x}{x^3}$

E. $\dfrac{2 \sin x - x \cos x}{x^3}$

19. (3 points) Using three rectangles of equal width, find the upper sum approximation (overestimate) of the area between the curve $y = 16 - x^2$ and the x-axis from $x = -4$ to $x = 2$.

A. 40

B. 44

C. 80

D. 88

E. 96

20. (3 points) Let $f(x) = \begin{cases} x^2, & x \leq 2 \\ 2x, & x > 2 \end{cases}$

Which of the following statements is true?

A. The function f is undefined at $x = 2$.

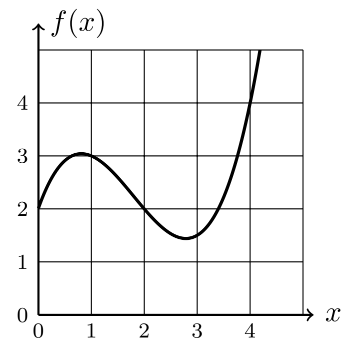
B. The function f is neither continuous nor differentiable at $x = 2$.

C. The function f is continuous, but not differentiable, at $x = 2$.

D. The function f is differentiable, but not continuous, at $x = 2$.

E. The function f is continuous and differentiable at $x = 2$.

21. (3 points) Consider the function $f(x)$ given by the graph on the right. Find a value c that satisfies the conclusion of the MVT on the interval $[1, 4]$.



A. $c = \frac{1}{3}$

B. $c = 1$

C. $c = 2$

D. $c = 3$

E. $c = 4$

22. (3 points) The graph of a function $f'(x)$ is shown below. If $f(0) = 2$, what is the value of $f(4)$?

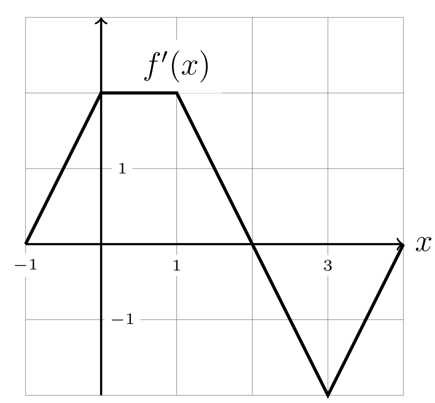
A. 0

B. 2

C. 3

D. 5

E. 6



More Challenging Question(s). Show all work to receive credit.

23. (4 points) Calculate the derivative of the function $f(x) = \sin(x^2 \cos x)$.

Solution: Think chain rule with a product rule inside:

$$f'(x) = \cos\left(x^2 \cos x\right) \cdot \left(2x \cos x - x^2 \sin x\right)$$

24. The following table lists the velocity $v(t)$ of a particle (in m/s):
Note that the time intervals are not equally spaced.

t (in sec)	0	1	3	4	6
v(t) (in m/s)	4	3	11	15	13

(a) (4 points) Estimate the acceleration of the particle at $t = 3$.

Solution: Acceleration is the rate of change of velocity. To approximate instantaneous rate of change (at $t = 3$) we can use average rate of change over a short interval nearby. I will use:

$$a(3) = v'(3) \approx \frac{v(4) - v(3)}{4 - 3}$$
$$\approx \frac{15 - 11}{1} \approx 4 \text{ m/s}^2$$

(b) (4 points) Estimate the total distance traveled by the particle from $t = 0$ to $t = 6$.

Solution:

$$\begin{aligned}\int_0^6 |v(t)|\, dt &= \int_0^6 v(t)\, dt && \text{(since } v(t) > 0 \text{ on this interval)}\\ &= (1 \text{ sec})(3 \text{ m/s}) + (2 \text{ sec})(11 \text{ m/s}) + (1 \text{ sec})(15 \text{ m/s}) + (2 \text{ sec})(13 \text{ m/s})\\ &= 3 \text{ m} + 22 \text{ m} + 15 \text{ m} + 26 \text{ m}\\ &= 66 \text{ m}\end{aligned}$$

This is using a right-hand sum. Other methods are also acceptable.