MTH 2311 Linear Algebra
Week 1 Resources

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Major Topics:

1. Systems of Linear Equations
2. Row Reductions and Echelon Forms
3. Vector and Matrix Equations

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 1.1-1.5

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1 Tutor Remarks:

Linear Algebra is a complex subject- sometimes it is made unnecessarily complex, and at other times it is drastically oversimplified. The upper division students who take this class are often driven to learn it well, and tend to ask more constructive and conceptual questions than in other math courses. In order to succeed in Linear Algebra, it is crucial that one has an understanding of how to interpret its major results from multiple perspectives. Thus, the focus of these resources will not necessarily be on the *calculations* that one can do with Linear Algebra, but the rather the *concepts* that it encompasses. Ultimately, my hope is that these resources will challenge you to see this discipline from a variety of perspectives, so that you will be best equipped to succeed in this intellectually demanding course.

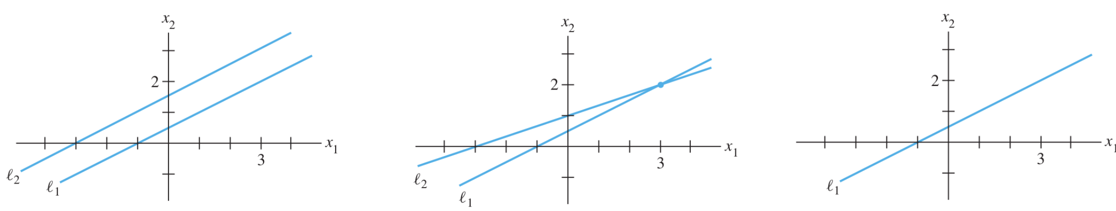
– Colin Burdine

1.1 Systems of Linear Equations

Fundamentally, Linear Algebra (as its name suggests) is concerned with linear equations. In particular, if we impose several linear equations and group them together, we call them a *system*. Consider the system of 2D linear equations below:

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$$

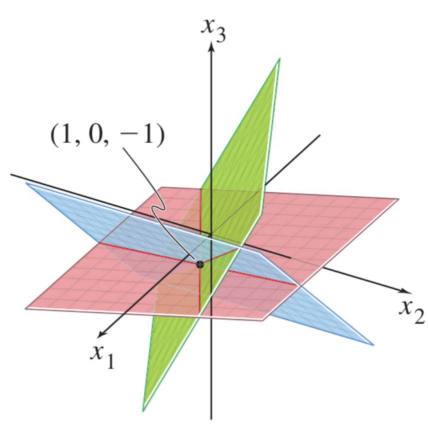
Note that if we were to graph these functions or solve the system using one of the methods we know from pre-calculus, we would see that the system has a single unique solution at the point $(0.5, 0.5)$. However, not all systems of equations have a single unique solution. Broadly speaking every system can be funneled into one of three categories, systems with *no solutions*, systems with *one unique solution*, and systems with *infinitely many solutions*:



Sytems with No Solution, One Solution, and Infinitely Many Solutions

Contrary to what the term 'Linear' seems to suggest, as we increase the number of variables, recall that in a 3-variable system, we are not working with lines in 3D space, but rather planes, such as we see below:

$$\begin{cases} x + 2y + z = 0 \\ 2y - 8z = 8 \\ 5x - 5z = 10 \end{cases}$$



(Image from Lay and McDonald *Linear Algebra and Its Applications, 5th Edition*, 2016)

To minimize the ammout of writing we do when solving linear systems, we usually put them in *matrix* form or *augmented matrix* form. Note that when doing this, it is common for students to initially see matrices as a pnemonic or organization tool- be careful not to give the wrong impression when explaining why this is done. As they will find out later in the course, a matrix is more than simply an organized grid of numbers. Usually this is represented in one of two ways:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \quad \textit{or} \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -8 & -8 \\ 5 & 0 & -1 & -10 \end{bmatrix}$$

Augmented Matrix notation and Homogeneous Matrix notation

While the textbook tends to use augmented matrix notation more often, it is good to expose students to both. When using augmented matrices, be sure to remind students that the vertical bar is useful for signifying equality. If the bar is omitted, sometimes, students can confuse the matrix for its homogeneous form and have the signs in their rightmost columns flipped.

1.2 Row Reductions and Echelon Forms

Row reduction is one of the hardest concepts for students to memorize because it is a complex algorithm that requires attention to detail to perform correctly. The reason we perform row reduction is to make solving systems of equations easier. There are three forms that can be generated from row reduction algorithms. These are *echelon form reduced echelon form*, and *reduced row-echelon form*. We will only be concerning ourselves with the first two forms this week. The forms are designed to allow one to solve a system of equations by performing row operations and then substituting the resulting quantities back to solve for the value of each variable (if it exists). Examples of these two forms in a 4x4 system are below:

$$\left[\begin{array}{cccc|c} 2 & ? & ? & ? & ? \\ 0 & 5 & ? & ? & ? \\ 0 & 0 & 0 & 6 & ? \\ 0 & 0 & 0 & 0 & ? \end{array}\right] \quad \textit{and} \quad \left[\begin{array}{cccc|c} 1 & ? & ? & ? & ? \\ 0 & 1 & ? & ? & ? \\ 0 & 0 & 0 & 1 & ? \\ 0 & 0 & 0 & 0 & ? \end{array}\right]$$

Examples of Echelon and Reduced Echelon form

In the interest of keeping this resource brief, I will not be going over the the algorithm for row-reducing a matrix, as any expedited description of the algorithm would not do it justice. If you need a review of row reducing, refer to the textbook on pages 15-17 in section 1.2 for a detailed example. I would also recommend reviewing pages 16-17 in section 1.2 on how to interpret these reduced forms and determining which of the three cases a linear system falls into.

1.3 Vector and Matrix Operations

There are a few vector space operations that students should be aware of when studying Linear Algebra. I have generated a short list below with examples for reference. While not all of these operations are covered in the opening sections of the text, they are important to know sooner rather than later, as they will all be used in the course:

1. scalar-vector product:

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$$

2. dot product (inner product):

$$\mathbf{xy} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = (x_1y_1 + x_2y_2 + x_3y_3)$$

3. vector norm (”length”):

$$\|\mathbf{x}\| = \sqrt{\mathbf{xx}} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

4. cross product:

$$\mathbf{x} \times \mathbf{y} = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = \begin{bmatrix} \det\begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} \\ -\det\begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix} \\ \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

5. matrix-vector product:

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

6. matrix-matrix product:

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

2 Frequently Asked Conceptual Questions

1. **What exactly *is* a matrix?**

   A matrix can be interpreted several different ways- that is why they are useful! However, a matrix is not just a neat way to organize numbers into rows and columns. A matrix can be thought of as representing a system of linear equations, a function that maps vectors to vectors, a linear transformation of space in an arbitrary number of dimensions, among other interpretations. Several of these will be introduced later in the semester. The general idea that you shoud emphasize early on is that a matrix is both a *list of vectors* and a *function that operates on vectors and other matrices.*

2. **When performing elimination, can I use any row in the system to eliminate any other row in the system?**

   Sometimes, yes, but be very careful when doing this. If you ony use the pivot row for elimination, you are guaranteed to always reduce the matrix into its proper echelon form. However, if you add a non-pivot row to any other non-pivot row to perform elimination, you run the risk of indavertenty cancelling out a row with itself later on in the calculation, even though it may appear that you are not doing so. For example, if I have rows r_1,and r_2, I could add them in the following way:

$$r_1 + 3r_2 - r_1 - 3r_2 = 0$$

   This would produce a row of zeroes in your reduced matrix that should not be there. This is why it is generally a bad idea to take 'shortcuts' when row-reducing a matrix.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.

1. **Suppose we have a system of 5 linear equations. Each linear equation involves 4 variables (i.e: of the form $ax_1 + bx_2 + cx_3 + dx_4 = w$). Determine which of the three cases (no solution, one solution, infinite solutions) the system can fall into.**

   The system can be in either of the three cases. Some possible shapes of the row-reduced system corresponding to each of these cases are below:

$$\left[\begin{array}{cccc|c} 1 & ? & ? & ? & ? \\ 0 & 1 & ? & ? & ? \\ 0 & 0 & 1 & ? & ? \\ 0 & 0 & 0 & 1 & ? \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & ? & ? & ? & ? \\ 0 & 1 & ? & ? & ? \\ 0 & 0 & 1 & ? & ? \\ 0 & 0 & 0 & 1 & ? \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & ? & ? & ? & ? \\ 0 & 0 & 1 & ? & ? \\ 0 & 0 & 0 & 1 & ? \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   Row-echelon systems for No Solution, One Solution, and Infinite Solutions

2. **Interpret the following row-reduced systems of equations. If a unique solution exists, find it. If infinite solutions exist, describe the solution set.**

(A) $\left[\begin{array}{ccc|c} 10 & 0 & 0 & -15 \\ 0 & -4 & 1 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right]$ **(B)** $\left[\begin{array}{cccc|c} 3 & -3 & 0 & 1 & -30 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

In (A), we extract the simplified system:
$3x_3 = 9 \rightarrow x_3 = 3$,
$-4x_2 + (3) = 2 \rightarrow x_2 = 1/4$,
$10x_1 = -15 \rightarrow x_1 = -3/2$.

Thus, the unique solution to (A) is $(-3/2, 1/4, 3)$.

In (B), we observe that columns 1, 3, and 4 are pivot columns, with the second column indicating that x_2 is free. Because we have 1 free variable in the solution set, the solution must be a line. More specifically, the solution set is all points of the form:

$$\begin{cases} x_1 - x_2 = -10 \\ x_3 = 2 \\ x_4 = 9 \end{cases} \quad \rightarrow \quad (-10 + x_2, x_2, 2, 9), \text{ where } x_2 \text{ is free.}$$

Assigning $t = x_2$, we can write the solution line $l(t)$ in vector notation as:

$$l(t) = \begin{bmatrix} -10 \\ 0 \\ 2 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t$$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page

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