Portfolio & Risk Analytics Research

Multi-Asset Risk Models

Overcoming the Curse of Dimensionality

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Outline

- Motivation
  - The “curse of dimensionality”
  - Portfolio optimization with noisy covariance matrices
  - Alpha and Hedge portfolios
- Candidate Multi-Asset Risk Models
- Ranking the Candidate Models
  - Evaluating the Accuracy of Correlation Forecasts (Factor-pair Portfolios)
  - Evaluating the Quality of Optimized Portfolios (Volatility and Turnover)
- MAC2 versus MAC1 Comparison
- Summary

Motivation and Overview

Multi-Asset Factor Covariance Matrices

- Portfolios may have exposure to multiple asset classes
- Each asset class is composed of multiple local markets
- Each local market is explained by many local factors

To obtain accurate risk forecasts for any portfolio requires a covariance matrix that combines all of the local factors

MAC Covariance Matrix

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{EQ} & \mathbf{F}_{FI}^{EQ} & \cdots & \mathbf{F}_{FX}^{EQ} \\ \mathbf{F}_{EQ}^{FI} & \mathbf{F}_{FI} & \cdots & \mathbf{F}_{FX}^{FI} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{EQ}^{FX} & \mathbf{F}_{FI}^{FX} & \cdots & \mathbf{F}_{FX} \end{bmatrix}$$

Global Equity Block

$$\mathbf{F}_{EQ} = \begin{bmatrix} \mathbf{F}_{USA} & \mathbf{F}_{JAP}^{USA} & \cdots & \mathbf{F}_{EUR}^{USA} \\ \mathbf{F}_{USA}^{JAP} & \mathbf{F}_{JAP} & \cdots & \mathbf{F}_{EUR}^{JAP} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{USA}^{EUR} & \mathbf{F}_{JAP}^{EUR} & \cdots & \mathbf{F}_{EUR} \end{bmatrix}$$

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The "Curse of Dimensionality"

- Forecasting accuracy requires a detailed factor structure spanning all markets and asset classes
  - Bloomberg MAC2 covariance matrix contains nearly 2000 factors
- Portfolio construction demands a *robust* covariance matrix
  - Risk model should not identify spurious hedges that fail out-of-sample
- With fewer observations than factors ($T<K$), sample covariance matrix contains one or more "zero eigenvalues"
  - Leads to spurious prediction of "riskless" portfolios
- This feature makes the sample covariance matrix unsuitable for portfolio construction

Special methods are required to simultaneously provide:

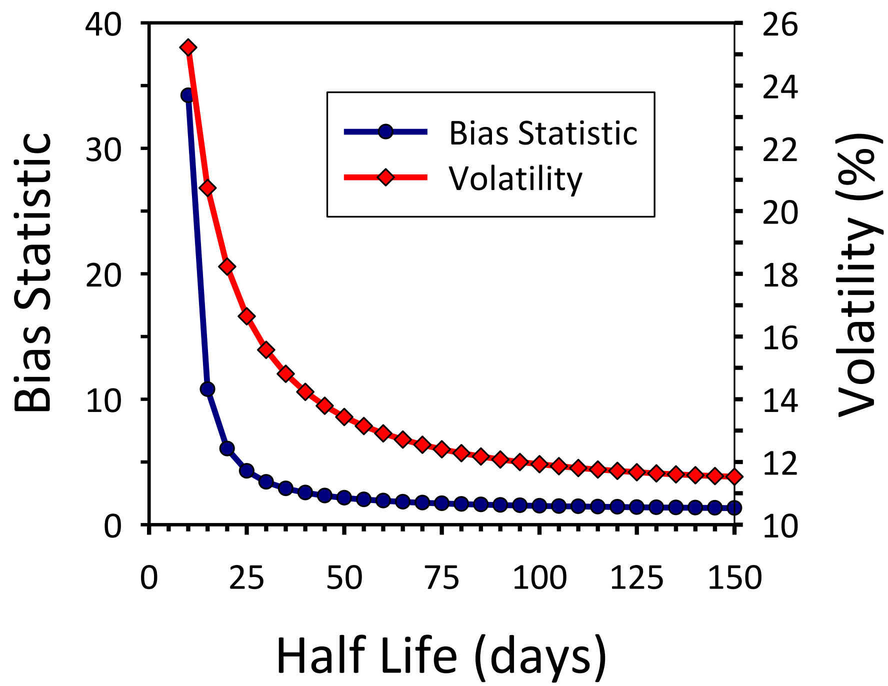
1. *Accurate volatility forecasts (risk management)*
2. *Robust risk models (portfolio construction)*

Empirical Study 1: The Perils of Noise

- Take the largest 100 US equities as of 31-Mar-2016, with complete daily return history to 13-Jan-1999
- Estimate family of asset covariance matrices using EWMA with a variable half-life τ
- Half-life provides convenient “knob” to control noise level
- Each day, construct the min-vol fully invested portfolio:

$$\mathbf{w}_\tau = \frac{\mathbf{\Omega}_\tau^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Omega}_\tau^{-1}\mathbf{1}}$$

- Bias statistic represents ratio of realized risk to forecast risk
- Biases and out-of-sample volatilities increase dramatically as the covariance matrix becomes noisier



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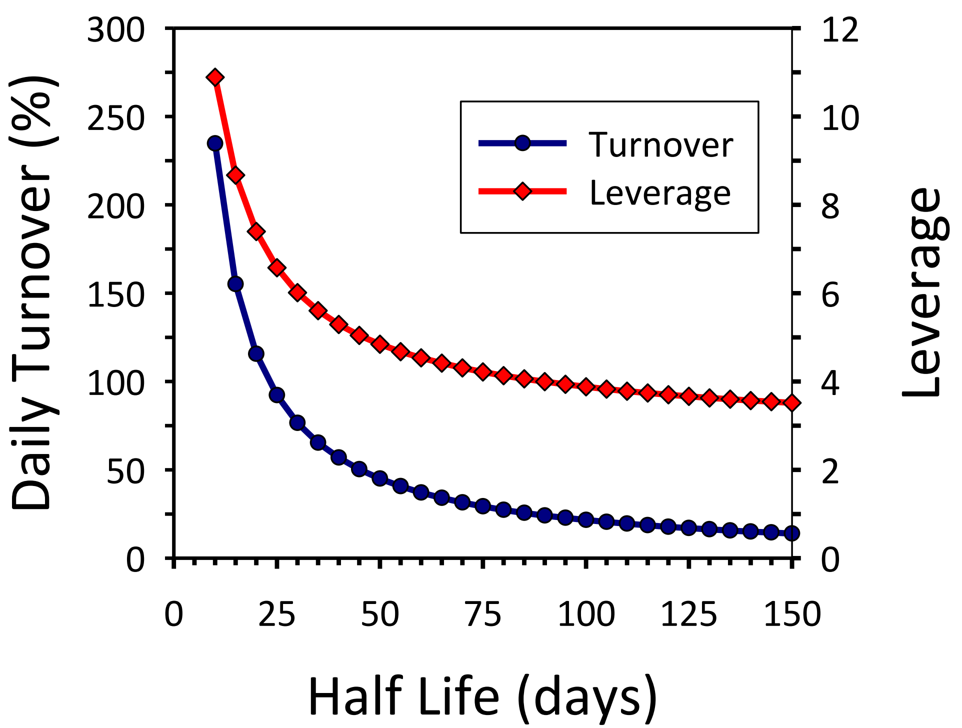
Turnover and Leverage

- Minimum-volatility portfolio is fully invested, but not long-only
- Compute mean turnover and leverage of optimized portfolios

$$TO_\tau = \frac{1}{T}\sum_{t=1}^{T}\sum_{n=1}^{N}\left|w_{t+1}^{\tau}(n) - w_t^{\tau}(n)\right|$$

$$L_\tau = \frac{1}{T}\sum_{t=1}^{T}\sum_{n=1}^{N}\left|w_t^{\tau}(n)\right|$$

- Leverage and turnover levels diverge at short HL
- At 10-day HL, leverage is 10x (550% long, 450% short)
- At 10-day HL, turnover exceeds 200% per day



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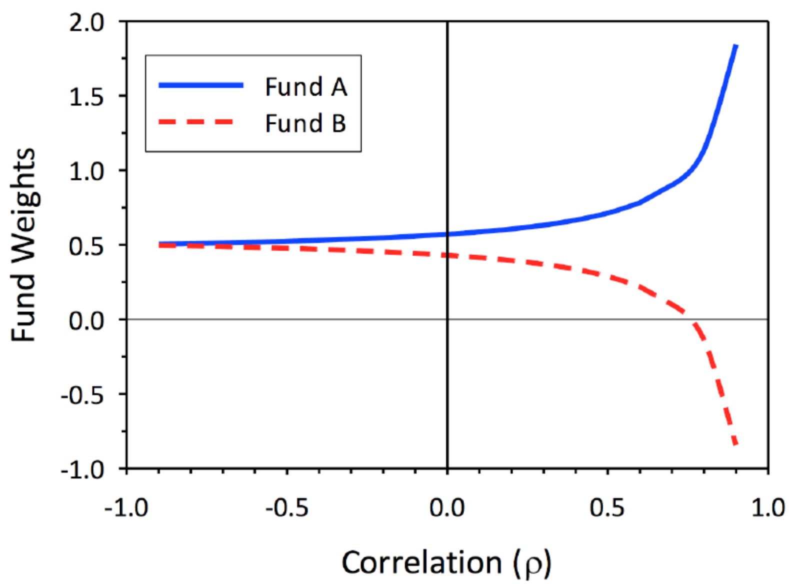
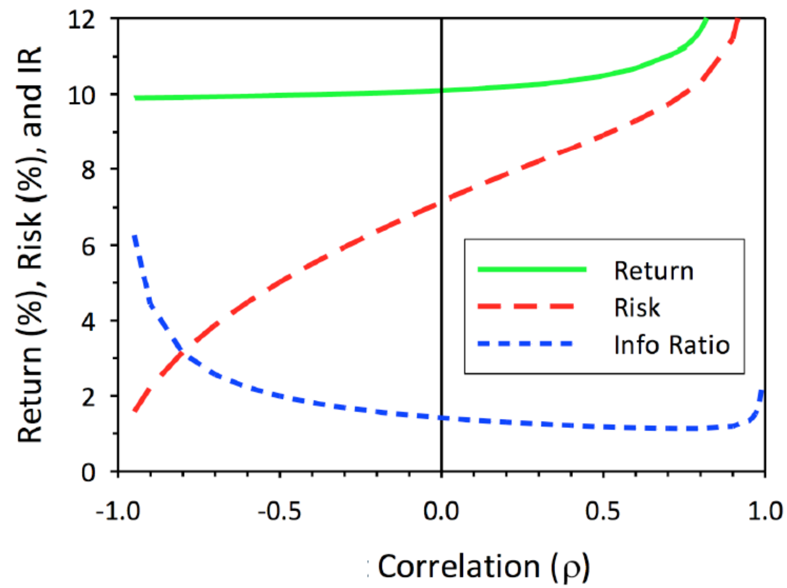
What Causes this Behavior?

- Consider two funds:
  - Fund A has volatility of 10%, and expected return of 11.3%
  - Fund B has volatility of 10% and expected return of 8.5%
- Compute optimal fund weights as a function of the fund correlation
- As correlation approaches 1, optimal portfolio goes long Fund A and short Fund B, resulting in large leverage
- Hedge may not perform well out-of-sample due to estimation error





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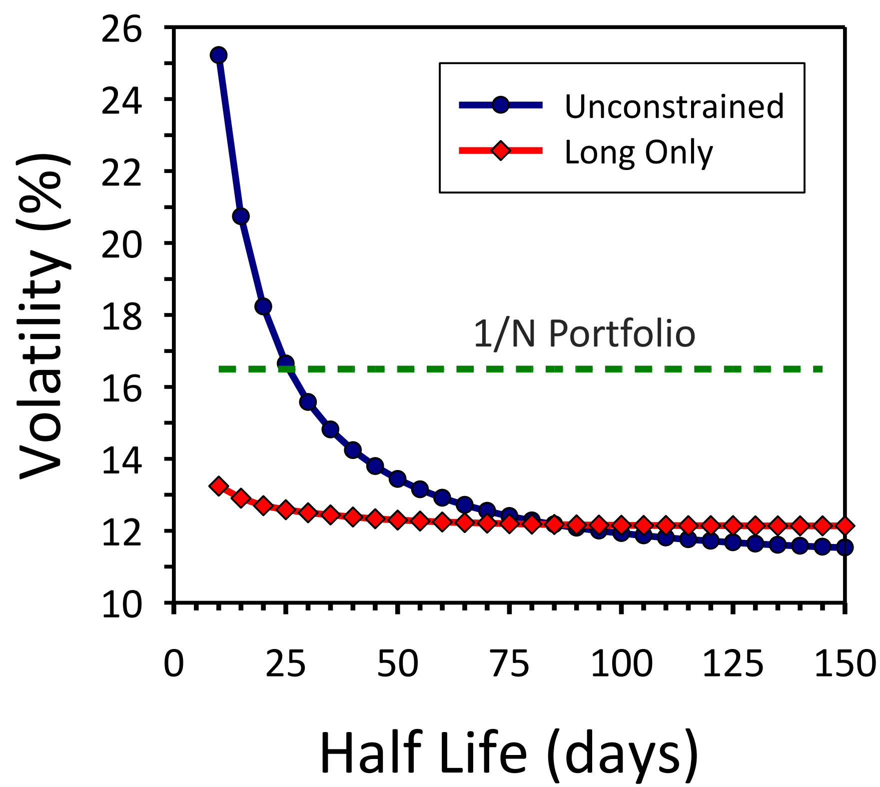
Alternative Portfolio-Construction Techniques

- Alternative portfolio construction techniques:
  - Use optimization but impose constraints (e.g., long only)
  - Forego optimization entirely and use 1/N portfolio (equal weights)
- For very short HL, the 1/N portfolio indeed outperforms the unconstrained optimal portfolio (but not the long-only portfolio)

Observations:

- For well-conditioned covariance matrices, optimized portfolios easily outperform 1/N portfolio
- For well-conditioned covariance matrices, unconstrained optimal outperforms long-only constraint
- For ill-conditioned matrices, imposing constraints is beneficial



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Portfolio Optimization (*Ex Ante*)

- Decompose optimal portfolio into alpha and hedge portfolios:

$$\mathbf{w} = \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\Omega}^{-1}\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha} + \mathbf{h}$$

- Hedge portfolio uncorrelated with the optimal portfolio:

$$\mathbf{h}'\boldsymbol{\Omega}\mathbf{w} = 0 \qquad \text{(Property 1)}$$

- The hedge portfolio has zero alpha

$$\mathbf{h}'\boldsymbol{\alpha} = 0 \qquad \text{(Property 2)}$$

- Hedge portfolio is negatively correlated with the alpha portfolio

$$\mathbf{h}'\boldsymbol{\Omega}\boldsymbol{\alpha} < 0 \qquad \text{(Property 3)}$$

$$\mathbf{w} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_N \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_N \end{bmatrix}$$

Alpha Hedge

Hedge portfolio reduces portfolio risk without changing the expected return

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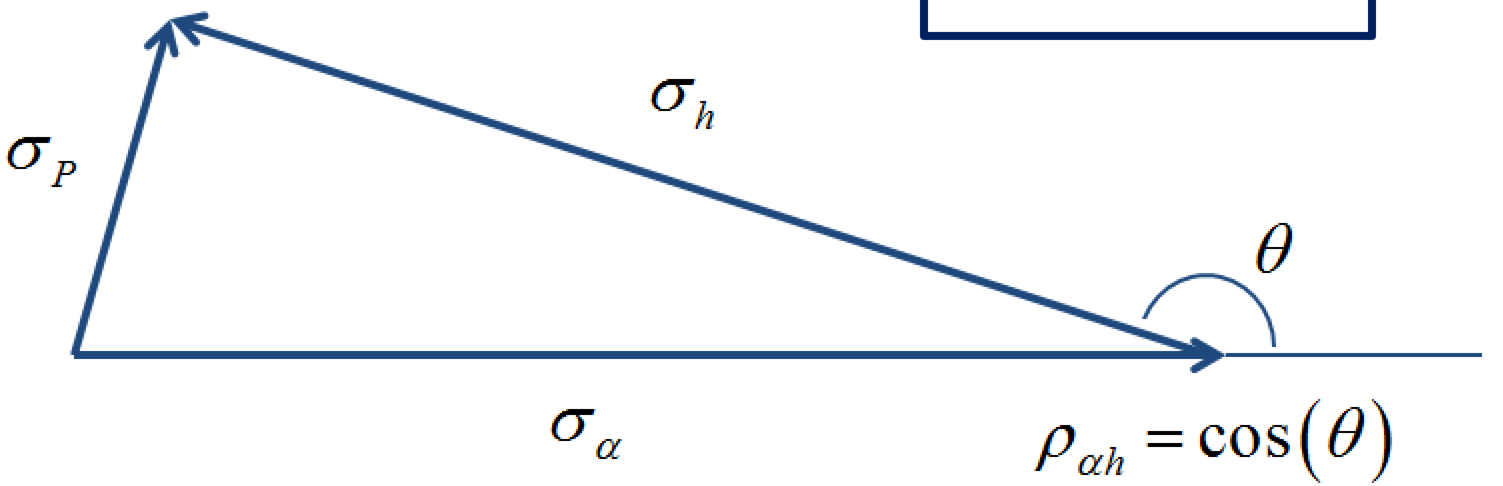
Geometry of Optimization (*Ex Ante*)

- Hedge portfolio is uncorrelated with optimal portfolio

$$\rightarrow \boxed{\sigma_P^2 = \sigma_\alpha^2 - \sigma_h^2}$$ Portfolio Variance

- Let $\rho_{\alpha h}$ denote the predicted correlation between $\boldsymbol{\alpha}$ and $\mathbf{h}$
- The magnitude of the correlation determines quality of hedge
- Optimal position in hedge portfolio: $$\boxed{\sigma_h = \sigma_\alpha \left|\rho_{\alpha h}\right|}$$



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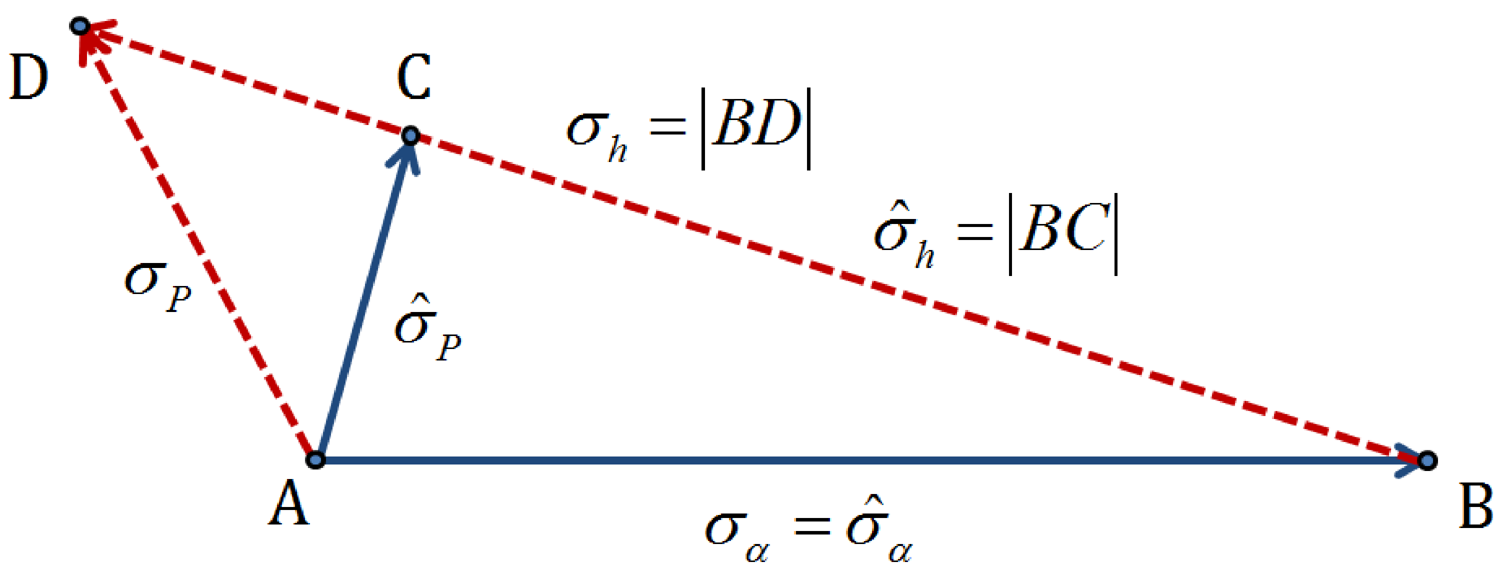
Potential Pitfalls of Optimization

- Optimization leads to superior *ex ante* performance, but is no guarantee of improvement *ex post*
- Estimation error within the covariance matrix represents a potential pitfall in portfolio optimization
- Estimation error in the volatility:
  - Risk models may underestimate the volatility of the hedge portfolio
- Estimation error in the correlation:
  - Risk models "paint an overly rosy picture" of the correlation between the alpha and hedge portfolios
- Estimation error gives rise to several detrimental effects:
  - Underestimation of risk of optimized portfolios
  - Higher out-of-sample volatility of optimized portfolios
  - Positive realized correlation between optimized and hedge portfolios

Estimation Error in the Volatility

- Suppose we correctly estimate correlation between hedge/alpha portfolios, but we under-estimate volatility of hedge portfolio
- Side effects:
  - Under-estimation of risk of optimal portfolio
  - Inefficient allocation of risk budget (hedge portfolio adds risk but no return)
  - Increased out-of-sample volatility of optimal portfolio



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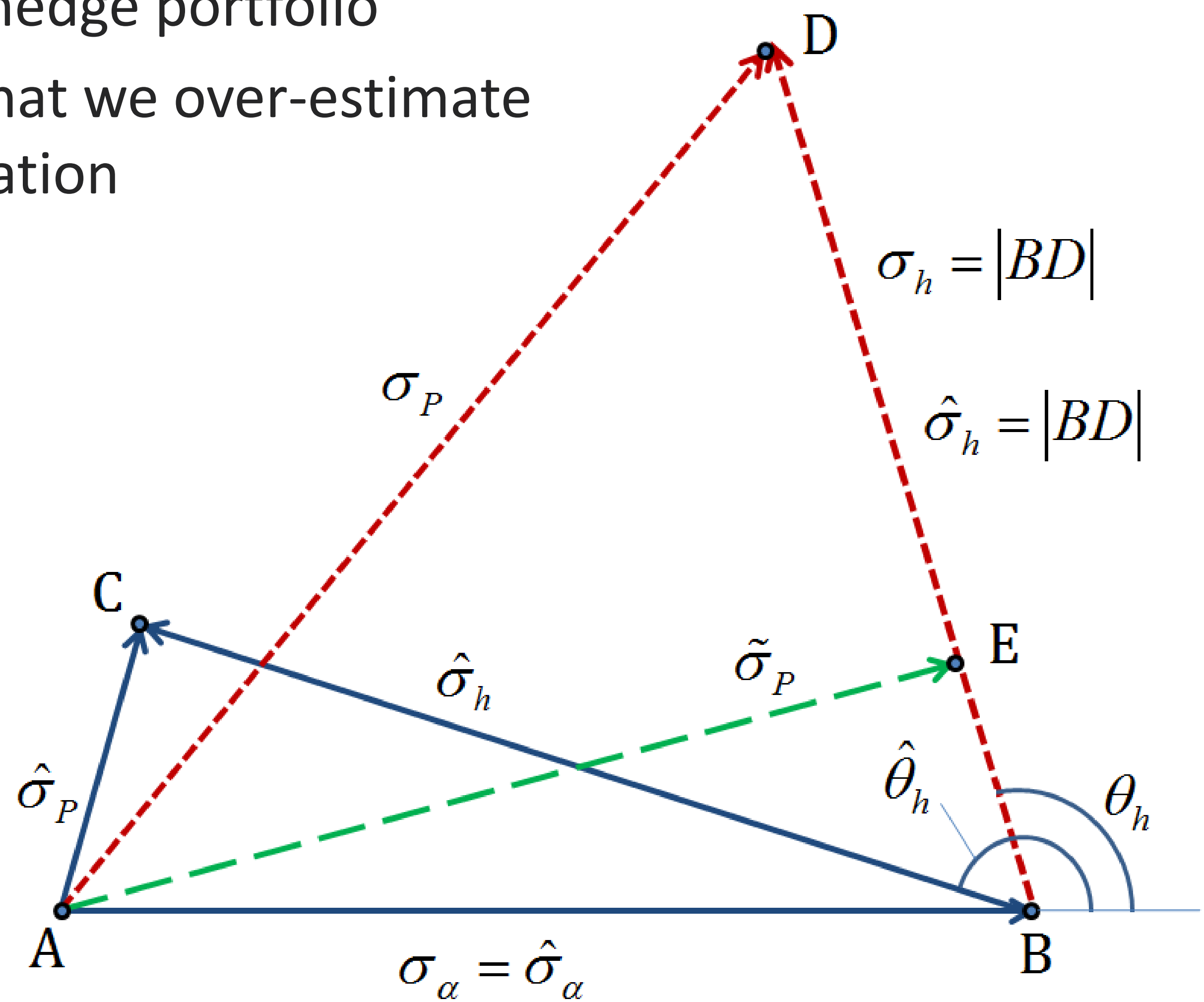
Estimation Error in the Correlation

- Now suppose that we correctly estimate the volatility of the hedge portfolio
- However, suppose that we over-estimate magnitude of correlation
- Side effects:
  - Under-estimation of portfolio risk
  - Inefficient allocation of risk budget
  - Increased volatility out-of-sample
- Portfolio E is true optimal portfolio



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Empirical Study 2

- As before, take the largest 100 US equities as of 31-Mar-2016, with complete daily return history to 13-Jan-1999
- Estimate a family of asset covariance matrices using EWMA with a variable half-life τ
- Each day, construct the minimum-volatility portfolio with unit weight in a particular stock:

$$\mathbf{w}_{n\tau} = \frac{\mathbf{\Omega}_{\tau}^{-1}\boldsymbol{\delta}_{n}}{\boldsymbol{\delta}'_{n}\mathbf{\Omega}_{\tau}^{-1}\boldsymbol{\delta}_{n}}$$

Optimal portfolio (stock *n*)

- We have a family of 100 portfolios for each HL parameter
- Each portfolio is 100 percent long stock a particular stock, and hedges the risk by shorting other stocks

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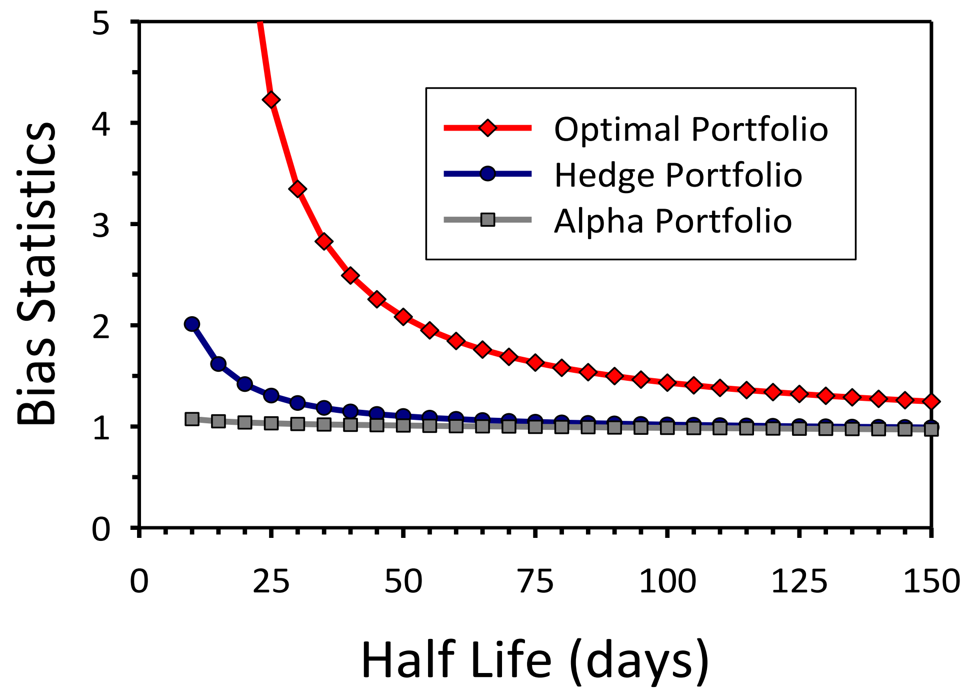
Biases in Portfolio Volatility

- Compute mean bias statistics for each of 100 alpha, hedge, and optimal portfolios
- Alpha portfolio is fully invested in a single stock
  - Alpha portfolio is independent of the covariance matrix
  - Covariance matrix makes unbiased forecasts for alpha portfolio
- Hedge/optimal portfolios depend on covariance matrix

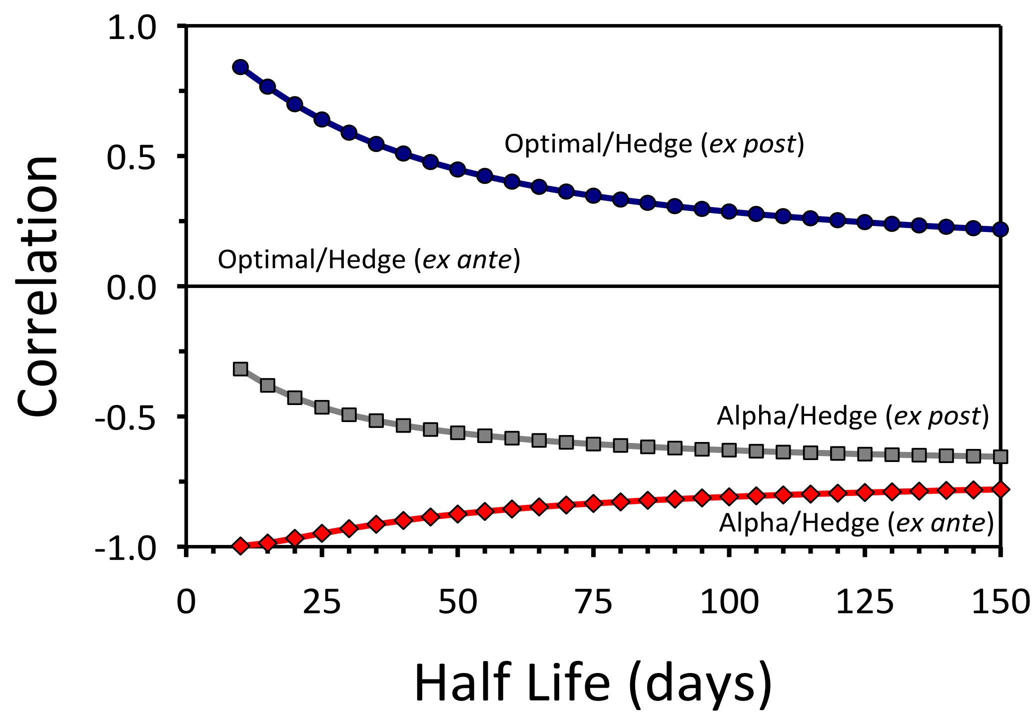
- Hedge portfolio risk is largely unbiased, except for very short HL
- Optimal portfolio is significantly under-forecast for full range of HL

Biases in correlations must be responsible for under-forecasting the volatility of optimal portfolio



Biases in Correlations

- Compute mean correlations *ex ante* and *ex post*
- Optimal/hedge correlation:
  - *Ex ante*, the correlation is exactly zero
  - *Ex post*, it is positive, indicating inefficient allocation of the risk budget
  - Problem is exacerbated for short HL
- Alpha/hedge correlation:
  - The hedge always appears better *ex ante* than it turns out to be *ex post*
  - Gap between *ex ante* and *ex post* grows larger for short HL parameters



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Alpha/Hedge Portfolios versus Efficient Frontier

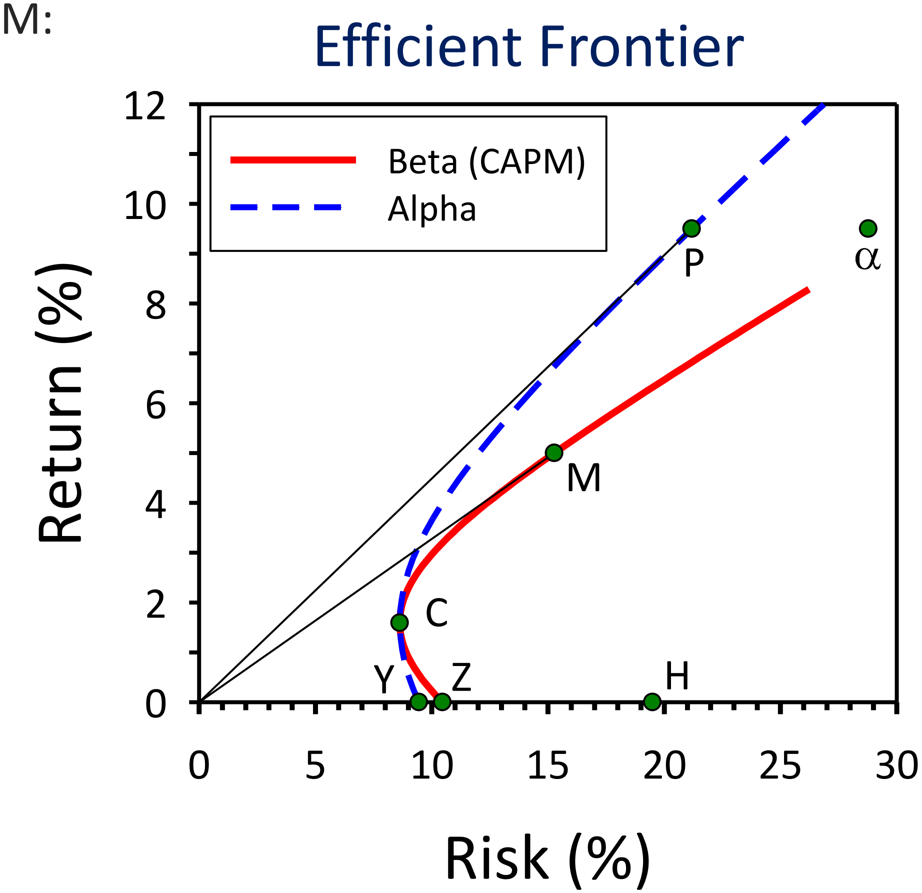
- Alpha/hedge decomposition is not the two-fund separation theorem
- Form the minimum-volatility fully invested portfolio with fixed expected return
  - Universe is largest 100 stocks with covariance matrix using 150-day HL on 31-Mar-2016
- Red line is efficient frontier assuming CAPM:

$$E\left[r_n\right] = \beta_n E\left[R_M\right]$$

- Blue line is efficient frontier with alphas:

$$E\left[r_n\right] = \beta_n E\left[R_M\right] + \delta_n$$

- C is the min-vol fully invested portfolio
- Y/Z are the zero-beta portfolios on the efficient frontier (relative to P/M)
- P/M are the efficient portfolios using respective return assumptions
- α is the alpha portfolio (177% long)
- H is the hedge portfolio (77% short)



Note: exceptional returns (δ_n) are drawn from a standard normal of width 30 bps

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Constructing the Efficient Frontier

- Portfolio optimization with equality constraints:

  Minimize: $\mathbf{w}'\boldsymbol{\Omega}\mathbf{w}$ Subject to: $\mathbf{A}\mathbf{w} = \mathbf{b}$

- Solution obtained using Lagrange multipliers:

$$\mathbf{w} = \boldsymbol{\Omega}^{-1}\mathbf{A}'\left(\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}$$

- Target return and full investment constraints:

$$\left.\begin{array}{ll} \boldsymbol{\alpha}'\mathbf{w} = \mu_P & \text{(Target return)} \\ \mathbf{1}'\mathbf{w} = 1 & \text{(Full investment)} \end{array}\right\} \longrightarrow \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_N \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} \mu_P \\ 1 \end{bmatrix}$$

- Terminology: alphas represent expected stock returns:

$$\alpha_n = \beta_n E\left[R_M\right] + \delta_n$$

The δ_n represent exceptional returns, commonly referred to as "alphas"

Candidate Models

Separating Volatilities and Correlations

- Divide the task of constructing a factor covariance matrix into two parts:
  - Estimating the factor volatilities **V** (diagonal matrix)
  - Estimating the factor correlation matrix **C**
- Factor volatilities are typically estimated using a relatively short half-life parameter (i.e., responsive forecasts)
- Factor correlations typically use longer half-life parameters
  - Reduces noise in the correlation matrix
  - Produces accurate risk forecasts
- The factor covariance matrix is easily reconstructed:

$$\mathbf{F} = \mathbf{VCV}$$

- Present study focuses on comparing the model quality of *correlation matrices* for the equity block

Sample Correlation Matrix

- Sample correlation matrix possesses many attractive properties:
  - Provides arguably the best estimate for any pairwise correlation
    - Best Linear Unbiased Estimate (BLUE) under standard econometric assumptions
  - Gives intuitive and transparent estimates, since it is based on the "textbook" definition of correlation coefficient
  - Produces accurate risk forecasts for most portfolios (with the notable exception of optimized portfolios)
- Sample correlation matrix also possesses an "Achilles heel":
  - If there are K factors and T periods, then sample correlation matrix contains zero eigenvalues (i.e., rank-deficient matrix) whenever $T<K$
  - Rank-deficient matrices predict the existence of "phantom" riskless portfolios that do not exist in reality
  - Sample correlation is *not* robust for portfolio construction

Objectives: (a) correlation estimates should closely mimic the sample, and (b) provide robust forecasts for portfolio construction purposes

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Other Techniques for Correlations

- Principal Component Analysis (PCA)
  - Statistical technique to extract global factors from the data
  - Assume a small number of global factors (principal components) fully capture correlations of local factors (i.e., uncorrelated residuals)
- Random Matrix Theory (RMT)
  - Statistical technique similar to PCA (factors extracted from data)
  - Eigenvalues beyond a cutoff point are simply averaged
- Time-series Approach
  - Specify "global" factor returns to explain "local" factor correlations
  - Estimate the exposures by time-series regression
- Eigen-adjustment Method
  - Eigenvalues of sample correlation matrix are systematically biased
  - Adjust the eigenvalues to remove biases

Menchero, Wang, and Orr. *Improving Risk Forecasts for Optimized Portfolios,* Financial Analysts Journal, May/June 2012, pp. 40-50

Blended Correlation Matrices

- Ledoit and Wolf (2003) showed that blending the sample covariance matrix with a one-factor model yielded optimized fully invested portfolios with lower out-of-sample volatility
- Blend sample correlation (using weight *w*) with PCA correlation using *J* principal components derived from *K* local factors
- Specify number of PCA factors by parameter μ , where $J = \mu K$
- Two-parameter model for correlation matrix:

$$\mathbf{C}_B(\mu, w) = w\mathbf{C}_0 + (1-w)\mathbf{C}_P(\mu)$$

Blended Matrix

- Optimal blending parameters are determined empirically
- Technique represents the new Bloomberg methodology

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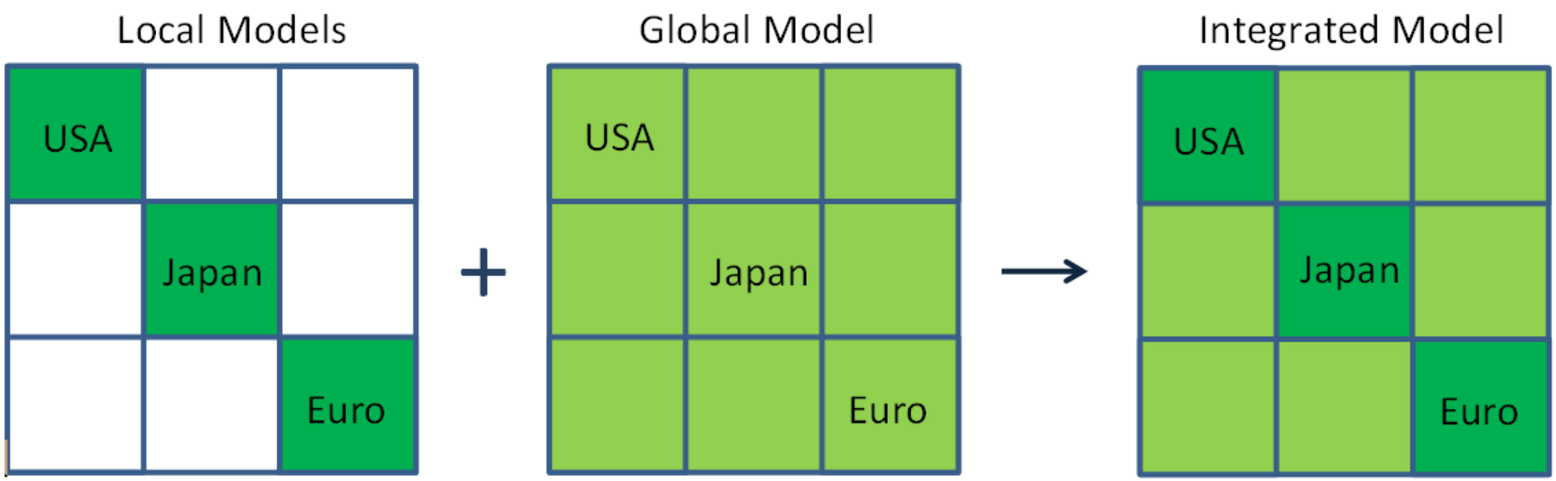
Ledoit and Wolf. *Improved Estimation of the Covariance matrix of Stock Returns,* Journal of Empirical Finance, December 2003, pp. 603-621

Adjusted Correlation Matrices

- Local models provide our "best" estimates of the correlation matrices for the diagonal blocks
- Global model is used to estimate the off-diagonal blocks
- Diagonal blocks of the global model differ from the correlation matrices obtained from the local models



- Integrated model is formed by "adjusting" the global model to replicate the local models along the diagonal blocks

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Ranking the Candidate Models

Measuring Biases in Risk Forecasts

- Bias statistic represents the ratio of forecast risk to realized risk

$$z_{nt} = \frac{r_{nt}}{\sigma_{nt}} \quad \rightarrow \quad B_n = \text{std}\left(z_{nt}\right) \quad \rightarrow \quad \bar{B} = \frac{1}{N}\sum_n B_n$$

Bias Statistic

- If the risk forecasts are *exactly correct*, the expected value of the bias statistic is precisely equal to 1
- If the risk forecasts are *unbiased* but noisy, the expected value of the bias statistic is *slightly greater* than 1
- Example: suppose we over-forecast volatility by 10% half of the time, and under-forecast by 10% half the time

$$E\left[B\right] = \sqrt{\frac{1}{2}\left(\frac{1}{0.9}\right)^2 + \frac{1}{2}\left(\frac{1}{1.1}\right)^2} = 1.02$$

- Typical bias statistic for unbiased risk forecasts

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Factor-Pair Portfolios

- Construct test portfolios capable of resolving minor differences in volatility forecasts due to differences in correlations
- Consider factor-pair portfolios

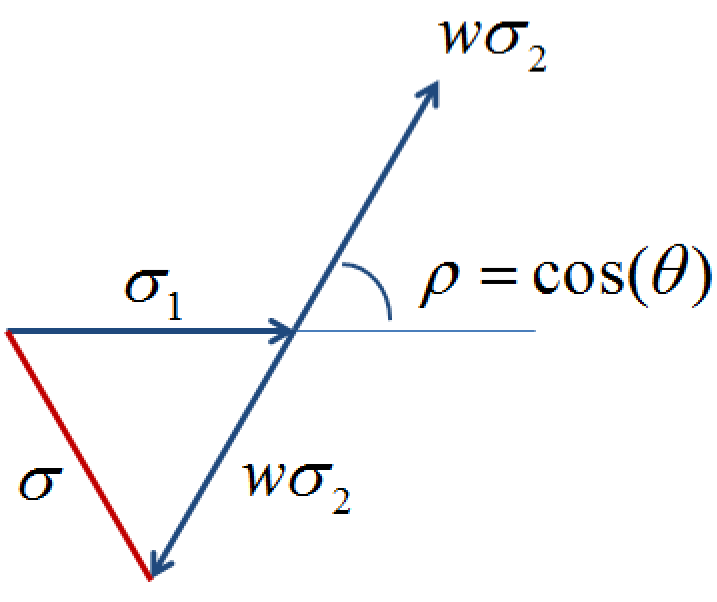
$$R = f_1 + wf_2 \quad \rightarrow \quad \sigma^2 = \sigma_1^2 + w^2\sigma_2^2 + 2\rho w\sigma_1\sigma_2$$

- Solve for the weight *w* that maximizes the percentage of risk due to the off-diagonal correlation
- Solution is given by:

$$w = \pm\left(\sigma_1 / \sigma_2\right)$$

- Portfolio volatility

$$\boxed{\sigma = \sqrt{2}\sigma_1\left(1 - |\rho|\right)^{1/2}}$$



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Description of Study

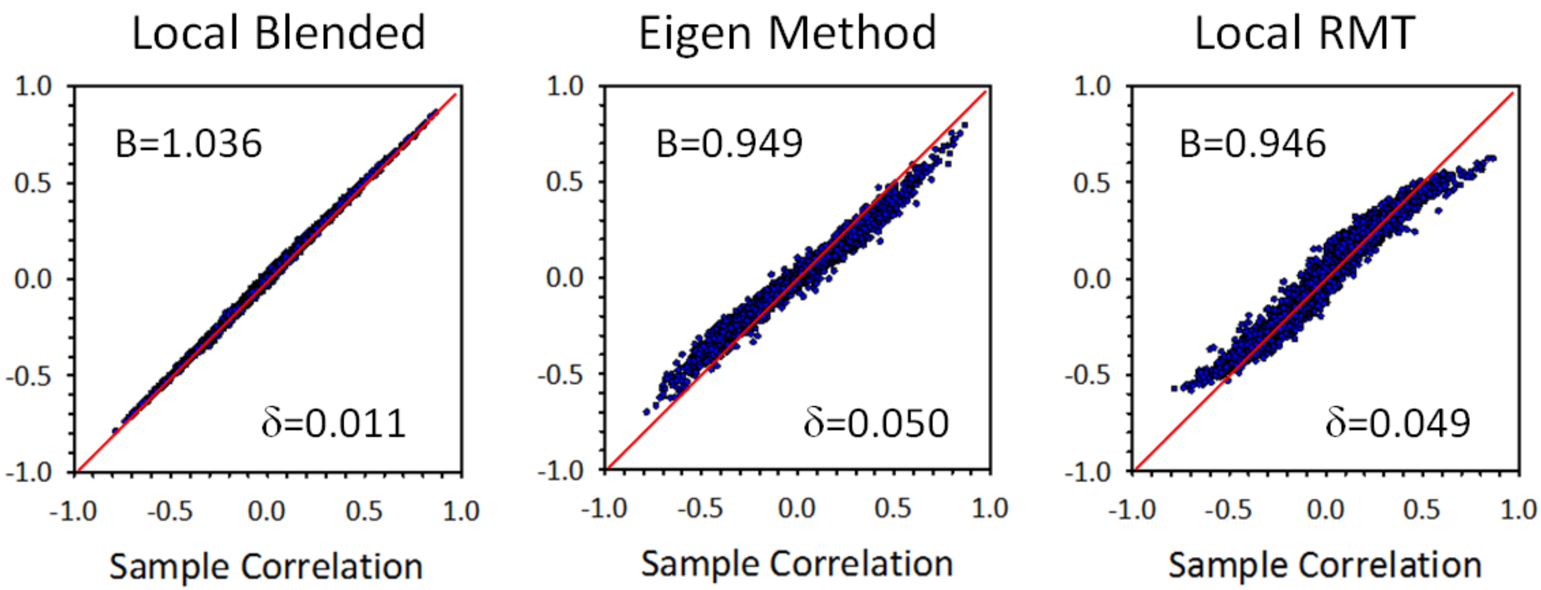
- Sample period contains 713 weeks (03-Jan-01 to 27-Aug-14)
- Model contains K=319 factors spanning nine equity blocks
- Evaluate accuracy of correlations using factor-pair portfolios

Parameters used in Study:

- Use T=200 weeks (equal weighted) as estimation window
- For PCA, RMT, and blended matrices
  - Use μ_L=0.25 for local blocks
  - Use μ_G=0.10 for global block
- For blended correlation matrices
  - Assign 80% weight to the sample (w=0.8) for local blocks
  - Assign 20% weight to the sample (w=0.2) for global blocks
- Blending parameter selection criteria:
  - Low out-of-sample volatility of optimized portfolios

Correlation Scatterplots (Diagonal Blocks)

- Local Blended provides a near "perfect fit" to the sample
- Eigen method and Local RMT exhibit systematic biases



Analysis date: 27-Aug-2014

- Compute B-stats for all factor-pairs with mean sample correlation above 0.40 (292 portfolios)
- Eigen method and Local RMT exhibit biases
- Sample and Local Blended are near ideal value

Model	Bias Stats
Sample	1.034
Local Blended	1.036
Eigen method	0.949
Local RMT	0.946

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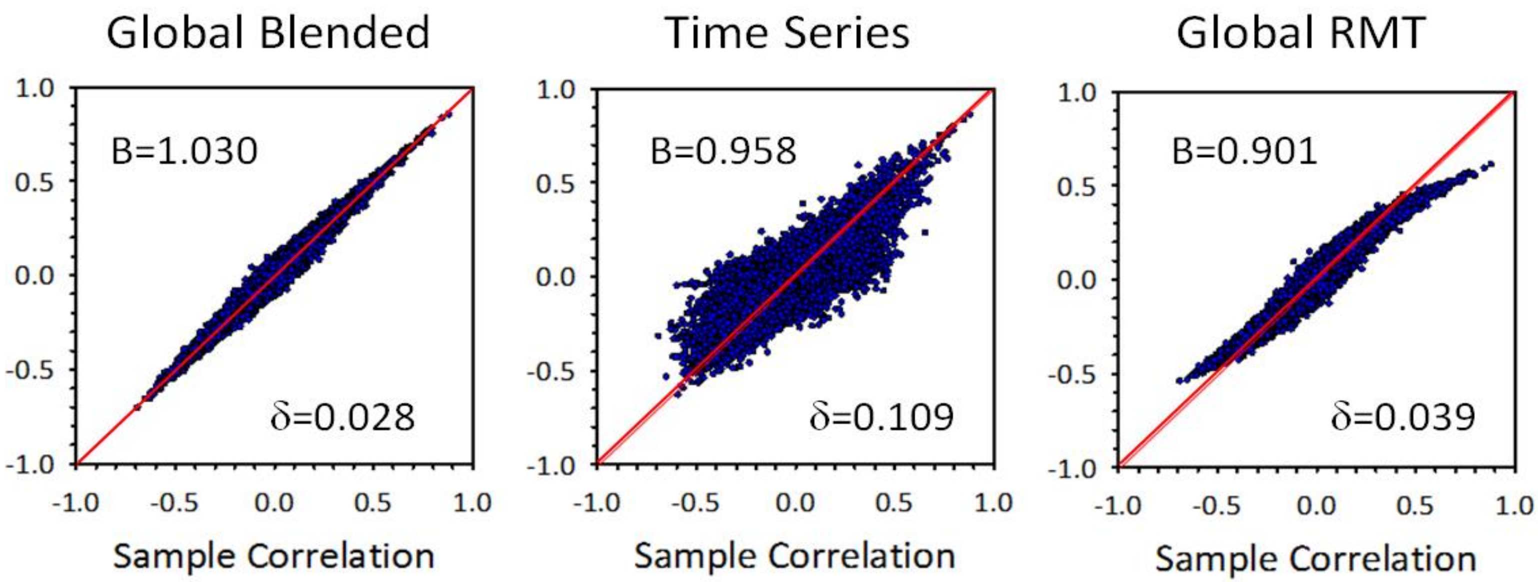
Analysis Period: 03-Jan-2001 to 27-Aug-2014

Correlation Scatterplots (Off-Diagonal)

- Global Blended provides an excellent fit to the sample
- Time Series and Global RMT exhibit systematic biases



Analysis date: 27-Aug-2014

- Compute B-stats for all factor-pairs with mean sample correlation above 0.50 (163 portfolios)
- Time Series and Global RMT exhibit biases
- Sample and Global Blended near ideal value

Model	Bias Stats
Sample	1.034
Global Blended	1.030
Time Series	0.958
Global RMT	0.901

Analysis Period: 03-Jan-2001 to 27-Aug-2014

Quality of Optimized Portfolios

- Construct optimal portfolios for each of *K*=319 factors and rebalance on a weekly basis (Jan-2001 to Aug-2014)
- Optimized unit-exposure portfolios have the maximum *ex ante* information ratio (i.e., minimum volatility)

$$\mathbf{w}_k^A = \frac{\mathbf{\Omega}_A^{-1}\boldsymbol{\alpha}_k}{\boldsymbol{\alpha}_k'\mathbf{\Omega}_A^{-1}\boldsymbol{\alpha}_k}$$

Optimal portfolio (Model A)

- Define the mean volatility ratio for Model A

$$v_A = \frac{1}{K}\sum_k \frac{\sigma_k^A}{\sigma_k^{\mathrm{Ref}}}$$

Model with lowest volatility ratio wins

- Compute factor turnover

$$TO_t = \sum_k \left| X_{k,t+1} - X_{k,\mathrm{t}} \right| \quad \rightarrow \quad \overline{TO} = \frac{1}{T}\sum_t TO_t$$

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Ranking the Models (Local Models)

- Allow hedging using only factors within the same block
- Convert STD, volatility, and turnover into z-scores
  - Positive z-scores represent “above average”
- Form composite z-score using weights (0.50, 0.25, 0.25)

Model	Standard Deviation	Volatility Ratio	Factor Turnover	z-score STDEV	z-score Vol Ratio	z-score Turnover	z-score Composite
Sample	0.000	0.955	0.686	1.104	0.451	-1.722	0.462
Global PCA	0.041	1.000	0.425	0.222	-0.243	-0.213	-0.005
Local PCA	0.054	1.030	0.380	-0.057	-0.706	0.042	-0.384
Global RMT	0.051	1.006	0.356	0.006	-0.327	0.181	-0.066
Local RMT	0.049	1.025	0.290	0.051	-0.630	0.563	0.017
Time Series	0.153	1.072	0.088	-2.228	-1.336	1.728	-2.002
Eigen-method	0.050	0.882	0.523	0.029	1.559	-0.782	0.411
Local Blended	0.011	0.903	0.352	0.872	1.233	0.203	1.567

- Local blended model scored above average on all measures
- Local blended had the highest composite z-score

Ranking the Models (Global Models)

- Allow hedging using all factors within the global block
- Convert STD, volatility, and turnover into z-scores
  - Positive z-scores represent “above average”
- Form composite z-score using weights (0.50, 0.25, 0.25)

Model	Standard Deviation	Volatility Ratio	Factor Turnover	z-score STDEV	z-score Vol Ratio	z-score Turnover	z-score Composite
Global PCA	0.035	1.000	1.095	0.474	-0.850	-0.702	-0.354
Global RMT	0.039	1.005	1.028	0.334	-0.933	-0.485	-0.438
Time Series	0.109	1.030	0.193	-2.146	-1.327	2.220	-1.987
Global Blended	0.028	0.977	0.864	0.717	-0.489	0.045	0.580
Global PCA (Adj)	0.037	0.884	1.125	0.400	1.000	-0.800	0.585
Global RMT (Adj)	0.037	0.876	0.998	0.405	1.125	-0.386	0.906
Time Series (Adj)	0.073	0.911	0.700	-0.869	0.568	0.577	-0.347
Global Blended (Adj)	0.029	0.890	1.023	0.684	0.905	-0.468	1.055

- Global blended (adjusted) had highest composite z-score
- Unified methodology applied throughout estimation process

Parameter Selection (US Equities)

- Leverage/Turnover minimized by few PC and low sample weight
- STD is minimized by taking many PC and high sample weight
- Volatility is minimized at intermediate values

Leverage

Mu (local)	w=0	w=0.20	w=0.40	w=0.60	w=0.80	w=1.00
0.10	2.90	3.10	3.63	4.36	5.52	15.73
0.20	3.95	4.04	4.42	5.04	6.21	15.73
0.30	4.69	4.72	5.02	5.58	6.77	15.73
0.40	5.79	5.68	5.83	6.29	7.50	15.73
0.50	6.75	6.47	6.51	6.92	8.20	15.73
0.60	8.32	7.64	7.51	7.91	9.31	15.73
0.70	9.69	8.63	8.40	8.86	10.34	15.73
0.80	12.41	10.59	10.43	11.03	12.47	15.73
0.90	15.61	14.60	14.53	14.74	15.13	15.73
1.00	15.73	15.73	15.73	15.73	15.73	15.73

Turnover

Mu (local)	w=0	w=0.20	w=0.40	w=0.60	w=0.80	w=1.00
0.10	0.12	0.12	0.16	0.21	0.28	0.51
0.20	0.25	0.22	0.22	0.26	0.31	0.51
0.30	0.38	0.31	0.29	0.30	0.33	0.51
0.40	0.55	0.43	0.38	0.36	0.36	0.51
0.50	0.67	0.52	0.44	0.40	0.38	0.51
0.60	1.01	0.71	0.56	0.46	0.41	0.51
0.70	1.20	0.81	0.61	0.49	0.43	0.51
0.80	1.76	1.01	0.70	0.54	0.47	0.51
0.90	1.37	0.83	0.65	0.56	0.51	0.51
1.00	0.51	0.51	0.51	0.51	0.51	0.51

Volatility Ratio

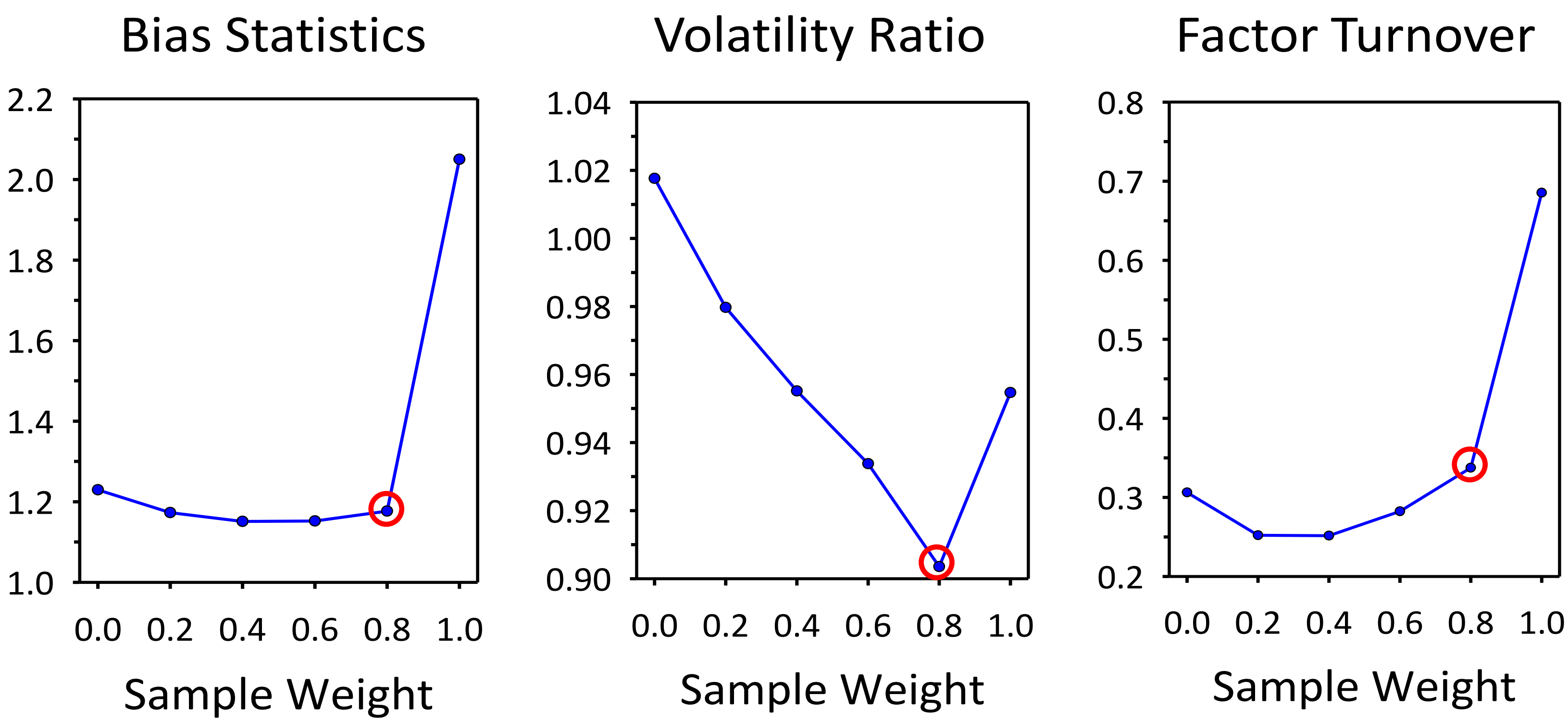
Mu (local)	w=0	w=0.20	w=0.40	w=0.60	w=0.80	w=1.00
0.10	1.087	1.020	0.989	0.971	0.957	1.118
0.20	1.040	1.004	0.984	0.970	0.953	1.118
0.30	1.045	1.010	0.989	0.971	0.948	1.118
0.40	1.078	1.034	1.004	0.977	0.948	1.118
0.50	1.119	1.057	1.015	0.979	0.949	1.118
0.60	1.199	1.096	1.032	0.984	0.958	1.118
0.70	1.289	1.133	1.044	0.989	0.973	1.118
0.80	1.438	1.157	1.051	1.011	1.021	1.118
0.90	1.376	1.157	1.103	1.091	1.097	1.118
1.00	1.118	1.118	1.118	1.118	1.118	1.118

Standard Deviation

Mu (local)	w=0	w=0.20	w=0.40	w=0.60	w=0.80	w=1.00
0.10	0.083	0.067	0.050	0.033	0.017	0.000
0.20	0.054	0.043	0.032	0.021	0.011	0.000
0.30	0.045	0.036	0.027	0.018	0.009	0.000
0.40	0.035	0.028	0.021	0.014	0.007	0.000
0.50	0.029	0.023	0.017	0.012	0.006	0.000
0.60	0.022	0.017	0.013	0.009	0.004	0.000
0.70	0.016	0.013	0.010	0.007	0.003	0.000
0.80	0.010	0.008	0.006	0.004	0.002	0.000
0.90	0.005	0.004	0.003	0.002	0.001	0.000
1.00	0.000	0.000	0.000	0.000	0.000	0.000

Benefits of Blending (Local Blocks)

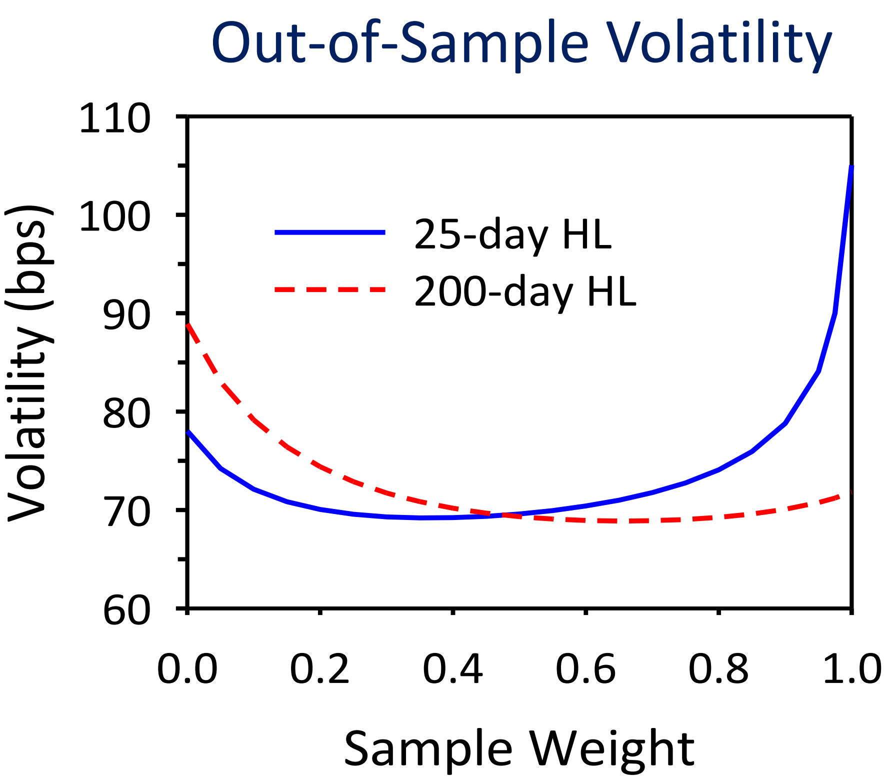
- Blending sample (w=0.8) with PCA model (μ=0.2) produced:
  - More accurate risk forecasts
  - Lower out-of-sample volatility
  - Lower turnover and leverage



Benefits of Blending

- Form minimum-volatility fully invested portfolio of 100 stocks
- Compute sample correlation (using 25d and 200d HL)
- Compute PCA model with one factor (25d and 200d HL)
- Blend the two covariance matrices

- Blending benefit exists for both models
- Blended 25d HL performs nearly as well as 200d HL
- Model with 25d HL assigns less weight to sample (0.4)
- Model with 200d HL assigns more weight to sample (0.8)



Portfolio & Risk Analytics Research

MAC2 versus MAC1 Comparison

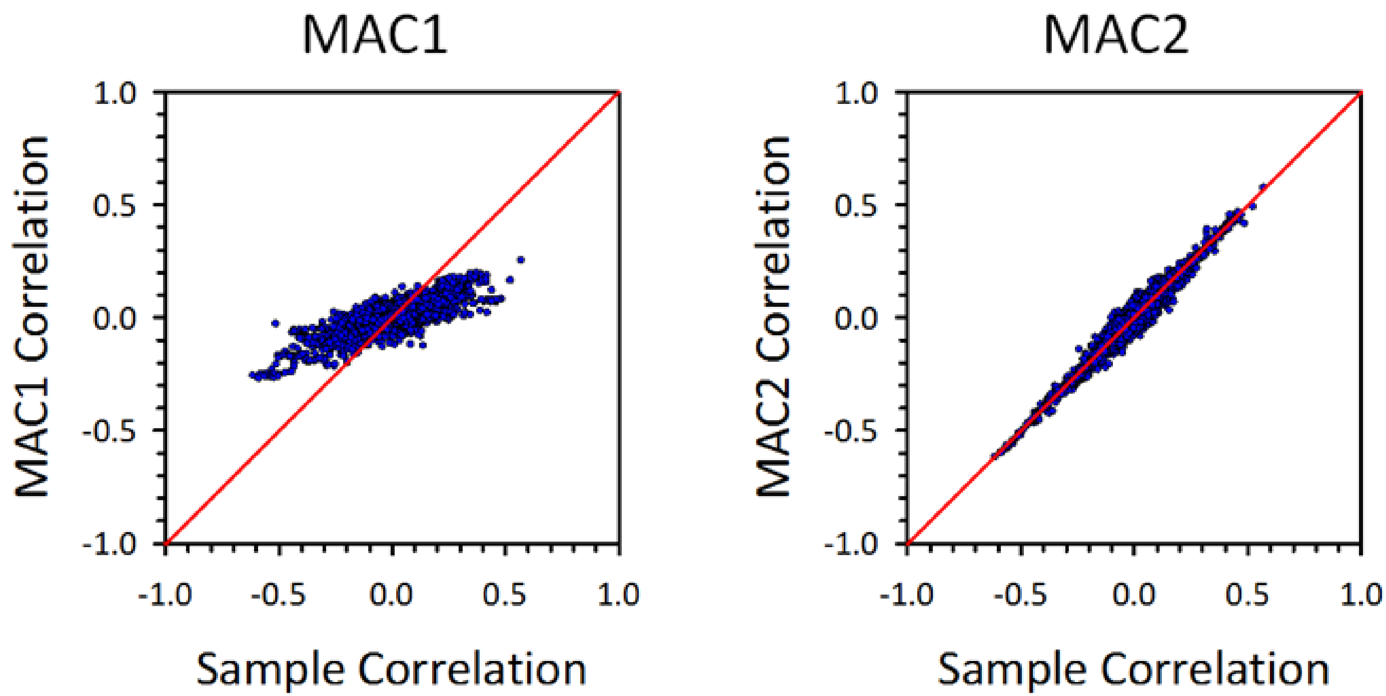
Bloomberg

Bloomberg MAC2 and MAC1 Models

- MAC1:
  - Computes diagonal blocks using RMT method with shrinkage
  - Computes off-diagonal blocks using the time-series method
  - For equities, “core” factors taken from global equity model
    - e.g., Japan autos is regressed on Japan factor and global auto factor
  - For other blocks, “core” factors are weighted average of local factors
    - e.g., Core factor for oil commodities is weighted average of Brent and WTI “shift”
  - Apply integration matrix to recover the diagonal blocks
- MAC2:
  - Uses blended methodology for both diagonal and off-diagonal blocks
  - Applies integration matrix to recover local models on diagonal blocks
- MAC1 and MAC2 models use EWMA with same HL parameters:
  - 26 weeks for volatilities
  - 52 weeks for correlations

US Equity versus US Fixed Income

- MAC2 estimates closely mimic the sample correlation
- MAC1 correlations exhibit considerable biases



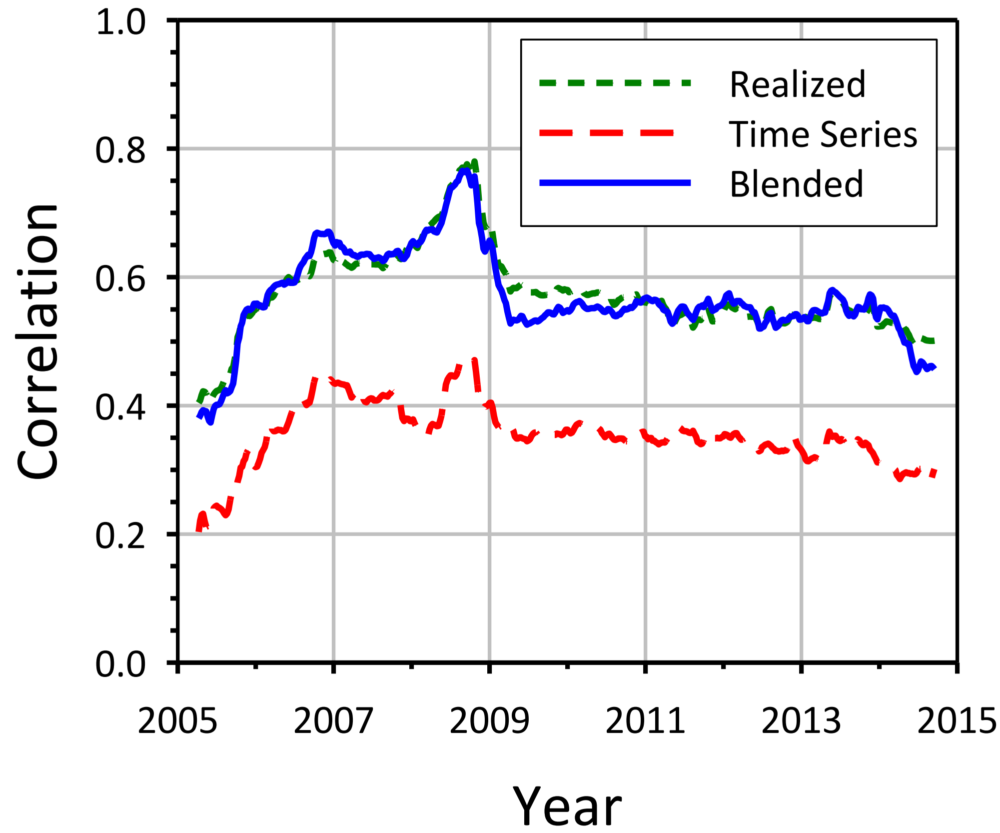
27-Aug-2014

- Results indicate that the time-series method does not fully capture the correlations across asset classes

Cross Asset-Class Correlations vs Time

- Consider the correlation between the US energy factor (equity) and the crude-oil commodity factor (Brent shift)
- Plot predicted and realized correlations (52w HL)

- Blended approach captures the observed relationship very closely
- Time-series method systematically under-predicts correlation
- Suggests missing factors in time-series approach



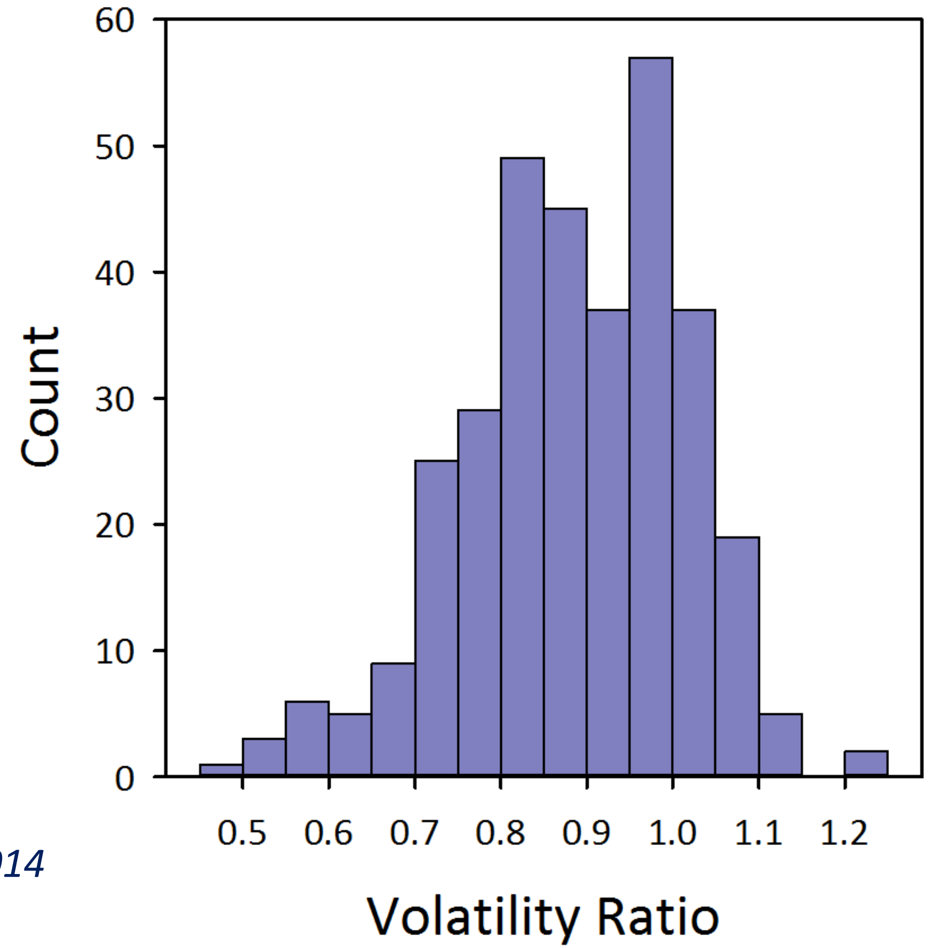
Bloomberg

Optimized Factor Portfolios (Off-Diagonal)

- For each of the 319 local equity factors, compute the volatility ratio between MAC2 and MAC1 for optimized factor portfolios

$$v = \frac{1}{K} \sum_k \frac{\sigma_k^{\mathrm{MAC2}}}{\sigma_k^{\mathrm{MAC1}}}$$

- MAC2 Model produced lower volatility in more than 80 percent of portfolios
- The average volatility ratio was 0.89
- Similar results hold for within-block optimizations



Sample Period: 30-Mar-2005 to 27-Aug-2014

Bloomberg

Summary

- Studied effect of noisy covariance matrices on optimization
- Introduction of second-generation Bloomberg model (MAC2)
- Adopted “blank-slate” approach to select the best model among a broad set of candidate models
- New methodology:
  - Two-parameter model uses blended correlations at all estimation levels
  - Parameters are empirically determined
  - Integration matrix is applied to recover local models on diagonal blocks
- New model closely mimics the sample correlation even across different asset classes (e.g., equity versus fixed income)
- New model guarantees full-rank covariance matrix to provide reliable forecasts for portfolio construction

Technical Appendix

Sample Correlation Matrix

- Sample period contains 713 weeks (01-03-01 to 8-27-14)
- Model contains *K*=319 local factors (for nine equity blocks)
- Compute sample covariance matrix (**F**$_0$) over *T*=200 weeks

$$F_{jk} = \frac{1}{T-1} \sum_t \left(f_{jt} - \bar{f}_j \right)\left(f_{kt} - \bar{f}_k \right)$$

- Let **S**$_0$ be a diagonal matrix of factor volatilities from **F**$_0$

$$\boxed{\mathbf{C}_0 = \mathbf{S}_0^{-1}\mathbf{F}_0\mathbf{S}_0^{-1}}$$

Sample Correlation Matrix

- **C**$_0$ provides an unbiased estimate of pairwise correlation
- However, **C**$_0$ is rank deficient (119 zero eigenvalues)
- **C**$_0$ falsely implies the existence of "riskless portfolios"
- **C**$_0$ is not suitable for portfolio optimization purposes

PCA Correlation Matrices (Global/Local)

- Transform the sample correlation matrix to diagonal basis

$$\mathbf{D}_0 = \mathbf{U}'\mathbf{C}_0\mathbf{U}$$

Columns of **U** are eigenvectors of $\mathbf{C}_0$

- Keep only the first *J* components, where *J*<*T* and *J*<*K*

$$\tilde{\mathbf{C}} = \tilde{\mathbf{U}}\tilde{\mathbf{D}}\tilde{\mathbf{U}}'$$

$\tilde{\mathbf{U}}$ is a *K*x*J* matrix

- Compute the “idiosyncratic” variance

$$\Delta_{kk} = 1 - \mathrm{diag}_k\left(\tilde{\mathbf{C}}\right) \rightarrow \boxed{\mathbf{C}_P = \tilde{\mathbf{U}}\tilde{\mathbf{D}}\tilde{\mathbf{U}}' + \boldsymbol{\Delta}}$$

Correlation Matrix

- Scale PCA correlation matrix with official factor volatilities

$$\boxed{\mathbf{F}_P = \mathbf{S}\mathbf{C}_P\mathbf{S}}$$

PCA covariance matrix

- Global PCA refers to PCA technique on all local factors (K=319)
- Local PCA refers to applying PCA on the diagonal blocks

Bloomberg

Random Matrix Theory (Global/Local)

- Consider the diagonal matrix $\hat{\mathbf{D}}$:
  - First *J* elements are largest eigenvalues of sample correlation matrix
  - Remaining *K-J* elements are the average of remaining eigenvalues
- Rotate back to the original basis

$$\hat{\mathbf{C}} = \mathbf{U}\hat{\mathbf{D}}\mathbf{U}'$$

Note: diagonal elements not equal to 1

- Scale rows and columns to recover 1 along the diagonals

$$\mathbf{C}_R = \hat{\mathbf{S}}^{-1}\hat{\mathbf{C}}\hat{\mathbf{S}}^{-1} \quad \text{where} \quad \hat{S}_{kk} = \sqrt{\hat{C}_{kk}}$$

- Scale RMT correlation matrix with official factor volatilities

$$\boxed{\mathbf{F}_R = \mathbf{S}\mathbf{C}_R\mathbf{S}}$$

RMT covariance matrix

- Global RMT refers to RMT technique on all local factors (K=319)
- Local RMT refers to applying RMT on the diagonal blocks

Time-Series Methods (Full Factor Set)

- Assume local factors are driven by a small set of global factors

$$\boxed{\mathbf{f} = \mathbf{g}\mathbf{B} + \mathbf{e}}$$

$\mathbf{f}$ is TxK, $\mathbf{g}$ is TxJ, $\mathbf{B}$ is JxK, $\mathbf{e}$ is TxK

local factors | global factors | factor loadings | purely local

- For equities, the full set of explanatory variables is given by the factor returns of a global equity multi-factor model
- Factor loadings are estimated by time-series regression

$$\mathbf{B} = \left(\mathbf{g}'\mathbf{g}\right)^{-1}\mathbf{g}'\mathbf{f}$$

- Define factor covariance matrices

$$\mathbf{G} = \frac{\mathbf{g}'\mathbf{g}}{T-1} \qquad \mathbf{E} = \frac{\mathbf{e}'\mathbf{e}}{T-1} \qquad \mathbf{D} = \mathrm{diag}\left(\mathbf{E}\right)$$

- Local factor correlation matrix

$$\mathbf{F}_{FS} = \mathbf{B}'\mathbf{G}\mathbf{B} + \mathbf{D} \quad \rightarrow \quad \boxed{\mathbf{C}_{FS} = \mathbf{S}_{FS}^{-1}\mathbf{F}_{FS}\mathbf{S}_{FS}^{-1}}$$

Time Series (Full Set)

Time-Series Methods (Partial Factor Set)

- Partial-set method mirrors full-set method, except each local factor is regressed on small subset of global factors
- For instance, the Japan Automobile factor might only be regressed on two global factors: Japan and Automobiles
- This results in a sparse factor loadings matrix, $\tilde{\mathbf{B}}$
- Local factor covariance matrix is given by

$$\mathbf{F}_{PS} = \tilde{\mathbf{B}}'\mathbf{G}\tilde{\mathbf{B}} + \mathbf{D}$$

- The correlation matrix is given by

$$\mathbf{C}_{PS} = \mathbf{S}_{PS}^{-1}\mathbf{F}_{PS}\mathbf{S}_{PS}^{-1}$$

Time Series (Partial Set)

- Selection of relevant global factors:
  - May contain a significant subjective element
  - Omission of important factors may lead to misestimation of risk

Eigen-Adjusted Correlation Matrices

- Menchero, Wang, and Orr (2012) showed that eigenvalues of sample covariance matrix are systematically biased

$$\mathbf{D}_0 = \mathbf{U}'\mathbf{C}_0\mathbf{U}$$ Columns of **U** are eigenvectors of $\mathbf{C}_0$

- Let $\tilde{\mathbf{D}}_0$ denote the diagonal matrix of de-biased eigenvalues
- Perform reverse rotation to original basis:

$$\tilde{\mathbf{C}} = \mathbf{U}\tilde{\mathbf{D}}_0\mathbf{U}'$$ Note: diagonal elements not equal to 1

- Scale rows and columns to recover 1 along the diagonals

$$\mathbf{C}_E = \tilde{\mathbf{S}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{S}}^{-1}$$ Eigen-adjusted correlation matrix

- Eigen-adjusted method is only applicable for the local blocks

Menchero, Wang, and Orr. *Improving Risk Forecasts for Optimized Portfolios,* Financial Analysts Journal, May/June 2012, pp. 40-50

Blended Correlation Matrices

- Ledoit and Wolf (2003) showed that blending the sample covariance matrix with a factor model yielded optimized portfolios with lower out-of-sample volatility
- We blend the sample correlation matrix (using weight *w*) with the PCA correlation matrix (using *J* factors)
- Specify number of PCA factors by parameter μ , where $J=\mu K$
- Two-parameter model for correlation matrix:

$$\mathbf{C}_B(\mu, w) = w\mathbf{C}_0 + (1-w)\mathbf{C}_P(\mu)$$

Blended Matrix

- Blending can be applied at either global or local level

Ledoit and Wolf. *Improved Estimation of the Covariance matrix of Stock Returns,* Journal of Empirical Finance, December 2003, pp. 603-621

Bloomberg

Adjusted Correlation Matrices

- Local portfolio managers (e.g., US equities) want the "best" correlation matrix, known as the *target* correlation matrix
- Factor correlation matrix may be adjusted so that diagonal blocks agree with the target correlation matrix

$$\mathbf{C}_T = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

Target correlation matrix

$$\hat{\mathbf{C}} = \begin{bmatrix} \hat{\mathbf{C}}_{11} & \hat{\mathbf{C}}_{12} \\ \hat{\mathbf{C}}_{21} & \hat{\mathbf{C}}_{22} \end{bmatrix}$$

Estimated correlation matrix

- Define adjustment matrix **A**

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_{11}^{1/2}\hat{\mathbf{C}}_{11}^{-1/2} & 0 \\ 0 & \mathbf{C}_{22}^{1/2}\hat{\mathbf{C}}_{22}^{-1/2} \end{bmatrix} \rightarrow \boxed{\hat{\mathbf{C}}_A = \mathbf{A}\hat{\mathbf{C}}\mathbf{A}'}$$

Adjusted matrix

- Diagonal blocks now agree with $\mathbf{C}_T$,
- Off-diagonal blocks given by:

$$\hat{\mathbf{C}}_A(2,1) = \mathbf{C}_{22}^{1/2}\hat{\mathbf{C}}_{22}^{-1/2}\hat{\mathbf{C}}_{21}\hat{\mathbf{C}}_{11}^{-1/2}\mathbf{C}_{11}^{1/2}$$