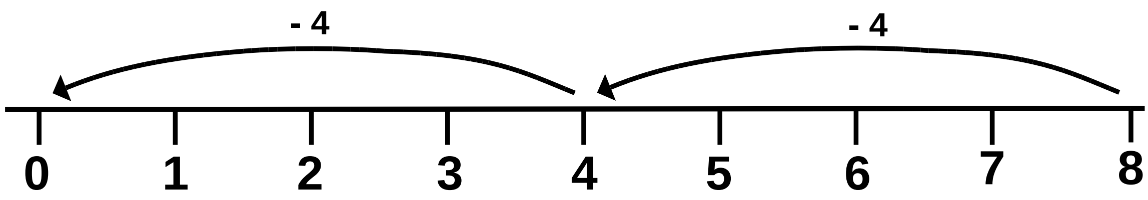
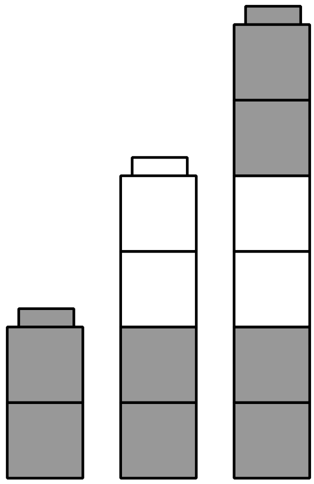
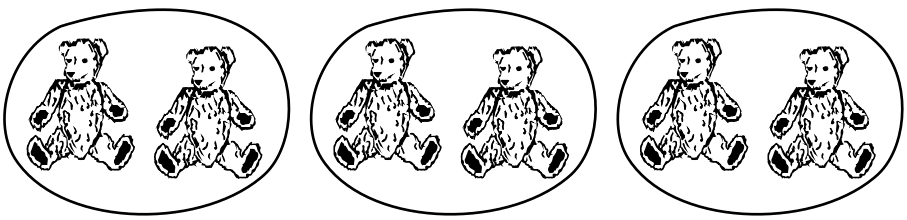


Multiplication & Division

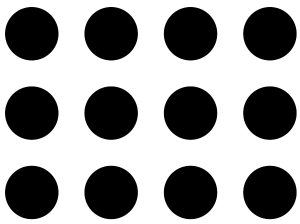
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100





3 sets of 2 = 6
3 lots of 2 = 6
3(2) → 6

4 × 3



3 × 4

Contents

Symbolic Encoding and Abstraction in Multiplication and Division

It is important for any teacher to realise that an apparently simple mathematical statement such as $3 \times 2 = 6$ has a multiplicity of meanings. For a start, the number 3 in the statement could mean many things, for example, 3 objects (toy cars or multilink cubes or shells etc.), 3 groups of objects, a position on a number line or simply an abstract number. The same is true of the numbers 2 and 6 in this statement. Thus, the numbers in the statement are generalisations of many different types of 3s, 2s and 6s found in different contexts.

In addition, the complete statement $3 \times 2 = 6$ itself is a generalisation which can represent many different situations.

e.g. John earns £3 each day for 2 days and so earns £6 in total;
A rectangle with dimensions 3cm and 2cm has an area of $6cm^2$;
A piece of elastic 3cm long is stretched until it has doubled in length to become 6cm long.

Thus, to young children the statement $3 \times 2 = 6$ often has little meaning when no context is given. A similar argument applies to simple division statements. Consequently, all early multiplication and division work should be done by means of practical tasks involving children themselves, 'real' objects or mathematical apparatus in which the context is entirely apparent. Similarly, recording of multiplication and division work should also, for the most part, contain some representation of the operations attempted. This can be in the form of the objects themselves or in pictorial form.

Moreover, the multiplication symbol, ×, and the division symbol, ÷, also both have a multiplicity of meanings. This can be seen in the conceptual structures for multiplication and division which follow. The manipulative actions undertaken by a child using real or mathematical objects, and which are represented by the $3 \times 2 = 6$ multiplication statement, vary according to context of the task. The outcomes of these potentially different physical processes resulting from multiplying 3×2 in different contexts can all be represented by the number 6 and so mathematicians can use the statement $3 \times 2 = 6$ to represent many different types of multiplication. This efficiency and economy of expression is one of the attractions of mathematics. However, for young children, the representation of very different physical manipulations of objects by the same set of mathematical symbols ($3 \times 2 = 6$) is often confusing because of the high degree of generalisation and abstraction involved. Therefore, mathematical symbols at this early stage should only be used alongside other forms of representation such as pictures or actual objects. A similar complexity applies to division which also has a multiplicity of meanings which are dependent upon context.

The different conceptual structures of multiplication and division that children will encounter are explained on the following pages. Note that the model of multiplication or division adopted for a calculation depends on the context and phrasing of the question asked.

Conceptual Structures for Multiplication

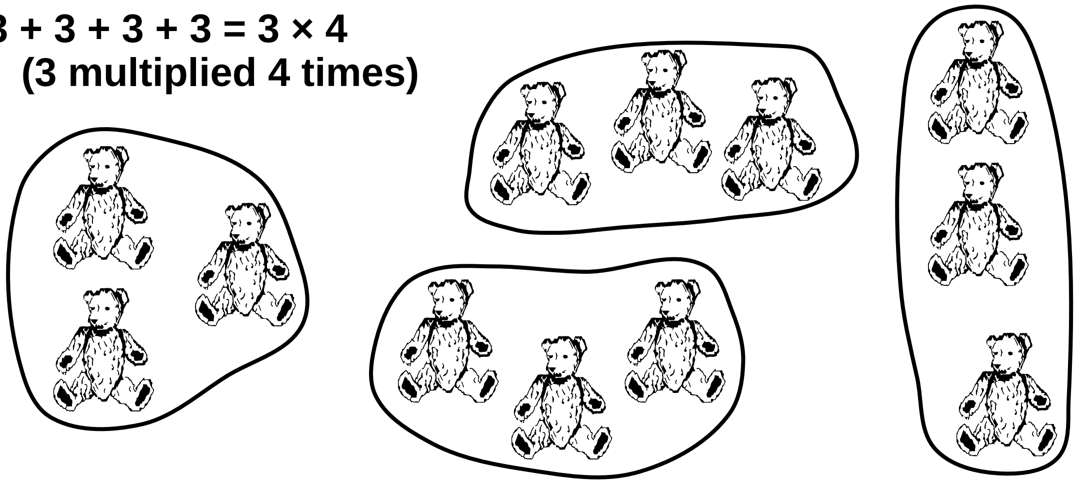
1. Repeated Addition

This is the first multiplication structure to which children should be introduced. It builds upon the already established understanding children have about addition but extends this from adding the contents of a grouping to adding the contents of one group and then using this to add the contents of several equally-sized groups. For understanding this multiplication structure, prior experience of equal groupings and of addition is important.

Some examples of this type of multiplication are:

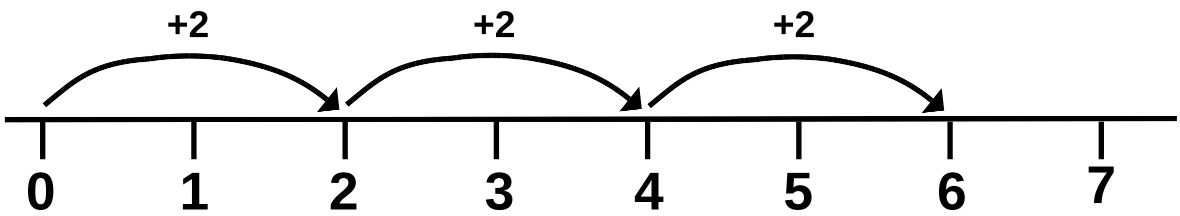
<u>With Objects</u>

$3 + 3 + 3 + 3 = 3 \times 4$
(3 multiplied 4 times)



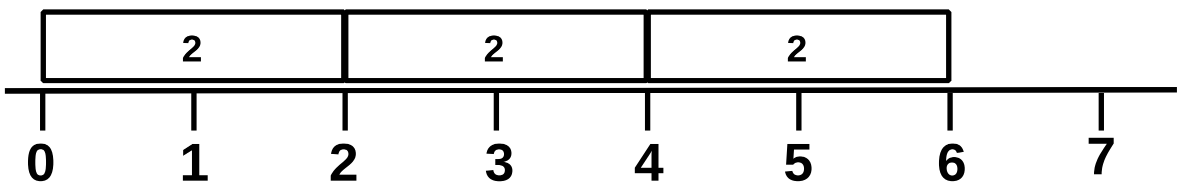
<u>On a Number Line</u>

$2 + 2 + 2 = 2 \times 3 = 6$



<u>Or with Number Rods</u> (e.g. Cuisenaire)

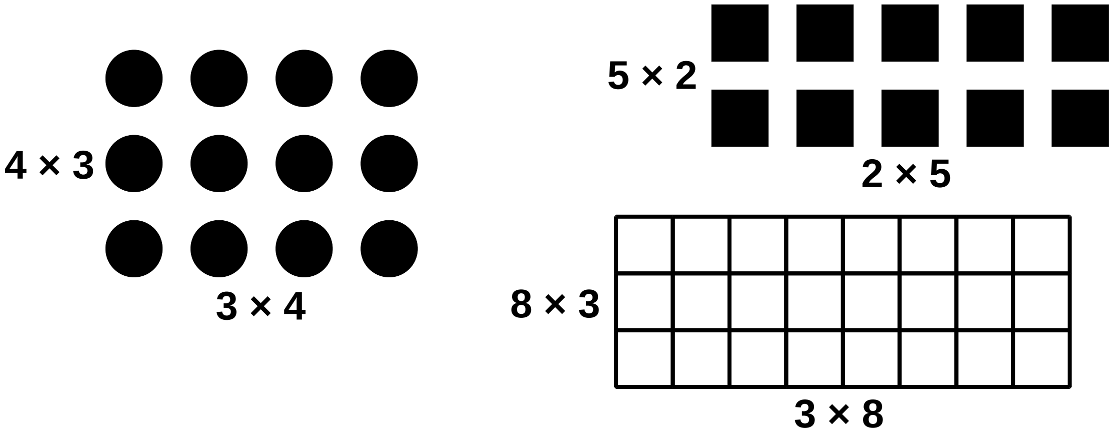
$2 + 2 + 2 = 2 \times 3 = 6$



2. Describing a Rectangular Array

This is likely to be the second multiplication structure to which children are introduced formally. It becomes useful when the commutative law for multiplication (i.e. $a \times b = b \times a$) is encountered since this provides a visual representation of this law. It is also encountered when the formula for the area of a rectangle (Area = length × breadth) is derived.

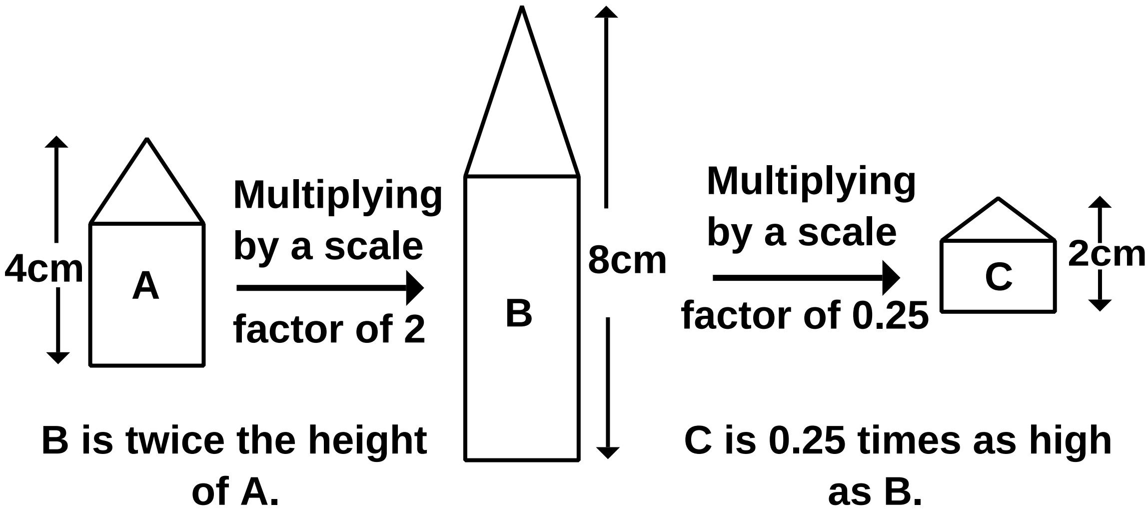
Some possible examples of this are:



3. Scaling

This is probably the hardest multiplication structure since it cannot be understood by counting though it is frequently used in everyday life in the context of comparing quantities or measurements and in calculations of the cost of multiple purchases, for example.

Some examples of this are:



Note that the result of such a scaling operation may result in a reduction in the quantity or measurement if the scale factor is less than 1.

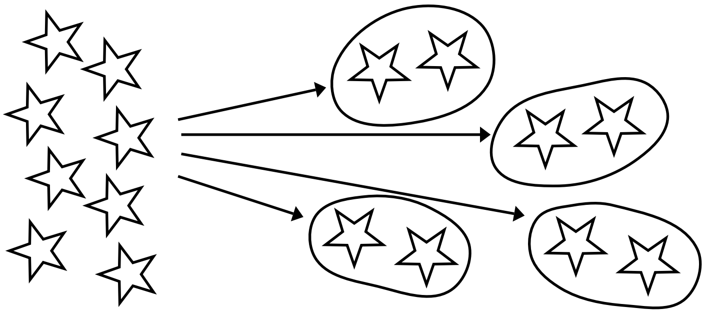
This multiplication structure also includes the idea of '**rate**'.
For example:

- the cost of 5 bars of chocolate @ 30p each is calculated as 30p × 5 = £1.50;
- if each tin of paint will cover $7m^2$ of wall space, the amount of wall space which can be covered using 10 tins of paint is calculated as $7m^2 \times 10 = 70m^2$.

Conceptual Structures for Division

1. Equal Sharing

8 ÷ 4 is interpreted as 'Share 8 equally between 4 groups' i.e. how many in each group?



The image here is of partitioning the set of 8 into 4 (equal) new sets.

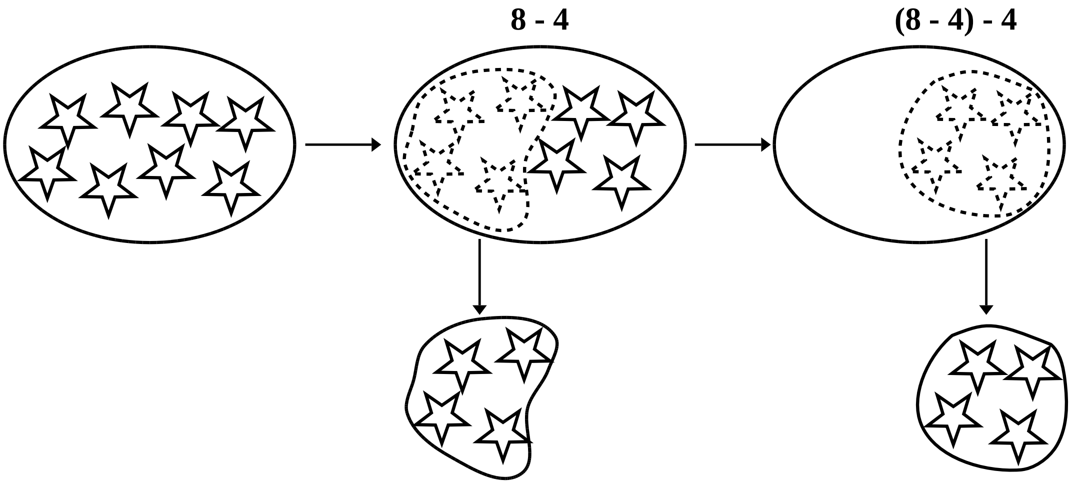
It is important that children are made aware that it is not always appropriate nor is it always possible to interpret division as equal sharing. For example, 8 ÷ 0·2 or 8 ÷ $\frac{1}{2}$ both become nonsense if interpreted as equal sharing because the number of groups into which items are shared would not be a whole number. Thus, in equal sharing situations, **the divisor must be a whole number** and less than the dividend and, consequently, the quotient will be smaller than the dividend.

2. Repeated Subtraction (or Equal Grouping)

This involves repeatedly subtracting the divisor from the dividend until either there is nothing left or the remainder is too small an amount from which to subtract the divisor again. It should be obvious that prior experience of subtraction is a prerequisite for understanding this structure.

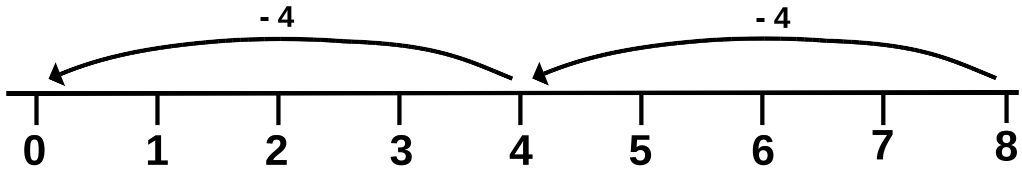
8 ÷ 4 is interpreted as 'How many sets of 4 can be subtracted from the original set of 8?' or as 'How many sets of 4 can be made from the original set of 8?'.

With Objects



2 sets of 4 can be subtracted so 8 ÷ 4 = 2.

On a Number Line



2 subtractive ‘jumps’ of 4 are possible so 8 ÷ 4 = 2.

Here, as with equal sharing, the divisor must be smaller than the dividend but in this case the divisor does not have to be a whole number. In consequence, the quotient obtained may be larger than the dividend.

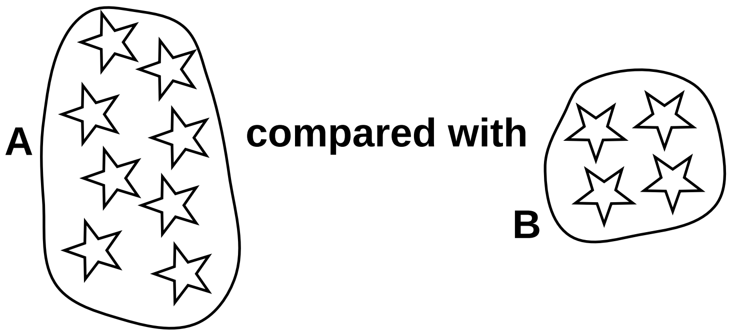
Repeated subtraction is the inverse of the repeated addition model for multiplication. Sometimes, questions of this type are solved by using the inverse of this structure thus making the question into a multiplication structure i.e. ‘how many sets of 4 are there in the original set of 8?’ or 4 × ■ = 8.

The repeated subtraction structure is the basis of many informal algorithms (methods) and of the standard short and long division algorithms.

3. Ratio

This is a comparison of the scale of two quantities or measurements in which the quotient is regarded as a scale factor. Children often find this structure difficult to understand and frequently confuse it with comparison by (subtractive) difference.

8 ÷ 4 is interpreted as ‘How many times more (or less) is 8 than 4?’



A is 2 times larger than B (because 8 ÷ 4 = 2).

This can also be written as the ratio 8 : 4 = 2 : 1 or as a fractional ratio $\frac{8}{4} = \frac{2}{1} = 2$ (the scale factor). Compare this division structure with the comparative structures for fractions. This is the inverse of the scaling structure for multiplication.

The Relationship between Multiplication and Division

Since multiplication and subtraction are **inverse operations** (i.e. one is the mathematical ‘opposite’ of the other) they should be taught alongside each other rather than as two separate entities. It is important that children are taught to appreciate and make use of this mathematical relationship when developing and using mental calculation strategies.

Progression in Teaching Multiplication and Division

General Summary of Progression

The National Numeracy Strategy recommends that teachers observe the following progression in teaching calculation strategies for multiplication and division:

1 Mental counting and counting objects (Years 2 and 3);
2 Early stages of mental calculation and learning number facts (with recording) (Years 2 and 3);
3 Working with larger numbers and informal jottings (Years 2, 3 and 4);
4 Non-standard expanded written methods, beginning in Year 4, first whole numbers;
5 Standard written methods for whole numbers then for decimals (beginning in Year 4 and extending through to Year 6);
6 Use of calculators (beginning in Year 5).

This summary of the required progression in learning for multiplication and division is outlined in more detail below.

Prior Experience Required for Learning about Multiplication & Division

The National Numeracy Strategy's 'Framework for Teaching Mathematics' recommends the introduction of multiplication and division in Year 2 of the primary school. It is important, however, that, before multiplication and division are taught formally, children should be familiar with the following experiences and skills.

Children should:

1 Be able to count securely;
2 Understand basic addition and subtraction;
3 Be able to form groupings of the same size
   - without 'remainders'
   - with 'remainders'.

Progression in Multiplication

1 Moving from unitary counting to counting in multiples;
2 The notion of repeated addition (including number line modelling) and doubling;
3 Introduction of × symbol and associated language;
4 Beginning the mastery of multiplication facts;
5 Simple scaling (by whole number scale factors) in the context of measurement or money;
6 Multiplication as an rectangular array;
7 The commutative law for multiplication;
8 Deriving new multiplication facts from known facts;
9 Factors, multiples, products
10 The associative law for multiplication;
11 Multiplying by powers of 10 (1, 10, 100, 1000 etc.);
12 The distributive law for multiplication;

13 Multiplying by multiples of powers of 10 (e.g. 30, 500);
14 Informal written methods (using partitioning);
15 Expanded Standard algorithm;
16 Contraction to Standard algorithm for multiplication by a single digit number;
17 Square numbers and simple indices;
18 Long multiplication by informal and then standard methods;
19 Multiplication of decimals by a single-digit number.
20 Prime and rectangular numbers

Progression in Division

1 The notion of 'fairness' and its application in equal sharing;
2 Refining equal sharing strategies;
3 Halving (including odd and even numbers as a by-product)
4 Division by repeated subtraction (including number line modelling);
5 Introduction of ÷ symbol and associated language;
6 Deriving division facts from multiplication facts (using division as the inverse of multiplication);
7 Informal written methods using repeated subtraction of the divisor;
8 Whole number remainders;
9 Dividing multiples of powers of 10 by powers of ten;
10 Informal written methods (using repeated subtraction of multiples of the divisor);
11 Expanded standard algorithm for short division;
12 Contraction to standard short division algorithm;
13 Fractional and decimal remainders;
14 Long division by informal methods (involving repeated subtraction of multiples of the divisor);
15 Expanded long division algorithm;
16 Standard long division algorithm;
17 Division of decimals by a single-digit number.

Children should be given opportunities to explore the operations of multiplication and division for both discrete objects (real and mathematical) and continuous quantities (such as measurements).

Establishing Initial Understanding of Multiplication

From Unitary Counting to Counting Multiples

Children will have been used to counting in ones. The first stage in learning about multiplication is to extend children's counting strategies so that they can count in intervals other than one.

Initially, this is done by using standard counting in 1s but with an emphasis given to multiples of a given number:

1 2 <u>**3**</u> 4 5 **6** 7 8 <u>**9**</u> 10 ...

The emphasised multiples can be marked in various ways as one would the rhythmic beat in music e.g. clapping, use of a drum, whispering the intermediate numbers etc. Children often can be seen marking the multiple numbers by physical movements such as nodding the head, using fingers repeatedly to count from one multiple number to the next.

However, to master such counting is more complicated than it may first appear. Julia Anghileri (1997) suggests that such counting involves **three concurrent counting sequences**:

Objects:	(●●●)	(●●●)	(●●●)	(●●●)	
Verbal Count:	1 2 **<u>3</u>**	4 5 **<u>6</u>**	7 8 **<u>9</u>**	10 11 **<u>12</u>**	....
Physical (e.g. fingers) or Internal Count :	1 2 3	1 2 3	1 2 3	1 2 3	....
Tally:	1	2	3	4	

adapted from p. 45-46, ‘Teaching and Learning Early Number’, Ed. Ian Thompson (1997), OUP

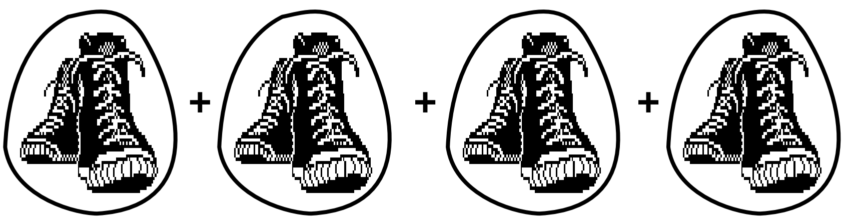
When counting objects, the physical or internal count constitutes a counting of the elements of each set while the tally keeps track of how many sets have been counted. If counting without objects, the physical or internal count monitors the progress towards the next multiple whereas the tally is a record of how many multiples of the chosen number have been encountered. Clearly, keeping track of all three counts at the same time is much more complex than ordinary counting.

Eventually, the intermediate numbers between the multiples can be dropped from the sequence leaving a count based on multiples of the given number only. Development of this type of counting should continue throughout the remaining primary school years by extending this as follows:

- counting backwards as well as forwards
- starting from numbers other than zero
- increasing the size of the multiple from single-digit numbers to multi-digit numbers e.g. counting in 100s, 25s etc.
- counting in multiples of a fraction e.g. 1, $1\frac{1}{2}$. 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, ...
- counting in multiples of a decimal e.g. £1·99, £3·98, £5·97, £7·96, ...

Introducing Repeated Addition

The idea of repeated addition is essentially an extension of the counting in multiples outlined above, namely, the recognition that if sets of objects are equal in numerical value, they can be more efficiently counted by counting the number of objects in one set and repeatedly adding this number to find the total value of all the sets.



2 + 2 + 2 + 2 = 8

Initial awareness of the idea of repeated addition should be developed through examples where such multiples occur naturally e.g.

for 2s: pairs of hands, feet, ears, eyes, pockets, shoes, bicycle wheels, Noah's Ark animals, butterfly wings

for 3s: place settings of knife, fork and spoon, clover leaves

for 4s: wheels on cars, lorries etc.

for 5s: digits on hands or feet

for 6s: insects' feet

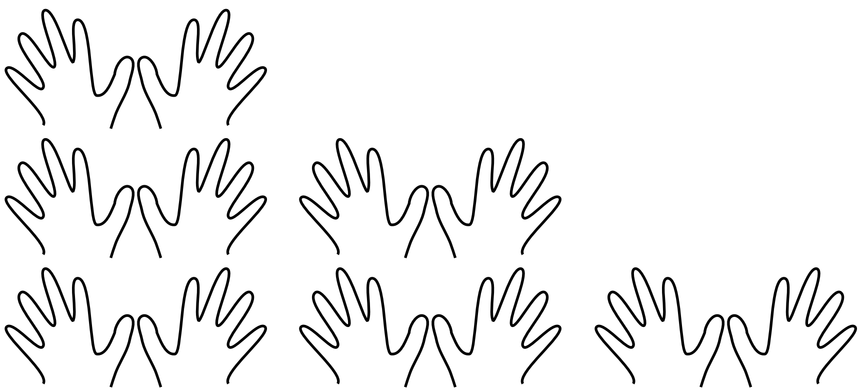
for 8s: spiders' feet

for 10s: digits on both hands or both feet.

Gradually, the scope of such activities should be broadened to include repeated addition of groups of natural objects (such as pebbles, shells etc.) and mathematical objects (such as counters, cubes, beads) where specific multiples do not naturally occur. When grouping some objects it often helps to provide the child with some kind of equipment to mark the boundary of a group, e.g. a hoop or a place mat, so that the contents of different groups remain separate and to emphasise the two levels of counting: counting the contents of a group and counting the number of groups.

At this stage any written notation is still that of **addition, not multiplication**. So we might record a typical activity as follows:

Pairs of Hands



2

2 + 2 = 4

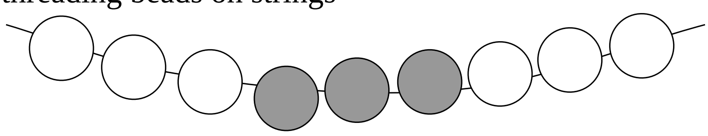
2 + 2 + 2 = 6

and so on ...

However, early multiplication language such as 'pairs of', 'sets/groups/lots of' etc. should be used.

<u>Typical activities include:</u>

- threading beads on strings

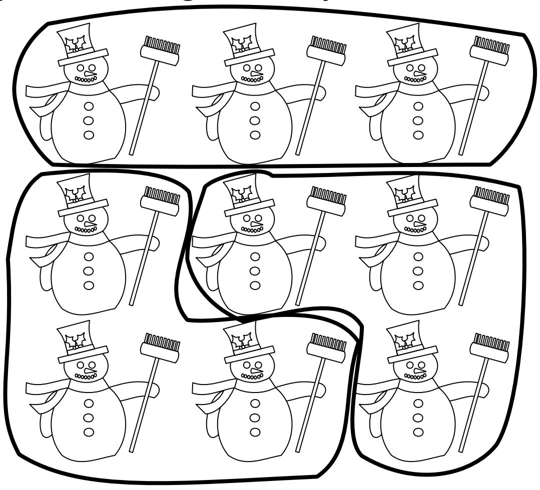


3 + 3 + 3 = 9

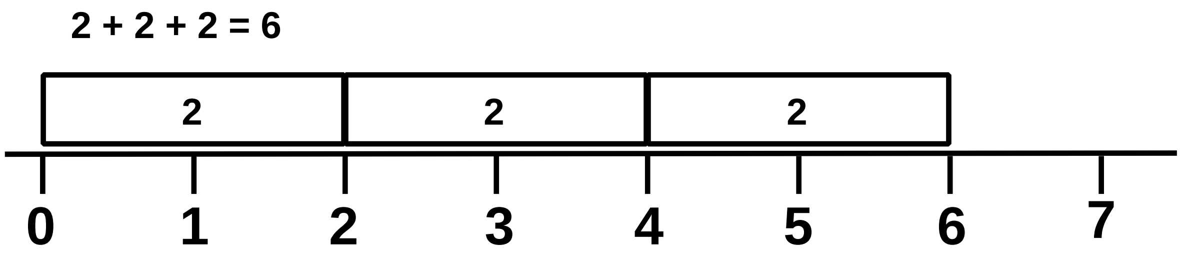
- building multilink/unifix trains

2 + 2 + 2 + 2 = 8

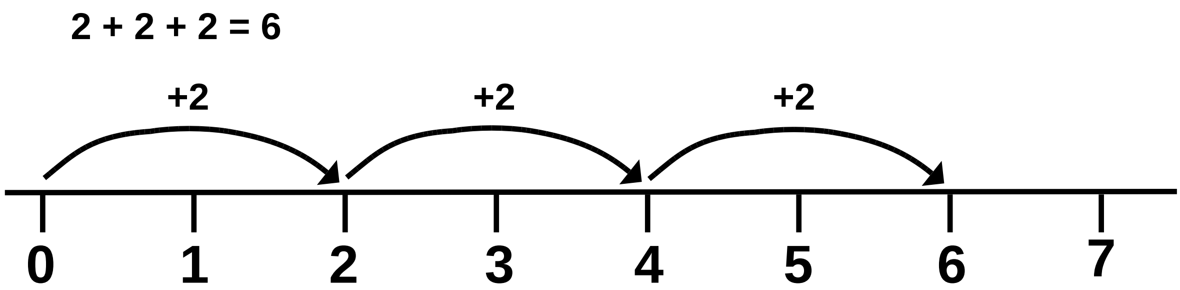
- circling a number repeated motifs on a wrapping paper motif and then using this as a way of calculating how many motifs are on the wrapping paper



Having established repeated addition with objects, we can begin the process of abstraction of this idea. The repeated addition idea can be applied to repeated number rods (such as Cuisenaire) on a number track or number line:

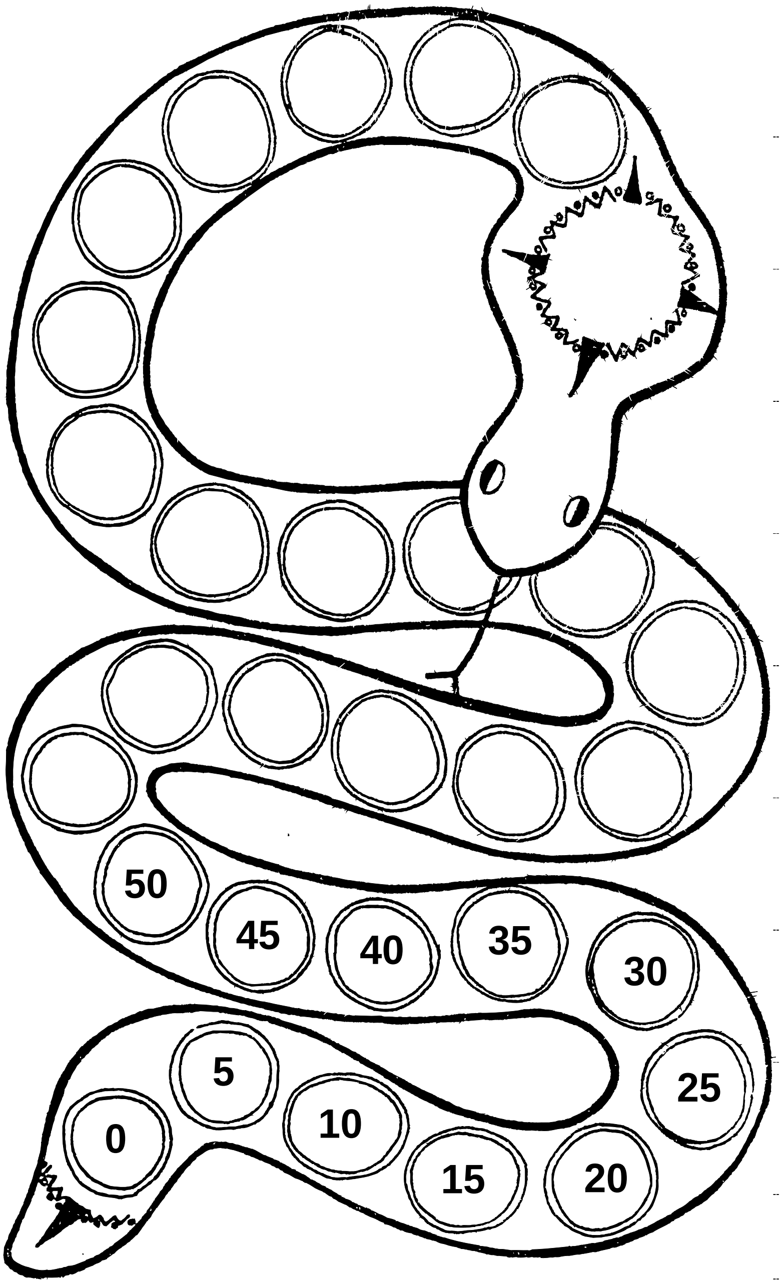


A further layer of abstraction can be applied by counting ‘jumps’ along the number line instead:



It is important to note that **for a number line** we are using **jumps of 2 intervals** (between numbers) **and not jumps of 2 numbers**. This avoids the common error of counting the starting number.

A final degree of abstraction is obtained on omission of the numbers between the multiples. For example, recording the multiples pattern obtained by repeated addition in the number snake:



Such repeated addition patterns can be generated by using the constant function on a calculator. The pattern above would be obtained by entering

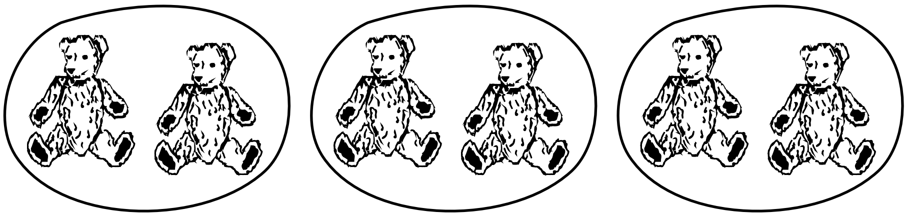
+ + 5 = = = ...

Each time the = key is pressed another 5 is added to the total thus generating the repeated addition pattern:

5 = 5
5 + 5 = 10
5 + 5 + 5 = 15
5 + 5 + 5 + 5 = 20 etc.

Introducing the × Symbol

The recording of multiplication using repeated addition notation of the form 2 + 2 + 2 + ... is long-winded. Alternative ways of recording can thus be sought. Some common intermediate recording methods are:



3 sets of 2 = 6
3 lots of 2 = 6
3(2) → 6

It is worth checking which, if any, of these recording methods are used in your school (e.g. in school mathematics scheme or in published schemes) before introducing these to children so as to avoid confusion.

The × symbol is variously interpreted by children (and teachers!). For example,

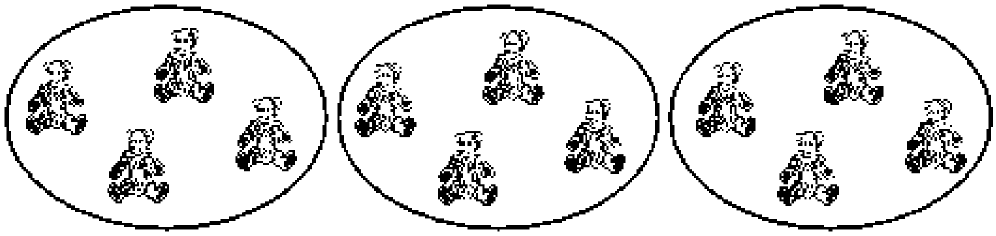
3 x 4 is often interpreted by children as one of the following:

3 multiplied by 4
3 times 4
3 fours
3 by 4
4 by 3
4 threes
3 lots of 4
4 lots of 3
3 'timesed' by 4

These interpretations are not all equivalent. In consequence, children are often confused about how to interpret the × symbol, particularly when they are asked to use apparatus to model a multiplication task such as 3 × 4 = ■. The confusion is about which of the two numbers describes the cardinal value of the set (the numerical value of its contents) and which refers to the number of sets i.e. Should there be 3 or 4 sets? Should there be 3 or 4 objects in a set?

Essentially, all of the 'possible' interpretations listed above fall into one of two categories:

(a) a set of 4 elements replicated 3 times



The interpretations **3 times 4**, **3 lots of 4**, **3 fours** and **4 by 3** all result in the above type of representation consisting of 3 sets, each of 4 objects.

(b) a set of 3 elements replicated 4 times

The interpretations **3 multiplied by 4**, **4 threes**, **3 by 4**, **4 lots of 3** and **3 'timesed by' 4** all result in a representation of 4 sets each containing 3 objects.

Strictly speaking, **the × symbol has only one correct interpretation, namely, 'multiplied by'** and so representation (b) above is the correct representation of 3 × 4. The ×4 is the multiplicative operation which is performed on the set of 3.

Unfortunately, parents, children and teachers are often inconsistent in their interpretations of multiplication tasks of the form 3 × 4 = ■, adopting different representations and different vocabulary on different occasions. Even some long-serving teachers are often unaware that they are causing major confusion because they are not consistent in their representation of multiplication tasks and in their use of mathematical language when reading or describing such tasks.

This is partly because of the use of the incorrect term 'times' as a substitute term for the correct term 'multiplied by' (these terms imply different representations) and partly because most adults know the commutative law for multiplication (and subconsciously use it without realising that they are doing so) whereas children at this stage do not possess this knowledge. It should be obvious from this that the confusion experienced by children tends to lessen on introduction of the commutative law for multiplication (in this case, 3 x 4 = 4 x 3). Equally, the need for consistency of interpretation on the part of the teacher is apparent.

Because of the potential for the confusion described above, some published schemes and some teachers avoid the introduction of the × symbol until after the commutative law has been mastered, preferring instead to persist with the brackets notation 4(3). Again, it may be advisable to check the published schemes and/or mathematics scheme of work in operation in a particular school before deciding what approach to take with any particular class of children in this regard.

Establishing Initial Understanding of Division

Fairness and Equal Sharing

<u>For Discrete Objects</u>

This is based on the notion of 'fairness' with which children are usually already familiar before division is first encountered in a formal sense. Experiences from home and from school contexts (other than formal mathematics), such as playground games, often involve this notion. It is also quite likely that the idea of equal sharing may be informally known from everyday activities such as sharing sweets, food or toys.

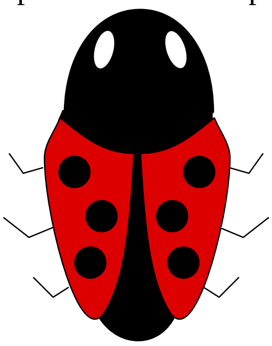
Initially, teacher-provided sharing activities are probably best limited to equal sharings which do not result in a remainder. The earliest equal sharing strategy adopted by most children is that of **distributing objects to people or groups one at a time** in a cyclical fashion until all the objects are exhausted. It is usually of assistance to the child if some kind of boundary is marked for each group (e.g. a place mat or a closed loop of string) within which the child places the distributed objects. This emphasises the individual groups and avoids errors which may occur if the child distributes objects in a manner that leaves him or her unsure as to which group particular objects have been assigned.

This early equal sharing strategy is not as straightforward as it may appear because the child has to keep track of the order of the cycle of distribution as well as deciding if there are enough left at the end of each cycle in order to distribute another object to each group or person. There is also a subtle difference between 'mathematical equal sharing' and 'real-life equal sharing' in that in real-life situations the child itself is usually one of the recipients in the distribution process (e.g. when sharing sweets or crisps with friends) but in mathematical tasks the child is usually merely the distributor and not a recipient (e.g. sharing 10 marbles between 2 bags).

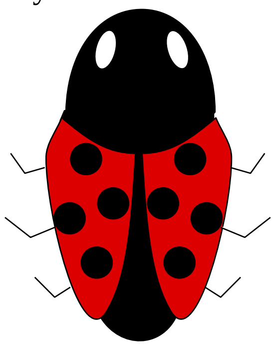
This initial 'one at a time' equal sharing strategy can be refined so that the child distributes more than one item at a time by deciding at the end of each cycle that this is both possible and faster. A further refinement might be to roughly share out the objects and then adjust afterwards by inspection until all groups or people have the same.

<u>Typical activities include:</u>

- sharing spots between two parts of a ladybird:

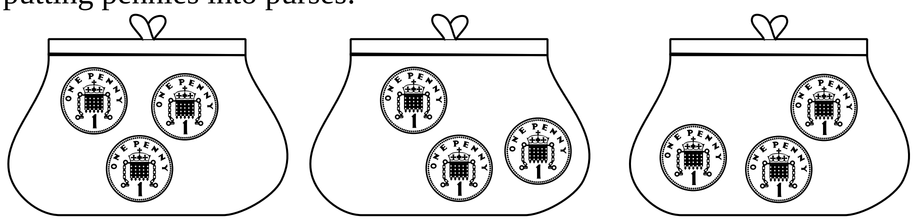


6 spots shared between two ladybird wings



8 spots shared between two ladybird wings

- forming teams on the playground or in P.E. to play games
- putting pennies into purses:



9 pennies shared between 3 purses

- sharing toy animal between fields;
- sharing coloured pencils between pots.

For Continuous Quantities

The idea of equal sharing should be applied to continuous quantities as well. In such cases, the equal sharing procedure may be physically different. Possible strategies that can be taught include equal sharing by:

- folding (e.g. to divide up paper or cloth);
- measuring (e.g. dividing a piece of plasticine by weighing)
- direct comparison (e.g. pouring water from one container to another identical container until the height of the water level in each is the same).

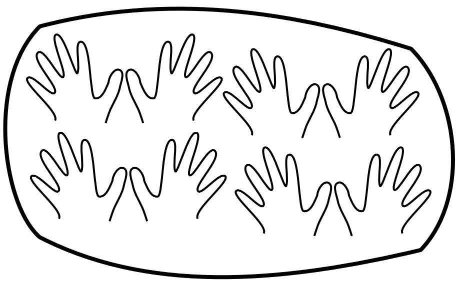
Note the close links here with finding simple fractions of quantities.

Introducing Repeated Subtraction

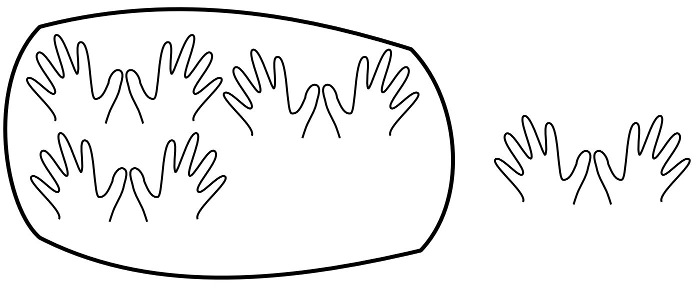
It is usually considered sensible to begin teaching repeated addition before repeated subtraction. Repeated subtraction is the inverse of the multiplicative structure known as repeated addition. Consequently, the same ideas should be taught as for repeated subtraction but making appropriate adjustments for the inverse (mathematically opposite) aspects.

So, initial work focuses on counting backwards (instead of forwards) in 1s emphasising the multiple numbers as described in the earlier section ‘From Unitary Counting to Counting Multiples’ until the intervening numbers can be dropped from the counting sequence. The same complexities apply but with the added factor of counting backwards (instead of forwards) with which most children are less confident.

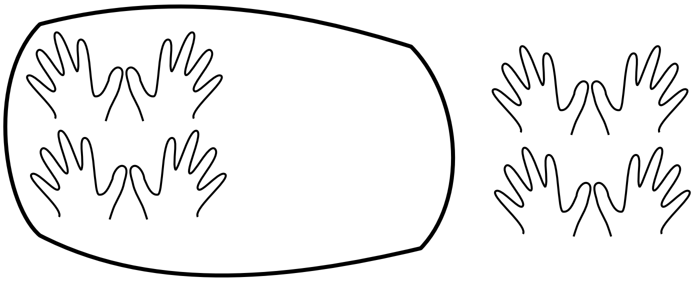
As described earlier for repeated addition, children should be introduced to practical repeated subtraction examples where multiples occur naturally. The child starts with a collection of such objects and repeatedly subtracts the given number while keeping track of how many of the multiple has been removed from the original group.



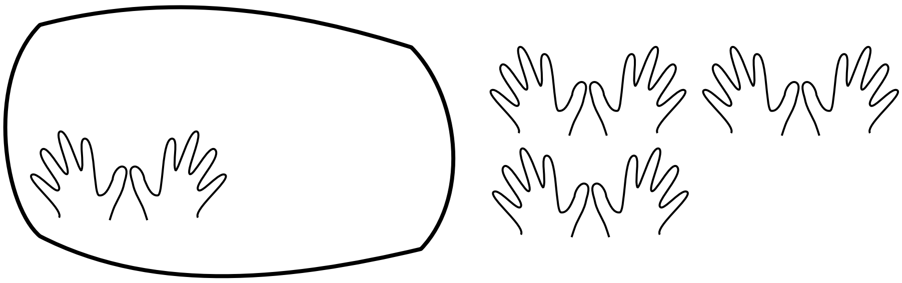
The cardinal value of the original set is 8.



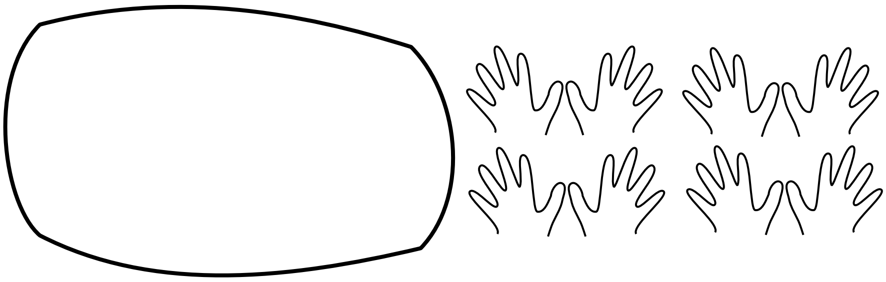
2 hands subtracted:
8 - 2 = 6.



2 more hands subtracted:
8 - 2 - 2 = 4



2 more hands subtracted:
8 - 2 - 2 -2 = 2

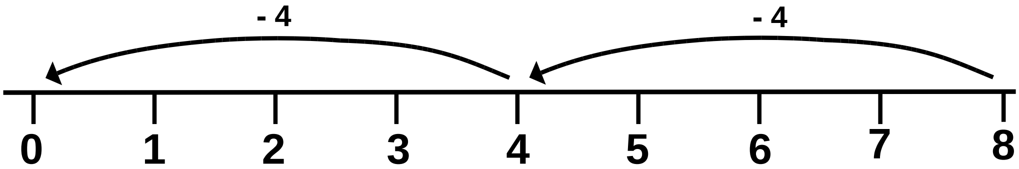


2 more hands subtracted:
8 - 2 - 2 -2 - 2 = 0.

4 pairs of hands are subtracted

Just as for repeated addition, activities should be broadened to include repeated subtraction of groups of natural objects (such as pebbles, shells etc.) and mathematical objects (such as counters, cubes, beads) where specific multiples do not naturally occur.

Having established the idea with actual objects, the process of abstraction towards using a number line with no objects can then begin.



Repeated Subtraction on a Number line: 8 - 4 - 4

Introducing the ÷ Symbol

The ÷ symbol is used to represent any of the possible division structures. Thus, the division structure to be adopted depends on contextual information (if this is supplied).

The division symbol, ÷, is often interpreted differently by children (and also by teachers). Just as for the × symbol, there is often a lack of consistency in the interpretation of the ÷ symbol. For example,

6 ÷ 3 is often interpreted as one of the following:

6 shared between 3

6 shared by 3

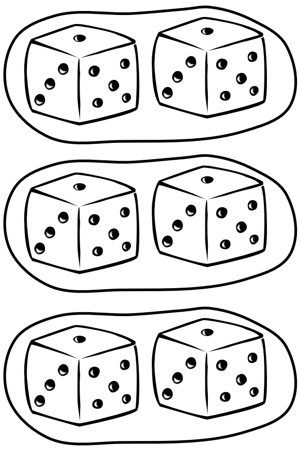
6 shared into 3s

6 shared into 3 groups
6 divided by 3
6 divided into 3s
6 divided into 3 groups
How many times does 3 ‘go into’ 6?
How many 3s in 6?

These interpretations are not equivalent. In fact, they each fall into one of two categories each of which has a different physical representation. Even though the wording is similar, the division process is visually different; one of them involves equal sharing and the other implies equal grouping or repeated subtraction.

However, because children, and often teachers, fail to perceive the distinction between these two categories children often experience confusion when trying to decide how to tackle a division task of the form 6 ÷ 3 = ■. In a similar way to the confusion for multiplication outlined earlier, the confusion arises because of uncertainty about whether the divisor number should be assigned to the number of sets resulting from the division or the number of objects in each set.

(a) The Divisor as the number of sets

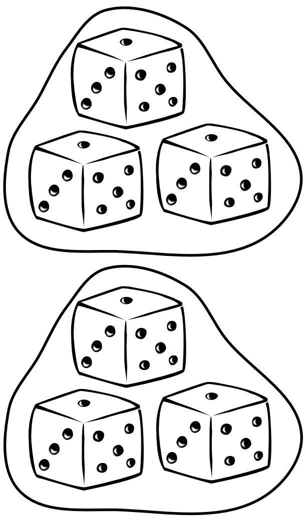


The physical representation of this category of interpretation of 6 ÷ 3 is as shown in the picture opposite. The following interpretations of 6 ÷ 3 result in this physical representation:

6 divided into 3;
6 shared by 3;
6 divided by 3;
6 shared between 3;
6 shared into 3 groups;
6 divided into 3 groups.

(The equal sharing structure)

(b) The Divisor as the content of each set



The physical representation of this category of interpretation of 6 ÷ 3 is as shown in the picture opposite. The following interpretations of 6 ÷ 3 result in this physical representation:

6 shared into 3s;
6 divided into 3s;
How many 3s in 6?
How many times does 3 ‘go into’ 6?

(The equal grouping or repeated subtraction structure)

[**Note that some interpretations (e.g. shared between) only apply in limited circumstances.** e.g. '0.4 shared between 0.2' has no meaning.]

When one notes the similarity between the expressions '6 divided into 3' and '6 divided into 3s', for example, it is probably unsurprising that the language associated with division causes many difficulties for children. Add to this the difference in physical representation and that '6 divided into 3' implies equal sharing while '6 divided into 3s' implies equal grouping or repeated subtraction and it becomes apparent that the potential for confusion is enormous. The situation may be exacerbated by use of some published schemes (and some teachers!) which do not always recognise these subtleties in meaning and so interchange these terms as if they were synonymous.

Learning Basic Multiplication and Division Facts

The initial emphasis is on progressing from reliance on counting, which rapidly becomes an inefficient strategy for calculation, to knowledge (i.e. instant recall) of certain frequently used number facts which will be of great assistance in harder calculations at a later stage in the learning process. It is important that children are given opportunities to use a wide variety of equipment in learning these basic number facts. This enables the teacher to provide learning opportunities of **the same number facts in different contexts**. This also allows the child to understand (eventually) that $3 \times 4 = 12$ is true whether he/she is using multilink cubes, pebbles or spots on a domino. In this way, the child learns to discard all irrelevant features of the equipment being used (e.g. colour, shape) and begins to recognise over a period of time the relevant feature which is common to all the apparatus used (in this case, the number of cubes, pebbles or spots) thus **abstracting** the number fact $3 \times 4 = 12$ from the range of contexts experienced.

Mastery of such facts is important since later multi-digit number calculations become much easier if children are secure in their knowledge of basic multiplication and division facts. Thus, within the National Numeracy Strategy great emphasis is placed on children learning such facts so as to be able to :

- recall them at need;
- use these known facts to derive new facts;
- appreciate the relationship between multiplication and division;
- build a repertoire of mental strategies for doing calculations;
- approximate calculations to determine whether answers obtained are reasonable;
- solve word problems involving multiplication and division.

In particular, children should, initially, be expected to master:

- doubling commonly-encountered numbers (e.g. numbers 1 - 20, multiples of 5, of 10, of 50, of 100 etc.);
- halving commonly-encountered numbers (e.g. even numbers up to 20, multiples of 10, of 100 etc.);
- multiplication facts for 2s, 5s and 10s up to 2×10, 5×10 and 10×10 respectively;
- the corresponding division facts;
- multiplication of single-digit numbers by 10;
- division of two-digit multiples of 10 by 10.

The **National Numeracy Strategy** gives examples in *The Framework for Teaching Mathematics* (Section 5, page 53) of those multiplication and division facts which children in Years 2 and 3 should be able to recall rapidly.

As children become confident with these known facts the range of recallable facts should be expanded to include:

- all remaining multiplication facts up to 10 × 10, including × 0 and × 1 facts;
- the corresponding division facts;
- doubles of all numbers from 1 - 100 and the corresponding halves;
- doubles of multiples of 10 up to 1000 and the corresponding halves;
- doubles of multiples of 100 up to 10 000 and the corresponding halves;
- doubles of two-digit numbers including decimals and the corresponding halves.

The **National Numeracy Strategy** gives examples in *The Framework for Teaching Mathematics* (Section 6, pages 58-59) of those multiplication and division facts which children in Years 4, 5 and 6 should be able to recall rapidly.

Teaching Strategies for Doubling and Halving

At first, the idea of doubling develops from learning simple addition facts such as 3 + 3, 9 + 9 etc. . As the numbers to be doubled become bigger, alternative strategies are required which make use of the idea of partitioning numbers into multiples of powers of ten (developed when learning about place value). For example, the number 32 is partitioned as 30 + 2. Developing these more advanced strategies should be considered as a two-stage process in learning:

- doubles facts which do not involve 'crossing tens boundaries'

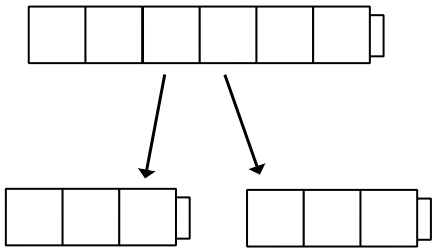
e.g. double 32	= double (30 + 2)	(partitioning 32 as 30 + 2)
	= double 30 + double 2	
	= 60 + 4	(the 4 is less than 10 so no tens boundary is crossed)
	= 64.	

- doubles facts which do require 'crossing tens boundaries'

e.g. double 36	= double (30 + 6)	(partitioning 36 as 30 + 6)
	= double 30 + double 6	
	= 60 + 12	(the 12 is more than ten so the total will exceed the 60s numbers and cross the 70 boundary)
	= 60 + 10 + 2	(either perceived as partitioning 12 or as 'carrying the ten' from the 12)
	= 70 + 2.	

The same partitioning strategy can be applied for doubling numbers with more than two digits.

Halving develops from the Partitioning subtraction structure (see *Addition and Subtraction* Booklet). Number cubes such as multilink or unifix are suitable for this purpose:



Splitting 6 cubes into 3 cubes and 3 cubes

From this idea, children can be introduced to **odd and even numbers** (as a by-product), even numbers being defined as those numbers which can be split into 2 equal whole-number amounts and odd numbers as those which cannot.

Effectively, halving of numbers less than 20 can be seen as the inverse of doubling:

e.g. 18 = 9 + 9 (splitting the 18 using knowledge of facts such as 9 + 9 = 18).

For halving larger even numbers, partitioning into multiples of powers of ten is required.

e.g. half of 42	= half of (40 + 2)	(partitioning 42 as 40 + 2)
	= half of 40 + half of 2	
	= 20 + 1	(using knowledge of 40 = 20 + 20 and 2 = 1 + 1)
	= 21.	

Strategies for Teaching Multiplication Facts up to 10 × 10

It is usual to teach multiplication facts/tables in the following order:

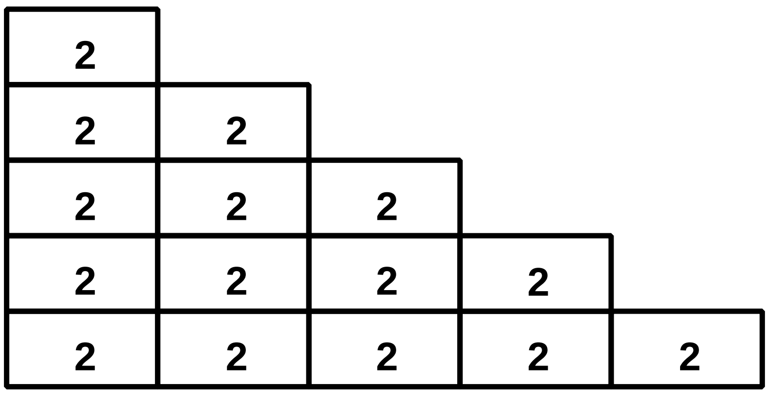
2× , 10× and then 5× tables
3× and 4× tables
6×, 8× and 9× tables
7× table.

It is generally accepted that the teaching of multiplication tables begins with children building representations of facts using concrete apparatus. The subsequent introduction of pictorial representations of multiplication tables increases the degree of abstraction with the use of symbols only being the final stage in the abstraction process (compare this with J. Bruner's three Modes of Learning: Enactive, Iconic and Symbolic).

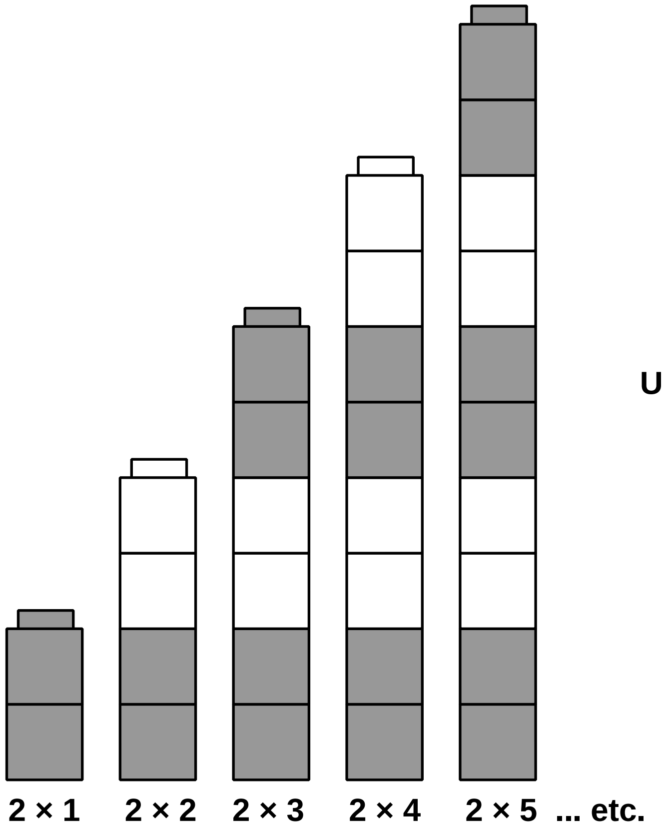
1. Use of Mathematical Apparatus to Build Multiplication Facts

It is common for children to be asked to use apparatus such as number rods or number cubes to build concrete representations of multiplication tables.

1 two or 2 × 1
2 twos or 2 × 2
3 twos or 2 × 3
4 twos or 2 × 4
5 twos or 2 × 5
etc.



Using Cuisenaire Rods to Build the Table of 2s

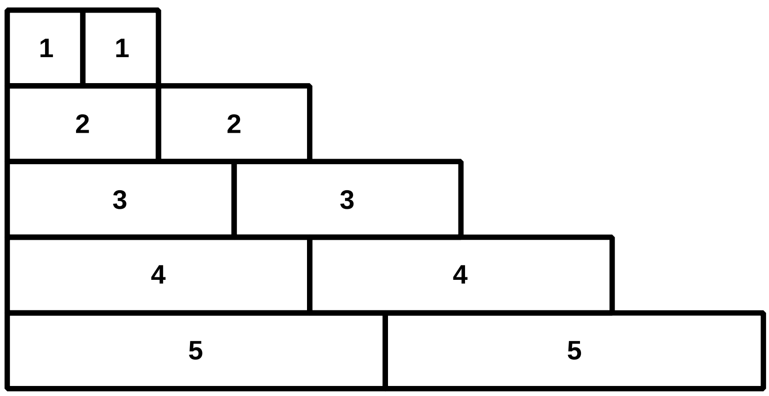


Using Unifix or Multilink Cubes to Build the Table of 2s

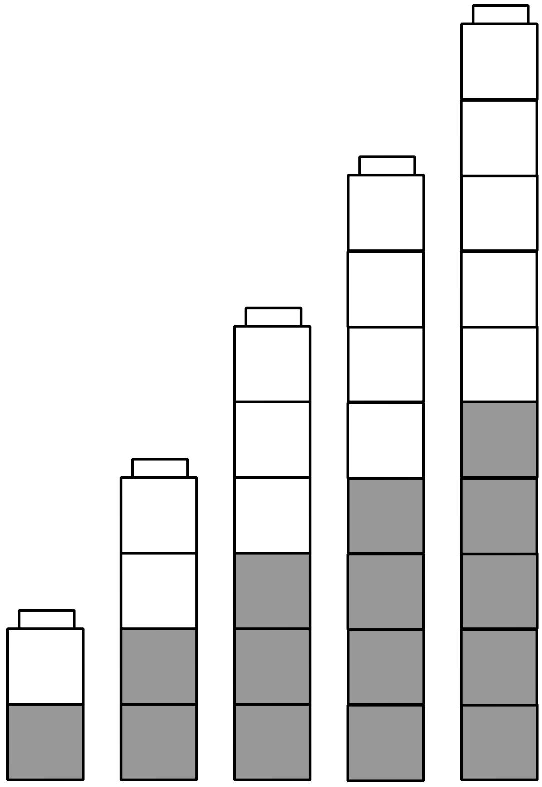
A distinction should be made between the **table of twos** and the **two times table**. The first is the collection of sets with a cardinal value of two (i.e. **sets of 2**) whereas the latter is concerned with the total of **two sets**. The representations shown above are for the table of twos. the corresponding representations for the two times table are as follows:

2 ones	**or**	**1 × 2**
2 twos	**or**	**2 × 2**
2 threes	**or**	**3 × 2**
2 fours	**or**	**4 × 2**
2 fives	**or**	**5 × 2**

... etc.



Using Cuisenaire Rods to Build the Two Times Table



Using Unifix or Multilink to Build the Two Times Table

1 × 2   2 × 2   3 × 2   4 × 2   5 × 2   ... etc.

2. Using Pictorial Representations

Similar representations to those above can be achieved by colouring squares on squared paper. The placing of number rods alongside number lines and the abstraction of this (showing jumps along a number line) are also common tasks for children.

3. Using Relationships between Multiplication Tables

Certain multiplication tables can be obtained from other multiplication tables by **doubling**. For example, the 4× table can be obtained by doubling the 2× table, the 6× table can be derived by doubling the 3× table etc. :

3× table	doubling the 3× table	6× table
1 × 3 = 3	1 × double 3 = double 3	1 × 6 = 6
2 × 3 = 6	2 × double 3 = double 6	2 × 6 = 12
3 × 3 = 9	3 × double 3 = double 9	3 × 6 = 18
4 × 3 = 12	4 × double 3 = double 12	4 × 6 = 24 etc.

Corresponding elements of different multiplication tables can also be added or subtracted to produce other multiplication tables. For example, the 7× table may be obtained by adding the corresponding elements of the 5× and 2× tables because
$7 = 5 + 2$ and so $n \times 7 = n \times (5 + 2) = (n \times 5) + (n \times 2)$, where n is any number:

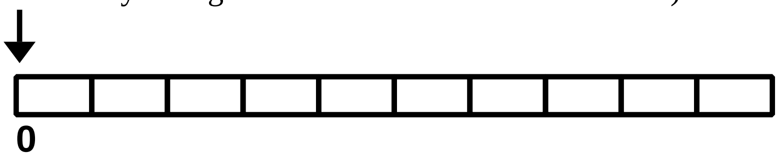
5× table	2× table	7× table
1 × 5 = 5	1 × 2 = 2	1 × 7 = 5 + 2 = 7
2 × 5 = 10	2 × 2 = 4	2 × 7 = 10 + 4 = 14
3 × 5 = 15	3 × 2 = 6	3 × 7 = 15 + 6 = 21
4 × 5 = 20	4 × 2 = 8	4 × 7 = 20 + 8 = 28 etc.

This works by making use of the **distributive law** for multiplication

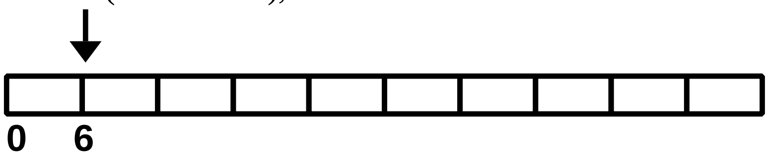
4. The Construction, from Known Facts, of Multiplication Tables

It is possible to help pupils to ‘construct’ a number line representation of any multiplication table using a counting stick marked in 10 divisions. The various elements of the multiplication table can be derived from those facts already known (usually by doubling or halving or by adding or subtracting multiples). For example, the 6 × table could be built up with pupils as follows:

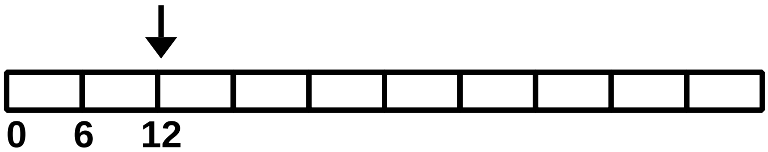
1 Label the left end of the counting stick with 0 (from knowing that multiplying any number by zero gives an answer of zero so 0 × 6 = 0):



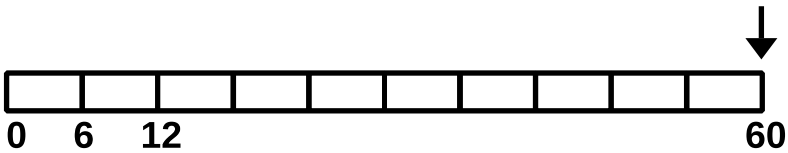
2 Using knowledge that multiplying any number by 1 does not alter the value of the number (so 1 × 6 = 6), label the next division with the number 6:



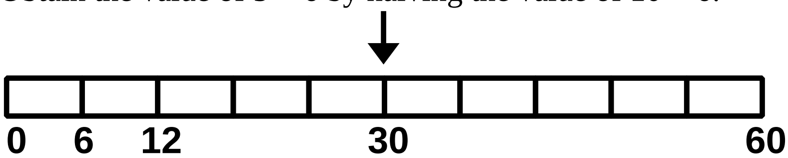
3 Double 6 to obtain 2 × 6 = 12:



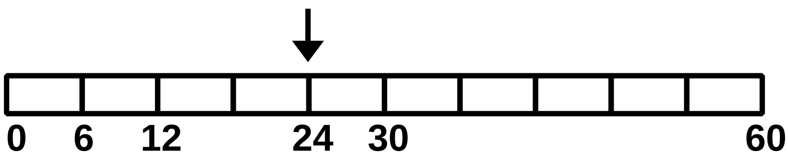
4 Use knowledge of multiples of 10 (e.g. from counting in tens) to calculate 10 × 6 = 60:



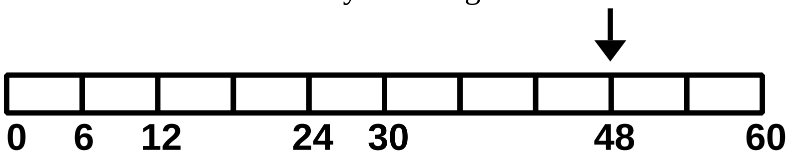
5 Obtain the value of 5 × 6 by halving the value of 10 × 6:



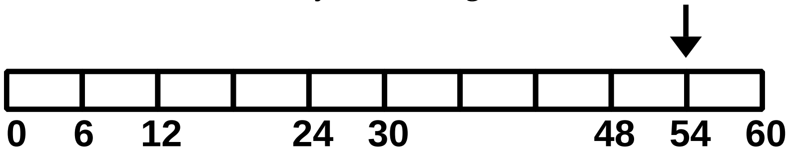
6 Obtain the value of 4 × 6 by either doubling the value of 2 × 6 or subtracting 6 from the value of 5 × 6:



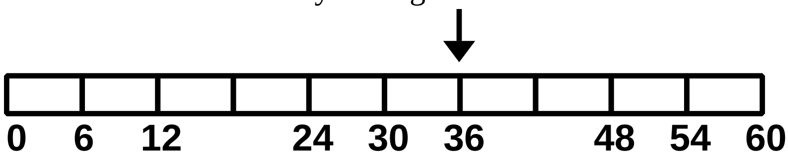
7 Obtain the value of 8 × 6 by doubling the value of 4 × 6:



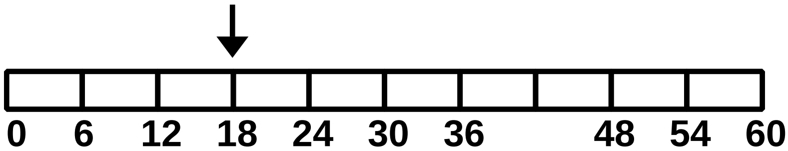
8 Find the value of 9 × 6 by subtracting 6 from the value of 10 × 6:



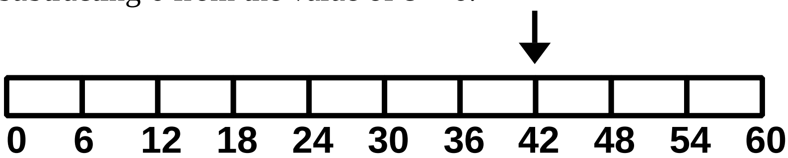
9 Find the value of 6 × 6 by adding 6 to the value of 5 × 6:



10 Calculate the value of 3 × 6 by either halving the value of 6 × 6 or adding 6 to the value of 2 × 6:



11 Finally, obtain the value of 7 × 6 by either adding 6 to the value of 6 × 6 or by subtracting 6 from the value of 8 × 6:



5. Using Number Squares, Number Lines and Number Tracks

The 100 square is frequently used as a means to explore multiplication facts. It is quite common for children to be asked to mark all the numbers from a specified multiplication table and look for patterns:

1	2	3	4	**5**	6	7	8	9	**10**
11	12	13	14	**15**	16	17	18	19	**20**
21	22	23	24	**25**	26	27	28	29	**30**
31	32	33	34	**35**	36	37	38	39	**40**
41	42	43	44	**45**	46	47	48	49	**50**
51	52	53	54	**55**	56	57	58	59	**60**
61	62	63	64	**65**	66	67	68	69	**70**
71	72	73	74	**75**	76	77	78	79	**80**
81	82	83	84	**85**	86	87	88	89	**90**
91	92	93	94	**95**	96	97	98	99	**100**

The 5× table pattern

1	2	3	4	5	6	7	8	**9**	10
11	12	13	14	15	16	17	**18**	19	20
21	22	23	24	25	26	**27**	28	29	30
31	32	33	34	35	**36**	37	38	39	40
41	42	43	44	**45**	46	47	48	49	50
51	52	53	**54**	55	56	57	58	59	60
61	62	**63**	64	65	66	67	68	69	70
71	**72**	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	**90**
91	92	93	94	95	96	97	98	**99**	100

The 9× table pattern

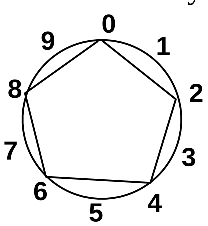
Children should be asked to explain why the patterns appear (and thereby explain the mathematics of the pattern). For example, the 9× table pattern can be explained by considering adding 9 repeatedly as adding (10 - 1) repeatedly. On the 100 square, adding 10 and subtracting 1 results in moving down a row and back 1 square. So, doing this repeatedly produces a diagonal pattern.

It is also worth asking children to predict what would happen to the pattern for a given multiplication table if the number of squares in each row of the 100 square is changed. So, for example, if using rows of 6 numbers the beginnings of the 6× table would produce a different pattern to that obtained on a normal 100 square.

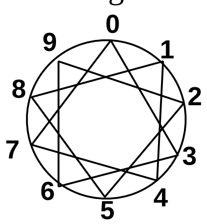
1	2	3	4	5	**6**
7	8	9	10	11	**12**
13	14	15	16	17	**18**

6. End Digit Patterns

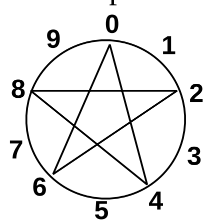
For many multiplication tables, the end digits of the multiples form a repeating pattern. These can be represented as a cyclical arrangement. For example:



2× table end digit pattern



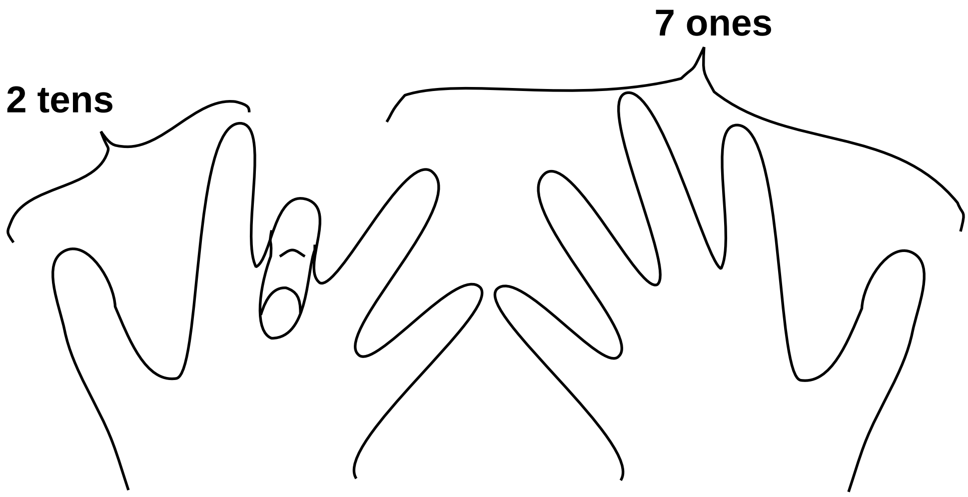
3× table end digit pattern



4× table end digit pattern

7. Finger Multiplication

The digits of both hands can be used as a means of practising the 9× table. For example, to show 3 × 9 fold down the 3rd digit from the left.



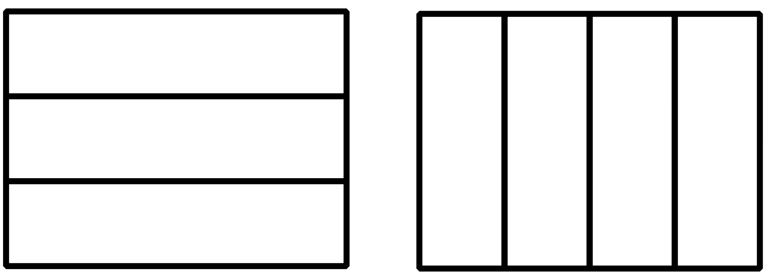
3 × 9 = 2 tens + 7 ones = 27

The number of digits to the left of the folded digit indicate the number of tens while the number of digits to the right of the folded digit indicate the number of units or ones.

8. Using the Commutative Law

The commutative law for multiplication states that $a \times b = b \times a$, where a and b are two numbers. So, for example, 4 × 3 = 3 × 4 = 12.

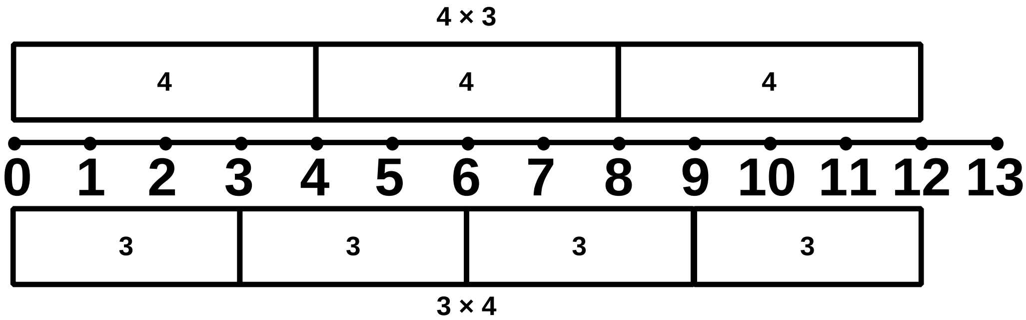
Learning the commutative law for multiplication (children do not need to know the name of this law but do need to learn that interchanging the numbers on either side of the multiplication sign does not affect the outcome) reduces the number of facts which children need to learn by almost half and because of this it is recommended that the commutative law be introduced as soon as children become comfortable with the meaning of the multiplication sign and have learned some of the easier multiplication facts. The commutative law can be introduced using Cuisenaire rods:



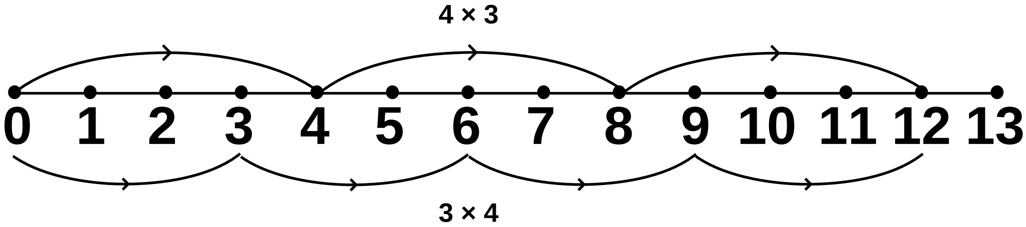
4 × 3 **3 × 4**

The argument here is either that 3 × 4 and 4 × 3 show ‘the same amount of wood’ and so are equal or that they are both equal to 12 unit cubes. A similar argument can be used with squared paper or rectangular arrays of counters.

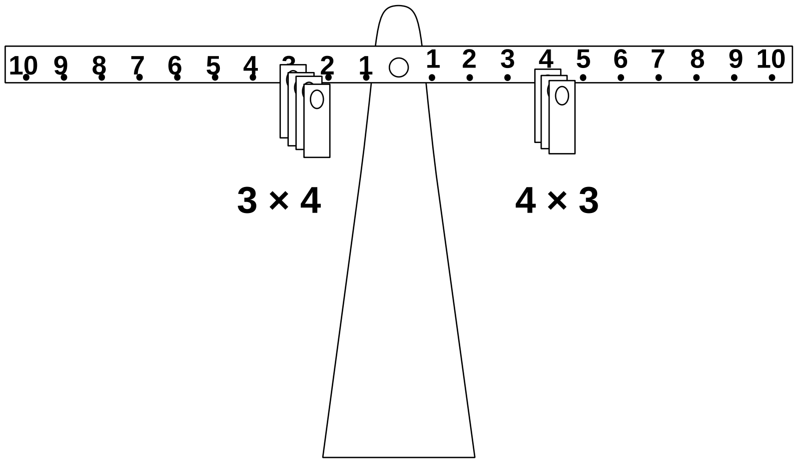
The commutative law can also be demonstrated using a number line with Cuisenaire rods:



or just using jumps on a number line:



It can also be demonstrated and explored using an equaliser balance:



The commutative law is the reason behind the symmetry of the multiplication square:

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

The commutative law is often used to make mental calculations easier.

Links between Multiplication and Division Facts

It is important to emphasise, wherever possible, the inverse relationship between multiplication and division and thus the links between corresponding multiplication and division facts. Children should be shown how to derive the remaining members of a related set of multiplication and division facts given only one of them. For example,

$$2 \times 5 = 10$$
$$5 \times 2 = 10$$
$$10 \div 2 = 5$$
$$10 \div 5 = 2$$

form such a set of related number facts. The ability to derive the remaining facts in the set is very helpful in learning number facts since the set of number facts can be learnt as a set rather than as 4 separate number facts. Thus, when learning multiplication tables children should be expected to recall (or quickly derive) them in any of the four configurations shown above.

Factors and Multiples

When some of the easier multiplication facts have been learnt children should be introduced to some additional mathematical vocabulary related to multiplication and division. Definitions of these follow:

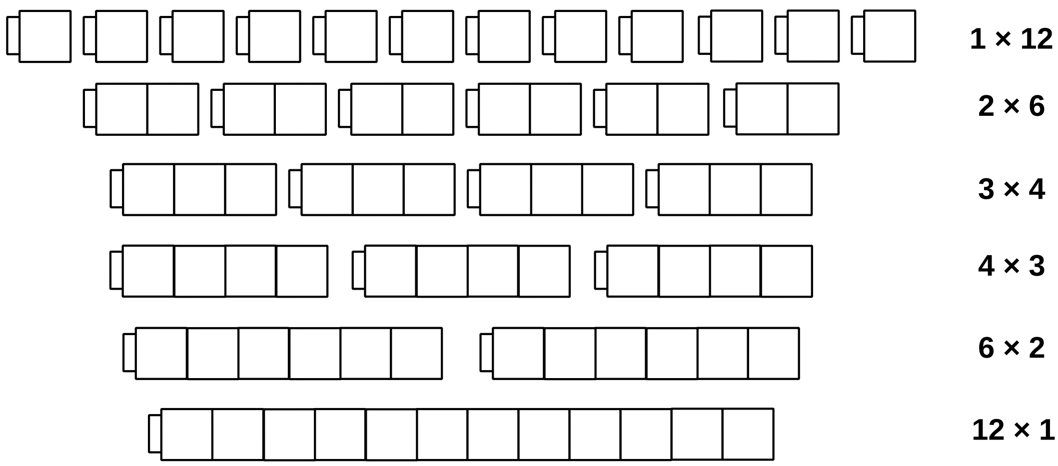
A number is a **factor** of another number if it will divide into the other exactly (i.e. without a remainder) e.g. 7 is a factor of 21 because 7 will divide into 21 exactly 3 times.

A number is a **multiple** of another number if it can be made by multiplying the second number by any other number e.g. 21 is a multiple of 7 because 7 × 3 = 21.

The terms ‘factor’ and ‘multiple’ are, in effect, opposite ways of saying the same thing e.g. 7 is a factor of 21 and 21 is a multiple of 7.

Finding Different Pairs of Factors of a Number

Breaking a multilink rod into equal-sized pieces in different ways is one way to explore the various factors of a particular number. So, for example, a 12-rod should produce the following pairs of factors of 12:



The multiplication square provides another means of exploring the factors of a given number.

For example, to find some of the factors of 20 a child can find all the locations in the multiplication square in which the number 20 appears.

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

By reading the numbers at the beginnings of the rows and columns in which the number 20 appears some of the factors of 20 (2, 4, 5 and 10) can be identified.

Knowledge of factors of numbers can often make a multiplication calculation much easier and thereby allowing a mental calculation to be performed instead of a pencil and paper calculation. For example, 16 × 14 can be calculated as 16 × 2 × 7 or even as 14 × 2 × 2 × 2 × 2 (repeated doubling).

Divisibility Rules

In order to decide if a number is a multiple of a smaller number it is often helpful to know some simple divisibility tests. Some common divisibility tests are listed below:

An integer is divisible:

by 2 ⇔ it ends in an even digit (0, 2, 4, 6 or 8);

by 3 ⇔ the sum of its digits is divisible by 3

[e.g. 345 has a digit sum of 3 + 4 + 5 = 12 and the digit sum of 12 is 1 + 2 = 3 which is divisible by 3];

by 4 ⇔ the number formed by the final two digits is divisible by 4

[e.g. for the number 1296 the final two digits are 96 and 96 is divisible by 4];

by 5 ⇔ the final digit is 0 or 5;

by 6 ⇔ the final digit is even and the digit sum is divisible by 3;

by 8 ⇔ the number formed by the final 3 digits is divisible by 8;

by 9 ⇔ the digit sum is divisible by 9;

by 10 ⇔ the final digit is 0;

by 11 ⇔ the difference between the sums of alternating digits is divisible by 11

[e.g. for 72479: difference = (7 + 4 + 9) - (2 + 7) = 20 - 9 = 11 which is divisible by 11]

Children should be expected to become familiar with these towards the end of Key Stage 2. Some of the more simple ones can be introduced much earlier as a means of helping children remember multiplication facts. The above tests can be extended to form other useful divisibility tests for larger divisors as shown below.

e.g. *An integer is divisible:*
by 20 $\Leftrightarrow$ the number formed by the final two digits is divisible by 20
[i.e. it ends in 00, 20, 40, 60 or 80];

by 25 $\Leftrightarrow$ the number formed by the final two digits is divisible by 25
[i.e. it ends in 00, 25, 50 or 75];

by 30 $\Leftrightarrow$ the final digit is 0 and the digit sum is divisible by 3;

by 40 $\Leftrightarrow$ the final digit is 0 and the number formed by the H and T digits is divisible by 4;

by 50 $\Leftrightarrow$ the number formed by the final two digits is divisible by 50;
[i.e. it ends in 00 or 50];

by 100 $\Leftrightarrow$ the final two digits are 00;

by 125 $\Leftrightarrow$ the number formed by the final three digits is divisible by 125
[i.e. it ends in 000, 125, 250, 375, 500, 625, 750, or 875].

These tests of divisibility can also be used as a method of checking some multiplication calculations (see Year 6 examples in NNS Section 6 page 73).

Teaching Mental Calculation Strategies for Multiplication and Division

Once some basic multiplication and division facts are known children should be encouraged to work out more complex facts from those which they already know. In this way, the number of known facts that children can instantly recall will gradually increase.

However, in order to derive new facts from known facts children will need to be aware of a range of possible strategies for making use of what they already know. While some more mathematically-able children will develop their own strategies without input from the teacher, it is important to note that, for many children, such mental strategies must be taught explicitly; it is not adequate merely to 'do some mental mathematics' and hope that all children will recognise the strategies involved. Consequently, the teacher must be pro-active in order to ensure that children acquire the necessary understanding of, and familiarity with, appropriate strategies. It is important that **the child should be taught mental calculation methods prior to those for written calculation**.

Some ways of teaching mental strategies are listed below. Using a mixture of these methods over a period of time should enable children to acquire the ability to use successfully those strategies which are taught.

Teachers can:

- explain to the children *how* to do something;
- use a child's error or a less efficient strategy as the starting point for a demonstration of a better strategy;
- encourage children to improve on their strategies;
- talk with the children about choosing strategies and the merits of each;
- show the children how to use something they already know in developing a new strategy;
- provide the children with a 'prompt' (a way of learning and then remembering a strategy or number fact);
- use apparatus to model a strategy;
- support the initial development of a strategy with rough jottings;
- build mental visualisation of a strategy.

The mental/oral starter and the plenary sections of daily mathematics lessons are good times to discuss and teach mental strategies with the whole class. In this way children will encounter the calculation methods used by other children as well as their own and those of their teacher.

Characteristics of mental multiplication and division strategies

In order for children to achieve success in mental multiplication and division tasks they will increasingly need familiarity with aspects of place value. In particular, it is crucial that children understand, and are able to, partition numbers into their constituent parts by recognising the value of the digits within the numbers, thus

$$36 = 30 + 6;\ 147 = 100 + 40 + 7 \text{ and, at later stages, } 5{\cdot}23 = 5 + 0{\cdot}2 + 0{\cdot}03.$$

For further information on how to teach partitioning see the *Place Value* Booklet.

They will also need to be familiar with how to multiply and divide by powers of ten and by multiples of powers of ten. The relationship between multiplication and division is also important (not least for checking answers).

It should be noted that, in general, mental methods often differ from pencil-and-paper methods. Many mental methods involve:

- working from left to right (starting with the most significant digits) whereas standard written methods often work from right to left (starting with the least significant digits);
- remembering the place value of each of the digits so that 83 is thought of as 80 + 3. In traditional written methods 83 is often (though wrongly) thought of as ‘an 8 and a 3’.
- tackling an easier calculation and then adjusting the answer appropriately, the which rarely occurs in written standard methods.

In a similar vein, it should also be noted that most calculations can be performed by more than one method. The method adopted by a child will depend on the particular numbers involved and the strategies with which the child is familiar and ‘comfortable’. Different children will inevitably adopt different methods. Children, when calculating mentally, should be taught to be selective about the strategies they employ.

Any strategy used should be:

- understood by the child
- reliable - the child can apply the strategy accurately and consistently

  and, preferably,
- efficient.

The **National Numeracy Strategy**’s *Framework for Teaching Mathematics* lists many examples of mental strategies for multiplication and division in **Section 5 pages 55 - 57 (Years 2 - 3) and Section 6 pages 60 - 65 (Years 4 - 6)**.

Common Mental Strategies for Multiplication and Division

1 *Rearranging Numbers (using the commutative law)*

e.g. 9×5 may be more easily calculated as 5×9 because the 5× table is more familiar than the 9× table.

2 *Rearranging Operations (using the associative law)*

e.g. changing the order of operations when multiplying so $(16 \times 2) \times 5$, for example, is more easily tackled as $16 \times (2 \times 5)$.

3 *Using Repeated Operations*

e.g. finding $128 \div 8$ by dividing by 2 three times

or multiplying by 16 by doubling four times.

4 *Compensation/Adjustment (using distributive law for multiplication)*

e.g. calculating 12×7 by working out 12×5 and then adding 24

5 *Using Doubling and Halving together*

e.g. calculating 15×6 as 30×3 (doubling the 15 and halving the 6)

or to multiply by 50, multiply by 100 and halve the answer obtained

6 *Partitioning into Multiples of Powers of Ten followed by use of the Distributive Law for Multiplication*

e.g. $27 \times 6 = (20 + 7) \times 6 = (20 \times 6) + (7 \times 6)$

or $27 \times 19 = 27 \times (20 - 1) = (27 \times 20) - (27 \times 1)$

7 *Using Inverse Relationships*

e.g. calculating $100 \div 5$ by knowing that $20 \times 5 = 100$ and that multiplication and division are inverse operations.

8 *Splitting numbers into factors*

e.g. $25 \times 32 = 5 \times 5 \times 4 \times 8 = 5 \times 4 \times 5 \times 8 = 20 \times 40 = 800$

or calculating $324 \div 18$ by calculating $324 \div 3 = 108$ and then $108 \div 6 = 18$.

9 *Moving All Digits in a Number to the Left or Right to Multiply or Divide by Powers of Ten*

e.g. $32 \times 20 = 32 \times 2 \times 10 = 64 \times 10 = 640$

or $643 \div 100 = 6{\cdot}43$

This strategy is a particularly powerful strategy for both mental and written calculations. Once pupils have the ability to multiply by 10, 100, 1000 etc. they can use known multiplication tables facts to derive harder facts.

e.g. knowing $7 \times 5 = 35$ we can derive:

$$70 \times 5 = 350$$
$$7 \times 50 = 350$$
$$70 \times 50 = 3500$$
$$7 \times 500 = 3500$$
$$700 \times 5 = 3500$$

$$70 \times 500 = 35000$$
$$700 \times 50 = 35000$$
$$700 \times 500 = 350000$$

The ability to derive harder facts like these is one of the foundations of long multiplication.

This strategy can be extended to decimals also:

$0{\cdot}7 \times 5 = 3{\cdot}5$

$0{\cdot}7 \times 0{\cdot}5 = 0{\cdot}35$ etc.

It is important that pupils should be taught that to multiply by powers of ten (10, 100, 1000 etc.) by **moving all the digits** in the number to be multiplied the appropriate number of places to the left **rather than the fallacious strategies of 'adding a zero' or 'moving the decimal point'**. Similarly, to divide by powers of 10 all of the digits in the number to be multiplied should be moved the appropriate number of places to the right.

The National Numeracy Strategy recommends that the focus should be on mental methods, supported by informal jottings where necessary, up to the end of Year 3. When written methods are introduced, mental skills should continue to be kept sharp and further developed. Any calculation should be done mentally if at all possible. Failing this, a written method may be adopted.

Using Recording to assist Mental Calculation

Children can:

- jot down numbers part-way to the answer;
- use boxes drawn on squared paper, Cuisenaire rods, or Dienes' blocks as a model for the calculation:

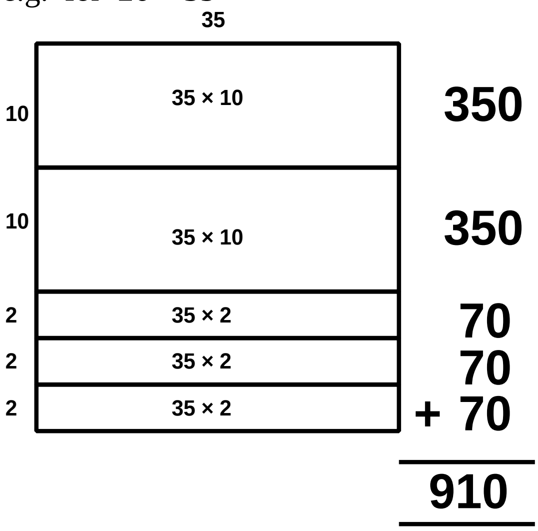
  e.g. for 24 × 3

10	10				
10	10				
10	10				

which can later be abbreviated to:

20	4
20	4
20	4

e.g. for 26 × 35



- read number sentences (equations) and record answers in a box,

  e.g. 5 × ☐ = 45;

- record and explain mental steps using numbers and symbols,

  e.g. for 27 × 18, calculated as a two-stage process in which the results of intermediate steps are recorded so as to reduce the number of items to be remembered:

  27 × 20 = 540

  27 × 2 = 54

  so 27 × 18 = 540 - 54 = 486.

Remainders and Rounding when Dividing

Remainders
35 ÷ 8 = 4 plus a remainder. This remainder can be written as a:

- **Whole Number: 35 ÷ 8 = 4 r 3** **(expected in Year 4)**
- **Fraction: 35 ÷ 8 = $4\frac{3}{8}$** **(expected from Year 5 onwards)**
- **Decimal: 35 ÷ 8 = 4.375** **(expected from Year 5 onwards)**

Remainders need to be interpreted according to the context of the division question.

For example:

- ***'Share 35 sweets equally between 8 people.'***
  4 sweets each and a whole-number remainder of 3 makes sense in this context (but $4\frac{3}{8}$ and 4.375 do not).

- ***'Divide a 35m plank into 8 equal pieces.'***
  $4\frac{3}{8}$m and 4.375m both make sense in this context (but 4 r3 does not).

We can summarize this by stating that when the context is about sharing then only a whole number remainder is suitable but in other contexts, fractional or decimal remainders should be chosen rather than than a whole-number remainder.

Rounding
In many contexts it is necessary to consider rounding up or rounding down the answer obtained. It is the context which must be used to decide whether rounding is necessary and, if so, in which direction (up or down). For example:

- ***'How many tables, each seating 8 people, are needed for 35 people?'***
  35 ÷ 8 = 4.375 so 5 tables will be needed (rounding up).

- ***'How many items costing 8p can be bought with 35p?'***
  35p ÷ 8 = 4.375p so 4 items can be bought (rounding down).

There are examples of the NNS expectations for dealing with remainders and rounding up or down on **pages 56 − 57 of Section 6 of the NNS 'Framework for Teaching Mathematics'**.

Developing Written Algorithms (Methods) for Multiplication and Division

Ultimately, the aim, as far as written methods are concerned, should be to ensure that as many children as possible can, by the age of 11, carry out a standard written method for both multiplication and division. The key principles to consider when guiding children through this developmental process are:

- Ensure that recallable facts (2×, 3×, 4×, 5× and 10× multiplication facts at least) are established first so children can concentrate on a written method without reverting to first principles.
- Children should know the outcome of multiplying by 0 and by 1 and of dividing by 1.
- Remember that children must understand partitioning and the idea of ‘zero as a place holder’ from place value work before beginning formal written methods.
- Ensure that children understand how to multiply mentally by 10, 100, 1000 etc..
- Once written methods are introduced, ensure that children continue to look out for and recognise the special cases that can be done mentally.
- Cater for children who progress at different rates; some may grasp a standard method readily while others may never do so without considerable help.
- Recognise that children tend to forget a standard method if they have no understanding of what they are doing.
- Ensure that those written methods to be taught are derived clearly from mental methods which are already established.
- Introduce expanded formats for written algorithms and gradually refine/contract these into the more compact standard format.
- For division, tackle examples without remainders before examples that do involve remainders.
- Eventually extend written methods developed for whole numbers to calculations involving decimals.

Progression in Written Methods for Multiplication

The Distributive Law

All written methods for multiplying two-digit (or larger) numbers rely on the use of the **distributive law for multiplication over addition** which states that the outcome of a multiplication is unchanged if the multiplication is applied to a partitioned number instead of the original number.

For example, 23×5 can be calculated as $(20 + 3) \times 5 = (20 \times 5) + (3 \times 5)$.

In general terms, the distributive law for multiplication over addition can be written algebraically as:

$$(a + b) \times c = (a \times c) + (b \times c) \quad \text{for any numbers a, b and c.}$$

Multiplying by Multiples of Powers of Ten

All written methods for multiplication also rely on pupils' ability to multiply mentally by multiples of powers of ten (see earlier section about common mental strategies).

Written Methods for Multiplication

The National Numeracy Strategy recommends the following progression of methods. They can be found on pages 66 - 67 of Section 6 of the NNS 'Framework for Teaching Mathematics'. Note that, for all of these methods, children should be taught to approximate answers before calculating.

1. Grid method

This relies on partitioning numbers into multiples of powers of ten. Each of the multiples of powers of ten are then multiplied by the multiplier and the numbers obtained are added together.

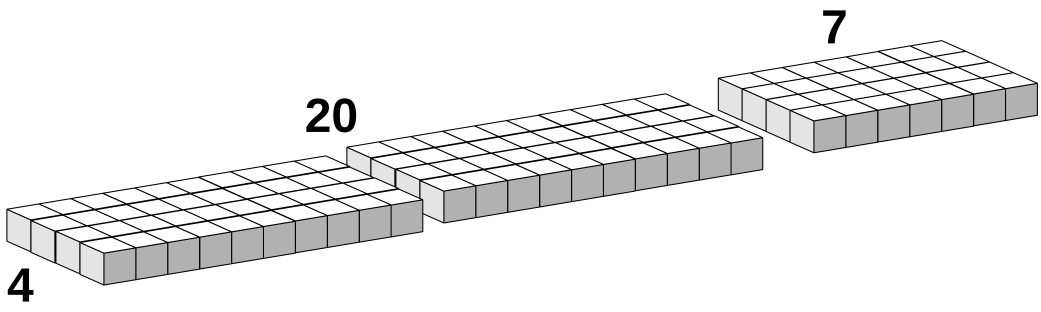
For TU × U (Year 4)

e.g. 27×4

	20	7
4	80	28

$80 + 28 = 108$

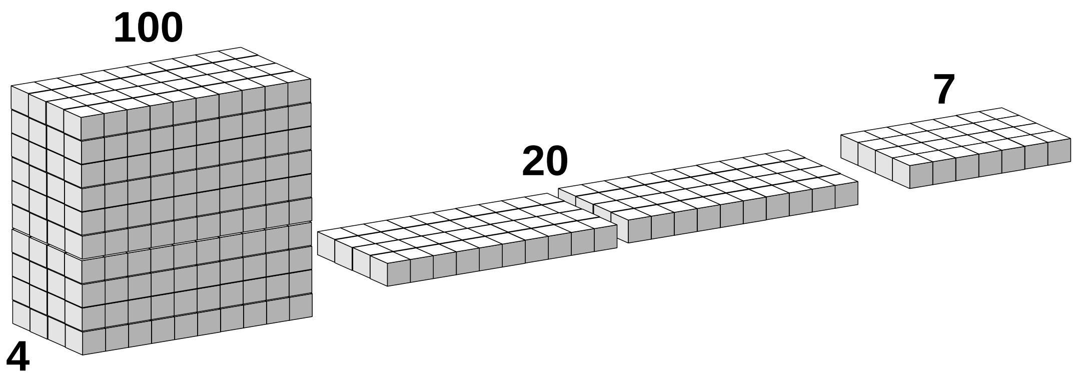
This can be modelled using Dienes' blocks or Cuisenaire (as an aid until an understanding of the method has been developed) as shown below:



For HTU × U (Year 5)
e.g. 127 × 4

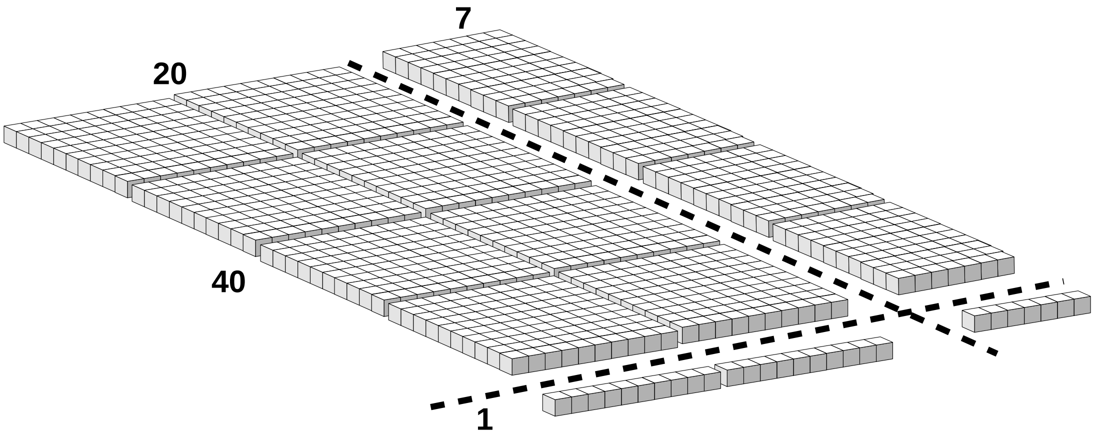
	100	20	7
4	400	80	28

400 + 80 + 28 = 508



For TU × TU (Year 5)
e.g. 27 × 41

	20	7	
40	800	280	1080
1	20	7	+ 27
			1107



ThHTU × U (Year 6)
e.g. 4127 × 4

	4000	100	20	7
4	16000	400	80	28

16000 + 400 + 80 + 28 = 16508

HTU × TU (Year 6)
e.g. 127 × 41

	100	20	7
40	4000	800	280
1	100	20	7

$$\begin{array}{r} 5080 \\ +\ 127 \\ \hline 5207 \end{array}$$

2. Expanded Standard Method for Short Multiplication

The multi-digit number is partitioned and, starting with the most significant digit in the partitioned number, each digit is multiplied by the multiplier. The expanded format avoids the need for carrying and the partial answers are equivalent to those which would be entered in the grid if using the grid method above. The expanded standard method can be modelled using Dienes' blocks if necessary in a similar manner to that for the grid method.

TU × U (Year 4)

$$\begin{array}{rl} 24 & \\ \times\ 3 & \\ \hline 60 & (20 \times 3) \\ 12 & (4 \times 3) \\ \hline 72 & \end{array}$$

HTU × U (Year 5)

$$\begin{array}{rl} 124 & \\ \times\ 3 & \\ \hline 300 & (100 \times 3) \\ 60 & (20 \times 3) \\ 12 & (4 \times 3) \\ \hline 372 & \end{array}$$

ThHTU × U (Year 6)

$$\begin{array}{rl} 2124 & \\ \times\ 3 & \\ \hline 6000 & (2000 \times 3) \\ 300 & (100 \times 3) \\ 60 & (20 \times 3) \\ 12 & (4 \times 3) \\ \hline 6372 & \end{array}$$

3. Standard Method for Short Multiplication

The multi-digit number is partitioned and multiplying begins with the least significant digit of the multi-digit number. Carrying may be necessary with the standard method.

TU × U (Year 4)

$$\begin{array}{r} 24 \\ \times\ 3 \\ \hline 72 \\ \hline {\scriptstyle 1} \end{array}$$

HTU × U (Year 5)

$$\begin{array}{r} 124 \\ \times\ 3 \\ \hline 372 \\ \hline {\scriptstyle 1} \end{array}$$

ThHTU × U (Year 6)

$$\begin{array}{r} 2124 \\ \times\ 3 \\ \hline 6372 \\ \hline {\scriptstyle 1} \end{array}$$

4. Long Multiplication

The multiplier is partitioned, and partial answers obtained which are then added.

TU × TU (Year 5)

$$\begin{array}{rl} 24 & \\ \times\ 35 & \\ \hline 720 & (24 \times 30) \\ 120 & (24 \times 5) \\ \hline 840 & \end{array}$$

HTU × TU (Year 6)

$$\begin{array}{rl} 124 & \\ \times\ 35 & \\ \hline 3720 & (124 \times 30) \\ 620 & (124 \times 5) \\ \hline 4340 & \\ \hline {\scriptstyle 1} & \end{array}$$

5. Multiplication of Decimals

Decimal points in the partial answers should be aligned one under the other to help ensure a correct answer is obtained.

U·t × U (Year 5)

$$\begin{array}{rl} 5.8 & \\ \times\ 3 & \\ \hline 15.0 & (5.0 \times 3) \\ 2.4 & (0.8 \times 3) \\ \hline 17.4 & \end{array}$$

U·t h × U (Year 6)

$$\begin{array}{rl} 5.82 & \\ \times\ 3 & \\ \hline 15.00 & (5.00 \times 3) \\ 2.40 & (0.80 \times 3) \\ 0.06 & (0.02 \times 3) \\ \hline 17.46 & \end{array}$$

There are examples of the NNS expectations for written multiplication calculations on **pages 66 - 67 of Section 6 of the NNS ‘Framework for Teaching Mathematics’**.

Progression in Written Methods for Division

There are 3 standard methods of writing division operations:

- $6 \div 3 = 2$
- $3\overline{)6}$
- $\frac{6}{3} = 2$

Be aware that children often find the various symbolic notations used for confusing because of the different locations of the numbers involved. In particular, the first and second of the notations above are particularly confusing because the divisor and dividend are written in the opposite order.

Written division methods use repeated subtraction as their basis. As for written multiplication methods, the progression begins with informal written methods which then form the basis of expanded standard methods which can be then contracted to the traditional, more efficient standard method.

1. Subtracting the Divisor Repeatedly

This is an informal method which can be modelled by counting equal, backward jumps along a number line or by using counters or Dienes' blocks (decomposing where necessary) and counting the number of times the divisor can be subtracted.

e.g. 14 ÷ 3

$$\begin{array}{r} \mathbf{14} \\ \underline{\mathbf{-\ 3}} \\ \mathbf{11} \\ \underline{\mathbf{-\ 3}} \\ \mathbf{8} \\ \underline{\mathbf{-\ 3}} \\ \mathbf{5} \\ \underline{\mathbf{-\ 3}} \\ \mathbf{2} \end{array}$$

Quotient is number of 3s subtracted plus the remainder

so 14 ÷ 3 = 4 r 2

2. Subtracting Multiples of the Divisor

(Year 4: TU ÷ U, Year 5: HTU ÷ U, Year 6: HTU ÷ TU)

Instead of repeatedly subtracting the divisor (as above), the process is now speeded up by subtracting known multiples of the divisor. Again, this can be modelled by using Dienes' blocks to represent the dividend and subtracting multiples of the divisor from it (decomposing where necessary).

e.g. 674 ÷ 3

674	
- 300	**100 groups of 3**
374	
- 300	**100 groups of 3**
74	
- 60	**20 groups of 3**
14	
- 12	**4 groups of 3**
2	**224 groups of 3**
	and remainder of 2

e.g. 1795 ÷ 56

1795	
- 560	**(56 × 10)**
1235	
- 560	**(56 × 10)**
675	
-560	**(56 × 10)**
115	
- 56	**(56 × 1)**
59	
- 56	**(56 × 1)**
3	

3. Expanded Short Division

(Year 4: TU÷ U, Year 5: HTU ÷ U)

This still uses the 'repeated subtraction of multiples of the divisor' idea but now uses an expanded form of the layout for the standard method.

e.g. 674 ÷ 3

2 2 4 r 2	
3) 6 7 4	
- 6 0 0	**(3 × 200)**
7 4	
- 6 0	**(3 × 20)**
1 4	
- 1 2	**(3 × 4)**
2	

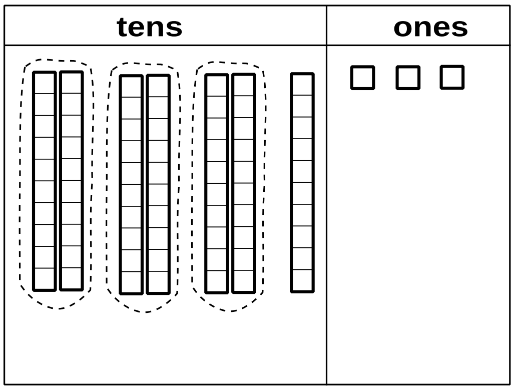
4. Standard Short Division

The above process ***could*** be contracted to produce the traditional, more efficient algorithm although this is not expected in the NNS ‘Framework for Teaching Mathematics’. The method for modelling this with Dienes’ blocks is shown alongside the process.

e.g. 73 ÷ 3

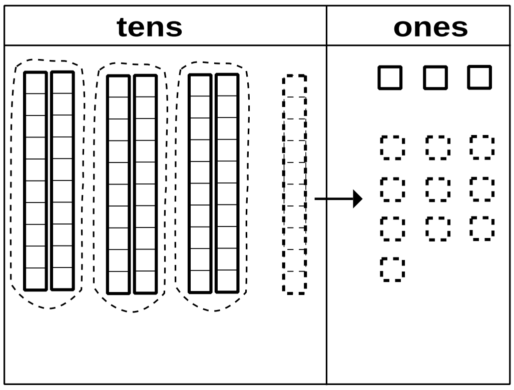
Share 7 tens between 3 groups → 2 tens in each group and 1 ten left over.

$$\begin{array}{r} 2 \\ 3\overline{)7\,3} \end{array}$$



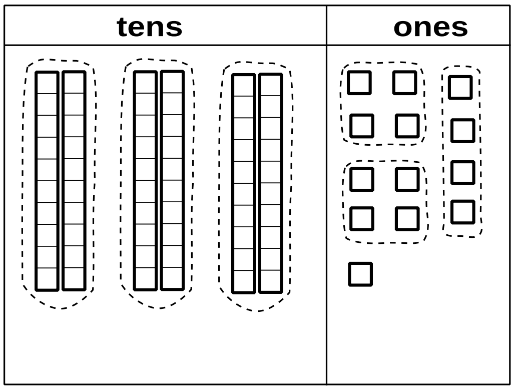
Exchange 1 ten for 10 ones.

$$\begin{array}{r} 2\phantom{\,{}^{1}3} \\ 3\overline{)7\,{}^{1}3} \end{array}$$



Share 13 ones between 3 groups → 4 ones in each group and 1 left over.

$$\begin{array}{r} 2\ \ 4\ \ \text{r }1 \\ 3\overline{)7\,{}^{1}3}\phantom{\ \text{r }1} \end{array}$$



5. Long Division

(Year 6: HTU ÷ TU)

Long division also uses the idea of repeatedly subtracting multiples of the divisor from the dividend. The difficulties in long division are usually in deciding what number should be subtracted.

e.g. 835 ÷ 26

$$
\begin{array}{rl}
\mathbf{3\,2\,r\,3} & \\
\mathbf{26)\overline{8\,3\,5}} & \\
\mathbf{-7\,8\,0} & \mathbf{(26 \times 30)} \\
\hline
\mathbf{5\,5} & \\
\mathbf{-5\,2} & \mathbf{(26 \times 2)} \\
\hline
\mathbf{3} &
\end{array}
$$

Deciding which multiple of the divisor to subtract at each stage is a process of trial and improvement to some extent. In the example above, the dividend is 836 so we cannot subtract 26 lots of a hundreds number (100, 200, 300, 400 ... etc.) because this would be too big to subtract from 835. So, instead we investigate subtracting 26 lots of a tens number (10, 20, 30, 40 ...etc.). By trial and improvement we find that the largest tens number we can use is 30 because 780 (= 26 × 30) can be subtracted from 835 but 1040 (= 26 × 40) is too much. This process is repeated at the second subtraction stage but using 26 lots of a units number. Here the largest units number which can be used is 2 because 52 (= 26 × 2) but 78 (= 26 × 3) is too big. So 52 (= 26 × 2) is subtracted from 55, the amount remaining from the first subtraction stage. The second subtraction leaves 3 to divide but no multiple of 26 is small enough to subtract from 3 so this is the remainder.

6. Short Division for Decimals

(Year 6: TU·t ÷ U)

This is identical to the method described above for Expanded Short Division with the proviso that decimal points should, throughout, be aligned under each other.

e.g. 67·4 ÷ 2

$$
\begin{array}{rl}
\mathbf{3\,3\,.\,7} & \\
\mathbf{2)\overline{6\,7\,.\,4}} & \\
\mathbf{-\ 6\,0\,.\,0} & \mathbf{(30 \times 2)} \\
\hline
\mathbf{7\,.\,4} & \\
\mathbf{-\ 6\,.\,0} & \mathbf{(3 \times 2)} \\
\hline
\mathbf{1\,.\,4} & \\
\mathbf{-\ 1\,.\,4} & \mathbf{(0.7 \times 2)} \\
\hline
\mathbf{0\,.\,0} &
\end{array}
$$

There are examples of the NNS expectations for written division calculations on **pages 68 - 69 of Section 6 of the NNS 'Framework for Teaching Mathematics'**.

Errors and Misconceptions in Multiplication and Division Calculations

Diagnosing Difficulties: General Points

Difficulties with calculations can occur for a number of reasons. Incorrect responses may be owing to:

- *computational error/careless mistake*
  The child uses the correct operation and procedure but incorrectly recalls a basic number fact(s).
- *misconceptions*
  The child has not grasped the concept of the operation being used (addition or subtraction) or, in the case of formal, vertically-written algorithms, fails to understand aspects of place value required in order to understand the algorithm being attempted.
- lack of understanding of relevant vocabulary
  The child misinterprets the language used in the question or task.
- *wrong operation*
  The child uses the wrong operation for the question.
- *defective procedure or method*
  The correct operation is chosen and number facts are recalled correctly but there are errors in the use of the procedure (algorithm) adopted.
- *over-generalisation*
  the child has learned a pattern, 'rule', or method and then has applied it to situations where it is not appropriate (hence the importance of real understanding and not just mechanical learning of a procedure).
  e.g. the child, having been introduced to decomposition in subtraction of two-digit numbers, then uses decomposition in every subtraction calculation even when decomposition is not necessary.
- *under-generalisation*
  The child has encountered insufficient examples or examples which have insufficient variation. A sufficiency of both examples and of variation in examples is required in order for the child to be able to abstract the conceptual or procedural understanding required and so, without this sufficiency, the child may generalise on the basis of inadequate knowledge or experience.
- *random response*
  There is no discernible relationship between the question and the response given.

(adapted from 'Guide for Your Professional Development Book 2, NNS)

It is important to distinguish between different kinds of difficulties experienced by children (as evidenced in their work) since the help offered to address such difficulties must be appropriate otherwise the difficulties will persist (and possibly worsen). The National Numeracy Strategy guidance materials offer the following general points about diagnosing errors and misconceptions:

- Children's errors are often due to misconceptions or misunderstood rules, rather than careless slips.
- It is important to diagnose each misconception rather than simply to re-teach the method.

- To make the diagnosis, ask the child to explain how they worked out the answer ...
- ... then deal with the misconception, helping the child to use another method or a simpler approach that she or he is confident with (e.g. an expanded layout for a written calculation).

(from ‘Guide for Your Professional Development Book 3’, NNS)

The plenary of the daily mathematics lesson is often a good time to address common misconceptions and errors because there may be several children who have the same difficulty. That said, it is incumbent upon the teacher to teach concepts and strategies in ways which will pre-empt or avoid misconceptions and errors developing.

The examples on the following pages indicate common misconceptions and errors in addition and subtraction calculations involving two-digit numbers. Many of the difficulties illustrated stem from:

- poor understanding of place value, or
- possessing a mechanical knowledge of the algorithm involved but not a conceptual understanding of the processes involved (e.g. carrying, exchange, decomposition), or
- poor understanding about when an algorithm is applicable and when it is not.

Common Errors and Misconceptions in Multiplication

A: $3 \times 0 = 3$ **Confusion between × and +**

$0 \times 3 = 3$

B: $\begin{array}{r} 3\ 6 \\ \underline{\times\ \ 3} \\ \underline{3\ 9} \end{array}$ **Adding instead of multiplying**

C: $\begin{array}{r} 3\ 6 \\ \underline{\times\ \ 3} \\ \underline{9\ 1\ 8} \\ {\scriptstyle 1}\ \ \end{array}$ **Carrying digit inserted in answer:**
does not know that 18 = 1 ten + 8 units
(place value difficulty)

D: $\begin{array}{r} 3\ 6 \\ \underline{\times\ 3} \\ \underline{9\ 8} \\ {\scriptstyle 1}\ \ \end{array}$ **Forgetting to add the carrying digit**

E: $\begin{array}{r} 3\ 6 \\ \underline{\times\ 3} \\ \underline{4\ 8} \\ {\scriptstyle 1}\ \ \end{array}$ **Addition of tens column including carrying digit**
(confusion with addition algorithm).

F: $\begin{array}{r} 3\ 6 \\ \underline{\times\ 3} \\ \underline{3\ 8} \\ {\scriptstyle 1}\ \ \end{array}$ **Multiplication of the tens digit with the carrying**
digit (working down column as in + and −
algorithm)

G: $\begin{array}{r} 3\ 6 \\ \underline{\times\ 3} \\ \underline{0\ 8} \\ {\scriptstyle 1}\ \ \end{array}$ **Failure to create a hundreds column**
(place value problem)

H: $\begin{array}{r} 3\ 0 \\ \underline{\times\ 3} \\ \underline{9\ 3} \end{array}$ **from 0 × 3 = 3 (as in A above)**

I: $\begin{array}{r} 5\ 1 \\ \underline{\times\ \ 4} \\ \underline{2\ 4} \end{array}$ **'5 × 4 = 20 and 20 is 2 tens so 2 goes in Tens**
column'

J: $\begin{array}{r} 7\ 2 \\ \underline{\times\ 3\ 6} \\ 4\ 3\ 2 \\ \underline{2\ 1\ 6} \\ \underline{6\ 4\ 8} \end{array}$ **⟵ Incorrect positioning of the tens**
multiplication (place value problem)

K: $\begin{array}{r} 72 \\ \underline{\times\ 36} \\ 12 \\ \underline{21} \\ \underline{33} \end{array}$	**U×U: 2 × 6 = 12** **T×T: 7 × 3 = 21** **then add - confusion with + and − algorithms in which one works downwards only (unaware of the need to also multiply diagonally)**
L: $\begin{array}{r} 72 \\ \underline{\times\ 36} \\ 4212 \\ \underline{2160} \\ \underline{6372} \end{array}$	⟵ **Carrying digits inserted at** ⟵ **the intermediate stages** **(place value problem)**
M: $\begin{array}{r} 72 \\ \underline{\times\ 36} \\ \underline{108} \end{array}$	**Adding instead of multiplying**
N: 32 × 10 = 320	**Failure to appreciate that × (and ÷) by powers of ‘you just add ten results in digits moving from one column to a zero’ another (place value problem). Note that the answer is correct in this case but the method is incorrect.**
O: 3·2 × 10 = 3·20	**Incorrect method is as in N above (but this time the answer is also wrong).**
P: 1·09 × 10 =10·090	**Variant of N and M using the incorrect ‘add 0’ method of multiplying by 10**

Common Division Errors and Misconceptions

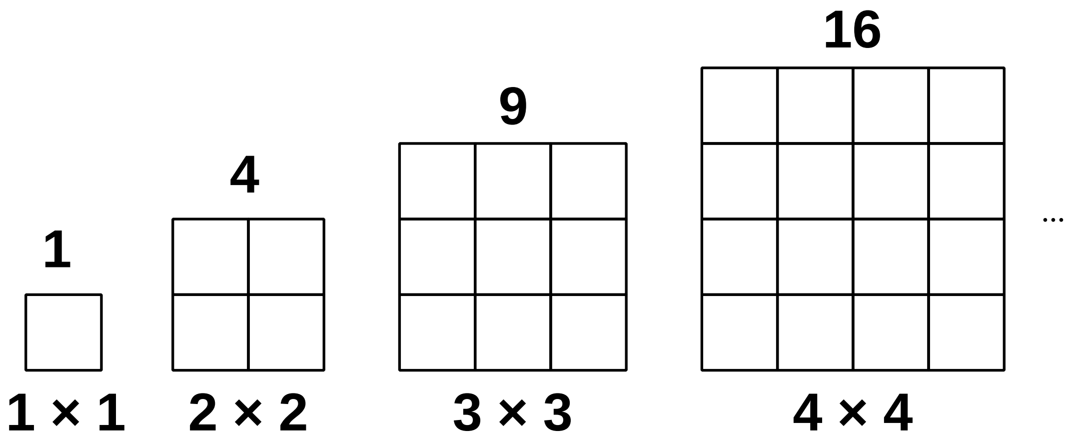
A: 18 ÷ 3 = 15	**Confusion between ÷ and − .**
B: 5 ÷ 1 = 0	**‘You can't share between one’.**
C: 3 ÷ 12 = 4	**Read as ‘3 divided into 12’ or ‘3 goes into 12’.**
D: 5 ÷ 5 = 0	**Either confusion with − or with 5 × 0 = 0.**
E: 4 ÷ 0 = 4	**Either confusion between ÷ and − or failure to recognise that ÷ by 0 is impossible (‘If I don't share any I still have 4 left’).**
F: 0 ÷ 3 = 3	**‘0 goes into 3 three times’ (i.e. 3 left).**
G: $\begin{array}{r} 22 \\ 2\overline{)54} \end{array}$	**Forgetting to carry over to the next digit.**

H: 4 2 **3)2 7**	**Starting with units instead of tens.**
I: 1 6 0 r 2 **3)5 8 2**	**As for type G but with a remainder.**
J: 3 5 **7)2 4 6**	**Forgetting the remainder or not knowing what to do with this.**
K: 2 4 **2)4 0 8**	**Starting with units and failing to put 0 in answer (so losing place value).**
L: 3 3 **3)9 0 9**	**Doesn't recognise the need to preserve place value in the answer or doesn't know how to do 0 ÷ 3.**
M: 4 0 r 7 **13)5 4 7** **5 2** **7**	**Forgetting to do the subtraction.**
N: 3 r 2 **1 8)5 4 2** **5 4** **0 0 2**	**'Can't do 2 ÷ 18' (losing the 0 in** the answer).
O: 3 2 r 6 **1 5)4 9 6** **4 6** **3 6** **3 0** **6**	**Multiplication error: 15 × 3 ≠ 46**
P: 3 7 r 11 **1 4)5 2 7** **4 2** **1 0 7** **9 8** **1 1**	**Subtraction error: 7 - 8 ≠ 1** **Check understanding of subtraction of 2-digit numbers.**

Square, Prime and Rectangular Numbers

Square Numbers

Square numbers are numbers which can be produced by multiplying a number by itself e.g. 4 is a square number because 4 = 2 × 2. The square of a number is the result obtained by multiplying it by itself e.g. the square of 4 = 4^2 = 4 × 4 = 16. The nomenclature is best explained by showing how the numbers relate to square arrays of increasing dimensions as shown below.



The square numbers appear on the diagonal (and line of symmetry) of the multiplication square.

Prime Numbers and Rectangular Numbers

One way to introduce prime and rectangular numbers is to ask children to each take a handful of multilink cubes and try to create a 'flat' (i.e. 1 cube thick) rectangular tile or array. Those numbers of cubes which can together form at least one rectangular tile (excluding a single row of cubes) are called rectangular numbers and thus we can define a **rectangular number** as being a number with more than two different positive whole number factors. For example, the number 6 is rectangular because it has factors 1, 2, 3 and 6 and can form rectangular arrays 2 × 3 and 1 × 6. Rectangular numbers are sometimes also called **composite numbers**.

In contrast, those numbers of cubes for which only a single row of cubes can be formed are prime numbers. Thus, a **prime number** is a number which has exactly two different positive whole number factors. For example, 2, 3 and 7 are prime numbers. Note that 1 is ***not*** a prime number (it has only 1 different positive whole number factor - the number 1). There is no known pattern which fully describes the prime numbers, though it is worth remembering that 2 is the only even prime number and that all prime numbers except 2 and 3 are one more or less than a multiple of 6.

The Sieve of Eratosthenes is a common Key Stage 2 activity involving prime numbers. The activity uses a standard hundred square and involves crossing out all the numbers which have more than 2 factors. The remaining numbers after completing the crossing out process are, of course, the prime numbers up to 100.

Any number can be written as a unique product of prime factors; this is known as the **prime factorisation** of a number. The prime factorisation of any number can be found by a process of repeatedly dividing the number by prime numbers until the number 1 is reached. For example, the prime factorisation of 60 can be found as follows:

Start with the smallest prime number 2 as the divisor

$60 \div 2 = 30$	we can still divide exactly by 2 so ....
$30 \div 2 = 15$	we cannot continue to divide exactly by 2 so try the next prime number 3 ...
$15 \div 3 = 5$	we cannot continue to divide exactly by 3 so try the next prime number 5 ...
$5 \div 5 = 1$	we have reached 1 so the process stops.

Looking back we can see that we divided 60 by 2, then by 2 again, then by 3 and finally by 5. This means that the prime factors of 60 are 2, 2, 3 and 5. So we can write $60 = 2 \times 2 \times 3 \times 5$ or $60 = 2^2 \times 3 \times 5$. This is known as the prime factorisation of 60.

Glossary of Mathematical Terms associated with Multiplication and Division

Inverse Operation

The operation which is 'opposite' mathematically to that being considered. Thus, division is the inverse of multiplication and vice versa.

Commutative Law

The commutative law applies to multiplication **but not to division**. It states that, for any multiplication statement, the ***numbers*** to be added ***may be interchanged*** around the multiplication sign without altering the product of the two numbers:

e.g. $3 \times 2 = 2 \times 3$

but $4 \div 2 \neq 2 \div 4$

Associative Law

The associative law applies to multiplication **but not to division**. It states that multiplication ***operations may be interchanged*** without altering the outcome. In the example below, the operation in the brackets are performed before those outside the brackets.

e.g. $3 \times (2 \times 4) = (3 \times 2) \times 4$

but $6 \div (4 \div 2) \neq (6 \div 4) \div 2$

Distributive Law

The distributive law applies to multiplication over addition or subtraction. It states that a multiplication ***operation may be applied to a number which has been partitioned*** without altering the outcome. In the example below, the operation in the brackets are performed before those outside the brackets.

e.g. $3 \times (2 + 4) = (3 \times 2) + (3 \times 4)$ (over addition)

$3 \times (4 - 2) = (3 \times 4) - (3 \times 2)$ (over subtraction)

Terms associated with Multiplication and Division Operations

$3 \times 2 = 6$

multiplicand, multiplier, product

$6 \div 2 = 3$

dividend, divisor, quotient

