Multivariable Calculus Study Guide: A LaTeX Version

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1 Disclaimer

It is not guaranteed that I have every single bit of necessary information for the course. This happened to be some of what I needed to know this specific semester in my course. For example, Stokes' Theorem is not even mentioned.

2 Vectors Between Two Points

$$Given: \ P(x_1, y_1) \ \& \ Q(x_2, y_2)$$

$$\overrightarrow{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

3 Vectors in the Plane

$$let \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \& \ \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3.1 Simple Operations

$$c\mathbf{v} = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$$

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$$

$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix}$$

3.2 Unit Vectors

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\&\, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$$

3.3 Vectors of a Specified Length

$$\left| \frac{c\mathbf{v}}{|\mathbf{v}|} \right| = |c|$$

$$\pm\frac{c\mathbf{v}}{|\mathbf{v}|} \parallel \mathbf{v}$$

4 Vectors in Three Dimensions

4.1 Notes

Everything in the above section can be expanded to three dimensions. Simply add another component.

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

4.2 Random Equations

$$xy\text{-plane } \{(x, y, z) : z = 0\}$$
$$xz\text{-plane } \{(x, y, z) : y = 0\}$$
$$yz\text{-plane } \{(x, y, z) : x = 0\}$$

$$\text{Sphere: } (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

5 Dot Product

5.1 Definitions

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}| \cos\theta$$
$$\mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$$
$$\mathbf{u} \parallel \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$$

5.2 Projections

The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is denoted $\text{scal}_{\mathbf{v}}\mathbf{u}$.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$
$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

6 Cross Product

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta \tag{1}$$

$$\mathbf{u} \parallel \mathbf{v} \Leftrightarrow \mathbf{u}\times\mathbf{v} = \mathbf{0}$$

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

Note: $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and the direction is defined by the right-hand rule.

7 Lines and Curves in Space

7.1 Vector-Valued Functions

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

7.2 Lines

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle,\ \text{for } -\infty < t < \infty$$

7.3 Line Segments

$$Given:\ P_1(x_1, y_1, z_1)\ \&\ P_2(x_2, y_2, z_2)$$
$$\overrightarrow{P_1P_2} = \langle x_1, y_1, z_1\rangle + t\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle,\ \text{for } 0 \le t \le 1$$

7.4 Curves in Space

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

Equation 1 is also equal to the area of the parallelogram created by the two vectors.

7.5 Limits

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

8 Calculus of Vector-Valued Functions

8.1 Derivative and Tangent Vector

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Note: $\mathbf{r}'(t)$ is the tangent vector to $\mathbf{r}(t)$ at the point $(f(t), g(t), h(t))$.

8.2 Indefinite Integral

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

Note: $\mathbf{C}$ is an arbitrary constant vector and $\mathbf{R} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$.

8.3 Definite Integral

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$

9 Motion in Space

9.1 Definitions

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

$$Speed = |\mathbf{v}(t)|$$

9.2 Two-Dimensional Motion in a Gravitational Field

$$Given:\ \mathbf{v}(0) = \langle u_0, v_0\rangle \text{ \& } \mathbf{r}(0) = \langle x_0, y_0\rangle$$

$$\mathbf{v}(t) = \langle x'(t), y'(t)\rangle = \langle u_0, -gt + v_0\rangle$$

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle$$

9.3 Two-Dimensional Motion

$$Given: \ \mathbf{v}(0) = \langle |\mathbf{v}_0| \cos\theta, |\mathbf{v}_0| \sin\theta \rangle \ \& \ \mathbf{r}(0) = \langle 0, 0 \rangle$$

$$Time = \frac{2|\mathbf{v}_0| \sin\theta}{g}$$

$$Range = \frac{|\mathbf{v}_0|^2 \sin 2\theta}{g}$$

$$Max\, Height = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\theta)^2}{2g}$$

10 Planes and Surfaces

10.1 Plane Equations

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a normal vector $\mathbf{n} = \langle a, b, c, \rangle$ is described by the equations:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0$$

In order to find the equation of a plane when given three points, simply create any two vectors out of the points and take the cross product to find the vector normal to the plane. Then use one of the above formulae.

10.2 Parallel and Orthogonal Planes

Two planes are parallel if their normal vectors are parallel. Two planes are orthogonal if their normal vectors are orthogonal.

10.3 Surfaces

10.3.1 Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

10.3.2 Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

It would be worth it to learn how to derive sections 9.2 and 9.3.

10.3.3 Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

10.3.4 Hyperboloid of Two Sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

10.3.5 Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

10.3.6 Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

11 Graphs and Level Curves

11.1 Functions of Two Variables

$$\mathbb{R}^2 \to \mathbb{R}$$

$$z = f(x, y)$$

$$F(x, y, z) = 0$$

11.2 Functions of Three Variables

$$\mathbb{R}^3 \to \mathbb{R}$$

$$w = f(x, y, z)$$

$$F(w, x, y, z) = 0$$

11.3 Level Curves

Imagine stepping onto a surface and walking along a path with constant elevation. The path you walk on is known as the contour curve, while the projection of the path onto the xy-plane is known as a level curve.

12 Limits and Continuity

12.1 Limits

The function f has the limit L as $P(x, y)$ approaches $P_0(a, b)$.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then the limit does not exist.

12.2 Continuity

The function f if continuous at the point (a, b) provided:

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

13 Partial Derivatives

13.1 Definitions

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

So basically just take the derivative of one (the subscript) given that the other one is a constant.

13.2 Notation for Higher-Order Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Note: $f_{xy} = f_{yx}$ for nice functions.

13.3 Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open region containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b). This also implies that it is continuous at (a, b).

14 Chain Rule

14.1 Examples

You can use a *tree diagram* to determine the equation for the chain rule. You can also just think about it. Refer to the following examples.

z is a function of x and y, while x and y are functions of t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

w is a function of x, y, and z, while x, y, and z are functions of t

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

z is a function of x and y, while x and y are functions of s and t

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

w is a function of z, z is a function of x and y, x and y are functions of t

$$\frac{dw}{dt} = \frac{dw}{dz}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)$$

14.2 Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

15 Directional Derivatives and Gradient

15.1 Definitions

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2\rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle u_1, u_2\rangle = \nabla f(a, b) \cdot \mathbf{u}$$

Gradient

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

15.2 Directions of Change

- f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$.
- f has its maximum rate of decrease at (a, b) in the direction of the gradient $-\nabla f(a, b)$. The rate of decrease in this direction is $-|\nabla f(a, b)|$.
- The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

15.3 Expanding to Three Dimensions

It's really intuitive how it expands into three dimensions. Just add another component or f_z where you think it should go.

16 Tangent Plane and Linear Approximation

16.1 Tangent Plane for $\mathbf{F(x, y, z) = 0}$

The tangent plane passes through the point $P_0(a, b, c)$.

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

16.2 Tangent Plane for $\mathbf{z = f(x, y)}$

The tangent plane passes through the point $(a, b, f(a, b))$.

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

16.3 Linear Approximation

Firstly, calculate the equation of the tangent plane of a point near the point you wish to approximate. Then simply plug in the point and you're done.

16.4 The differential dz

The change in $z = f(x, y)$ as the independent variables change from (a, b) to $(a + dx, b + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy$$

17 Max-Min Problems

17.1 Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

17.2 Critical Points

A critical point exists if either

- $f_x(a, b) = f_y(a, b) = 0$
- one (or both) of f_x or f_y does not exist at (a, b)

17.3 Second Derivative Test

Let $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$

- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D(a, b) < 0$, then f has a saddle point at (a, b).
- If $D(a, b) = 0$, then the test is inconclusive.

17.4 Absolute Maximum/Minimum Values

Let f be continuous on a closed bounded set R in $\mathbf{R}^2$. To find absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum values of f on R.

18 Double Integrals

18.1 Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

18.2 Double Integrals over Nonrectangular Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and by the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy$$

18.3 Areas of Regions by Double Integrals

$$\text{area of } R = \iint\limits_R dA$$

19 Polar Double Integrals

19.1 Double Integrals over Polar Rectangular Regions

Let f be continuous on the region in the xy-plane $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta$$

19.2 Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$$

where $\beta - \alpha \leq 2\pi$. Then.

$$\iint\limits_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

If f is nonnegative on R, the double integral gives the volume of the solid bounded by the surface $z = f(r, \theta)$ and R.

19.3 Area of Polar Regions

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\,dr\,d\theta$$

20 Triple Integrals

Let $D = \{(x, y, z) : a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\}$, where g, h, G, H are continuous functions. The triple integral of a continuous function f on D is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\,dz\,dy\,dx$$

21 Cylindrical and Spherical Coordinates

21.1 Definitions

21.1.1 Cylindrical Coordinates

(r, θ, z) An extension of polar coordinates into $\mathbf{R}^3$. Simply add a z component.

21.1.2 Spherical Coordinates

(ρ, φ, θ)

- ρ is the distance from the origin to a point P.
- φ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates; it measure rotation about the z-axis relative to the positive x-axis.

21.2 Rectangular to Cylindrical

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$z = z$$

21.3 Cylindrical to Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

21.4 Integration in Cylindrical Coordinates

$$\iiint\limits_D f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

21.5 Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2$$

You have to solve for φ and θ with trigonometry.

21.6 Spherical to Rectangular

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$

21.7 Integration in Spherical Coordinates

$$\iiint\limits_D f(\rho,\varphi,\theta)\,dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho,\varphi,\theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

22 Change of Variables

22.1 Jacobian Determinant of a Transformation of Two Variables

Given a transformation $T: x = g(u,v), y = h(u,v)$, where g and h are differentiable on a region of the uv-plane, the Jacobian determinant of T is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

22.2 Change of Variables for Double Integrals

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(g(u,v),h(u,v))|J(u,v)|\,dA$$

22.3 Change of Variables for Triple Integrals

I am *SO* not typing out the expansion of the above into triple integrals. It's intuitive. Just add stuff where you think it should go.

22.4 YOU have to Choose the Transformation

Just cry.

23 Vector Fields

23.1 Vector Fields in Two Dimensions

$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle$$

23.2 Radial Vector Fields in $\mathbf{R}^2$

Let $\mathbf{r} = (x, y)$. A vector field of the form $\mathbf{F} = f(x, y)\mathbf{r}$, where f is a scalar-valued function, is a radial vector field.

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p}$$

p is a real number. At every point (sans origin), the vectors of this field are directed outward format he origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$. You can also apply all of this to $\mathbf{R}^3$ by just adding a z component.

23.3 Gradient Fields and Potential Functions

Let $z = \varphi(x, y)$ and $w = \varphi(x, y, z)$ be differentiable functions on regions of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is a gradient field, and the function φ is a potential function for $\mathbf{F}$.

24 Line Integrals

24.1 Evaluating Scalar Line Integrals in $\mathbf{R}^2$

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int\limits_C f\,ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

24.2 Evaluating Scalar Line Integrals in $\mathbf{R}^3$

Simply add a z component to the above where it obviously belongs.

24.3 Line Integrals of Vector Fields

24.3.1 Definition

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parametrized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

24.3.2 Different Forms

$\mathbf{F} = \langle f, g, h \rangle$ and C has a parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (fx'(t) + gy'(t) + hz'(t))\, dt = \int\limits_C f\, dx + g\, dy + h\, dz = \int\limits_C \mathbf{F} \cdot d\mathbf{r}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parametrized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b (fx'(t) + gy'(t))\, dt = \int\limits_C f\, dx + g\, dy = \int\limits_C \mathbf{F} \cdot d\mathbf{r}$$

24.4 Work

$\mathbf{F}$ is a force field

$$W = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

24.5 Circulation

$\mathbf{F}$ is a vector field

$$Circulation = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds$$

24.6 Flux

$$Flux = \int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (fy'(t) - gx'(t))\, dt$$

$\mathbf{n} = \mathbf{T} \times \mathbf{k}$, and a positive answer means a positive outward flux.

25 Conservative Vector Fields

25.1 Test for Conservative Vector Field

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbf{R}^3$, where f, g, and h have continuous first partial derivatives on

D. Then, $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

- $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$
- $\dfrac{\partial f}{\partial z} = \dfrac{\partial h}{\partial x}$
- $\dfrac{\partial g}{\partial z} = \dfrac{\partial h}{\partial y}$

For vector fields in $\mathbf{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

25.2 Finding Potential Functions

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases. This method can also be used to check if a vector field is conservative by seeing if there is a potential function.

25.3 Fundamental Theorem for Line Integrals

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

25.4 Line Integrals on Closed Curves

Let R in $\mathbf{R}^2$ (or D in $\mathbf{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R.

26 Green's Theorem

26.1 Circulation Form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f\, dx + g\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

26.2 Area of a Plane Region by Line Integrals

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

26.3 Flux Form

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

27 Divergence and Curl

27.1 Divergence of a Vector Field

$$\text{div}(\mathbf{F}) = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

27.2 Divergence of Radial Vector Fields

$$\text{div}(\mathbf{F}) = \frac{3-p}{|\mathbf{r}|^p}$$

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{p/2}}$$

27.3 Curl

$$\text{curl}(\mathbf{F}) = \nabla\times\mathbf{F}$$

Just derive the curl by doing the cross product.

27.4 Divergence of the Curl

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0$$

28 Surface Integrals

28.1 Parameterization

28.1.1 z is Explicitly Defined

Use $x = x$, $y = y$, and since z is explicitly defined, you already have what z equals.

28.1.2 Cylinder

Simply use cylindrical coordinates to parameterize the surface in terms of θ and z.

28.1.3 Sphere

Simply use spherical coordinates to parameterize the surface in terms of φ and θ.

28.1.4 Cone

Use:

- $x = v \cos u$
- $y = v \sin u$
- $z = v$

$0 \leq u \leq 2\pi$ and $0 \leq v \leq h$

28.2 Surface Integrals of Parameterized Surfaces

$$\iint_{\Sigma} f(x,y,z)\, d\sigma = \iint_{R} f(x(u,v), y(u,v), z(u,v)) \left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| dA$$

29 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D enclosed by a smooth oriented surface S. Then

$$\iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{D} \nabla \cdot \mathbf{F}\, dV$$

where $\mathbf{n}$ is the outward normal vector on S.