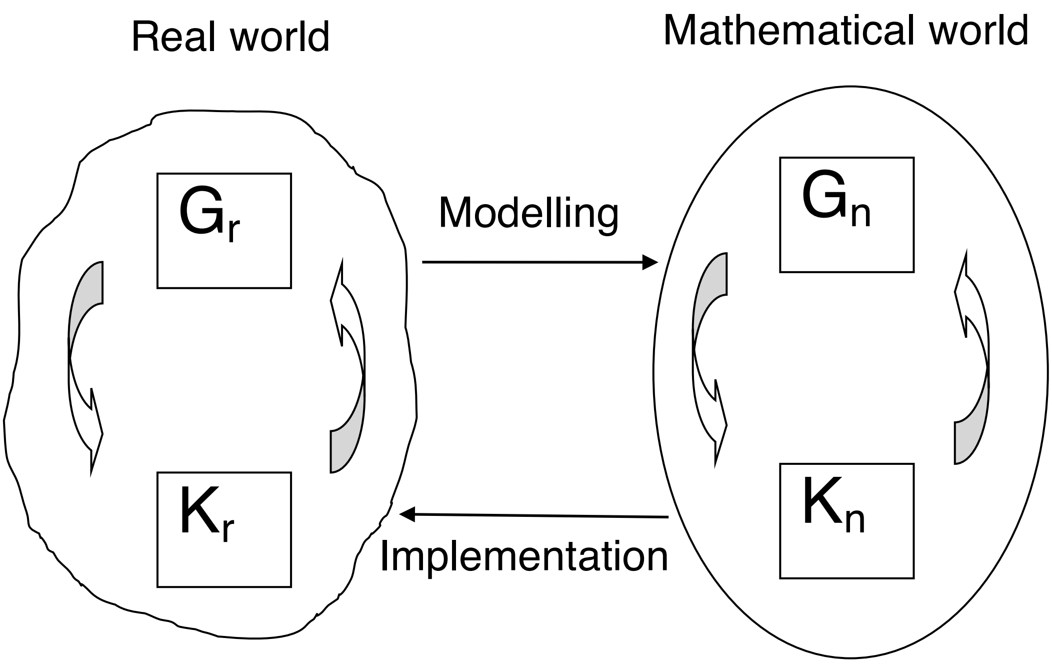
MULTIVARIABLE CONTROL SYSTEMS DESIGN*°

by Ian K. Craig

* These viewgraphs are based on notes prepared by Prof. Michael Athans of MIT for the course "Multivariable Control Systems 1 & 2"
° These viewgraphs should be read in conjunction with the textbook: S Skogestad, I Postlehwaite, Multivariable Feedback Control, Second Edition, Wiley, Chichester, 2005.

General Control Problem (I)



General Control Problem (II)

Summary

- Analyze the plant for control purposes
- Obtain an adequate mathematical model for the plant
- Controller design and control system analysis
- Controller implementation
- Controller evaluation

Analyze the plant (G_r) for control purposes

- obtain process knowledge
- perform an initial cost benefit analysis
- determine initial control objectives
- determine measurements, manipulated and control variables
- determine the role of the operator, before and after the implementation of the controller

Obtain an adequate mathematical model (G_n) for the plant

- use first principles and empirical relationships and/or plant input-output data
- simplify the model to fit purpose
- *analyse the resulting model (input-output analysis)*

General Control Problem (III)

Controller design (K_n) and control system analysis

- select controller configuration and type
- decide performance specifications
- *design controller and analyse it to see if specifications are satisfied*

Controller implementation (K_r)

- simulate controller (hardware-in-loop; pilot plant)
- select hardware and software and implement

Controller evaluation

- test and validate for functional and economic specifications

References:

- S. Skogestad, I. Postlehwaite, *Multivariable Feedback Control*, Second Edition, Wiley, Chichester, 2005. Page 1.
- Craig I.K., and Henning, R.G.D., Evaluation of advanced industrial control projects: a framework for determining economic benefits, *Control Engineering Practice*, Vol. 8, No. 7, 2000, pp. 769-780.

Main thrust of the course

- Description of:
  - Analysis tools (block 1)
  - Synthesis tools (block 2)

- Analysis and synthesis as part of the engineering design process

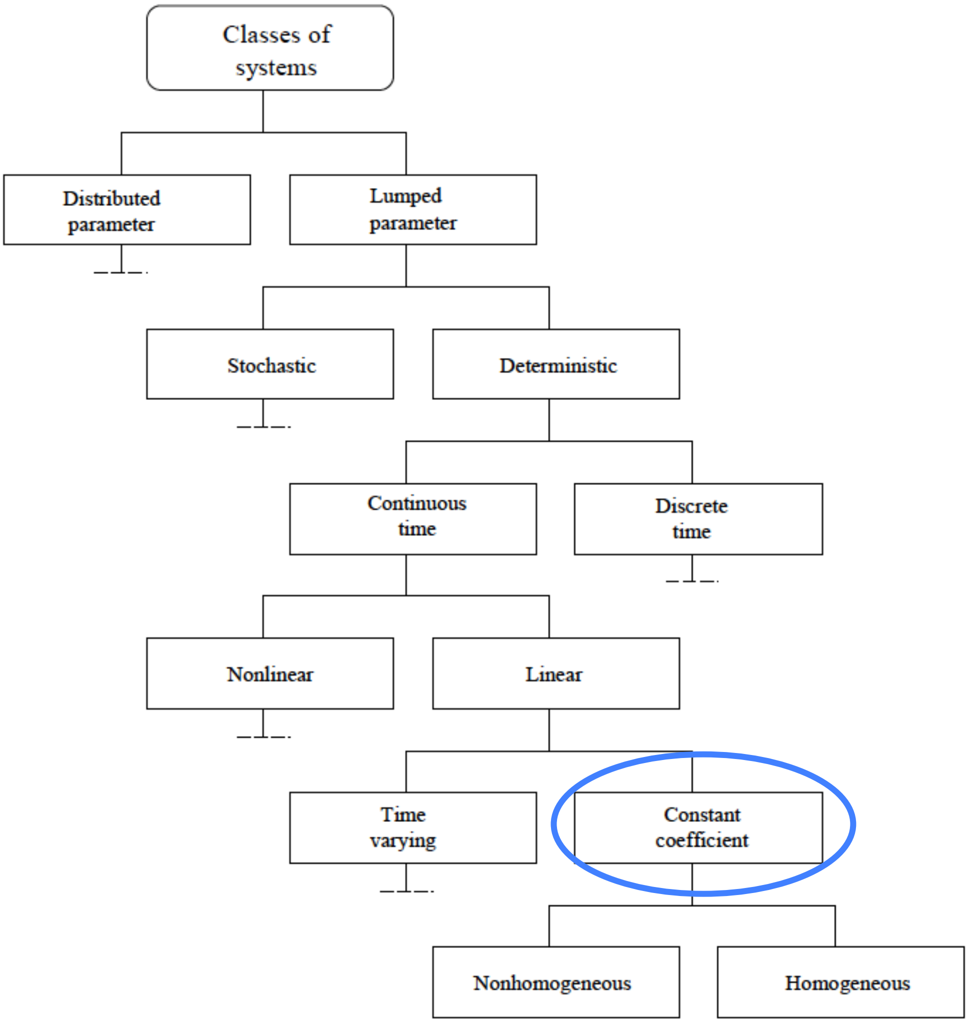
- Analysis: the process of determining whether a given system has the desired characteristics

  Tools
  - linear systems theory
  - linear algebra
  - functional analysis

- Synthesis: the process of finding a particular system component to achieve desirable characteristics

  Tools
  - dynamic optimal control theory
  - optimal estimation theory

Classes of system equations



This couse deals with:
Lumped parameter, deterministic, continuous-time, linear, constant coefficient systems

What is a dynamic system?

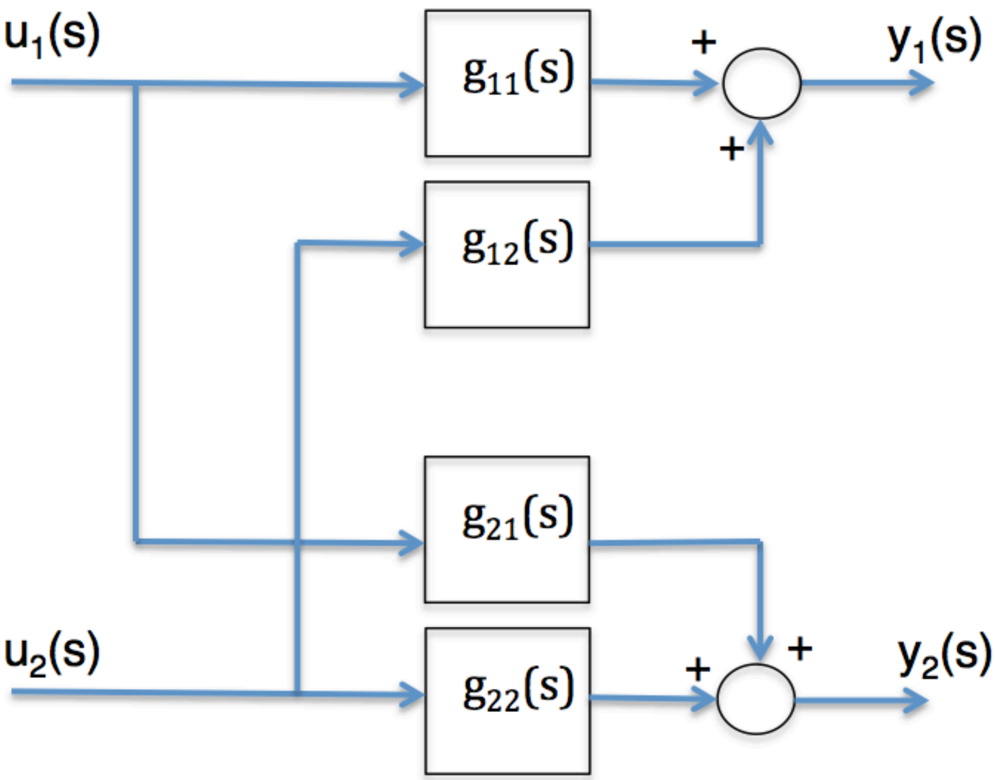
- Physical property
  - physical dynamic systems contain energy-storage elements
  - energy-storage elements are usually interconnected

- Physical energy changes as a function of time
  - Potential energy ⇔ kinetic energy
  - Dissipation of energy
  - Change in energy from external inputs

- Dynamic evolution of energy is the key to understanding the behaviour of dynamic systems

What is a MIMO system?

- MIMO: Multi-Input Multi-Output
- Plant model example



- Mathematical description

$$\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{pmatrix} \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix}$$

y(s) = G(s)u(s)

G(s): Transfer function matrix

Physical MIMO systems

- Aerospace
  - aircraft
  - missiles
  - satellites
  - space platforms

- Biological
  - immune system
  - glucose–insulin feedback system
  - disease transmission

- Chemical
  - reactors
  - distillation columns

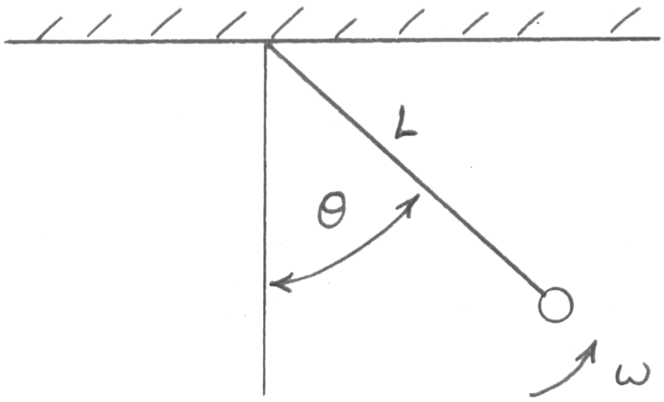
- Electrical
  - power systems
  - motors

- Mechanical
  - robots
  - automotive

- Metallurgical
  - grinding milling circuits
  - furnances
  - rolling machines

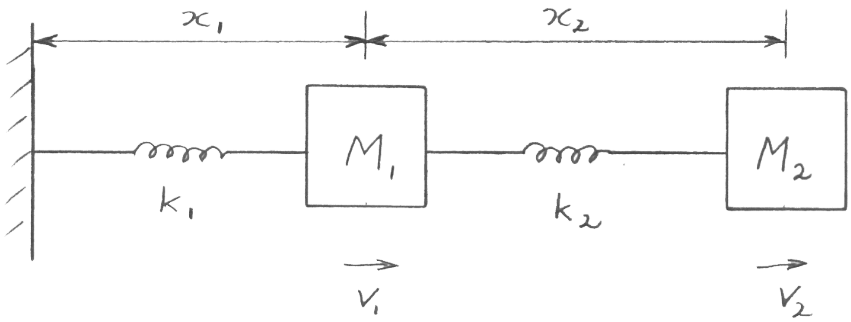
Simple dynamic systems

- Pendulum



θ $\Rightarrow$ potential energy
ω $\Rightarrow$ kinetic energy

- Mass-spring system



x_1, x_2 $\Rightarrow$ potential energy
v_1, v_2 $\Rightarrow$ kinetic energy

Physical state variables

- Associate one state with each energy storage element

- Mechanical system state variables

Positions ⇒ potential energy
Velocities ⇒ kinetic energy

- Electrical circuit state variables

Inductor currents ⇒ kinetic energy
Capacitor voltages ⇒ potential energy

- Thermodynamic systems

Pressures ⇒ "potential energy"
Temperatures ⇒ "kinetic energy"

Properties of dynamic systems

- Physical systems are interconnections of energy storage and dissipative elements
- State dynamics are described by coupled first order differential equations
- Unforced dynamic systems (no external inputs)
    - Assume valid model
    - If we know the numerical values of the state variables now, then we can calculate all future values of:
        - all state variables
        - other variables of interest
    - State variables are continuous functions of time

Forced dynamic systems

- External inputs (controls, disturbances)
  - can modify temporal evolution of energy (state) variables
  - add or subtract finite energy to/from system
  - bounded, piece-wise continuous

- If we know
  - state variables now
  - inputs from now on

Then we can calculate:
  - future values of all state variables
  - future values of all output variables

Forced state variable models (I)

- Finite dimensional time invariant models

- Notation

- State vector: $x'(t) = [x_1(t), x_2(t), ..., x_n(t)]'$
- Input vector: $u'(t) = [u_1(t), u_2(t), ..., u_m(t)]'$
- Output vector: $y'(t) = [y_1(t), y_2(t), ..., y_p(t)]'$

- Nonlinear model description

- State dynamics

$$dx_1(t)/dt = f_1(x(t),u(t))$$
$$dx_2(t)/dt = f_2(x(t),u(t))$$
$$\vdots$$
$$dx_n(t)/dt = f_n(x(t),u(t))$$

Collectively: $dx(t)/dt = f(x(t),u(t))$

- Output equations

$$y_1(t) = g_1(x(t),u(t))$$
$$y_2(t) = g_2(x(t),u(t))$$
$$\vdots$$
$$y_p(t) = g_p(x(t),u(t))$$

Collectively: $y(t) = g(x(t),u(t))$

Forced state variable models (II)

- Linear model description

- State dynamics: i = 1, 2, ..., n

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{k=1}^{m} b_{ik} u_k(t)$$

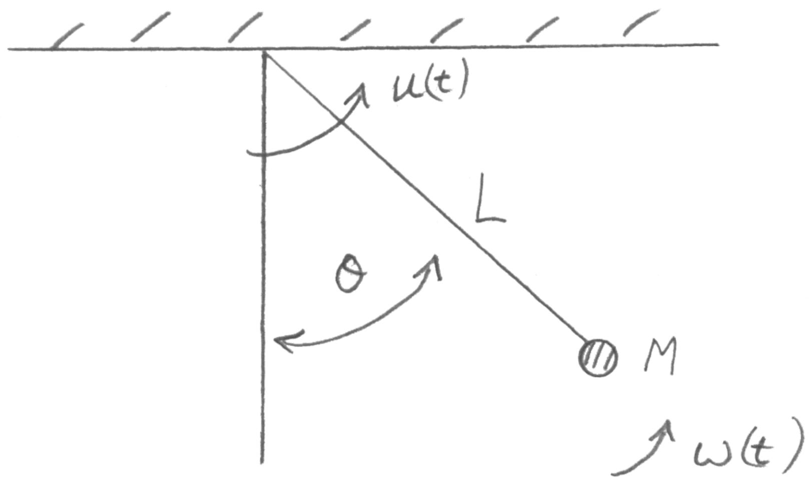
$\Rightarrow$ dx(t)/dt = Ax(t) + Bu(t)

- Output equation: q = 1, 2, ..., p

$$y_q(t) = \sum_{i=1}^{n} c_{qi} x_i(t) + \sum_{k=1}^{m} d_{qk} u_k(t)$$

$\Rightarrow$ y(t) = Cx(t) + Du(t)

Pendulum example



- State variables

  $\theta(t) \equiv x_1(t)$ = angular position
  $\omega(t) \equiv x_2(t)$ = angular velocity

- Control variable: torque u(t)

- State dynamics (without friction)

  $dx_1(t)/dt = x_2(t)$

  $dx_2(t)/dt = (g/L)\sin(x_1(t)) + (1/ML)u(t)$

Linearization of nonlinear dynamics

- Most dynamic systems are nonlinear
  - quantitative and qualitative properties not transparent

- Motivation for linearizing
  - lack of systematic design methodology for direct design of nonlinear feedback control systems
  - linearized dynamic models are useful
    - analysis: qualitative and quantitative insight
    - design: general and integrated CAD methodologies

- Linearized models have obvious limitations

- Pendulum example
  Linear approximation for small $x_1(t)$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \mathrm{K} \approx \theta$$

Thus $dx_2(t)/dt = (g/L)x_1(t) + (1/ML)u(t)$

Linear Dynamic Systems: Issues to be discussed

- Open-loop stability : MIMO poles

- Transient response : MIMO zeros

  - nature provides poles
  - control engineers regulate the zeros

- Modes of dynamic systems

- Solutions: time- and frequency-domain

Structure

- Work with finite dimensional linear time invariant (FDLTI) models

- LTI model describes some nonlinear system near steady-state equilibrium

- Notation

  State: $x(t) \in R^n$

  Control: $u(t) \in R^m$

  Output: $y(t) \in R^p$

- Dynamics

$$dx(t)/dt = Ax(t) + Bu(t) \qquad ;x(0) = \xi$$

$$y(t) = Cx(t) + Du(t)$$

Transfer function matrix

- Time-domain state-space model

$$dx(t)/dt = Ax(t) + Bu(t) \qquad ;x(0) = 0$$
$$y(t) = Cx(t) + Du(t)$$

- Vector Laplace transforms

$$x(s) = \mathcal{L}\{x(t)\} \quad \text{with} \quad \mathcal{L}\{dx(t)/dt\} = sIx(s) - 0$$
$$u(s) = \mathcal{L}\{u(t)\}$$
$$y(s) = \mathcal{L}\{y(t)\}$$

- Laplace transform of state-space

$$sIx(s) = Ax(s) + Bu(s)$$
$$y(s) = Cx(s) + Du(s)$$

or $$x(s) = (sI - A)^{-1}Bu(s)$$

- Input-output description

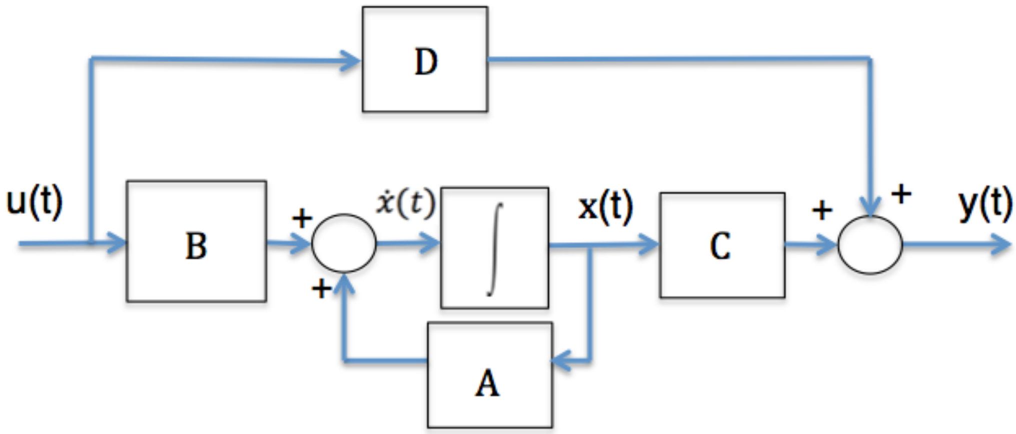
$$y(s) = [C(sI - A)^{-1}B + D]u(s)$$
$$y(s) = G(s)u(s)$$

G(s) is a p x m TFM

Finite dimensional LTI models

- State-space block diagram

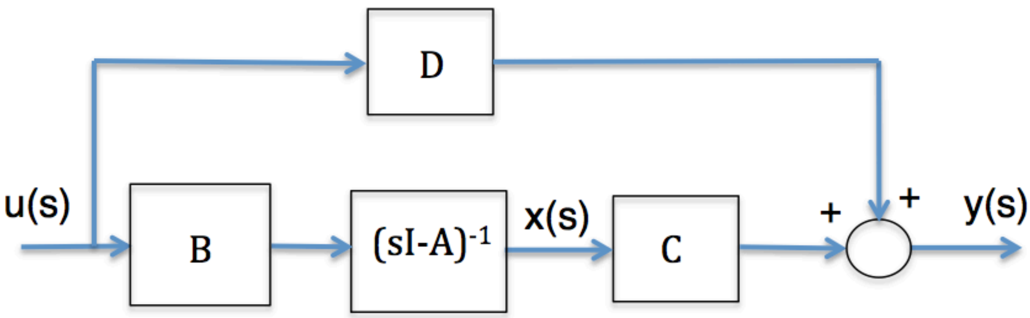
$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$



- Frequency-domain block diagram

$$y(s) = G(s)u(s)$$

$$G(s) = C(sI - A)^{-1}B + D$$



Unforced LTI systems

- Set inputs to zero: ignore outputs

$dx(t)/dt = Ax(t) \qquad ;x(0) = \xi$

- Time-domain solution

$x(t) = e^{At}\,\xi$

with $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(At\right)^{k}$

- Laplace transform of state-space model

$sIx(s) - \xi \qquad = Ax(s)$

or $\quad x(s) \qquad = (sI - A)^{-1}\xi$

thus $\quad \mathcal{L}\{e^{At}\} \qquad = (sI - A)^{-1} \qquad ;(n \times n)$

$\mathcal{L}^{-1}\{(sI - A)^{-1}\} \qquad = e^{At}$

Modal analysis (I)

- Eigenstructure of A ; i = 1, 2, ... , n

  $Av_i = \lambda_i v_i$; $w'_i A = \lambda_i w'_i$

- Dyadic formula

$$A = \sum_{i=1}^{n} \lambda_i v_i w_i'$$

- Fact:

  $A^k v_i = \lambda^k_i v_i$; $w'_i A^k = \lambda^k_i w'_i$

$$A^k = \sum_{i=1}^{n} \lambda_i^k v_i w_i'$$

- Fact:

$$e^{At} = \sum_{j=0}^{\infty} \frac{1}{j!}(At)^j = \sum_{j=0}^{\infty}\sum_{i=1}^{n} \frac{1}{j!}(\lambda_i t)^j v_i w'_i$$

thus $$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w'_i$$

with $$e^{\lambda_i t} = \sum_{j=0}^{\infty} \frac{1}{j!}(\lambda_i t)^j$$

Modal analysis (II)

- Unforced dynamics solution

$$x(t) = e^{At}\xi = \sum_{i=1}^{n} e^{\lambda_i t} v_i [w_i' \xi]$$

- Definition of i^{th} mode:

$$e^{\lambda_i t}\ v_i$$

  - State consists of sum of modes associated with right eigenvector directions
  - $[w'_i\ \xi]$: Degree that initial state ξ excites ith mode

- Important relation

$w'_j\ v_i \quad = \partial_{ij} \qquad$;(Kronecker delta)

$w'_j\ v_i \quad = 1 \quad ; i = j$

$w'_j\ v_i \quad = 0 \quad ; i \neq j$

- Laplace transform

$$\mathcal{L}\left\{e^{\lambda_i t}\right\} = \frac{1}{s - \lambda_i}$$

$$x(s) = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} v_i [w_i' \xi]$$

thus $\qquad (sI - A)^{-1} = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} v_i w_i'$

Eigenvector interpretation

- Unforced dynamics

  dx(t)/dt = Ax(t) ;x(0) = ξ

- To excite single mode

pick initial state ξ colinear with eigenvector v_j,
i.e.: $\xi = kv_j$

- Solution

$$
\begin{aligned}
x(t) &= \sum_{i=1}^{n} e^{\lambda_i t} v_i [w_i' \xi] \\
&= \sum_{i=1}^{n} k e^{\lambda_i t} v_i [w_i' v_j] \\
&= k e^{\lambda_j t} v_j
\end{aligned}
$$

- Solution contains single mode!

Stability and multivariable poles

- Open-loop dynamics

  $dx(t)/dt = Ax(t) \qquad ;x(0) = \xi$

$$x(t) = e^{At}\xi = \sum_{i=1}^{n} e^{\lambda_i t} v_i [w'_i \xi]$$

- If $\mathrm{Re}\{\lambda_i\} > 0$ then i^{th} mode $e^{\lambda_i t} v_i$ is unstable

- System is stable iff

  $\mathrm{Re}\{\lambda_i[A]\} < 0$ for all i = 1, 2, ..., n

  thus $x(t) \rightarrow 0$ for all $\xi \neq 0$

- Definition: multivariable poles are the eigenvalues of A
  - definition makes sense in terms of natural frequencies
  - MIMO poles are roots of characteristic polynomial: $\det(\lambda I - A)$

Forced LTI systems

- Model

  dx(t)/dt = Ax(t) + Bu(t) ;x(0) = ξ

  y(t) = Cx(t) + Du(t)

- Standing assumption:

Components of control vector u(t)
- bounded
- piece-wise continuous functions of time

- Complete solution

$$x(t) = e^{At}\xi + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$$

- Proof: use vector calculus facts

$(d/dt)e^{At} = Ae^{At}$

$(d/dt) \int_0^t f(\tau)d\tau = f(t)$

and differentiate $x(t) = e^{At}[\xi + \int_0^t e^{-A\tau} Bu(\tau)d\tau]$

to get dx(t)/dt = Ax(t) + Bu(t)

The complete solution

- Dynamics

$$dx(t)/dt = Ax(t) + Bu(t) \qquad ;x(0) = \xi$$
$$y(t) = Cx(t) + Du(t)$$

- Laplace transform

$$sIx(s) - \xi = Ax(s) + Bu(s)$$

thus $$x(s) = (sI - A)^{-1}\xi + (sI - A)^{-1}Bu(s)$$

- Compare with time-domain solution

$$\mathcal{L}\left\{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right\} = (sI - A)^{-1}Bu(s)$$

- Complete solution

Time-domain

$$y(t) = Ce^{At}\xi + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Frequency-domain

$$y(s) = C(sI - A)^{-1}\xi + [C(sI - A)^{-1}B + D]u(s)$$

Modal forms (I)

- State dynamics : x(0) = 0

$$dx(t)/dt = Ax(t) + Bu(t).$$
$$= Ax(t) + \sum_{k=1}^{m} b_k u_k(t)$$

with B = [b_1 b_2 ... b_m]

Solution $x(t)=\sum_{k=1}^{m}\int_0^t e^{A(t-\tau)} b_k u_k(\tau)d\tau$

Recall $e^{A(t-\tau)}=\sum_{i=1}^{n} e^{\lambda_i(t-\tau)} v_i w_i'$

- Time-domain solution

$$x(t)=\sum_{i=1}^{n}\sum_{k=1}^{m} v_i(w_i' b_k)\int_0^t e^{\lambda_i(t-\tau)} u_k(\tau)d\tau$$

- Frequency-domain solution

$$x(s)=\sum_{i=1}^{n}\sum_{k=1}^{m} v_i(w_i' b_k)\frac{1}{s-\lambda_i} u_k(s)$$

Modal forms (II)

- Solution with $x(0) = \xi \neq 0$

$$x(t) = \sum_{i=1}^{n} (w_i' \xi) v_i e^{\lambda_i t} + \sum_{i=1}^{n} \sum_{k=1}^{m} v_i (w_i' b_k) \int_0^t e^{\lambda_i (t-\tau)} u_k(\tau) d\tau$$

- Alternate form

$$x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} \left[(w_i' \xi) + \sum_{k=1}^{m} (w_i' b_k) \int_0^t e^{-\lambda_i \tau} u_k(\tau) d\tau \right]$$

- Modal directions preserved

- Natural time evolution ($e^{\lambda_i t}$) is changed by controls

- Insight

$(w'_i \xi)$: degree that initial state ξ excites i^{th} mode

$(w'_i b_k)$: degree that k^{th} control u_k influences i^{th} mode

Output response

- LTI model

  $$dx(t)/dt = Ax(t) + \sum_{k=1}^{m} b_k u_k(t) \qquad ;x(0) = \xi.$$

  $$y(t) = Cx(t) + Du(t)$$

- Individual outputs: $q = 1, 2, \ldots , p$

  $$y_q(t) = c'_q x(t) + d'_q u(t)$$

- Output response

  $$y_q(t) = \sum_{i=1}^{n} (c'_q v_i)(w'_i \xi) e^{\lambda_i t} + \sum_{i=1}^{n} \sum_{k=1}^{m} (c'_q v_i)(w'_i b_k) \int_0^t e^{\lambda_i (t-\tau)} u_k(\tau) d\tau + d'_q u(t)$$

- Insight

$(c'_q v_i)$: degree to which i^{th} mode will be visible in q^{th} output

Frequency domain solutions

- Recall

  $(sI - A)^{-1} = \sum_{i=1}^{n} \frac{1}{s-\lambda_i} v_i w'_i$

- State

$$x(s) = \sum_{i=1}^{n} \frac{1}{s-\lambda_i} v_i [w'_i \xi] + \sum_{i=1}^{n} \frac{1}{s-\lambda_i} v_i w'_i B u(s)$$

- Output

$$y(s) = \sum_{i=1}^{n} \frac{1}{s-\lambda_i} C v_i [w'_i \xi]$$

$$+ \sum_{i=1}^{n} \frac{1}{s-\lambda_i} C v_i w'_i B u(s) + D u(s)$$

Controllability

Background

- Formalized by R.E. Kalman in 1960
- Key concept in dynamic systems and control theory
- Formalizes intuitive notions about being able to control state variables and modes
- Will present "modern" and classical controllability tests
- Used with observability to understand MIMO input-output properties
  - MIMO pole-zero cancellations
  - minimum realizations

Controllability definition

- Deals only with state dynamics
- Applicable to both linear and nonlinear systems
- Definition:

The system $dx(t)/dt = f(x(t),u(t))$;$x(0) = \xi$

is called controllable if for any initial state $\xi \in R^n$ and any terminal state $\theta \in R^n$, we can find a piece-wise continuous function $u(t)$, $0 \leq t \leq T$, with T finite, such that

$$x(T) = \theta$$

Otherwise the system is called uncontrollable

Remarks

- No easy test for general nonlinear systems
- Easy test exist for finite-dimensional linear-time-invariant (FDLTI) dynamic systems
- Two tests

"modern" - modal approach

"classical" - Via Caley-Hamilton theorem

- Warning: A dynamic model which is mathematically controllable, might be uncontrollable from a practical point of view
  - Test does not say how states behave, e.g. might not be possible to hold states at given value θ
  - The required inputs u(t) may be very large
  - Some of the states may be of no practical importance
  - Definition does not provide a degree of controllability
- What does the controllability result tell us?
  - If our model includes states that we cannot affect
  - If we can save on computer time by deleting uncontrollable (stable!) states

Modal solutions

- State dynamics with $x(0) = \xi$

$$dx(t)/dt = Ax(t) + Bu(t)$$

$$dx(t)/dt = Ax(t) + \sum_{k=1}^{m} b_k u_k(t)$$

- Recall eigenstructure

$Av_i = \lambda_i v_i$; $w'_iA = \lambda_i w'_i$; $w'_j v_i = \partial_{ij}$

- State response

$$x(t) = \sum_{i=1}^{n} (w'_i \xi) v_i e^{\lambda_i t}$$
$$+ \sum_{i=1}^{n} \sum_{k=1}^{m} v_i (w'_i b_k) \int_0^t e^{\lambda_i (t-\tau)} u_k(\tau) d\tau$$

- Modal uncontrollability

If $w'_i b_k = 0$ for <u>some</u> k, then mode i is uncontrollable from control $u_k(t)$

Modal uncontrollability

- Mode i is uncontrollable (from all inputs) iff for all k = 1, 2, ..., m

  $w'_i b_k = 0$ or $w'_i B = 0$

- System is uncontrollable iff one or more of its modes are uncontrollable

- Reasoning using state response

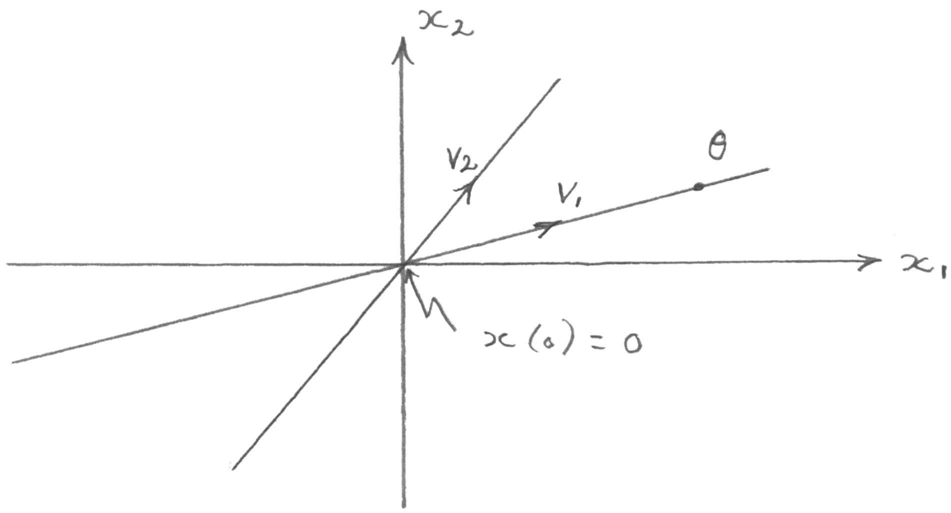
  Pick initial state $\xi = 0$
  Pick terminal state θ colinear to uncontrollable mode i, i.e. $\theta = kv_i$

$$x(T) = \sum_{j=1}^{n} \sum_{k=1}^{m} v_j (w_j' b_k) \int_0^T e^{\lambda_j (t-\tau)} u_k(\tau) d\tau$$

$x(T) \neq kv_i$ - directions do not match

Visualization

- 2 modes and 2 controls: n = 2; k = 2
- Suppose mode $v_1 e^{\lambda_1 t}$ is uncontrollable



- Response restricted along v_2 direction. θ cannot be reached
- Time-domain solution

$$
\begin{aligned}
x(t) = {} & v_1(w_1' b_1 = 0)\int_0^t e^{\lambda_1(t-\tau)} u_1(\tau)\,d\tau \\
& + v_1(w_1' b_2 = 0)\int_0^t e^{\lambda_1(t-\tau)} u_2(\tau)\,d\tau \\
& + v_2(w_2' b_1)\int_0^t e^{\lambda_2(t-\tau)} u_1(\tau)\,d\tau \\
& + v_2(w_2' b_2)\int_0^t e^{\lambda_2(t-\tau)} u_2(\tau)\,d\tau
\end{aligned}
$$

Modal controllability

- The i^{th} mode is controllable (from one or more inputs) iff

$$w'_i B \neq 0$$

w'_i: left eigenvector associated with i^{th} mode,

$$w'_i A = \lambda_i w'_i$$

- System is controllable iff all the modes of the system are controllable

Test: $w'_i B \neq 0$ for all $i = 1, 2, ..., n$

- Notation

Refer to the controllability of a matrix pair: [A, B]

A = n x n matrix
B = n x m matrix

Complex modes

- If $\lambda_i = \lambda^*_j$

then $v_i = v^*_j$ and $w_i = w^*_j$

- i^{th} mode is uncontrollable from k^{th} input

$$\Rightarrow \quad w'_i b_k = 0$$

let $w_i = \alpha_i + j\beta_i$

thus $(\alpha_i + j\beta_i)' b_k = 0$

$$\Rightarrow \quad \alpha'_i b_k = 0 \quad \text{and} \quad \beta'_i b_k = 0$$

but $w_j = \alpha_i - j\beta_i = w^*_i$

$$\Rightarrow \quad (\alpha_i - j\beta_i)' b_k = 0$$

$$\Rightarrow \quad w'_j b_k = 0$$

$\Rightarrow$ j^{th} mode is also uncontrollable from k^{th} input

- Complex mode is thus uncontrollable

Stabilizability

- Useful concept for design

- If mode $v_i e^{\lambda_i t}$ is uncontrollable but $Re\{\lambda_i\} < 0$, then mode i is stabilizable

- If all uncontrollable modes are stabilizable, then [A, B] is called stabilizable

- Notes

- If [A, B] is controllable, then it is stabilizable

- If every unstable mode is controllable, then [A, B] is stabilizable

Classical controllability test

- Form the n x (m x n) controllability matrix M_c

M_c = [B AB A^2B ... $A^{n-1}B$]

- If out of the m x n columns M_c there are n that are linearly independent, i.e.

Rank (M_c) = n

Then [A, B] is controllable

- If Rank (M_c) < n

Then [A, B] is uncontrollable

- it may be stabilizable

- No modal information
- No stabilizability information