Munkres - Topology - Chapter 2 Solutions

Section 13

Problem 13.1. *Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.*

Solution: Let $\mathcal{C}_A$ the collection of open sets U where $x \in U \subseteq A$ for some $x \in A$. Suppose $U_0 = \bigcup_{U \in \mathcal{C}_A} U$. Since X is a topological space, U_0 is open in X. Clearly if $x \in A$, then $x \in U_0$, so $U_0 \subseteq A$. Conversely, if $x \in U_0$, then there is a $U \subseteq U_0$ such that $x \in U_0 \subseteq A$, so $x \in A$. Therefore $U_0 \subseteq A$, from which it follows that $U_0 = A$. Consequently, A is open in X.

Problem 13.2. *Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.*

Solution: I label the topologies in Example 1 as follows:

1	2	3
4	5	6
7	8	9

The table below compares them. The number indicates the **finer** of the two topologies. "N" indicates that the two topologies are incomparable.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1		2	3	4	5	6	7	8	9
2	2		N	N	N	N	2	8	9
3	3	N		3	N	6	3	N	9
4	4	N	3		N	6	N	8	9
5	5	N	N	N		N	N	N	9
6	6	N	6	6	N		6	N	9
7	7	2	3	N	N	6		8	9
8	8	8	N	8	N	N	8		9
9	9	9	9	9	9	9	9	9	

It's clear that 1 and 9 are comparable to all the topologies, with 1 the coarsest and 9 the finest. Also note that 5 is only comparable to 1 and 9.

Problem 13.2. *(a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X? (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$. (c) If $X = \{a, b, c\}$, let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.*

Solution: Part (a) Let $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$. Obviously X and $\emptyset$ are element sof $\mathcal{T}$ since they are contained in each α. Now suppose $\{U_\beta\}$ is an indexed family of non-empty open sets in $\mathcal{T}$. It follows that each $U_\beta \in \mathcal{T}_\alpha$ for each α, so $U = \bigcup_\beta U_\beta$ is an element of each $\mathcal{T}_\alpha$. Consequently, U is open $\mathcal{T}$. Now suppose $\{U_1, \ldots, U_n\}$ is a collection of non-empty open sets in $\mathcal{T}$. Since each U_k is open in each $\mathcal{T}_\alpha$, it follows that $U' = \bigcap_{k=1}^n U_k$ is open in each $\mathcal{T}_\alpha$. Therefore U' is open in $\mathcal{T}$, establishing that $\mathcal{T}$ is a topology on X.

The set $\bigcup_\alpha \mathcal{T}_\alpha$ is not necessarily a topology on X. Suppose $X = \mathbb{Z}$. Let $\mathcal{T}_0, \mathcal{T}_1$ be topologies on X where $\mathbb{Z}_{>0} \in \mathcal{T}_0$ and $\mathbb{Z}_{<0} \in \mathcal{T}_1$ but $\mathbb{Z}_{>0} \cup \mathbb{Z}_{<0}$ is not in either. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$. We have $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{<0}$ open in $\mathcal{T}$, but $\mathbb{Z}_{>0} \cup \mathbb{Z}_{<0} = \mathbb{Z} \backslash \{0\}$ is not open in $\mathcal{T}$. It follows that $\mathcal{T}$ is not a topology on X.

Part (b) The smallest toplogy on X is $\mathcal{T}_s$ generated by the subbasis $\mathcal{S} = \bigcup_\alpha \mathcal{T}_\alpha$. As we have seen, $\mathcal{T}_s$ is a topology, and it contains every $\mathcal{T}_\alpha$. Now suppose there is a topology $\mathcal{T}'$ that is strictly coarser than $\mathcal{T}_s$ (i.e., $\mathcal{T}' \subset \mathcal{T}_s$). Let $U \in \mathcal{T}_s \backslash \mathcal{T}'$, which must exist. If U is open in any $\mathcal{T}_\alpha$, then $\mathcal{T}_\alpha$ cannot be contained in $\mathcal{T}'$. It follows that every element in the subbasis $\mathcal{S}$ must be in $\mathcal{T}'$. On the other hand, suppose U is not contained in the subbasis $\mathcal{S}$, in which case it is not in any $\mathcal{T}_\alpha$. It follows that U must the union or finite intersection of a subcollection of elements in $\mathcal{S}$. If the former, there is an indexed family $\{S_\beta\} \subseteq \mathcal{S}$ whose union is not in $\mathcal{T}'$, which contradicts that $\mathcal{T}'$ is a topology. Alternatively, there is a subcollection $\{S_1, \ldots, S_n\}$ whose intersection is not in $\mathcal{T}'$, also contradicting that $\mathcal{T}'$ is a topology. Because there are no other possible elements of $\mathcal{T}_s$ that are not in $\mathcal{T}'$, we conclude that $\mathcal{T}_s = \mathcal{T}'$. As a result, $\mathcal{T}$ is the unique smallest topology on X that contains all $\mathcal{T}_\alpha$.

The largest topology on X containing all $\mathcal{T}_\alpha$ is $\mathcal{T}_l = \bigcap_\alpha \mathcal{T}_\alpha$, which (by part(a)) is a topology. Clearly $\mathcal{T}_l \subseteq \mathcal{T}_\alpha$ for all α. Suppose there is a $\mathcal{T}''$ that is strictly finer than $\mathcal{T}_l$ and is contained in all $\mathcal{T}_\alpha$. Let $U = \mathcal{T}_l \backslash \mathcal{T}''$. It follows that U is not an element of some $\mathcal{T}_\alpha$ (otherwise, it would be the intersection of the topologies). As a result, we reach the contradiction that $\mathcal{T}''$ is not contained in $\mathcal{T}_\alpha$. Consequently, $\mathcal{T}_l$ is the largest topology on X contained in all $\mathcal{T}_\alpha$.

Part (c) The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Notice that this is the topology generated by the subbasis equal to $\mathcal{T}_1 \cup \mathcal{T}_2$.

The smallest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, X, \{a\}\}$.

Problem 13.5. *Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.*

Solution: Suppose $\mathcal{A}$ is a basis for a topology $\mathcal{T}_A$ on X. Let $\{\mathcal{T}_\alpha\}$ be an indexed collection of all topologies on X where $\mathcal{A}$ is contained in each $\mathcal{T}_\alpha$, and let $\mathcal{T}_B = \bigcap_\alpha \mathcal{T}_\alpha$ (which is a topology by exercise 13.4(a)). Suppose that U_0 is an open set in $\mathcal{T}_A$. We infer from Lemma 13.1 that U_0 is the union of some subcollection of $\mathcal{A}$. Since this subcollection is contained in each topology $\mathcal{T}_\alpha$, it follows that their union is in each, so U_0 is open in $\mathcal{T}_B$. Therefore $\mathcal{T}_A \subseteq \mathcal{T}_B$. Conversely, since $\mathcal{T}_A$ is a topology on X containing $\mathcal{A}$ by definition, it follows that $\mathcal{T}_B \subseteq \mathcal{T}_A$. Thus $\mathcal{T}_A = \mathcal{T}_B$.

The argument for a given subbasis $\mathcal{S}$ for a topology on X is very similar. Let $\mathcal{T}_S$ be the topology generated by the subbasis. Let $\{\mathcal{T}_\gamma\}$ be an indexed collection of all topologies on X containing $\mathcal{S}$, and let $\mathcal{T}_C = \bigcap_\gamma \mathcal{T}_\gamma$. If U is open in $\mathcal{T}_S$, it is equal to either the union or finite intersection of some subcollection in $\mathcal{S}$. Since $\mathcal{S}$ is contained in each topology $\mathcal{T}_\gamma$, we infer that U is contained in each and is therefore open in $\mathcal{T}_C$. Therefore $\mathcal{T}_S \subseteq \mathcal{T}_C$. Conversely, since $\mathcal{T}_S$ is a topology containing $\mathcal{S}$, it follows that $\mathcal{T}_C \subseteq \mathcal{T}_S$. We conclude that $\mathcal{T}_S = \mathcal{T}_C$.

Problem 13.6. *Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.*

Solution: It suffices to show that neither of the topologies is finer than the other. Let $\mathcal{T}_l$ and $\mathcal{T}_K$ be the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$, respectively. A given $x \in \mathbb{R}$ is contained in the basis element $[x, b)$ of $\mathcal{T}_l$. However, every basis element of $\mathcal{T}_K$ is an open interval (in some cases, minus the set K). There is no open interval (a, b) that contains x and is contained in $[x, b]$ because a must be less than x. By Lemma 13.3, $\mathcal{T}_K$ is not finer than $\mathcal{T}_l$.

Conversely, 0 is contained in the basis element $(-1, 1)\backslash K$ of $\mathcal{T}_K$. Any basis element (a, b) of $\mathcal{T}_l$ contains x where $a < 0$ and $b > 0$, but this basis element cannot be contained in $(-1, -1)\backslash K$. Given $b > 0$, let $k \in \mathbb{N}$ where $k > 1/b$. It follows that $0 < 1/k < b$, so $1/k \in (a, b)$ but $1/k \notin (-1, 1)\backslash K$. Again by Lemma 13.3, $\mathcal{T}_l$ is not finer than $\mathcal{T}_l$. Hence $\mathcal{T}_l$ and $\mathcal{T}_K$ are not comparable.

Problem 13.8. *(a) Apply Lemma 13.2 to show that the countable collection $\mathcal{B} = \{(a, b) \mid a < b,$ a and b rational$\}$ is a basis that generates the standard topology on $\mathbb{R}$. (b) Show that the collection $\mathcal{C} = \{[a, b) \mid a < b,$ a and b rational$\}$ is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.*

Solution: Part (a) Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$ and $\mathcal{T}_\mathbb{R}$ be the standard topology on $\mathbb{R}$. Let U_0 be an open set in $\mathcal{T}$. It follows that U_0 is the union of some subcollection $\{B_k\}$ of $\mathcal{B}$. Since $\mathcal{B}$ is a subset of the basis of $\mathcal{T}_\mathbb{R}$, it follows that U_0 is open in $\mathcal{T}_\mathbb{R}$. Conversely, given a set U_1 open in $\mathcal{T}_\mathbb{R}$, there is a subcollection $\{B'_l\}$ of $\mathcal{T}_\mathbb{R}$'s basis who union equals U_1. Given $x \in U_1$, there is a $B'_l = (a', b')$ containing x. By the density of rational numbers, there are $a, b \in \mathbb{Q}$ where $a' \leq a < x < b \leq b'$ such that $(a, b) \in B \in \mathcal{B}$ and B is contained in B'_l. It follows that U_1 is open in $\mathcal{T}$, from which we conclude that $\mathcal{T}$ and $\mathcal{T}_\mathbb{R}$ are equal.

Part (b) Let $\mathcal{T}_C$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}_l$ be the lower-limit topology on $\mathbb{R}$. We will show that $\mathcal{T}_l$ is finer than $\mathcal{T}_C$. Given $x \in \mathbb{R}$, we see that x is contained in any basis element $[a, b)$ of $\mathcal{T}_C$ where $a, b, \in \mathbb{Q}$ and $a \leq x$ and $x < b$ (which must exist by the density of rational numbers). But $[a, b)$ is also a basis element of $\mathcal{T}_l$, so by Lemma 13.3 $\mathcal{T}_l$ is finer than $\mathcal{T}_C$.

Now let x be irrational. It follows that x is contained in the basis element $[x, y)$ of $\mathcal{T}_l$ where $y \in \mathbb{R}$ and $x < y$. However, there is no basis element of $\mathcal{T}_C$ that contains x and is contained by $[x, y)$. If $[\alpha, \beta)$ contains x where α, β are rational, then necessarily $\alpha < x$, so $[\alpha, \beta)$ is not a subset of $[x, y)$. By Lemma 13.3, $\mathcal{T}_C$ is not finer than $\mathcal{T}_l$. Accordingly, the two topologies are not equal.

Section 16

Problem 16.1. *Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.*

Solution: Let $\mathcal{B}_X$ be a basis of the topology of X, let $\mathcal{T}_Y^X$ be the subspace topology of Y inherited from X, and let $\mathcal{T}_A^X$ and $\mathcal{T}_A^Y$ be the subspace topologies of A inherited from X and Y, respectively. By Lemma 16.1, a basis of $\mathcal{T}_Y^X$ is $\mathcal{B}_Y^X = \{B_0 \cap Y : B_0 \in \mathcal{B}_X\}$ and a basis of $\mathcal{T}_A^Y$ is $\mathcal{B}_A^Y = \{B_1 \cap A : B_1 \in \mathcal{B}_Y^X\} = \{(B_0 \cap Y) \cap A : B_0 \in \mathcal{B}_X\}$. Since A is a subset of Y, it follows $(B_0 \cap Y) \cap A = B_0 \cap (Y \cap A) = B_0 \cap A$. Therefore $\mathcal{B}_A^Y = \{B_0 \cap A : B_0 \in \mathcal{B}_X\}$. But this set is also a basis for the subspace $\mathcal{T}_A^X$ inherited from X. Since the bases for $\mathcal{T}_A^X$ and $\mathcal{T}_A^Y$ are equal, it follows that the two topologies are equal.

Problem 16.2. *If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?*

Solution: Let U_0 be open in the subspace topology $\mathcal{T}_Y$ of $\mathcal{T}$. There is a $W_0 \in \mathcal{T}$ where $U_0 = W_0 \cap Y$. But since $\mathcal{T}'$ is finer than $\mathcal{T}$, it follows that W_0 is open in $\mathcal{T}'$, so $W_0 \cap Y = U_0$ is open in the subspace topology $\mathcal{T}'_Y$ of $\mathcal{T}'$. Therefore $\mathcal{T}'_Y$ is finer than $\mathcal{T}_Y$.

We cannot say that $\mathcal{T}'_Y$ is necessarily strictly finer than $\mathcal{T}_Y$. Suppose $X = \mathbb{N}$, and $\mathcal{T}' = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \mathbb{N}\}$, and $\mathcal{T} = \{\emptyset, \{2\}, \mathbb{N}\}$. Let $Y_0 = \{1, 2\}$. As a result, $\mathcal{T}'_{Y_0} = \mathcal{T}_{Y_0} = \{\emptyset, \{2\}\}$. On the other hand, if $Y_1 = \{1, 2, 3\}$, then $\mathcal{T}_{Y_1} = \mathcal{T}_{Y_0}$ but $\mathcal{T}'_{Y_1} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$, so $\mathcal{T}'_{Y_1}$ is strictly finer than $\mathcal{T}_{Y_1}$.

Problem 16.9. *Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$ where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.*

Solution: We'll designate the dictionary-order topology as $\mathcal{T}_d$ and the product topology $\mathcal{U}_p$. From Example 14.1, the basis of $\mathcal{T}_d$ is the collection of intervals $(a \times b, c \times d)$ where $a < c$ or where $a = c$ and $b < d$. The text indicates that collection of intervals $\mathcal{B}_d = \{(a, b) \times (a, d) : b < d\}$ (i.e., where only the second condition is true) is also a basis of $\mathcal{T}_d$. We'll show that here. A given set W that is open in $\mathcal{T}_d$ is equal to the union of of a subcollection of basis elements. Let $W = \bigcup_\alpha (a \times b, c \times d)_\alpha$. For each $(x, y) \in W$, it follows that (x, y) is contained in some $(a \times b, c \times d)_\alpha$. If $a = c$, this basis element is in $\mathcal{B}_d$. Otherwise, $a < x < c$, and there is a basis element $B = (x, c) \times (x, d) \in \mathcal{B}_d$ where $(x, y) \in B \subseteq (a \times b, c \times d)_\alpha$. (If (x, y) is in multiple basis elements in the union, you can choose c and d to be small enough to contain all of them.) By Lemma 13.2, $\mathcal{B}_d$ is a basis of $\mathcal{T}_d$.

A basis for the discrete topology $\mathbb{R}_d$ is $\mathcal{B}_d = \{\{x\} : x \in \mathbb{R}\}$ (i.e., the collection of all singletons of $\mathbb{R}$). Given $V \in \mathbb{R}_d$, for $x \in V$, it follows that $x \in \{x\} \subseteq V$. Therefore $\mathcal{B}_d$ is a basis of $\mathbb{R}_d$. The standard basis for $\mathbb{R}$ is, of course, the collection of intervals (a, b) where $a < b$.

With the foregoing results and Theorem 15.1, a basis $\mathcal{B}_p$ for $\mathcal{U}_p$ is the collection of sets $\{x\} \times \{a, b\}$ where $a, b, x \in \mathbb{R}$ and $a < b$. But $\{x\} \times \{a, b\} = ((x, a), (x, b)) = (x, a) \times (x, b)$, so $\mathcal{B}_p$ and $\mathcal{B}_p$ are identical. Since they generate the same topology, $\mathcal{T}_d$ and $\mathcal{U}_p$ are equal.

Next we will compare this topology with the standard topology of $\mathbb{R}^2$. The basis of the standard topology is the collection of sets $(a, b) \times (c, d)$ where $a < b$ and $c < d$. Any $(x, y) \in \mathbb{R}^2$ is contained in the basis element $(a, b) \times (c, d)$ where $a < x < b$ and $c < y < d$. At the same time, $(x, y) \in \{x\} \times (c, d)$, which is a basis element of $\mathcal{U}_p$ contained in $(a, b) \times (c, d)$. By Lemma 13.3, $\mathcal{T}_d$ and $\mathcal{U}_p$ are finer than the standard topology. On the other hand, the standard topology is not finer than $\mathcal{T}_d$ and $\mathcal{U}_p$. For example, $\{1\} \times \mathbb{R}$ is open in $\mathcal{T}_d$ and $\mathcal{U}_p$ but is not open in the standard topology of $\mathbb{R}^2$. As a result, $\mathcal{T}_d$ and $\mathcal{U}_p$ are strictly finer than the standard topology on $\mathbb{R}^2$.

Section 17

Problem 17.1. *Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\emptyset$ and X are in $\mathcal{C}$, and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection $\mathcal{T} = \{X - C | C in \mathcal{C}\}$ is a topology on X.*

Solution: Clearly $X \backslash X = \emptyset$ and $x \backslash \emptyset = X$ are open in $\mathcal{T}$. Suppose $U = \bigcup_\alpha U_\alpha$ for some indexed collection $\{U_\alpha\}$ of open sets in $\mathcal{T}$. Since each $U_\alpha = X \backslash C_\alpha$ for some $C_\alpha \in \mathcal{C}$, it follows that:

$$U = \bigcup_\alpha (X \backslash C_\alpha) = X \backslash \bigcap_\alpha C_\alpha = X \backslash C',$$

where C' is the intersection of $\{C_\alpha\}$. By definition, $C' \in \mathcal{C}$, so U is open in $\mathcal{T}$.

Now suppose $V = \bigcap_{k=1}^N V_k$ for some finite collection $\{V_1, \ldots, V_N\}$ of open sets in $\mathcal{T}$. Since each $V_k = X \backslash D_k$ for some $D_k \in \mathcal{C}$, it follows that:

$$V = \bigcap_{k=1}^N (X \backslash V_k) = X \backslash \bigcup_{k=1}^N D_k = X \backslash D',$$

where D' is the finite union of $\{D_1, \ldots, D_N\}$. By definition, $D' \in \mathcal{C}$, so V is open in $\mathcal{T}$. We conclude that $\mathcal{T}$ is a topology on X.

Problem 17.2. *Show that if A is closed in Y and Y is closed in X, then A is closed in X.*

Solution: This question requires us to prove Theorem 17.3. By hypothesis, $Y \backslash A$ is open in Y, and there is some U open in X where $U \cap Y = Y \backslash A$. Therefore $X \backslash U$ is closed in X. Consequently, since $A \subseteq Y$:

$$Y \cap (X \backslash U) = (Y \cap X) \backslash (Y \cap U) = Y \backslash (Y \backslash A) = A.$$

By Theorem 17.1, A is closed in X.

Problem 17.4. *Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.*

Solution: The set $X \backslash A$ is open in X. We have:

$$U \cap (X \backslash A) = (U \cap X) \backslash (U \cap A) = U \backslash (U \cap A) = (U \backslash U) \cup (U \backslash A) = U \backslash A.$$

By the definition of a topology, $U \backslash A$ is open in X.

The set $X \backslash U$ is closed in X. We have:

$$A \cap (X \backslash U) = (A \cap X) \backslash (A \cap U) = A \backslash (A \cap U) = (A \backslash A) \cup (A \backslash U) = A \backslash U.$$

By Theorem 17.1(3), $A \backslash U$ is closed in X.

Problem 17.5. *Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions what equality hold?*

Solution: We can use Theorem 17.5(a), which implies that $x \in \overline{(a,b)}$ if and only if every neighborhood of x intersects (a,b). First we will show that $\overline{(a,b)}$ is a subset of $[a,b]$ in all cases. Suppose $x \in \overline{(a,b)}$. It cannot be that $x < a$ or $x > b$. If $x < a$, then (α, a) with $\alpha < x$ contains x but does not intersect (a,b). Similarly, if $x > b$, then (b, β) with $x < \beta$ contains x but does not intersect (a,b). We infer then that either $\overline{(a,b)}$ is empty or any $x \in \overline{(a,b)}$ must be on the interval $[a,b]$. Hence $\overline{(a,b)} \subseteq [a,b]$.

Next we will show that $\overline{(a,b)} = [a,b]$ if and only if a has no immediate successor and b has no immediate predecessor. Suppose $x \in (a,b)$. For any $\gamma < x$ and $\delta > x$, the interval (γ, δ) (or analogous half-open interval if γ or δ or smallest or largest elements of X) contains x and intersects (a,b). It follows that if $x \in (a,b)$, then $x \in \overline{(a,b)}$.

Assume that a has no immediate successor and b has no immediate predecessor. It follows that any open interval containing a must be of the form (γ, δ) (or if γ or δ is the smallest or largest element, an analogous half-open interval) where $\gamma < a$ and $\delta > a$. There is a $\zeta \in X$ where $a < \zeta < \delta$. Consequently, $\zeta \in (a, b)$ and $\zeta \in (\gamma, \delta)$. Therefore (γ, δ) contains a and intersects (a, b), so $a \in \overline{(a,b)}$. A similar argument shows $b \in \overline{(a,b)}$.

Now assume instead that a has an immediate successor or b has an immediate predecessor.. If $x = a$, it follows that there is an element $\delta \in X$ such that (a, δ) is empty. As a result, if $\gamma < a$, then (γ, δ) contains a but does not intersect (a, b); hence, $a \notin \overline{(a,b)}$. A similar argument shows $b \notin \overline{(a,b)}$.

We conclude that $\overline{(a,b)} = [a, b]$ if and only if a has no immediate successor and b has no immediate predecessor.

Problem 17.6. *Let A, B, and A_α be subsets of space X. Prove the following: (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$. (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. (c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A}_\alpha$; give an example where equality fails.*

Solution: Part (a) This is an important result. We have $A \subseteq B \subseteq \overline{B}$. Since $\overline{B}$ is a closed set containing A, it contains the closure of A. Consequently, $\overline{A} \subseteq \overline{B}$.

Part (b) Since $\overline{A}$ and $\overline{B}$ are closed in X, it follows from Theorem 17.1(3) that $\overline{A} \cup \overline{B}$ is closed in X. Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, it follows that $A \cup B \subseteq \overline{A} \cup \overline{B}$. Therefore, $\overline{A} \cup \overline{B}$ contains the closure of $A \cup B$, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Conversely, $A \subseteq A \cup B \subseteq \overline{A \cup B}$. By part (a), $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$. Taking the union, we have $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

It follows that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Part (c)

Problem 17.8. *Let A, B, and A_α denote subsets of space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds. (a) $\overline{A \cap B}$. (b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A}_\alpha$. (c) $\overline{A - B} = \overline{A} - \overline{B}$.*

Solution: Part (a) See part (b).

Part (b) Since $\bigcap A_\alpha \subseteq A_\beta$ for all A_β in the collection $\{A_\alpha\}$, it follows from exercise 17.6(a) that $\overline{\bigcap A_\alpha} \subseteq \overline{A_\beta}$. Intersecting all $\overline{A_\beta}$, we have $\overline{\bigcap A_\alpha} \subseteq \bigcap \overline{A_\alpha}$.

It is not necessary true that $\overline{\bigcap A_\alpha} \supseteq \bigcap \overline{A_\alpha}$. From example 6, $\overline{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{0\}$, and $\overline{\mathbb{R}}_{<0} = \mathbb{R}_{<0} \cup \{0\}$. It follows that $\overline{\mathbb{R}}_{>0} \cap \overline{\mathbb{R}}_{<0} = \{0\}$; however, $\overline{\mathbb{R}_{>0} \cap \mathbb{R}_{<0}} = \overline{\emptyset} = \emptyset$ (since $\emptyset$ is a closed set that contains $\emptyset$).

Part (a) is just this general result applied to the collection $\{A, B\}$, so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Part (c) Theorem 17.5 implies if $x \in \overline{A} \backslash \overline{B}$, every neighborhood U of x intersects A (n.b., but there exists some neighborhood U' of x that does not intersect B; otherwise, $x \in \overline{B}$, so $x \notin \overline{A} \backslash \overline{B}$). Consequently, every neighborhood of x intersects $A \backslash B$, so $x \in \overline{A \backslash B}$. We conclude that $\overline{A} \backslash \overline{B} \subseteq \overline{A \backslash B}$.

It may not be true that $\overline{A \backslash B} \subseteq \overline{A} \backslash \overline{B}$. Suppose there is an $x \in X$ such that every neighborhood U of x intersects both A and B. Therefore U intersects $A \backslash B$, so $x \in \overline{A \backslash B}$. However, $x \in \overline{A}$ and $x \in \overline{B}$, from which it follows that $x \notin \overline{A} \backslash \overline{B}$.

Problem 17.9. *Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$, $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Solution: If $(x, y) \in \overline{A} \times \overline{B}$, then $x \in \overline{A}$ and $y \in \overline{B}$. Applying Theorem 17.5, every basis element B_x of X containing x intersects A. Similarly, every basis element B_y containing y intersects B. Let $\mathcal{C}_x$ and $\mathcal{C}_y$ be the collections of all B_x and B_y that contain x or y, respectively. By Theorem 15.1, the Cartesian product of the elements in $\mathcal{C}_x$ and $\mathcal{C}_y$ is contained in a basis of $X \times Y$. Any basis element containing (x, δ) for any $\delta \in Y$ must be of the form $B_x \times B_Y$, where B_Y is a basis element of Y. Similarly, any basis element containing (γ, y) for any $\gamma \in X$ is of the form $B_X \times B_y$ where B_X is a basis element of X. If $B_x \times B_Y$ is a basis element where $B_Y \notin \mathcal{C}_y$, then $y \notin B_Y$, so $(x, y) \notin B_x \times B_Y$. By the same logic, $x \notin B_X \times B_y$ where $B_X \notin \mathcal{C}_x$. It follows that any basis element containing (x, y) is of the form $B_x \times B_y$, which must intersect $A \times B$. Therefore, $x \in \overline{A \times B}$ and $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$.

Conversely, suppose $(x, y) \in \overline{A \times B}$. If B_x is a basis element of A that intersects x and B_y is a basis element of B the intersects y, then $B_x \times B_y$ is a basis of $\overline{A \times B}$ that contains (x, y). Consequently, B_x must intersect A and B_y must intersect B. Consequently $(x, y) \in \overline{A} \times \overline{B}$, and $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$. We conclude that $\overline{A \times B} = \overline{A} \times \overline{B}$.

Problem 17.10. *Show that every order topology is Hausdroff.*

Solution: Suppose $\mathcal{T}$ is an order topology on a given set X. Let x_1, x_2 be distinct points in X where $x_1 < x_2$. If x_2 is not the immediate successor of x_1, there is some $c \in (x_1, x_2)$. If x_1 and x_2 are not the smallest or largest elements of X, respectively, then there is some $a < x_1$ and $b > x_2$. It follows that (a, c) and (c, b) are neighborhoods of x_1 and x_2 that are disjoint. On the other hand, if (x_1, x_2) is empty, then (a, x_2) and (x_1, c) are the appropriate disjoint neighborhoods.

If x_1 is the smallest element of X, we can follow the same argument as above but let the neighborhood of x_1 be $[x_1, c)$ or $[x_1, x_2)$, as appropriate. Similarly, if x_2 is the largest element of X, let the neighborhood of x_2 be $(c, x_2]$ or $(x_1, x_2]$, as appropriate.

We infer that $\mathcal{T}$ is a Hausdorff space.

Problem 17.11. *Show that the product of two Hausdorff spaces is Hausdroff.*

Solution: Suppose X and Y are Hausdorff spaces. Given distinct $(x_0, y_0), (x_1, y_1)$ of $X \times Y$, if $x_0 \neq x_1$ and $y_0 \neq y_1$, there are neighborhoods A_0 of x_0 and A_1 of x_1 and neighborhoods B_0 of y_0 and B_1 of y_1 that are disjoint. We then have:

$$(A_0 \times B_0) \cap (A_1 \times B_1) = (A_0 \cap A_1) \times (B_0 \cap B_1) = \emptyset \cap \emptyset = \emptyset.$$

Therefore $A_0 \times B_0$ and $A_1 \times B_1$ are disjoint neighborhoods of (x_0, y_0) and (x_1, y_1).

On the other hand, if $x_0 = x_1$ (in which case $y_0 \neq y_1$), let A be any neighborhood of x_0 and B_0 and B_1 be as above. We have:

$$(A \times B_0) \cap (A \times B_1) = (A \cap A) \times (B_0 \cap B_1) = A \cap \emptyset = \emptyset.$$

Therefore $A \times B_0$ and $A \times B_1$ are disjoint neighborhoods of (x_0, y_0) and (x_0, y_1). A similar argument establishes the existence of disjoint neighborhoods for (x_0, y_0) and (x_1, y_0) where $x_0 \neq x_1$.

The product space of X and Y is therefore Hausdorff.

Problem 17.12. *Show that the subspace of a Hausdorff space is Hausdorff.*

Solution: Let X be a Hausdorff space and Y a subset of X. Given any distinct x_0, x_1 in Y, there are neighborhoods U of x_0 and V of x_1 in X that are disjoint. By definition, $U' = U \cap Y$ and $V' = V \cap V$ are open in Y, so:

$$U' \cap V' = (U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset.$$

Hence, U' and V' are disjoint neighborhoods of x_0 and x_1 in Y. The subspace Y is Hausdorff.

Problem 17.13. *Show that X is Hausdorff if and only if the* **diagonal** $\Delta = \{x \times x | x \in X\}$ *is closed in $X \times X$.*

Solution: Suppose X is a Hausdorff space. Given $x \in X$, from Theorem 17.8 the singleton $\{x\}$ is closed in X, so $\overline{\{x\}} = \{x\}$. Accordingly, for every $y \neq x$, there is some neighborhood of y in X that does not intersect $\{x\}$. Next we will show that $\overline{\Delta} = \Delta$. Clearly $(x, x) \in \overline{\Delta}$ since $\Delta \subseteq \overline{\Delta}$. Now let $(x, y) \in X \times Y$ where $x \neq y$. For any (x', x') there is a either a neighborhood of y that does not intersect x' (if $y \neq x'$) or a neighborhood of x that does not intersect x' (if $x \neq x'$). Hence, there is a neighborhood in $X \times X$ that does not intersect Δ, so $(x, y) \notin \overline{\Delta}$ if $x \neq y$. We conclude that $\overline{\Delta} = \Delta$, so Δ is closed in $X \times X$.

Conversely, suppose Δ is closed in $X \times X$, so $\overline{\Delta} = \Delta$. As just shown, for any y, there is some neighborhood of it that does not intersect any $x \neq y$ (otherwise $(x, y) \in \overline{\Delta}$ for some x). Accordingly, for any distinct $x, y \in X$, there exists a neighborhood of x and a neighborhood of y that are disjoint. Therefore X is a Hausdorff space.

Problem 17.14. *In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?*

Solution: On page 99 of the text, Munkres states without proof that $\mathbb{R}$ in the finite complement toplogy is not Hausdorff. We'll quickly prove this result here. Given space X with the finite complement topology, assume X is Hausdorff. It follows that for distinct $u, v \in X$, there are neighborhoods U of u and V of v that are disjoint. Since $U \subset X \backslash V$, it follows that U is finite since V is open. Similarly, $X \backslash U$ must be finite since U is open. It follows that $U \cup (X \backslash U) = X$ is finite. Since $\mathbb{R}$ is infinite, it is necessarily not Hausdorff.

Since this space is not Hausdorff, we cannot rely on Theorem 17.10 that any sequence converges to at most one point. Let A be a neighborhood of c. It follows that the intervals $(0, a)$ for some $a > 0$ is not contained in $X \backslash A$; otherwise, $X \backslash A$ would be infinite, contradicting that A is open. Therefore $(0, a) \subseteq A$. If $K \in \mathbb{N}$ and $K > 1/b$, then if $n \geq K$, then $0 < x_n = 1/n < 1/(1/b) = b$. Therefore for any point in $\mathbb{R}$, every neighborhood of that point contains (x_n) for some $n \geq K$. Hence, (x_n) converges to every point in $\mathbb{R}$.

Problem 17.15. *Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.*

Solution: Suppose a space X satisfies the T_1 axiom. Given $A = \{a\}$ and $B = \{b\}$ where $a \neq b$, each singleton is closed. Therefore $A = \overline{A}$ and $B = \overline{B}$. Since $b \notin \overline{A}$, we infer from Theorem 17.5 that there is a neighborhood of b that does not intersect A. Similarly, there is a neighborhood of a than does not intersect B. As a result, any two distinct points each has a neighborhood that does not contain the other.

Conversely, suppose each pair of distinct points in X each has a neighborhood that does not contain the other. Given $c \in X$, we infer that if $d \neq c$, some neighborhood of d does not contain $\{c\}$. It follows that $\overline{\{c\}} = \{c\}$, so $\{c\}$ is closed. Therefore every finite point set in X is closed.

In sum, a space satisfies the T_1 axiom if and only if each pair of distinct points in X each has a neighborhood that does not contain the other.

Problem 17.16. *Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13. (a) Determine the closure of the set $K = \{1/n | n \in \mathbb{Z}_+\}$ under each of these topologies. (b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?*

Solution: Part (a)
(i) Standard Topology of $\mathbb{R}$. Applying Theorem 17.6, we can find the closure of K by finding all of its limit points. Clearly $0 \notin K$. Any neighborhood U of 0 contains an interval (a, b) where $a < 0$ and $b > 0$. If $n \in \mathbb{N}$ where $n > 1/b$, then $1/n < b$ and is contained in U. Therefore 0 is a limit point of K. No other point is $\mathbb{R}$ is a limit point of K. Obviously if $c < 0$ or $c > 1$, then $(-\infty, 0)$ and $(1, \infty)$ do not intersect K. Suppose $c \in (0, 1]$. If $c = 1$, then $(3/4, \infty)$ but does not intersect $K \backslash \{1\}$. If $c = 1/n$ for some $n > 1$, then $(1/n + 1, 1/n - 1)$ contains c but does not intersect $K \backslash \{c\}$. Finally, if $c \neq 1/n$ for any n, then by the density of rational numbers there are $\alpha, \beta \in \mathbb{Q}$ where $1/n + 1 < \alpha < c < \beta < 1/n$ for the appropriate n, so (α, β) does not intersect $K \backslash \{c\}$. It follows that the only limit point of K is 0. Applying Theorem 17.6, $\overline{K} = K \cup \{0\}$.
(ii) Topology of $\mathbb{R}_K$. For the same reasons as with the standard topology of $\mathbb{R}$ in (i), no $c < 0$ or $c \in (0, \infty)$ can be a limit point of K. Any neighborhood U of 0 cannot include $1/n$ for any $n \in \mathbb{N}$, so U cannot intersect K. Since K has no limit points, $\overline{K} = K$.
(iii) Finite Complement Topology of $\mathbb{R}$. Given $c \in \mathbb{R}$, any neighborhood U of c must contain $(0, b)$ for some $b > 0$; otherwise, $X \backslash U$ would be infinite. Accordingly, $1/n \in (0, b) \subseteq U$ for all $n > 1/b$, so every real number is a limit point of K. The closure of K is $\mathbb{R}$. (This is similar to the result that the sequence $x_n = 1/n$ converges to every point in $\mathbb{R}$ under this topology.)
(iv) Upper Limit Topology of $\mathbb{R}$. There are no limit points of K, so $\overline{K} = K$. For $c \leq 0$ or $c > 1$, the neighborhoods $(\infty, c]$ and $(1, c]$ of c do not intersect K. If $c \in (0, 1]$, there is some $n \in \mathbb{N}$ such that $1/(n + 1) < c \leq 1/n$. The interval $(1/(n + 1), c]$ is a neighborhood of c that does not intersect K other than at c (if $c \in K$, of course).
(v) Topology of All Sets $(-\infty, a) = \{x : x < a\}$ as a Basis. Every number in $\mathbb{R}_{\geq 0}$ is a limit point of K, so $\overline{K} = \mathbb{R}_{\geq 0}$. If $c < 0$, then the interval $(-\infty, 0)$ is a neighborhood of c that does not intersect K. If $c \geq 0$, then any neighborhood U of c must contain the interval $(-\infty, \gamma)$ where $\gamma > c$ (since every open set of the space is the union of some subcollection of basis elements). If $n \in \mathbb{N}$ and $n > 1/\gamma$, then $0 < 1/n < \gamma$, and U intersects K at an infinite number of points.

Part (b)
(i) Standard Topology of $\mathbb{R}$. This is Hausdorff (and therefore satisfies T_1 by Theorem 17.8). Given $x, y \in \mathbb{R}$ where $x < y$, let $\epsilon = (y - x)/2$. It follows that $(x - \epsilon, x + \epsilon)$, and $(y - \epsilon, y + \epsilon)$ are disjoint neighborhoods of each point.
(ii) Topology of $\mathbb{R}_K$. Since the K-topology is finer than the standard topology (which is Hausdorff), the K-topology is Hausdorff (see the Theorem 1 below). Consequently the K-topology necessarily satisfies that T_1 axiom.

Theorem 1. *Suppose T is a Hausdorff space. If U is finer than T, then U is Hausdorff.*

Proof. For any distinct points x and y, there are neighborhoods A of x and B of y in T that are disjoint. Since $T \subseteq U$, it follows that A and B are open in U. Hence, U is Hausdorff. □

(iii) Finite Complement Topology of $\mathbb{R}$. As shown in exercise 17.14, this topology is not Hausdorff. The topology does, however, satisfy the T_1 axiom. Given any real number x, the set $\{x\}$ has the open complement $\mathbb{R} \backslash \{x\}$ (which is finite). Therefore every single-point set is closed.
(iv) Upper Limit Topology of $\mathbb{R}$. The upper limit topology is Hausdorff and therefore satisfies the T_1 axiom. Let $x, y \in \mathbb{R}$ where $x < y$. The intervals $(-\infty, x]$ and $(x, y]$ are disjoint neighborhoods of x and y. (Note that this topology is also finer than the standard topology, so it is Hausdorff for that reason, as well.)
(v) Topology of All Sets $(-\infty, a) = \{x : x < a\}$ as a Basis. As an initial matter, any neighborhood S of $x \in \mathbb{R}$ must contain $(-\infty, \xi)$ where $\xi > x$. By Lemma 13.1, any open set of the topology is the union of some subcollection of basis

elements. Since $x \in S$, the set S must contain some basis element containing x. By definition of the basis of this topology, this basis element must be $(-\infty, \xi)$ where $\xi > x$.

This topology is not Hausdorff. Let $x, y \in \mathbb{R}$ where $x < y$. Any neighborhood U of x must contain $(-\infty, \gamma)$ where $\gamma > x$ and any neighborhood V of y must contain $(-\infty, \delta)$ where $\delta > y$. Therefore $U \cap V \subseteq (-\infty, \inf\{\gamma, \delta\})$, which is non-empty.

This topology does not satisfy the T_1 axiom. Let $c \in \mathbb{R}$ where $c > x$. Every neighborhood of c must contain some interval $(-\infty, \gamma)$ where $\gamma > c$, which intersects $\{x\}$. Therefore $c \in \overline{\{x\}}$, so $\overline{\{x\}} \neq \{x\}$. We conclude that no finite-point set in $\mathbb{R}$ is closed in this topology.

Problem 17.17. *Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.*

Solution: Lower-Limit Topology. Given $c \in [0, \sqrt{2})$, any neighborhood U of c must contain a $[c, d)$ where $d > c$. Since $\mathbb{R}$ is a linear continuum, U intersects A at some point on (c, d) and is therefore a limit point. If $c \geq \sqrt{2}$, then the interval $[c, d)$ where $d > c$ is a neighborhood of c that does not intersect A. A similar argument shows no $c < 0$ is a limit point of A. By Theorem 17.4, $\overline{A} = A \cup [0, \sqrt{2}) = [0, \sqrt{2})$.

By the same reasoning, $\overline{B} = B \cup [\sqrt{2}, 3) = [\sqrt{2}, 3)$.

The Basis $\mathcal{C}$. Any $c \in [0, \sqrt{2}]$ is a limit point of A. Let U be any neighborhood U of c. If c is rational, U must contain $[\alpha, \beta)$ where $\alpha = c$ and β is rational and greater than c. If c is irrational, U contains $[\alpha, \beta)$ where α, β are rational and $\alpha < c$ and $\beta > c$. It follows that U intersects A at some point in (α, β). Any $c < 0$ cannot be a limit point because if α is rational and less than c, then $[\alpha, 0)$ is a neighborhood of c that does not intersect A. The same argument shows that no $c > \sqrt{2}$ can be a limit point of A. Therefore $\overline{A} = A \cup [0, \sqrt{2}] = [0, \sqrt{2}]$.

By contrast, the closure of B is $[\sqrt{2}, 3)$. If U is a neighborhood of $\sqrt{2}$, then U must contain $[\gamma, \delta)$ for some γ, δ are rational and $\gamma < \sqrt{2}$ and $\delta > \sqrt{2}$. It follows that U intersects B at some point in (γ, δ). Therefore $\sqrt{2}$ is a limit point. However, the interval $[3, 4]$ is a neighborhood of 3 that does not intersect B, so 3 is not a limit point. Results for the remainder of $\mathbb{R}$ can be found using the same arguments as with A. Therefore $\overline{B} = B \cup [\sqrt{2}, 3) = [\sqrt{2}, 3)$.

Problem 17.19. *[See problem.]*

Solution: Observe that a point x is in the boundary of a set A if every neighborhood of x intersects both A and $X \backslash A$.

Part (a) Suppose $u \in \text{Int } A \cap \text{Bd } A$. Since $u \in \text{Int } A$, it follows that there is some neighborhood U of u contained in A, it which case U cannot intersect $X \backslash A$. Therefore $u \notin \overline{X \backslash A}$, leading to the contradiction that $u \notin \text{Bd } A$. We conclude that $\text{Int } A$ and $\text{Bd } A$ must be disjoint.

Next we will show that $\overline{A} = \text{Int } A \cup \text{Bd } A$. Given $v \in \text{Int } A \cup \text{Bd } A$, either $v \in \text{Int } A \subseteq \overline{A}$ or $v \in \text{Bd } A$. If the latter, since $\text{Bd } A = \overline{A} \cap \overline{X \backslash A}$, it follows that $v \in \overline{A}$. Therefore $\text{Int } A \cup \text{Bd } A \subseteq \overline{A}$. Conversely, suppose $w \in \overline{A}$. If $w \in \text{Int } A$, then obviously $w \in \text{Int } A \cup \text{Bd } A$. If $w \notin \text{Int } A$, then every neighborhood of w cannot be contained in A. Consequently, every neighborhood of w must intersect $X \backslash A$, from which it follows that $w \in \overline{X \backslash A}$. Since $\overline{A} \subseteq \text{Int } A \cup \text{Bd } A$, we conclude that $\overline{A} = \text{Int } A \cup \text{Bd } A$.

Part (b) If $\text{Bd } A = \emptyset$, then from part (a) $\overline{A} = \text{Int } A \cup \text{Bd } A = \text{Int } A$. Since $\text{Int } A \subseteq A \subseteq \overline{A}$, it follows that $\text{Int } A = A$ (so A is open) and $\overline{A} = A$ (so A is closed).

If A is both open and closed, then $\text{Int } A = A$ and (since $X \backslash A$ is closed) $\overline{X \backslash A} = X \backslash A$. We then have:

$$\text{Bd } A = \overline{A} \cap (\overline{X \backslash A}) = A \cap (X \backslash A) = (A \cap X) \backslash (A \cap A) = A \backslash A = \emptyset.$$

Part (c) Suppose U is open. It follows that $\text{Int } U = U$ and $X \backslash U$ is closed, so $\overline{X \backslash U} = X \backslash U$. Accordingly:

$$\text{Bd } U = \overline{U} \cap (\overline{X \backslash U}) = \overline{U} \cap (X \backslash U) = (\overline{U} \cap X) \backslash (\overline{U} \cap U) = \overline{U} \backslash U.$$

Conversely, suppose $\text{Bd } U = \overline{U} \backslash U$. Since $U \subseteq \overline{U} \subseteq X$, it follows that:

$$\overline{U} \backslash U = (\overline{U} \cap X) \backslash (\overline{U} \cap U) = \overline{U} \cap (X \backslash U).$$

Since $\text{Bd } U = \overline{U} \cap \overline{X \backslash U}$ by definition, we infer that $X \backslash U = \overline{X \backslash U}$, so $X \backslash U$ is closed. Therefore $X \backslash (X \backslash U) = U$ is open.

Part (d) This proposition is not valid. The set $U = (\infty, 0) \cup (0, \infty) = \mathbb{R} \backslash \{0\}$ is open in $\mathbb{R}$ under the standard topology. The closure of U is $\mathbb{R}$, which is also open, so $\text{Int } \overline{U} = \mathbb{R} \neq U$.

Problem 17.20. *Find the boundary and the interior of the following subsets of* $\mathbb{R}^2$*:*

(a) $A = \{x \times y | y = 0\}$
(b) $B = \{x \times y | x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \{x \times y | x \text{ is rational}\}$
(e) $E = \{x \times y | 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{x \times y | x \neq 0 \text{ and } y \leq 1/x\}$

Solution: We assume the standard topology on $\mathbb{R}$ for this entire problem.

Part (a) Under the standard topology of $\mathbb{R}$, every basis element $(a,b)\times(c,d)$ is not contained in any interval $(\alpha,\beta)\times\{0\}$. Therefore, no open set is contained in A, from which it follows that Int A is empty. Now let $x = (\gamma, 0)$ where $\gamma \in \mathbb{R}$. Any neighborhood of x must intersect A, so $(\gamma, 0) \in \overline{A}$. On the other hand, if $y = (\gamma, \delta)$ where $\delta < 0$ or $\delta > 0$, there are neighborhoods of y that do not intersect A (e.g., $(-\infty, 2\delta) \times (\infty, 0)$. Therefore y is not an element of $\overline{A}$. It follows that $\overline{A} = \{x \times \{0\} : x \in \mathbb{R}\} = A$. By exercise 19(a), $\overline{A} = \text{Int } A \cup \text{Bd } A = \text{Bd } A$. Therefore Bd $A = A$.

Part (b) If an open set U is contained in B, it cannot contain any basis element $B' = \{(a,b) \times (c,d) : a \leq 0\}$ or $B'' = \{(a,b) \times (c,d) : b < 0 \text{ or } d > 0\}$. The set U may, however, contain any other basis element. Accordingly, Int $B = \{x \times y : x > 0 \text{ and } y \neq 0\} = B$. Therefore B is open.

By exercise 19(c), since B is open it follows that Bd $B = \overline{B} \backslash B$. Any neighborhood V of tuples $(0,b)$ or $(a,0)$ intersects B, so $(0,b),(a,0) \in \overline{B}$ for all $a,b \in \mathbb{R}$. No other neighborhood of any other point outside B intersects B. Therefore $\overline{B} = \{x \times y : x \geq 0\}$. It follows that Bd $B = \{(a,0) : a \in \mathbb{R}\} \cup \{(0,b) : b \in \mathbb{R}\}$.

Part (c) We have $C = A \cup B = \{x \times y : x, y \in \mathbb{R} \text{ and } (y \neq 0 \text{ or } x > 0)\}$. By exercise 6(b):

$$\overline{C} = \overline{A} \cup \overline{B} = \{x \times y : x, y \in \mathbb{R} \text{ and } (y = 0 \text{ or } x > 0)\} = (\mathbb{R}_{<0} \times \{0\}) \cup (\mathbb{R}_{\geq 0} \times \mathbb{R}).$$

Further, Int $C = \{x \times y : x, y \in \mathbb{R} = \mathbb{R}_{>0} \times \mathbb{R} \text{ and } x > 0\}$. Since $\overline{C} = \text{Int } C \cup \text{Bd } C$ and Int C and Bd C are disjoint, it follows that:

$$\text{Bd } C = \{x \times \{0\} : x \in \mathbb{R}_{<0}\} \cup \{\{0\} \times y : y \in \mathbb{R}\} = (\mathbb{R}_{<0} \times \{0\}) \cup (\{0\} \times \mathbb{R}).$$

Part (d) By the density of rational numbers, any interval $(a,b) \times (c,d)$ contains some $\alpha \times (b,d)$ where α is irrational, so every open interval in $\mathbb{R} \times \mathbb{R}$ is not contained in D. Therefore Int B is empty.

Given the tuple $(x,y) \in \mathbb{R} \times \mathbb{R}$, by the density of rational numbers any neighborhood of (x,y) intersects D at some (β, y) where β is rational. Because every $(x,y) \in \overline{D}$, it follows that $\overline{D} = \mathbb{R} \times \mathbb{R}$. Hence Bd $D = \overline{D} = \mathbb{R} \times \mathbb{R}$.

Part (e) The set E consists of all points between and including the two curves of the hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$ but excluding any points on the 45-degree angle lines (x', y') where $|x'| = |y'|$ (since $(x')^2 - (y')^2 = 0 \notin E$). All points within the hyperbola have some neighborhood contained by E. However, any (x,y) on curves $x^2 - y^2 = 1$ has no neighborhood contained by E (since any such neighborhood must include some (α, β) such that $\alpha^2 - \beta^2 > 1$). Therefore Int $E = \{x \times y : 0 < x^2 - y^2 < 1\}$.

Every neighborhood of any (x,y) on $x^2 - y^2 = 1$ intersects E. In addition, given $x', y' \in \mathbb{R}$ where $|x'| = |y'|$, any neighborhood U of (x', y') contains the all the points $(x' + \epsilon, y')$ where $0 < \epsilon < R$. Let $(x')^2 + 2x'\epsilon + \epsilon^2 - (y')^2 = 2x'\epsilon + \epsilon^2 \leq 1$. Solving the quadratic, we find that if $0 < \epsilon < R$ and $\epsilon < -x' + \sqrt{(x')^2 + 1}$, then $(x' + \epsilon, y)$ intersects E. Therefore $(x', y') \in \overline{E}$.

On the other hand, any (x', y') on $x'^2 - y'^2 > 1$ has a neighborhood that does not intersect E. Accordingly, $\overline{E} = \{x \times y : 0 \leq x^2 - y^2 \leq 1\}$.

The boundary of E is therefore Bd $E = \{x \times y : x^2 - y^2 = 1\} \cup \{x \times y : |x| = |y|\}$.

Part (f) Observe that F consists of all points on and **below** the hyperbola curves $y = 1/x$ but excluding any point in $\{0\} \times \mathbb{R}$.

For any point not in $\{0\} \times \mathbb{R}$ that is below $y = 1/x$, there is some neighborhood of that point that is contained by F. Every point on or above $y = 1/x$ has no neighborhood contained by F. The same is true for any point on $\{0\} \times \mathbb{R}$. As a result, Int $F = \{x \times y : x \neq 0 \text{ and } y < 1/x\}$.

The closure of F clearly includes all points on $y = 1/x$ for $x \neq 0$. In addition, every neighborhood of any point on $\{0\} \times \mathbb{R}$ intersects F. Any point above $y = 1/x$ has some neighborhood that does not intersect F. Consequently, $\overline{F} = \{x \times y : x \neq 0 \text{ and } y \leq 1/x\} \cup (\{0\} \times \mathbb{R})$.

Since $\overline{F} = \text{Int } F \cup \text{Bd } F$ and Int F and Bd F are disjoint, it follows that Bd $F = \{x \times y : x \neq 0 \text{ and } y = 1/x\} \cup (\{0\} \times \mathbb{R})$.

Section 18

Problem 18.1. *Prove that for functions* $f : \mathbb{R} \to \mathbb{R}$*, the* ϵ*-*δ *definition of continuity implies the open set definition.*

Solution: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ by the ϵ-δ definition. Given an open set W of $\mathbb{R}$, if $f^{-1}(W)$ is empty, then the inverse image of W is trivially open. Otherwise, if $x_0 \in f^{-1}(W)$, then there is some $\epsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq W$. Since f is continuous at x_0, there is some $\delta(\epsilon) > 0$ such that if $|x - x_0| < \delta(\epsilon)$, then $|f(x) - f(x_0)| < \epsilon$. Letting $C = (x_0 - \delta(\epsilon), x_0 + \delta(\epsilon))$, it follows that if $x \in C$, then $f(x) \in W$; therefore, $C \subseteq f^{-1}(W)$. Since every $x_0 \in f^{-1}(W)$ is contained in some open set $C \subseteq f^{-1}(W)$, we infer that $f^{-1}(W)$ is a union of open sets in $\mathbb{R}$ and is therefore open.

Problem 18.2. *Suppose that $f : X \to Y$ is continuous. If f is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?*

Solution: No, this isn't necessarily true. Suppose f is a constant function where $f(X) = \{\gamma\}$ for some $\gamma \in Y$. It follows that $f(x) = \gamma$, which is not a limit point of $f(A)$ because any neighborhood of $f(x)$ cannot intersect $f(A)\backslash\{f(x)\} = f(A)\backslash\{\gamma\}$.

Problem 18.3. *Let X and X' denote a single set in two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function. (a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$. (b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.*

Solution: Part(a) Suppose i is continuous. Given a basis element B of X, the set $f^{-1}(B) = V'$ is open in X'. Since i is the identity function, $x \in B$ if and only if $x \in V'$, so $B = V'$. Further, if $x \in V'$, there is a basis element B' such that $x \in B' \subseteq V' = B$. By Theorem 13.3, $\mathcal{T}'$ is finer than $\mathcal{T}$.

Conversely, suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. If U is an open set of X, then $i^{-1}(U) = U$. Because $\mathcal{T}'$ is finer than $\mathcal{T}$, the set U is open in $\mathcal{T}'$. By definition, i is continuous.

Part(b) If i is a homeomorphism, then i and i^{-1} are each continuous. Applying the result in part (a), it follows that $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is finer than $\mathcal{T}'$. Hence $\mathcal{T}' = \mathcal{T}$.

Conversely, suppose $\mathcal{T}' = \mathcal{T}$, in which case a set is one topology is open if and only if the set is open in the other topology. Since i is obviously bijective, it has an inverse $i^{-1} : X \to X'$, which we will denote as j. If U is an open set in X, then $i^{-1}(U) = U$ is open in X'. Similarly, if U' is an open set in X', then $j^{-1}(U') = U'$ is open in $XtopoT$. Consequently, i and i^{-1} are continuous, so i is a homeomorphism.

Problem 18.4. *Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by $f(x) = x \times y_0$ and $g(y) = x_0 \times y$ are imbeddings.*

Solution: The function f is an injection. Given $a, b \in X$, we have $f(a) = (a, y_0) = (b, y_0) = f(b)$, so $a = b$. The set $X \times y_0$ is a subspace of $X \times Y$. Let $f' : X \to X \times y_0$ be obtained from f. Clearly f' is a bijection, so it has an inverse $(f')^{-1} : X \times y_0 \to X$. Observe that $f(x) = (f_1(x), f_2(x))$ where f_1 is the identity function and f_2 is a constant function (where every x is mapped to y_0). Since f_1 and f_2 are continuous, by Theorem 18.4 f' is continuous. On the other hand, by Theorem 15.1 and Lemma 16.1 a basis of $X \times Y$ is the collection $\{\ B_x \times y_0 : B_x$ is a basis element of $B\}$. Accordingly, $(f')^{-1}(B_x, y_0) = B_x$, which is open in X, so $(f')^{-1}$ is continuous. Since f' is a homeomorphism, we conclude that f is an imbedding of X in $X \times Y$.

Following the same argument, y is an imbedding of Y in $X \times Y$.

Problem 18.6. *Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.*

Solution: We can use a variation of the Dirichlet function:

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

This function is continuous only at $x = 0$. Let U be a neighborhood of $f(0) = 0$, which must contain a basis element $V = (a, b)$ where $a < 0 < b$. We have $f(V) = \{x \in \mathbb{Q} : a < x < b\}$, which is a subset of U. Thus f is continuous at $x = 0$.

On the other hand, f is not continuous at any $c \neq 0$. We may assume without loss of generality than $c > 0$. If c is rational, the interval $(0, 2c)$ is a neighborhood of $f(c) = c$. Any neighborhood V of c must contain the basis element (α, β)

where $\alpha < c < \beta$. By the density of irrational numbers, there is some irrational number in (α, β); therefore $0 \in f(V)$, so $f(V)$ cannot be a subset of $(0, 2c)$. by Theorem 18.1(4), f is not continuous at any rational number other than 0. If c is irrational, then the interval $(-c/2, c/2)$ is a neighborhood of $f(c) = 0$. Any neighborhood V' of c will include some rational number $d > c$, so $d \in f(V')$, so $f(V')$ cannot be a subset of $(-c/2, c/2)$. Again by Theorem 18.1(4), f is not continuous at any irrational number.

We conclude that f is continuous only at $x = 0$.

Problem 18.9. *Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$ is continous for each α. (a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous. (b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous. (c) An indexed family of sets $\{A_\alpha\}$ is said to be* **locally finite** *if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.*

Solution: Part (a) Since $f|A_\alpha$ is continuous, by Theorem 18.1 for every closed set B of Y, the set $(f|A_\alpha)^{-1}(B)$ is closed. It follows that:

$$f^{-1}(B) = \bigcup_\alpha (f|A_\alpha)^{-1}(B),$$

where by hypothesis the union is finite. Since each $(f|A_\alpha)^{-1}(B)$ is closed, this union is closed. By Theorem 18.1, f is continuous.

Part (b) Let (x_n) be a strictly increasing sequence of all rational numbers greater than or equal to zero, and let $A_n = [x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Clearly each A_n is closed; further, $\bigcup_n A_n = \mathbb{R}_{\geq 0}$. Define $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that each restriction $f|A_n(\alpha) = x_n$. Since $f|A_n$ is constant on each A_n, it is continuous. But f is clearly not continuous on $\mathbb{R}_{\geq 0}$; there is a discontinuity at every rational number greater than zero.

Part (c) Suppose $\{A_\alpha\}$ is locally finite and $\bigcup_\alpha A_\alpha = X$; further assume each A_α is closed and each $f|A_\alpha$ is continuous. Let B be a closed set in Y. We know $f^{-1}(B) = \bigcup_\alpha (f|A_\alpha)^{-1}(B)$. We will show that $f^{-1}(B)$ is closed because it equals its closure. Given $x_0 \notin f^{-1}(B)$, there is a neighborhood U of x_0 that intersects a finite number of A_α. Designate N as the set of all these α. We have:

$$C_{x_0} = \bigcup_{\alpha \in N} (f|A_\alpha)^{-1}(B).$$

By Theorem 18.1 each $(f|A_\alpha)^{-1}(B)$ is closed, so this finite union is closed. Therefore $V = X \backslash C_{x_0}$ is open in X. Since $x_0 \notin f^{-1}(B)$, we infer that $x_0 \in V$. Then $U \cap V$ is a neighborhood of x_0 that does not intersect R, from which it follows that $x \notin \overline{f^{-1}(B)}$. Hence $f^{-1}(B) = \overline{f^{-1}(B)}$ and $f^{-1}(B)$ is closed. By Theorem 18.1 f is continuous.

Problem 18.10. *Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation $(f \times g)(a \times c) = f(a) \times g(c)$. Show that $f \times g$ is continuous.*

Solution: A basis for $B \times D$ is the collection $\{U \times V : U$ is a basis element of B and V is a basis element of $D\}$. Given a basis element $U \times V$, it follows that $f^{-1}(U)$ are $g^{-1}(V)$ are open. By the definition of a product topology, $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$ is open in $A \times C$. Hence $f \times g$ is continuous.

Problem 18.11. *Let $F : X \times Y \to Z$. We say that F is* **continuous in each variable separately** *if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.*

Solution: Suppose F is continuous. Given $y_0 \in Y$, for each $x_0 \in$, let U be a neighborhood of $h(x_0) = F(x_0, y_0)$. It follows that $F^{-1}(U) = V \times W$ is open in $X \times Y$, and $(x_0, y_0) \in V \times W$. By the definition of the product topology, there is a basis element B of X such that $x_0 \in B$. Therefore $h(B) = F(B \times \{y_0\}) \subseteq U$. By Theorem 18.1(4), h is continuous for every $y_0 \in Y$. A similar argument shows k is continuous for every $x_0 \in X$. We conclude F is continuous in each variable separately.

Problem 18.12. *Let $F : \mathbb{R} \times \mathbb{R}$ be defined by the equation:*

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if} \quad x \times y \neq 0 \times 0. \\ 0 & \text{if} \quad x \times y = 0 \times 0. \end{cases}$$

(a) Show that F is continuous in each variable separably. (b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times X)$. (c) Show that F is not continuous.

Solution: Part (a) To make things easy, I'll use basic analysis to prove continuity. For each $y_0 \in Y$, define:

$$h_{y_0} = F(x \times y_0) = \begin{cases} xy_0/(x^2 + y_0^2) & \text{if} \quad x_0 \neq 0 \text{ or } y_0 \neq 0, \\ 0 & \text{if} \quad x \times y = 0 \times 0. \end{cases}$$

If $y_0 \neq 0$, for any $c \in \mathbb{R}$:

$$\lim_{x \to c} h_{y_0}(x) = \lim_{x \to c} \frac{xy_0}{x^2 + y_0^2} = \frac{cy_0}{c^2 + y_0^2} = h_{y_0}(c),$$

showing that h_{y_0} is continuous on $\mathbb{R}$. The foregoing also shows that if $y_0 = 0$, then h_0 is continuous for all $x \neq 0$. If $y_0 = 0$, then:

$$\lim_{x \to 0} h_0(x) = \lim_{x \to 0} \frac{0}{x^2} = 0 = h_0(0).$$

Therefore h_0 is continuous at $x = 0$, as well. It follows that h_{y_0} is continuous for all $y_0 \in \mathbb{R}$. A similar argument shows $k_{x_0}(y) = F(x_0 \times y)$ is continuous for all $x_0 \in \mathbb{R}$. Thus F is continuous in each variable separately.

Part (b)

$$g(x) = F(x \times x) = \begin{cases} \frac{x^2}{x^2 + x^2} = \frac{1}{2} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Note that g is not continuous. Let $U = (-1/4, 1/4)$, which is a neighborhood of $g(0) = 0$. Any neighborhood V of 0 will contain a basis element (a, b) where $a < 0 < b$. Thus $g(B) = \{0, 1/2\} \subseteq g(V)$, so $g(V)$ cannot be a subset of U. By Theorem 18.1(4), g is not continuous on $\mathbb{R}$ (but it is continuous on $\mathbb{R} \backslash \{0\}$.

Part (c) Observe that if $x = y \neq 0$, then $F(x, y) = 1/2$. Let $B = \{1/2\}$, which is closed. Thus $F^{-1}(B) = \{(x, x) : x \in \mathbb{R} \backslash \{0\}\}$. The set $F^{-1}(B)$ is not closed because $(0, 0) \notin F^{-1}(B)$ but is in its closure. By Theorem 18.1(3), F is not continuous.

Section 19

Problem 19.3. *Prove Theorem 19.4.*

Solution: Given distinct $\mathbf{x}, \mathbf{y} \in \prod_\alpha X_\alpha$, there is some δ such that $\mathbf{x}_\delta \neq \mathbf{y}_\delta$. Because X_δ is Hausdorff, there is some neighborhood U'_δ of $\mathbf{x}_\delta$ and V'_δ of $\mathbf{y}_\delta$ that are disjoint. Define $U = \prod_\alpha A_\alpha$ and $V = \prod_\alpha B_\alpha$ as:

$$A_\alpha = \begin{cases} X_\alpha & \text{if} \quad \alpha \neq \delta, \\ U' & \text{if} \quad \alpha = \delta, \end{cases} \qquad B_\alpha = \begin{cases} X_\alpha & \text{if} \quad \alpha \neq \delta, \\ V' & \text{if} \quad \alpha = \delta. \end{cases}$$

In either the box or the product topology, U and V are neighborhoods of $\mathbf{x}$ and $\mathbf{y}$, respectively, in $\prod_\alpha X_\alpha$. Since U' and V' are disjoint, U and V are disjoint. We conclude that $\prod_\alpha X_\alpha$ is Hausdorff.

Problem 19.4. *Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.*

Solution: Designate $Y = \prod_{j=1}^{n-1} X_n$ and $Z = \prod_{j=1}^{n} X_n$. Define $f : Y \times X_n \to Z$ as $f((a_1, \ldots, a_{n-1}), a_n) = (a_1, \ldots, a_n)$. A simple verification shows that f is a bijection. Therefore f has an inverse $f^{-1} : Z \to Y \times X_n$ where $f^{-1}(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n)$.

If f and f^{-1} are continuous, then $Y \times X_n$ is homeomorphic with Z. Since both are finite products, their topologies are the same under both the box and product topologies. Therefore any set of the form $\prod_{j=1}^{n} B_j$ where B_j is a basis element of X_j is a basis of Z. Similarly, any set of the form $\prod_{j=1}^{n-1} B_j$ is a basis elements of Y. Obviously B_n is a basis element of X_n. It follows that:

$$f^{-1}(B_1, \ldots, B_n) = ((B_1, \ldots, B_{n-1}), B_n),$$

is open in $Y \times X_n$, so f is continuous.

A similar argument shows that f^{-1} is continuous. Therefore $Y \times X_n$ is homeomorphic with Z.

Problem 19.5. *One of the implications stated in Theorem 19.6 holds for the box topology. Which one?*

Solution: Since the box topology is finer than the product topology, any open set in the latter is open in the former. Therefore if f is continuous in the box topology, it must be continuous in the product topology. The topologies of each X_α are unchanged. Applying Theorem 19.6, each f_α is continuous.

Problem 19.6. *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?*

Solution: Suppose each $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$. Let U be a neighborhood of $\mathbf{x}$ in $\prod_\alpha X_\alpha$. There is some basis element $V = \prod_\alpha B_\alpha$ contained in U that contains $\mathbf{x}$. For each α designate $K_\alpha \in \mathbb{N}$ such that if $n \geq K_\alpha$, then $\pi_\alpha(\mathbf{x}_n) \in B_\alpha$. Note that for each α either $B_\alpha = X_\alpha$ or B_α is some other basis element of X_α. If the former, $K_\alpha = 1$ since every element in $(\mathbf{x}_n)$ is in X_α; otherwise, since $(\pi_\alpha(\mathbf{x}_n))$ converges to $\pi_\alpha(\mathbf{x})$, there is such a K_α. Since the product space is in the product topology, there can be only a finite number of $K_\alpha > 1$. As a result, $K = \sup\{K_\alpha : K_\alpha > 1\}$ exists. Therefore, if $n \geq K$, then $\mathbf{x}_n \in V$. Hence $(\mathbf{x}_n)$ converges to $\mathbf{x}$.

Conversely, suppose $(\mathbf{x}_n)$ converges to $\mathbf{x}$. Let U' be a neighborhood of $\pi_\alpha(\mathbf{x})$. There is some basis element B' of X_α contained in U' that contains $\pi_\alpha(\mathbf{x})$. By the definition of the product topology, there is basis element $\prod_\beta B''_\beta$ where $B''_\beta = X_\beta$ for $\beta \neq \alpha$ and $B''_\alpha = B'_\alpha$. By hypothesis, there is some $K' \in \mathbb{N}$ such that if $n \geq K'$, then $\pi_\alpha(\mathbf{x}_n) \in B''_\alpha = B'$. Therefore $\pi_\alpha(\mathbf{x}_n) \in U'$ if $n \geq K_\alpha$. We conclude that $\pi_\alpha(\mathbf{x}_n)$ converges to $\pi_\alpha(\mathbf{x})$ for all α.

We argued the converse without reference to anything specific to the product topology, and it will hold equally true for the box topology. Therefore if $(\mathbf{x}_n)$ converges to $\mathbf{x}$, each $\pi_\alpha(\mathbf{x}_n)$ converges to $\pi_\alpha(\mathbf{x})$ in the box topology. In the other direction, however, we relied upon the property of product topologies that any basis element of $\prod_\alpha X_\alpha$ has a finite number of elements in the product that are proper subsets of X_α. It turns out that is not necessarily true that in the box topology if each $(\pi_\alpha(\mathbf{x}_n))$ converges to $\pi_\alpha(\mathbf{x})$ then $(\mathbf{x}_n)$ converges to $\mathbf{x}$. Let $(\mathbf{x}_n)$ be a sequence over $\mathbb{R}^\omega$ in the box topology where $\mathbf{x}_n = (1/n, 1/n^{1/2}, 1/n^{1/3}, \ldots)$ for all $n \in \mathbb{N}$. Obviously each $(\pi_\alpha(\mathbf{x}_n))$ converges to 0. Let U be the neighborhood $((-1/e, 1/e), (-1/e, 1/e), \ldots)$, which is a neighborhood of $\mathbf{0}$. Suppose that there is a $K \in \mathbb{N}$ such that if $n \geq K$, then $\mathbf{x}_n \in U$. Therefore for any $b \in \mathbb{N}$, it must be that $1/k^{1/b} < 1/e$. This requires, however, that $b < \ln K$, so if $b \geq \ln K$ then $1/K^{1/b} \geq 1/e$ Since this is true for any K, it follows then there can be no such K for the neighborhood U. Therefore $(\mathbf{x}_n)$ does not converge to $\mathbf{x}$ even though each of its coordinate sequences does.

Problem 19.7. *Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.*

Solution: In the product topology, any sequence (x_n) is in the closure of $\mathbb{R}^\infty$. If U' is a neighborhood of (x_n), a basis element of U must be of the form $B = \bigcup_\alpha B_\alpha$ where only a finite number of $B_\alpha \neq \mathbb{R}$. As a result, a finite number of B_α do not contain 0. Therefore B intersects some $(z_n) \in \mathbb{R}^\infty$, so U' intersects (x_n). Hence the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\omega$.

Now we will examine the box topology. Let (x_n) be a sequence in R^ω that is **not** ultimately zero (i.e., $(x_n) \notin \mathbb{R}^\infty$). It follows that there are an infinite number of j where $x_j \neq 0$. Let U be a set in the form $(U_1, U_2, \ldots)$ where U_k is a neighborhood of x_k and, for any $x_j \neq 0$, the neighborhood U_k does not contain 0 (which must exist because $(x_j/2, x_j)$ is open for positive x_j and $(x_j, x_j/2)$ is open for negative x_j). By the definition of the basis for the box topology, U is open. Clearly then, U is a neighborhood of (x_n) with an infinite number of constituent neighborhoods that do not contain 0 . Since any member of $\mathbb{R}^\infty$ has a finite number of non-zero terms, U cannot intersect $\mathbb{R}^\infty$. Therefore $(x_n) \notin \overline{\mathbb{R}^\infty}$. We conclude that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. In other words, $\mathbb{R}^\infty$ is closed.

Problem 19.8. *Given the sequence $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation $h(x_1, x_2, \ldots) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots)$. Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology.*

Solution: Let $\phi_k : \mathbb{R} \to \mathbb{R}$ where, for a given real a_k, b_k where $a_k > 0$, we define $\phi_k(x_k) = a_kx_k + b_k$. This function is a bijection. If $\phi_k(x_k) = a_kx_k + b_k = a_ky_k + b_k = \phi_k(y_k)$, then $x_k = y_k$. Given any $z_k \in \mathbb{R}$, we have $\phi_k((z_k - b_k)/a_k) = z_k$. Thus ϕ_k is injective and surjective.

The function ϕ_k is also order-preserving. If $x_k < y_k$, then $\phi_k(x_k) = a_kx_k + b_k < a_ky_k + b_k = \phi_k(y_k)$. It is then easily shown that ϕ_k is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. If A is a basis element of $\mathbb{R}$, it is of the form $A = (\alpha, \beta)$. Because ϕ_k is order-preserving, $\phi_k(A) = (a_k\alpha + b_k, a_k\beta + b_k)$, which is open. Similarly $\phi_k^{-1}(A) = ((\alpha - b_k)/a_k, (\beta - b_k)/a_k$, which is also open. Therefore ϕ_k and ϕ_k^{-1} are continuous.

The solution for the product topology is then trivial. Observe that:

$$h(x_1, x_2, \ldots) = (\phi_1(x_1), \phi_2(x_2), \ldots) \quad \text{and} \quad h^{-1}(x_1, x_2, \ldots) = (\phi_1^{-1}(x_1), \phi_2^{-1}(x_2), \ldots).$$

The function h is bijective because it is the product of bijective functions. By Theorem 19.6, such each ϕ_α and ϕ_α^{-1} is continuous, so h and h^{-1} are continuous. Therefore h is a homeomorphism from $\mathbb{R}^\omega$ to $\mathbb{R}^\omega$.

The function h is also homeomorphic from $\mathbb{R}^\omega$ to $\mathbb{R}^\omega$ in the box topology. If U is any basis element of $\mathbb{R}^\omega$ in this topology, it is of the form $U = \bigcup_j U_j$ where U_j is basis element of $\mathbb{R}$ comprising some (γ_j, δ_j). Therefore $h^{-1}(U) = (\phi_1^{-1}(U_1), \phi_2^{-1}(U_2), \ldots) = (V_1, V_2, \ldots)$ where each V_k is open in $\mathbb{R}$. Thus $h^{-1}(U)$ is open and h is continuous. Using an analogous argument, h^{-1} is continuous. The conclusion follows.

Problem 19.9. *Show that the choice axiom is equivalent to the statement that for any index family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets with $J \neq 0$, the cartesian product $\prod_{\alpha \in J}$ is not empty.*

Solution: Let's assume the validity of the axiom of choice. Let $\mathcal{A} = \prod_{\alpha \in J} A_\alpha$ be any non-empty collection of non-empty sets. By Lemma 9.2, there is a function $\phi : \mathcal{A} \to \bigcup_{a \in \alpha} A_\alpha$ where $\phi(A_\alpha) \in A_\alpha$ for all α. Then define $\psi : J \to \mathcal{A}$ where $\psi(\alpha) = A_\alpha$. Clearly ψ is bijection. The function $\phi \circ \psi : J \to \bigcup_{a \in \alpha} A_\alpha$ therefore exists. By the definition of a generalized Cartesian product, $\phi \circ \psi \in \prod_{\alpha \in J} A_\alpha$. The product is not empty.

Conversely, suppose $\mathcal{A}$ is not empty. Let ζ be an element of the product, so $\zeta : J \to \bigcup_{a \in \alpha} A_\alpha$. Since ψ is bijective, its inverse $\psi^{-1} : \mathcal{A} \to J$ exists. The composition $\zeta \circ \psi^{-1} : \mathcal{A} \to \bigcup_{a \in \alpha} A_\alpha$ therefore exists and maps each A_α to some element of A_α. A choice function exists thus exists. Accordingly, given a non-empty collection $\{D_\alpha\}_{\alpha \in J}$ of disjoint, non-empty sets, we can generate the set:

$$C = \{\zeta \circ \psi^{-1}(D_\alpha) : \alpha \in J\},$$

which contains exactly one element from each D_α. The axiom of choice follows.

Problem 19.9. *[See problem.]*

Solution: Part(a) For each α, let $C_\alpha = \{D \subseteq A : f_\alpha^{-1}(B_\alpha) = D \text{ for some basis element } B_\alpha \text{ of } X_\alpha\}$. Let $\mathcal{U} = \bigcup_{\alpha \in J} C_\alpha \cup \{A\}$, which is a subbasis of A since every element of A must be in some element of $\mathcal{U}$. We will show that the topology $\mathcal{T}$ generated by $\mathcal{U}$ must be a subset of any other topology $\mathcal{T}'$ on A such that each f_α is continuous with respect to this topology. Let U be an element of $\mathcal{U}$, which must equal $f_\alpha^{-1}(B_\alpha)$ for some basis element B_α of some X_α. But since f_α is continuous with respect to $\mathcal{T}'$, it follows that U must be open in $\mathcal{T}'$. Consequently, $\mathcal{T} \subseteq \mathcal{T}'$. Therefore $\mathcal{T}'$ must be the uniquely coarsest topology on A such that every f_α is continuous.

Part(b) Let $\mathcal{T}''$ be the topology generated by the subbasis $\mathcal{S}$. Given an open set D in $\mathcal{T}''$, it must be that D is the union or finite intersection of elements of $\mathcal{S}$. Let $V \in \mathcal{S}$ be one of these elements. It follows that V is the preimage of some open set W in X_α for some α. Since W is the union some collection of basis elements in X_α, it follows that V is the union of the preimage of these basis elements under f_α, each of which must be a member of $\mathcal{U}$. The union or finite intersection, as applicable, of these preimages is then D. Therefore D is open in $\mathcal{T}$, so $\mathcal{T}' \subseteq \mathcal{T}$.

Conversely, since every basis element in any X_α is open in X_α, it follows that each $C \in \mathcal{U}$ is also in $\mathcal{S}$. Hence $\mathcal{T} \subseteq T'$. Therefore the two topologies are equal, so $\mathcal{S}$ is a subbasis of A.

Part(c) Suppose $g : Y \to A$ is continuous in relation to the topology $\mathcal{T}$ of A. By hypothesis, if U is open X_α, then $f_\alpha^{-1}(U)$ is open in A. Therefore, $g^{-1}(f^{-1}(U)) = g^{-1} \circ f_\alpha^{-1}(U)$ is open. Therefore $f_\alpha \circ g$ is continuous.

Conversely, suppose $f_\alpha \circ g$ is continuous for all α. Let U be open in A under $\mathcal{T}$. We will show that $g^{-1}(U)$ is open in Y. Since $\mathcal{S}$ is a subbasis of $\mathcal{T}$, the set U is either the union or finite intersection of elements of $\mathcal{S}$:

$$U = \bigcup_{\beta \in J} B_\beta \quad \text{or} \quad U = \bigcap_{\beta \in J} B_\beta,$$

where each $B_\beta \in \mathcal{S}$. In the case of the intersection, J must be a finite set. Each B_β belongs to some S_α, so we can "partition" $\{B_\beta\}$ as follows:

$$T_\alpha = \{B_\beta : f_\alpha^{-1}(U_\alpha) = B_\beta \text{ for some open } U_\alpha \in X_\alpha\}.$$

Designate:

$$V_\alpha = \{U_\alpha : f_\alpha^{-1}(U_\alpha) \in T_\alpha \text{ and } V_\alpha \text{ is open in} X_\alpha\}.$$

The preimage $f_\alpha^{-1}(V_\alpha)$ is open because each f_α is continuous with respect to $\mathcal{T}$. Since $f_\alpha \circ g$ is continuous, it must be that $g^{-1}(f_\alpha^{-1}(V_\alpha() = g^{-1}(T_\alpha)$ is open. Now observe that:

$$U = \bigcup_{\alpha \in J} \bigcup_{B_\beta \in T_\alpha} B_\beta \quad \text{or} \quad U = \bigcap_{\alpha \in J} \bigcap_{B_\beta \in T_\alpha} B_\beta$$

Therefore:

$$g^{-1}(U) = g^{-1}(\bigcup_{\alpha\in J} \bigcup_{B_\beta \in T_\alpha} B_\beta) = \bigcup_{\alpha \in J} g^{-1}\left(\bigcup_{B_\beta\in T_\alpha} B_\beta\right),$$

or

$$g^{-1}(U) = g^{-1}(\bigcap_{\alpha\in J} \bigcap_{B_\beta \in T_\alpha} B_\beta) = \bigcap_{\alpha \in J} g^{-1}\left(\bigcap_{B_\beta\in T_\alpha} B_\beta\right),$$

Since $g^{-1}\left(\bigcup_{B_\beta\in T_\alpha} B_\beta\right)$ and $g^{-1}\left(\bigcap_{B_\beta\in T_\alpha} B_\beta\right)$ are open, it follows that $g^{-1}(U)$ is open in Y. Because this is true for every open set U in A under topology $\mathcal{T}$, we conclude g is continuous.

Part(d) We will prove the proposition by generating an open set in the subspace whose inverse image under f is a given open set in A. Let U be open in A under the topology $\mathcal{T}$. Each basis element B_β contained in U is the finite intersection of elements of $\mathcal{S}$ since the latter is the subspace that generates $\mathcal{T}$. Therefore:

$$B_\beta = \bigcap_{k=1}^{N} S_k = \bigcap_{k=1}^{N} f_{\gamma_k}^{-1}(V_{\gamma_k}),$$

where $S_k \in \mathcal{S}$ and V_{γ_k} is open in X_{γ_k}. Designate K as the set of all γ_k (i.e., a finite set of the indices of the spaces that contribute the subspace elements that generate B_β. For each α, define:

$$T_\alpha = \begin{cases} X_\alpha & \text{for} \quad \alpha \notin K, \\ \bigcap_{\gamma_k=\alpha} V_{\gamma_k} & \text{for} \quad \alpha \in K. \end{cases}$$

Now let $W' = \prod_\alpha T_\alpha$. Observe that W' must be open in $\prod_\alpha X_\alpha$ because each T_α is either all of X_α (which it trivially open) or is the finite intersection of open sets in X_α. The set $W = W' \cap f(A)$ is then open in the subspace $f(A)$ of $\prod_\alpha X_\alpha$ by the definition of the subspace topology. Since $f_\alpha^{-1}(A) = f_\alpha^{-1}(X_\alpha \cap f_\alpha(A)) = A$, we have:

$$f^{-1}(W) = f^{-1}\left(f(A) \cap \prod_\alpha T_\alpha\right) = A \cap \bigcap_\alpha f_\alpha(T_\alpha) = A \cap \bigcap_{k=1}^{N} f_{\gamma_k}^{-1}(V_{\gamma_k}) = B_\beta.$$

Thus $f(B_\beta) = W$, which is open the subspace $f(A)$ of $\prod_\alpha X_\alpha$. It follows that U is the union of all basis elements B_β contained in it; thus $f(U)$ is the union of open sets in the subspace $f(A)$ and must be open. We conclude that the image of every open set in A under $\mathcal{T}$ is an open set in the subspace $f(A)$ of $\prod_\alpha X_\alpha$.

Section 20

Problem 20.1. *[See problem.]*

Solution: Part (a) We will prove part (b) first. The solution to part (a) is a specific case of the result there.

Part (b) Assume, as prompted, that d' is a metric on $\mathbb{R}^N$. Let $U = \prod_{j=1}^{N}(a_j, b_j)$ be a basis element of the product topology on $\mathbb{R}^N$ and $\mathbf{x}$ be an element of U. For each j there is some ϵ_j such that $(x - \epsilon_j, x + \epsilon_j) \subseteq (a_j, b_j)$. We can find it by letting $j = \inf\{|a_j - x_j|, |b_j - x_j|\}$. Now let $\epsilon = \inf\{\epsilon_j\}_{j=1}^{N}$. Given $\mathbf{y} \in B_{d'}(\mathbf{x}, \epsilon)$, it must be that:

$$\left[\sum_{j=1}^{N} |x_j - y_j|^p\right]^{1/p} < \epsilon.$$

Hence $|x_j - y_j| < \epsilon \le \epsilon_j$, so $\mathbf{y} \in U$ and $\mathbf{x} \in B_{d'}(\mathbf{x}, \epsilon) \subseteq U$. Therefore the metric topology induced by d' is finer than the product topology.

Conversely, let $B_{d'}(\mathbf{x}, \epsilon)$ be a basis elemnet of the metric topology induced by d'. Given $\mathbf{y} \in B_{d'}(\mathbf{x}, \epsilon)$, let $\Delta\epsilon = \epsilon - d'(\mathbf{x}, \mathbf{y})$. We may generate:

$$V = \prod_{j=1}^{N}\left(y_j - \frac{\Delta\epsilon}{2N^{1/p}}, y_j + \frac{\Delta\epsilon}{2N^{1/p}}\right),$$

which is a basis element of the product topology. Given $\mathbf{z} \in \frac{\epsilon}{2}$, we have:

$$d'(\mathbf{x},\mathbf{z}) \le d'(\mathbf{x},\mathbf{y}) + d'(\mathbf{y},\mathbf{z})$$

$$= d'(\mathbf{x},\mathbf{y}) + \left[\sum_{j=1}^{N} |y_j - z_j|^p\right]^{1/p} < d'(\mathbf{x},\mathbf{y}) + \left[\sum_{j=1}^{N}\left(\frac{\Delta\epsilon}{2N^{1/p}}\right)^p\right]^{1/p} = d'(\mathbf{x},\mathbf{y}) + \left[N\frac{\Delta\epsilon^p}{2^p N}\right]^{1/p}$$

$$= d'(\mathbf{x},\mathbf{y}) + \frac{\Delta\epsilon}{2} = d'(\mathbf{x},\mathbf{y}) + \frac{\epsilon}{2} - \frac{d'(\mathbf{x},\mathbf{y})}{2} = \frac{\epsilon}{2} + \frac{d'(\mathbf{x},\mathbf{y})}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\mathbf{z}$ is contained in $B_{d'}(\mathbf{x},\epsilon)$, so $V \subseteq B_{d'}(\mathbf{x},\epsilon)$ and the product topology is finer than the metric topology induced by d'. We conclude that the product topology and metric topology are equal.

Part (a) Redux We prove part (a) by letting $p = 1$ in the result in part (b). In $\mathbb{R}^2$, each basis element $B_{d'}(\mathbf{x},\epsilon)$ corresponds to the points within a diamond-shaped region centered at the $\mathbf{x}$ and tips at $(x+\epsilon, y), (x, y-\epsilon), (x-\epsilon, y)$, and $(x, y+\epsilon)$. For example with $B_{d'}(0,1)$, this diamond has tips at $(1,0), (0,1), (-1,0)$, and $(0,-1)$.

Problem 20.2. *Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.*

Solution: Define the function $d : (\mathbb{R}\times\mathbb{R}) \times (\mathbb{R}\times\mathbb{R}) \to \mathbb{R}$ as:

$$d(x_1 \times x_2, y_1 \times y_2) = \begin{cases} 1 & \text{for} \quad x_1 \ne y_1, \\ \inf\{|x_2 - y_2|, 1\} & \text{for} \quad x_1 = y_1. \end{cases}$$

It is easily shown that d is a metric based on the examples in the text, so we'll skip that step here.

A basis of the dictionary-order topology on $\mathbb{R}\times\mathbb{R}$ consists of all sets $(a \times b, a \times d)$ where $b < d$. Let U be such a basis element and let $\mathbf{x}$ be an element of U. There is some $\epsilon \in (0,1)$ such that:

$$(x_2 - \epsilon, x_2 + \epsilon) \subseteq (b, d).$$

Let $x_1 = a$. The set $(x_1 \times (x_2 - \epsilon), x_1 \times (x_2 + \epsilon))$ equals to the ϵ-ball centered at $\mathbf{x}$, which is contained in U. If $\mathbf{y} \in B_d(x,\epsilon)$, then since y_1 must equal x_1 because $\epsilon < 1$. Therefore $d(\mathbf{x},\mathbf{y}) = |x_2 - y_2| < \epsilon$, so $y_1 \times y_2 \in U$ by the definition of ϵ. As a result, the metric topology induced by d is finer than the dictionary-order topology.

Conversely, let $B = B_d(\mathbf{x},\epsilon)$. If $\epsilon \ge 1$, then $B = \mathbb{R}\times\mathbb{R}$, which is open in the dictionary-order topology. Otherwise, $\epsilon \in (0,1)$, so $B = (x_1 \times x_2, x_1 \times y_2)$. But B is also a basis element of the dictionary-order topology. Thus the dictionary-order topology is finer than the metric topology induced by d. We conclude that the metric topology and dictionary-order topology are equal, and thus the dictionary-order topology is metrizable.

Problem 20.3. *Let X be a metric space with metric d. (a) Show that $d : X \times X \to \mathbb{R}$ is continuous. (b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.*

Solution: Part (a) Let U be a basis element of $\mathbb{R}$, so $U = (a,b)$ for some $a < b$. It follows that $d^{-1}(U) = \{x \times y \in X \times X : d(x,y) \in (a,b)\}$ (in other word, each pair of points in $X \times X$ that are a distance between a and b of each other). Given $x \times y \in d^{-1}(U)$, let $\epsilon = \inf\{b - d(x,y), d(x,y) - a\}$ and $V_{xy} = B_d(x,\epsilon/2) \times B_d(y,\epsilon/2)$. If $u \times v \in V_{xy}$ then:

$$d(u,v) \le d(u,x) + d(x,v) \le d(u,x) + d(x,y) + d(y,v) < \frac{\epsilon}{2} + d(x,y) + \frac{\epsilon}{2} = b - d(x,y) + d(x,y) = b,$$

so the points u and v must be within b of each other. Further,

$$d(x,y) \le d(x,u) + d(u,v) + d(v,y) < \epsilon + d(u,v) = d(x,y) - a + d(u,v).$$

It follows that $a < d(u,v)$. Accordingly, $u \times v \in d^{-1}(U)$. Since this is true for every $x \times y \in d^{-1}(U)$, we infer that:

$$d^{-1}(U) = \bigcup_{x\times y \in d^{-1}(U)} V_{xy}.$$

As this is a union of open sets in $X \times X$, we conclude that $d^{-1}(U)$ is open. Thus d is continuous.

Part (b) Suppose $d : X' \times X' \to \mathbb{R}$ is continuous. From exercise 18.11, d is continuous in each variable separately over both domains, as well. As a result, for any $y_0 \in X'$ (which is also an element of X), the functions $h_{y_0} : X' \to \mathbb{R}$ where $h_{y_0}(x) = d(x, y_0)$ and $h'_{y_0} : X \to \mathbb{R}$ where $h'_{y_0}(x) = d(x, y_0)$ are continuous. Any basis element of X is of the form $B = B_d(y_0, \epsilon)$. Observe that $h'^{-1}_{y_0}((0, \epsilon)) = B$ since a point is in B if and only if it is within a distance of ϵ from y_0. But since h and h' are the same mappings, $h^{-1}_{y_0}((0, \epsilon)) = B$, so the set B is open in X' by the continuity of h_{y_0}. Because every basis element of X is open in X', it follows that X' is finer than X.

Problem 20.4. *[See problem.]*

Solution: Part (b) Skipped.

Part (b) In summary, these sequences converge in the various topologies as follows:

	Box Topology	Product Topology	Uniform Topology
(w_n)		✓	
(x_n)		✓	✓
(y_n)		✓	✓
(z_n)	✓	✓	✓

Observe that each of these sequences has elements with an arbitrary number of zeroes. Accordingly, each must converge, if at all, to 0. We'll confine ourselves then to looking at neighborhoods of 0 in all topologies.

Sequence (w_n). Let U be a neighborhood of $\mathbf{0}$ in the box topology corresponding to $\prod_{k=1}^{\infty}(-\alpha, \alpha)$ where $\alpha > 0$. For $n \in \mathbb{N}$, if $n \geq \alpha$, then:

$$w_n = (\underbrace{0, \ldots, 0}_{\text{n zeroes}}, n, n, \ldots),$$

which cannot be contained in U. Therefore (w_n) does not converge in the box topology.

Let V be a neighborhood of $\mathbf{0}$ in the product topology. It follows that $V = \prod_{k=1}^{\infty} V_k$ where a finite number of V_k are a proper subset of $\mathbb{R}$ that contains 0 (since V is a neighborhood of zero). Let N be the largest index of V_k that is a proper subset of $\mathbb{R}$. If $n > N$, then for $1 \leq k \leq N$ we have $(w_n)_k = 0 \in V_k$, for for $k > N$, we have $(w_n)_k \in \mathbb{R} = V_k$. Hence (w_n) converges in the product topology.

Given $B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ for any $\epsilon \in (0, 1)$ in the uniform topology, if $n > \epsilon$ (which is every valid n), then $(w_n)_k = n$ for $k \geq n$. Thus $|(w_n)_k - 0| = n > \epsilon$, so $w_n \notin B_{\bar{\rho}}(\mathbf{0}, \epsilon)$. The sequence (w_n) does not converge in the uniform topology.

Sequence (x_n). Let $U = \prod_{k=1}^{\infty}(-1/2^k, 1/2^k)$ in the box topology. For any $n \in \mathbb{N}$, if $k > \log_2 n$, then $(x_n)_k) = 1/n > 1/2^k$, so $x_n \notin U$. Therefore (x_n) does not converge in the box topology.

In the product topology, (x_n) converges for the same reason that (w_n) converges there.

Any neighborhood W of $\mathbf{0}$ in the uniform topology contains $B_{\bar{\rho}}(\mathbf{0}, \delta)$ for some $\delta \in (0, 1)$. If $n > 2/\delta$, then $(x_n)_k = 0$ if $k < n$, so $|(x_n)_k - 0| = 0 < \delta$. If $k \geq n$, then $|(x_n)_k - 0)| = 1/n < \delta/2$. Accordingly, $\sup\{|(x_n)_k|\}_{k=1}^{\infty} \leq \delta/2 < \delta$, so $x_n \in B_{\bar{\rho}}(\mathbf{0}, \delta) \subseteq W$. We conclude (x_n) converges in the uniform topology.

Sequence (y_n). In the box topology, let $U = \prod_{k=1}^{\infty}(-1/2^k, 1/2^k)$. For any $n \in \mathbb{N}$, it follows that $1/n > 1/2^n$, so $(y_n)_n \notin (-1/2^n, 1/2^n)$. Hence $(y_n) \notin U$ for any n, so (y_n) does not converge in the box topology.

Let V be a neighborhood of $\mathbf{0}$ in the product topology. Hence V contains $V' = \prod_{k=1}^{\infty} V'_k$ where, for some $\delta > 0$, we have $V'_k = (-\delta, \delta)$ a finite number of k. All other $V_k = \mathbb{R}$. For $n > 1/\delta$ and $1 \leq k \leq n$, it follows that $0 < (y_n)_k = 1/n < \delta$, so $(y_n)_k \in V_k$. Since $V_k = 0 \in V_k = \mathbb{R}$ for $k > n$, we conclude that $y_n \in V$. Therefore (y_n) converges in the product topology.

Let W be a neighborhood of $\mathbf{0}$ in the uniform topology, which must contain $B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ for some $\epsilon \in (0, 1)$. If $n > 2/\epsilon$, then for $1 \leq k \leq n$, we have $|(y_n)_k - 0| = 1/n < \epsilon/2$; for $k > n$, we have $|(y_n)_k - 0| = 0$. As a result, $\sup\{|(z_n)_k|\}_{k=1}^{\infty} \leq \epsilon/2 < \epsilon$. Therefore (z_n) converges in the uniform topology.

Sequence (z_n). Let U be a neighborhood of $\mathbf{0}$ in the box topology. This set must contain $\prod_{k=1}^{\infty}(-\gamma, \gamma)$ for some $\gamma > 0$. If $n > 1/\gamma$, then for $k \in \{1, 2\}$ it follows that $(z_n)_k = 1/n < \gamma$ and $(z_n)_k = 0$ for all $k > 2$. Accordingly, $z_n \in U$, so (z_n) converges in the box topology.

We infer that (z_n) converges in the product topology because it is coarser than the box topology.

The sequence (z_n) converges in the uniform topology for the same reason as (y_n), but with $(z_n)_k \neq 0$ for only $k \in \{1, 2\}$.

Problem 20.6.

Solution: Part (a) Let $\mathbf{y} = \prod_{k=1}^{\infty}(x_k + \epsilon(1 - 1/k))$ where $\epsilon \in (0, 1)$. Obviously $y_k \in (x_k - \epsilon, x_k + \epsilon)$ for all $k \in \mathbb{N}$, so $\mathbf{y} \in U$. Further $|x_k - y_k| = |x_k - x_k - \epsilon(1 - 1/k)| = \epsilon(1 - 1/k)$. It follows that:

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\epsilon(1 - 1/k)\}_{k=1}^{\infty} = \epsilon.$$

Therefore $\mathbf{y} \notin B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. The two sets are not equal.

Part (b) If U is open in the uniform topology, it is the union of some collection of ϵ-balls. Using $\mathbf{y}$ defined in part (a), there must be an ϵ-ball centered at $\mathbf{y}$ that is contained in U. Let $V = B_{\bar{\rho}}(\mathbf{y}, 2\delta)$ for some $\delta > 0$. Let $\mathbf{z}$ be a point in this 2δ-ball where $z_n = y_n + \delta$. For $K \in \mathbb{N}$, if $K > \epsilon/\delta$, then for $n \geq K$:

$$\bar{d}(x_n, z_n) = |x_n - (y_n + \delta)| = \left|x_n - x_n - \delta - \epsilon\left(1 - \frac{1}{n}\right)\right| = \left|\delta + \epsilon\left(1 - \frac{1}{n}\right)\right| > |\delta + \epsilon - \delta| = \epsilon.$$

Accordingly, $\bar{\rho}(\mathbf{x}, \mathbf{z} \geq \epsilon$, so $B_{\bar{\rho}}(\mathbf{y}, 2\delta)$ is not contained in U for any $\delta > 0$. We conclude that U cannot be open in the uniform topology.

Part (c) Given $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$, it follows that $\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{|x_k - y_k|\}_{k=1}^{\infty} < \epsilon$. There is some δ where $\sup\{|x_k - y_k|\}_{k=1}^{\infty} < \delta < \epsilon$, hence $|x_k - y_k| < \delta$ for each $k \in \mathbb{N}$. Accordingly $y_k \in (x_k - \delta, x_k + \delta)$ for each k. Since $\mathbf{y} \in U(\mathbf{x}, \delta)$, it follows that $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ is a subset of $\bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta)$.

Now let $\mathbf{z} \in \bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta)$. Given δ where $\mathbf{z} \in U(\mathbf{x}, \delta)$, for each k we have $|x_k - z_k| < \delta$. Consequently, δ is an upper bound on this set, so $\sup\{|x_k - z_k|\}_{k=1}^{\infty} \leq \delta$. It follows that $\bar{\rho}(\mathbf{x}, \mathbf{z}) \leq \delta < \epsilon$ and $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. Therefore $B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta)$.

Section 21

Problem 21.1. *Let $A \subset X$. If d is a metric for the topology of X, show that $d|A \times A$ is a metric for the subspace topology on A.*

Solution: Designate $f : A \times A \to \mathbb{R}$ where $f = d|A \times A$ and the topology generated in A by it $\mathcal{T}$. Clearly if $\mathbf{a}, \mathbf{b} \in A$, then $f(\mathbf{a}, \mathbf{b}) = d(\mathbf{a}, \mathbf{b})$. Suppose $B = B_d(\mathbf{a}, \epsilon) \cap A$ in subspace A and $C = B_f(\mathbf{a}, \epsilon)$ in A (not the subspace!). If $\mathbf{c} \in B$, then $\mathbf{c} \in A$ and $d(\mathbf{a}, \mathbf{c}) = f(\mathbf{a}, \mathbf{c}) < \epsilon$, so $c \in C$. On the other hand, if $\mathbf{d} \in C$, then $\mathbf{d} \in A$ and $f(\mathbf{a}, \mathbf{c}) = d(\mathbf{a}, \mathbf{c}) < \epsilon$, so $\mathbf{d} \in B$. Hence B equals C.

Given a basis element U of subspace A, it follows that $U = B_d(\mathbf{x}, \epsilon) \cap A$ for some $\mathbf{x} \in X$ and $\epsilon > 0$. For any $\mathbf{y} \in U$, it must be that $\mathbf{y} \in A$ and there is $B_f(\mathbf{y}, \delta) \cap A$ contained in A. By the previous result, $B_d(\mathbf{y}, \delta) \cap A = B_f(\mathbf{y}, \delta)$, which is a basis element in A that is contained in U. Hence $\mathcal{T}$ is finer than the subspace A. Conversely, given the basis element $U' = B_f(\mathbf{y}', \delta')$ in A, the ball $B_f(\mathbf{y}', \delta') \cap A$ equals U' and is a basis element of the subspace A. Therefore the subspace A is finer than $\mathcal{T}$. We conclude that the two topologies are equal. Hence f is a metric that induces the subspace A's topology.

Problem 21.2. *Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. Show that f is an imbedding. It is called a* **isometric imbedding** *of X in Y.*

Solution: Let $g : X \to f(X)$ be a restriction of f where $g(x) = f(x)$ for all $x \in X$. Obviously g is surjective. For any $x_1, x_2 \in X$, if $g(x_1) = g(x_2) = y$, then $d_Y(g(x_1), g(x_2)) = d_Y(y, y) = 0 = d_X(x_1, x_2)$. By the definition of the metric, $x_1 = x_2$, so g is injective. It follows that g is bijective.

Next we will show that g is continuous. Let $x_1 \in X$ and $\epsilon > 0$ be given. If $x_2 \in X$ where $d_X(x_1, x_2) < \epsilon$, then $d_Y(g(x_1), g(x_2)) = d_X(x_1, x_2) < \epsilon$. By Theorem 21.1, g is continuous.

Now we will show that g^{-1} (which exists since g is bijective) is continuous. Let $y_1 \in Y$ and $\epsilon > 0$ be given. Suppose $y_2 \in Y$ and $d_Y(y_1, y_2) < \epsilon$. If $g(x_1) = y_1$ and $g(x_2) = y_2$, then $d_Y(y_1, y_2) = d_Y(g(x_1), g(x_2)) = d_X(x_1, x_2) < \epsilon$. So g^{-1} is continuous as well. We conclude that f is an imbedding of X in Y.

Problem 21.3.

Solution: Part (a) Let U be a basis element for the product space, so $U = \prod_{k=1}^{n} U_k$ where $U_k = B_{d_k}(x_k, \epsilon_k)$ for some $x_k \in X_k$ and $\epsilon_k > 0$. Given $\mathbf{y} \in U$, for each k there is a δ_k-ball $B_{d_k}(y_k, \delta_k)$ for some $\delta_k > 0$ contained in U_k. Let $M = \inf\{\delta_k\}_{k=1}^{n}$. We will show that $B_\rho(\mathbf{y}, M)$ is contained in U. If $\mathbf{z} \in B_\rho(\mathbf{y}, M)$, then $d_k(y_k, z_k) < M \le \delta_k$, so $z_k \in B_{d_k}(y_k, \delta_k) \subseteq U_k$. Therefore $\mathbf{z} \in U$. It follows that the metric space with metric ρ is finer than the product space.

Now let $V = B_\rho(\mathbf{x}, \epsilon)$ be a basis element of the metric topology for some $\epsilon > 0$ and $\mathbf{x} \in \prod_{k=1}^{n} X_k$. It follows that $V = \prod_{k=1}^{n} V_k$ where $V_k = B_{d_k}(x_k, \epsilon)$ (it is easy to show that each formulation is a subset of the other). But V is open in the product topology because each V_k is open in the metric space X_k. We infer that the product topology is finer than the metric topology, so the two topologies are equal.

Part (b) This question requires us to generalize Theorem 20.4. Following the arguments in Theorem 20.4, D satisfies that properties of the metric. Next we will show that the metric topology and product topology are equal.

Let U be open in the metric space with metric D. Given $\mathbf{x} \in U$, there is an ϵ-ball $B_D(\mathbf{x}, \epsilon)$, Let $N \in \mathbb{N}$ where $0 < 1/N < \epsilon/2$ and define the set V as:

$$V = B_{D_1}(x_1, \epsilon/2) \times \cdots \times B_{D_N}(x_N, \epsilon/2) \times X_{N+1} \times X_{N+2} \times \cdots .$$

Clearly V is open in the product topology since there are only a finite number of elements in the product that are a proper subset of X_k. We will show that V is contained in U. For $k > N$:

$$\frac{\bar{d}_k(x_k, y_k)}{k} \le \frac{1}{k} \le \frac{1}{N} < \epsilon/2.$$

If $1 \le k \le N$, then $\bar{d}_k(x_k, y_k)/k < \epsilon/2$. Accordingly $\sup\{\bar{d}_j(x_j, y_j)/j\}_{j=1}^{\infty} \le \epsilon/2 < \epsilon$, so $\mathbf{y} \in U$. Therefore the product topology is finer than the metric topology.

Next, suppose W' is basis element in the product topology. For any $\mathbf{w} \in W'$, we can find another basis element $W = \prod_{k=1}^{\infty} W_k$ contained in W' where, for a finite set $\{\alpha_j\}$, the set $W_{\alpha_j} = B_{\bar{d}}(w_{\alpha_j}, \delta_{\alpha_j})$ for some $\delta_j > 0$, and for all other k the set $W_k = X_k$. Let $\delta = \inf\{\delta_{\alpha_j}/\alpha_j\}$, which must be greater than zero since this set is finite. Now designate $Z = B_D(\mathbf{w}, \delta)$. Given $\mathbf{z} \in Z$, if $j \in \{\alpha_j\}$, then $\bar{d}(w_j, z_j)/j < \delta \le \delta_j/j$, so $d(w_k, z_j) < \delta_j$; thus $z_j \in W_k$. For all other j, we have $z_j \in X_k = W_k$. Thus Z is contained in W, so the metric topology is finer than the product topology. We conclude that the two topologies are equal.

Problem 21.4. *Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)*

Solution: $\mathbb{R}_\ell$ Topology. Let $x \in \mathbb{R}$. Define $V_x = \{[x, x + 1/2^n) : n \in \mathbb{N}\}$. We will show that V_x is countable. Let $\phi_x : \mathbb{N} \to V_x$ where $\phi_x(n) = [x, x + 1/2^n)$. If $W = [x, x + 1/2^n) \in V_x$ for some $n \in \mathbb{N}$, then clearly $\phi_x(n) = W$. Since ϕ_x is surjective, by Theorem 7.1 it is countable. Now suppose U is a neighborhood of x in the $\mathbb{R}_\ell$ topology. The set U must contain the basis element $[\alpha, \beta)$ for some $\alpha \le x < \beta$. Choose $N \in \mathbb{N}$ such that $1/2^N < \beta - x$. Accordingly, $[x, x + 1/2^N)$ is an element of V_x that is contained in $[\alpha, \beta) \subseteq U$. The topology $\mathbb{R}_\ell$ satisfies the first countability axiom.

Ordered-Square Topology. Let $x \times y \in I \times I = [0,1] \times [0,1]$. If $y \in (0,1)$, define:

$$T_{x \times y} = \left\{ \{x\} \times \left(y(1 - \frac{y}{n}), y + \frac{1-y}{n} \right) : n \in \mathbb{N} \right\}.$$

It's easily shown that $T_{x\times y}$ is countable. Let $\psi : \mathbb{N} \to T_{x \times y}$ where $\psi n = \{x\} \times (y(1 - y/n), y + (1-y)/n)$. Given any $\{x\} \times (y(1 - y/n), y + (1-y)/n)$, the image of n under ψ equals this set, so ψ is an injection. The countability of $T_{x\times y}$ follows from Theorem 7.1.

Any neighborhood U_0 of $x \times y$ contains the basis element $\{x\} \times (a_0, b_0)$ where $a_0 < y < b_0$. Simple algebra shows that if $1/N < \inf\{1/y - a_0/y^2, (b_0 - y)/(1-y)\}$, then $y(1 - y/N) > a_0$ and $y + (1-y)/n) < b_0$. Accordingly, $\{x\} \times (y(1 - y/n), y + (1-y)/n) \in T_{x\times y}$ is contained in U_0. Therefore the first countability axiom is satisfies for any $x \times y$ where $y \in (0,1)$.

If $y = 0$ and $x \neq 0$, define:

$$T'_{x \times y} = \left\{ \left(x\left(1 - \frac{x}{n}\right) \times 1, x \times \frac{1}{n} \right) : n \in \mathbb{N} \right\}.$$

Similar to the arguments above, $T'_{x\times y}$ is countable. Given a neighborhood U_1 of $x \times y$, it must be that U_1 contains $(a_1 \times b_1, c_1 \times d_1)$ where $a_1 < x \le c_1$ and $d_1 > 0 = y$. If $1/N < \inf\{1/x - a_1/x^2, y + (d_1 - y)/(1-y)\}$, then $a_1 < x(1 - x/N) < x$ and $0 < y + (1-y)/n) < d_1$. Thus $(x(1 - x/N) \times \{1\}, \{x\} \times (y + (1-y)/N))$ is contained in U_1. The first countability axiom is satisfied for any $x \times y$ where $y = 0$ and $x \neq 0$.

If $x \times y = 0 \times 0$, then it is the least element of $I \times I$. Define:

$$T''_{x\times y} = \left\{ \{0\} \times [0, \frac{1}{n}) : n \in \mathbb{N} \right\}.$$

Again, $T_{x\times y}$ is countable. Any neighborhood U_2 of 0×0 must contain the basis element $\{0\} \times [0, b_2)$ where $b_2 \in (0, 1]$. If $1/N < b_2$, then $(\{0\} \times [0, 1/N))$ is contained in U_2, so the first countability axiom is satisfied.

We can use similar arguments and countable sets where $y = 1$. In particular:

$$T_{x\neq 1\times 1} = \left\{ \left(x \times \left(1 - \tfrac{1}{n}\right), \left(x + \tfrac{1-x}{n}\right) \times 0\right) : n \in \mathbb{N} \right\},$$

$$T_{1\times 1} = \left\{ \{1\} \times [1, 1 - \tfrac{1}{n}) : n \in \mathbb{N} \right\}.$$

We conclude that $I \times I$ meets the first countability axiom.

Problem 21.7. *Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.*

Solution: Suppose (f_n) converges uniformly to f in $\mathbb{R}$. Define $\mathbf{y}, \mathbf{z}_n \in \mathbb{R}^X$ such that $\mathbf{y}(x) = f(x)$ and $\mathbf{z}_n(x) = f_n(x)$ for all $n \in \mathbb{N}$ and any $x \in X$. Let U be a neighborhood of $\mathbf{y}$, which must contain the ball $B_{\bar{\rho}}(\mathbf{y}, \epsilon)$ for some $\epsilon \in (0, 1)$. From the uniform convergence of (f_n), there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $d(f(x), f_n(x)) < \epsilon/2$ for all $x \in X$. Since $\epsilon < 1$, it follows that $d(f(x), f_n(x)) = \bar{d}(f_n(x), f(x))$. Consequently, if $n \geq N$, then:

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup\{\bar{d}(f(x), f_n(x))\}_{x\in X} \leq \frac{\epsilon}{2} < \epsilon.$$

Accordingly $\mathbf{z}_n = \in B_{\bar{\rho}}(\mathbf{y}, \epsilon) \subseteq U$. It follows that (f_n) converges to f as elements of the metric space $\mathbb{R}^X$.

Conversely, suppose (f_n) converges to f as elements of the metric space $\mathbb{R}^X$. Given $\epsilon > 0$, let $\delta = \inf\{\epsilon/2, 1/2\}$. There is an $M \in \mathbb{N}$ such that if $n \geq M$, then $\mathbf{z}_n \in B_{\bar{\rho}}(\mathbf{y}, \delta)$. Therefore given $x \in X$, since $\mathbf{z}_n(x) = f_n(x)$ and $\mathbf{y}(x) = f(x)$, we have:

$$d(f(x), f_n(x)) = \bar{d}(f(x), f_n(x)) = \bar{d}(\mathbf{y}(x), \mathbf{z}_n(x)) \leq \delta < \epsilon,$$

for $n \geq M$. Consequently, (f_n) uniformly converges to f in $\mathbb{R}$.