Florentin Smarandache

My High School Math Notebook

Vol. 1

(Arithmetic, Plane Geometry, and Space Geometry)

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(1973 – 1974)
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Preface

Since childhood I got accustomed to study with a pen in my hand.

I extracted theorems and formulas, together with the definitions, from my text books.

It was easier, later, for me, to prepare for the tests, especially for the final exams at the end of the semester.

I kept (and still do today) small notebooks where I collected not only mathematical but any idea I read in various domains.

These two volumes reflect my 1973-1974 high school studies in mathematics.

Besides the textbooks I added information I collected from various mathematical books of solved problems I was studying at that time.

In Romania in the 1970s and 1980s the university admission exams were very challenging. Only the best students were admitted to superior studies. For science and technical universities, in average, one out of three candidates could succeed, since the number of places was limited. For medicine it was the worst: only one out of ten!

The first volume contains: Arithmetic, Plane Geometry, and Space Geometry.

The second volume contains: Algebra (9^{th} to 12^{th} grades), and Trigonometry.

Florentin Smarandache

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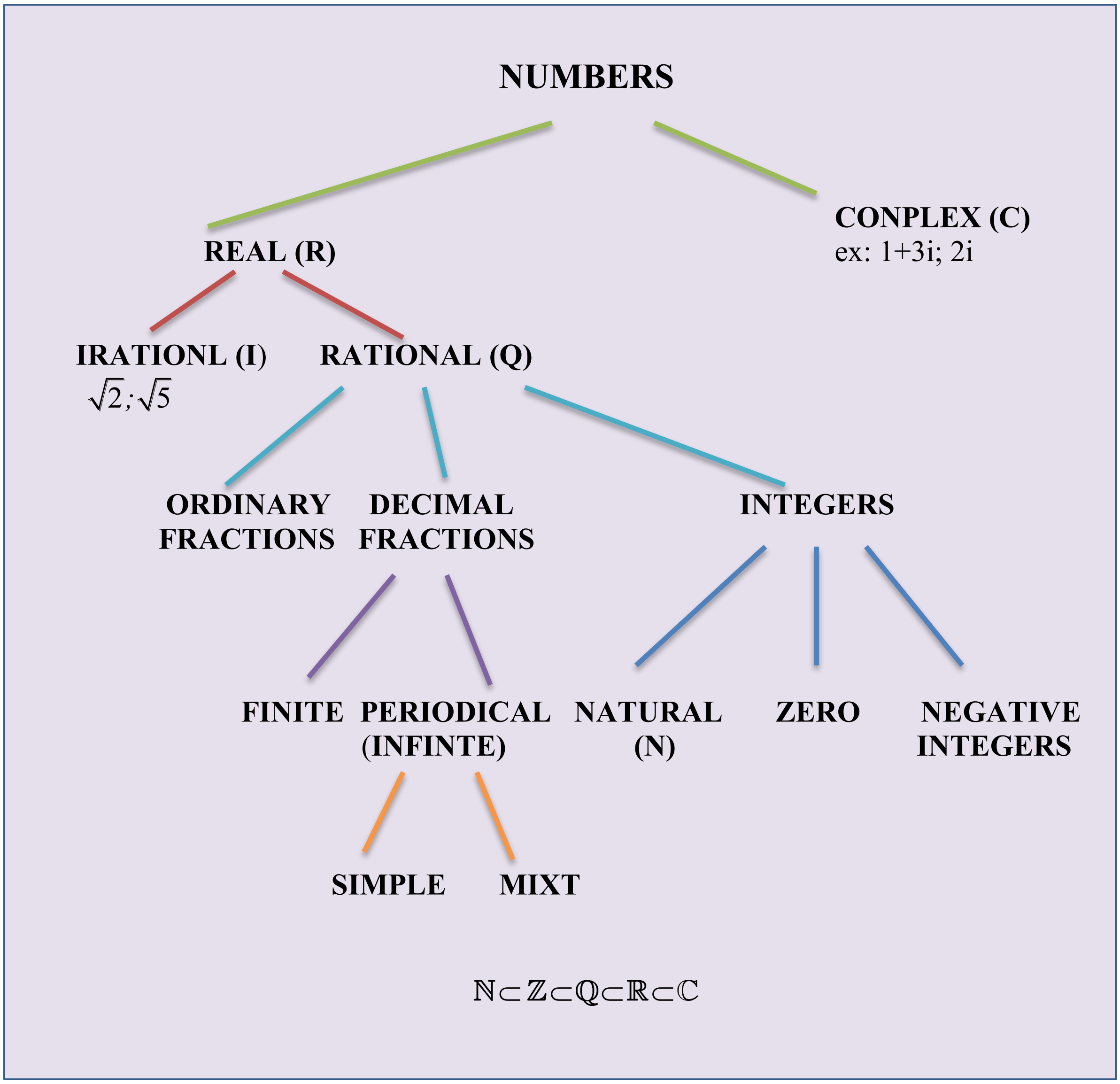
ARITHMETIC



Numbers

- Ordinary (shows the order of elements; first, second, etc.)
- Cardinal (shows how many elements are in a set; seven, six, etc.)

Numbers:

- Concrete (when it is specified the nature of the counted elements: 14 apples)
- Abstract (when there is no specification of the elements: 14, 3, 100, etc.)

Systems of numeracy

a) Primitive numeration
   - The numbers are written with lines I; II, III, IIII, IIIII
   - Are connected with the fingers from the hand

b) Using the letters (Romans)

I V X L C D M

1 5 10 50 100 500 1000

For numbers larger than four thousands it is indicated the number of the thousands by the letter m or we put above the number indicating the thousands a line:

XXIMDCXII = 21,612

IVC = 4100

$\overline{\overline{XI}}\ \overline{CDX}\ DC = 11,410,600$

Numeration systems in the decimal base

- The digits: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 (ten digits)
- In a number a digit shows two things:
  - Through the position where it is written shows the group to which it belongs;
  - Through its value its shows how many groups are there.

Positional system

Definition

The positional system is the numeration system in which through the place where it is written a digit, indicates the type of the group.

In a numeration system with the base B there is the need of B digits from 1 to $B-1$ and the digit 0 (zero). The grouping is done in B units. B units of one order form a unit of a superior order.

Dozen is a group of 12 units.

The Babylonians used the base 60; from those times it remained the division of the hour or the degrees, as 60 minutes in an hour, and the minute in 60 seconds.

Transformation of a number from a given base to another base

Transformation of a number from a given base to base 10.

$$0.234_{(5)} = \frac{2}{5^1} + \frac{3}{5^2} + \frac{4}{5^3}$$

$$6053_{(8)} = 6 \cdot 8^3 + 0 \cdot 8^2 + 5 \cdot 8^1 + 3 = 3115_{(10)}$$

Transformation of a number from base 10 to another base

$$1253_{(10)} = N_{(7)} = 3440_{(7)}$$

Successive divisions

```
1253 | 7
 55  | 179 | 7
  63 |  39 | 25 | 7
   0 |   4 |  4 | 3
```

Transformation of a number from a random base to another base different of 10

The number will be transformed first in base 10 and then in the required base.

Addition

Definition

The addition is a binary operation in which to any pair (ordered) of elements of a set corresponds one element of the same set.

$a+b=c$

a,b - Addition terms

c - The result of the sum (total)

Properties of Additions

1) Commutability $a+b=b+a$. There exists an element neutral such that $a+0=a$
2) Associativity $(a+b)+c=a+(b+c)$

The sum of numbers written in other bases

$$
\begin{array}{r}
8756_{(9)} \\
+2045_{(9)} \\
\hline
11812_{(9)}
\end{array}
$$

During the sum it must be taken into consideration that the numbers should have the same measure.

Subtraction

The operation of subtraction is the invers operation of addition.

$a-b=c$

a,b The subtraction's terms: (a is called the minuend, b is called the subtrahend)

c is called the difference

Properties of subtraction

1) If the minuend is increased (or diminished) with a number, the difference increases (or diminishes) with the same number.
2) If the subtrahend is increased (or decreased) with a number, the difference decreases (or increases) with the same number.
3) If both terms of a subtraction are increased (or decreased) by the same number, the difference will not change.
4) To subtract a number from a sum it is sufficient to subtract it from one of the terms of the sum.

The subtraction in a given positional number system

$$\begin{array}{r} 28504_{(9)} \\ -\ 3615_{(9)} \\ \hline 24778_{(9)} \end{array}$$

Definition

The arithmetic the complement of a number, which does not end in zero, is the difference between the power of 10 that is immediately superior to that number.

Multiplication

To multiply a number with another number it means to repeat adding this number as many times shown in the second number. The multiplication is a repeated addition

$a \cdot b = c$

a,b Are called the multiplication's factors

c Is called the product

a Is called the multiplicand

b Is called the multiplier

The properties of multiplication

1) Commutative: $a \cdot b = b \cdot a$
2) Associative $(a \cdot b) \cdot c = (a \cdot b) \cdot c$
3) Distributive with respect to addition $a \cdot (b + c) = ab + ac$

To multiply a number with a product, we'll multiply the number with a factor of the product.

If a factor of the product increases (or diminishes) m times, then the product itself increases (or diminishes) by the same number of times.

The multiplication of the numbers written in a given positional number system

$$\begin{array}{r} 234_{(5)} \\ \cdot 402_{(5)} \\ \hline 1023 \\ 2101 \\ \hline 211123_{(5)} \end{array}$$

The number of the digits of a product

If the number a has m digits and the number b has n digits, then the product $a \cdot b$ has $m + n - 1$ or $m + n$ digits.

Division

Division is the inverse operation to multiplication.

The division can be exact ($r = 0$), or with reminder $0 < r < divisor$

The properties of division

1) To divide a product by a number it is sufficient to divide by that number only one of the product's factors.
2) If we multiply the dividend and the divisor by the same number, the quotient doesn't change.
3) If we divide the dividend and the divisor by the same number, the quotient doesn't change.
4) To divide a number by a product, we'll divide each factor of the product by that number.

$$\frac{a}{b} = c$$

a Is called the dividend
b Is called the divisor
c Is called the quotient

To divide a sum or a difference by a number, we'll divide each term of the sum or difference by that number.

$$\frac{a+b+c}{m} = \frac{a}{m} + \frac{b}{m} + \frac{c}{m} \text{ or } \frac{a-b-c}{m} = \frac{a}{m} - \frac{b}{m} - \frac{c}{m}$$

If the dividend increases (or diminishes) by a number, the quotient will increase (or decrease) by the same number.

$$\frac{a}{b} = q; \ \frac{a}{b \cdot m} = \frac{q}{m}; \ \frac{a}{\frac{b}{m}} = qm$$

Division with remainder

$$a = b \cdot q + r, \ r < b$$

If we divide the dividend and the divisor by the same number, the quotient doesn't change, but the remainder is divided by that number.

Partial remainders

$$\begin{array}{r|l} 4847 & 23 \\ \underline{46} & 210 \\ R_1 = 24 & \\ \underline{23} & \\ R_2 = 17 & \\ R_3 & \end{array}$$

R_1 =2 is the partial remainder

R_2 =1 is the partial remainder

R_3 =17 is the final reminder

Algorithm

Definition

An algorithm is a method that consists of a succession of computations, all fallowing the same process.

For example: the Euclid's algorithm, the division is also called the division algorithm.

Measuring quantities

The fundamental measures: length, mass, time. The fundamental units of measurement: meter, gram, second.

Derived measures: speed, area, etc.

Etalon meter is a meter build from platinum, which is kept in Paris (is length is constant).

The prefixes used in the multiples and submultiples of certain units:

Kilo
Hecto
Deka
Meter
Deci
Centi
Mili

Debit = capacity/time

Norm= quantity/time

Ratio=quantity/time

Density=mass/volume $\varphi = \frac{m}{V}$

Acceleration = $\frac{v_2 - v_1}{t}$

Relation between three fundamental units: 1 liter-1 kg= $1dm^3$

Arithmetic mean

$$a = \frac{a_1 + a_2 + ... + a_n}{n}$$

$a_k - a$ = deviation in plus with respect to the mean

$a - a_k$ = deviation in minus with respect to the mean

$1 \le k \le n$

The sum of deviations in plus is equal to the sum of deviations in minus.

Another mode to compute the arithmetic mean

Let the numbers 200, 260, 290. These numbers can be written as 200, 200, 200 + 60+90;

$$\frac{60+90}{3}=50;\ 200+50=250$$

Properties of arithmetic's mean

1) The arithmetic average is contained between the smaller and the largest given values, it is said that it is internal.
2) If one of the given values increases and then the average increases; it is said that the arithmetic average is monotone. If a_1 increases by h, then he average grows by $\frac{h}{n}$.
3) If we substitute two or more of the given values by their partial average, the general average does not change, it is said that the arithmetic average is associative.
$$\frac{a_1+a_2+...+a_n}{n}=\frac{a'+a'+a'+a_4+...+a_n}{n}$$
$$\frac{a_1+a_2+a_3}{3}=a'\Rightarrow a_1+a_2+a_3=a'+a'+a'$$
4) If all given values increase by the same number h and the arithmetic average grows also by h; it is said that the arithmetic average is translated.
5) If all given values are multiplied by the same number h the arithmetic average is multiplied by h; it is said that the arithmetic average is an homogenous function of first degree in the given values.
6) The amount of deviation from the mean is zero.

The square of deviations

$$(a_1-A)^2+...+(a_n-A)^2=a_1^{\ 2}+...+2naA+nA^2=a_1^{\ 2}+...+a_n^{\ 2}-na^2+n(a-A)^2$$

The deviations square with respect to a number A is minim when this number is the arithmetic mean ($A=a$).

Pondered average

$$a=\frac{a_1k_1+a_2k_2+...+a_nk_n}{k_1+k_2+...+k_n}$$

The pondered average has the same properties as the arithmetic average

The ratio of mixed quantities is inversely equal to the ratio between the averages deviations:

$$\frac{k_2}{k_1}=\frac{t-t_1}{t_2-t}$$

Problems of mixtures and alloy

1) The concentration of a solution
2) The title of an alloy
3) Caloric balance

Geometric mean

$$\frac{x}{a} = \frac{b}{x} \Rightarrow x^2 = ab \Rightarrow x = \sqrt{ab}$$

Harmonic mean

$$\frac{2}{x} = \frac{1}{a} + \frac{1}{b} \Leftrightarrow x = \frac{2ab}{a+b}$$

Methods of solving arithmetic problems

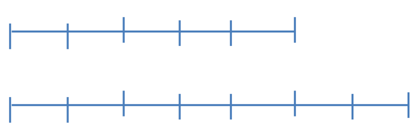
a) Algebraic
b) Through arithmetic rational

The figurative method

The figurative method consists in representing, through a design, the unknown elements, and by fixing into the design the relations between the elements or the relations between the given elements and the quantities given by the problem.

Example 1

A container has $\frac{7}{5}$ more water than another container, and together the containers have 240 liters. How many liters have each container?



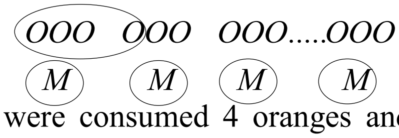
First container = 5 parts

Second container has 7 parts

5 parts +7 parts = 12 parts (both containers)
120 liters $\div$ 12 parts = 20 liters each part
20 liters $\times$ 5 = 100 liters in the first container
20 liters $\times$ 7 = 140 liters in the second container.

Example 2

In a basket with fruits are three times more oranges than apples. At the table are four people; each person takes one orange and an apple. In the basket remained four times more oranges than apples. How many oranges and how many apples were in the basket at the beginning?



At the table were consumed 4 oranges and 4 apples. There remained 3 + 3 + 2 = 8 oranges (from the 4 groups, and 0 apples (4 – 4 = 0)); these are added to the 8 groups formed at the beginning (because the oranges have to be 4 times more).

8 group + 4 groups = 12 groups (initially)

$12 \times 1 = 12$ apples
$12 \times 3 = 36$ oranges

Types of problems

1) Find two numbers knowing their sum and their difference.
2) Find two numbers knowing their sum and their ratio.
3) Find two numbers knowing their difference and their ratio.
4) Problems of elimination of an unknown through sum and difference.

The method of reduction to the same comparative term

This method derives from resolving a system through the method of reduction.

Example

For 9 meters of fabric and 5 meters of silk one person spent $1,950.00. For 3 meters of fabric and 7 meters of silk (same qualities) one spent $1,290.00. How much cost 1 meter of fabric and how much cost 1 meter of silk?

9 m fabric…………….5 m silk………………$1,950.00
9 m fabric…………….7 m silk………………$1,290.00/ $\times$ 3

---

9 m fabric…………….5 m silk………………$1,950.00
9 m fabric……………21 m silk………………$3,870.00

/ 16 m silk………………$.1,950.00

…………..1m silk = $\frac{\$1,920.00}{16m}$ = $120/m silk

$120.00/m $\times$ 5 m = $600.00
…$1,950.00 - $600.00 = $1,350 – the cost of 9 m of fabric
$1,350.00 $\div$ 9 m = $150.00/m fabric.

The method of the false hypothesis

Problem

There are 100 glasses, some have the capacity of 20 cl, and others have the capacity of 35 cl. If all glasses are filled, the total quantity of liquid will be 2,870 cl. How many glasses of each kind are there?

We suppose that we fill up only the glasses with the capacity of 20 cl:
20 cl $\times$ 100 = 2,000 cl.
The liquid left is: 2,870 cl -2,000 cl = 870 cl
The small glasses are now all full.
35 cl – 20 cl = 15 cl
The quantity of liquid left has to be added to the larger glasses.
870 $\div$ 15 = 58 large glasses
100 glasses – 58 large glasses = 42 small glasses.

The method of resolving by starting from the end to the beginning of a problem

The problems are resolved from the end to the beginning.

Example

A number is selected; it is multiplied by 5. To the result is added 42, the sum obtained is divided by 7, and from the quotient it will be subtracted 11, obtaining 200. What is the selected number?

$(x \cdot 5 + 42) \div 7 - 11 = 200$

- Which is the last operation? "from the quotient subtract 11 to obtain 200"; 200+11=211
- "The sum obtained has been divided by 7 and it has been obtained 211"; 211$\times$ 7=1477"
- We added 42 and we obtained 1477"
- 1477-42=1435
- "A number multiplied by 5 gives 1435"
- 1435 $\div$ 5 = 287; this is the selected number.

Combined problems

Are used several methods: a complex problem is decomposed in a couple of simple problems; it is used a process of creative, progressive thinking.

Divisibility of the natural numbers

$a \vdots b$ (a is divisible by b)

$b | a$ (b divides a)

The set of multiples of a number

$M_a(0; a; 2a; 3a; ...ia; ...)$

The set of the divisors of a number

$D_{24} = \{1,2,3,4,6,8,12,24\}$

The set of the common divisors of two numbers is the set of the divisors of those two numbers

Two or more numbers which don't have a common divisor besides 1 are called prime numbers between them $(a,b) = 1$, example $(10,21) = 1$.

The set of the common divisors of n numbers is the set of the common divisors of those numbers.

Three numbers are prime between them if $(a,b,c) = 1$.

The numbers are prime between them two by two if $(a,b) = 1; (b,c) = 1; (c,a) = 1$.

The rules of divisibility

1) The sum (difference) of several multiples of a number is itself a multiple of that number.
2) The sum of several numbers, some multiples of b, others which give a reminder different of zero, when divided by b, is a multiple of b only when the sum of reminders is a multiple of b.

3) If two numbers a,a' gives the same reminder when divided by b, then their difference is a multiple of b. In this case the numbers are called congruent modulo b $a \equiv a' \left(mod\, b \right)$
4) If a is a multiple of b, and b is a multiple of c, then a is a multiple of c.

Divisibility criteria

a) Divisibility by 2: A number is divisible by 2 if its last digit is zero or is divisible by 2 (0,2,4,6,8).
b) Divisibility by 5: A number is divisible by 5 if its last digit is zero or 5.
c) Divisibility by 4: A number is divisible by 4 if its last two digits are divisible by 4.
d) Divisibility by 25: A number is divisible by 25 if its last two digits are divisible by 25. (the last two digits are 00, 25,50, 75, etc.)
e) Divisibility by 8: A number is divisible by 8 if its last three digits are divisible by 8.
f) Divisibility by 125: A number is divisible by 125 if its last three digits are divisible by 125.
g) Divisibility by 3: A number is divisible by 3 if the sum of its digits is a multiple of 3.
h) Divisibility by 9: A number is divisible by 9 if the sum of its digits is a multiple of 9.
i) Divisibility by 11: A number is divisible by 11 if and only if the difference between the sum of its digits, located on even places and the sum of its digits, located on odd places is a multiple of 11. Example: 76.035

$$\begin{matrix} S_1 = 5+0+7=12 \\ S_2 = 3+6=9 \end{matrix} ;\ 76035 = \mathcal{M}11 + S_1 - S_2 = \mathcal{M}11 + 12 - 9 = \mathcal{M}11 + 3$$

j) Divisibility by 7: A number is divisible by 7 if the difference between the sum of classes of odd order and the sum of the classes of even order is a multiple of 7. Example 72.813.605

$$S_1 = 606 + 72 = 677$$

$$S_2 = 813$$

$$72.813.605 = \mathcal{M}7 + S_1 - S_2 = \mathcal{M}7 + 677 - 813 = \mathcal{M}7 + \left(136\right) = \mathcal{M}7 - 3 = \mathcal{M}7 + 4$$

k) Divisibility by 13: A number is divisible by 13 if the difference between the classes of odd order and the sum of the classes representing the even order is a multiple of 13. Example 81.670.200

$$S_1 = 200 + 81 = 281$$

$$S_2 = 670$$

$$81.670.200 = \mathcal{M}13 + 281 - 670 = \mathcal{M}13 - 389 = \mathcal{M}13 - 12 = \mathcal{M}13 + 1$$

l) The general criteria of divisibility:
Let
b the base of the numeration system
d the number for which we need to determine the divisibility criteria
r_i the remainders
Let $a = a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + ...a_1 \cdot b^1 + a_0$ the number for which we need to find the criteria to be divisible by d.

$$b^1 = \mathcal{M}d + r_1$$
$$b^2 = \mathcal{M}d + r_2$$
$$b^3 = \mathcal{M}d + r_3$$
....
$$b^n = \mathcal{M}d + r_n$$
$$a = \mathcal{M}d + a_n r_n + a_{n-1} r_{n-1} + \ldots + a_1 r_1 + a_0$$

Theorems referring to Euclid's algorithm

Let a and b two natural numbers $a > b$, and $a = b \cdot q + r \quad (r < b)$

1) If a number d divides b and r, then d divides also a;
2) If a number d divides a and b, it will divide also r;
3) If a number d divides a and r, then it will divide also b;
4) If $r = 0$, that is $a = b \cdot q$, then $(a,b) = b$ and $D_{a,b}$ = the divisors of b.

Euclid's algorithm (Euclidean Algorithm)

To find the greatest common divisor (GCD) of two given numbers a and b, $a > b$, we'll divide a by b, and we'll denote the remainder by r_1 that is $(a,b) = (b,r_1)$. We'll repeat the process with the numbers b and r_1, and so on, until will reach to a division of a remainder zero (assuming that there will be a finite number of operations).

The last's division's divider will be the GCD (the greatest common divisor) of he given numbers.

Example:

$(932,425)$

032=425x2+82
425=82x5+15
82=15x5+7
15=7x2+1
7=1x7

Therefore $(932,425) = 1$

Consequence of Euclid's algorithm

Any common divisor of two numbers divides the greatest common divisor of the given numbers.

$D_{a,b,c} = D_{a,b} \cap D_c$

To find the GCD of multiple numbers, we'll randomly group them, and find the GCD of each group, $d_1, d_2, \ldots, d_n$, then we'll determine the GDC of the $d_1, d_2, \ldots, d_n$.

Properties

1) If two given numbers are multiplied by the same number, then the GCD of the given numbers will be multiplied by the same number;
2) Let a and b two given numbers, whose one of the divisors is d. If a and b are divided by d, then the GCD will be divided by d. (If $(a,b)=D$, then $\left(\frac{a}{d},\frac{b}{d}\right)=\frac{D}{d}$;
3) The reminders of the division of two number by their GCD are prime numbers between them;
4) If a number divides a product of two numbers and it is prime with one of them, then it will divide also the other number in the product.

Common multiples

Theorems

1) $[a,b]=\frac{a\cdot b}{d}(=a_1\cdot b=a\cdot b_1)$.
2) The set of common multiples of two numbers is equal to the set of multiples of the least common multiple.
3) The least common multiple (LCM) of two prime numbers between them is the product of those numbers.
4) If a number n is divisible with each of the numbers a,b (prime between them), then it is also divisible with their product.
5) The quotient of LCM $[a,b]$ by the numbers a and b are two numbers prime between them.
6) $[a,b]=a_1\cdot b_1\cdot d$; $a=a_1\cdot d$ and $b=b_1\cdot d$
7) $\mathcal{M}_{a,b,c}=\mathcal{M}_{a,b}\cap\mathcal{M}_c$

To find the LCM of several numbers we will form randomly selected groups. We'll determine the LCM of each group $m_1,m_2,...,m_n$, then we'll find the LCM of the groups formed from $m_1,m_2,...,m_n$, and so on.

Diophantine equations of first degree with two unknowns

This type of equations has been studied for the first time by the Greek mathematician Diophantus (IVth century)

General form $ax+by=c$, $(a,b)=1;$ $(a,b,c)\in\mathbb{Z}$.

The computation of these equations is done in the set of integer numbers $x,y\in\mathbb{Z}$.

Example: Find the solution of the following equation $15x-26y=7$ in the set of integer numbers.

$$15x-26y=7\Rightarrow x=\frac{7+26y}{15}=y+\frac{7+11y}{15}$$

$$\frac{7+11y}{15}=p\Rightarrow 11y=15p-7$$

$$\frac{4p-7}{11}=s\Rightarrow p=\frac{11S+7}{4}=2S+1+\frac{3S+3}{4}\quad s\in\mathbb{Z}$$

$$\frac{3S+3}{4} = r \Rightarrow s = \frac{3r-3}{3} = r-1+\frac{r}{3},\ r\in\mathbb{Z}$$

$$\frac{r}{3} = v \Rightarrow r = 3v,\ v\in\mathbb{Z},$$

Therefore, $r=3v$; $r=3v \Rightarrow s=3v-1+v=4v-1 \Rightarrow p=8v-2+1+3v=11v-1$

$\Rightarrow y=11v-1+4v-1=15v-2$

$\Rightarrow x=15v-2+11v-1=26v-3$.

The solutions are $\begin{cases} x=26v-3 \\ y=15v-2 \end{cases}\ v\in\mathbb{Z}$

There exists, therefore, an infinite pairs of solutions, which are obtained by giving to v different values $\{\ldots,-2,-1,0,+1,+2,\ldots\}$.

If we want that x and y to meet certain conditions, we'll form a system of inequalities giving to x and y the required conditions, and then determine the v values.

How is the method applied: We determine first the unknown with the smaller coefficient; extract the whole part from the ratio; the remaining ratio is denoted with a letter; the process is repeated until there is no ratio left. Then starting from the end we determine the values used in notations until the unknown x and y.

Theorems

1) Given the equation $ax=by+c$, where $(a,b)=1$ and $0<c<b$, through the numbers $x=1,2,\ldots,b$ exits one and only one number which verifies the equation; the corresponding value for y is less than a.
2) The solution for the equation $ax+by=c$ is given by the formula:
   $\begin{cases} x=x_1+\lambda b \\ y=y_1+\lambda a \end{cases}$ $\lambda\in\mathbb{Z}$; x_1, y_1 are a particular solution for the Diophantine equation.

The equation $ax+by=c$, where $(a,b)=d>1$

Case I: If d divides also c, then the equation has solutions. The equation is divided by d, and then it is computed normal.

Case II: If d does not divide c, then the equation does not have solutions (it is called impossible).

Prime Numbers

- Divisors:
  a) Improper (1 and the number itself)
  b) Proper (divisors other than 1 and the number itself)
- Numbers
  a) Prime (don't have proper divisors)
  b) Composed (have proper divisors)

Number 1 is not prime or composed.

Theorem

The smaller divisor proper of a number is prime.

Properties of the divisors of a composed number

a) If b is a divisor whose square is bigger (smaller) than a, then $b_1 = \frac{a}{b}$ is a divisor whose square is smaller (bigger) than a.

b) The sequence of the prime numbers is infinite.

The sieve of Eratosthenes (III century BC)

Algorithm to find the sequence of prime numbers smaller than a given number a. It is a method of obtaining prime numbers by sifting out the composite numbers from the set of natural numbers so that only prime numbers remain.

In a table are written all number up to a.

Cross out all multiples of 2, starting with 2^2.

Then we cross out the multiples of 3, starting with 3^2.

Then we cross out the multiples of 5, staring with 5^2 and so on until we cross out $\mathcal{M}p$ starting with $\mathcal{M}p^2$, p prime. All numbers that remain in the table are prime.

Theorems

1) If a number a has a common divisor, which is a composed number b, then a has also a divisor prime smaller than b.
2) If a number does not have any divisor prime, ten it does not have any prime divisor composed number, then it will not have any divisor composed.

How to determine that a number is prime

Let a a given number, first will find the biggest prime number p which satisfies the condition $p^2 \leq a$ and write all prime numbers until p; then we verify if each prime number it is a divisor for a.

If a is not divisible by any of these numbers, then a is a prime number.

Example:

Let consider the number 281, which is prime

We have $17^2 = 289 > 281$; we consider the numbers 2, 3, 5, 7, 11, 13. We verify that none of these numbers divides 281; therefore 281 is a prime number.

Decomposition of a number in prime factors

360	2
180	2
90	2
45	5
9	3^2
1	

therefore $360 = 2^3 \cdot 3^2 \cdot 5$

A number cannot have two distinct decompositions in prime factors.

Given two numbers A and B, decomposed in prime factors, the necessary and sufficient condition that A would divide B is that all prime factors of A can be found in the decomposition of B with the exponents at most equal.

The number of divisors of a number

Given a number

$$N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n},$$

the number of its divisors is $(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_n + 1)$.

To find the least common divisor of two or more numbers, each decomposed in factors, we'll consider the common factors with the smaller exponent.

To find the greatest common multiple of two or more numbers, each decomposed in factors, we'll consider the common ad not common factors at the greatest exponent.

$[a,b] \times (a,b) = a \cdot b$

Ordinary fractions

Sizes

- Commensurable (can be measured with a measurement unit)
- Incommensurable (cannot be measured)

$\frac{a}{b} = \frac{numerator}{denominator}$, a,b are called the fraction's terms

The fractions can be:

- Sub unitary fractions $\frac{3}{4}, \frac{7}{10}$ $(a < b)$
- Equi unitary fractions $\frac{1}{1}, \frac{5}{5}$ $(a = b)$
- Super unitary fractions $\frac{5}{4}, \frac{6}{2}$ $(a > b)$

Comparison of fractions

a) From two fractions, which have the same denominator, the bigger is that whose numerator is larger. $\frac{3}{4} > \frac{2}{4} > \frac{1}{4}$

b) From two fractions, which have the same numerator, the bigger is that whose denominator is smaller. $\frac{3}{2} > \frac{3}{3} > \frac{3}{4}$.

Amplification of fractions

To amplify a fraction means to multiply its numerator and denominator by the same number: $\frac{a}{b} = \frac{ma}{mb}$.

Simplification of a fraction

To simplify a fraction means to divide its numerator and denominator by the same number: $\frac{a}{b}=\frac{\frac{a}{m}}{\frac{b}{m}}$.

Solved problems

1) Prove that the greatest common divisor of the numbers $5n+3$ and $11n+8$ is 1 or 7. Find for what values of n the fraction $\frac{5n+3}{11n+8}$ can be simplified.

We'll denote $\begin{cases} A=5n+3 \\ B=11n+8 \end{cases}$, if d divides A and B it will divide also:

$$11A+5B=-\cancel{55n}-33+\cancel{55n}+40=7,$$

therefore, d is 1 or 7 because $d/7$, and $\frac{5n+3}{11n+8}$ can be simplified by 7.

$5n+3=\mathcal{M}7$, therefore $5n=\mathcal{M}7-3$, but $n=\mathcal{M}7\pm k;0\le k\le 6$; $k\in\mathbb{Z}$

a) $n=\mathcal{M}7\Rightarrow 5n=\mathcal{M}7\neq\mathcal{M}7-3;$
b) $n=\mathcal{M}7+1\Rightarrow 5n=\mathcal{M}7+5\neq\mathcal{M}7-3;$
c) $n=\mathcal{M}7+2\Rightarrow 5n=\mathcal{M}7+3\neq\mathcal{M}7-3;$
d) $n=\mathcal{M}7+3\Rightarrow 5n=\mathcal{M}7+1\neq\mathcal{M}7-3;$
e) $n=\mathcal{M}7+4\Rightarrow 5n=\mathcal{M}7+6\neq\mathcal{M}7-3;$
f) $n=\mathcal{M}7+5\Rightarrow 5n=\mathcal{M}7+4\neq\mathcal{M}7-3;$
g) $n=\mathcal{M}7+6\Rightarrow 5n=\mathcal{M}7+2\neq\mathcal{M}7-3;$

therefore $n=\mathcal{M}7+5=7k+5;k\in\mathbb{Z}$

2) Find for what values of a natural number n the fraction $\frac{5n+12}{2n+1}$ is an integer number.

We extract the integer part $\frac{5n+12}{2n+1}=\frac{4n+2+n+10}{2n+1}=2+\frac{n+10}{2n+1}$;

therefore $\frac{n+10}{2n+1}$ must be an integer.

Therefore $n+10\ge 2n+1\Rightarrow n\le 9$. We'll try one at the time all values of n, $n\in\{0,1,2,...,9\}$, and we find $n=9,\ n=0$.

If $\frac{a}{b}$ is not an integer, then $\frac{a^n}{b^n}$ is not an integer.

3) A boat rides on a river's watercourse with a speed of $14\frac{1}{2}Km$ per hour, and against the watercourse with a speed of 12Km per hour. What is the speed of the boat in stagnant water and what is the speed of the watercourse?

We denote v_1 the boat's speed in a still water and v_2 the speed of the moving water.

$$\begin{cases} 14\frac{1}{2} Km/h = v_1 + v_2 \\ 12 Km/h = v_1 - v_2 \end{cases}$$

$$14\frac{1}{2} - 12 = 2\frac{1}{2} = \frac{5}{2}$$

$$\frac{5}{2} \div 2 = \frac{5}{4} Km/h = v_2$$

$$12 + \frac{5}{4} = 13\frac{1}{4} Km/h = v_1$$

Decimal fractions Decimal numbers)

In general, a decimal number is equal to a fraction which has as denominator the natural number, which we have in mind, when we delete the decimal point, and as numerator 10^n, n being the number of digits of the decimal part.

How to transform a decimal number to a number of another numeration bases

$$0.326_8 = 3\cdot 8^{-1} + 2\cdot 8^{-2} + 6\cdot 8^{-3} = \frac{3}{8} + \frac{2}{8^2} + \frac{6}{8^3} = \frac{326_8}{8^3}$$

The division can be

- Exact $(r = 0)$
- With reminder ($r \neq 0$ and $r <$ than the divider.

The quotient can be:

- Exact
- Approximate
  - In minus $\frac{4.7}{8} \approx 0.5$
  - In plus $\frac{4.7}{8} \approx 0.6$

The division of decimal numbers

$21.35 \div 4.3$

```
213500 |43
 415   |4.965
  280  |
   220 |
     5 |
```

To find the remainders we'll write:

$213500 = 43\cdot 4.965 + 5 \ / \div 1000$

$213.5 = 43\cdot 4.965 + 0.005 / \div 10$

$21.35 = 4.3\cdot 4.965 + 0.0005$

Therefore, $r = 0.0005$.

To divide two decimal numbers, we'll multiply the numerator and the denominator with 10^n, n being the number of decimals of the denominator. We are now in the case of the division of a decimal number (or whole) to an integer. In this way the quotient doesn't change; the remainder is multiplied also by 10^n.

Transformation of ordinary fractions in decimal fractions

The irreducible fraction $\frac{a}{b}$ is transformed in a decimal fraction finite id the numerator is of the form $2^\alpha \cdot 5^\beta$.

Simple periodical fractions

If $(b,10)=1$, the irreducible fraction $\frac{a}{b}$ can be transformed in a simple periodical fraction.

Mixed periodical functions

If the denominator b of a fractions of the form $b=2^\alpha \cdot 5^\beta \cdot b_1$, where b_1 is prime with 10, then $\frac{a}{b}$ is transformed in mixt periodic fraction. The non-periodical part of n digits, where $n=max(\alpha,\beta)$; the period contains at most (b_1-1) digits. (The period contains λ digits, where λ is the least divisor of $\lambda_M = Q(b)$ for which $10^2 \equiv 1(mod\, b)$).

The transformation of the periodical decimal fractions in ordinary fractions

The generator fraction of a periodical simple decimal fraction (sub-unitary) has as denominator the number formed of the period and as numerator the number $10^n-1=\underset{n}{99...9}$, where 9 is the number of digits of the period.

$$0.(351)=\frac{351}{999}$$

The generator fraction of a periodic mixt decimal fraction (sub-unitary) has as numerator the difference between the number formed of the number formed by the non-periodic part followed by the periodic part and the number formed by the non-periodic part. As denominator we'll have the number written with digits 9 that are as many as the number of digits in the periodic part, followed by as many zeroes as the number of digits in the non-periodical part.

$$0.31(456)=\frac{31456-31}{99900}$$

Resolved problems

1) The period of fraction $\frac{a}{7},(7,10)=1,(a,7)=1$ is determined from the period of fraction $\frac{1}{7}$ moving a group of digits from the beginning to the end.

Example:

$$\frac{1}{7}=0,(142\ 857)$$

$$\frac{2}{7}=0,(285714)$$

$$\frac{3}{7}=0,(428571)$$

2) Prove that if in the sequence of the reminders of the division of the numbers $a, a\cdot 10, a\cdot 10^2, \ldots,$ by the number b, where $(a,b)=1$ and $(b,10)=1$, there is the reminder $r_h=b-r_a$, then $r_{h+1}=b-r_1; r_{h+2}=b-r_2; \ldots; r_{h+n}=r_n$.

Deduct from here that the fraction's $\frac{a}{b}$ period has an even number of digits equal to $2h$, and that $\alpha+\beta=10^h-1$, where α is the number formed by the first h digits of the period, and β is the number formed by the last h digits of the period.

Example

$a=17; b=13$; the remainders are $r_a, r_1, r_2, \ldots, r_i, \ldots$.

We have $a=b\cdot c+r_a$,

$$10r_a=b\cdot c_1+r_1$$
$$10r_1=b\cdot c_2+r_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$10r_i=b\cdot c_{i+1}+r_{i+1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

c_i are the period's digits.

But $r_h=b-r_a \Rightarrow 10r_h=10b-10r_a=10b-b\cdot c_1-r_1=b(9-c_1)+b-r_1$;

$r_{h+1}=b-r_1 \Rightarrow 10r_{h+1}=10b-10r_a=10b-b\cdot c_2-r_2=b(9-c_2)+b-r_2$;

Therefore

$r_{h+1}=b-r_2$, etc.

We have

$$\begin{matrix} c_1+c_{n+1}=9 \\ c_2+c_{n+2}=9 \end{matrix} \quad \text{etc.}$$

$$\alpha+\beta=\underbrace{99\ldots9}_{h\ times}=10^h-1$$

Example

$\frac{17}{13}$, its successive remainders are

$5;\ 11;\ 6;\ 8(=13-5);\ 2(=13-11);\ 7(=13-6)$

The period is $384\ 615$, and $384+615=999$.

3) Prove that the sum of the periods of the decimal fractions in which $\frac{a}{b},\frac{b-a}{b};(a<b;(b,10)=1;(a,b)=1)$ are transformed, is a number formed only from the digit 9.

$$\frac{a}{b}=\frac{P}{99...9}$$

$$\frac{b-a}{b}=\frac{P'}{99...9}$$

$$\frac{a}{b}+\frac{b-a}{b}=1+\frac{P+P'}{99...9}\Rightarrow P+P'=99...9$$

4) What ordinary fractions transform in decimal fractions with a one digit in the non-periodical part and two digits in the periodical part?

$$\frac{a}{b}=\overline{0,n(c_1c_2)}=\frac{\overline{n(c_1c_2)}-n}{990}$$

At the denominator it can be any divisor of 990, except those for 90 and 99:

$$\left.\begin{array}{l}990=2\cdot 3^2\cdot 5-11\\ 90=2\cdot 3^2\cdot 5\\ 99=3^2-11\end{array}\right\}\text{ the numerator 22.55 or 110, etc. } (a,b)=1$$

5) Prove that in the case of fraction $\frac{a}{b}$, through the division of a,b we obtain only prime numbers with b (smaller than b) as partial remainders $10^i a=b\cdot c+r_i$, if r_i and b would have a common divisor, then this divisor would divide also $10^i\cdot a$, which is impossible (because of the hypothesis).

Through the division of $\frac{a}{b}$, where $(a,)=1,\ (b,10)=1$, the period has at most $\varphi(b)$ digits; if the period has k digits, then $k/\varphi(b)$; where $\varphi(b)$ is the number of all prime numbers with b and inferior to it.

$$\varphi(b)=b\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)...\left(1-\frac{1}{p_n}\right),\text{ or}$$

$$\varphi(b)=\left(p_1^{\alpha_1}-p_1^{\alpha_1-1}\right)\left(p_2^{\alpha_2}-p_2^{\alpha_2-1}\right)...\left(p_n^{\alpha_n}-p_n^{\alpha_n-1}\right),\text{ where } b=p_1^{\alpha_1}p_2^{\alpha_2}...p_n^{\alpha_n}.$$

Fractional units or principal fractions

Principal fractions are those fractions which have the numerator 1.

Example: $\frac{1}{13},\frac{1}{18},\frac{1}{2}$.

A set is called numerable if its elements can be arranged in a sequence $a_1,a_2,...,a_n$, that is, if it can be established a bi-univocal correspondence between the given set and the set of the natural numbers.

Between the set of rational numbers and the set of the points on an axis does not exist a bi-univocal correspondence (due to the fact that on the axis there are also the irrational numbers). Therefore the set of irrational numbers is not numerable.

Between the set of real numbers and the points on the axis there is a bi-univocal correspondence.

The union of two numerable sets is a numerable set.

Approximate calculations

Absolute error

The absolute error is the difference between the exact value of a number and its approximate value.

The error can go two ways: on the left side of the zero, and to the right side of the zero on the number's line.

When segments are measured, there we see errors.

Example: $AB = 6.35m \Rightarrow 6.35 < AB < 6.36$

To round a number (integer or decimal) of a certain order, the digits of lower order are substituted by zeroes (or we neglect them, if these are decimals), we leave the digit of that order unchanged if after it follows a smaller digit smaller than 5 or we increase it with one more unit.

Example 432.516 rounded to the hundredths will be 432.52.

Approximate values resulted from computations

1) When working with exact values, but the computation leads to an approximate value.
2) When working with approximate values, and the computations lead to an approximate result.

Solved problems

1) Between what limits is the aria of a rectangle knowing that $L = 10 - 11m$ and $l = 5 - 6m$?

$$\begin{array}{l} 10 < L < 11 \\ 5 < l < 6 \\ \hline 50 < L \cdot l < 66 \end{array}$$

2) Between what limits is the difference of two segments $AB = 4 - 5cm$ and $CD = 1 - 3cm$?

$$\begin{array}{l} 4 < AB < 5 \\ 1 < CD < 3 \\ \hline 4 - 3 = 1 < AB - CD < 4 = 5 - 1 \end{array}$$

Operations with approximated numbers

Addition

1) When we don't know the direction of the error

$$
\begin{array}{l}
a-\alpha < A < a+\alpha \\
b-\beta < B < b+\beta \\
c-\gamma < C < c+\gamma \\
\hline
(a+b+c)-(\alpha+\beta+\gamma) < A+B+C < (a+b+c)+(\alpha+\beta+\gamma)
\end{array}
$$

A,B,C are the exact values

a,b,c are the approximate values

α,β,γ are the approximations (the limits of errors)

The error limit of the sum is equal with the sum of the errors of the terms.

2) When we know the direction of the error

$$
\begin{array}{l}
a < A < a+\alpha \\
b < B < b+\beta \\
c < C < c+\gamma \\
\hline
a+b+c < A+B+C < (a+b+c)+(\alpha+\beta+\gamma)
\end{array}
$$

a,b,c are the values approximated in minus.

The sum of the approximate values in minus of the terms represents an approximate value in minus also, the limit of the error being equal to the errors' sum.

3) When we know the terms' errors

$A = a + l_1$

$B = b + l_2$

$C = c - l_3$

$A + B + C = (a+b+c) + l_1 + l_2 - l_3$

a,b,c are the values approximated in minus

l_1, l_2, l_3 are the errors

The exact value of a sum is equal to the sum of the approximate values of the terms plus the algebraic sum of the errors.

The most probable value of a measurement is the arithmetic average of the resultants found through several measurements, after the visible error have been eliminates.

Subtraction

If the direction of the error is unknown in the case of the difference's terms, then we'll not know the sense's error of the result, but is sure that the result's error is smaller than the sum of the terms' errors.

Multiplication

1) When we know the sense of errors

$$a-\alpha < A < a+\alpha$$
$$b-\beta < B < b+\beta$$
$$ab-(a\beta+b\alpha)+\alpha\beta < AB < ab+(a\beta+b\alpha)+\alpha\beta$$

$\alpha\beta$ is neglected being very small, then

$$ab-(a\beta+b\alpha) < AB < ab+(a\beta+b\alpha)$$
$$E(ab)=a\cdot E(b)+b\cdot E(a)$$

where

$E(a)=\alpha$, the error of A

$E(b)=\beta$, the error of B

2) When a and b represent approximate values in minus

$$a < A < a+\alpha$$
$$b < B < b+\beta$$
$$ab < AB < ab+(a\beta+b\alpha)+\alpha\beta$$
$$E(ab)=a\cdot E(b)+b\cdot E(a)$$
$$E(abc)=ab\cdot E(c)+ac\cdot E(b)+bc\cdot E(a)$$

Division

If the numerator is less that its exact value, and the denominator is greater than its exact value, the quotient will be smaller than the exact value, and if the numerator is larger than the exact value and the denominator is less than its exact value, the we'll obtain the quotient will be larger than the exact value.

$$12 < a < 16$$
$$2 < b < 3$$
$$\frac{12}{3}=4<\frac{a}{b}<\frac{16}{2}=8$$

The quotient's error

$$a-\alpha < A < a+\alpha$$
$$b-\beta < B < b+\beta$$
$$\frac{a-\alpha}{b+\beta}<\frac{A}{B}<\frac{a+\alpha}{b-\beta}$$
$$E\left(\frac{a}{b}\right)=\frac{aE(b)+bE(a)}{b\left[b-E(b)\right]} \qquad (3)$$

If a and b are approximate vales of the A and B with the approximate $E(a), E(b)$, for which we don't know the error's direction, then the $\frac{a}{b}$ is an approximate value of the exact quotient $\frac{A}{B}$, without knowing the error's direction, and the error limit is given by the relation (3)

The quotient's error when we know the approximate values in negative direction of the terms.

$$a < A < a + \alpha$$
$$b < B < b + \beta$$
$$\frac{a}{b+\beta} < \frac{A}{B} < \frac{a+\alpha}{b}$$

Relative errors

The relative error is the quotient between the absolute error and the exact value of a number:

$$l = \frac{A-a}{A}$$

Example:

Let $A = 0.325$ and let $a = 0.3$ its tenth rounded value.

The absolute value is $A - a = 0.025$; the relative error is $\frac{0.025}{0.325} = \frac{1}{13}$.

If a number has m exact significant digits, the relative error is smaller than $\frac{1}{k \cdot 10^{m-1}}$ or smaller than $\frac{1}{10^{m-1}}$.

If is given that the relative error is smaller than $\frac{1}{(k+1) \cdot 10^{m-1}}$, where k is the first digit of the number, we deduct that the number has m exact digits.

If given $l < \frac{1}{10^m}$, because $\frac{1}{10^m} < \frac{1}{(k+1) \cdot 10^{m-1}}$, we deduct that m digits are exact.

The relative error of a product

$$l(a \cdot b) = l(a) + l(b)$$
$$l(abc) = l(a) + l(b) + l(c)$$

The relative error of a product is equal with the sum of the relative errors of the factors of the product.

The relative error of a ratio

$$l\left(\frac{a}{b}\right) \approx l(a) + l(b)$$

The relative error of a ratio is approximate equal to the sum of the relative errors of the terms.

Ratio and proportions

The irrational numbers are incommensurables; these cannot be measured exactly; in the decimal writing these have an infinite number of decimals, which are not is a periodic succession. Example is π

The fundamental property of a ratio

The ratio of two values is equal to the ratio of their measures in which it has been used the same measurement unit and it does not depend of the measurement unit used.

Proportions

The equality of two ratios is called proportion

$$\frac{a}{b}=\frac{c}{d}$$

a,d are the outer terms and are called extremes

b,c are the middle terms and are called means

The fundamental property of a proportion

The product of the extremes is equal to the product of the means.

Derived proportions

$$\frac{a}{b}=\frac{c}{d}$$

I) Derived proportions with the same terms

1) Interchange the means between them $\frac{a}{c}=\frac{b}{d}$
2) Interchange the extremes between them $\frac{d}{b}=\frac{c}{a}$
3) Invers the ratios $\frac{b}{a}=\frac{d}{c}$
4) Swap the ratios' places $\frac{c}{d}=\frac{a}{b}$
5) Simultaneously swap the extremes and the means between them $\frac{d}{c}=\frac{b}{a}$

II) Derived proportions with the terms swapped

1) Multiply the numerators with the same number
2) Divide the numerators by the same number
3) Multiply the denominators with the same number
4) Divide the denominators by the same number
5) Amplify the both ratios or only one ratio by the same number
6) Simplify the ratios or only ratio with the same number

7) Multiply the two ratios by another ratio $\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{am}{bn}=\frac{cm}{dn}$
8) The ratio of the sum of the numerators and the sum of the denominators is equal to each of the given ratios $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$
9) The ratio of the difference of the numerators and the difference of the denominators is equal to each of the given ratios $\frac{a+c}{b+d}=\frac{a-c}{b-d}$
10) The ratio of the sum of the numerators and the sum of denominators is equal to the ratio of the difference of the numerators and he difference of the denominators $\frac{a+c}{b+d}=\frac{a-c}{b-d}$

III) Derived proportions by changing in the same mode each ratio, through linear combinations between terms

1) The ratio formed by adding the numerator with the denominator $\frac{a+b}{b}=\frac{c+d}{d}$ and denominators unchanged
2) The ratio formed by subtracting from the numerator the denominator, and leave the denominators unchanged $\frac{a-b}{b}=\frac{c-d}{d}$
3) The ratio formed by adding the numerator with the denominator and replace the denominator by the numerator $\frac{a+b}{a}=\frac{c+d}{c}$
4) The ratio formed by subtracting from the numerator the denominator, and replace the denominator by the numerator $\frac{a-b}{a}=\frac{c-d}{c}$
5) The ratio formed by the unchanged numerators and as denominators we add to the denominator the numerator $\frac{a}{b+a}=\frac{c}{d+c}$
6) The ratio formed by the unchanged numerators and as denominators we subtract from the denominator the numerator $\frac{a}{b-a}=\frac{c}{d-c}$
7) The ratio formed by replacing the numerators with the denominators and as denominators we have the sum of the numerator and denominator $\frac{b}{b+a}=\frac{d}{d+c}$
8) The ratio formed by replacing the numerators with the denominators and as denominators we have the difference between the numerator and denominator $\frac{b}{b-a}=\frac{d}{d-c}$

$$\frac{a}{b}=\frac{a'}{b'}\Rightarrow\frac{ma+nb}{pa+qb}=\frac{ma'+nb'}{pa'+qb'}$$

$$\frac{a}{b}=\frac{a'}{b'}\Rightarrow\frac{m_1a^2+m_2ab+m_3b^2}{n_1a^2+n_2ab+n_3b^2}=\frac{m_1a'^2+m_2a'b'+m_3b'^2}{n_1a'^2+n_2a'b'+n_3b'^2}$$

Average proportions

The average of two given numbers a and b is the number m determined such that the ratios $\frac{a}{m}$ and $\frac{m}{b}$ have the same value $\frac{a}{m}=\frac{m}{b}\Rightarrow m^2=ab\Rightarrow ab\Rightarrow m=\sqrt{ab}$

The averages of the form $\frac{a}{a}=\frac{a'}{a'}$ or $\frac{a}{b}=\frac{a}{b}$ are not considered true proportions.

Multiple equal proportions

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\frac{a_1+a_2+a_3}{b_1+b_2+b_3}$$

If two or more ratios are equal, the ratio of the sum of the numerators and the sum of the denominators is equal to the given ratios.

The division of a number in proportional parts with given numbers

Example: Divide 176 in proportional parts with the numbers 7; 10; 15.

$$\frac{x}{7}=\frac{y}{10}=\frac{z}{15}=\frac{x+y+z}{7+10+15}=\frac{176}{32}$$

$$\frac{x}{7}=\frac{176}{32}\Rightarrow x=38\frac{1}{2}$$

$$\frac{y}{10}=\frac{176}{32}\Rightarrow y=55$$

$$\frac{z}{15}=\frac{176}{32}\Rightarrow z=82\frac{1}{2}$$

The division of a number in invers proportional parts with given numbers

Example: Divide 315 in invers proportional parts with the numbers 6; 10; 12

$$\frac{x}{\frac{1}{6}}=\frac{y}{\frac{1}{10}}=\frac{z}{\frac{1}{12}}=\frac{x+y+z}{\frac{1}{6}+\frac{1}{10}+\frac{1}{12}}=\frac{315}{\frac{21}{60}}$$

$$\frac{x}{\frac{1}{6}}=\frac{315}{\frac{21}{60}}\Rightarrow x=15$$

$$\frac{y}{\frac{1}{10}}=\frac{315}{\frac{21}{60}}\Rightarrow y=9$$

$$\frac{z}{\frac{1}{12}}=\frac{315}{\frac{21}{60}}\Rightarrow z=7.5$$

Direct proportional measures

Direct proportional measures are those measures that depend one of the other such that if one increases n times ($n \in N$) the other increases by the same number of times, (the same when we decrease).

Theorem

The ratio of two values of one of the measures is equal to the ratio between the corresponding values of the other measure.

Invers proportional measures

Invers proportional measures are those measures that depend one of the other such that if one increases n times ($n \in N$) the other decreases by the same number of times.

Theorem

The ratio of two values of one of the measures is equal to the invers of the ratio between the corresponding values of the other measure.

The fundamental rule of proportions

Example: In a cylindrical recipient there is stored gas under pressure; its volume is $8cm^3$ and the pressure is 5atm. If the containers' piston is raised such that the gas volume becomes $20cm^3$, what will be its pressure?

$8cm^3 \overset{i}{\ldots\ldots\ldots} 4atm$

$20cm^3 \ldots\ldots\ldots x\ atm$

$$x = \frac{8 \cdot 5}{20} = 2atm$$

(The V and P are invers proportional by the law of Boyle-Mariotte).

In general, the measures can be dependent or independent.

The compound proportions' fundamental rule

At a farm, the quantity of $1400Kg$ stored grasses is needed to feed 10 cows for 7 days. In how many days 2 cows will consume $8000Kg$ of grasses?

1)

$1400Kg \overset{d}{\ldots\ldots} 10cows \overset{i}{\ldots\ldots\ldots} 7days$

$8000Kg \ldots\ldots 2cows \ldots\ldots\ldots xdays$

$$\frac{x}{7} = \frac{10}{2} \cdot \frac{8000}{1400} \Rightarrow x = 200days$$

2)

$1400Kg \overset{d}{\ldots\ldots} 10cows \overset{i}{\ldots\ldots\ldots} 7days$

$8000Kg \ldots\ldots 2cows \ldots\ldots\ldots xdays$

$$x = \frac{10 \cdot 7 \cdot 8000}{1400 \cdot 2} \Rightarrow x = 200days$$

3)

$$1400Kg..^{d}....10cows..^{i}......7days$$
$$8000Kg.......10cows..........x'\,days$$

$$x' = \frac{8000 \cdot 7}{1400} days$$

$$1400Kg..^{d}....10cows..^{i}......\frac{8000 \cdot 7}{1400} days$$
$$8000Kg.......2cows..........xdays$$

$$x = \frac{10\frac{8000 \cdot 7}{1400}}{2} \Rightarrow x = 200days$$

4) **The method of reduction to the unit**

$$1400Kg..^{d}....10cows..^{i}......7days$$
$$1400Kg..^{d}....1cow........7 \cdot 10days$$

$$1Kg......1cow........\frac{7 \cdot 10}{1400} days$$

$$8000Kg.......1cow..........\frac{7 \cdot 10}{1400} \cdot 8000days$$
$$8000Kg.......2cows..........\frac{7 \cdot 10 \cdot 8000}{1400 \cdot 2} days = 200days$$

Percentages

$$p\% = \frac{p}{100}$$
$$p\ ‰ = \frac{p}{1000}$$

Ratio to percentage

A percentage is another way of describing a ratio with respect to 100

a) Finding the percentage from a number:

$$p\% \text{ from } A = A \cdot \frac{p}{100}$$

b) Finding the ratio percentage of a number in relation to another number A

$$A..........a$$
$$100.......x$$

$$x = 100 \cdot \frac{a}{A}$$

Successive percentages

If

$$\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{ka+hb}{k'a+h'b}=\frac{kc+hd}{k'c+h'd},\ \left(\forall k,h,k',h'\notin N\right)$$

and

$$\frac{ka^2+hab+lb^2}{k'a^2+h'ab+l'b^2}=\frac{kc^2+hcd+ld^2}{k'c^2+h'cd+l'd^2}$$

Periodical simple fractions

If $(a,b)=1$ and $(b,10)=1\Rightarrow\frac{a}{b}$ is a periodical simple fraction.

The maximum number of digits of a period is $\lambda_M=\varphi(b)$, where $\varphi(b)$ is Euler's function

The exact number of digits of a period is λ, which is the smaller among the divisors λ_M, which, as exponent of the base 10, gives the remainder by the division to the numerator b; therefore λ/λ_M, $\left(10^2\equiv1(mod\,b)\right)$.

Example: Find the number of digits of the period of $\frac{1}{7}$, and then write directly the remaining divisions: $\frac{2}{7},\ldots,\frac{6}{7}$.

$(1,7)=1$ and $(7,10)=1\Rightarrow\frac{1}{7}$ the periodical fraction simple

$\lambda_M=\varphi(7)=6\Rightarrow\lambda=1,2,3$ or 6

$$\left.\begin{array}{l}10^1\equiv3(mod\,7)\\10^2\equiv2(mod\,7)\\10^3\equiv6(mod\,7)\\10^6\equiv1(mod\,7)\end{array}\right\}\Rightarrow\lambda=6=\text{the number of digits of the period.}$$

$$\frac{1}{7}=0.(142857)$$

Table:

$$\frac{r\ |1\ |3\ |2\ |6\ |4\ |5\ \text{ partial remainders}}{c\ |1\ |4\ |2\ |8\ |5\ |7\ \text{ corresponding ratios}}\Rightarrow\frac{2}{7}=0.(264513),\ \frac{5}{7}=0.(714285)$$

There are the same digits in the period, but in other positions.

The number of tables necessary to write all divisions of the form $\frac{x}{b}$, where $(x,b)=1\left((b,10)=1\right)$ is equal to $\frac{\lambda_M}{\lambda}$; in the case when we have multiple tables, it will be considered another division $\frac{y}{x}$ such that the remainder $y < x$ not to be among the remainders of the first table (or on other tables). In our case $\frac{6}{6} = 1$ (only one table)

The division of a number in direct proportional parts by a number and invers proportional with another number (simultaneously)

The construction of a road between two small towns was valued to \$190,000. Find the contribution of each town knowing that the amount was proportional with the population of each town and invers proportional to the distance of each village to the road. First town has 5000 residents and the distance to the road is 3Km. Second town has 3000 residents and the distance to the road is 2Km

$$\frac{x}{5000 \cdot \frac{1}{3}} = \frac{y}{3000 \cdot \frac{1}{2}}$$

$$\frac{x}{\frac{10000}{6}} = \frac{y}{\frac{9000}{6}}$$

$$\frac{x}{10000} = \frac{y}{9000} = \frac{190000}{19000} = 10 \Rightarrow x = 100000; y = 90000$$

Show that the product of three consecutive numbers cannot be the perfect cube of a natural number.

$$n^3 < n^3 + 3n^2 + 2n < n^3 + 3n^2 + 3n + 1.$$

PLANE GEOMETRY

Fundamental notions of the elementary geometry (Euclidean):

a) The point
b) The line
c) The plane

The axioms are truths which need to be proved.
The postulates are evident truths, which are not proved.
Theorem
Lemmas are helping statements that will help understand a theorem.
Corollary is an immediate consequence of a theorem.
Proposition is a statement.
The theorems can be

a) Direct $I \Rightarrow C$
b) Converse $C \Rightarrow I$
c) Contrary $nonI \Rightarrow nonC$
d) Contrary converse $nonC \Rightarrow nonI$

The method of reduction ad absurdum consists in the process of supposing the contrary, and based on the law of the excluded third (a proposition cannot be other than false or true).

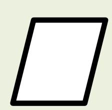
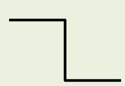
Example:
Direct theorem: the diagonals of a rhombus are perpendicular (true)
The converse theorem: If the diagonals of a quadrilateral are perpendicular, then the quadrilateral is rhombus (false)
Contrary theorem: If a quadrilateral is not a rhombus, its diagonals are not perpendicular (false)
Contrary converse theorem: If the diagonals of a quadrilateral are not perpendicular, then the quadrilateral is not a rhombus (true).

Lines

1) Line segment
2) Semi line
3) Line

Lines:

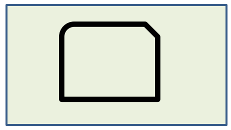
a) Polygonal line

- Closed – polygons 
- Open 

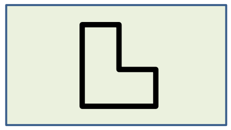
b) Curves (circle, ellipse, etc.)

Polygons

1. Convex – when the extension of any of its sides does not intersect any other of its sides

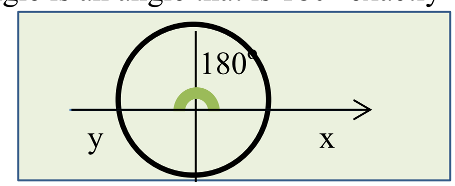


2. Concave – when the extension of at least one of its sides intersects another side of the polygon.

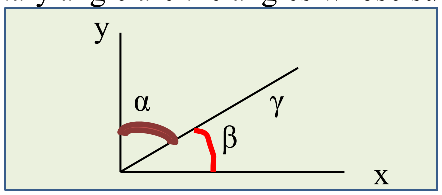


Angles

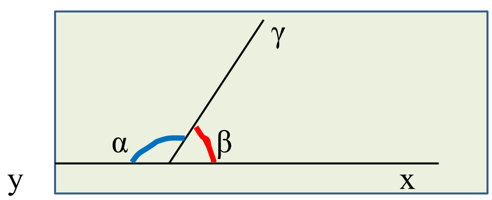
a) Straight angle is an angle that is 180° exactly



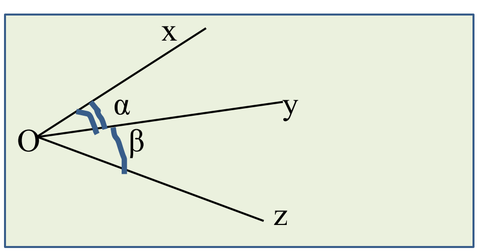
b) Complementary angle are the angles whose sum is 90^0



c) Supplementary angles are the angles whose sum is equal to 180^0



d) Adjacent angles are the angles which have a common side, same vertex and the other sides are on different part of the common side.



Theorem

On a point of a line it can be constructed only one perpendicular line.

From an external point of a line only one perpendicular line can be constructed on a given line.

All right angles are equal.

All straight angle are equal

The circle is the set of point from a plane whose distances to fixed point are equal.

The angle at the center is the angle whose vertex is in the center of the circle.

Theorem

In one circle or several equal circles to equal angles at the center correspond equal arcs, and to unequal angles at the center correspond unequal arcs; to the larger angle at the center corresponds the larger arc.

The measure of the angle at the center is equal to the measure of arc between its sides.

The sum of the angles formed around one point is 360^{o}.

Theorem

The angles whose vertexes are opposite are (called also vertically opposite angles) equal.

Converse

If a line forms equal angles with two semi lines, which have the same origin on that line, and are placed on the sides of line, and are not adjacent, then the semi lines are straight.

Triangles

A triangle is the polygon with three sides.

1) Triangles can be classified by the relative lengths of their sides:
   a) Equilateral ($L_1 = L_2 = L_3$)
   b) Isosceles $L_1 = L_2$
   c) Scalene (all sides are different)

2) Triangles can be classified by the internal angles:
   a) Right-angled triangle (rectangle triangle)
   b) Obtuse-angled triangle(obtuse triangle)
   c) Acute angled triangle (oblique triangles)

The exterior angle of a triangle is the angle formed by a side and the extension of another side; it is equal to the sum of the other two interior angles of the triangle.

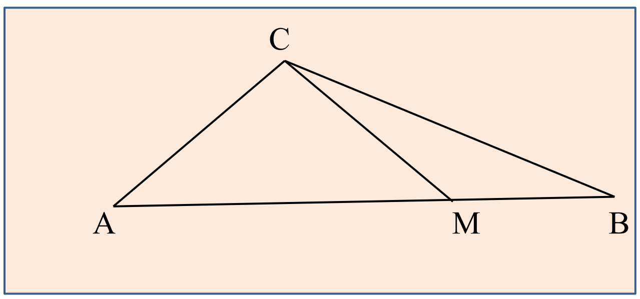
Important lines in a triangle

1) The height or the altitude in a triangle is the perpendicular segment from a vertex to its opposite side. The altitudes are concurrent and their intersection point is called **orthocenter**.
2) The median in a triangle is a line segment from a vertex of the triangle to the midpoint of the side opposite that vertex. There are three medians and all are

concurrent in point called **centroid**. The centroid is on the median at $\frac{2}{3}$ from the vertex and $\frac{1}{3}$ from the base.

3) The bisector in a triangle is the line segment that equally divides the angle of the triangle. There are three bisectors, which are concurrent in a point called the center of the inscribed circle in that triangle.
4) The perpendicular bisector in a triangle is the perpendicular line on the middle side of the triangle. There are three perpendicular bisectors in a triangle and they are concurrent in a point which is the center of the circumscribed circle to the triangle.

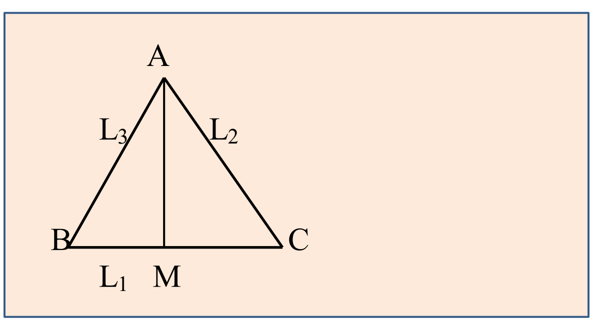
A cevian is any line segment in a triangle with one endpoint on a vertex of the triangle and the other endpoint on the opposite side.



CM = cevian from the mathematician Giovanni Ceva.

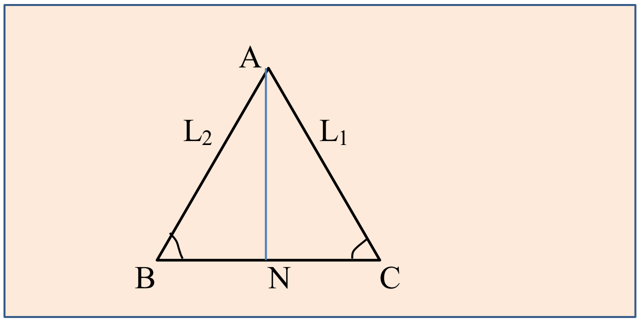
The properties of an equilateral triangle

a) $L_1 = L_2 = L_3$
b) $\sphericalangle A = \sphericalangle B = \sphericalangle C = 60^o$
c) The altitudes are also medians, bisectors, and .perpendicular bisector.



The properties of the isosceles triangle

a) $L_1 = L_2$
b) The angles opposite to the equal sides are equal.
c) The altitude corresponding to the base of the triangle is median and perpendicular bisector



The cases of equality of two random triangles

First Case (A.S.A) a side and two joint angles respectively equal

Second Case (S. A. S) two sides and the angle between them respectively equal

Third Case (S.S.S) all sides respectively equal

The cases of equality of two rectangle triangles

a) A cathetus and an acute angle respectively equal
b) Hypotenuse and an acute angle respectively equal
c) The two catheti are respectively equal
d) The hypotenuse and a cathetus are respectively equal
e)

An exterior angle of a triangle is greater than any interior angle not joint to it.

Relations between the sides and the angles of a triangle

In a triangle the greater side is opposite to the greater angle.

Converse: the greater angle is opposite to a greater side.

Inequalities between the sides of a triangle

a) In a triangle a side is greater than the difference of the other two sides and smaller than their sum:

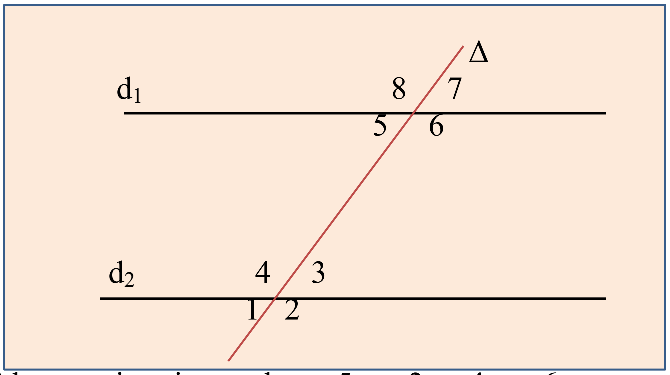
$a > b - c$ $\qquad a < b + c$

$b > a - c$ $\qquad b < a + c$

$c > a - b$ $\qquad c < a + b$

If two triangles have each two equal sides and the angles between them not equal, then to the greater angle opposes the greater side.

Angles formed by two parallel lines intersected by a secant.



1) Alternate interior angles $\sphericalangle 5 = \sphericalangle 3;\ \sphericalangle 4 = \sphericalangle 6$
2) Alternate external angles $\sphericalangle 2 = \sphericalangle 8;\ \sphericalangle 1 = \sphericalangle 7$
3) Corresponding angles

$$\sphericalangle 1 = \sphericalangle 5;\ \sphericalangle 2 = \sphericalangle 6$$
$$\sphericalangle 4 = \sphericalangle 8;\ \sphericalangle 3 = \sphericalangle 7$$

4) Interior of the same side $\sphericalangle 4 + \sphericalangle 5 = 180^0$
5) External of the same side $\sphericalangle 1 + \sphericalangle 8 = 180^0$

If two lines intersected by a secant forms alternate interior equal angles or alternate external or corresponding equal angles then the two lines are parallel.

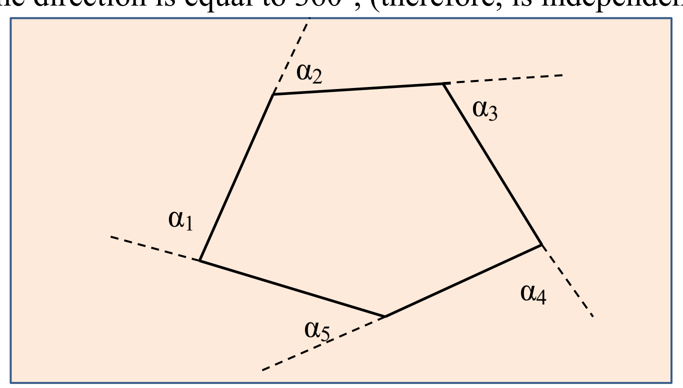
Two angles with the sides respectively parallel are equal only if both angles are acute or supplementary if one of them is obtuse and the second one is acute.

Two angles with the sides respectively perpendicular are equal if both are acute or obtuse, or supplementary if one is acute and the other is obtuse.

The sum of the angles of a triangle is $180^{0.}$

The sum of the interior angles of polygon convex with n sides is equal to $180^0(n-2)$.

The sum of the external angles of a convex polygon formed by the extensions of the sides in the same direction is equal to 360^0, (therefore, is independent of the number of the sides)



$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 360^0$$

Geometrical loci

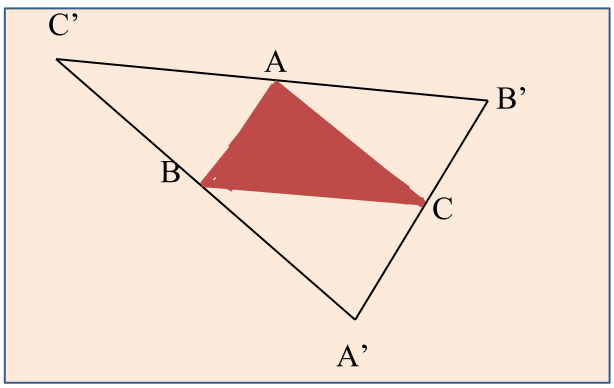
Any point on the perpendicular bisector is equally positioned from the segment's extremities and conversely: any point equally positioned from the extremities of a segment, will be on the perpendicular bisector of that segment.

A geometrical locus is a geometric figure formed by the set of points which have the same property.

Any point on the bisector of an angle is at equal distance from the sides of the angle, and conversely: any point at the equal distance from the sides of an angle it will be on the bisector of that angle.

The geometrical locus of the points at an equal distance of two concurrent lines is formed by two perpendicular lines, the bisectors of the angle formed by the lines.

The anticomplementary triangle to a given triangle ABC is the triangle which is formed by constructing parallels to the opposed sides through each vertex of the given triangle.



$\Delta A'B'C'$ is the anticomplementary triangle of ΔABC.

Two points C, D which divide the side AB in the same ratio are called harmonic conjugate (one interior and the other exterior)

Quadrilaterals

A quadrilateral is a polygon with four sides.

Parallelogram

A parallelogram is a quadrilateral whose sides are parallel two by two.

Properties

The opposite angles of a parallelogram are equal.
The adjacent angles of a parallelogram are supplementary.
The opposite sides of a parallelogram are equal two by two.
(The parallel lines contained between parallel lines are equal)

The diagonals of a parallelogram intersect forming on each diagonal equal segments.

The sufficient and necessary conditions for a quadrilateral to be a parallelogram:

a) The opposite sides to be parallel two b two
b) The opposite sides to be equal two by two
c) The opposite angles to be equal two by two
d) The diagonals to split in equal segments
e) Two sides (opposed to each other) to be equal between them and parallel

Rhombus

The rhombus is a parallelogram with two adjacent equal sides.

Properties

The diagonals of a rhombus are perpendicular
The altitudes of a rhombus are equal.

Rectangle

A rectangle is a parallelogram with a right angle.

Properties

The diagonals of a rectangle are equal

The median relative to the hypotenuse of a right angle triangle is equal to half of the hypotenuse.

In right triangle the cathetus which is opposite to the angle of 30^0 is equal to half of the hypotenuse.

Converse: If in a right angle triangle a cathetus is equal to half of the hypotenuse, then it opposes to an angle of $30^{0.}$

Square

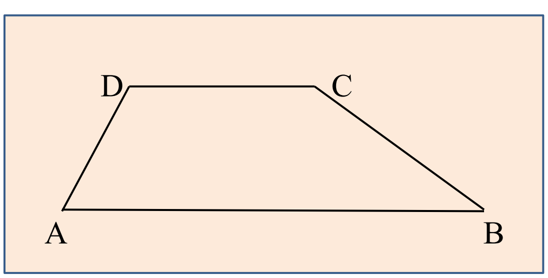
A square is the rhombus with a right angle or a rectangle with two adjacent sides equal.

A polygon regular is a polygon which has all sides equal and all angles equal.

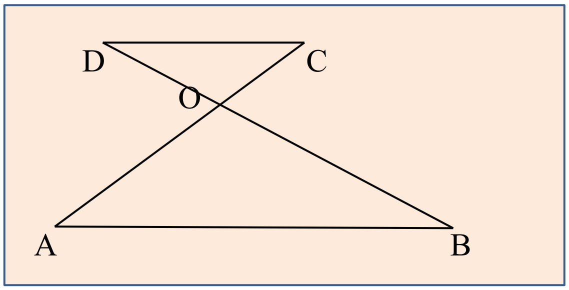
Trapezoid

A trapezoid is a quadrilateral with at least two parallel sides.

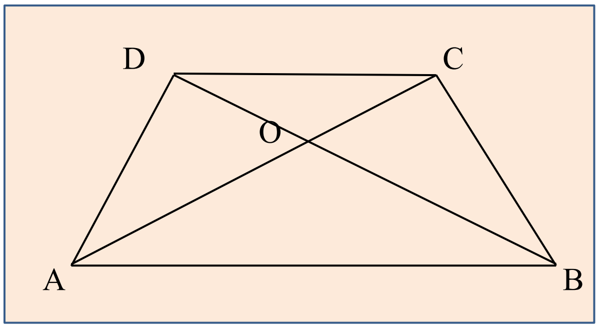
A trapezoid can be convex



or concave

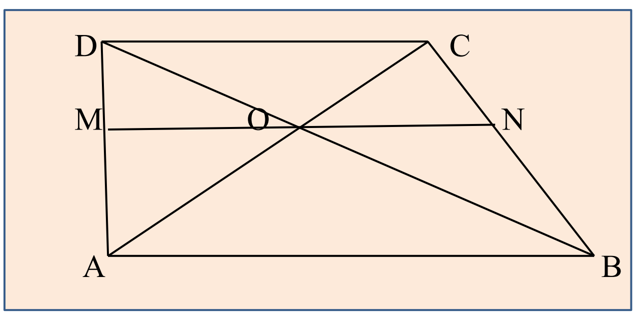


The isosceles trapezoid is the trapezoid in which the sides which are not parallel are equal.



$AO = OB; DO = OC$

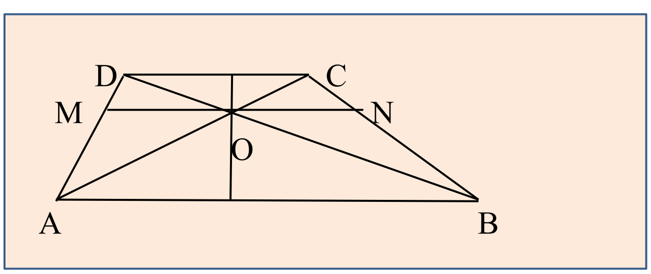
The **right angle trapezoid** is trapezoid which has a right angle (this implies that there is another right angle in that quadrilateral)



The angles adjacent to the bases of an isosceles trapezoid are equal, and the remaining two are supplementary, and converse.

The diagonals of an isosceles trapezoid make with each of the bases equal angles.

Relations in a random trapezoid



$$Aria = \frac{(B+b)\cdot I}{2}$$

If $MON \parallel AB$ then

1) $OM = ON$ 2) $\frac{CN}{NB} = \frac{DM}{MA} = \frac{DC}{AB}$

Geometrical loci

The geometric locus of the set of points at given distance from a line is formed by a parallel line to the given line and at the given distance.

The geometric locus of the set of points for which the ratio of the distances to two parallel lines is constant is formed of two parallel lines to the given lines.

The geometric locus of the set of points for which the ratio of the distances to two concurrent lines is constant and it is formed by two lines which are concurrent to the given lines.

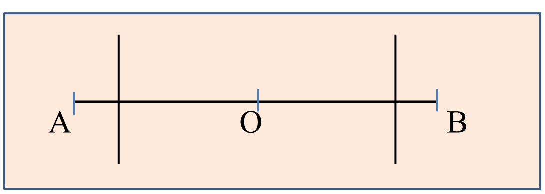
The geometric locus of the set of points from which a line segment can be seen under an angle of 90^0 is a circle which has as diameter the given segment.

The geometric locus from which a given segment can be viewed under a given angle is formed by two arcs of circle of the given angle and which has their extremities coincide to the ends of the segment and are symmetric with respect to the given segment.

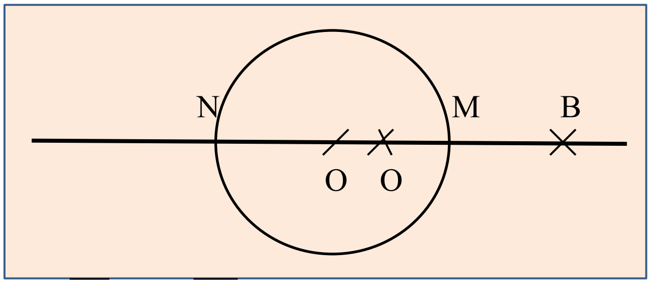
The geometric locus of the points for which the sum of the squared distances to two given fixed points is a constant is formed of a circle whose diameter is the line which connects the given points.

The geometric locus of the points which have equal powers with respect to two given circles is the radical axis of the two circles.

The geometric locus of the points for which the absolute value of the difference of their squared distances to two fixed points is constant, is formed from two perpendicular lines which connect the two points and are symmetric in relation to the segment's middle point which connects the two points.



The geometric locus of the points for which the ratio of the distances to two given points is equal to a given ratio k, is the circle whose diameter is the segment whose extremities are in the point which divides the given segment in the ratios $\pm k$ (the Greek mathematician Apollonius' circle).



$\frac{\overline{NA}}{\overline{NB}} = k$, $\frac{\overline{MA}}{\overline{MB}} = -k$

Circle

The position of point with respect to a circle

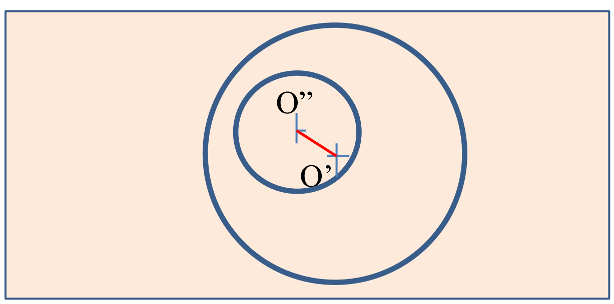
1) Interior to the circle
2) On the circle
3) Exterior to the circle

The position of line with respect to a circle

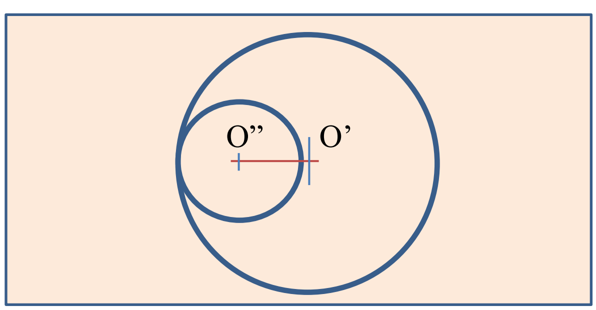
1) Secant (intersects the circle in two distinct points $d < R$
2) Tangent (has just one point in common with the circle $d = R$)
3) Exterior (it does not have any points in common with the circle $d > R$)

The position of two circles

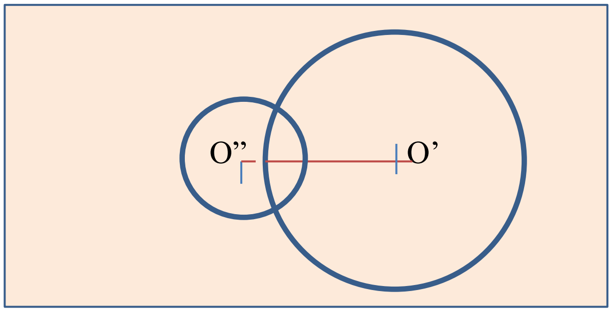
1) Interior circles $d > R_2 - R_1$



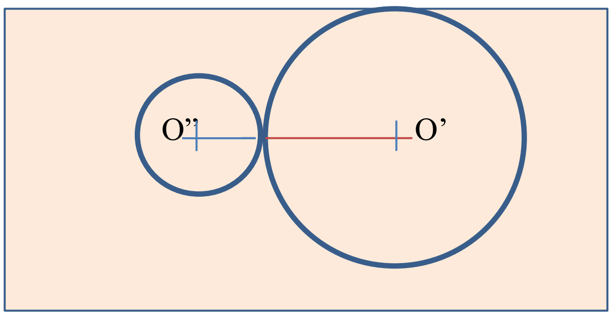
2) Tangent interior $d = R_2 - R_1$



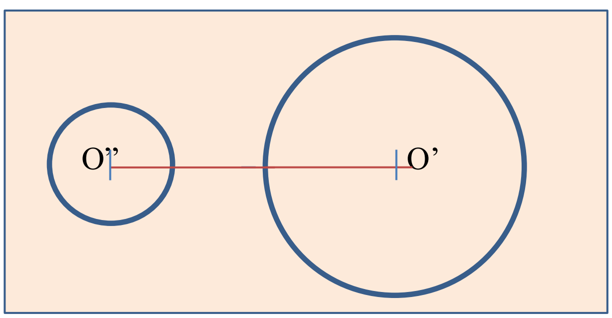
3) Secant $R_2 - R_1 < d < R_2 + R_1$



4) Tangent exterior $d = R_2 + R_1$



5) Exterior $d > R_2 + R_1$



The centers’ line is the line that connects the centers of two circles.
Through a point pass an infinity of circles

Through two points pass an infinity of circles

Through three non collinear points passes only one circle whose center is at an equal distance of the given points, and whose center is at the intersection of the perpendicular bisectors determined by these points.

The perpendicular diameter on a cord bisects it (bisects also the corresponding arc of the cord)

The normal to a circle in one of its points is the perpendicular constructed on the tangent at the circle in that point; therefore it is one of circle's radiuses.

In the same circle or in equal circles, the cords at the same distance to the center are equal.

In the same circle or in equal circles, the unequal cords are at an unequal distance from the center, the largest cord being closest to the center.

The diameter of a circle is the largest cord in the circle.

In the same circle or in equal circles, to any equal arcs correspond equal cords.

In the same circle or in unequal circles to unequal arcs (larger than the semicircle) correspond unequal cords and to the largest arc corresponds the largest cord.

The arcs contained between parallel cords are equal.

The common point of two circles is on the circle's line.

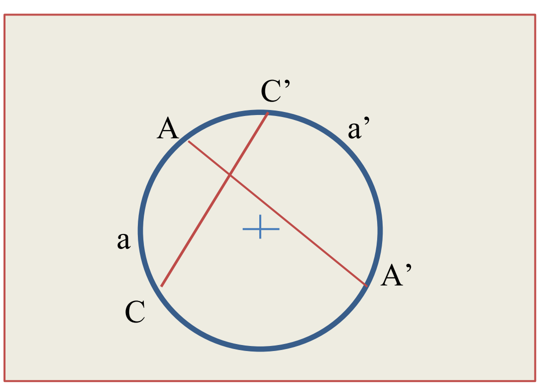
The tangents constructed from the same point to the circle are equal.

The line that connects the middle points of two parallel cords is the diameter perpendicular on the cords.

Angle inscribed in a circle is the angle whose vertex is on the circle and its sides are cords in the circle (or a cord and a tangent).

The measure of angle interior to a circle is equal to half of the sum of the measures of the arcs contained between the sides of the angle and their extensions.

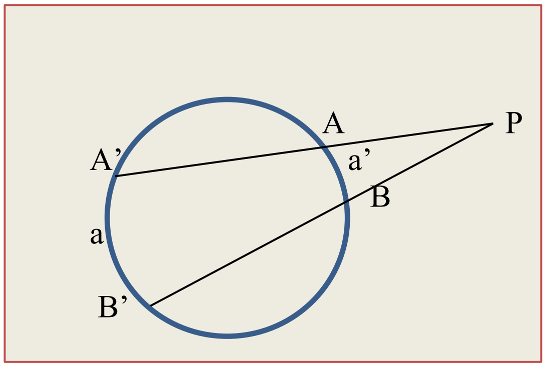
$$m(\sphericalangle ABC)=\frac{m(AaC)}{2}+\frac{m(A'a'C')}{2}=\frac{m(AaC)+m(A'a'C')}{2}$$



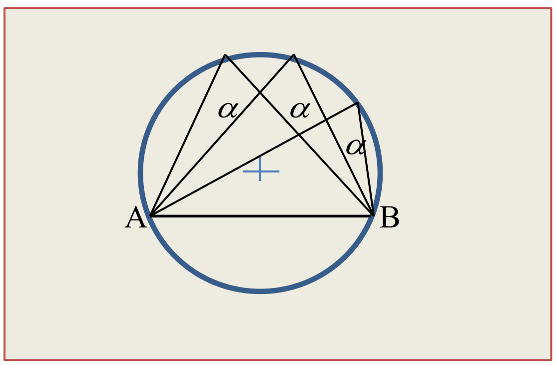
Angle exterior to a circle is the angle whose vertex is in the exterior of the circle, and its sides are secant or tangent to the circle.

The measure of an angle exterior to a circle is equal to half of the difference of the measures of the arcs contained between the sides of the angle.

$$m\left(\sphericalangle A'PB'\right)=\frac{m\left(A'a'B'\right)-m\left(AaB\right)}{2}$$



Arc subtended by a given angle



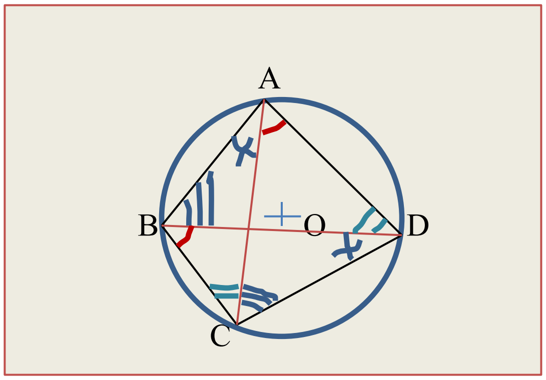
From the points of this arc, a segment can be viewed by the same angle α **.**

Inscribable quadrilaterals

The quadrilaterals which have all their vertexes on a circle are called inscribable quadrilaterals.

In a inscribable quadrilateral the angle formed by one of its diagonal with a side is equal to the angle formed by the other diagonal with the opposite side.

$ab=cd$



The opposed angles of an inscribable quadrilateral make together 180^0 (are supplementary).

First Theorem of Ptolemy

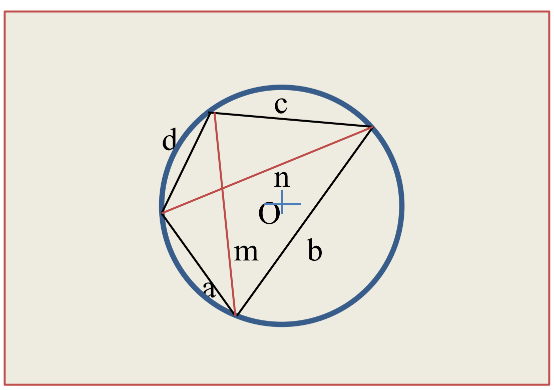
In an inscribable quadrilateral the sum between the products of the opposite sides is equal to the product of the diagonals:

$$a \cdot c + b \cdot d = m \cdot n$$

Second Theorem of Ptolemy

In an inscribable quadrilateral takes place the following relation:

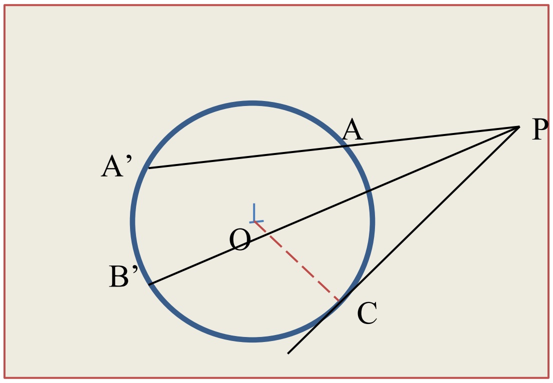
$$\frac{m}{n} = \frac{a \cdot b + c \cdot d}{a \cdot d + b \cdot c}$$



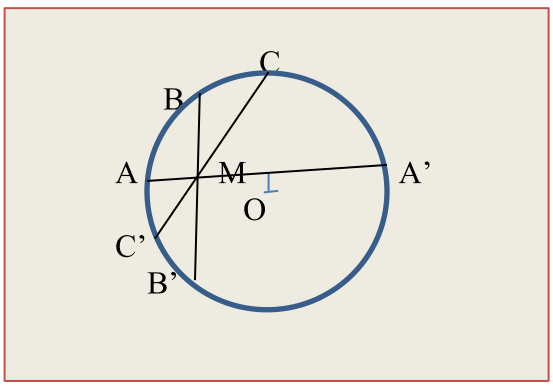
The power of a point in relation to circle

a) When the point is exterior to the circle:

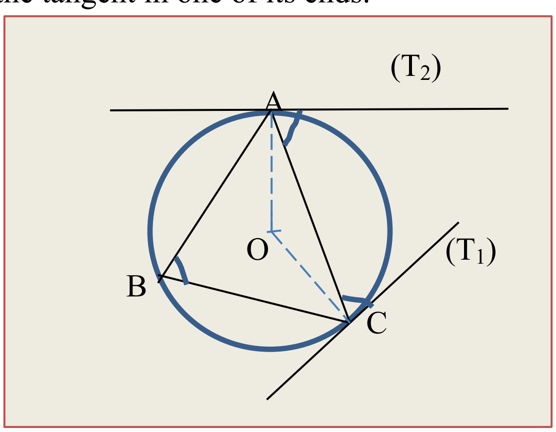
$$PA \cdot PA' = PB \cdot PB' = ... + PC^2 = \text{constant}$$



b) The point is interior to the circle
$MA \cdot MA' = MB \cdot MB' = MC \cdot MC' = \ldots = \text{constant}$

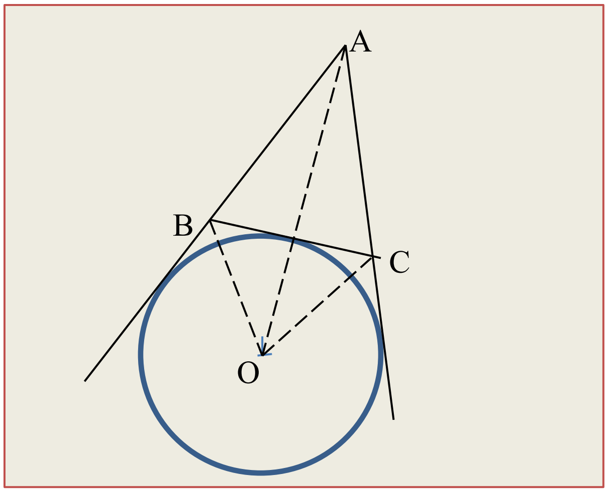


In an inscribed triangle in a circle, an angle is equal to the angle formed by the opposed side to the tangent in one of its ends.



Excircle or escribed circle to triangle

Excircle or escribed circle to triangle is the circle tangent to all sides of the triangle; to a triangle can be escribed three triangles.



$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{R}$$

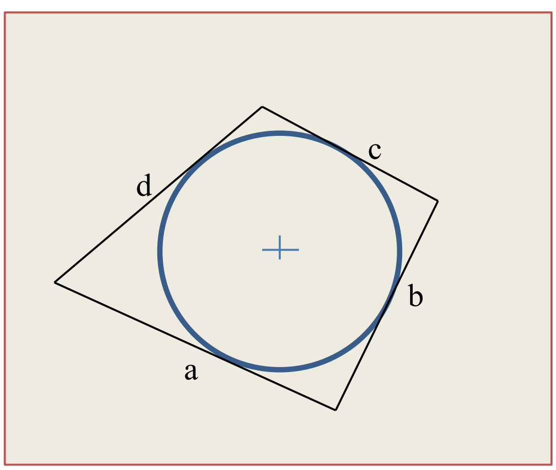
The center of this circle is at the intersection of the interior bisector with the external bisectors of the other angles.

The center of the circumscribed circle to a polygon is at the intersection of all perpendicular bisectors of the polygon. If these are not concurrent, then the polygon cannot be inscribed in a circle.

The center of the circumscribed circle to a right angle triangle is at the middle point on the hypotenuse.

The center of the inscribed circle in a polygon is at the intersection of the interior bisectors of all angles. If bisectors are not concurrent then the polygon is not circumscribable.

If a quadrilateral is circumscribed to a circle, the sum of two opposed sides is equal to the sum of the other two sides and converse.

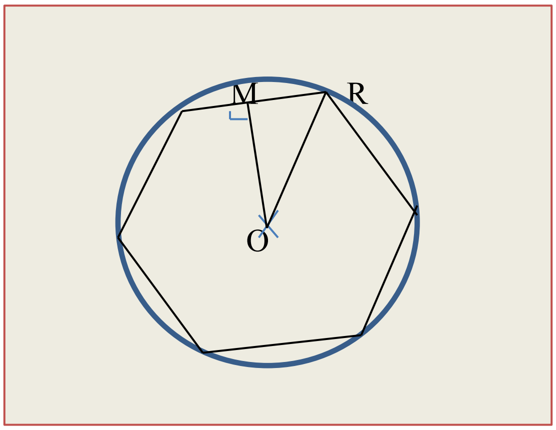


Regular polygons

Regular polygon is a polygon convex with all sides equal and all angles equal.

If a circle is divided in n equal arcs and then connect these consecutive points, we obtain a regular polygon with n sides.

For any regular polygon there is a circumscribable circle, and in any regular polygon we can inscribe a circle.



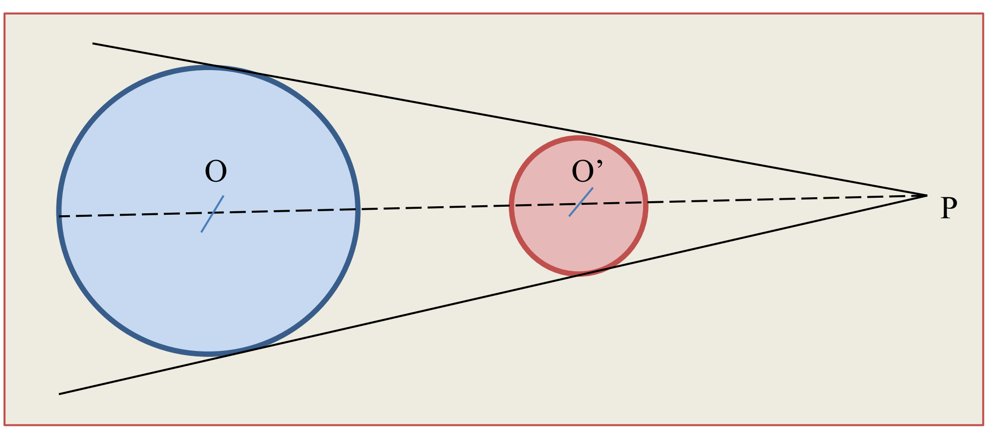
$$l_n = 2R\sin\frac{180°}{n}$$

$$a_n = R\cos\frac{180°}{n}$$

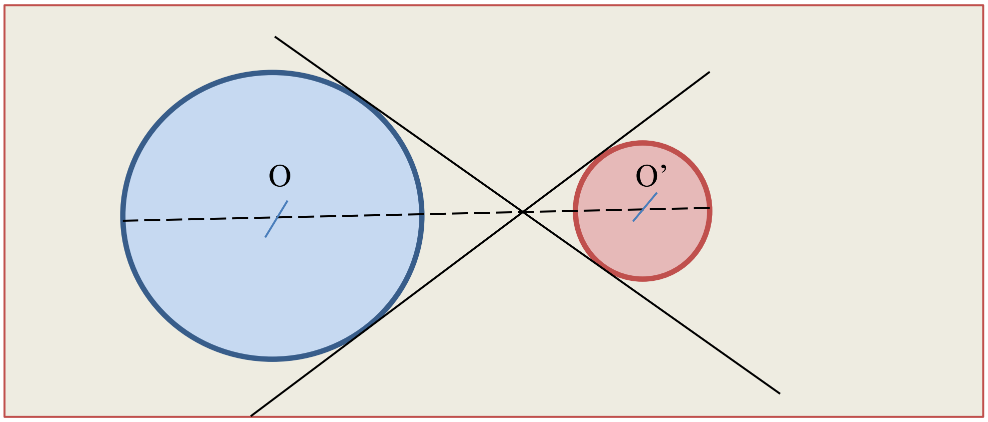
OR is the radius of the circumscribed circle

OM is the apothem of the regular polygon

The common tangent exterior of two circles



The common tangent interior of two circles



For the polygon with n sides (n>3) the sides' equality does not implies their angles' equality.

Dividing a circle in n equal arcs (n>2) and constructing tangents to the circle in these points, it is obtained a regular polygon circumscribed to the circle.

The centers of the inscribed and circumscribed circles to a regular polygon coincide.

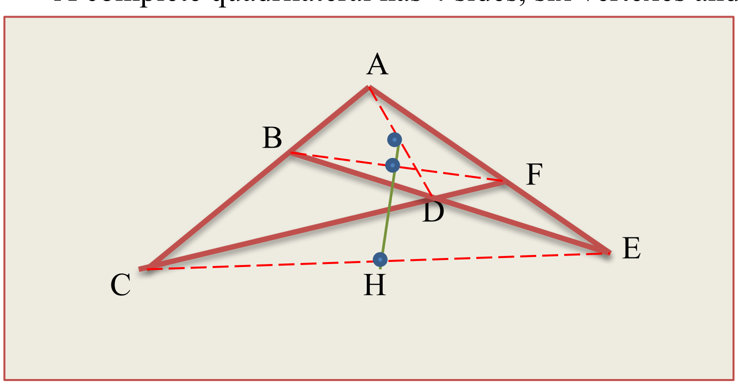
The regular polygons can be:

a) Convex
b) Concave

Quadrilateral complete

A polygon formed by four coplanar lines (three of them should not be concurrent) is called a complete quadrilateral.

A complete quadrilateral has 4 sides, six vertexes and three diagonals



The middle points of the three diagonals are collinear.

Gauss' Theorem

It can be constructed a polygon only with the compass and the straight ruler, being sufficient that the number n of the polygon's sides to be a prime and to have an expression of the following form:

$n = f(m) = 2^{2^m} + 1$, where $m \in N$

Relations

a) $$L_{2n} = \sqrt{2R(R - a_n)}$$

$$a_{2n} = \frac{1}{2}\sqrt{2R(R + a_n)}$$

b) $$L^2 + 4a^2 = 4R^2$$

or

$$L_{2n}^2 = R\left(2R - \sqrt{4R^2 - l_n^2}\right)$$

$$A_{2n}^2 = \frac{1}{4}R\left(2R + \sqrt{4R^2 - l_n^2}\right)$$

where

$$\frac{L}{l} = \frac{R}{a} = \frac{2R}{\sqrt{4R^2 - l^2}}$$

l is the side of the inscribed polygon in O

L is the side of the circumscribed polygon in O

Equilateral triangle inscribed in the circle $C(O)$ and radius R

$$L_3 = \sqrt{3}$$

$$a_3 = \frac{R}{2}$$

$$Aria_3 = \frac{l^2\sqrt{3}}{4} = \frac{3R^2\sqrt{3}}{4}$$

$$h = \frac{l\sqrt{3}}{2} = \frac{3R}{2}$$

The hexagon inscribed in the circle $C(O)$ and radius R

$$L_6 = R$$

$$a_6 = \frac{R\sqrt{3}}{2}$$

$$Aria_6 = \frac{p \cdot a}{2} = \frac{6R \cdot \frac{R\sqrt{3}}{2}}{2} = \frac{3R^2\sqrt{3}}{2}$$

Geometric transformations

Measures:

a) Scalar (is a quantity): length, temperature, mass
b) Vectorial (size, direction, sense): speed, force

Vectors classification:

1) free – have an arbitrary position in space)
2) sliding – have the same line of support
3) Bound – have the same determined origin

Two vectors are equal if they have the same dimension, direction and sense.

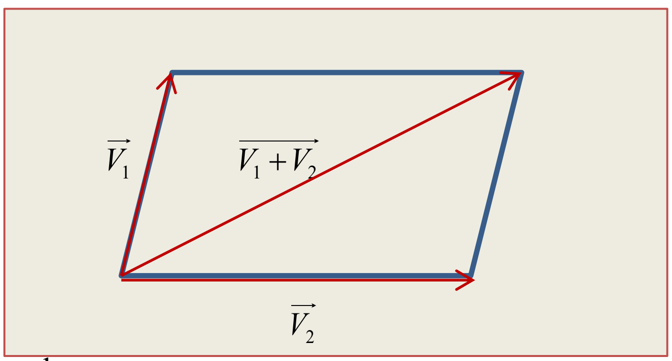
Properties:

a) Reflexive: $\overrightarrow{AB} = \overrightarrow{AB}$
b) Symmetric: if $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow \overrightarrow{CD} = \overrightarrow{AB}$
c) Transitive: if $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{CD} = \overrightarrow{EF} \Rightarrow \overrightarrow{AB} = \overrightarrow{EF}$

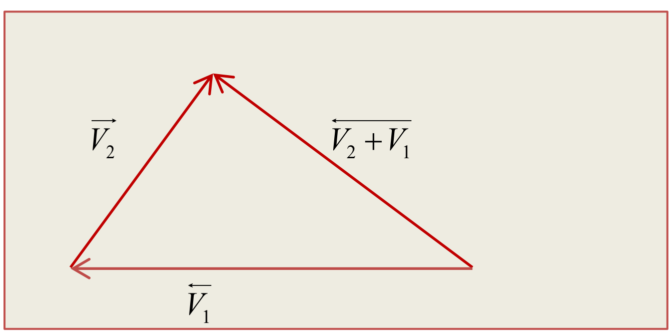
Vectors addition

To add two vectors we use the parallelogram's rule:

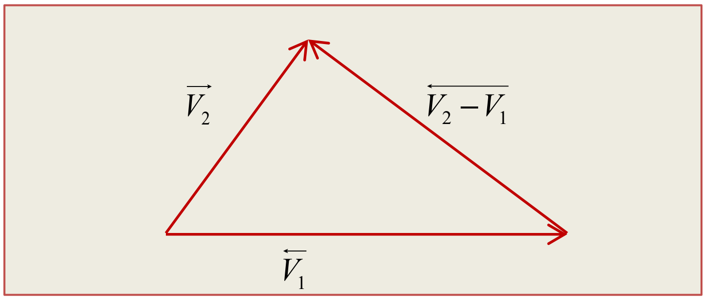
$$\left|\overrightarrow{V_1 + V_2}\right| \leq \left|\overrightarrow{V_1}\right| + \left|\overrightarrow{V_2}\right|$$



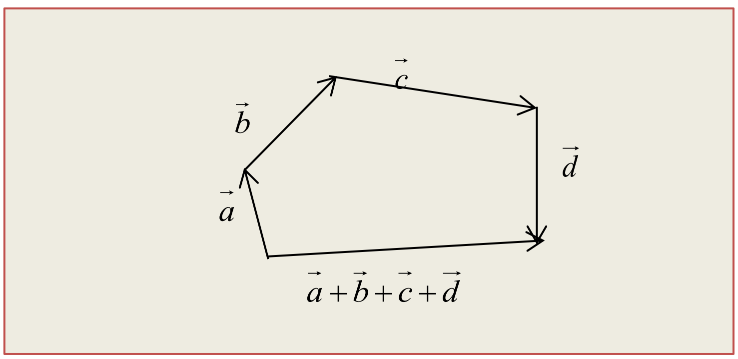
Or the triangle's rule



The difference of two vectors



To add several vectors it is used the polygon's rule: To sum any number of vectors we construct equal vectors with the given ones such that the origin of one coincides with the extremity of the next one. The vector whose origin coincides with the origin of the first vector, and the extremity coincides to the extremity of the last vector represents the sum of the given vectors.

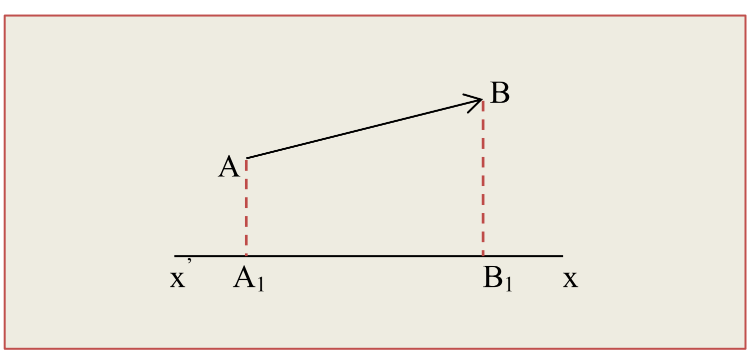


Multiplication of a vector with a number

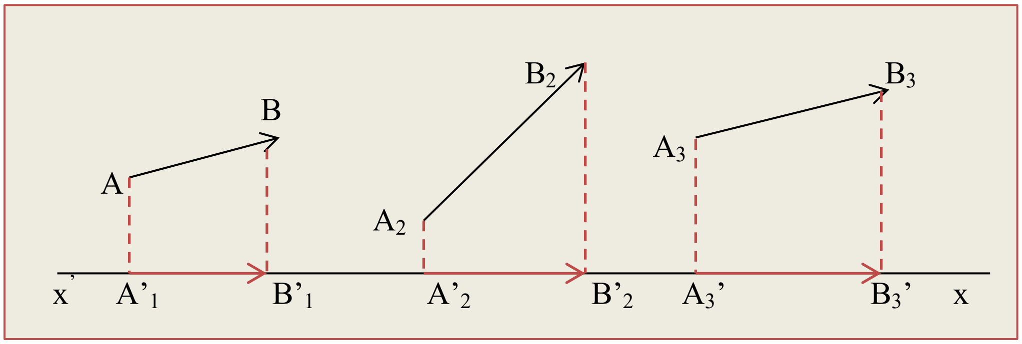
The product between a vector $\vec{a}$ and number λ is a vector whose module is equal to the product between the number's λ module and the module of vector $\vec{a}$, and whose sense is the sense of $\vec{a}$ or -$\vec{a}$, dependent of the sense of λ ($\lambda > 0$ or $\lambda < 0$). For $\lambda = 0$ the product is zero.

The projection of a vector on axis

The projection of a vector $\overrightarrow{AB}$ on the axis x is the size of vector $\overrightarrow{A_1B_1}$. It is denoted by pr. $x\overrightarrow{AB}$.

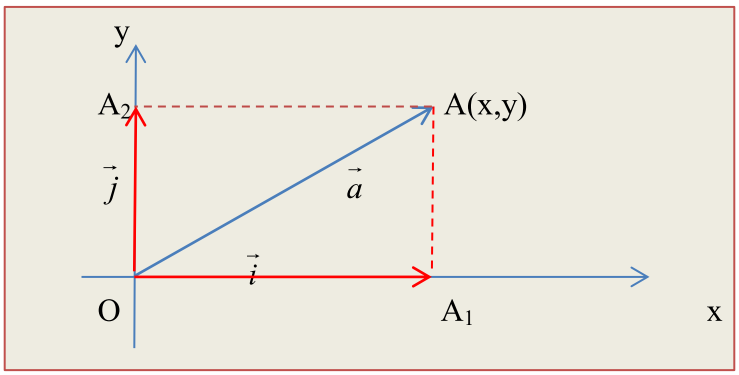


The projections of the vectors which have the same sense, on the same axis are proportional with their module.



$$\frac{A'_1B'_1}{\overrightarrow{A_1B_1}} = \frac{A'_2B'_2}{\overrightarrow{A_2B_2}} = \frac{A'_3B'_3}{\overrightarrow{A_3B_3}}$$

Decomposition of vectors in plane



$$\vec{a} = \overrightarrow{xi} + \overrightarrow{yj}$$

$\vec{i}$ and $\vec{j}$ are the axes' versors.

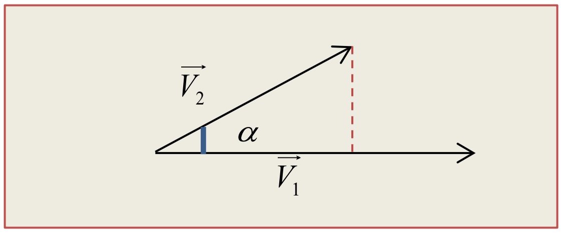
A versor or unit vector is the vector whose module is one (the unity).

The projection of vector sum on an axis is equal to the sum of the projections of the vector terms on the same axis.

The product of two vectors

The product of two vectors is equal to the product between their modules and the cosine of the angle formed by their directions.

$$\overrightarrow{V_1} \cdot \overrightarrow{V_2} = \left|\overrightarrow{V_1}\right| \cdot \left|\overrightarrow{V_2}\right| \cdot \cos\alpha = \overrightarrow{V_1} \cdot pr_{\overrightarrow{V_1}}\overrightarrow{V_2} = \overrightarrow{V_2} \cdot pr_{\overrightarrow{V_2}}\overrightarrow{V_1}$$



Translation

If to a point M in plane corresponds, through a certain rule (algebraic, geometric, mechanical, optical, etc.) a point M' well determined of the plan, we say that this correspondence defines a plane punctual transformation.

The plane translation defined by a vector $\vec{V}$ is the punctual transformation through which to each point A from plane corresponds a point A' from the plane such that $\overrightarrow{AA'} = \vec{V}$ (that is the segment AA' is equal, parallel and of the same sense with the oriented segment $\vec{V}$).

The translation is, therefore characterized by a vector and vice versa.

Through this transformation a figure F is transformed to an equal figure F' equal to it.

The translation conserves the lengths, angles (and the order of the points).

The set of translations form a group commutative.

Symmetry

The symmetry can be:

a) In relation to point
b) In relation to a line
c) In relation to a plane

The symmetry conserves the length and angles; transforms a figure F in an equal figure F' (its identical images)

Homothetic transformation

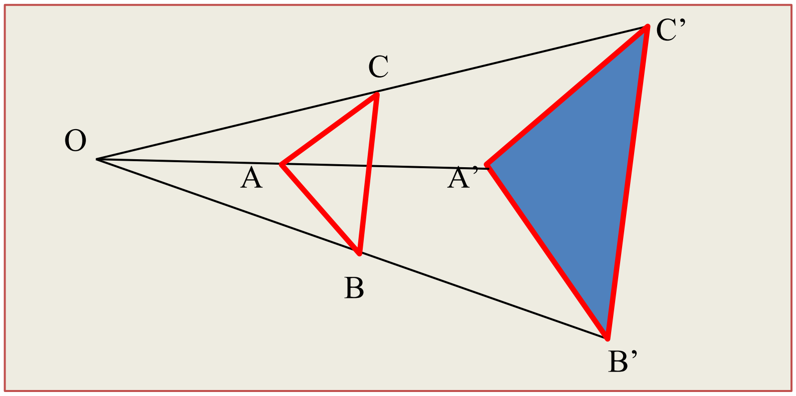
$$\frac{\overline{OC}}{\overline{OC'}} = \frac{\overline{OA}}{\overline{OA'}} = \frac{\overline{OB}}{\overline{OB'}} = \ldots = k$$

The point O is called the homothetic center of the homothetic transformation

F' is called the homothetic transformation of F

The homothetic transformation conserves the angles

k is the homothetic ration (here we have $k = \frac{2}{3}$)



Inversion transformation

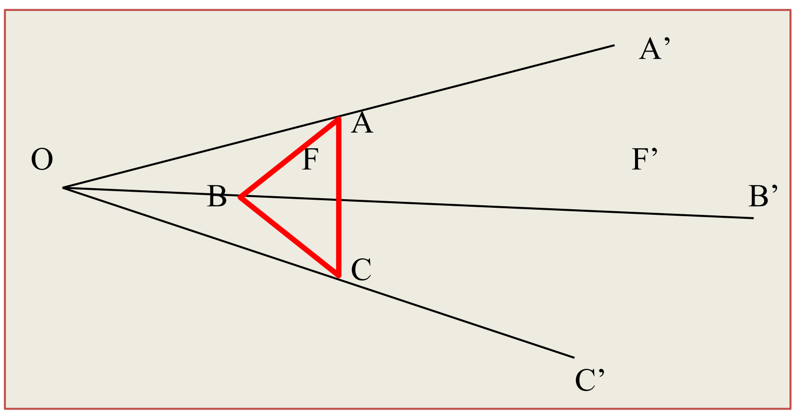
The inversion transformation conserves the angles

$$\overline{OA}\cdot\overline{OA'}=\overline{OB}\cdot\overline{OB'}=\overline{OC}\cdot\overline{OC'}=...=k$$

k is called the inversion power

The point O is called the inversion pole

F' is the inversion transformation of F



Rotation transformation

The rotation transformation of an angle α of the points in a plane in relation to a point (rotation center) is the punctual transformation which makes that to each point A of the plane to correspond a point A', such that $OA'=OA$ and $\sphericalangle AOA'=\alpha$ as measure and sense.

The rotation transformation conserves the lengths and the angles.

The rotation transforms a figure F to other equal figure F'

Conformal transformation

A conformal transformation is a succession of transformations that preserve the local angles.

Any segment constructed through the intersection point of the diagonals of a parallelogram and bordered by the parallel sides of the parallelogram is divided in two equal parts by this point.

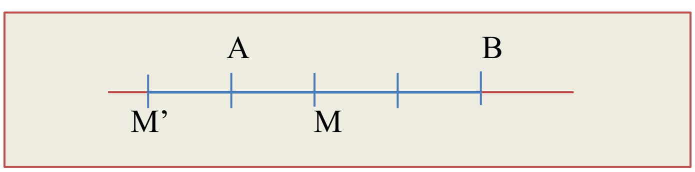
A line segment has as symmetry axis its perpendicular bisector.
An angle's symmetry axis is its bisector.
The isosceles triangle's axis of symmetry is its altitude.
The equilateral triangle has 3 axes of symmetry.
The rhombus has two axes of symmetry, these are its diagonals.
The rectangle has two axes of symmetry: the lines which connect the middle points of the opposite sides.
The parallelogram does not have any axis of symmetry.
The square has four axes of symmetry: two diagonals and the two segments that connect the middle points of the parallel sides.
The isosceles trapezoid has one axis of symmetry: the segment which connects the middle points of its bases.
The circle has an infinite number of axes (its diameters).

Proportional segments

Two segments are proportional with other two segments if the ratio of the first two is equal to the ratio of the other two segments.

The harmonic conjugate points

Two points are harmonic conjugate (one interior, the other exterior to a segment) when they divide this segment in the same ratio.



$$\frac{MA}{MB} = \frac{M'A}{M'B} = k$$

The points M', A, M, B form a harmonic division.

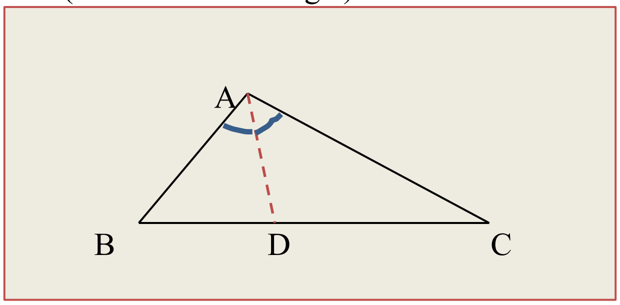
Thales' Theorem

In a triangle, a parallel to one of the sides will divide the other two sides in proportional segments.

Several parallel lines determine on two secant lines proportional segments.

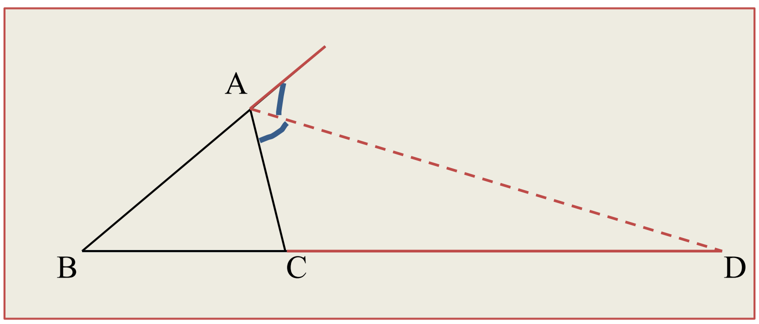
Bisector Theorem

Any bisector interior to a triangle divides the opposite side in proportional segments to the other sides (the sides of the angle).



$$\frac{BD}{DC} = \frac{AB}{AC}$$

Any exterior bisector of an angle of a triangle divides the opposite side of the triangle in proportional segments with the sides that form the angle.



$$\frac{BD}{CD} = \frac{AB}{AC}$$

Similarity of polygons

Two polygons with the same number of sides, whose angles are respectively equal, and the homological sides proportional are similar.

Homological sides are the sides which connect the vertexes of two angle respectively equal.

The cases of similarity of two triangles

Two triangles are similar if any of the following equivalent conditions occur:

1) If their angles are respectively equal
2) The homological sides are proportional
3) An angle equal between proportional sides.

The fundamental theorem of similarity

In a given triangle a parallel line to one of its sides forms with the other two sides a triangle similar to given triangle.

All equilateral triangles are similar.

The isosceles triangles which have the angle from the vertex equal or an angle from the base equal are similar.

The two right triangles, whose ratio of their hypotenuses is equal to the ratio of two catheti, are similar.

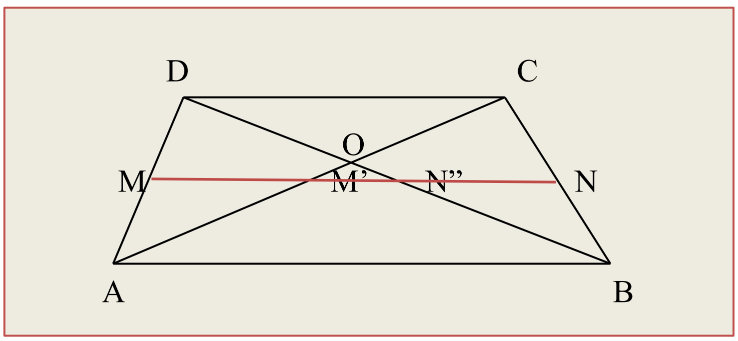
In a triangle ABC two cevians AA_1, AA_2 are isometric if the points A_1, A_2 are symmetric respective to the middle of the segment AB.

Proposition

The isotomic of three concurrent cevians are concurrent (Traian Lalescu).

The middle line of a triangle is the line segment which connects the middle points of two sides; it is parallel to the third line and equal to its half.

The middle line of a trapezoid is the line segment which connects the middle points of the non-parallel sides. It is equal to the half of the sum of the bases; it is parallel to the bases, being, therefore perpendicular on the middle of the height of the trapezoid.



MN is the middle line

$$MN = \frac{AB + CD}{2}$$

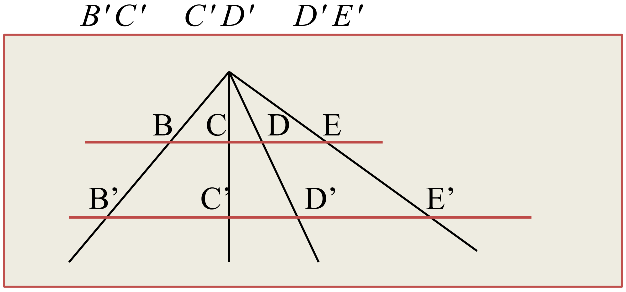
$$M'N' = \frac{AB - CD}{2}$$

Polar triangle

Polar triangle is the triangle formed by the feet of the perpendiculars constructed from a point on the sides of a triangle.

A fascicle of line determines on two parallel lines proportional segments.

$$\frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'}$$



The similarity ratio of two similar polygons is equal to the ratio of their perimeters.

Two similar polygons decompose in the same number of similar triangles, the same arranged.

Metric relations in a right triangle

The cathetus' theorem

In a right triangle a cathetus is the proportional average between the hypotenuse and its projection on the hypotenuse.

The altitude's theorem

In a right triangle, the altitude from the right angle is the proportional average between the segments determined on the hypotenuse.

Pythagoras Theorem

In a right triangle, the sum of the squared catheti is equal to the hypotenuse squared.

Conversely: If in a triangle the square of a side is equal to the sum of the square of the other two sides, then the triangle is a right triangle.

Pythagorean numbers – are the numbers which satisfy the Pythagoras' theorem

b	3	5	7	8	20	9	24	16
c	4	12	24	15	21	40	10	30
a	5	13	25	17	29	41	26	34

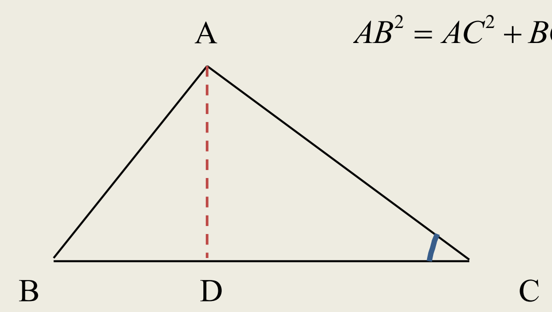
Metric relations in arbitrary triangles

Generalized Pythagoras Theorem

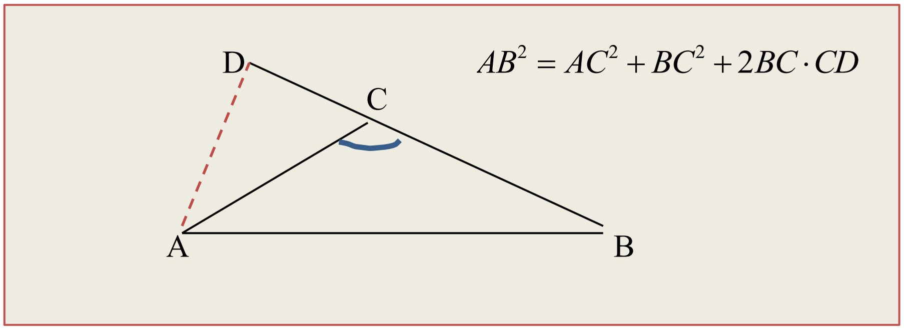
In an arbitrary triangle the square of a side is equal to the sum of the square of the other sides to which is added or subtract twice the product between one side and the projection of the other side on it, depending of the angle opposed to the side is obtuse or acute.

Case I

$$AB^2 = AC^2 + BC^2 - 2BC \cdot CD$$



Case II



Adjacent polygons

Adjacent polygons are the polygons which have one or more sides of line segments in common (without interior points in common).

Two equal polygons have equal area.

Two polygons which have the same area are called equivalent.

The ratio of the areas of two rectangles which have the same base is equal to the ratio of their height (altitudes).

The ratio of the areas of two rectangles is equal to the ratio between the products of their dimensions.

Heron (or Hero) Formula

$$S_{ABC} = \sqrt{p(p-a)(p-b)(p-c)}$$

where $p = \dfrac{a+b+c}{2}$

$$h_a = \frac{2}{a}\sqrt{p(p-a)(p-b)(p-c)}$$

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

$$\sin A \sin B \sin C = \frac{5}{2R^2}$$

$$r_a + r_b + r_c = 4R + r$$

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$tgA + tgB + tgC = tgAtgBtgC$$

Area of a rhombus

$$S_{Rhombus} = \frac{D \cdot d}{2} = L \cdot h$$

Area of a trapezoid

$$S_{Trapezoid} = \frac{(B+b)\cdot h}{2}$$

Aria of regular polygon

Aria of a regular polygon is equal to the semi product between the polygon's perimeter and its apothem

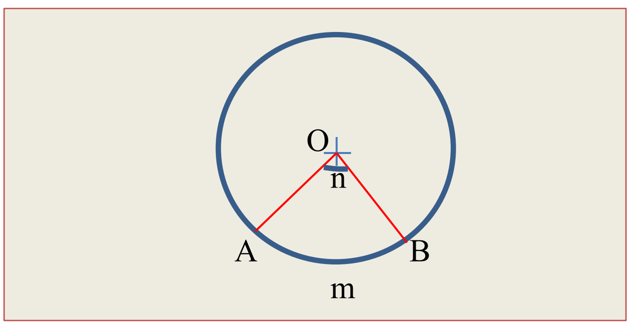
$$S = \frac{p\cdot a}{2}$$

Aria of a circle

$$S = \pi R^2$$

Aria of a circular sector

$$S = \frac{\pi R^2}{360}\cdot n = \frac{len\, AmB\cdot R}{2}$$



The radius of a circle inscribed in a triangle is $r = \frac{S}{p}$, where S is the area of the triangle, and p is triangle's perimeter.

The radius of the circle circumscribed to a triangle is $R = \frac{abc}{4S}$, where a,b,c are the lengths of the triangle, and S is the area of the triangle.

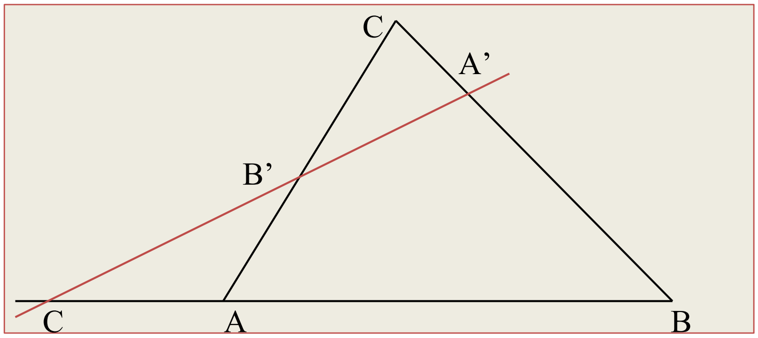
The ratio of the areas of two similar triangles is equal to the similarity ratio squared.

$$\frac{S}{S'} = \left(\frac{a}{a'}\right)^2$$

The ratio of the areas of two similar polygons is equal to the similarity ratio squared.

Menelaus' Theorem (Transversal's theorem)

If a line intersects the sides of a triangle ABC in the points A', B' and C', then between the segments determined on the sides of the triangle takes place the following relation



$$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C'A}{C'B}=1$$

Converse:

If two points are on the triangle's sides and the third on the extension of the third side of the triangle, or all the points are on the extensions of the sides of the triangle.

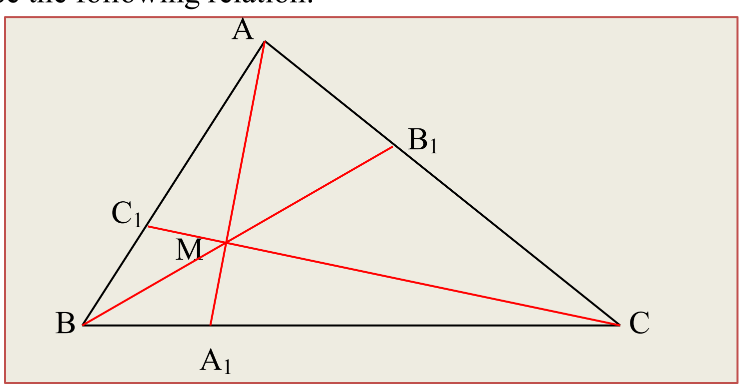
The generalization of Menelaus's theorem (Carnot)

If a line intersects the sides A_i, A_{i+1} of a polygon in M_i, then

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{M_2A_2}{M_2A_3}\cdot\frac{M_3A_3}{M_3A_4}\cdot\ldots\cdot\frac{M_nA_n}{M_nA_1}=1$$

Ceva's theorem

If through the vertexes of a triangle we'll construct three concurrent cevians AA_1, BB_1, CC_1, then between the segments determined by the points A_1, B_1, C_1 on the opposite sides takes place the following relation:



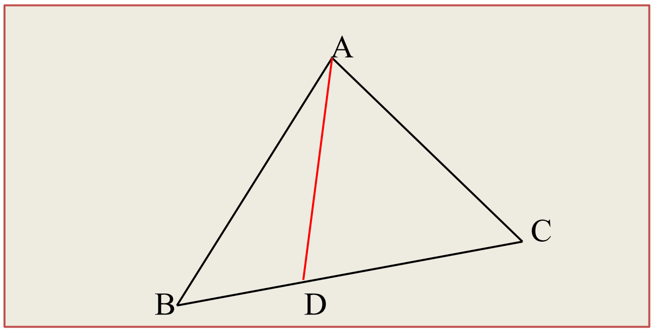
$$\frac{A_1B}{A_1C}\cdot\frac{B_1C}{B_1A}\cdot\frac{C_1A}{C_1B}=-1$$

$$M\in\mathcal{P}(ABC)$$

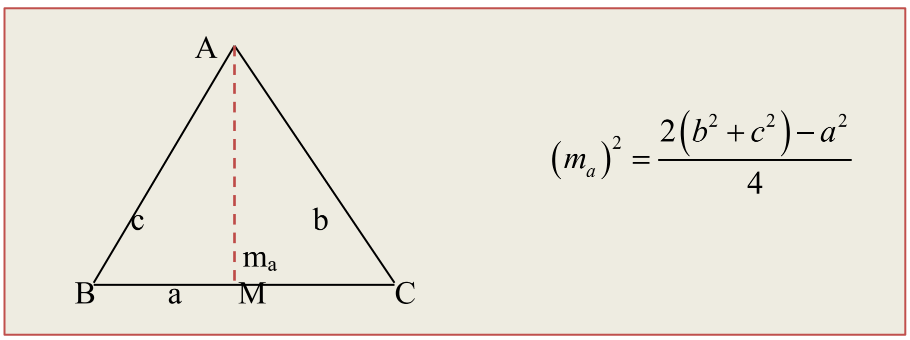
Van Aubel's theorem

In a triangle ABC, let D a point on BC. Between the points D, and B,C takes place the following relation:

$$AB^2\cdot DC+AC^2\cdot BD-AD^2\cdot BC=BC\cdot BD\cdot CD$$

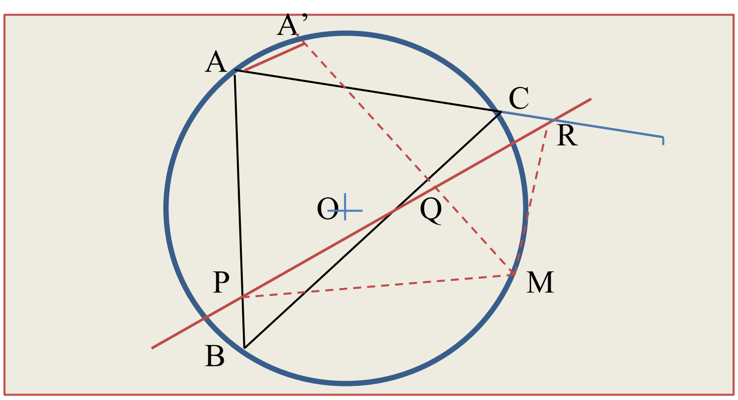


The median's theorem



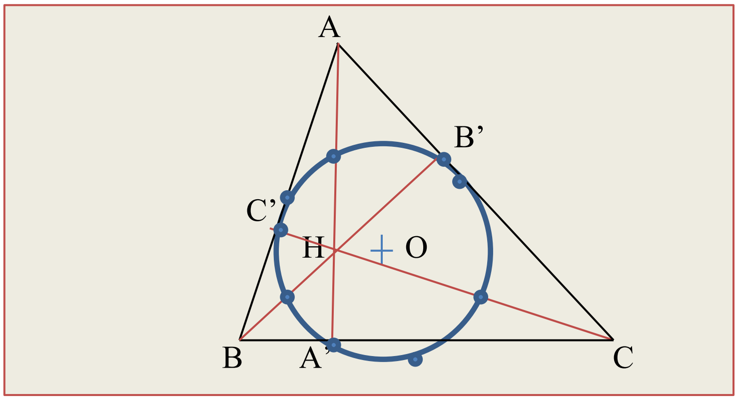
Simson's line

Let ABC a triangle inscribed in a circle and M a point on the circle. The points of the intersection of the perpendiculars constructed from M on the sides of the triangle are three collinear points P,Q,R. The line is called Simson's line



If A' is the intersection of the point M on BC, as above then AA' is parallel with the Simson's line PQR.

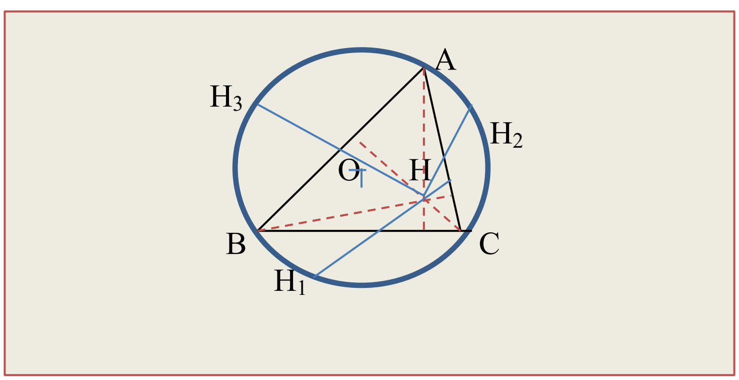
Euler's circle (the circle of the 9 points)



In a triangle, the points where the heights of the triangle intersect, the middle points of the middle of the sides of the given triangle and the middle of the segments AH, BH, CH .are situated on a circle.

The center of direct homothety of the circumscribed and that of the circle of the 0 points is the orthocenter H , and the homothety ration is equal to $\frac{1}{2}$.

Euler's Theorem



The symmetric of the orthocenter in relation to the middle points of the sides are on the center of the triangle's circumscribed circle.

Isogonal lines

Isogonal lines are two lines which pass through the vertex of an angle and form angles equal with the sides of the triangle (are symmetric in rapport to the angle's bisector.

Symmedian

Symmedian is the median's isogonal (the line that connects the vertex with the intersection of the tangents at the edge of the opposite side).

The point of Lemoine

The three symmedians of a triangle are concurrent in appoint called the Lemoine's point –it is isogonal to the center of mass of the triangle.

The distances from the point of Lemoine to the sides of the triangle are proportional.

Antiparallel lines

Two lines AB, CD are antiparallel with respect with the sides of an angle XOY, if angle OAB of the first line with OX is equal to the angle OCD made by the second line with OY.

Two vertexes of a triangle, the center of the inscribed circle I and the center of the ex-inscribed circle of the triangle I_a are on a circle whose diameter is II_a, with the center in the middle or the arc BC of the circumscribed circle. A side BC is viewed from the center I of the inscribed circle under an angle of $90° + \frac{A}{2}$ and from the center I_a of the ex-inscribed circle under an angle of $90° - \frac{A}{2}$.

The cevian of rank k

The cevian of rank k is the line AD with the property that $\frac{BD}{CD} = \left(\frac{c}{b}\right)^k$.

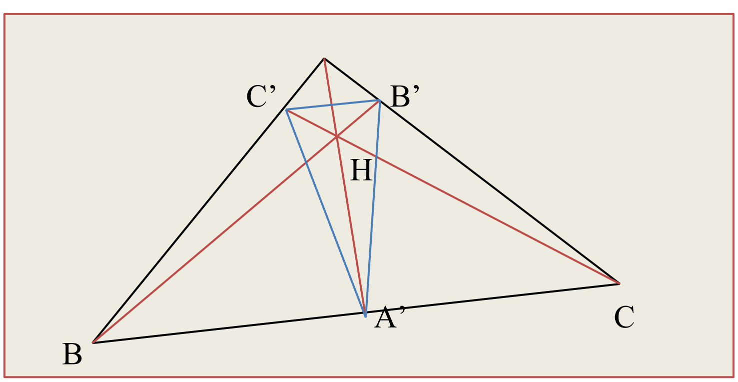
The median = the cevian of rank 0, bisector of rang 1, symmedian of rank 2

The cevians of rank k are concurrent.

If a quadrilateral is circumscribed to a circle, the sum of two opposite sides is equal to the sum of the other two sides and conversely.

The circle which passes through the points B, C and through the orthocenter H of a triangle is the symmetric of the circumscribed circle of the triangle with respect to the side BC.

Orthic triangle

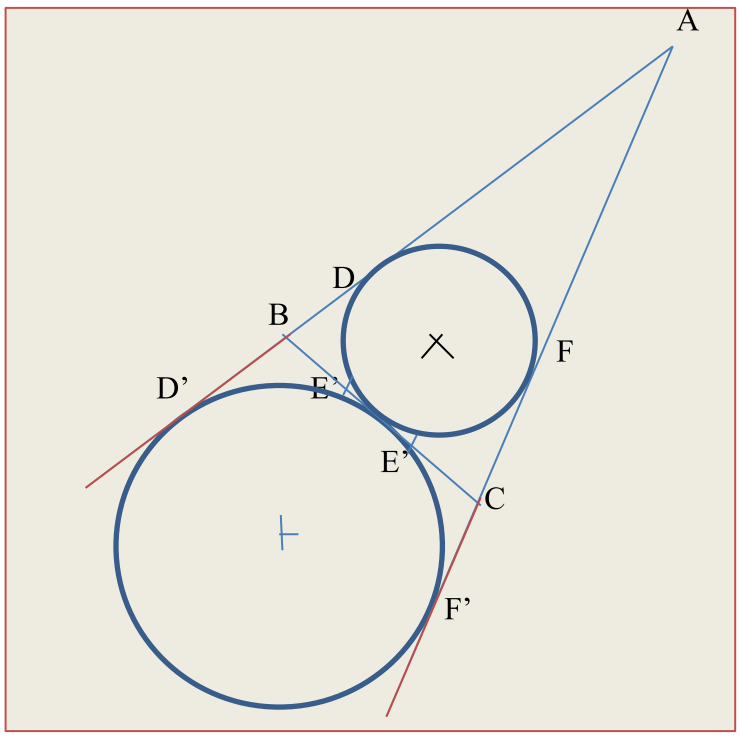


The altitudes of a triangle ABC are bisectors for the triangle orthic $\sphericalangle B'A'C' = 180° - 2A$; the perpendiculars from the vertexes of triangle ABC on the sides of the orthic triangle $A'B'C'$ are concurrent in the center of the circumscribed circle to triangle ABC.

The distance from a vertex A to the orthocenter is twice the distance from the center of the circumscribed circle to triangle ABC o the opposite side.

The symmetric of the orthocenter relative to the middle of the sides are on the circumscribed circle; the segments HA_1 (A_1 diametric opposed to A) and intersect in the middle.

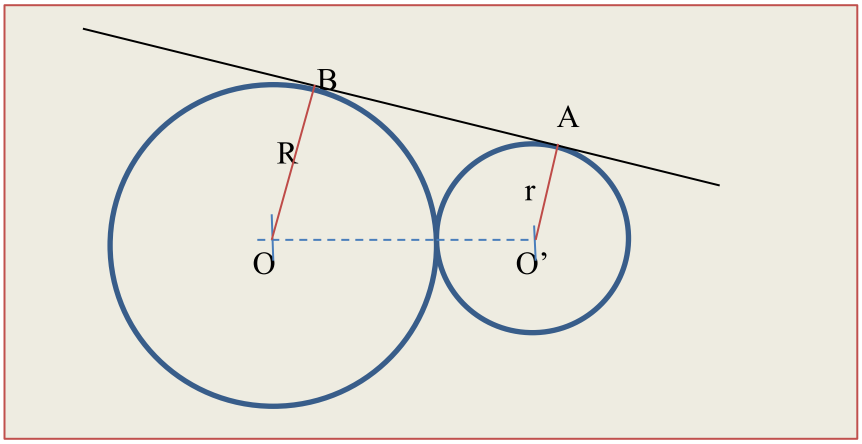
The inscribed triangle ABC - sides a,b,c and perimeter p - touches the sides in the points D,E,F; the ex-inscribed circle in the angle BAC touches the sides in D',E',F'; $AE' = AF' = p$, $AE = AF = p - a$; $BD = BF = CD' = CE' = p - b$



The radiuses of the circles inscribed and ex-inscribed of a write angle triangle are $r = p - a,\ r_a = p,\ r_b = p - c,\ r_c = p - b$

The product of the sides of a triangle is equal to twice the product between the triangle's aria and the diameter of the circumscribed circle $abc = 2Sd$.

The length of the common tangent of two tangent circles is twice of the geometric average of their radiuses: $abc = 2\sqrt{R \cdot r}$



Orthogonal circles

The circles which intersect forming an angle of 90 (that is the tangents to the circles constructed in one of the common points are perpendicular)

$$d^2 - r^2 + r'^2$$

where d is the distance between the centers of the two circles, r and r' are radiuses of the two circles.

The power of a circle whose radius r to an orthogonal circle to is r^2, and conversely.

The circles with the centers on the same line and which have the same radical axis form a fascicle. There is an infinity of circles with the centers of orthogonal axis, orthogonal to the circles in a fascicle; these form a conjugate fascicle to the given fascicle.

Two circles are twice homothetic.

d'Alembert theorem

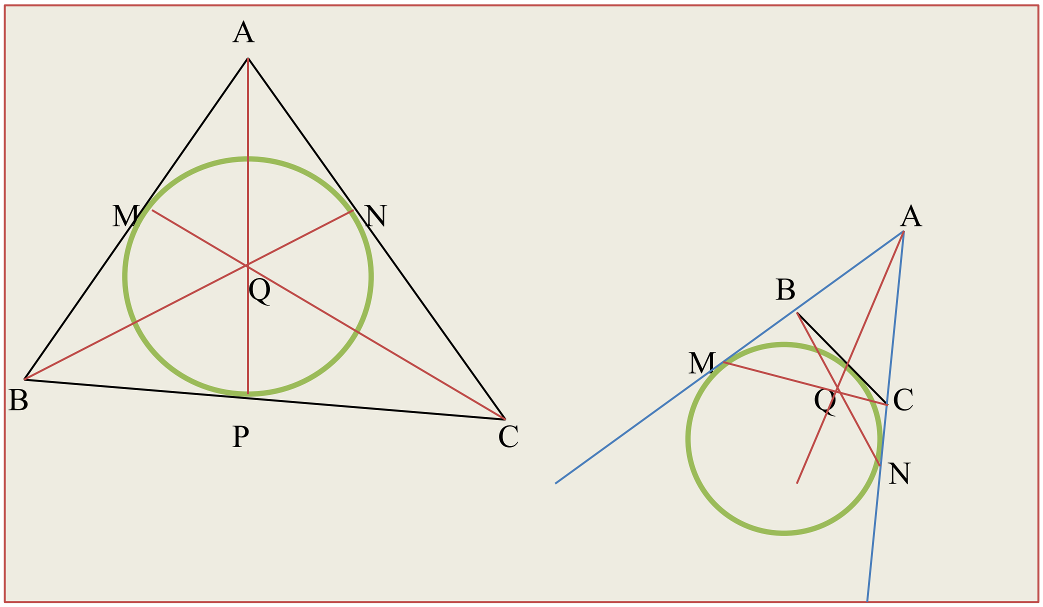
The centers of a direct homothety of three circles taken two by two, are collinear; a center of direct homothety and two of inverse homothety are collinear.

Gergonne's theorem

The lines which connect the vertexes of a triangle with the contact points of the inscribed circle on the opposite sides are concurrent.

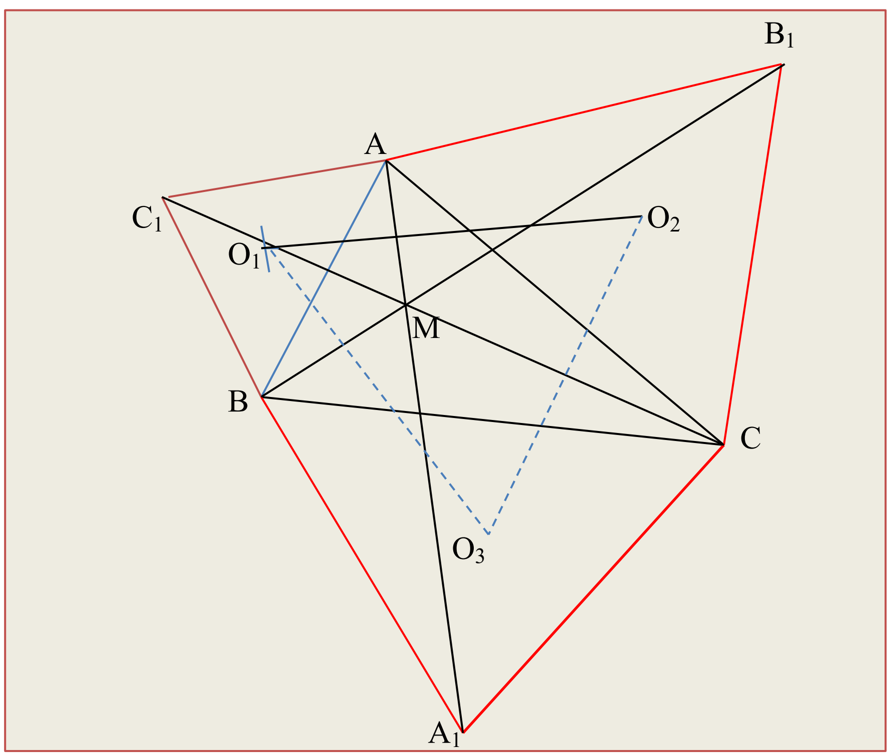
$$AP \cap BN \cap CM = Q$$

Conversely we can say about the lines which connect the vertexes of a triangle with the contact points of the circle ex-inscribed



Torricelli's theorem

If on the sides of a triangle ABC we'll construct equilateral triangles in exterior)or all in interior)l the lines AA_1, BB_1, CC_1, which connect the vertexes of the given triangle to the corresponding vertexes of the equilateral triangles, are equal and concurrent in a point M .



The circle BCA_1, and the similar ones pass through M ; the sides of the triangle are viewed under equal angels from M ; the centers of the equilateral triangles form an equilateral triangle. If M is in the interior of the given triangle, then $MA + MB + MC$ is minimum.

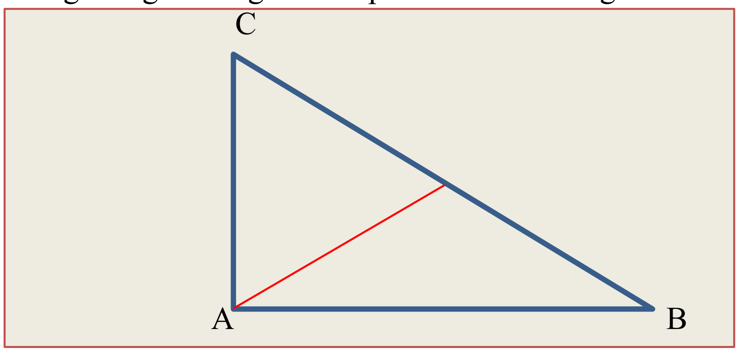
M is called Torricelli's point, and the concurrent circle are called the Torricelli's circles
$O_1O_2O_3$ is the Napoleon triangle

$$O_1O_2 = \frac{a^2+b^2+c^2+4S\sqrt{3}}{6}$$

$$O'_1O'_2 = \frac{a^2+b^2+c^2-4S\sqrt{3}}{6}$$

The relation of Van Aubel

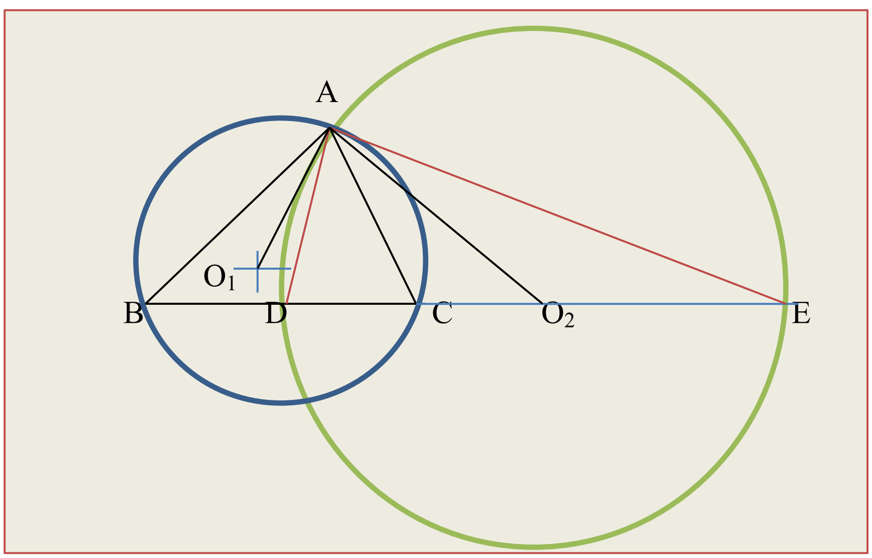
In a right angle triangle takes place the following relation



$$AB^2 \cdot MC^2 + AC^2 \cdot MB^2 = BC^2 \cdot MA^2$$

Apollonius circle

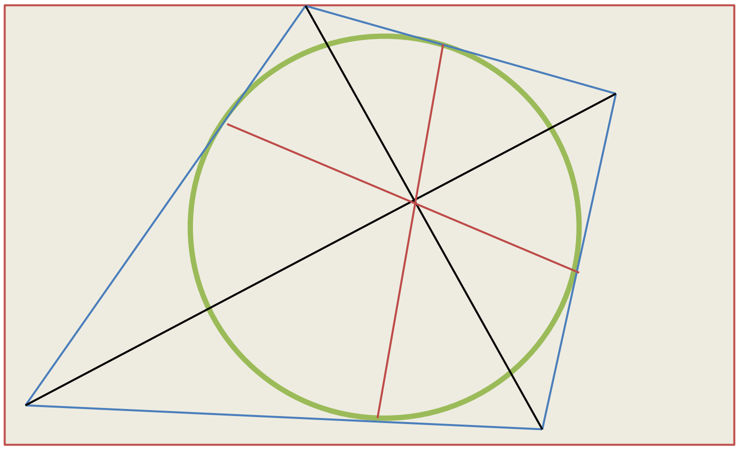
The circle whose diameter is the segment DE formed by the legs of the bisectors of angle A on the opposite side is orthogonal to the circle circumscribed to triangle ABC.



The center of the Apollonius circle is at the intersection of the tangent from vertex A with the side BC. The ratio of the distances of the point of the Apollonius circle to the vertexes B,C is constant and equal to $\frac{AB}{AC}$.

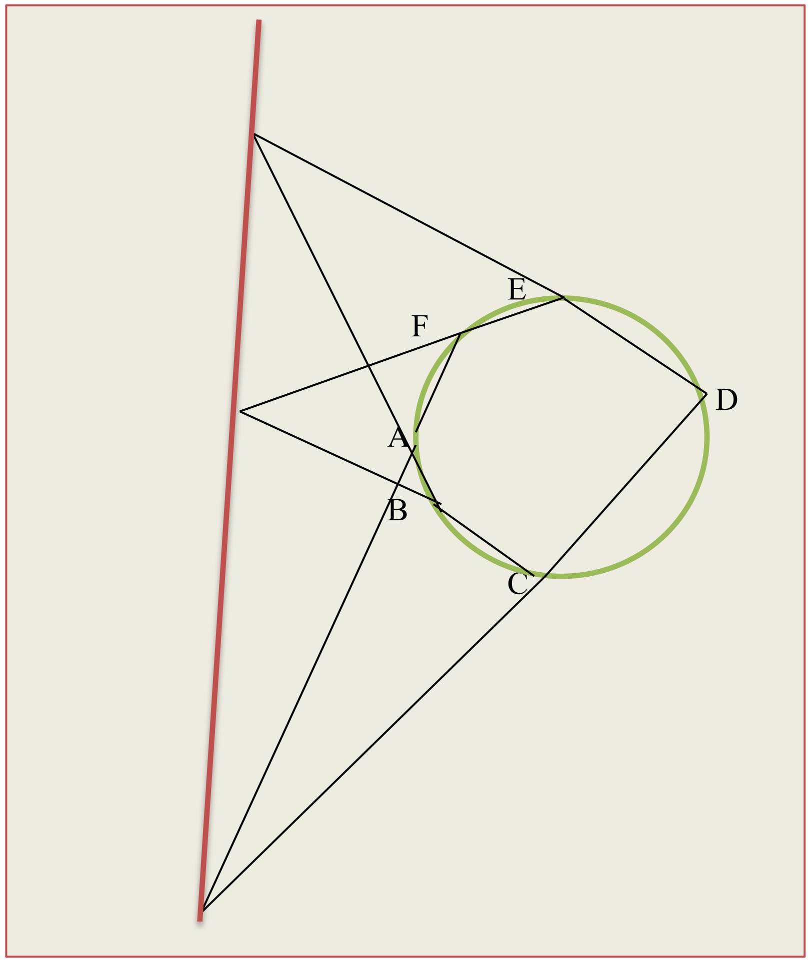
Newton's theorem

In a circumscribed quadrilateral, the diagonals and the lines which connect the contact points of the opposite sides are concurrent



Pascal's theorem

The opposite sides of an inscribed hexagon intersect in collinear points.



Miquel's point

The circumscribed circles to the triangles formed by four lines (complete quadrilateral) have a common point.

The centers of these circles and the Miquel's point are on the circle.

The Lemoine line

The symmedians of a triangle are concurrent (the Lemoine's point). The symmedian divides the opposite line in the ratio of the adjacent sides.

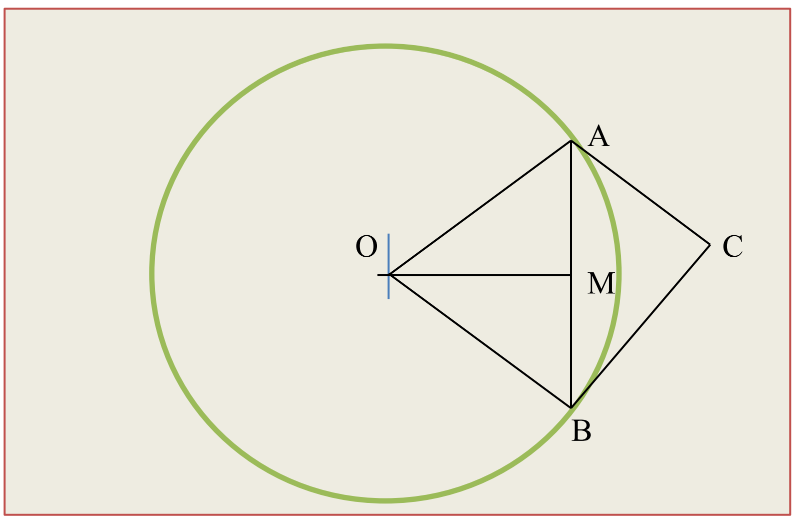
The tangent in a vertex of the triangle is conjugated to the symmedian; divides the opposite sides in the same ratio.

The points of intersection of the tangents with the opposite sides are situated on a line called Lemoine line.

In a right angle triangle the symmedian of the right triangle coincide with the altitude and the Lemoine's point with the middle point of this altitude.

The distances of the Lemoine's point to the sides are proportional to the triangle's sides.

An adjoin circle $\overline{AB}$ is the circle that passes through A and is tangent in B to the side BC of a triangle.



Brocard theorem

The adjoin circles $\overline{AB}, \overline{BC}, \overline{CA},$ are concurrent.

The circles $\overline{BA}, \overline{CB}, \overline{AC}$ are concurrent (the first point of Brocard (retrograde point).

The point of Nagel: The lines which connect the vertex of a triangle with the contact points on the opposite sides of the ex-inscribed circles are concurrent. The points N, G, I are collinear.

Quadrilateral harmonic

The product of two opposite sides is equal to the product of the other opposite sides (quadrilateral inscribable); this equality takes place when the vertices A, B, C, D are in a harmonic ratio (e.g. M being arbitrary on a circle, the fascicle $M(ABCD) = -1$.

A mobile tangent sections four fixed tangents in a given harmonic ratio.

The locus of the conjugates of a point M on a mobile secant constructed through M, in relation to the ends of the determined cord, is a line, called the polar of M, perpendicular on the diameter of the point M. The pole of a tangent is the contact point. The pole of a secant is the intersection of the tangents in the points of the intersection with the circle.

The polar of three collinear points are three concurrent lines, and conversely.

If a line d passes through the pole of a line d', the property is converse The lines d and d' are called conjugate.

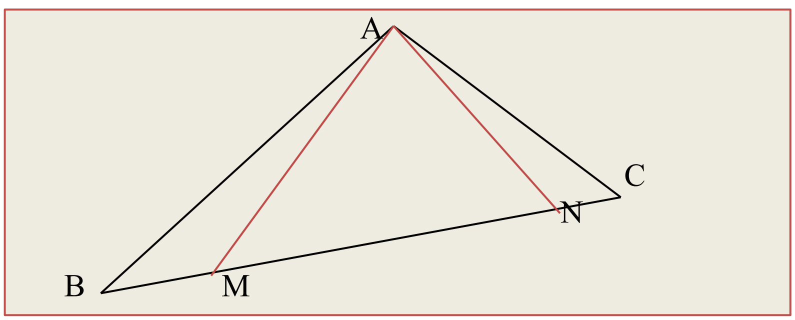
To any figure formed of points and lines corresponds to figure called the dual – formed of lines (the polar of the points) and of the points (the lines pole).

Pappus' theorem

We divide the sides $\overline{BC}, \overline{CA}, \overline{AB}$ of a triangle in the same ratio, through the points M, N, P. The triangle MNP has the same mass center as the given triangle ABC.

Steiner's theorem

Through the vertex A of a triangle, we construct two isogonal cevians, which intersect the side BC in M, N. Then the following relation takes place:



$$\frac{CM}{BM} \cdot \frac{CN}{BN} = \frac{AC^2}{AB^2}$$

Pompeiu's theorem

If the triangle ABC is equilateral and we take a point P which belongs to the plane $P(ABC)$, then the lines PA, PB, PC can determine a triangle. If $P \in \mathcal{C}(A,B,C)$ then the triangle is degenerate (the sum of two sides is equal to the third side)

Generalization (Viorel Gh. Vodă)

$PA \sin A, PB \sin C$ form a random triangle.

The tangents to a circle constructed from the ends of two perpendicular cords form an inscribable quadrilateral.

The tangent in A to the circle ABC is parallel to the line $B'C'$, which is the line obtained by connecting the legs of the altitudes from B, C.

The Geometric locus of the middle point M_1 of segment AM, when A is fixed and M describes a given circle, is a circle od radius $r = \frac{R}{2}$ of whose center is on the middle of the segment OA (O is the center of the circle described by M; R is the radius of this circle).

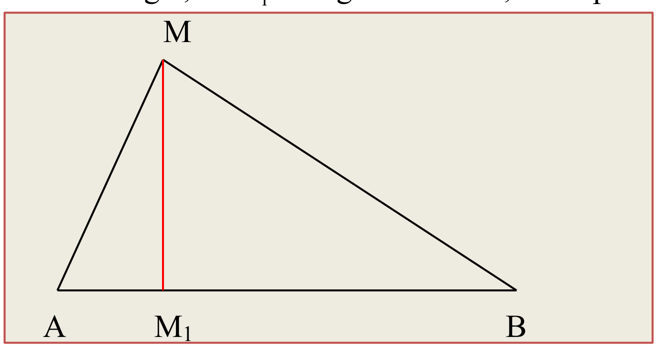
The geometric locus of the center of the inscribed circle to the triangle ABC when B,C are fixed and A is mobile on the circumscribed circle to the triangle ABC is an arc of $90° + \frac{A}{2}$.

Homological triangles

If the lines AA_1, BB_1, CC_1 are concurrent, then the triangles ABC and $A_1B_1C_1$ are called homological. If $BC \cap B_1C_1 = \alpha, CA \cap C_1A_1 = \beta, AB \cap A_1B_1 = \gamma$, then the points α, β, γ are collinear (Desargues' theorem) (the Gergonne point). (The generalization is not true.)

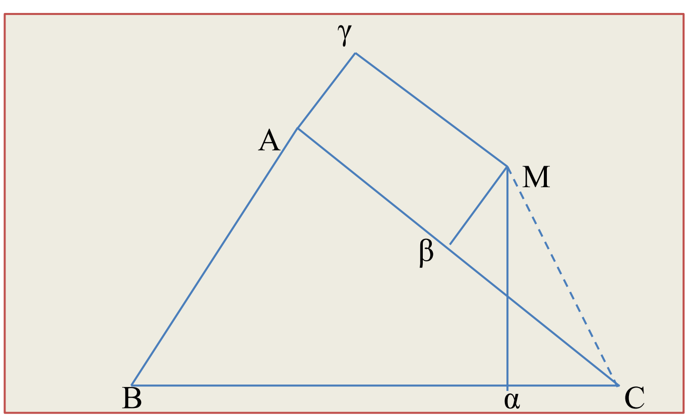
The sum of the squared sides of a parallelogram is equal to the sum of its squared diagonals.

In a random triangle, MM_1 being the altitude, takes place the following relation:



$$MA^2 - MB^2 = M_1A^2 - M_1B^2$$

If the projections of a point M on the sides of a triangle ABC are α, β, γ, takes place the following relation:



$$B\alpha^2 - C\alpha^2 + C\beta^2 - A\beta^2 + A\gamma^2 - B\gamma^2 = 0$$

The orthopole theorem

Let a triangle ABC and a line (d), let A',B',C' the projections of the vertexes A,B,C on (d). The perpendicular from A' on BC and the analogues are three concurrent lines [the concurrency point is called the line's orthopole to respect to the triangle (the Neuberg's point)].

Feuerbach theorem

The circle of the nine points is tangent to the inscribed and ex-inscribed circles.

Geodesic

A geodesic is the smallest length between two points on a surface.

$\frac{1}{r}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}$ (r is the radius of the inscribed circle, h_a is the altitude from point A.

The by-ratio of four points non collinear M_1,M_2,M_3,M_4; $\frac{\overline{M_3M_1}}{\overline{M_3M_2}}:\frac{\overline{M_4M_1}}{\overline{M_4M_2}}$

If $R=-1$, then the by-ratio is harmonic.

Bretschneider's theorem

In a convex quadrilateral $ABCD$ takes place the following relation:

$$d_1^2d_2^2=a^2c^2+b^2d^2-2abcd\cos(A+C)$$

SPACE GEOMETRY

The fundamental elements of space geometry

a) The point $(A,B,C,...)$
b) The line $(a,b,c,...)$
c) The plane $(\alpha,\beta,\gamma,...)$

Definition
The position relations are the relations between the fundamental elements.

Definition
A geometric figure is formed of points, lines, or planes.
Figures can be:
1. Plane, which are figures that can be entirely laid (with all points) on a plane
2. In space, which cannot be laid entirely (with all points) on a plane

The axioms of space geometry

1. Three non-collinear points determine a plane.
2. If two points of a line are on a plane, then the line is entirely contained in that plane.
3. If two planes have a common point, then these have at least one more point in common.
4. In a space there exist at least four non-coplanar points (which are not on the same plane).

Theorem
Two distinct planes, which have a common point, intersect by a line that passes through that point. (The intersection of two distinct planes is a line).

Determination of a plane

a) Three non-collinear points;
b) A line and an exterior point;
c) Two concurrent lines;
d) Two parallel lines.
e) The mobile lines which pass through a point and are supported on a fixed line
f) Mobile lines that are supported on two concurrent lines
g) Mobile lines that are supported on two parallel lines
h) Mobile lines that are supported on a line and are parallel with another line.

The positions of a point with respect to a plane
a) The point is on the plane;
b) The point is exterior of the plane.

The relative positions of two lines
a) Concurrent;

b) Parallel
c) Coincide
d) Arbitrary

Euclid Postulate

Through a given point we can construct only one parallel line to the given line (this is true in the plane geometry as well as in the space geometry).

Theorem

Two lines parallel to a third line are parallel.

Theorem

Two planes which have a common point and which contain two parallel lines respectively intersect in a line which is parallel with the two given parallel lines.

Theorem

Two angles, that are not on the same plane, and which have their sides parallel, are equal.

Theorem

If a line is parallel to plane, the parallel to the given line constructed through a point of the plane it will be contained in the plane.

Definition

A line in a plane divides it in two regions called semi-planes.

Definition

A plane separates the space in two regions; to pass from a region to the other, the plane must be traversed. These regions are called semi-spaces.

Theorem

If two lines are parallel, any plane which intersects one of them will intersect the second line also.

Theorem

A line which is parallel to two planes is parallel with their intersection.

Theorem

Through an exterior point of a plane one can construct an infinity of parallel lines to the given plane; all these lines being situated in the plane which is parallel to the given plane constructed through the exterior point

Definition

The angle of two semi-lines in space is defined as the angle whose vertex is a random point and the sides are parallel and oriented in the same sense as the given semi-lines.

The relative positions of the line in relation to a plane

a) The line parallel with the plane;
b) The line intersects the plane in a point. The line is contained in the plane.

Theorem

If a line is parallel with a line contained in a plane, it is parallel with the plane.

Theorem

If a line is parallel with a plane, the intersection of a plane constructed through the line with the given plane is parallel line to the given line.

Theorem

If two planes have a common point and are parallel with the same line, then their intersection is parallel with this line.

Theorem

If two lines are situated such that any plane which intersects one of them, intersects the second also, their intersection is parallel to this line.

Theorem

The parallel lines to two secant lines of a plane, constructed through an exterior point of the plane, determine a plane parallel to the given plane.

Definition

A class is a subset of all the elements that satisfy the following properties:

a) Symmetry: if $a \parallel b$ then $b \parallel a$;
b) Transitivity: If $a \parallel b$ and $b \parallel c$, then $a \parallel c$.

Definition

The elements of a class are equivalent with respect to the defined relation, which is called relation of equivalence.

Examples of relations of equivalence:

1) The relation of equality
2) The similarity relation

Theorem

All parallel lines between them have the same direction (a line is parallel to itself)

Theorem

The middle points of the sides of a quadrilateral form a parallelogram.

Theorem

If a plane is parallel to the intersection of two planes, then all planes intersections are parallel by two.

Theorem

A line is perpendicular on a plane if it is perpendicular on all the lines of that plane.

Theorem

If a line is perpendicular on two lines of a plane, which are not parallel, then it is perpendicular on all of the lines of the plane, therefore it is perpendicular on the plane.

Theorem

In a point of a line it can be constructed an infinity of perpendicular lines on the given line; all these perpendicular lines are contained in a plane which is perpendicular on the line in that point (called normal plan of the line in the given point).

Therefore, in a point of a line we can construct only one perpendicular plane on that line.

Theorem

Through a point we can construct only one perpendicular line on a plane.

Theorem

Any parallel line to a perpendicular line on a plane is perpendicular on the plan.

Theorem

Two perpendicular lines on the same plane are parallel lines.

Theorem

A line and a plane perpendicular on the same line are parallel.

Theorem

From a point exterior to a plane it can be constructed only one perpendicular line on a plane and multiple oblique lines.

Theorem

The perpendicular line is shorter than any oblique line.

Theorem

Two oblique lines, whose legs are at an equal distance from the leg of the perpendicular, are equal.

Theorem

From the oblique lines, the shortest is that whose leg is closest to the perpendicular line's leg.

Theorem

If two oblique lines constructed from the same point on the same plane are equal, then their legs are at equal distance from the perpendicular line's leg; between two unequal oblique lines, the leg of the shortest is the closest to the perpendicular line's leg.

Symmetry (Punctual transformation)

a) Symmetry with respect to a point (central)
b) Symmetry with respect to a line
c) Symmetry with respect to a plane

Theorem

Two symmetric figures are equal; if the distances of the homological points are equal, but the figures cannot overlap, it results that these are <u>invers equal.</u>

The theorem of the three perpendiculars

If from point A of plane P is constructed a perpendicular line AB on the plane and the perpendicular line AC on a line (D) from the plane, then the line which connects the point C with a random point on the line AB is perpendicular on (D).

Converse I

If from the point B, exterior to plane P, is constructed a perpendicular BA on the plane and the perpendicular BC on line (D) from the plane, then the line AC which connects the legs of the two perpendiculars constructed from B is perpendicular on (D).

Converse II

If in the point C of the line (D) from the plane P are constructed two perpendiculars on it, namely CA in plane and CB exterior, then the perpendicular constructed from a point of CB on CA is perpendicular on the plane.

Theorem

The set of points at an equal distance from two given points is called the mediator plane of the segment formed by the two given points.

Theorem

The set of points at equal distance of three given non-collinear points is a perpendicular line on the triangle's plane formed by the given three points, in the center of the circumscribed circle of the triangle.

Theorem

With respect to four non-coplanar points there exists only one point situated at equal distance from them.

Theorem

The set of points of a plane for which the ratio of the distances to two points is constant is a circle with the center on the line determined by the given two points (The circle of Apollonius).

The relative positions of two planes

a) Parallel
b) Intersect
c) Identical

Theorem
Two planes perpendicular on the same line are parallel.

Theorem
If two concurrent lines in a plane are respectively parallel with two lines from another plane, then the two planes are parallel.

Theorem
If two lines concurrent in a plane are parallel with another plane, then the two planes are parallel.

Theorem
Through an exterior point exterior to a plane it can be constructed only one plane parallel to the given plane.

Theorem
The intersections of two parallel planes through a third plane are parallel lines.

Theorem
Two parallel planes to a third plane are parallel.

Theorem
If two planes are parallel, then any line perpendicular on one of the planes will be perpendicular on the second plane also.

Theorem
The parallel line segments between parallel planes are equal.

Theorem
The distance from a point in a plane to another plane is the distance between the planes.

Theorem
Multiple parallel planes determine on a fascicle of lines proportional segments.

Theorem
If two planes are parallel, then any line which intersects one of them will intersect the second as well.

Theorem
The set of points which are at an equal distance from a plane consists of two parallel planes to the given plane, which are situated on both sides of the plane at the given distance.

Theorem
The set of points at an equal distance from two parallel planes is a parallel plane to the given planes at an equal distance (equidistant plane)

Theorem

Through a point of a line it can be constructed only one plane perpendicular on that line.

Translation

Theorem

The segments which correspond through a translation are equal and parallel.
The translation transforms a figure to another equal figure which is parallel to the given figure; in particular transforms a line into another line, and a plane to another plane.

Definition

Parallel figures are the figures which have the property that the homological sides are parallel.

Rotation

Theorem

In a rotation around a line, the distance between two random points does not change.

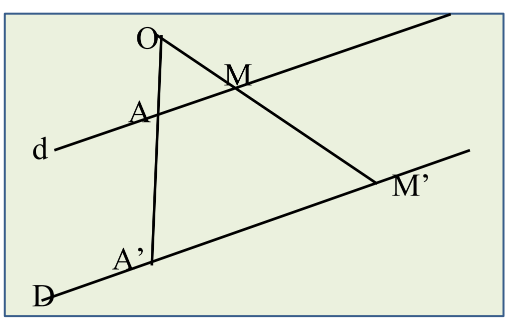
Theorem

Through a rotation a figure is transformed into an equal figure; in particular, a line is transformed into a line, a plane is transformed into a plane.

Homothety

Definition

The homothetic center (O) is the point with respect to which is executed the punctual transformation.



$\frac{OM'}{OM} = \frac{OA'}{OA} k$, where k is the ratio between the distances from the center (O) to the homological points of the figure.

The homothetic of a figure f is the figure which is obtained through the transformation.

Theorem
The homothetic of a line is a line parallel to the given line.

Theorem
The homothetic of a plane is a plane.

Theorem
In a homothety two homological segments are in the same ratio.

Theorem
The figures obtained through a homothety are parallel.

Theorem
The set of the middle points of the segments that rest on two give planes is a plane equidistant from the two given planes.

Theorem
The set of points which divide in the same ratio a line which rests on two given planes is a plane parallel to the given planes.

The set of the middle points of the segments that rest on two given lines is a plane parallel to the given plane.

Theorem
The set of points which divide in the same ratio a line, which rests on two given lines is a plane parallel with them.

Dihedral angles

Definition
A dihedral angle is a geometrical figure formed by two semi-planes bounded by the same line.

Elements of a dihedron

a) The edge (the common line of the semi-planes)
b) The faces of the dihedron (the semi-planes)

Two dihedrons are equal if they coincide when super positioned.

Theorem
The corresponding angle plane of a dihedron is the angle formed by the perpendiculars constructed on the two faces on the edge in the same point.

Theorem
Two equal dihedrons have equal plane angles.

Converse

If two dihedrons have equal plane angles, then these are equal.

Perpendicular planes

Definition
Two planes are perpendicular if their dihedral angle is right.

Theorem
A plane which contains a perpendicular line on another plan will be perpendicular on it.

Theorem
The dihedron angles with parallel faces are equal or supplementary.

Polyhedral angle

Definition
The polyhedral angle is the angle formed by the intersection of multiple planes.

Theorem
If two planes are perpendicular, then any line contained on one of them and perpendicular on their line of intersection is perpendicular on the other plane.

Converse
If two planes are perpendicular, then the perpendicular line from a point of a plane on the other plane belongs to the first plane.

Theorem
The locus of the perpendiculars constructed through a point on a line, is a plane which is perpendicular on the line in that point.

Theorem
Given two lines, in order that through one of them to pass a perpendicular plane on the other line it is necessary that the two given lines to be perpendicular.

Theorem
A plane perpendicular on two planes, it is perpendicular on their intersection.

Converse
A plane perpendicular on the edge of a dihedron is perpendicular on its faces.

Definition
The distance from a point to a plane is the perpendicular constructed from that point on the plane.

Theorem
Two dihedrons are adjacent if these have the same edge and a common face.

Bisector plane

Definition

The bisector plane of a dihedron is the plane that divides it in two equal dihedrons.

Theorem

All plane angles of a dihedron are equal.

Theorem

If two secant planes are perpendicular on a third plane, then their line of intersection is also perpendicular on this plane.

Theorem

Through a line which is not perpendicular on a plane it can be constructed only one perpendicular plan on the given plane.

The set of all equidistant points of two planes consists of other two planes perpendicular between them, which are the bisecting planes of the respective dihedron angle.

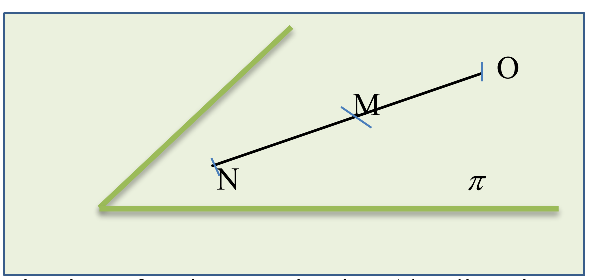
Theorem

The bisecting planes of a trihedron have a line in common.

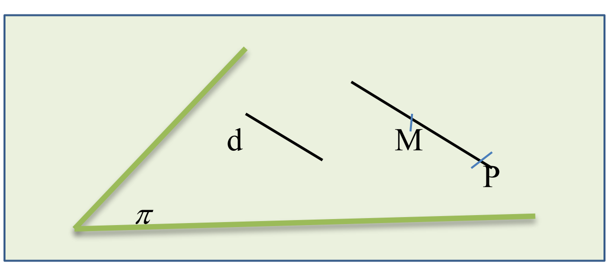
A trihedron has six bisecting planes and four bisectors.

Projections

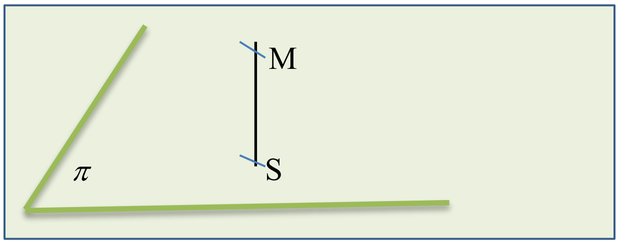
a) Central projection (of center (O), of point M is the intersection of the line ON with the plane π)



b) Projection of a given projection (the direction d is given)



c) Orthogonal projection (we take the intersection of the perpendicular with the plane π - in short we call it projection)



Definition

It is called orthogonal projection of a point A on a plane π the leg of the perpendicular constructed from A on the plane π.

Theorem

The projection of a figure is the locus of the all the projections of all the points of the figure.

The projection on a plane of a line, which is not perpendicular on the plane, is a line.

If the line is perpendicular on the plane, its projection is a point (the point in which it pricks (intersects) the plane).

If the line is parallel to the plane, its projection is equal with the given line.

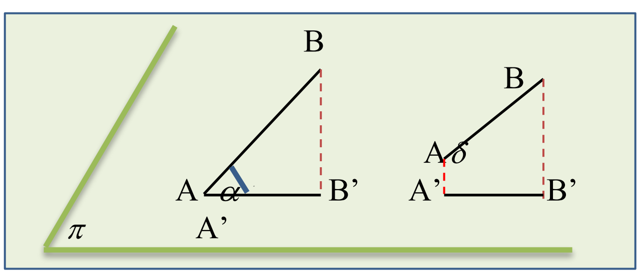
If the line is oblique with respect to the plane, the projected line is smaller in length.

The projection of a geometric figure is also a geometric figure.

The projection in plane of a space figure is a plane geometric figure.

Projected plane δ

The plane from which a line is projected on another plane (these are perpendicular among them)

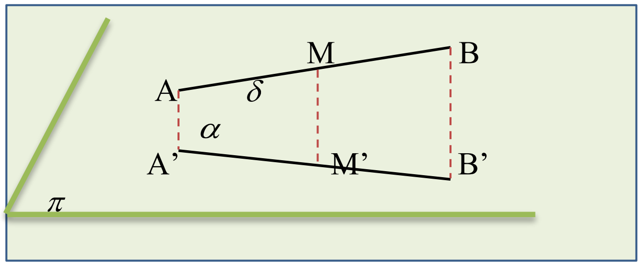


$A'B' = AB\cos\alpha$

The projection of a segment is equal to the segment’s length multiplied by the cosine of the angle of the line with the plane.

If the projection of a line on a plane is perpendicular on a line from the plane, then the line itself is perpendicular on the line in the plane.

In a projection the ratio of two segments on the same line is the same (maintained).



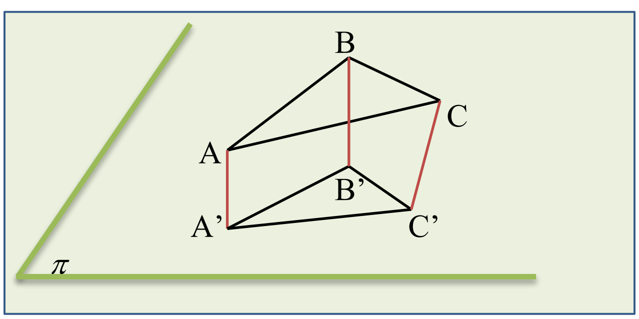
$$\frac{MA}{MB} = \frac{M'A'}{M'B'}$$

Definition

The angle formed by a line with a plane is the angle formed by the line with its projection on the plane.

The angle formed by an oblique line with its projection is the smallest angle formed by it with any other line contained in that plane.

The area of the projection of a triangle on a plane is equal to the area of the triangle multiplied by the cosine of the dihedron angle formed by the triangle’s plane with the plane of projection.



$$S' = S\,cos\,\alpha$$

Theorem

A right angle with a side parallel to a plane is projected in this plane by a right angle.

Theorem

An angle which has a side parallel to a plane and its projection with a right angle is a right angle.

The planes which project the edges of a trihedron on the opposite faces have a line in common.

Theorem

If two lines are not coplanar, then there exists only one line which rests on both lines and is perpendicular on both. This line is called common perpendicular of the two lines. It is the smallest distance between two points which belong respectively to the two lines.

If the lines are secant coplanar, their common perpendicular is the perpendicular constructed on their point of intersection on the plane determined by them.

If the lines are parallel, then all perpendicular lines to these lines constructed in their plane, are common perpendicular lines.

Polyhedron

Definition

The polyhedron is a geometric figure formed of many faces (polygons).

There are:

a) Regular polyhedron (its faces are regular equal polygons and its dihedron angles are equal)
b) Non-regular

There are five types of regular polyhedrons:

1) Tetrahedron –solid with four faces (triangles)
2) Hexahedron – solid with six faces (squares)
3) Octahedron - solid with eight faces (triangles)
4) Dodecagon – solid with 12 faces (pentagons)
5) Icosahedron – solid with 20 faces (triangles)

Theorem

Two polyhedrons are equal if by overlapping coincide.

Theorem

The faces of a convex polyhedron are convex polygons.

A line intersects a convex polyhedron at most in two points (if it is not contained on a face).

Definition

A prism is a polyhedron bounded by two bases, equal polygons situated in parallel planes, and by the parallel faces whose number is equal to the number of the sides of the base.

Definition

The prismatic surface is the set of all parallel lines with a given line, and which pass through a closed polygonal line.

Definition

The closed polygonal line is the directional line of the prismatic surface.

Definition

The parallel lines constructed to the given line through the points of the directional line are called the generators of the prismatic surface.

Definition

The generator lines which contain the vertexes of the directional line are called the edges of the prism.

Definition

The intersection of a prismatic surface with n faces with plane which is not parallel to the generators is a polygonal line with n sides.

The classification of prisms

a) The name of the polygon at the base:
   - Triangular
   - Quadrilateral; Example: the parallelepiped whose faces are all parallelograms (random, rectangular)
   - Pentagonal
   - Hexagonal
   - Etc.
   - The polygon with n sides

b) The position of the lateral edges with respect to the base plans
   - Right (the lateral edges are perpendicular on the base plan), the lateral faces are rectangles.
   - Oblique (the lateral edges are oblique with respect to the base plan)

The rectangular parallelepiped is the prism whose base is a rectangle (therefore all faces are rectangles)

The cube is a right prism with three equal concurrent edges in the same vertex. All the faces of a cube are squares.

Definition

The prism height is the perpendicular constructed from a plan to another.

Definition

The prism diagonal is the line segment which connects two vertices which don't belong to a lateral face of the prism.

Definition

The prism's section is the surface obtained by intersecting the prism with a plane.

The section parallel to the prism's base is a polygon equal to the base.

Two parallel planes which section a prismatic surface determine two equal polygons.

Definition

The interior points with respect to a close line P are the points from which any semi line intersects this line.

Definition

The exterior points with respect to a close line P are the points from which it can be constructed semi lines which intersect this line.

Parallelepiped

The parallelepiped is a prism with the base a parallelogram.

Definition

The right parallelepiped is the prism with the base a rectangle.

Properties

1) The four diagonals of a right parallelepiped are concurrent in a point, which is the middle point of each of them, and which is called the symmetry center.
2) The diagonal planes pass also through the symmetry center.
3) The parallelepiped diagonals are equal: $d^2 = a^2 + b^2 + c^2$, in particular for the cubic prism: $d = a\sqrt{3}$.
4) The lateral area of a right prism $A_l =$ The base perimeter x height; the total area $A_t = A_l +$ the area of the two bases; the volume $V =$ the base area x height = the base area x side.

Definition

A polyhedral surface is formed of the polygons on different plans, and which have common sides.

Observation

Two equal solids have the same volume.

Theorem

If two solids (equal or not equal) have the same volume, then these are equivalent.

A pyramidal surface α is determined by a point V and of a polygonal closed plane line $A_1A_2...A_nA_1$, V does not belong to the plane. The surface is made of the set of all the points of the lines which contain the point V and one point of the line $A_1A_2...A_nA_1$.

The edges of the pyramid are the generator lines which pass through the base vertices.

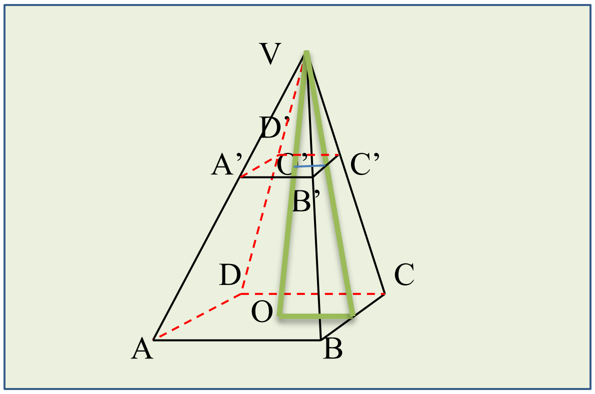
The height is the perpendicular line constructed from the point V on the base plan.

Theorem

A plan which intersects all the edges of a pyramidal surface with n edges intersects the pyramidal surface by a polygonal closed line with n sides.

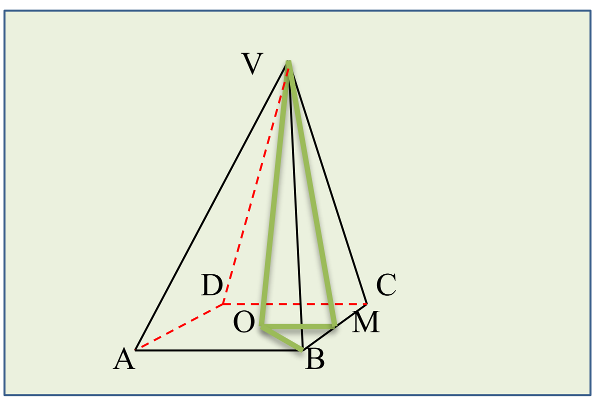
Theorem

The ratio between the section's area and the base's area are equal to the ratio between the squared heights (when the section is parallel to the base).



$$\frac{AriaA'B'C'D'}{AriaABCD} = \frac{VO'^2}{VO^2}$$

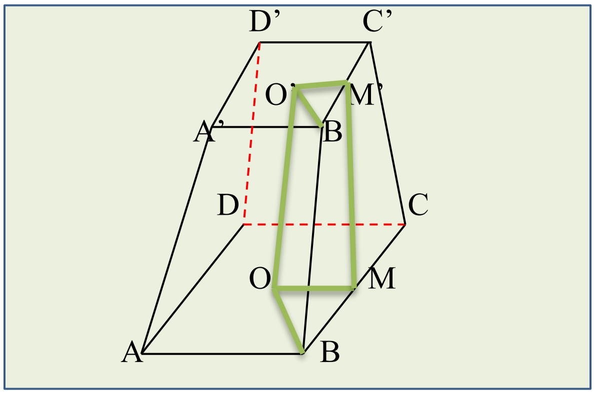
Right triangles which form in a regular pyramid are:



1) ΔVOM (The pyramid height, the base apothem, and apothem of the pyramid).
2) ΔVOB (The pyramid height, the radius of the circumscribed circle to the base, the lateral edge of the pyramid).
3) ΔVMC (The pyramid height, the lateral edge of the pyramid, and half of the base's side).
4) ΔOMC (The base apothem, the circumscribed circle to the base, and half of the base's side).

The section produced by a plane parallel to the base, which intersects one of the lateral edges of the pyramid, is a polygon whose sides are parallel to the sides of the base and which is similar with the base.

Trapezoids which are formed in regular truncated pyramid are:



1) Rectangular trapezoid $OO'M'M$ (the height of the truncated pyramid, the apothems of the bases, apothem of the truncated pyramid).
2) Rectangular trapezoid $OO'B'B$ (the height of the truncated pyramid, the radiuses of the circumscribed circles to the bases, the lateral edge of the truncated pyramid).
3) The isosceles trapezoid $AA'B'B$
4) He isosceles trapezoid $AA'C'C$

The pyramids can be triangular or quadrilateral, etc.

The pyramids can be regular (when the base is a regular polygon and the leg of the height is in the center of the base) or non-regular.

Definition

The median of a tetrahedron is the segment that connects a vertex with the center of mass (barycenter) of the opposite face.

Theorem

The median are concurrent in a point G situated on each of them at the distance of ¾ from the vertex and ¼ from the corresponding face.

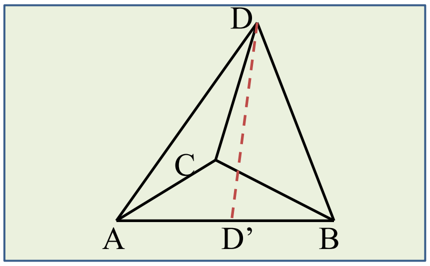
Definition

The median plane of a tetrahedron is the plane which passes through an edge and the middle of the opposite side; these are concurrent in a point G (the point of intersection of the medians).

Definition

<u>The bimedian</u> of a tetrahedron is the segment that connects the middle points of two opposite edges, these are concurrent in a point that is the middle of each of them (G).

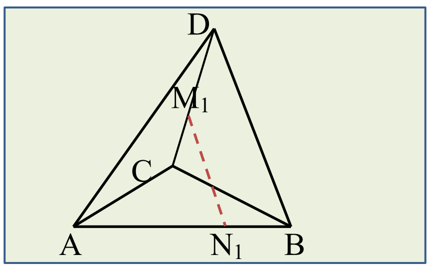
The bisector planes of the dihedrons formed by two adjacent faces of a tetrahedron are concurrent in a point which is the center of the circumscribed sphere to the tetrahedron



DD' median

$$DD'^2 = \frac{1}{3}\left(DA^2 + DB^2 + DC^2\right) - \frac{1}{9}\left(AB^2 + BC^2 + CA^2\right);$$

$$\left.\begin{matrix} AA' \\ BB' \\ CC' \end{matrix}\right\} medians;\ AA'^2 + BB'^2 + CC'^2 + DD'^2 = \frac{4}{9}\left(AB^2 + AC^2 + CA^2 + DA^2 + DB^2 + DC^2\right);$$



M_1N_1 bimedian

$$M_1N_1^{\ 2} = \frac{AD^2 + BD^2 + AC^2 + BC^2 - AB^2 - CD^2}{4};$$

$$\left.\begin{matrix} M_2N_2 \\ M_3N_3 \end{matrix}\right\} = bimedians;\ M_1N_1^{\ 2} + M_2N_2^{\ 2} + M_3N_3^{\ 2} = \frac{1}{4}\left(AB^2 + AC^2 + CA^2 + DA^2 + DB^2 + DC^2\right) =$$

$$= \frac{9}{16}\left(AA'^2 + BB'^2 + CC'^2 + DD'^2\right).$$

An orthogonal tetrahedron is a tetrahedron in which the three pairs of opposed edges are perpendicular: $AB \perp CD;\ BC \perp AD;\ AC \perp BD$.

A spherical sector is the figure generated by the rotation of a circular sector around a diameter which doesn't traverse it.

The locus of the points for which their projections on the sides of a triangle is

a) In plane: the circumscribed circle to the triangle (the Simpson line)
b) In space: the rotation cylinder whose base is the circumscribed circle to the triangle.

Round figures

Definition

A cylindrical surface is the set of the points of all the lines in space constructed through each of the points of line γ, parallel to a given line.

Definition

The line γ is called directing line.

Definition

The parallel lines to the given line constructed through the points of line γ are called the generators of the cylindrical surface.

Classification of the cylinders

a) By the name of the directing line (parabolic, elliptic, circular, etc.)
b) By the angle between the plane of the base with the generator:
   - Right, when the base plane is perpendicular on the generator;
   - Oblique.

Definition

The cylinder circular right is the cylinder which has the base a circle contained in a plane perpendicular on the generator. This results from a complete rotation of a rectangle around an edge.

Definition

The tangent plane to a cylindrical surface is the plane which contains the generator of the cylinder and it is tangent in each point of contact to the circles (curves) of the cylindrical surface.

The generators of the cylinder are equal and parallel, as segments of parallel lines between parallel planes.

Definition

The prism inscribed in a circular cylinder is a prism whose bases are inscribed in the circles of the cylinder bases and having a lateral edge as generator of the cylinder.

All lateral edges of a prism inscribed in a circular cylinder are generators of the cylinder.

In a circular cylinder it can be inscribed a prism with the base a regular polygon with n sides.

Definition

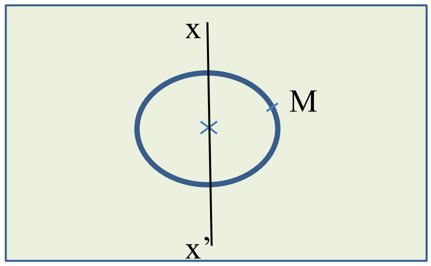
The prism circumscribed to circular cylinder is the prism whose bases are polygons circumscribed to the base circles of the cylinder, and having a lateral edge parallel to a generator of the cylinder.

The planes which contain the lateral faces of a prism circumscribed to a circular cylinder are tangent to the cylindrical surface.

To any circular cylinder one can circumscribe a prism, which has as bases regular polygons with n sides.

Rotations

Rotation axis xx'



Through the rotation of a figure around an axis, the distances of the points from the axis to the figure will be preserved, and the resulted figures are cold rotational figures.

Through the rotation of a rectangle around one of its sides it is obtained a cylinder circular right.

Through the rotation of a right triangle around one of its cathetus it is obtained a cone circular right.

Through the rotation of a random triangle around one of its sides we obtain two unequal cones, which have the same base.

Through the rotation of an isosceles (or equilateral) triangle around its heath from the vertex of the two equal sides it is obtained a cone.

Through the rotation of a rectangle around the line which connects the middle points of the two opposite sides (the symmetry axis) it is obtained a cylinder.

Through the rotation of an isosceles trapezoid around its symmetry axis, it is obtained a truncated cone.

Any circular cone and right cylinder are called rotational figures.

Definition

Conical surface is the surface which is generated by a line called generator, which moves passing through a fixed point and supports itself on a curve.

The cones are circular (the base is a circle), parabolic, elliptic.

The cones can be right (the height has its leg in the center of the curve), or oblique.

The section performed in a circular cone through a plane parallel to the base which intersects a generator n of the cone, is a circle.

The ratio between the area of the base of a cone circular right and the area of the section performed in the cone through a plane parallel to the base is equal to the square of the ratio between the height of the given cone and the height of the cone formed through section.

In a right circular cone the generators are equal:

$$G^2 = R^2 + I^2$$

Through the section of a right circular cone with a plane parallel to the base it will form a cone that has the radius of the base proportional to the height and the radius of the base of the given cone (and the generator).

Definition

The inscribed pyramid in a circular cone is the pyramid whose base is a polygon inscribed in the circle of the base of the cone and whose vertex coincides with the cone's vertex.

In a cone it can be inscribed a pyramid having as base a regular polygon with n sides.

Definition

The pyramid circumscribed to cone circular is the pyramid, which has as a base a polygon circumscribed to the cone's base circle and its vertex coincides with the cone's vertex.

To any cone it can be circumscribed a pyramid having as base a regular polygon with n sides.

Through a rotation of $180°$ of an isosceles triangle, around a line which contains the base's corresponding height, it will generate a right circular cone.

The truncated cone

$$G^2 = (R - r)^2 + h^2$$

In a truncated cone right circular the generators are equal.

Definition

A truncated circular right cone is the portion of a cone circular right between the base and a parallel section with the base.

Sphere

Definition

A spherical surface is the set of all points from a space, which are at the same distance from a fixed point (a positive number). The fixed point is called the center of the sphere. The diameter of a sphere is the line segment which connects two points on the sphere and passes through the center of the sphere. The chord in a sphere is the line which connects two points on the sphere.

The position of a line with respect to a sphere

a) The line intersects the sphere in two points when $d < R$
b) The line is tangent to the sphere (it has only one common with the sphere) $d = R$.
c) The line is external to the sphere (it does not have any common point with the sphere) $d > R$

The tangent to a sphere is the perpendicular line on the radius on the contact point.

The diameter is the longest chord in a sphere.

The position of a plane with respect to a sphere

a) The secant plane to a sphere (intersects the sphere in a circle) $d < R$
b) The plane is tangent to the sphere (it has a single point in common with the sphere) $d = R$
c) The plane is exterior to the sphere (it does not have any point in common with the sphere)

The plane tangent to a sphere is perpendicular on a radius on its contact point.

Any plane intersects the sphere by a circle.

The largest circle of a sphere is the circle obtained through the section of a sphere with a plane which passes through the center of the sphere; there are an infinite number of such circles.

The sphere is a rotational figure because it results from the rotation of a semicircle around its diameter.

Determination of a sphere

a) The center and the radius
b) The sphere diameter
c) A circle, whose center coincides with the sphere's center (the largest circle of the sphere)
d) Four points, which are not in the same plane) of a sphere.

Definition

The spherical cap is a portion of the sphere's surface limited by a plane which intersects the sphere.

Definition

The cap's base is the circle by which the plane intersects the sphere.

Definition

The cap's height is the distance between the center of the cap's base to the pole of the sphere determined by the diameter which passes through the center of the cap's base.

Definition

The spherical zone is the portion between the sphere's surfaces, limited by two parallel secant planes.

Definition

The zones' bases are the circles by which the two parallel planes intersect the sphere.

Definition

The sphere, as a solid, is formed by the union of all the points of a sphere's surface and of its interior points.

Definition

The spherical sector is the figure obtained through the rotation of a circular sector around a line $n'n$ which contains the center of the circular sector and which does not contain the interior points of the sector.

Definition

The line $n'n$ is called the sector's axis.

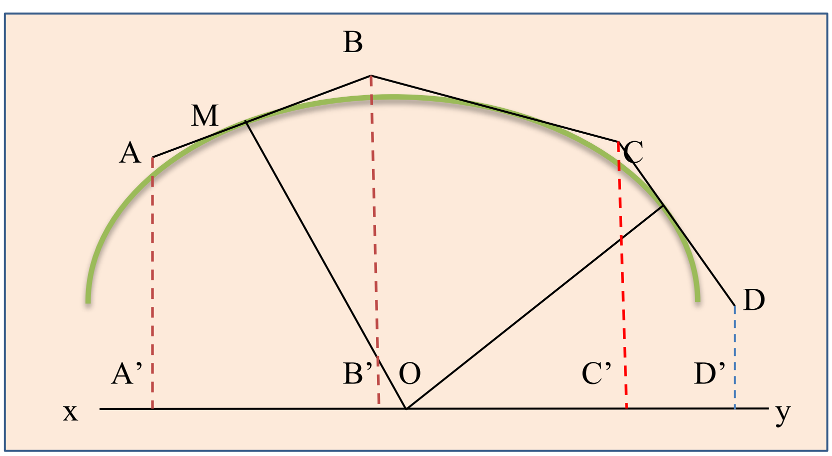
Definition

The radius of the spherical sector is the radius of the circular sector.

Definition

The height of the spherical sector is the distance between the centers of the circles traced by the extremities of the circle of the circular sector.

The area of the surface generated by the rotation of a polygonal regular line around of an axis which contains the center, and which doesn't traverse it, but it is in the same plane with the polygonal regular line, is equal to the projection of the polygonal line on the axis multiplied by the length of the circle whose radius is the apothem of the polygonal regular line.

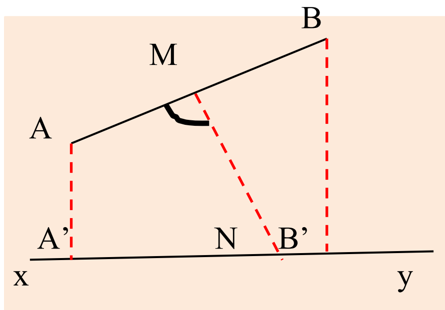


$S = A'D' \cdot 2\pi \cdot OM$

The locus of the tangents which can be constructed in a point to a sphere is the tangent plane to the sphere in that point.

The area of the surface generated through the rotation of a segment around of an axis which is in the same plane with the line segment, and which doesn't traverse it, is equal with the segment's projection on the axis multiplied with the length of the circle whose radius is the mediator of the segment whose boundaries are the axis and the segment.

$$S = A'D' \cdot 2\pi \cdot MN$$



Solids inscribed in a sphere

a) Cylinder inscribed in a sphere is the right circular cylinder whose bases are circles of the sphere.
b) Prism inscribed in a sphere is the right prism whose bases are inscribed polygons in the sphere
c) Circular cone inscribed in a sphere is the con whose base is a circle of the sphere and its vertex is on sphere.
d) Pyramid inscribed in a sphere is the prism whose base is a polygon inscribed in a circle of the sphere and its vertex is on the sphere.
e) A truncated cone right circular inscribed in a sphere is the truncated circular cone whose bases are circles of the sphere.
f) A truncated pyramid inscribed in a sphere is the right truncated pyramid whose bases are polygons inscribed in circles of the sphere.

Inscribed spheres

a) Sphere inscribed in a cylinder is the sphere tangent to the bases planes of a cylinder and to the cylinder's generators.
b) Sphere inscribed in a cone is the sphere tangent to the base's plane and to the cone's generators.

Definition

Spherical segment is the part of a sphere contained between two parallel planes which intersect the sphere.

Euclid's theorem

In any polyhedron convex the sum between the number of the faces and the number of the vertexes is bigger by two the number of its edges.

$$F + V = M + 2$$

The areas and the volumes of the most used polyhedrons

1) The right prisms

a) The right prism

$$A_l = P \cdot I$$
$$A_t = A_l + 2 \cdot A_b$$
$$V = A_b \cdot I$$

Where P is the base perimeter; I is the height; A_l is the lateral area; A_t is the total area; A_b is the area of a base; V is the volume

b) The oblique prism

$A_l = P \cdot I$; A_l is the perimeter of the right section multiplied by the edge

$$A_t = A_l + 2 \cdot A_b$$

V = area of the right section multiplied by the edge, therefore equal to the area of the base multiplied by the height.

c) Random parallelepiped

$$A_l = P \cdot I$$
$$A_t = A_l + 2 \cdot A_b$$
$$V = A_b \cdot I$$

d) Rectangular parallelepiped

$A_l = 2(a+b)c$; a,b are the sides of the base, c is the height

$$A_t = 2(ab+bc+ca)$$

$V = abc$; $d^2 = a^2 + b^2 + c^2$

e) Cube

$A_l = 9a^2$; a is the side of the cube

$$A_t = 6a^2$$

$V = a^3$; $d = a^2\sqrt{3}$; d is the diagonal of the cube

2) Regular pyramid

a) Regular pyramid

$A_l = \dfrac{P \cdot A_p}{2}$; A_p is the pyramid apothem

$$A_t = A_l + A_b$$
$$V = \frac{A_b \cdot I}{3}$$

b) Irregular pyramid

$A_l =$ the sum of the areas of the lateral faces.

$$A_t = A_l + A_b$$
$$V = \frac{A_b \cdot I}{3}$$

By sectioning a pyramid with a plane parallel to the base it is obtained a pyramid; the ratio of the volume of the two pyramids is equal to the ratio of their heights cubed:

$$\frac{V'}{V} = \left(\frac{h'}{h}\right)^3$$

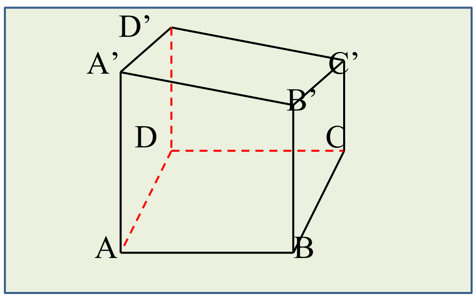
The ratio of the area of the bases is equal to the ratio of their heights squared:

$$\frac{A_b'}{A_b} = \left(\frac{h'}{h}\right)^2$$

Truncated prism is the solid generated through the sectioning of a prism with a plan which is not parallel to the base

$A'B'C'D'$ Parallelogram

$V =$ area of the right section multiplied by the arithmetic media of its edges.



c) Truncated pyramid

$A_l = \frac{(P+p)\cdot A_p}{2}$; P is the perimeter of the big base; p is the perimeter of the small base; A_p is the apothem of the truncated pyramid

$A_t = A_l + A_B + A_b$; A_B is the area of the big base; A_b is the area of the small base

$$V = \frac{I}{3}\left(A_B + A_b + \sqrt{A_B \cdot A_b}\right)$$

The areas and the volumes of the rotation surfaces

1) The circular right cylinder

$$A_l = 2\pi RG$$

$$A_t = 2\pi R(G+R)$$

$$V = \pi R^2 I$$

Where

R = the base radius

G = the generator

I = the height

2) The circular right cone

$$A_l = \pi RG$$

$$A_t = \pi R(G+R)$$

$$V = \frac{\pi R^2 I}{3}$$

3) The truncated circular right cone

$$A_l = \pi G(R+r)$$

$$A_t = \pi G(R+r) + \pi\left(R^2 + r^2\right)$$

$$V = \frac{\pi I}{3}\left(R^2 + r^2 + Rr\right)$$

4) Sphere

a)

$$Aria = 4\pi R^2$$

$$V = \frac{4\pi R^3}{3}$$

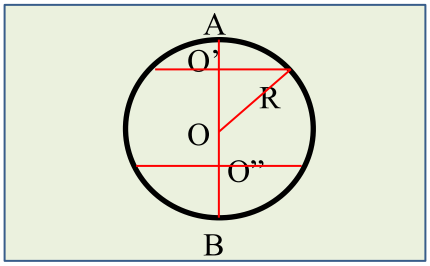
$D = 2R$; $V = A\frac{R}{3}$; $D =$ the sphere's diameter

b)The spherical zone

$$Aria = 2\pi RI$$

R = the sphere radius

I = the height of the zone $(O'O'')$

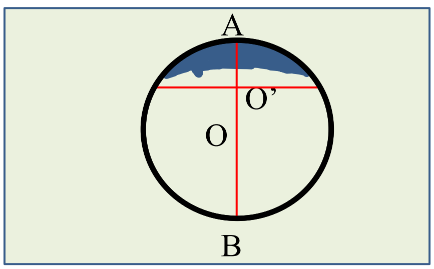


c)The spherical calotte

$$Aria = 2\pi RI$$

R = the sphere radius

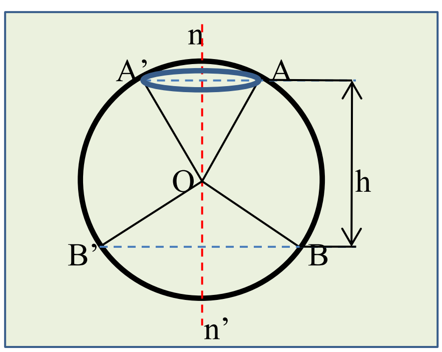
I = the height of the spherical calotte $(O'A)$



d)The spherical sector

$$V = \frac{2\pi R^3 I}{3}$$

R = the sphere radius

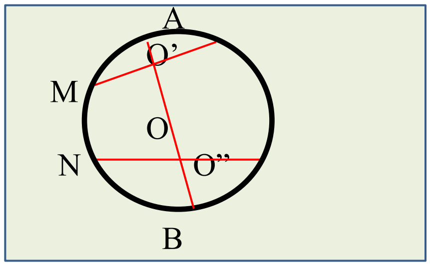


e)The spherical segment

$$V = \frac{4}{3}\pi\left(\frac{h}{2}\right)^3 + \frac{1}{2}\left(\pi r_1^2 h + \pi r_2^2 h\right)$$

r_1, r_2 = the radiuses of the section circles

h = height of the spherical segment (the distance between the planes $(O'O'')$.



When a plane is tangent to the sphere the volume becomes:

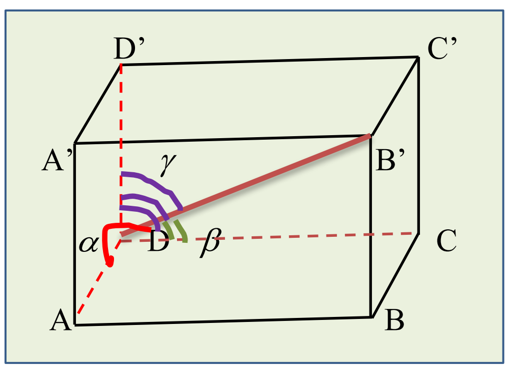
$$V = \frac{\pi h^2}{3}(3R - h)$$

In a pyramid whose edges are equal, the height's leg is in the center of the circumscribed circle of the base polygon.

If in a pyramid the height's leg is in the center of the inscribed circle of the base polygon, then the apothems of the lateral faces are equal (the apothem of the pyramid).

In a right parallelepiped takes place the following relation:

$$cos^2 \alpha + cos^2 \beta + cos^2 \gamma = 1$$



If in a pyramid the vertex is moved parallel to the base, the volume does not change (therefore, it is obtained a pyramid equivalent to the first).

A prismatoid is a polyhedron which has two bases parallel (equal or not equal), eventually, one of them is reduced to a point or a line.

$$V = \frac{h}{6}(Q_1 + Q_2 + 4Q_3)$$

Q_1, Q_2 are the bases areas

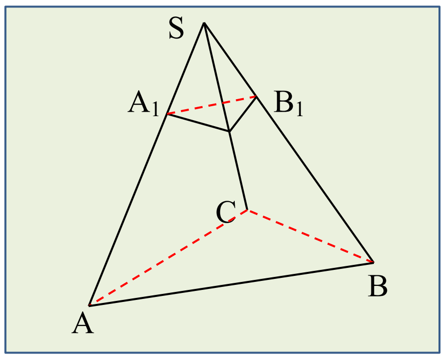
Q_3 is the section's area

The harmonic ratio (bi-ratio) of four points A, B, C, D on a line (taken on this order) is:

$$\left(ABCD\right)=\frac{CA}{CB}\div\frac{DA}{DB}$$

In a triangular pyramid given A',B',C' three random points on SA,SB,SC, the following relation takes place (Eugen Rusu).

$$\frac{Vol\left(SABC\right)}{Vol\left(SA'B'C'\right)}=\frac{SA\cdot SB\cdot SC}{SA'\cdot SB'\cdot SC'}$$



The heights of a given tetrahedron denoted h_1,h_2,h_3,h_4 and the distances from an interior point M to the corresponding faces denoted by d_1,d_2,d_3,d_4, the following relation takes place (Eugen Rusu).

$$\frac{d_1}{h_1}=\frac{d_2}{h_2}=\frac{d_3}{h_3}=\frac{d_4}{h_4}=1$$

If in a right pyramid $SABCD$ with the base a rectangle, a random plane cuts the edges in the points $A',B',C'D'$, then

$$\frac{1}{SA'}+\frac{1}{SC'}=\frac{1}{SB'}+\frac{1}{SD'}$$

Since childhood I got accustomed to study with a pen in my hand. I extracted theorems and formulas, together with the definitions, from my textbooks.
It was easier, later, for me, to prepare for the tests, especially for the final exams at the end of each semester.

I kept (and still do today) small notebooks where I collected not only mathematical but any idea I read from various domains.

These two volumes reflect my 1973-1974 high school studies in mathematics at the Pedagogical High School of Rm. Vâlcea, Romania.
Besides the textbooks I added information I collected from various mathematical books of solved problems I was studying at that time.

The first volume contains: *Arithmetic, Plane Geometry*, and *Space Geometry*.
The second volume contains: *Algebra (9th to 12th grades),* and *Trigonometry*.

The Author

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