

$\mathcal{N} = 4$ Super-Yang-Mills Theory, QCD and Collider Physics

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Abstract

We review how (dimensionally regulated) scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory provide a useful testing ground for perturbative QCD calculations relevant to collider physics, as well as another avenue for investigating the AdS/CFT correspondence. We describe the iterative relation for two-loop scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory found in *C. Anastasiou et al., Phys. Rev. Lett. 91:251602 (2003)*, and discuss recent progress toward extending it to three loops.

1 Introduction and Collider Physics Motivation

Maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory (MSYM) is unique in many ways. Its properties are uniquely specified by the gauge group, say $SU(N_c)$, and the value of the gauge coupling g . It is conformally invariant for any value of g . Although gravity is not present in its usual formulation, MSYM

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is connected to gravity and string theory through the AdS/CFT correspondence [1]. Because this correspondence is a weak-strong coupling duality, it is difficult to verify quantitatively for general observables. On the other hand, such checks are possible and have been remarkably successful for quantities protected by supersymmetry such as BPS operators [2], or when an additional expansion parameter is available, such as the number of fields in sequences of composite, large R -charge operators [3,4,5,6,7,8].

It is interesting to study even more observables in perturbative MSYM, in order to see how the simplicity of the strong coupling limit is reflected in the structure of the weak coupling expansion. The strong coupling limit should be even simpler when the large- N_c limit is taken simultaneously, as it corresponds to a weakly-coupled supergravity theory in a background with a large radius of curvature. There are different ways to study perturbative MSYM. One approach is via computation of the anomalous dimensions of composite, gauge invariant operators [1,3,4,5,6,7,8]. Another possibility [9], discussed here, is to study the scattering amplitudes for (regulated) plane-wave elementary field excitations such as gluons and gluinos.

One of the virtues of the latter approach is that perturbative MSYM scattering amplitudes share many qualitative properties with QCD amplitudes in the regime probed at high-energy colliders. Yet the results and the computations (when organized in the right way) are typically significantly simpler. In this way, MSYM serves as a testing ground for many aspects of perturbative QCD. MSYM loop amplitudes can be considered as components of QCD loop amplitudes. Depending on one's point of view, they can be considered either "the simplest pieces" (in terms of the rank of the loop momentum tensors in the numerator of the amplitude) [10,11], or "the most complicated pieces" in terms of the degree of transcendentality (see section 6) of the special functions entering the final results [12]. As discussed in section 6, the latter interpretation links recent three-loop anomalous dimension results in QCD [13] to those in the spin-chain approach to MSYM [5].

The most direct experimental probes of short-distance physics are collider experiments at the energy frontier. For the next decade, that frontier is at hadron colliders — Run II of the Fermilab Tevatron now, followed by startup of the CERN Large Hadron Collider in 2007. New physics at colliders always contends with Standard Model backgrounds. At hadron colliders, *all* physics processes — signals and backgrounds — are inherently QCD processes. Hence it is important to be able to predict them theoretically as precisely as possible. The cross section for a "hard," or short-distance-dominated processes, can be factorized [14] into a partonic cross section, which can be computed order by order in perturbative QCD, convoluted with nonperturbative but *measurable* parton distribution functions (pdfs). For example, the cross section for producing a pair of jets (plus anything else) in a $p\bar{p}$ collision is given by

$$\sigma_{p\bar{p}\rightarrow jjX}(s) = \sum_{a,b} \int_0^1 dx_1 dx_2 f_a(x_1; \mu_F) \bar{f}_b(x_2; \mu_F) \times \hat{\sigma}_{ab\rightarrow jjX}(sx_1x_2; \mu_F, \mu_R; \alpha_s(\mu_R)), \quad (1)$$

where s is the squared center-of-mass energy, $x_{1,2}$ are the longitudinal (light-cone) fractions of the p, \bar{p} momentum carried by partons a, b , which may be quarks, anti-quarks or gluons. The experimental definition of a jet is an involved one which need not concern us here. The pdf $f_a(x, \mu_F)$ gives the probability for finding parton a with momentum fraction x inside the proton; similarly \bar{f}_b is the probability for finding parton b in the antiproton. The pdfs depend logarithmically on the factorization scale μ_F , or transverse resolution with which the proton is examined. The Mellin moments of $f_a(x, \mu_F)$ are forward matrix elements of leading-twist operators in the proton, renormalized at the scale μ_F . The quark distribution function $q(x, \mu)$, for example, obeys $\int_0^1 dx x^j q(x, \mu) = \langle p | [\bar{q} \gamma^+ \partial_+^j q]_{(\mu)} | p \rangle$.

2 Ingredients for a NNLO Calculation

Many hadron collider measurements can benefit from predictions that are accurate to next-to-next-to-leading order (NNLO) in QCD. Three separate ingredients enter such an NNLO computation; only the third depends on the process:

- (1) The experimental value of the QCD coupling $\alpha_s(\mu_R)$ must be determined at one value of the renormalization scale μ_R (for example m_Z), and its evolution in μ_R computed using the 3-loop β -function, which has been known since 1980 [15].
- (2) The experimental values for the pdfs $f_a(x, \mu_F)$ must be determined, ideally using predictions at the NNLO level, as are available for deep-inelastic scattering [16] and more recently Drell-Yan production [17]. The evolution of pdfs in μ_F to NNLO accuracy has very recently been completed, after a multi-year effort by Moch, Vermaseren and Vogt [13] (previously, approximations to the NNLO kernel were available [18]).
- (3) The NNLO terms in the expansion of the partonic cross sections must be computed for the hadronic process in question. For example, the parton cross sections for jet production has the expansion,

$$\hat{\sigma}_{ab\rightarrow jjX} = \alpha_s^2 (A + \alpha_s B + \alpha_s^2 C + \dots). \quad (2)$$

The quantities A and B have been known for over a decade [19], but C has not yet been computed.

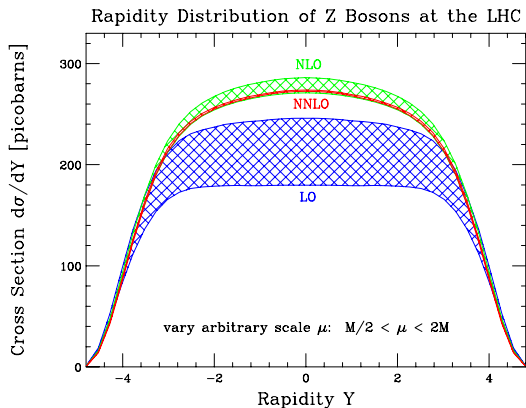


Figure 1. LHC Z production [22].

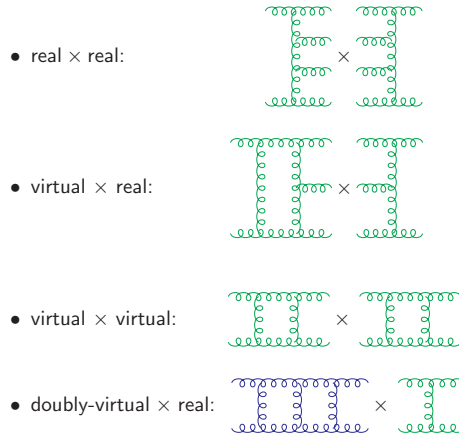


Figure 2. Purely gluonic contributions to $\hat{\sigma}_{gg \rightarrow jjX}$ at NNLO.

Indeed, the NNLO terms are unknown for all but a handful of collider processes. Computing a wide range of processes at NNLO is the goal of a large amount of recent effort in perturbative QCD [20]. As an example of the improved precision that could result from this program, consider the production of a virtual photon, W or Z boson via the Drell-Yan process at the Tevatron or LHC. The total cross section for this process was first computed at NNLO in 1991 [21]. Last year, the rapidity distribution of the vector boson also became available at this order [17,22], as shown in fig. 1. The rapidity is defined in terms of the energy E and longitudinal momentum p_z of the vector boson in the center-of-mass frame, $Y \equiv \frac{1}{2} \log \left(\frac{E+p_z}{E-p_z} \right)$. It determines where the vector boson decays within the detector, or outside its acceptance. The rapidity is sensitive to the x values of the incoming partons. At leading order in QCD, $x_1 = e^Y m_V / \sqrt{s}$, $x_2 = e^{-Y} m_V / \sqrt{s}$, where m_V is the vector boson mass.

The LHC will produce roughly 100 million W s and 10 million Z s per year in detectable (leptonic) decay modes. LHC experiments will be able to map out the curve in fig. 1 with exquisite precision, and use it to constrain the parton distributions — in the same detectors that are being used to search for new physics in other channels, often with similar $q\bar{q}$ initial states. By taking ratios of the other processes to the “calibration” processes of single W and Z production, many experimental uncertainties, including those associated with the initial state parton distributions, drop out. Thus fig. 1 plays a role as a “partonic luminosity monitor” [23]. To get the full benefit of the remarkable experimental precision, though, the theory uncertainty must approach the 1% level. As seen from the uncertainty bands in the figure, this precision is only achievable at NNLO. The bands are estimated by varying the arbitrary renormalization and factorization scales μ_R and μ_F (set to a common value μ) from $m_V/2$ to $2m_V$. A computation to all orders in α_s would have no dependence on μ . Hence the μ -dependence of a fixed order computation is related to the size of the missing higher-order terms in the series. Although

sub-1% uncertainties may be special to W and Z production at the LHC, similar qualitative improvements in precision will be achieved for many other processes, such as di-jet production, as the NNLO terms are completed.

Even within the NNLO terms in the partonic cross section, there are several types of ingredients. This feature is illustrated in fig. 2 for the purely gluonic contributions to di-jet production, $\hat{\sigma}_{gg \rightarrow jjX}$. In the figure, individual Feynman graphs stand for full amplitudes interfered (\times) with other amplitudes, in order to produce contributions to a cross section. There may be 2, 3, or 4 partons in the final state. Just as in QED it is impossible to define an outgoing electron with no accompanying cloud of soft photons, also in QCD sensible observables require sums over final states with different numbers of partons. Jets, for example, are defined by a certain amount of energy into a certain conical region. At leading order, that energy typically comes from a single parton, but at NLO there may be two partons, and at NNLO three partons, within the jet cone.

Each line in fig. 2 results in a cross-section contribution containing severe infrared divergences, which are traditionally regulated by dimensional regularization with $D = 4 - 2\epsilon$. Note that this regulation breaks the classical conformal invariance of QCD, and the classical and quantum conformal invariance of $\mathcal{N} = 4$ super-Yang-Mills theory. Each contribution contains poles in ϵ ranging from $1/\epsilon^4$ to $1/\epsilon$. The poles in the real contributions come from regions of phase-space where the emitted gluons are soft and/or collinear. The poles in the virtual contributions come from similar regions of virtual loop integration. The virtual \times real contribution obviously has a mixture of the two. The Kinoshita-Lee-Nauenberg theorem [24] guarantees that the poles all cancel in the sum, for properly-defined, short-distance observables, after renormalizing the coupling constant and removing initial-state collinear singularities associated with renormalization of the pdfs.

A critical ingredient in any NNLO prediction is the set of two-loop amplitudes, which enter the doubly-virtual \times real interference in fig. 2. Such amplitudes require dimensionally-regulated all-massless two-loop integrals depending on at least one dimensionless ratio, which were only computed beginning in 1999 [25,26,27]. They also receive contributions from many Feynman diagrams, with lots of gauge-dependent cancellations between them. It is of interest to develop more efficient, manifestly gauge-invariant methods for combining diagrams, such as the unitarity or cut-based method successfully applied at one loop [10] and in the initial two-loop computations [28].



Figure 3. Illustration of soft-collinear (left) and pure-collinear (right) one-loop divergences.

3 $\mathcal{N} = 4$ Super-Yang-Mills Theory as a Testing Ground for QCD

$\mathcal{N} = 4$ super-Yang-Mills theory serves an excellent testing ground for perturbative QCD methods. For n -gluon scattering at tree level, the two theories in fact give identical predictions. (The extra fermions and scalars of MSYM can only be produced in pairs; hence they only appear in an n -gluon amplitude at loop level.) Therefore any consequence of $\mathcal{N} = 4$ supersymmetry, such as Ward identities among scattering amplitudes [29], automatically applies to tree-level gluonic scattering in QCD [30]. Similarly, at tree level Witten's topological string [31] produces MSYM, but implies twistor-space localization properties for QCD tree amplitudes. (Amplitudes with quarks can be related to supersymmetric amplitudes with gluinos using simple color manipulations.)

3.1 Pole Structure at One and Two Loops

At the loop-level, MSYM becomes progressively more removed from QCD. However, it can still illuminate general properties of scattering amplitudes, in a computationally simpler arena. Consider the infrared singularities of one-loop massless gauge theory amplitudes. In dimensional regularization, the leading singularity is $1/\epsilon^2$, arising from virtual gluons which are both soft *and* collinear with respect to a second gluon or another massless particle. It can be characterized by attaching a gluon to any pair of external legs of the tree-level amplitude, as in the left graph in fig. 3. Up to color factors, this leading divergence is the same for MSYM and QCD. There are also purely collinear terms associated with individual external lines, as shown in the right graph in fig. 3. The pure-collinear terms have a simpler form than the soft terms, because there is less tangling of color indices, but they do differ from theory to theory.

The full result for one-loop divergences can be expressed as an operator $\mathbf{I}^{(1)}(\epsilon)$ which acts on the color indices of the tree amplitude [32]. Treating the L -loop amplitude as a vector in color space, $|\mathcal{A}_n^{(L)}\rangle$, the one-loop result is

$$|\mathcal{A}_n^{(1)}\rangle = \mathbf{I}^{(1)}(\epsilon)|\mathcal{A}_n^{(0)}\rangle + |\mathcal{A}_n^{(1),\text{fin}}\rangle, \quad (3)$$

where $|\mathcal{A}_n^{(1),\text{fin}}\rangle$ is finite as $\epsilon \rightarrow 0$, and

$$\mathbf{I}^{(1)}(\epsilon) = \frac{1}{2} \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \sum_{i=1}^n \sum_{j \neq i}^n \mathbf{T}_i \cdot \mathbf{T}_j \left[\frac{1}{\epsilon^2} + \frac{\gamma_i}{\mathbf{T}_i^2} \frac{1}{\epsilon} \right] \left(\frac{\mu_R^2}{-s_{ij}} \right)^\epsilon, \quad (4)$$

where γ is Euler's constant and $s_{ij} = (k_i + k_j)^2$ is a Mandelstam invariant. The color operator $\mathbf{T}_i \cdot \mathbf{T}_j = T_i^a T_j^a$ and factor of $(\mu_R^2/(-s_{ij}))^\epsilon$ arise from soft gluons exchanged between legs i and j , as in the left graph in fig. 3. The pure $1/\epsilon$ poles terms proportional to γ_i have been written in a symmetric fashion, which slightly obscures the fact that the color structure is actually simpler. We can use the equation which represents color conservation in the color-space notation, $\sum_{j=1}^n \mathbf{T}_j = 0$, to simplify the result. At order $1/\epsilon$ we may neglect the $(\mu_R^2/(-s_{ij}))^\epsilon$ factor in the γ_i terms, and we have $\sum_{j \neq i}^n \mathbf{T}_i \cdot \mathbf{T}_j \gamma_i / \mathbf{T}_i^2 = -\gamma_i$. So the color structure of the pure $1/\epsilon$ term is actually trivial. For an n -gluon amplitude, the factor γ_i is set equal to its value for gluons, which turns out to be $\gamma_g = b_0$, the one-loop coefficient in the β -function. Hence the pure-collinear contribution vanishes for MSYM, but not for QCD.

The divergences of two-loop amplitudes can be described in the same formalism [32]. The relation to soft-collinear factorization has been made more transparent by Sterman and Tejeda-Yeomans, who also predicted the three-loop behavior [33]. Decompose the two-loop amplitude $|\mathcal{A}_n^{(2)}\rangle$ as

$$|\mathcal{A}_n^{(2)}\rangle = \mathbf{I}^{(2)}(\epsilon) |\mathcal{A}_n^{(0)}\rangle + \mathbf{I}^{(1)}(\epsilon) |\mathcal{A}_n^{(1)}\rangle + |\mathcal{A}_n^{(2),\text{fin}}\rangle, \quad (5)$$

where $|\mathcal{A}_n^{(2),\text{fin}}\rangle$ is finite as $\epsilon \rightarrow 0$ and

$$\begin{aligned} \mathbf{I}^{(2)}(\epsilon) = & -\frac{1}{2} \mathbf{I}^{(1)}(\epsilon) \left(\mathbf{I}^{(1)}(\epsilon) + \frac{2b_0}{\epsilon} \right) + \frac{e^{-\epsilon\gamma} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{b_0}{\epsilon} + K \right) \mathbf{I}^{(1)}(2\epsilon) \\ & + \frac{e^{\epsilon\gamma}}{4\epsilon \Gamma(1-\epsilon)} \left[-\sum_{i=1}^n \sum_{j \neq i}^n \mathbf{T}_i \cdot \mathbf{T}_j \frac{H_i^{(2)}}{\mathbf{T}_i^2} \left(\frac{\mu^2}{-s_{ij}} \right)^{2\epsilon} + \hat{\mathbf{H}}^{(2)} \right]. \end{aligned} \quad (6)$$

Here K and $H_i^{(2)}$ depend on the theory, and $H_i^{(2)}$, like γ_i , also depends on the external leg i . For QCD, K has long been known from soft-gluon resummation [34], while $H_i^{(2)}$ were found by explicit computation of four-parton two-loop scattering amplitudes [35,36,37]. For MSYM, the quantities are naturally simpler,

$$K^{\mathcal{N}=4} = -\zeta_2 C_A, \quad (7)$$

$$H_i^{(2),\mathcal{N}=4} = \frac{\zeta_3}{2} C_A^2, \quad (8)$$

where $C_A = N_c$ is the adjoint Casimir value. The quantity $\hat{\mathbf{H}}^{(2)}$ has non-trivial, but purely subleading-in- N_c , color structure. It is associated with soft, rather than collinear, momenta [37,33], so it is theory-independent, up to color factors. An ansatz for it for general n has been presented recently [38].

3.2 Recycling Cuts in MSYM

An efficient way to compute loop amplitudes, particularly in theories with a great deal of supersymmetry, is to use unitarity and reconstruct the amplitude from its cuts [10,38]. For the four-gluon amplitude in MSYM, the two-loop structure, and much of the higher-loop structure, follows from a simple property of the one-loop two-particle cut in this theory. For simplicity, we strip the color indices off of the four-point amplitude $\mathcal{A}_4^{(0)}$, by decomposing it into color-ordered amplitudes $A_4^{(0)}$, whose coefficients are traces of $SU(N_c)$ generator matrices (Chan-Paton factors),

$$\mathcal{A}_4^{(0)}(k_1, a_1; k_2, a_2; k_3, a_3; k_4, a_4) = g^2 \sum_{\rho \in S_4/Z_4} \text{Tr}(T^{a_{\rho(1)}} T^{a_{\rho(2)}} T^{a_{\rho(3)}} T^{a_{\rho(4)}}) \times A_4^{(0)}(k_{\rho(1)}, k_{\rho(2)}, k_{\rho(3)}, k_{\rho(4)}) . \quad (9)$$

The two-particle cut can be written as a product of two four-point color-ordered amplitudes, summed over the pair of intermediate $\mathcal{N} = 4$ states S, S' crossing the cut, which evaluates to

$$\sum_{S, S' \in \mathcal{N}=4} A_4^{(0)}(k_1, k_2, \ell_S, -\ell_{S'}) \times A_4^{(0)}(\ell_{S'}, -\ell_S, k_3, k_4) = i s_{12} s_{23} A_4^{(0)}(k_1, k_2, k_3, k_4) \times \frac{1}{(\ell' - k_1)^2} \frac{1}{(\ell - k_3)^2} , \quad (10)$$

where $\ell' = \ell - k_1 - k_2$. This equation is also shown in fig. 4. The scalar propagator factors in eq. (10) are depicted as solid vertical lines in the figure. The dashed line indicates the cut. Thus the cut reduces to the cut of a scalar box integral, defined by

$$\mathcal{I}_4^{D=4-2\epsilon} \equiv \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2 (\ell - k_1)^2 (\ell - k_1 - k_2)^2 (\ell + k_4)^2} . \quad (11)$$

One of the virtues of eq. (10) is that it is valid for arbitrary external states in the $\mathcal{N} = 4$ multiplet, although only external gluons are shown in fig. 4. Therefore it can be re-used at higher loop order, for example by attaching yet another tree to the left.

$$\sum_{N=4} = i s_{12} s_{23} \times \text{tree amplitude} = \text{tree amplitude}$$

Figure 4. The one-loop two-particle cuts for the four-point amplitude in MSYM reduce to the tree amplitude multiplied by a cut scalar box integral (for any set of four external states).

$$= i^2 s_{12} s_{23} \left[\text{diagram 1} + s_{12} \text{diagram 2} + \text{perms} \right]$$

Figure 5. The two-loop $gg \rightarrow gg$ amplitude in MSYM [11,39]. The blob on the right represents the color-ordered tree amplitude $A_4^{(0)}$. (The quantity $s_{12}s_{23}A_4^{(0)}$ transforms symmetrically under gluon interchange.) In the the brackets, black lines are kinematic $1/p^2$ propagators, with scalar (ϕ^3) vertices. Green lines are color δ^{ab} propagators, with structure constant (f^{abc}) vertices. The permutation sum is over the three cyclic permutations of legs 2,3,4, and makes the amplitude Bose symmetric.

At two loops, the simplicity of eq. (10) made it possible to compute the two-loop $gg \rightarrow gg$ scattering amplitude in that theory (in terms of specific loop integrals) in 1997 [11], four years before the analogous computations in QCD [36,37]. All of the loop momenta in the numerators of the Feynman diagrams can be factored out, and only two independent loop integrals appear, the planar and nonplanar scalar double box integrals. The result can be written in an appealing diagrammatic form, fig. 5, where the color algebra has the same form as the kinematics of the loop integrals [39].

At higher loops, eq. (10) leads to a “rung rule” [11] for generating a class of $(L + 1)$ -loop contributions from L -loop contributions. The rule states that one can insert into a L -loop contribution a rung, *i.e.* a scalar propagator, transverse to two parallel lines carrying momentum $\ell_1 + \ell_2$, along with a factor of $i(\ell_1 + \ell_2)^2$ in the numerator, as shown in fig. 6. Using this rule, one can construct recursively the external and loop-momentum-containing numerators factors associated with every ϕ^3 -type diagram that can be reduced to trees by a sequence of two-particle cuts, such as the diagram in fig. 7a. Such diagrams can be termed “iterated 2-particle cut-constructible,” although a more compact notation might be ‘Mondrian’ diagrams, given their resemblance to Mondrian’s paintings. Not all diagrams can be computed in this way. The diagram in fig. 7b is not in the ‘Mondrian’ class, so it cannot be determined from two-particle cuts. Instead, evaluation of the three-particle cuts shows that it appears with a non-vanishing coefficient in the subleading-color contributions to the three-loop MSYM amplitude.

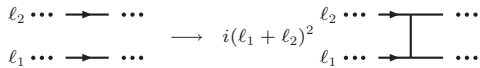


Figure 6. The rung rule for MSYM.

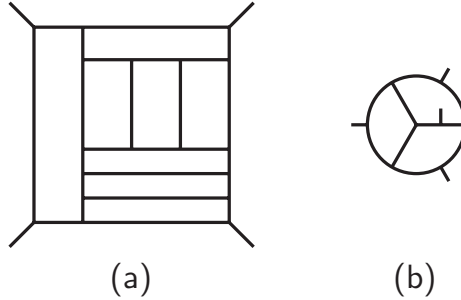


Figure 7. (a) Example of a ‘Mondrian’ diagram which can be determined recursively from the rung rule. (b) The first non-vanishing, non-Mondrian diagrams appear at three loops in nonplanar, subleading-color contributions.

4 Iterative Relation in $\mathcal{N} = 4$ Super-Yang-Mills Theory

Although the two-loop $gg \rightarrow gg$ amplitude in MSYM was expressed in terms of scalar integrals in 1997 [11], and the integrals themselves were computed as a Laurent expansion about $D = 4$ in 1999 [25,26], the expansion of the $\mathcal{N} = 4$ amplitude was not inspected until last fall [9], considerably after similar investigations for QCD and $\mathcal{N} = 1$ super-Yang-Mills theory [36,37]. It was found to have a quite interesting “iterative” relation, when expressed in terms of the one-loop amplitude and its square.

At leading color, the L -loop $gg \rightarrow gg$ amplitude has the same single-trace color decomposition as the tree amplitude, eq. (9). Let $M_4^{(L)}$ be the ratio of this leading-color, color-ordered amplitude to the corresponding tree amplitude, omitting also several conventional factors,

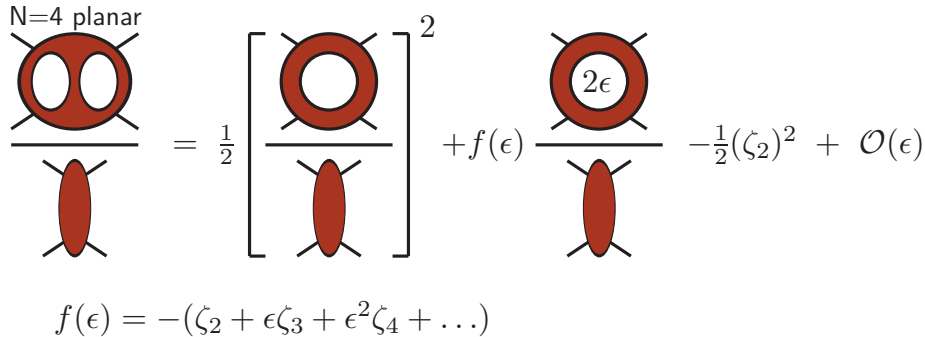
$$A_4^{(L), \mathcal{N}=4 \text{ planar}} = \left[\frac{2e^{-\epsilon\gamma} g^2 N_c}{(4\pi)^{2-\epsilon}} \right]^L A_4^{(0)} \times M_4^{(L)}. \quad (12)$$

Then the iterative relation (see also fig. 8) is

$$M_4^{(2)}(\epsilon) = \frac{1}{2} \left(M_4^{(1)}(\epsilon) \right)^2 + f(\epsilon) M_4^{(1)}(2\epsilon) - \frac{1}{2} (\zeta_2)^2 + \mathcal{O}(\epsilon), \quad (13)$$

where $f(\epsilon) \equiv (\psi(1-\epsilon) - \psi(1))/\epsilon = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \dots)$.

The analogous relation for gluon-gluon scattering at two loops in QCD takes a similar form at the level of the pole terms in ϵ , thanks to the general result (5). But the finite remainder $-\frac{1}{2}(\zeta_2)^2$ is replaced by approximately six pages of formulas (!), including a plethora of polylogarithms, logarithms and



$$f(\epsilon) = -(\zeta_2 + \epsilon\zeta_3 + \epsilon^2\zeta_4 + \dots)$$

Figure 8. Schematic depiction of the iterative relation (13) between two-loop and one-loop MSYM amplitudes.

polynomials in ratios of invariants s/t , s/u and t/u [37]. The polylogarithm is defined by

$$\text{Li}_m(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^m} = \int_0^x \frac{dt}{t} \text{Li}_{m-1}(t), \quad \text{Li}_1(x) = -\ln(1-x). \quad (14)$$

It appears with degree m up to 4 at the finite, order ϵ^0 , level; and up to degree $4-i$ in the $\mathcal{O}(\epsilon^{-i})$ terms. In the case of MSYM, identities relating these polylogarithms are needed to establish eq. (13).

Although the $\mathcal{O}(\epsilon^0)$ term in eq. (13) is miraculously simple, as noted above the behavior of the pole terms is not a miracle. It is dictated in general terms by the cancellation of infrared divergences between virtual corrections and real emission [24]. Roughly speaking, for this cancellation to take place, the virtual terms must resemble lower-loop amplitudes, and the real terms must resemble lower-point amplitudes, in the soft and collinear regions of loop or phase-space integration.

At the level of the finite terms, the iterative relation (13) can be understood in the Regge/BFKL limit where $s \gg t$, because it then corresponds to exponentiation of large logarithms of s/t [40]. For general values of s/t , however, there is no such argument.

The relation is special to $D = 4$, where the theory is conformally invariant. That is, the $\mathcal{O}(\epsilon^1)$ remainder terms cannot be simplified significantly. For example, the two-loop amplitude $M_4^{(2)}(\epsilon)$ contains at $\mathcal{O}(\epsilon^1)$ all three independent Li_5 functions, $\text{Li}_5(-s/u)$, $\text{Li}_5(-t/u)$ and $\text{Li}_5(-s/t)$, yet $[M_4^{(1)}(\epsilon)]^2$ has only the first two of these [9].

The relation is also special to the planar, leading-color limit. The subleading color-components of the finite remainder $|\mathcal{A}_n^{(2),\text{fin}}\rangle$ defined by eq. (5) show no significant simplification at all.

For planar amplitudes in the $D \rightarrow 4$ limit, however, there is evidence that an identical relation also holds for an arbitrary number n of external legs, at least for certain “maximally helicity-violating” (MHV) helicity amplitudes. This evidence comes from studying the limits of two-loop amplitudes as two of the n gluon momenta become collinear [9,38,41]. (Indeed, it was by analyzing these limits that the relation for $n = 4$ was first uncovered.) The collinear limits turn out to be consistent with the same eq. (13) with M_4 replaced by M_n everywhere [9], *i.e.*

$$M_n^{(2)}(\epsilon) = \frac{1}{2} \left(M_n^{(1)}(\epsilon) \right)^2 + f(\epsilon) M_n^{(1)}(2\epsilon) - \frac{1}{2} (\zeta_2)^2 + \mathcal{O}(\epsilon). \quad (15)$$

The collinear consistency does not constitute a proof of eq. (15), but in light of the remarkable properties of MSYM, it would be surprising if it were not true in the MHV case. Because the direct computation of two-loop amplitudes for $n > 4$ seems rather difficult, it would be quite interesting to try to examine the twistor-space properties of eq. (15), along the lines of refs. [31,42]. (The right-hand-side of eq. (15) is not completely specified at order $1/\epsilon$ and ϵ^0 for $n > 4$. The reason is that the order ϵ and ϵ^2 terms in $M_n^{(1)}(\epsilon)$, which contribute to the first term in eq. (15) at order $1/\epsilon$ and ϵ^0 , contain the $D = 6 - 2\epsilon$ pentagon integral [43], which is not known in closed form. On the other hand, the differential equations this integral satisfies may suffice to test the twistor-space behavior. Or one may examine just the finite remainder $M_n^{(L),\text{fin}}$ defined via eq. (5).)

It may soon be possible to test whether an iterative relation for planar MSYM amplitudes extends to three loops. An ansatz for the three-loop planar $gg \rightarrow gg$ amplitude, shown in fig. 9, was provided at the same time as the two-loop result, in 1997 [11]. The ansatz is based on the “rung-rule” evaluation of the iterated 2-particle cuts, plus the 3-particle cuts with intermediate states in $D = 4$; the 4-particle cuts have not yet been verified. Two integrals, each beginning at $\mathcal{O}(\epsilon^{-6})$, are required to evaluate the ansatz in a Laurent expansion about $D = 4$. (The other two integrals are related by $s \leftrightarrow t$.) The triple ladder integral on the top line of fig. 9 was evaluated last year by Smirnov, all the way through $\mathcal{O}(\epsilon^0)$ [44]. Evaluation of the remaining integral, which contains a factor of $(\ell + k_4)^2$ in the numerator, is in progress [45]; all the terms through $\mathcal{O}(\epsilon^{-2})$ agree with predictions [33], up to a couple of minor corrections.

5 Significance of Iterative Behavior?

It is not yet entirely clear why the two-loop four-point amplitude, and probably also the n -point amplitudes, have the iterative structure (15). However, one can speculate that it is from the need for the perturbative series to

$$\begin{aligned}
\text{N=4 planar} &= i^3 s_{12} s_{23} \left[\text{red oval} \left[s_{12}^2 \text{ (diagram)} + s_{23}^2 \text{ (diagram)} \right. \right. \\
&\quad \left. \left. + 2s_{12}(\ell + k_4)^2 \text{ (diagram)} + 2s_{23}(\ell + k_1)^2 \text{ (diagram)} \right] \right]
\end{aligned}$$

Figure 9. Graphical representation of the three-loop amplitude for MSYM in the planar limit.

be summable into something which becomes “simple” in the planar strong-coupling limit, since that corresponds, via AdS/CFT, to a weakly-coupled supergravity theory. The fact that the relation is special to the conformal limit $D \rightarrow 4$, and to the planar limit, backs up this speculation. Obviously it would be nice to have some more information at three loops. There have been other hints of an iterative structure in the four-point correlation functions of chiral primary (BPS) composite operators [46], but here also the exact structure is not yet clear. Integrability has played a key role in recent higher-loop computations of non-BPS spin-chain anomalous dimensions [4,5,6,8]. By imposing regularity of the BMN ‘continuum’ limit [3], a piece of the anomalous dimension matrix has even been summed to all orders in $g^2 N_c$ in terms of hypergeometric functions [7]. The quantities we considered here — gauge-invariant, but dimensionally regularized, scattering amplitudes of color non-singlet states — are quite different from the composite color-singlet operators usually treated. Yet there should be some underlying connection between the different perturbative series.

6 Aside: Anomalous Dimensions in QCD and MSYM

As mentioned previously, the set of anomalous dimensions for leading-twist operators was recently computed at NNLO in QCD, as the culmination of a multi-year effort [13] which is central to performing precise computations of hadron collider cross sections. Shortly after the Moch, Vermaseren and Vogt computation, the anomalous dimensions in MSYM were extracted from this result by Kotikov, Lipatov, Onishchenko and Velizhanin [12]. (The MSYM anomalous dimensions are universal; supersymmetry implies that there is only one independent one for each Mellin moment j .) This extraction was non-trivial, because MSYM contains scalars, interacting through both gauge and Yukawa interactions, whereas QCD does not. However, Kotikov *et al.* noticed, from comparing NLO computations in both leading-twist anomalous dimensions and BFKL evolution, that the “most complicated terms” in the QCD

computation always coincide with the MSYM result, once the gauge group representation of the fermions is shifted from the fundamental to the adjoint representation. One can define the “most complicated terms” in the x -space representation of the anomalous dimensions — *i.e.* the splitting kernels — as follows: Assign a logarithm or factor of π a transcendentality of 1, and a polylogarithm Li_m or factor of $\zeta_m = \text{Li}_m(1)$ a transcendentality of m . Then the most complicated terms are those with leading transcendentality. For the NNLO anomalous dimensions, this turns out to be transcendentality 4. (This rule for extracting the MSYM terms from QCD has also been found to hold directly at NNLO, for the doubly-virtual contributions [38].) Strikingly, the NNLO MSYM anomalous dimension obtained for $j = 4$ by this procedure agrees with a previous result derived by assuming an integrable structure for the planar three-loop contribution to the dilatation operator [5].

7 Conclusions and Outlook

$\mathcal{N} = 4$ super-Yang-Mills theory is an excellent testing ground for techniques for computing, and understanding the structure of, QCD scattering amplitudes which are needed for precise theoretical predictions at high-energy colliders. One can even learn something about the structure of $\mathcal{N} = 4$ super-Yang-Mills theory in the process, although clearly there is much more to be understood. Some open questions include: Is there any AdS/CFT “dictionary” for color non-singlet states, like plane-wave gluons? Can one recover composite operator correlation functions from any limits of multi-point scattering amplitudes? Is there a better way to infrared regulate $\mathcal{N} = 4$ supersymmetric scattering amplitudes, that might be more convenient for approaching the AdS/CFT correspondence, such as compactification on a three-sphere, use of twistor-space, or use of coherent external states? Further investigations of this arena will surely be fruitful.

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