Name ______________________ Class ______________ Date ______________

8-1 Additional Vocabulary Support

Inverse Variation

Choose the expression from the list that best matches each sentence.

combined variation	constant of variation	inverse variation	joint variation

1. equations of the form $xy = k$ inverse variation
2. when one quantity varies with respect to two or more quantities combined variation
3. when one quantity varies directly with two or more quantities joint variation
4. the product of two variables in an inverse variation constant of variation

Choose the expression from the list that best matches each sentence.

combined variation	constant of variation	inverse variation	joint variation

5. When one quantity increases and the other quantity decreases proportionally, the relationship is an inverse variation.
6. The function $z = kxy$ is an example of a joint variation.
7. The constant of variation is represented by the variable k.
8. Both functions $z = \frac{kx}{wy}$ and $z = kxy$ are examples of combined variation.

Multiple Choice

9. Which function would be used to model the relationship "x and y vary inversely"? C

   (A) $y = kx$ (B) $z = kxy$ (C) $y = \frac{k}{x}$ (D) $y = \frac{x}{k}$

10. Which function would be used to model the relationship "z varies jointly with x and y"? G

   (F) $y = kxz$ (G) $z = kxy$ (H) $y = \frac{k}{xz}$ (I) $y = \frac{xz}{k}$

Name ______________________ Class ____________ Date ____________

8-1 Think About a Plan

Inverse Variation

The spreadsheet shows data that could be modeled by an equation of the form $PV = k$. Estimate P when $V = 62$.

	A	B
1	P	V
2	140.00	100
3	147.30	95
4	155.60	90
5	164.70	85
6	175.00	80
7	186.70	75

Understanding the Problem

1. The data can be modeled by **$PV = k$**.

2. What is the problem asking you to determine?

an estimate of the value of P when $V = 62$

Planning the Solution

3. What does it mean that the data can be modeled by an inverse variation?

The product of each pair of P and V values is approximately the same constant.

4. How can you estimate the constant of the inverse variation?

Find PV for each row of the data. It should be approximately the same for each row.

5. What is the constant of the inverse variation? **about 14,000**

6. Write an equation that you can use to find P when $V = 62$. **$62P = 14{,}000$**

Getting an Answer

7. Solve your equation.

$62P = 14{,}000$

$P = \frac{14{,}000}{62} \approx 226$

8. What is an estimate for P when $V = 62$? **about 226**

Name ______________________ Class ____________ Date ____________

8-1 Practice

Form G

Inverse Variation

Is the relationship between the values in each table a *direct variation*, an *inverse variation*, or *neither*? Write equations to model the direct and inverse variations.

1.

x	2	4	5	20
y	10	5	4	1

inverse; $y = \frac{20}{x}$

2.

x	1	3	7	10
y	2	8	20	29

neither

3.

x	1	2	5	7
y	6	12	30	42

direct; $y = 6x$

4.

x	0.2	0.5	2	3
y	25	62.5	250	375

direct; $y = 125x$

5.

x	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{3}{2}$	2
y	31	7	3	$\frac{5}{2}$

neither

6.

x	3	1.5	0.5	0.3
y	5	10	30	50

inverse; $y = \frac{15}{x}$

Suppose that x and y vary inversely. Write a function that models each inverse variation. Graph the function and find y when $x = 10$.

7. $x = 7$ when $y = 2$
   $y = \frac{14}{x}$; $\frac{7}{5}$

8. $x = 4$ when $y = 0.2$
   $y = \frac{4}{5x}$; 0.08 or $\frac{2}{25}$

9. $x = \frac{1}{3}$ when $y = \frac{9}{10}$
   $y = \frac{3}{10x}$; 0.03 or $\frac{3}{100}$

10. The students in a school club decide to raise money by selling hats with the school mascot on them. The table below shows how many hats they can expect to sell based on how much they charge per hat in dollars.

Price per Hat (p)	5	6	8	9
Hats Sold (h)	72	60	45	40

a. What is a function that models the data? $ph = 360$ or $h = \frac{360}{p}$

b. How many hats can the students expect to sell if they charge $7.50 per hat? 48

11. The minimum number of carpet rolls n needed to carpet a house varies directly as the house's square footage h and inversely with the square footage r in one roll. It takes a minimum of two 1200-ft^2 carpet rolls to cover 2300 ft^2 of floor. What is the minimum number of 1200-ft^2 carpet rolls you would need to cover 2500 ft^2 of floor? Round your answer up to the nearest half roll. 2.5

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8-1 Practice (continued)

Form G

Inverse Variation

12. On Earth, the mass m of an object varies directly with the object's potential energy E and inversely with its height above the Earth's surface h. What is an equation for the mass of an object on Earth? (*Hint:* $E = gmh$, where g is the acceleration due to gravity.) $m = \frac{kE}{h}$, where $k = \frac{1}{g}$

Each ordered pair is from an inverse variation. Find the constant of variation.

13. $\left(3, \frac{1}{3}\right)$ 1

14. $(0.2, 6)$ 1.2

15. $(10, 5)$ 50

16. $\left(\frac{5}{7}, \frac{2}{5}\right)$ $\frac{2}{7}$

17. $(-13, 22)$ -286

18. $\left(\frac{1}{2}, 10\right)$ 5

19. $\left(\frac{1}{3}, \frac{6}{7}\right)$ $\frac{2}{7}$

20. $(4.8, 2.9)$ 13.92

21. $\left(\frac{5}{8}, -\frac{2}{5}\right)$ $-\frac{1}{4}$

22. $(4.75, 4)$ 19

Write the function that models each variation. Find z when $x = 6$ and $y = 4$.

23. z varies jointly with x and y. When $x = 7$ and $y = 2$, $z = 28$. $z = 2xy$; 48

24. z varies directly with x and inversely with the cube of y. When $x = 8$ and $y = 2$, $z = 3$.

$z = \frac{3x}{y^3}$; $\frac{9}{32}$

Each pair of values is from an inverse variation. Find the missing value.

25. $(2, 4), (6, y)$ $\frac{4}{3}$

26. $\left(\frac{1}{3}, 6\right), \left(x, -\frac{1}{2}\right)$ -4

27. $(1.2, 4.5), (2.7, y)$ 2

28. One load of gravel contains 240 ft^3 of gravel. The area A that the gravel will cover is inversely proportional to the depth d to which the gravel is spread.

a. Write a model for the relationship between the area and depth for one load of gravel. $A = \frac{240}{d}$

b. A designer plans a playground with gravel 6 in. deep over the entire play area. If the play area is a rectangle 40 ft wide and 24 ft long, how many loads of gravel will be needed? 2 loads

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8-1 Practice

Form K

Inverse Variation

Is the relationship between the values in each table a *direct variation*, an *inverse variation*, or *neither*? Write an equation to model the direct and inverse variations.

1.

x	y
0.1	3
3	0.1
6	0.05
24	0.0125

inverse variation; $y = \frac{0.3}{x}$

2.

x	y
1	3
2	6
5	15
6	18

direct variation; $y = 3x$

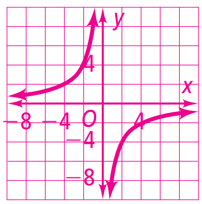
3.

x	y
0	1
2	5
4	7
6	8

neither

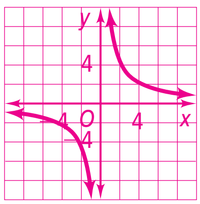
Suppose that x and y vary inversely. Write a function that models each inverse variation. Graph the function and find y when $x = 10$.

4. $x = 2$ when $y = -4$



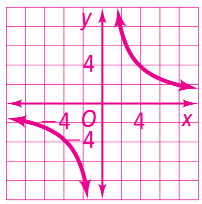
$y = -\frac{8}{x}$; $-\frac{4}{5}$

5. $x = -9$ when $y = -1$



$y = \frac{9}{x}$; $\frac{9}{10}$

6. $x = 1.5$ when $y = 10$



$y = \frac{15}{x}$; 1.5

7. Suppose the table at the right shows the time t it takes to drive home when you travel at various average speeds s.

a. Write a function that models the relationship between the speed and the time it takes to drive home. **$s = \frac{10}{t}$**

b. At what speed would you need to drive to get home in 50 min or $\frac{5}{6}$ h? **12 mi/h**

Time t (h)	Speed s (mi/h)
$\frac{1}{6}$	60
$\frac{1}{4}$	40
$\frac{1}{3}$	30
$\frac{3}{4}$	13.3

Name ______________________ Class ____________ Date ____________

8-1 Practice (continued) Form K

Inverse Variation

Use combined variation to solve each problem.

8. The height h of a cylinder varies directly with the volume of the cylinder and inversely with the square of the cylinder's radius r with the constant equal to $\frac{1}{\pi}$.

a. Write a formula that models this combined variation. $h = \frac{V}{\pi r^2}$

b. What is the height of a cylinder with radius 4 m and volume 500 m^3? Use 3.14 for π and round to the nearest tenth of a meter. **10.0 m**

9. Some students volunteered to clean up a highway near their school. The amount of time it will take varies directly with the length of the section of highway and inversely with the number of students who will help. If 25 students clean up 5 mi of highway, the project will take 2 h. How long would it take 85 students to clean up 34 mi of highway? **4 h**

Write the function that models each variation. Find z when $x = 2$ and $y = 6$.

10. z varies inversely with x and directly with y. When $x = 5$ and $y = 10$, $z = 2$. $z = \frac{y}{x}$; 3

11. z varies directly with the square of x and inversely with y. When $x = 2$ and $y = 4$, $z = 3$. $z = \frac{3x^2}{y}$; 2

Each ordered pair is from an inverse variation. Find the constant of variation.

12. (2, 2)
$k = 4$

13. (1, 8)
$k = 8$

14. (9, 4)
$k = 36$

Each pair of values is from an inverse variation. Find the missing value.

15. (9, 5), (x, 3)
$x = 15$

16. (8, 7), (5, y)
$y = 11.2$

17. (2, 7), (x, 1)
$x = 14$

Name ______________________________ Class ______________ Date ______________

8-1 Standardized Test Prep

Inverse Variation

Multiple Choice

For Exercises 1–5, choose the correct letter.

1. Which equation represents inverse variation between x and y? **B**

(A) $4y = kx$ (B) $xy = 4k$ (C) $y = 4kx$ (D) $4k = \frac{x}{y}$

2. The ordered pair (3.5, 1.2) is from an inverse variation. What is the constant of variation? **H**

(F) 2.3 (G) 2.9 (H) 4.2 (I) 4.7

3. Suppose x and y vary inversely, and $x = 4$ when $y = 9$. Which function models the inverse variation? **A**

(A) $y = \frac{36}{x}$ (B) $x = \frac{y}{36}$ (C) $y = \frac{x}{36}$ (D) $\frac{x}{y} = 36$

4. Suppose x and y vary inversely, and $x = -3$ when $y = \frac{1}{3}$. What is the value of y when $x = 9$? **H**

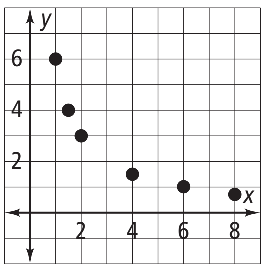
(F) -9 (G) -1 (H) $-\frac{1}{9}$ (I) $\frac{1}{9}$

5. In which function does t vary jointly with q and r and inversely with s? **D**

(A) $t = \frac{kq}{rs}$ (B) $t = \frac{ks}{qr}$ (C) $t = \frac{s}{kqr}$ (D) $t = \frac{kqr}{s}$

Short Response

6. A student suggests that the graph at the right represents the inverse variation $y = \frac{3}{x}$. Is the student correct? Explain.



No; for each point on the graph $xy = 6$, not 3.

[2] correct answer with explanation

[1] correct answer, without explanation

[0] no answer given

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8-1 Enrichment

Inverse Variation

Each situation below can be modeled by a direct variation, inverse variation, joint variation, or combined variation equation. Decide which model to use and explain why.

1. The circumference C of a circle is about 3.14 times the diameter d.
   Direct variation; answers may vary. Sample: This relationship can be modeled by the equation $C = 3.14d$, where 3.14 is the constant of variation.

2. The number of cavities that develop in a patient's teeth depends on the total number of minutes spent brushing.
   Inverse variation; answers may vary. Sample: As the number of minutes spent brushing increases, the number of cavities should decrease. This suggests an inverse variation.

3. The time it takes to build a bridge depends on the number of workers.
   Inverse variation; answers may vary. Sample: As the number of workers increases, the time it takes to build a bridge should decrease. This suggests an inverse variation.

4. The number of minutes it will take to solve a problem set depends on the number of problems and the number of people working on the problem set.
   Combined variation; answers may vary. Sample: The time to solve a problem set increases as the number of problems increases, but decreases as the number of people working on the set increases. This suggests a combined variation.

5. The current I in an electrical circuit decreases as the resistance R increases.
   Inverse variation; answers may vary. Sample: As one variable increases, the other decreases. This suggests an inverse variation.

6. Charles's Gas Law states the volume V of an enclosed gas at a constant pressure will increase as the absolute temperature T increases.
   Direct variation; answers may vary. Sample: As one variable increases, so does the other. This suggests a direct variation.

7. Boyle's Law states that the volume V of an enclosed gas at a constant temperature is related to the pressure. The pressure of 3.45 L of neon gas is 0.926 atmosphere (atm). At the same temperature, the pressure of 2.2 L of neon gas is 1.452 atm.
   Inverse variation; answers may vary. Sample: As the pressure increases, the volume decreases. This suggests an inverse variation.

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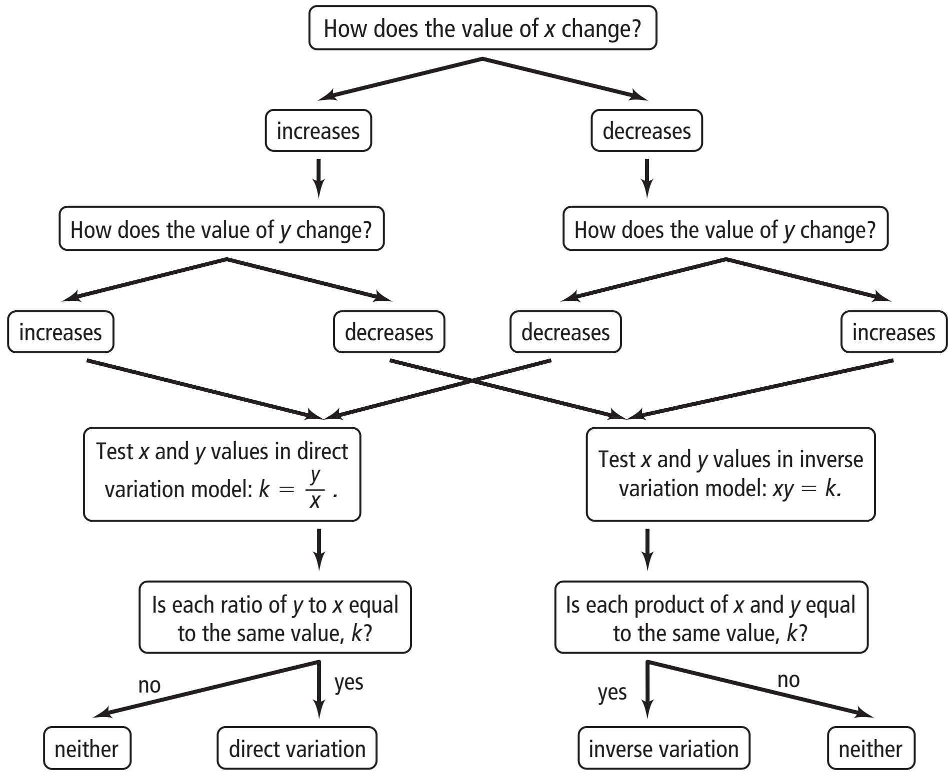
8-1 Reteaching

Inverse Variation

The flowchart below shows how to decide whether a relationship between two variables is a direct variation, inverse variation, or neither.



Problem

Do the data in the table represent a direct variation, inverse variation, or neither?

x	1	2	4	5
y	20	10	5	4

As the value of x increases, the value of y decreases, so test the table values in the inverse variation model: $xy = k$: $1 \cdot 20 = 20$, $2 \cdot 10 = 20$, $4 \cdot 5 = 20$, $5 \cdot 4 = 20$. Each product equals the same value, 20, so the data in the table model an inverse variation.

Exercises

Do the data in the table represent a direct variation, inverse variation, or neither?

1.

x	5	10	15	20
y	10	20	30	40

direct variation

2.

x	1	3	4	6
y	12	4	3	2

inverse variation

Name ______________________ Class ____________ Date ____________

8-1 Reteaching (continued)

Inverse Variation

To solve problems involving inverse variation, you need to solve for the constant of variation k before you can find an answer.

Problem

The time t that is necessary to complete a task varies inversely as the number of people p working. If it takes 4 h for 12 people to paint the exterior of a house, how long does it take for 3 people to do the same job?

$t = \frac{k}{p}$ Write an inverse variation. Because time is dependent on people, t is the dependent variable and p is the independent variable.

$4 = \frac{k}{12}$ Substitute 4 for t and 12 for p.

$48 = k$ Multiply both sides by 12 to solve for k, the constant of variation.

$t = \frac{48}{p}$ Substitute 48 for k. This is the equation of the inverse variation.

$t = \frac{48}{3} = 16$ Substitute 3 for p. Simplify to solve the equation.

It takes 3 people 16 h to paint the exterior of the house.

Exercises

3. The time t needed to complete a task varies inversely as the number of people p. It takes 5 h for seven men to install a new roof. How long does it take ten men to complete the job? **3.5 h**

4. The time t needed to drive a certain distance varies inversely as the speed r. It takes 7.5 h at 40 mi/h to drive a certain distance. How long does it take to drive the same distance at 60 mi/h? **5 h**

5. The cost of each item bought is inversely proportional to the number of items when spending a fixed amount. When 42 items are bought, each costs $1.46. Find the number of items when each costs $2.16. **about 28 items**

6. The length l of a rectangle of a certain area varies inversely as the width w. The length of a rectangle is 9 cm when the width is 6 cm. Determine the length if the width is 8 cm. **6.75 cm**

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8-2 Additional Vocabulary Support

The Reciprocal Function Family

For Exercises 1–5, draw a line from each word in Column A to its definition in Column B.

Column A	Column B
1. reciprocal	A. function that models inverse variation
2. branch	B. stretches, compressions, reflections, and horizontal and vertical translations
3. reciprocal function	C. multiplicative inverse
4. reflection of the reciprocal function	D. the graph of $y = -\frac{1}{x}$
5. transformations	E. each part of the graph of a reciprocal function

For Exercises 6–9, the graph of each function is a transformation of the parent graph of $f(x) = \frac{1}{x}$. Draw a line from each function to its transformation.

6. $f(x) = \frac{2}{x}$	A. a horizontal translation
7. $f(x) = -\frac{1}{x}$	B. a reflection over the x-axis
8. $f(x) = \frac{1}{x - 2}$	C. a vertical translation
9. $f(x) = \frac{1}{x} + 4$	D. a stretch

Name ______________________ Class ____________ Date ____________

8-2 Think About a Plan

The Reciprocal Function Family

a. Gasoline Mileage Suppose you drive an average of 10,000 miles each year. Your gasoline mileage (mi/gal) varies inversely with the number of gallons of gasoline you use each year. Write and graph a model for your average mileage *m* in terms of the gallons *g* of gasoline used.

b. After you begin driving on the highway more often, you use 50 gal less per year. Write and graph a new model to include this information.

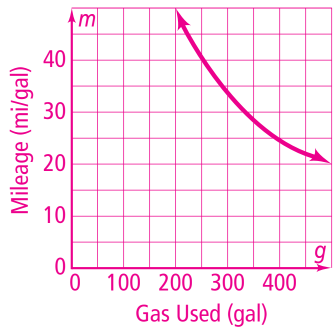
c. Calculate your old and new mileage assuming that you originally used 400 gal of gasoline per year.

1. Write a formula for gasoline mileage in words.

The mileage is equal to the number of miles divided by the number of gallons.

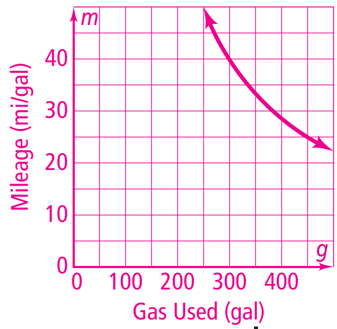
2. Write and graph an equation to model your average mileage *m* in terms of the gallons *g* of gasoline used.

$m = \frac{10{,}000}{g}$



3. Write and graph an equation to model your average mileage *m* in terms of the gallons *g* of gasoline used if you use 50 gal less per year.

$m = \frac{10{,}000}{g - 50}$



4. How can you find your old and your new mileage from your equations?

Evaluate each equation at $g = 400$.

5. What is your old mileage? **25 mi/gal**

6. What is your new mileage? **about 28.6 mi/gal**

Name ______________________________ Class ____________ Date ____________

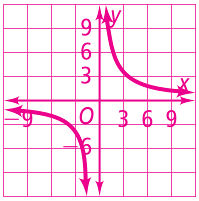
8-2 Practice

Form G

The Reciprocal Function Family

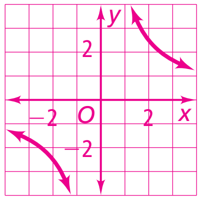
Graph each function. Identify the *x*- and *y*-intercepts and the asymptotes of the graph. Also, state the domain and the range of the function.

1. $y = \frac{12}{x}$



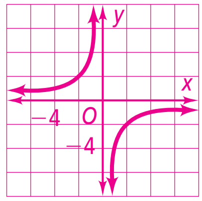
no intercepts; *x* = 0, *y* = 0; all real numbers except *x* = 0; all real num. except *y* = 0

2. $y = \frac{5}{x}$



no intercepts; *x* = 0, *y* = 0; all real numbers except *x* = 0; all real num. except *y* = 0

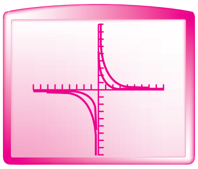
3. $y = -\frac{4}{x}$



no intercepts; *x* = 0, *y* = 0; all real numbers except *x* = 0; all real num. except *y* = 0

Use a graphing calculator to graph the equations $y = \frac{1}{x}$ and $y = \frac{a}{x}$ using the given value of *a*. Then identify the effect of *a* on the graph.

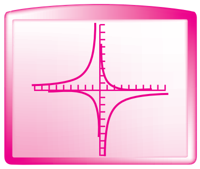
4. $a = 3$



x scale: 1 *y* scale: 1

Stretch by a factor of 3.

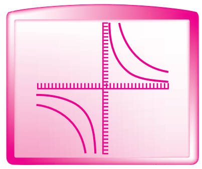
5. $a = -5$



x scale: 1 *y* scale: 1

Reflect over *x*-axis and stretch by a factor of 5.

6. $a = 0.4$

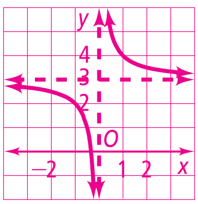


x scale: 0.1 *y* scale: 0.1

Shrink by a factor of 0.4.

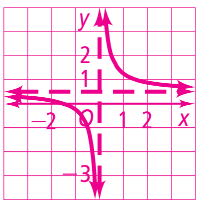
Sketch the asymptotes and the graph of each function. Identify the domain and range.

7. $y = \frac{1}{x} + 3$



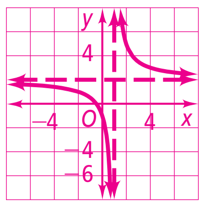
all real numbers except *x* = 0; all real numbers except *y* = 3

8. $y = \frac{3}{4x} + \frac{1}{2}$



all real numbers except *x* = 0; all real numbers except $y = \frac{1}{2}$

9. $y = \frac{3}{x - 1} + 2$



all real numbers except *x* = 1; all real numbers except *y* = 2

Write an equation for the translation of $y = -\frac{3}{x}$ that has the given asymptotes.

10. $x = -1; y = 3$

$y = -\frac{3}{x + 1} + 3$

11. $x = 4; y = -2$

$y = -\frac{3}{x - 4} - 2$

12. $x = 0; y = 6$

$y = -\frac{3}{x} + 6$

8-2 Practice (continued)

Form G

The Reciprocal Function Family

13. The length of a pipe in a panpipe ℓ (in feet) is inversely proportional to its pitch p (in hertz). The inverse variation is modeled by the equation $p = \frac{495}{\ell}$. Find the length required to produce a pitch of 220 Hz. **2.25 ft**

Write each equation in the form $y = \frac{k}{x}$.

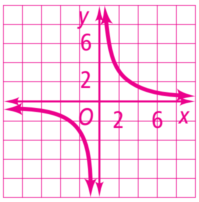
14. $y = \frac{4}{5x}$ $y = \frac{0.8}{x}$

15. $y = -\frac{7}{2x}$ $y = \frac{-3.5}{x}$

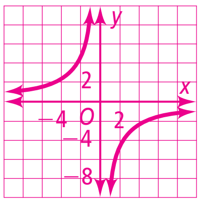
16. $xy = -0.03$ $y = \frac{-0.03}{x}$

Sketch the graph of each function.

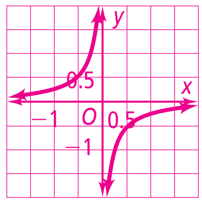
17. $xy = 6$



18. $xy + 10 = 0$



19. $4xy = -1$



20. The junior class is buying keepsakes for Class Night. The price of each keepsake p is inversely proportional to the number of keepsakes s bought. The keepsake company also offers 10 free keepsakes in addition to the class's order. The equation $p = \frac{1800}{s + 10}$ models this inverse variation.

a. If the class buys 240 keepsakes, what is the price for each one? **$7.20**

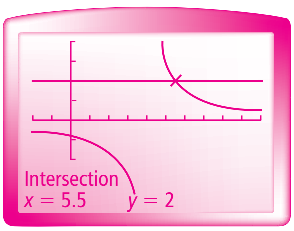
b. If the class pays $5.55 for each keepsake, how many can they get, including the free keepsakes? **324**

c. If the class buys 400 keepsakes, what is the price for each one? **$4.39**

d. If the class buys 50 keepsakes, what is the price for each one? **$30**

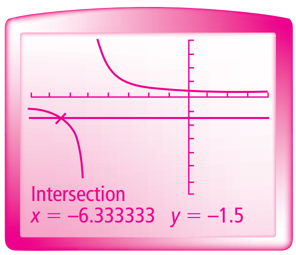
Graph each pair of functions. Find the approximate point(s) of intersection.

21. $y = \frac{3}{x - 4}$; $y = 2$ **(5.5, 2)**



x scale: 1 y scale: 1

22. $y = \frac{2}{x + 5}$; $y = -1.5$ **(−6.3, −1.5)**



x scale: 1 y scale: 1

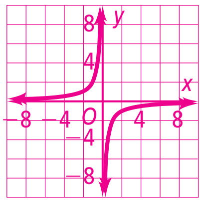
Name ______________________ Class ____________ Date ____________

8-2 Practice
Form K

The Reciprocal Function Family

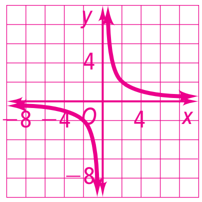
Graph each function. Identify the *x*- and *y*-intercepts and asymptotes of the graph. Also, state the domain and range of the function.

1. $y = -\frac{2}{x}$



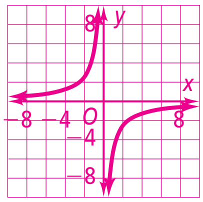
x-intercept: none; *y*-intercept: none; horizontal asymptote: *x*-axis; vertical asymptote: *y*-axis; domain: all real numbers except $x = 0$; range: all real numbers except $y = 0$

2. $y = \frac{4}{x}$



x-intercept: none; *y*-intercept: none; horizontal asymptote: *x*-axis; vertical asymptote: *y*-axis; domain: all real numbers except $x = 0$; range: all real numbers except $y = 0$

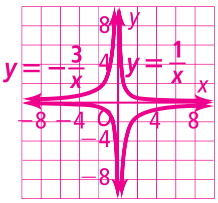
3. $y = -\frac{5}{x}$



x-intercept: none; *y*-intercept: none; horizontal asymptote: *x*-axis; vertical asymptote: *y*-axis; domain: all real numbers except $x = 0$; range: all real numbers except $y = 0$

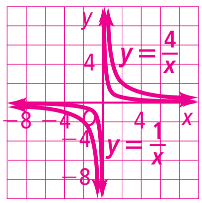
Graph the equations $y = \frac{1}{x}$ and $y = \frac{a}{x}$ using the given value of a. Then identify the effect of a on the graph.

4. $a = -3$



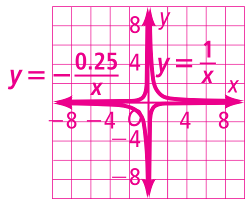
reflected across the *x*-axis and stretched by a factor of 3

5. $a = 4$



stretched by a factor of 4

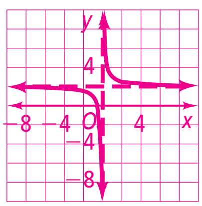
6. $a = -0.25$



reflected across the *x*-axis and compressed by a factor of 0.25

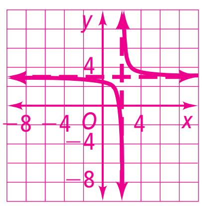
Sketch the asymptotes and the graph of each function. Identify the domain and range.

7. $y = \frac{1}{x} + 2$



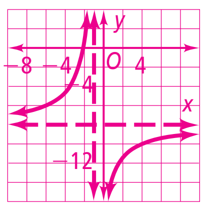
domain: all real numbers except 0; range: all real numbers except 2

8. $y = \frac{1}{x - 2} + 3$



domain: all real numbers except 2; range: all real numbers except 3

9. $y = \frac{-10}{x + 1} - 8$



domain: all real numbers except −1; range: all real numbers except −8

Name ______________________ Class ____________ Date ____________

8-2 Practice (continued) — Form K

The Reciprocal Function Family

Write an equation for the translation of $y = \frac{3}{x}$ that has the given asymptotes.

10. $x = 0$ and $y = 2$

$y = \frac{3}{x} + 2$

11. $x = -2$ and $y = 4$

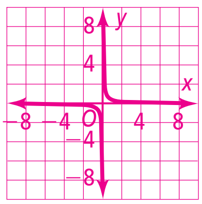
$y = \frac{3}{x+2} + 4$

12. $x = 5$ and $y = -3$

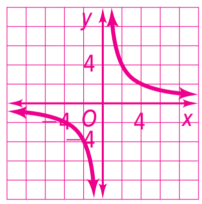
$y = \frac{3}{x-5} - 3$

Sketch the graph of each function.

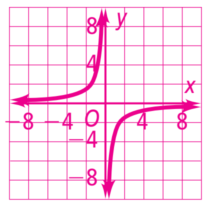
13. $3xy = 1$



14. $xy - 8 = 0$



15. $2xy = -6$



16. Writing Explain the difference between what happens to the graph of the parent function of $y = \frac{a}{x}$ when $|a| > 1$ and what happens when $0 < |a| < 1$.

When $|a| > 1$, the parent function is stretched by the factor of a. When $0 < |a| < 1$, the parent function is compressed by the factor of a.

17. Suppose your class wants to get your teacher an end-of-year gift of a weekend package at her favorite spa. The package costs \$250. Let c equal the cost each student needs to pay and s equal the number of students.

a. If there are 22 students, how much will each student need to pay? **\$11.37**

b. Using the information, how many total students (including those from other classes) need to contribute to the teacher's gift, if no student wants to pay more than \$7? **36 students**

c. Reasoning Did you need to round your answers up or down? Explain.

up; because if you round down, the total contribution by the students would be less than \$250.

Name ______________________ Class ____________ Date ____________

8-2 Standardized Test Prep

The Reciprocal Function Family

Multiple Choice

For Exercises 1–3, choose the correct letter.

1. What is an equation for the translation of $y = -\frac{4.5}{x}$ that has asymptotes at $x = 3$ and $y = -5$? **A**

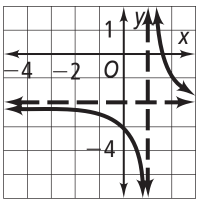
(A) $y = -\frac{4.5}{x - 3} - 5$ (C) $y = -\frac{4.5}{x - 5} + 3$

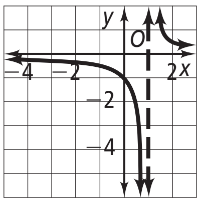
(B) $y = -\frac{4.5}{x + 3} - 5$ (D) $y = -\frac{4.5}{x + 5} + 3$

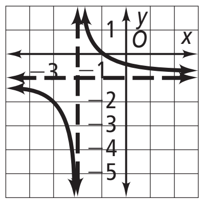
2. What is the equation of the vertical asymptote of $y = \frac{2}{x - 5}$? **I**

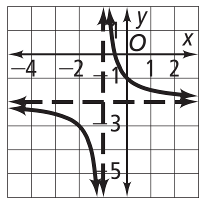
(F) $x = -5$ (G) $x = 0$ (H) $x = 2$ (I) $x = 5$

3. Which is the graph of $y = \frac{1}{x + 1} - 2$? **D**

(A) 

(B) 

(C) 

(D) 

Extended Response

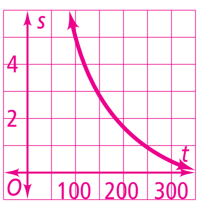
4. A race pilot's average rate of speed over a 720-mi course is inversely proportional to the time in minutes t the pilot takes to fly a complete race course. The pilot's final score s is the average speed minus any penalty points p earned.

a. Write a function to model the pilot's score for a given t and p. (*Hint:* $d = rt$)

b. Graph the function for a pilot who has 2 penalty points.

c. What is the maximum time a pilot with 2 penalty points can take to finish the course and still earn a score of at least 3?

[4] $s = \frac{720}{t} - p$;



144 min

[3] correct answer with most of work shown and appropriate strategies used OR incorrect answer with all work shown and appropriate strategies used

[2] correct answer with little work shown OR incorrect answer but work shown reflects some understanding of problem

[1] answer is incomplete or incorrect and no work is shown

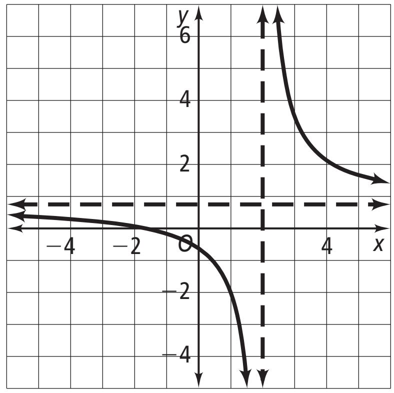
[0] no answer given

Name ______________________ Class ____________ Date ____________

8-2 Enrichment

The Reciprocal Function Family

Understanding Horizontal Asymptotes



The line $y = \frac{3}{4}$ is a horizontal asymptote for the graph of the function $y = \frac{3x + 5}{4x - 8}$. By using long division, you can rewrite this function in the form quotient + remainder divided by the divisor: $y = \frac{3}{4} + \frac{11}{4x - 8}$.

Examine what happens to the remainder divided by the divisor and the value of *y* as the value of *x* gets larger. Fill in the following table to four decimal places.

	x	$\frac{11}{4x - 8}$	$y = \frac{3}{4} + \frac{11}{4x - 8}$
1.	3	**2.7500**	**3.5000**
2.	10	**0.3438**	**1.0938**
3.	100	**0.0281**	**0.7781**

Note that as *x* gets larger, both the remainder and the value of *y* get smaller. Although the value of *y* is always greater than $\frac{3}{4}$, it gets closer to $\frac{3}{4}$ as *x* gets larger. As *x* gets infinitely large, *y* approaches $\frac{3}{4}$ from above. Write this as: As $x \to +\infty$, $y \to \frac{3}{4}$ from above.

Examine what happens as *x* gets smaller. Fill in the following table to four decimal places.

	x	$\frac{11}{4x - 8}$	$y = \frac{3}{4} + \frac{11}{4x - 8}$
4.	−3	**−0.5500**	**0.2000**
5.	−10	**−0.2292**	**0.5208**
6.	−100	**−0.0270**	**0.7230**

Here the value of *y* is always less than $\frac{3}{4}$, but it gets closer to $\frac{3}{4}$ as *x* gets smaller (more negative). Write this as: As $x \to -\infty$, $y \to \frac{3}{4}$ from below.

In both cases, *y* approaches $\frac{3}{4}$, so the horizontal asymptote is $y = \frac{3}{4}$.

Name ______________________ Class ____________ Date ____________

8-2 Reteaching

The Reciprocal Function Family

A Reciprocal Function in General Form

The *general form* is $y = \frac{a}{x - h} + k$, where $a \neq 0$ and $x \neq h$.

The graph of this equation has a horizontal asymptote at $y = k$ and a vertical asymptote at $x = h$.

Two Members of the Reciprocal Function Family

When $a \neq 1$, $h = 0$, and $k = 0$, you get the *inverse variation function,* $y = \frac{a}{x}$.

When $a = 1$, $h = 0$, and $k = 0$, you get the *parent reciprocal function,* $y = \frac{1}{x}$.

Problem

What is the graph of the inverse variation function $y = \frac{-5}{x}$?

Step 1 Rewrite in general form and identify a, h, and k.

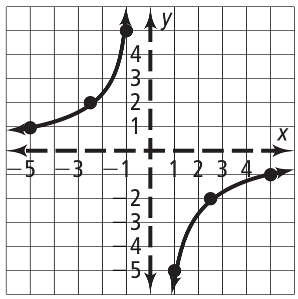
$y = \frac{-5}{x - 0} + 0$ $\qquad a = -5, h = 0, k = 0$

Step 2 Identify and graph the horizontal and vertical asymptotes.

horizontal asymptote: $y = k$
$y = 0$
vertical asymptote: $x = h$
$x = 0$

Step 3 Make a table of values for $y = \frac{-5}{x}$. Plot the points and then connect the points in each quadrant to make a curve.

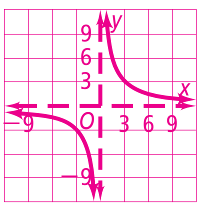
x	−5	−2.5	−1	1	2.5	5
y	1	2	5	−5	−2	−1



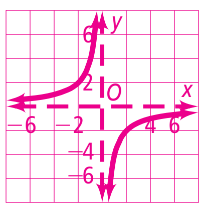
Exercises

Graph each function. Include the asymptotes.

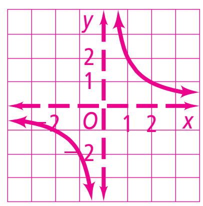
1. $y = \frac{9}{x}$



2. $y = -\frac{4}{x}$



3. $xy = 2$



Name ______________________ Class ____________ Date ____________

8-2 Reteaching (continued)

The Reciprocal Function Family

A reciprocal function in the form $y = \frac{a}{x - h} + k$ is a *translation* of the inverse variation function $y = \frac{a}{x}$. The translation is h units horizontally and k units vertically. The translated graph has asymptotes at $x = h$ and $y = k$.

Problem

What is the graph of the reciprocal function $y = -\frac{6}{x + 3} + 2$?

Step 1 Rewrite in general form and identify a, h, and k.

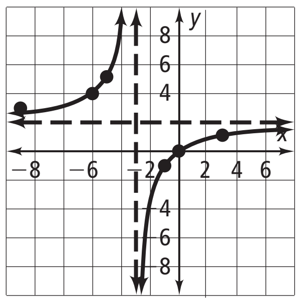
$y = \frac{-6}{x - (-3)} + 2$ $\quad\quad a = -6,\ h = -3,\ k = 2$

Step 2 Identify and graph the horizontal and vertical asymptotes.

horizontal asymptote: $y = k$
$y = 2$
vertical asymptote: $x = h$
$x = -3$

Step 3 Make a table of values for $y = \frac{-6}{x}$, then *translate* each (x, y) pair to $(x + h, y + k)$. Plot the translated points and connect the points in each quadrant to make a curve.

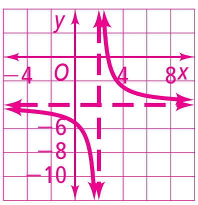
x	−6	−3	−2	2	3	6
y	1	2	3	−3	−2	−1
x + (−3)	−9	−6	−5	−1	0	3
y + 2	3	4	5	−1	0	1



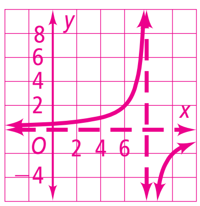
Exercises

Graph each function. Include the asymptotes.

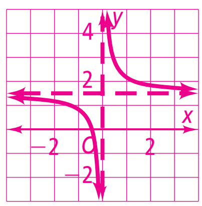
4. $y = \frac{3}{x - 2} - 4$



5. $y = -\frac{4}{x - 8}$



6. $y = \frac{2}{3x} + \frac{3}{2}$



Name ______________________ Class ____________ Date ____________

8-3 Additional Vocabulary Support

Rational Functions and Their Graphs

Concept List

continuous	discontinuous	factors
horizontal asymptote	non-removable discontinuity	point of discontinuity
rational function	removable discontinuity	vertical asymptote

Choose the concept from the list above that best represents the item in each box.

1. the line that a graph approaches as y increases in absolute value vertical asymptote	**2.** In the denominator, these reveal the points of discontinuity. factors	**3.** This type of discontinuity appears as a hole in the graph. removable discontinuity
4. This type of graph has no jumps, breaks, or holes. continuous	**5.** a function that you can write in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions rational function	**6.** a graph that has a one-point hole or a vertical asymptote discontinuous
7. This type of discontinuity appears as a vertical asymptote on the graph. non-removable discontinuity	**8.** The graph of $f(x)$ is not continuous at this point. point of discontinuity	**9.** the line that a graph approaches as x increases in absolute value horizontal asymptote

Name ______________________ Class ____________ Date ____________

8-3 Think About a Plan

Rational Functions and Their Graphs

Grades A student earns an 82% on her first test. How many consecutive 100% test scores does she need to bring her average up to 95%? Assume that each test has equal impact on the average grade.

Understanding the Problem

1. One test score is **82%**.

2. The average of all the test scores is **95%**.

3. What is the problem asking you to determine?
   the number of tests with scores of 100% the student needs to have an average of 95%

Planning the Solution

4. Let x be the number of 100% test scores. Write an expression for the total number of test scores.
   $x + 1$

5. Write an expression for the sum of the test scores.
   $100x + 82$

6. How can you model the student's average as a rational function?
   $A = \frac{100x + 82}{x + 1}$

Getting an Answer

7. How can a graph help you answer this question?
   Answers may vary. Sample: I can graph the rational function and the function $y = 95$ at the same time. The intersection is the solution.

8. What does a fractional answer tell you? Explain.
   I would have to round a fractional answer up to the nearest whole number, because the number of test scores must be a whole number and the average must be 95% or greater.

9. How many consecutive 100% test scores does the student need to bring her average up to 95%? **3**

Name ______________________ Class ____________ Date ____________

8-3 Practice

Form G

Rational Functions and Their Graphs

Find the domain, points of discontinuity, and *x*- and *y*-intercepts of each rational function. Determine whether the discontinuities are removable or nonremovable.

1. $y = \frac{(x-4)(x+3)}{x+3}$

all real num except $x = -3$; (4, 0), (0, −4); removable

2. $y = \frac{(x-3)(x+1)}{x-2}$

all real num except 2; $x = 2$; (−1, 0) (3, 0), $(0, \frac{3}{2})$; non-removable

3. $y = \frac{2}{x+1}$

all real num except −1; $x = -1$, no x-intercept, (0, 2); non-removable

4. $y = \frac{4x}{x^4 + 16}$

all real num; none; (0, 0)

Find the vertical asymptotes and holes for the graph of each rational function.

5. $y = \frac{5-x}{x^2-1}$

vertical asymptotes at $x = 1$ and $x = -1$

6. $y = \frac{x^2-2}{x+2}$

vertical asymptote at $x = -2$

7. $y = \frac{x}{x(x-1)}$

vertical asymptote at $x = 1$; hole at $x = 0$

8. $y = \frac{x+3}{x^2-9}$

vertical asymptote at $x = 3$; hole at $x = -3$

9. $y = \frac{x-2}{(x+2)(x-2)}$

vertical asymptote at $x = -2$; hole at $x = 2$

10. $y = \frac{x^2-4}{x^2+4}$

no vertical asymptotes or holes

11. $y = \frac{x^2-25}{x-4}$

vertical asymptote at $x = 4$

12. $y = \frac{(x-2)(2x+3)}{(5x+4)(x-3)}$

vertical asymptotes at $x = -\frac{4}{5}$ and $x = 3$

Find the horizontal asymptote of the graph of each rational function.

13. $y = \frac{2}{x-6}$

$y = 0$

14. $y = \frac{x+2}{x-4}$

$y = 1$

15. $y = \frac{2x^2+3}{x^2-6}$

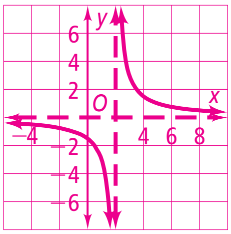
$y = 2$

16. $y = \frac{3x-12}{x^2-2}$

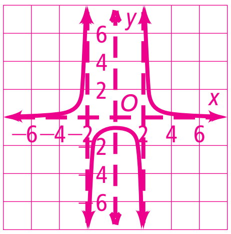
$y = 0$

Sketch the graph of each rational function.

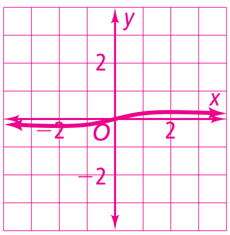
17. $y = \frac{3}{x-2}$



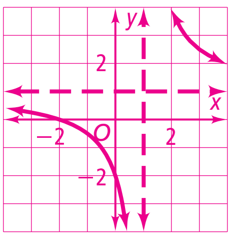
18. $y = \frac{3}{(x-2)(x+2)}$



19. $y = \frac{x}{x^2+4}$



20. $y = \frac{x+2}{x-1}$



Name ______________________ Class ____________ Date ____________

8-3 Practice (continued) Form G

Rational Functions and Their Graphs

21. How many milliliters of 0.75% sugar solution must be added to 100 mL of 1.5% sugar solution to form a 1.25% sugar solution? **50 mL**

22. A soccer player has made 3 of his last 24 shots on goal, or 12.5%. How many more consecutive goals does he need to raise his shots-on-goal average to at least 20%? **3**

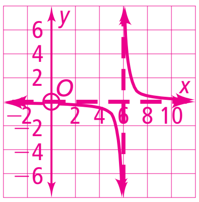
23. Error Analysis A student listed the asymptotes of the function $y = \frac{x^2 + 5x + 6}{x(x^2 + 4x + 4)}$ as shown at the right. Explain the student's error(s). What are the correct asymptotes?
The horizontal asymptote should be $y = 0$, because the degree of the numerator is less than the degree of the denominator. The zeros of the denominator are $x = 0$ and $x = -2$, so there should also be a vertical asymptote at $x = -2$.

horizontal asymptote
none

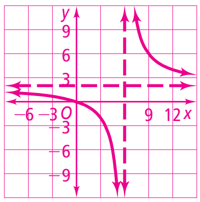
vertical asymptote
x = 0

Sketch the graph of each rational function.

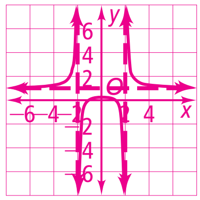
24. $y = \frac{x}{x(x-6)}$



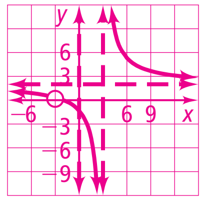
25. $y = \frac{2x}{x-6}$



26. $y = \frac{x^2 - 1}{x^2 - 4}$

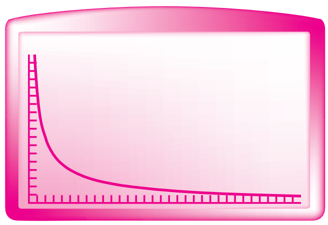


27. $y = \frac{2x^2 + 10x + 12}{x^2 - 9}$



28. You start a business word-processing papers for other students. You spend \$3500 on a computer system and office furniture. You figure additional costs at \$.02 per page.

a. Write a rational function modeling the total average cost per page. Graph the function. **$y = \frac{0.02x + 3500}{x}$, where x = number of pages**



b. What is the total average cost per page if you type 1000 pages? If you type 2000? **\$3.52; \$1.77**

c. How many pages must you type to bring your total average cost to less than \$1.50 per page? **at least 2365 pages**

d. What are the vertical and horizontal asymptotes of the graph of the function? **$x = 0$; $y = 0.02$**

Name ____________________ Class __________ Date __________

8-3 Practice

Form K

Rational Functions and Their Graphs

Find the domain, points of discontinuity, and x- and y-intercepts of each rational function. Determine whether the discontinuities are removable or non-removable. To start, factor the numerator and denominator, if possible.

1. $y = \frac{x + 5}{x - 2}$

domain: all real numbers except $x = 2$; non-removable point of discontinuity at $x = 2$; x-intercept: $x = -5$; y-intercept: $y = -\frac{5}{2}$

2. $y = \frac{1}{x^2 + 2x + 1}$

domain: all real numbers except $x = -1$; non-removable point of discontinuity at $x = -1$; x-intercept: none; y-intercept: $y = 1$

3. $y = \frac{x + 4}{x^2 + 2x - 8}$

domain: all real numbers except $x = 2$ and $x = -4$; non-removable point of discontinuity at $x = 2$ and removable point of discontinuity at $x = -4$; x-intercept: none; y-intercept: $y = -\frac{1}{2}$

Find the vertical asymptotes and holes for the graph of each rational function.

4. $y = \frac{x + 6}{x + 4}$

vertical asymptote: $x = -4$

5. $y = \frac{(x - 2)(x - 1)}{x - 2}$

vertical asymptote: none; hole at $x = 2$

6. $y = \frac{x + 1}{(3x - 2)(x - 3)}$

vertical asymptotes: $x = \frac{2}{3}$ and $x = 3$

Find the horizontal asymptote of the graph of each rational function. To start, identify the degree of the numerator and denominator.

7. $y = \frac{x + 1}{x + 5}$

$\frac{x + 1}{x + 5}$ ← degree 1 (numerator), ← degree 1 (denominator)

$y = 1$

8. $y = \frac{x + 2}{2x^2 - 4}$

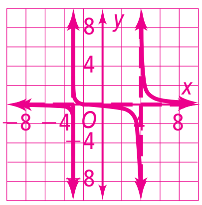
$y = 0$

9. $y = \frac{3x^3 - 4}{4x + 1}$

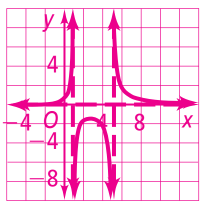
no horizontal asymptote

Sketch the graph of each rational function.

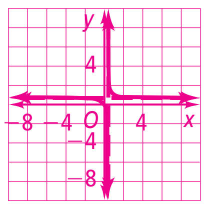
10. $y = \frac{x + 2}{(x + 3)(x - 4)}$



11. $y = \frac{x + 3}{(x - 1)(x - 5)}$



12. $y = \frac{2x}{3x - 1}$



Name ______________________________ Class ____________ Date ____________

8-3 Practice (continued)

Form K

Rational Functions and Their Graphs

13. The CD-ROMs for a computer game can be manufactured for $.25 each. The development cost is $124,000. The first 100 discs are samples and will not be sold.

a. Write a function for the average cost of a disc that is not a sample. $y = \frac{0.25x + 124{,}000}{x - 100}$

b. What is the average cost if 2000 discs are produced? If 12,800 discs are produced? **$65.53; $10.02**

c. Reasoning How could you find the number of discs that must be produced to bring the average cost under $8?

d. How many discs must be produced to bring the average cost under $8? **16,104 discs**

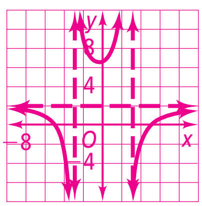
c. The intersection of $y = \frac{0.25x + 124{,}000}{x - 100}$ and $y = 8$ rounded to the next whole number gives the number of discs that must be produced.

14. Error Analysis For the rational function $y = \frac{x^2 - 2x - 8}{x^2 - 9}$, your friend said that the vertical asymptote is $x = 1$ and the horizontal asymptotes are $y = 3$ and $y = -3$. Without doing any calculations, you know this is incorrect. Explain how you know.

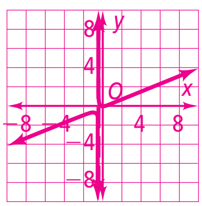
A rational function can have only 1 horizontal asymptote. Your friend must have switched the horizontal and vertical asymptote answers.

Sketch the graph of each rational function.

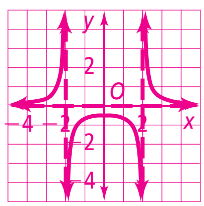
15. $y = \frac{4x^2 - 100}{2x^2 + x - 15}$



16. $y = \frac{2x^2}{5x + 1}$



17. $y = \frac{2}{x^2 - 4}$



18. Multiple Choice What are the points of discontinuity for the graph of $y = \frac{(2x + 3)(x - 5)}{(x + 5)(2x - 1)}$? **C**

(A) $-5, 1$ (B) $-\frac{3}{2}, 5$ (C) $-5, \frac{1}{2}$ (D) $5, -\frac{1}{2}$

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8-3 Standardized Test Prep

Rational Functions and Their Graphs

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. What function has a graph with a removable discontinuity at $\left(5, \frac{1}{9}\right)$? A

(A) $y = \frac{(x - 5)}{(x + 4)(x - 5)}$

(B) $y = \frac{4}{x - 5}$

(C) $y = \frac{4x - 1}{5x + 1}$

(D) $y = \frac{x + 1}{5x - 4}$

2. What is the vertical asymptote of the graph of $y = \frac{(x + 2)(x - 3)}{x(x - 3)}$? H

(F) $x = -3$ (G) $x = -2$ (H) $x = 0$ (I) $x = 3$

3. What best describes the horizontal asymptote(s), if any, of the graph of $y = \frac{x^2 + 2x - 8}{(x + 6)^2}$? C

(A) $y = -6$

(B) $y = 0$

(C) $y = 1$

(D) The graph has no horizontal asymptote.

4. Which rational function has a graph that has vertical asymptotes at $x = a$ and $x = -a$, and a horizontal asymptote at $y = 0$? G

(F) $y = \frac{(x - a)(x + a)}{x}$

(G) $y = \frac{1}{x^2 - a^2}$

(H) $y = \frac{x^2}{x^2 - a^2}$

(I) $y = \frac{x - a}{x + a}$

Short Response

5. How many milliliters of 0.30% sugar solution must you add to 75 mL of 4% sugar solution to get a 0.50% sugar solution? Show your work.

$$\frac{75(0.04) + 0.003x}{75 + x} = 0.005$$

$$3 + 0.003x = 0.375 + 0.005x$$

$$x = 1312.5 \text{ mL}$$

[2] correct answer with work shown
[1] incorrect work OR correct answer, without work shown
[0] no answer given

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8-3 Enrichment

Rational Functions and Their Graphs

Other Asymptotes

Recall that a rational function does not have a horizontal asymptote if the degree of the numerator is greater than the degree of the denominator. If, however, the degree of the numerator is exactly one more than the degree of the denominator, then the graph of the function has a **slant asymptote**.

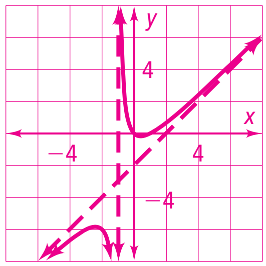
You can use long division to find the equation of a slant asymptote. The equation of the slant asymptote is given by the quotient, disregarding the remainder.

1. Use long division to find the slant asymptote of $f(x) = \frac{x^2 - x}{x + 1}$.

The slant asymptote of the function is $y =$ **$x - 2$**.

2. Graph the slant asymptote on a coordinate grid.

3. Graph the vertical asymptote for the equation in Exercise 1 on the grid.



4. Copy and complete the table, and plot the points accordingly.

x	−3	−2	0	2	3
f(x)	−6	−6	0	$\frac{2}{3}$	$\frac{3}{2}$

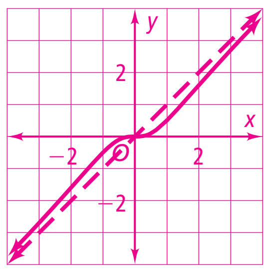
Connect with a smooth curve, being sure to draw near to all asymptotes.

5. Use long division to find the slant asymptote of $f(x) = \frac{x^3}{x^2 + 1}$.

The slant asymptote of the function is $y =$ **x**.

6. Graph the slant asymptote on a new coordinate grid.

7. Find any vertical asymptotes for the equation in Exercise 5 and graph on the grid.
no vertical asymptotes



8. Copy and complete the table, and plot the points accordingly.

x	−3	−2	0	2	3
f(x)	$-\frac{27}{10}$	$-\frac{8}{5}$	0	$\frac{8}{5}$	$\frac{27}{10}$

Connect with a smooth curve, being sure to draw near to all asymptotes.

9. The technique used to find slant (linear) asymptotes works for rational functions in which the degree of the numerator is one more than the degree of the denominator. Use long division to find the non-linear asymptote of the rational function given by $f(x) = \frac{x^3 + 1}{x}$.

The asymptote of the function is $y =$ **x^2**. Can you guess the shape of this asymptote? **parabola**

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8-3 Reteaching

Rational Functions and Their Graphs

A rational function may have one or more types of discontinuities: holes (removable points of discontinuity), vertical asymptotes (non-removable points of discontinuity), or a horizontal asymptote.

If	Then	Example
a is a zero with multiplicity m in the numerator and multiplicity n in the denominator, and $m \geq n$	hole at $x = a$	$f(x) = \frac{(x-5)(x+6)}{(x-5)}$ hole at $x = 5$
a is a zero of the denominator only, or a is a zero with multiplicity m in the numerator and multiplicity n in the denominator, and $m < n$	vertical asymptote at $x = a$	$f(x) = \frac{x^2}{x-3}$ vertical asymptote at $x = 3$

Let p = degree of numerator.
Let q = degree of denominator.

• $m < n$	horizontal asymptote at $y = 0$	$f(x) = \frac{4x^2}{7x^2 + 2}$ horizontal asymptote at $y = \frac{4}{7}$
• $m > n$	no horizontal asymptote exists	
• $m = n$	horizontal asymptote at $y = \frac{a}{b}$, where a and b are coefficients of highest degree terms in numerator and denominator	

Problem

What are the points of discontinuity of $y = \frac{x^2 + x - 6}{3x^2 - 12}$, if any?

Step 1 Factor the numerator and denominator completely. $y = \frac{(x-2)(x+3)}{3(x-2)(x+2)}$

Step 2 Look for values that are zeros of both the numerator and the denominator. The function has a hole at $x = 2$.

Step 3 Look for values that are zeros of the denominator only. The function has a vertical asymptote at $x = -2$.

Step 4 Compare the degrees of the numerator and denominator. They have the same degree. The function has a horizontal asymptote at $y = \frac{1}{3}$.

Exercises

Find the vertical asymptotes, holes, and horizontal asymptote for the graph of each rational function.

1. $y = \frac{x}{x^2 - 9}$ vertical asymptotes: $x = 3$, $x = -3$; horizontal asymptote: $y = 0$

2. $y = \frac{6x^2 - 6}{x - 1}$ hole: $x = 1$

3. $y = \frac{4x + 5}{3x + 2}$ vertical asymptote: $x = -\frac{2}{3}$; horizontal asymptote: $y = \frac{4}{3}$

8-3 Reteaching (continued)

Rational Functions and Their Graphs

Before you try to sketch the graph of a rational function, get an idea of its general shape by identifying the graph's holes, asymptotes, and intercepts.

Problem

What is the graph of the rational function $y = \frac{x + 3}{x + 1}$?

Step 1 Identify any holes or asymptotes.

no holes; vertical asymptote at $x = -1$; horizontal asymptote at $y = \frac{1}{1} = 1$

Step 2 Identify any x- and y-intercepts.

x-intercepts occur when $y = 0$. y-intercepts occur when $x = 0$.

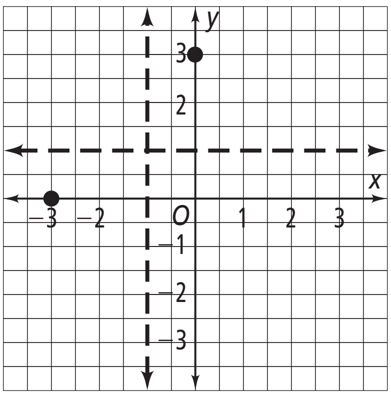
$\frac{x + 3}{x + 1} = 0$ $\quad\quad$ $y = \frac{0 + 3}{0 + 1}$

$x + 3 = 0$ $\quad\quad$ $y = 3$

$x = -3$

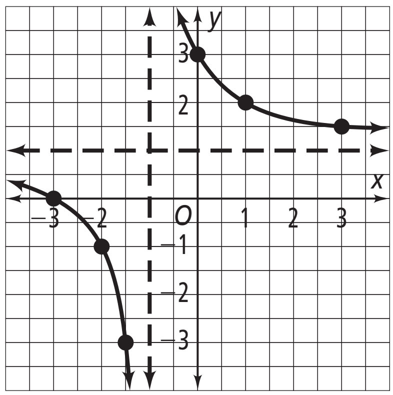
x-intercept at -3 $\quad\quad$ y-intercept at 3

Step 3 Sketch the asymptotes and intercepts.



Step 4 Make a table of values, plot the points, and sketch the graph.

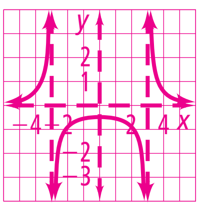
x	y
−2	−1
−1.5	−3
1	2
3	1.5



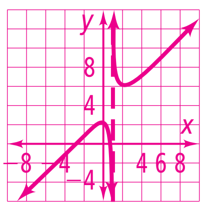
Exercises

Graph each function. Include the asymptotes.

4. $y = \frac{4}{x^2 - 9}$



5. $y = \frac{x^2 + 2x - 2}{x - 1}$



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8-4 Additional Vocabulary Support

Rational Expressions

There are two sets of cards that show how to simplify $\frac{x^2-4}{x^2-2x+1}\cdot\frac{x^2+2x-3}{2x^2-3x-2}$. The set on the left explains the thinking. The set on the right shows the steps. Write the steps in the correct order.

Think Cards	Write Cards
Factor the numerators and denominators.	$\frac{\cancel{(x-2)}(x+2)}{(x-1)\cancel{(x-1)}}\cdot\frac{(x+3)\cancel{(x-1)}}{(2x+1)\cancel{(x-2)}}$
Write the problem.	$\frac{(x+2)(x+3)}{(x-1)(2x+1)}$
Write the remaining factors.	$\frac{x^2-4}{x^2-2x+1}\cdot\frac{x^2+2x-3}{2x^2-3x-2}$
Divide out common factors.	$\frac{(x-2)(x+2)}{(x-1)(x-1)}\cdot\frac{(x+3)(x-1)}{(2x+1)(x-2)}$

Think	Write
First, write the problem.	**Step 1** $\frac{x^2-4}{x^2-2x+1}\cdot\frac{x^2+2x-3}{2x^2-3x-2}$
Second, factor the numerators and denominators.	**Step 2** $\frac{(x-2)(x+2)}{(x-1)(x-1)}\cdot\frac{(x+3)(x-1)}{(2x+1)(x-2)}$
Third, divide out common factors.	**Step 3** $\frac{\cancel{(x-2)}(x+2)}{(x-1)\cancel{(x-1)}}\cdot\frac{(x+3)\cancel{(x-1)}}{(2x+1)\cancel{(x-2)}}$
Fourth, write the remaining factors.	**Step 4** $\frac{(x+2)(x+3)}{(x-1)(2x+1)}$

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8-4 Think About a Plan

Rational Expressions

Manufacturing A toy company is considering a cube or sphere-shaped container for packaging a new product. The height of the cube would equal the diameter of the sphere. Compare the ratios of the volumes to the surface areas of the containers. Which packaging will be more efficient? For a sphere, $SA = 4\pi r^2$.

Understanding the Problem

1. Let x be the height of the cube. What are expressions for the cube's volume and surface area?

   Volume: x^3 Surface area: $6x^2$

2. Let x be the diameter of the sphere. What are expressions for the sphere's volume and surface area?

   Volume: $\frac{4}{3}\pi\left(\frac{x}{2}\right)^3$ or $\frac{\pi x^3}{6}$ Surface area: $4\pi\left(\frac{x}{2}\right)^2$ or πx^2

3. What is the problem asking you to do?

   Find the ratios of volume to surface area for the cube and the sphere. Compare the ratios to decide which is a more efficient package.

Planning the Solution

4. Write an expression for the ratio of the cube's volume to its surface area. Simplify your expression.

   $\frac{x^3}{6x^2} = \frac{x}{6}$

5. Write an expression for the ratio of the sphere's volume to its surface area. Simplify your expression.

   $\frac{\frac{\pi x^3}{6}}{\pi x^2} = \frac{x}{6}$

Getting an Answer

6. Compare the ratios of the volumes to the surface areas of the containers. Which packaging will be more efficient?

   The ratios are the same. The package shapes are equally efficient.

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8-4 Practice

Form G

Rational Expressions

Simplify each rational expression. State any restrictions on the variables.

1. $\frac{4x + 6}{2x + 3}$ **$2;\ x \neq -\frac{3}{2}$**

2. $\frac{2y}{y^2 + 6y}$ **$\frac{2}{y+6};\ y \neq -6, 0$**

3. $\frac{20 + 40x}{20x}$ **$\frac{2x+1}{x};\ x \neq 0$**

4. $\frac{7x - 28}{x^2 - 16}$ **$\frac{7}{x+4};\ x \neq \pm 4$**

5. $\frac{3y^2 - 3}{y^2 - 1}$ **$3;\ y \neq \pm 1$**

6. $\frac{3x^2 - 12}{x^2 - x - 6}$ **$\frac{3x-6}{x-3};\ x \neq -2, 3$**

7. $\frac{x^2 + 3x - 18}{x^2 - 36}$ **$\frac{x-3}{x-6};\ x \neq \pm 6$**

8. $\frac{x^2 + 13x + 40}{x^2 - 2x - 35}$ **$\frac{x+8}{x-7};\ x \neq -5, 7$**

Multiply. State any restrictions on the variables.

9. $\frac{5a}{5a + 5} \cdot \frac{10a + 10}{a}$ **$10;\ a \neq -1, 0$**

10. $\frac{2x + 4}{10x} \cdot \frac{15x^2}{x + 2}$ **$3x;\ x \neq 0, -2$**

11. $\frac{x^2 - 5x}{x^2 + 3x} \cdot \frac{x + 3}{x - 5}$ **$1;\ x \neq -3, 0, 5$**

12. $\frac{x^2 - 6x}{x^2 - 36} \cdot \frac{x + 6}{x^2}$ **$\frac{1}{x};\ x \neq 0, \pm 6$**

13. $\frac{5y - 20}{3y + 15} \cdot \frac{7y + 35}{10y + 40}$ **$\frac{7(y-4)}{6(y+4)};\ y \neq -5, -4$**

14. $\frac{x - 2}{(x + 2)^2} \cdot \frac{x + 2}{2x - 4}$ **$\frac{1}{2x+4};\ x \neq \pm 2$**

15. $\frac{3x^3}{x^2 - 25} \cdot \frac{x^2 + 6x + 5}{x^2}$ **$\frac{3x^2+3x}{x-5};\ x \neq 0, \pm 5$**

16. $\frac{y^2 - 2y}{y^2 + 7y - 18} \cdot \frac{y^2 - 81}{y^2 - 11y + 18}$ **$\frac{y}{y-2};\ y \neq 2, \pm 9$**

Divide. State any restrictions on the variables.

17. $\frac{7x^4}{24y^5} \div \frac{21x}{12y^4}$ **$\frac{x^3}{6y};\ x, y \neq 0$**

18. $\frac{6x + 6}{7} \div \frac{4x + 4}{x - 2}$ **$\frac{3(x-2)}{14};\ x \neq -1, 2$**

19. $\frac{5y}{2x^2} \div \frac{5y^2}{8x^2}$ **$\frac{4}{y};\ x, y \neq 0$**

20. $\frac{3y + 3}{6y + 12} \div \frac{18}{5y + 5}$ **$\frac{5(y+1)^2}{36(y+2)};\ y \neq -2, -1$**

21. $\frac{y^2 - 49}{(y - 7)^2} \div \frac{5y + 35}{y^2 - 7y}$ **$\frac{y}{5};\ y \neq 0, \pm 7$**

22. $\frac{x^2 + 10x + 16}{x^2 - 6x - 16} \div \frac{x + 8}{x^2 - 64}$ **$x + 8;\ x \neq -2, \pm 8$**

23. $\frac{y^2 - 5y + 4}{y^2 - 1} \div \frac{y^2 - 9}{y^2 + 5y + 4}$

$\frac{y^2-16}{y^2-9};\ y \neq \pm 1, \pm 3, -4$

24. $\frac{x^2 - 4}{x^2 + 6x + 9} \div \frac{x^2 + 4x + 4}{x^2 - 9}$

$\frac{x^2-5x+6}{x^2+5x+6};\ x \neq -2, \pm 3$

8-4 Practice (continued) Form G

Rational Expressions

25. A farmer must decide whether to build a cylindrical grain silo or a rectangular grain silo. The cylindrical silo has radius r. The rectangular silo has width r and length $2r$. Both silos have the same height h.

a. Write and simplify an expression for the ratio of the volume of the cylindrical silo to its surface area, including the circular floor and ceiling. $\frac{rh}{2r + 2h}$

b. Write and simplify an expression for the ratio of the volume of the rectangular silo to its surface area, including the rectangular floor and ceiling. $\frac{rh}{2r + 3h}$

c. Compare the ratios of volume to surface area for the two silos. $\frac{rh}{2r + 2h} > \frac{rh}{2r + 3h}$

d. Compare the volumes of the two silos. $V_{cyl} > V_{rect}$

e. **Reasoning** Assume the average cost of construction materials per square foot of surface area is the same for either silo. How can you measure the cost-effectiveness of each silo? **Answers may vary. Sample: The surface area of a silo determines the cost to build the silo. Compare the ratios of the volume to the surface area of the silos.**

Simplify each rational expression. State any restrictions on the variables.

26. $\frac{2x^2 + 11x + 5}{3x^2 + 17x + 10}$ $\frac{2x + 1}{3x + 2}$; $x \neq -5, -\frac{2}{3}$

27. $\frac{6x^2 + 5xy - 6y^2}{3x^2 - 5xy + 2y^2}$ $\frac{2x + 3y}{x - y}$; $x \neq y, \frac{2}{3}y$

Multiply or divide. State any restrictions on the variables.

28. $\frac{x^2 + 2x + 1}{x^2 - 1} \cdot \frac{x^2 + 3x + 2}{x^2 + 4x + 4}$

$\frac{x^2 + 2x + 1}{x^2 + x - 2}$; $x \neq -2, \pm 1$

29. $\frac{x^2 - 3x - 10}{2x^2 - 11x + 5} \div \frac{x^2 - 5x + 6}{2x^2 - 7x + 3}$

$\frac{x + 2}{x - 2}$; $x \neq \frac{1}{2}, 2, 3, 5$

30. **Reasoning** A rectangle has area $\frac{10b}{6b - 6}$ and length $\frac{b + 2}{2b - 2}$. Write an expression for the width of the rectangle. $\frac{10b}{3b + 6}$

31. **Open-Ended** Write three rational expressions that simplify to $\frac{x + 1}{x - 1}$.

Answers may vary. Sample: $\frac{x^2 + 2x + 1}{x^2 - 1}$, $\frac{x^2 + 3x + 2}{x^2 + x - 2}$, $\frac{x^2 - x - 2}{x^2 - 3x + 2}$

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8-4 Practice — Form K

Rational Expressions

Simplify each rational expression. State any restrictions on the variables.

1. $\dfrac{-27x^3y}{9x^4y}$

$-\frac{3}{x}$; $x \neq 0, y \neq 0$

2. $\dfrac{-6 + 3x}{x^2 - 6x + 8}$

$\frac{3}{x - 4}$; $x \neq 2$ or 4

3. $\dfrac{2x^2 - 3x - 2}{x^2 - 5x + 6}$

$\frac{2x + 1}{x - 3}$; $x \neq 3$ or 2

Multiply. State any restrictions on the variables.

To start, factor all polynomials.

4. $\dfrac{4x^2 - 1}{2x^2 - 5x - 3} \cdot \dfrac{x^2 - 6x + 9}{2x^2 + 5x - 3}$

$\dfrac{(2x + 1)(2x - 1)}{(2x + 1)(x - 3)} \cdot \dfrac{(x - 3)(x - 3)}{(2x - 1)(x + 3)}$

$\frac{x - 3}{x + 3}$; $x \neq \frac{1}{2}, -\frac{1}{2}, 3, -3$

5. $\dfrac{2x^2 + 7x + 3}{x - 4} \cdot \dfrac{x^2 - 16}{x^2 + 8x + 15}$

$\frac{(2x + 1)(x + 4)}{x + 5}$;
$x \neq 4, -5, -3$

6. $\dfrac{4x^2}{5y} \cdot \dfrac{7y}{12x^4}$

$\frac{7}{15x^2}$; $x \neq 0, y \neq 0$

Divide. State any restrictions on the variables.

To start, rewrite the division as multiplication by the reciprocal.

7. $\dfrac{16x^5}{3y^3} \div \dfrac{8x^3}{9y^2}$

$\dfrac{16x^5}{3y^3} \cdot \dfrac{9y^2}{8x^3}$

$\frac{6x^2}{y}$; $x \neq 0, y \neq 0$

8. $\dfrac{x^2 + 2x - 15}{x^2 - 16} \div \dfrac{x + 1}{3x - 12}$

$\frac{3(x + 5)(x - 3)}{(x + 4)(x + 1)}$;
$x \neq 4, -1$, or -4

9. $\dfrac{3y - 12}{2y + 4} \div \dfrac{6y - 24}{4y + 8}$

1; $y \neq -2$ or 4

Name ______________________ Class ____________ Date ____________

8-4 Practice (continued) — Form K

Rational Expressions

10. Your school wants to build a courtyard surrounded by a low brick wall. It wants the maximum area for a given amount of brick wall. The courtyard can be either a circle or an equilateral triangle. Which shape would have the greater area to perimeter ratio?

circle

Simplify each rational expression. State any restrictions on the variables.

11. $\frac{x^2 - 2x - 8}{3x^2 + 4x - 4}$

$\frac{x-4}{3x-2}$; $x \neq -2$ or $\frac{2}{3}$

12. $\frac{6x + 15}{2x^2 + 3x - 5}$

$\frac{3}{x-1}$; $x \neq 1$ or $-\frac{5}{2}$

13. $\frac{x^2 - y^2}{6x^2 + 6xy}$

$\frac{x-y}{6x}$; $x \neq 0, -y$

14. **Writing** How can you tell whether a rational expression is in simplest form? Include an example with your explanation.

Answers may vary. Sample: A rational expression is in simplest form when the numerator and denominator have no common factors; $\frac{x+1}{x-1}$.

15. The width of a rectangle is given by the expression $\frac{x + 10}{3x + 24}$ and the area can be represented by $\frac{2x + 20}{6x + 15}$. What is the length of the rectangle?

$\frac{2(x+8)}{2x+5}$

16. **Multiple Choice** Which expression can be simplified to $\frac{x - 1}{x - 3}$? **D**

A. $\frac{x^2 - x - 6}{x^2 - x - 2}$

B. $\frac{x^2 - 2x + 1}{x^2 + 2x - 3}$

C. $\frac{x^2 - 3x - 4}{x^2 - 7x + 12}$

D. $\frac{x^2 - 4x + 3}{x^2 - 6x + 9}$

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8-4 Standardized Test Prep

Rational Expressions

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. Which expression equals $\frac{x^2 - 4x - 5}{x^2 + 6x + 5}$? C

(A) $x + 1$ (B) $-10x - 10$ (C) $\frac{x - 5}{x + 5}$ (D) $\frac{4x - 5}{6x + 5}$

2. Which expression equals $\frac{42a^2b^4}{12a^5b^{-2}}$? F

(F) $\frac{7b^6}{2a^3}$ (G) $\frac{30a^7}{b^2}$ (H) $\frac{7ab^3}{2}$ (I) $\frac{30b^2}{a^3}$

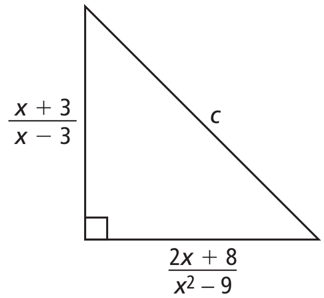
3. Which expression equals $\frac{t^2 - 1}{t - 2} \cdot \frac{t^2 - 3t + 2}{t^2 + 4t + 3}$? A

(A) $\frac{t^2 - 2t + 1}{t + 3}$ (B) $\frac{t^2 - 1}{t + 3}$ (C) $\frac{(t + 1)^2(t + 3)}{(t - 2)^2}$ (D) $\frac{2t^2 - 3t + 1}{t^2 + 5t + 1}$

4. What is the area of the triangle shown at the right? H

(F) $\frac{2x + 8}{x^2 - 6x + 9}$ (H) $\frac{x + 4}{x^2 - 6x + 9}$

(G) $\frac{x^2 + 6x + 9}{x + 4}$ (I) $\frac{2x^2 + 12x + 18}{x + 4}$



Short Response

5. What is the quotient $\frac{y + 2}{2y^2 - 3y - 2} \div \frac{y^2 - 4}{y^2 + y - 6}$ expressed in simplest form? State any restrictions on the variable. Show your work.

$$\frac{y + 2}{2y^2 - 3y - 2} \div \frac{y^2 - 4}{y^2 + y - 6} = \frac{y + 2}{2y^2 - 3y - 2} \cdot \frac{y^2 + y - 6}{y^2 - 4}$$

$$= \frac{\cancel{y + 2}}{(2y + 1)(y - 2)} \cdot \frac{(y + 3)\cancel{(y - 2)}}{\cancel{(y + 2)}\cancel{(y - 2)}} = \frac{y + 3}{2y^2 - 3y - 2}; y \neq -3, -2, -\frac{1}{2}, 2$$

[2] correct answer with complete work shown

[1] incorrect or incomplete work shown OR correct answer, without work shown

[0] no answer given

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8-4 Enrichment

Rational Expressions

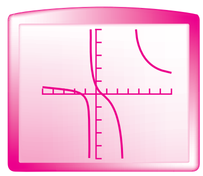
In previous lessons, you learned how to graph reciprocal functions of the form $f(x) = \frac{a}{x - h} + k$. You learned that graphs of reciprocal functions have a horizontal asymptote at $y = k$ and a vertical asymptote at $x = h$. For other types of rational functions, the asymptotes are not as easily determined.

1. Explain why the reciprocal function $f(x) = \frac{1}{x}$ is not the parent graph of $f(x) = \frac{x^2 + 2x - 3}{x^2 - 5x - 6}$.

Answers may vary. Sample: The function cannot be simplified so there is no x^2-term in the denominator.

2. For rational functions, vertical asymptotes are lines located at the value(s) of x that make the denominator 0. Write the equations of the vertical asymptotes for the rational function $f(x) = \frac{x^2 + 2x - 3}{x^2 - 5x - 6}$.

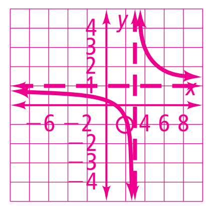
$x = 6$ and $x = -1$



x scale: 2 y scale: 2

3. Use a graphing calculator to graph the rational function $f(x) = \frac{x^2 + 2x - 3}{x^2 - 5x - 6}$. Use your graph to check if your vertical asymptotes are correct. How does your graph confirm that the reciprocal function $f(x) = \frac{1}{x}$ is not the parent graph?

Answers may vary. Sample: The graph of the rational function is not a transformation of the graph of the reciprocal function.



4. What are the vertical asymptotes for the rational function $f(x) = \frac{x^2 - 3x + 2}{x^2 - 5x + 6}$? Confirm that these are the vertical asymptotes by graphing the function. What do you notice? **$x = 3$; the graph has a vertical asymptote at $x = 3$, not at $x = 2$.**

5. From the graph of the function in Exercise 4, you saw that even though $x = 2$ made the denominator equal 0, it was not an asymptote. A value such as this creates a hole in the graph, and is called a removable discontinuity. The value is not part of the domain, yet it is not a vertical asymptote. Factor the rational expression $\frac{x^2 - 3x + 2}{x^2 - 5x + 6}$ and use your results to explain why $x = 2$ is not an asymptote.

Answers may vary. Sample: The expression simplifies to $\frac{x - 1}{x - 3}$. The domain of the original expression does not include 2, but it can be included in the final simplified expression.

6. For the rational function $f(x) = \frac{2x^2 - 7x - 4}{x^2 + x - 20}$, algebraically determine the location of the vertical asymptote and the value at which there is a removable discontinuity.

vertical asymptote at $x = -5$; removable discontinuity at $x = 4$

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8-4 Reteaching

Rational Expressions

Simplest form of a rational expression means the numerator and the denominator have no factors in common. You may have to restrict certain values of the variable(s) when you write in simplest form, because division by zero is undefined.

Problem

What is the expression $\frac{6x^3y^2 + 6x^2y^2 - 12xy^2}{3x^2y^3 - 12y^3}$ written in simplest form? State any restrictions on the variables.

$\frac{6xy^2(x^2 + x - 2)}{3y^3(x^2 - 4)}$ Factor $6xy^2$ out of the numerator and $3y^3$ out of the denominator.

$\frac{6xy^2(x + 2)(x - 1)}{3y^3(x + 2)(x - 2)}$ Factor $(x^2 + x - 2)$ and $(x^2 - 4)$.

$\frac{(2 \cdot \cancel{3} \cdot x \cdot \cancel{y} \cdot \cancel{y})\cancel{(x + 2)}(x - 1)}{(\cancel{3} \cdot y \cdot \cancel{y} \cdot \cancel{y})\cancel{(x + 2)}(x - 2)}$ Divide out the common factors.

$\frac{2x(x - 1)}{y(x - 2)}$ Write the remaining factors.

Look at the **original** expression.

$\frac{6xy^2(x + 2)(x - 1)}{3y^3(x + 2)(x - 2)}$ is undefined if

$3y^3 = 0$, $x + 2 = 0$, or $x - 2 = 0$.

So, $y \neq 0$, $x \neq -2$, and $x \neq 2$.

Look at the **simplified** expression.

$\frac{2x(x - 1)}{y(x - 2)}$ is undefined if

$y = 0$ or $x - 2 = 0$.

So, $y \neq 0$ and $x \neq 2$.

In simplest form, the expression is $\frac{2x(x - 1)}{y(x - 2)}$, where $y \neq 0$, $x \neq -2$, and $x \neq 2$.

Exercises

Simplify each rational expression. State any restrictions on the variable.

1. $\frac{x^2 + x}{x^2 + 2x}$ $\frac{x + 1}{x + 2}$; $x \neq -2, 0$

2. $\frac{x^2 - 5x}{x^2 - 25}$ $\frac{x}{x + 5}$; $x \neq \pm 5$

3. $\frac{x^2 + 3x - 18}{x^2 - 36}$ $\frac{x - 3}{x - 6}$; $x \neq \pm 6$

4. $\frac{4x^2 - 36}{x^2 + 10x + 21}$ $\frac{4x - 12}{x + 7}$; $x \neq -3, -7$

5. $\frac{3x^2 - 12}{x^2 - x - 6}$ $\frac{3x - 6}{x - 3}$; $x \neq -2, 3$

6. $\frac{x^2 - 9}{2x + 6}$ $\frac{x - 3}{2}$; $x \neq -3$

Name ______________________ Class ____________ Date ____________

8-4 Reteaching (continued)

Rational Expressions

To find the quotient $\frac{a}{b} \div \frac{c}{d}$, multiply by the reciprocal of the divisor. Invert the divisor and then multiply: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

To divide any rational expression by a second rational expression, follow this pattern.

Problem

What is the quotient $\frac{x^2 - 2x - 35}{2x^3 - 3x^2} \div \frac{7x - 49}{4x^3 - 9x}$, written in simplest form? State any restrictions on the variable.

Step 1 Invert the divisor, which is the second expression.

Reciprocal of $\frac{7x - 49}{4x^3 - 9x}$ is $\frac{4x^3 - 9x}{7x - 49}$

Step 2 Write the quotient as one rational expression.

$$\frac{x^2 - 2x - 35}{2x^3 - 3x^2} \div \frac{7x - 49}{4x^3 - 9x} = \frac{x^2 - 2x - 35}{2x^3 - 3x^2} \cdot \frac{4x^3 - 9x}{7x - 49} = \frac{(x^2 - 2x - 35)(4x^3 - 9x)}{(2x^3 - 3x^2)(7x - 49)}$$

Step 3 Factor each polynomial in the numerator and denominator.

$$\frac{(x^2 - 2x - 35)(4x^3 - 9x)}{(2x^3 - 3x^2)(7x - 49)} = \frac{[(x - 7)(x + 5)][x(2x + 3)(2x - 3)]}{[x^2(2x - 3)][7(x - 7)]}$$

Step 4 Divide out the common factors and simplify.

$$\frac{[\cancel{(x - 7)}(x + 5)][\cancel{x}(2x + 3)\cancel{(2x - 3)}]}{[(x \cdot \cancel{x})\cancel{(2x - 3)}][7\cancel{(x - 7)}]} = \frac{(x + 5)(2x + 3)}{7x} = \frac{2x^2 + 13x + 15}{7x}$$

Therefore, $\frac{x^2 - 2x - 35}{2x^3 - 3x^2} \div \frac{7x - 49}{4x^3 - 9x} = \frac{2x^2 + 13x + 15}{7x}$, where $x \neq 0, \pm\frac{3}{2}, 7$.

Exercises

Divide. State any restrictions on the variables.

7. $\frac{3x + 12}{2x - 8} \div \frac{x^2 + 8x + 16}{x^2 - 8x + 16}$ **$\frac{3x - 12}{2x + 8}$; $x \neq \pm 4$**

8. $\frac{2x^2 - 16x}{x^2 - 9x + 8} \div \frac{2x}{5x - 5}$ **5; $x \neq 0, 1, 8$**

9. $\frac{2x - 10}{3x - 21} \div \frac{x - 5}{4x - 28}$ **$\frac{8}{3}$; $x \neq 5, 7$**

10. $\frac{4x - 16}{4x} \div \frac{x^2 - 2x - 8}{3x + 6}$ **$\frac{3}{x}$; $x \neq -2, 0, 4$**

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8-5 Additional Vocabulary Support

Adding and Subtracting Rational Expressions

The column on the left shows the steps used to subtract two rational expressions. Use the column on the left to answer each question in the column on the right.

Problem **What is the difference of the two rational expressions in simplest form? State any restrictions on the variable.** $$\frac{2}{x^2 - 36} - \frac{1}{x^2 + 6x}$$	**1.** Read the problem. What process are you going to use to solve the problem? Find the difference of two rational expressions.
Factor the denominators. $$\frac{2}{(x - 6)(x + 6)} - \frac{1}{x(x + 6)}$$	**2.** Why do you factor the denominators? To determine the common denominator.
Rewrite each expression with the LCD. $$= \frac{2}{(x - 6)(x + 6)} \cdot \frac{x}{x} - \frac{1}{x(x + 6)} \cdot \frac{(x - 6)}{(x - 6)}$$ $$= \frac{2x}{x(x - 6)(x + 6)} - \frac{x - 6}{x(x + 6)(x - 6)}$$	**3.** What does LCD stand for? Least Common Denominator
Add the numerators. Combine like terms. $$= \frac{x + 6}{x(x - 6)(x + 6)}$$	**4.** Why is the numerator $x + 6$? $2x - (x - 6) = 2x - x + 6 = x + 6$
Divide out the common factors. $$= \frac{\cancel{x + 6}}{x(x - 6)(\cancel{x + 6})}$$ $$= \frac{1}{x(x - 6)}$$ The difference of the expressions is $\frac{1}{x(x - 6)}$ for $x \neq 0, x \neq 6, x \neq -6$.	**5.** Why is the numerator equal to 1? When simplifying, you divide $x + 6$ by itself, so the answer is 1. **6.** Why are 0, 6, and -6 restrictions on the variable? These values make at least one of the original denominators equal zero.

Name ______________________ Class ____________ Date ____________

8-5 Think About a Plan

Adding and Subtracting Rational Expressions

Optics To read small font, you use the magnifying lens with the focal length 3 in. How far from the magnifying lens should you place the page if you want to hold the lens at 1 foot from your eyes? Use the thin-lens equation.

Know

1. The focal length of the magnifying lens is **3 in.**.

2. The distance from the lens to your eyes is **12 in.**.

3. The thin-lens equation is $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$.

Need

4. To solve the problem I need to find:

   the distance from the page to the lens.

Plan

5. What variables in the thin-lens equation have values that are known?

   f is the focal length of the lens, or 3 in.; d_i is the distance from the lens to the eyes, or 12 in.

6. Solve the thin-lens equation for the variable whose value is unknown. $d_o = \frac{fd_i}{d_i - f}$

7. Substitute the known values into your equation and simplify. $d_o = \frac{fd_i}{d_i - f} = \frac{3 \cdot 12}{12 - 3} = \frac{36}{9} = 4$

8. How far from the page should you hold the magnifying lens? **4 in.**

Name ______________________ Class ____________ Date ____________

8-5 Practice

Form G

Adding and Subtracting Rational Expressions

Find the least common multiple of each pair of polynomials.

1. $3x(x + 2)$ and $6x(2x - 3)$
$6x(x + 2)(2x - 3)$

2. $2x^2 - 8x + 8$ and $3x^2 + 27x - 30$
$6(x - 1)(x - 2)^2(x + 10)$

3. $4x^2 + 12x + 9$ and $4x^2 - 9$
$(2x + 3)^2(2x - 3)$

4. $2x^2 - 18$ and $5x^3 + 30x^2 + 45x$
$10x(x + 3)^2(x - 3)$

Simplify each sum or difference. State any restrictions on the variables.

5. $\frac{x^2}{5} + \frac{x^2}{5}$ $\frac{2x^2}{5}$

6. $\frac{6y - 4}{y^2 - 5} + \frac{3y + 1}{y^2 - 5}$ $\frac{3(3y - 1)}{y^2 - 5}$; $y \neq \pm\sqrt{5}$

7. $\frac{2y + 1}{3y} + \frac{5y + 4}{3y}$ $\frac{7y + 5}{3y}$; $y \neq 0$

8. $\frac{12}{xy^3} - \frac{9}{xy^3}$ $\frac{3}{xy^3}$; $x, y \neq 0$

9. $-\frac{2}{n + 4} - \frac{n^2}{n^2 - 16}$ $\frac{2 - n}{n - 4}$; $n \neq \pm 4$

10. $\frac{3}{8x^3y^3} - \frac{1}{4xy}$ $\frac{3 - 2x^2y^2}{8x^3y^3}$; $x, y \neq 0$

11. $\frac{6}{5x^2y} + \frac{5}{10xy^2}$ $\frac{12y + 5x}{10x^2y^2}$; $x, y \neq 0$

12. $\frac{x + 2}{x^2 + 4x + 4} + \frac{2}{x + 2}$ $\frac{3}{x + 2}$; $x \neq -2$

13. $\frac{4}{x^2 - 25} + \frac{6}{x^2 + 6x + 5}$
$\frac{10x - 26}{(x + 5)(x - 5)(x + 1)}$; $x \neq -1, \pm 5$

14. $\frac{y}{4y + 8} - \frac{1}{y^2 + 2y}$
$\frac{y - 2}{4y}$; $y \neq -2, 0$

Simplify each complex fraction.

15. $\dfrac{\frac{2}{x}}{\frac{3}{y}}$ $\frac{2y}{3x}$

16. $\dfrac{1 + \frac{2}{x}}{4 - \frac{6}{x}}$ $\frac{x + 2}{4x - 6}$

17. $\dfrac{\frac{1}{x - 2}}{2 + \frac{1}{x}}$ $\frac{x}{2x^2 - 3x - 2}$

18. $\dfrac{\frac{3}{x + 1}}{\frac{5}{x - 1}}$ $\frac{3x - 3}{5x + 5}$

19. $\dfrac{\frac{4}{x^2 - 1}}{\frac{3}{x + 1}}$ $\frac{4}{3x - 3}$

20. $\dfrac{1 + \frac{2}{3}}{\frac{4}{9}}$ $\frac{15}{4}$

21. $\dfrac{\frac{2}{x} + 6}{\frac{1}{y}}$ $\frac{2y + 6xy}{x}$

22. $\dfrac{\frac{x + 3}{x - 3}}{\frac{x^2 - 9}{3x - 9}}$ $\frac{3}{x - 3}$

23. $\dfrac{\frac{5}{x + 3}}{2 + \frac{1}{x + 3}}$ $\frac{5}{2x + 7}$

8-5 Practice (continued) Form G

Adding and Subtracting Rational Expressions

24. The total resistance for a parallel circuit is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

a. If $R = 1$ ohm, $R_2 = 6$ ohms, and $R_3 = 8$ ohms, find R_1. **$\frac{24}{17}$ ohms**

b. If $R_1 = 3$ ohms, $R_2 = 4$ ohms, and $R_3 = 6$ ohms, find R. **$\frac{4}{3}$ ohms**

Add or subtract. Simplify where possible. State any restrictions on the variables.

25. $\frac{3}{7x^2y} + \frac{4}{21xy^2}$

$\frac{9y + 4x}{21x^2y^2}$; $x, y \neq 0$

26. $\frac{xy - y}{x - 2} - \frac{y}{x + 2}$

$\frac{x^2y}{x^2 - 4}$; $x \neq \pm 2$

27. $\frac{3}{x^2 - x - 6} + \frac{2}{x^2 + 6x + 5}$

$\frac{(5x + 1)(x + 3)}{(x - 3)(x + 2)(x + 5)(x + 1)}$; $x \neq -5, -2, -1, 3$

28. $\frac{6}{y^2 + 5y} + \frac{3y}{4y + 20} - \frac{1}{4}$

$\frac{2y^2 - 5y + 24}{4y(y + 5)}$; $y \neq -5, 0$

29. A teacher uses an overhead projector with a focal length of x cm. She sets a transparency $x + 20$ cm below the projector's lens. Write an expression in simplest form to represent how far from the lens she should place the screen to place the image in focus. Use the thin-lens equation $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$. **$\frac{x^2 + 20x}{20}$ cm**

30. Open-Ended Write two complex fractions that simplify to $\frac{x + 5}{x^2}$.

Check students' work.

31. Writing Explain the differences in the process of adding two rational expressions using the lowest common denominator (LCD) and adding them using a common denominator that is not the LCD. Include an example in your explanation.

Answers may vary. Sample: Adding by using a common denominator that is not the LCD, means you will have to divide out the common factor(s) after adding. For example,

$$\frac{1}{2x^2 + 4} + \frac{1}{3x^2 + 6} = \frac{3x^2 + 6}{(2x^2 + 4)(3x^2 + 6)} + \frac{2x^2 + 4}{(2x^2 + 4)(3x^2 + 6)} = \frac{5x^2 + 10}{(2x^2 + 4)(3x^2 + 6)}$$

$$= \frac{5\cancel{(x^2 + 2)}}{(2)\cancel{(x^2 + 2)}(3)(x^2 + 2)} = \frac{5}{6(x^2 + 2)}$$

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8-5 Practice *Form K*

Adding and Subtracting Rational Expressions

Find the least common multiple of each pair of polynomials.

To start, completely factor each expression.

1. $4x^2 - 36$ and $6x^2 + 36x + 54$

$(2)(2)(x - 3)(x + 3)$ and $(2)(3)(x + 3)(x + 3)$

$12(x - 3)(x + 3)^2$

2. $(x - 2)(x + 3)$ and $10(x + 3)^2$

$10(x - 2)(x + 3)^2$

Simplify each sum or difference. State any restrictions on the variables.

To start, factor the denominators and identify the LCD.

3. $\dfrac{6x - 1}{x^2y} + \dfrac{3y + 2}{2xy}$

$\dfrac{6x - 1}{(x)(x)(y)} + \dfrac{3y + 2}{(2)(x)(y)}$

$\dfrac{14x + 3xy - 2}{2x^2y}$; $x \neq 0, y \neq 0$

4. $\dfrac{1}{x^2 - 4x - 12} - \dfrac{3x}{4x + 8}$

$\dfrac{-3x^2 + 18x + 4}{4(x + 2)(x - 6)}$; $x \neq 6, -2$

5. $\dfrac{2x}{x^2 + 5x + 4} + \dfrac{2x}{3x + 3}$

$\dfrac{2x^2 + 14x}{3(x + 1)(x + 4)}$; $x \neq -4, -1$

Add or subtract. Simplify where possible. State any restrictions on the variables.

6. $\dfrac{x + 2}{x - 1} + \dfrac{x - 3}{2x + 1}$

$\dfrac{3x^2 + x + 5}{(2x + 1)(x - 1)}$; $x \neq 1$ or $-\frac{1}{2}$

7. $\dfrac{x}{x^2 - x} + \dfrac{1}{x}$

$\dfrac{2x - 1}{x(x - 1)}$; $x \neq 0$ or 1

8. $4y - \dfrac{y + 2}{y^2 + 3y}$

$\dfrac{4y^3 + 12y^2 - y - 2}{y^2 + 3y}$; $y \neq 0$ or -3

9. Error Analysis A classmate said that the sum of $\dfrac{4}{x^2 - 9}$ and $\dfrac{7}{x + 3}$ is $\dfrac{7x + 25}{x^2 - 9}$. What mistake did your classmate make? What is the correct sum?

The classmate multiplied the second term by $\frac{x + 3}{x + 3}$ instead of $\frac{x - 3}{x - 3}$. The correct sum is $\frac{7x - 17}{x^2 - 9}$.

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8-5 Practice (continued) *Form K*

Adding and Subtracting Rational Expressions

Simplify each complex fraction.

To start, multiply the numerator and the denominator by the LCD of all the rational expressions.

10. $\dfrac{\frac{1}{x} + 3}{\frac{5}{y} + 4}$

$\dfrac{(\frac{1}{x} + 3)xy}{(\frac{5}{y} + 4)xy}$

$\dfrac{y + 3xy}{5x + 4xy}$

11. $\dfrac{-3}{\frac{5}{x} + y}$

$\dfrac{-3x}{5 + xy}$

12. $\dfrac{\frac{4}{x + 2}}{\frac{3}{x - 1}}$

$\dfrac{4(x - 1)}{3(x + 2)}$

13. Reasoning What real numbers are not in the domain of the function $f(x) = \dfrac{\frac{x + 1}{x + 2}}{\frac{x + 3}{x + 4}}$? Explain.

$x \neq -2, -3, -4$; any of these values of x make a denominator zero, causing a fraction to be undefined, so they are not in the domain of the function.

14. If you jog 12 mi at an average rate of 4 mi/h and walk the same route back at an average rate of 3 mi/h, you have traveled 24 mi in 7 h and your overall rate is $\frac{24}{7}$ mi/h. What is your overall average rate if you travel d mi at 3 mi/h and d mi at 4 mi/h?

$\frac{24}{7}$ mi/h

15. Multiple Choice Simplify: $\dfrac{\frac{2}{x} - 5}{\frac{6}{x} - 3}$. **A**

Ⓐ $\dfrac{2 - 5x}{6 - 3x}$ Ⓑ $\dfrac{2 + 5x}{6 - 3x}$ Ⓒ $\dfrac{2x - 5}{6x + 3}$ Ⓓ $\dfrac{6 + 3x}{2 - 5x}$

8-5 Standardized Test Prep

Adding and Subtracting Rational Expressions

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. Which is the least common denominator of fractions that have denominators $5x + 10$ and $25x^2 - 100$? **C**

Ⓐ $5(x - 2)$

Ⓑ $5(x^2 - 20)$

Ⓒ $25(x^2 - 4)$

Ⓓ $75(x + 2)(x^2 - 4)$

2. Which expression equals $\dfrac{\frac{2}{m} + 6}{\frac{1}{n}}$? **G**

Ⓕ $\dfrac{12n}{m}$

Ⓖ $\dfrac{2n + 6mn}{m}$

Ⓗ $\dfrac{6m + 2}{mn}$

Ⓘ $\dfrac{m}{2n + 6mn}$

3. Which expression equals $\dfrac{4}{x^2 - 3x} + \dfrac{6}{3x - 9}$? **A**

Ⓐ $\dfrac{2(x + 2)}{x(x - 3)}$

Ⓑ $\dfrac{10}{x^2 - 9}$

Ⓒ $\dfrac{4x + 18}{3x(x - 3)}$

Ⓓ $\dfrac{2}{x}$

4. The harmonic mean of two numbers a and b equals $\dfrac{2}{\frac{1}{a} + \frac{1}{b}}$. Which expression equals the harmonic mean of x and $x + 1$? **I**

Ⓕ $\dfrac{2}{x^2 + x}$

Ⓖ $\dfrac{4x + 2}{x^2 + x}$

Ⓗ $2x + 1$

Ⓘ $\dfrac{2x^2 + 2x}{2x + 1}$

Short Response

5. Subtract $3 - \dfrac{1}{x^2 + 5}$. Write your answer in simplest form. State any restrictions on the variable. Show your work.

$$3 - \frac{1}{x^2 + 5} = \frac{3(x^2 + 5)}{x^2 + 5} - \frac{1}{x^2 + 5} = \frac{3(x^2 + 5) - 1}{x^2 + 5} = \frac{3x^2 + 14}{x^2 + 5}$$; **no restrictions**

[2] correct answer with complete work shown

[1] incorrect or incomplete work shown OR correct answer, without work shown

[0] no answer given

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8-5 Enrichment

Adding and Subtracting Rational Expressions

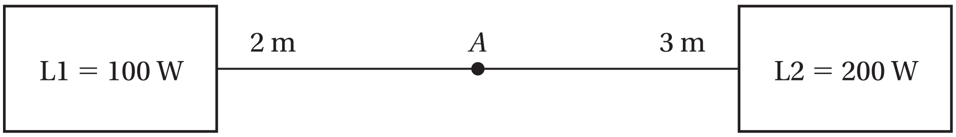
The Superposition Principle

The illumination received from a light source is given by the formula

$$I = S \cdot D^{-2} \text{ or } I = \frac{S}{D^2}$$

where I is the illumination at a certain point, S is the strength of the light source, measured in watts or kilowatts, and D is the distance of the point from the light source. The superposition principle states that the total illumination received at a given point from two sources is equal to the sum of the illuminations from each of the sources.

Suppose a plant is positioned at point A. Copy and complete the following to find the total illumination received by the plant when both lights are on.



$I_{\text{total}} = I_{L1} + I_{L2}$

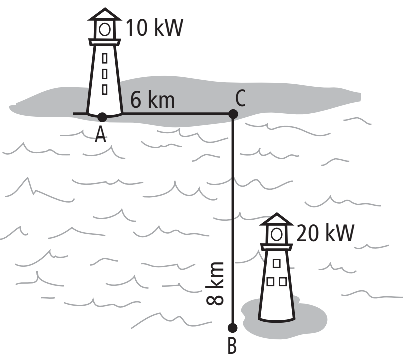
$= \frac{100}{2^2} + \frac{200}{3^2}$

$= 25 + 22.\overline{2}$

$= 47.2$ Round to the nearest tenth.

1. The amount of illumination received by the plant is about 47.2 watts/m^2.

Lighthouse A, located on an ocean shore, uses a 10-kW light. Lighthouse B uses a 20-kW light and is located 8 km out to sea from a point 6 km down the beach from Lighthouse A.



2. A man is walking down the beach away from lighthouse A and toward point C. When he is x kilometers away from lighthouse A and has not yet reached point C, write the illumination he receives as a function of x in simplest form.

   $\frac{30x^2 - 120x + 1000}{x^2(x^2 - 12x + 100)}$ watts/m^2

3. Now suppose that the man is x kilometers beyond point C as he walks down the beach. What illumination does he receive, written as a function of x in simplest form?

   $\frac{30x^2 + 240x + 1360}{(x^2 + 12x + 36)(x^2 + 64)}$ watts/m^2

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8-5 Reteaching

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions is a lot like adding and subtracting fractions. Before you can add or subtract the expressions, they must have a common denominator. The easiest common denominator to work with is the *least common denominator*, or LCD.

Problem

What is the LCD of $\frac{6x}{x^3 + 2x^2}$ and $\frac{5}{x^3 + x^2 - 2x}$?

$$\left.\begin{aligned} x^3 + 2x^2 &= x^2(x + 2) \\ x^3 + x^2 - 2x = x(x^2 + x - 2) &= x(x + 2)(x - 1) \end{aligned}\right\}$$

Completely factor each denominator.

$x^2, (x + 2), x, (x + 2), (x - 1)$ Make a list of all the factors.

$x^2, (x + 2), x, \cancel{(x + 2)}, (x - 1)$ Cross off any repeated factors.

$x^2, (x + 2), \cancel{x}, \cancel{(x + 2)}, (x - 1)$ When the only difference between factors is the exponent (like x^2 and x), cross off all but the factor with the greatest exponent.

$x^2(x + 2)(x - 1)$ Multiply the remaining factors on the list. The product is the LCD.

The LCD of $\frac{6x}{x^3 + 2x^2}$ and $\frac{5}{x^3 + x^2 - 2x}$ is $x^2(x + 2)(x - 1)$.

Exercises

Assume that the polynomials given are the denominators of rational expressions. Find the LCD of each set.

1. $x + 3$ and $2x + 6$ **$2(x + 3)$**

2. $2x - 1$ and $3x + 4$ **$(2x - 1)(3x + 4)$**

3. $x^2 - 4$ and $x + 2$ **$(x + 2)(x - 2)$**

4. $x^2 + 7x + 12$ and $x + 4$ **$(x + 3)(x + 4)$**

5. $x^2 + 5$ and $x - 25$ **$(x^2 + 5)(x - 25)$**

6. x^3 and $6x^2$ **$6x^3$**

7. x, $2x$, and $4x^3$ **$4x^3$**

8. $x^2 + 8x + 16$ and $x + 4$ **$(x + 4)^2$**

9. $x^2 + 4x - 5$ and $x^3 - x^2$
$x^2(x - 1)(x + 5)$

10. $x^2 - 9$ and $x^2 + 2x - 3$
$(x + 3)(x - 3)(x - 1)$

8-5 Reteaching (continued)

Adding and Subtracting Rational Expressions

To find the sum or difference of rational expressions with unlike denominators:

- completely factor each denominator
- identify the least common denominator, or LCD
- multiply each expression by the factors needed to produce the LCD
- add or subtract numerators, and put the result over the LCD

Problem

What is the difference of $\frac{2x}{3x^2 + 5x} - \frac{14}{3x^2 + 26x + 35}$ in simplest form? State any restrictions on the variable.

$$\left.\begin{aligned} 3x^2 + 5x &= x(3x + 5) \\ 3x^2 + 26x + 35 &= (3x + 5)(x + 7) \end{aligned}\right\}$$ Completely factor each denominator.

$$x(3x + 5)(x + 7)$$ Identify the LCD.

$$\left[\frac{2x}{x(3x + 5)} \cdot \frac{(x + 7)}{(x + 7)}\right] - \left[\frac{14}{(3x + 5)(x + 7)} \cdot \frac{x}{x}\right]$$ Multiply to produce the LCD.

$$= \frac{2x(x + 7)}{x(3x + 5)(x + 7)} - \frac{14x}{x(3x + 5)(x + 7)}$$

$$= \frac{2x(x + 7) - 14x}{x(3x + 5)(x + 7)}$$ Subtract the numerators.

$$= \frac{2x^2 + 14x - 14x}{x(3x + 5)(x + 7)}$$ Distribute.

$$= \frac{2x}{3x^2 + 26x + 35}$$ Simplify.

Therefore, $\frac{2x}{3x^2 + 5x} - \frac{14}{3x^2 + 26x + 35} = \frac{2x}{3x^2 + 26x + 35}$, where $x \neq -7, -\frac{5}{3}, 0$.

Exercises

Simplify each sum or difference. State any restrictions on the variable.

11. $\frac{y}{y - 1} + \frac{2}{1 - y}$ $\frac{y - 2}{y - 1}$; $y \neq 1$

12. $\frac{3}{x + 2} + \frac{2}{x^2 - 4}$ $\frac{3x - 4}{(x + 2)(x - 2)}$; $x \neq \pm 2$

13. $\frac{x}{x^2 + 5x + 6} - \frac{2}{x^2 + 3x + 2}$

$\frac{x - 3}{(x + 1)(x + 3)}$; $x \neq -1, -2, -3$

14. $\frac{4x + 1}{x^2 - 4} - \frac{3}{x - 2}$

$\frac{x - 5}{(x + 2)(x - 2)}$; $x \neq \pm 2$

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8-6 Additional Vocabulary Support

Solving Rational Equations

Problem

What are the solutions of the rational equation? Justify your steps.

$\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2 - 6x + 8}$	Write original equation.
$\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{(x-2)(x-4)}$	Factor the denominators to find the LCD.
$(x-2)(x-4)\left[\frac{x}{x-2} + \frac{1}{x-4}\right] = (x-2)(x-4)\left[\frac{2}{(x-2)(x-4)}\right]$	Multiply each side by the LCD to clear the denominators.
$x(x-4) + 1(x-2) = 2$	Distribute and simplify.
$x^2 - 4x + x - 2 = 2$	Distribute.
$x^2 - 3x - 4 = 0$	Simplify.
$(x-4)(x+1) = 0$	Factor the quadratic.
$x = 4$ or $x = -1$	Solve for x.
$x = 4$ causes division by 0, so $x = 4$ is an extraneous solution.	Check for extraneous solutions.

Because $\frac{-1}{-1-2} + \frac{1}{-1-4} = \frac{2}{(-1)^2 - 6(-1) + 8}$, the solution is $x = -1$.

Exercise

What are the solutions of the rational equation? Justify the steps.

$\frac{5}{x} + \frac{4}{x+3} = \frac{8}{x^2 + 3x}$	Write the original equation.
$\frac{5}{x} + \frac{4}{x+3} = \frac{8}{x(x+3)}$	Factor the denominator to find the LCD.
$x(x+3)\left[\frac{5}{x} + \frac{4}{(x+3)}\right] = x(x+3)\left[\frac{8}{x(x+3)}\right]$	Multiply each side by the LCD.
$9x + 15 = 8$	Distribute and simplify.
$x = -\frac{7}{9}$	Solve.

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8-6 Think About a Plan

Solving Rational Equations

Storage One pump can fill a tank with oil in 4 hours. A second pump can fill the same tank in 3 hours. If both pumps are used at the same time, how long will they take to fill the tank?

Understanding the Problem

1. How long does it take the first pump to fill the tank? **4 h**

2. How long does it take the second pump to fill the tank? **3 h**

3. What is the problem asking you to determine?

the length of time it takes for both pumps to fill the tank when used at the same time

Planning the Solution

4. If V is the volume of the tank, what expressions represent the portion of the tank that each pump can fill in one hour?

First pump: $\frac{1}{4}V$ Second pump: $\frac{1}{3}V$

5. What expression represents the part of the tank the two pumps can fill in one hour if they are used at the same time?

$\frac{1}{4}V + \frac{1}{3}V$

6. Let t be the number of hours. Write an equation to find the time it takes for the two pumps to fill one tank.

$\left(\frac{1}{4}V + \frac{1}{3}V\right)t = V$

Getting an Answer

7. Solve your equation to find how long the pumps will take to fill the tank if both pumps are used at the same time.

$\left(\frac{1}{4}V + \frac{1}{3}V\right)t = V$

$\left(\frac{1}{4} + \frac{1}{3}\right)t = 1$

$\frac{7}{12}t = 1$

$t = \frac{12}{7} = 1\frac{5}{7}$ hours

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8-6 Practice

Form G

Solving Rational Equations

Solve each equation. Check each solution.

1. $\frac{x}{3} + \frac{x}{2} = 10$ **12**
2. $\frac{1}{x} - \frac{x}{9} = 0$ **±3**
3. $-\frac{4}{x+1} = \frac{5}{3x+1}$ **$-\frac{9}{17}$**
4. $\frac{4}{x} = \frac{x}{4}$ **±4**
5. $\frac{3x}{4} = \frac{5x+1}{3}$ **$-\frac{4}{11}$**
6. $\frac{3}{2x-3} = \frac{1}{5-2x}$ **$\frac{9}{4}$**
7. $\frac{x-4}{3} = \frac{x-2}{2}$ **−2**
8. $\frac{2x-1}{x+3} = \frac{5}{3}$ **18**
9. $\frac{2y}{5} + \frac{2}{6} = \frac{y}{2} - \frac{1}{6}$ **5**
10. $\frac{1}{2x+2} + \frac{5}{x^2-1} = \frac{1}{x-1}$ **7**
11. $\frac{2}{x+3} + \frac{5}{3-x} = \frac{6}{x^2-9}$ **−9**
12. An airplane flies from its home airport to a city 510 mi away and back. The total flying time for the round-trip flight is 3.9 h. The plane travels the first half of the trip at 255 mi/h with no wind.
    - **a.** How strong is the wind on the return flight? Round your answer to the nearest tenth. **about 13.4 mi/h**
    - **b.** Is the wind on the return flight a headwind or a tailwind? **tailwind**

Use a graphing calculator to solve each equation. Check each solution.

13. $\frac{x-1}{6} = \frac{x}{4}$ **−2**
14. $\frac{x-2}{10} = \frac{x-7}{5}$ **12**
15. $\frac{4}{x+3} = \frac{10}{2x-1}$ **−17**
16. $\frac{3}{3-x} = \frac{4}{2-x}$ **6**
17. $\frac{3y}{5} + \frac{1}{2} = \frac{y}{10}$ **−1**
18. $5 - \frac{4}{x+1} = 6$ **−5**
19. $\frac{2}{3} + \frac{3x-1}{6} = \frac{5}{2}$ **4**
20. $\frac{4}{x-1} = \frac{5}{x-2}$ **−3**
21. $\frac{1}{x} - \frac{2}{x+3} = 0$ **3**

Solve each equation for the given variable.

22. $h = \frac{2A}{b}$; b **$b = \frac{2A}{h}$**
23. $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$; d_o **$d_o = \frac{fd_i}{d_i - f}$**
24. $\frac{h}{t} + 16t = v_o$; h **$h = v_o t - 16t^2$**
25. $m = \frac{y_2 - y_1}{x_2 - x_1}$; x_1 **$x_1 = x_2 - \frac{y_2 - y_1}{m}$**
26. $\frac{xy}{z} + 2x = \frac{z}{y}$; x **$x = \frac{z^2}{y^2 + 2yz}$**
27. $\frac{S - 2wh}{2w + 2h} = \ell$; S **$S = 2\ell w + 2wh + 2\ell h$**

8-6 Practice (continued) *Form G*

Solving Rational Equations

28. One delivery driver can complete a route in 6 h. Another driver can complete the same route in 5 h.

a. Let N be the total number of deliveries on the route. Write expressions to represent the number of deliveries each driver can make in 1 hour. $\frac{N}{6}; \frac{N}{5}$

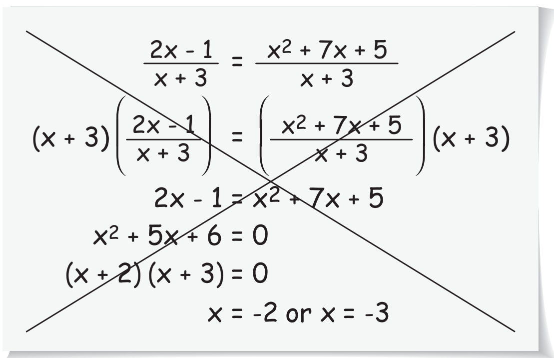
b. Write an expression to represent the number of hours needed to make N deliveries if the drivers work together. $\frac{N}{\left(\frac{N}{6} + \frac{N}{5}\right)}$

c. If the drivers work together, about how many hours will they take to complete the route? Round your answer to the nearest tenth. **2.7 h**

29. A fountain has two drainage valves. With the first valve open, the fountain drains completely in 4 h. With only the second valve open, the fountain drains completely in 5.25 h. About how many hours will the fountain take to drain with both valves open? Round your answer to the nearest tenth. **2.3 h**

30. A pen factory has two machines making pens. Together, the machines make 1500 pens during an 8-h shift. Machine A makes pens at 2.5 times the rate of Machine B. About how many hours would Machine A need to make 1500 pens by itself? Round your answer to the nearest tenth. **11.2 h**

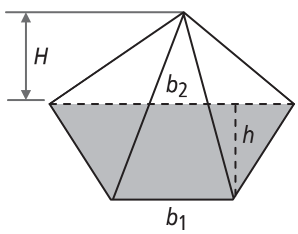
31. Error Analysis Describe and correct the error made in solving the equation.
The student did not check the solutions in the original equation. The solution $x = -3$ is extraneous.



32. The formula $V = hH\left(\frac{b_1 + b_2}{6}\right)$ gives the volume of a pyramid with a trapezoidal base.

a. Solve this equation for b_2. $b_2 = \frac{6V}{hH} - b_1$

b. Find b_2 if $b_1 = 5$ cm, $h = 8$ cm, $H = 9$ cm, and $V = 216\text{ cm}^3$. **13 cm**



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8-6 Practice

Form K

Solving Rational Equations

Solve each equation. Check each solution.

To start, multiply each side by the LCD.

1. $\frac{x}{4} - \frac{3}{x} = \frac{1}{4}$

$4x\left(\frac{x}{4} - \frac{3}{x}\right) = (4x)\left(\frac{1}{4}\right)$

$x = -3$ or 4

2. $x + \frac{6}{x} = -5$

$x = -3$ or -2

3. $\frac{5}{2x - 2} = \frac{15}{x^2 - 1}$

$x = 5$

4. The aerodynamic covering on a bicycle increases a cyclist's average speed by 10 mi/h. The time for a 75-mi trip is reduced by 2 h.

a. Using t for time, write a rational equation you can use to determine the average speed using the aerodynamic covering. $\frac{75}{t-2} = \frac{75}{t} + 10$

b. What is the average speed for the trip using the aerodynamic covering? 25 mi/h

Using a graphing calculator, solve each equation. Check each solution.

5. $\frac{4}{2x - 3} = \frac{x}{5}$

$x = -2.5$ or 4

6. $x + 5 = \frac{6}{x}$

$x = -6, 1$

7. $\frac{2}{x + 7} = \frac{x}{x^2 - 49}$

$x = 14$

Solve each equation for the given variable.

8. $F = \frac{mv^2}{r}$ for v

$v = \sqrt{\frac{Fr}{m}}$

9. $\frac{c}{dt} = Qm$ for d

$d = \frac{c}{Qmt}$

10. $\frac{F}{Gm_1} = \frac{m_2}{r^2}$ for r

$r = \sqrt{\frac{Gm_1m_2}{F}}$

8-6 Practice (continued) Form K

Solving Rational Equations

11. You can travel 40 mi on your motorbike in the same time it takes your friend to travel 15 mi on his bicycle. If your friend rides his bike 20 mi/h slower than you ride your motorbike, find the speed for each bike.
rate for motorbike: 32 mi/h; rate for bicycle: 12 mi/h

12. A passenger train travels 392 mi in the same time that it takes a freight train to travel 322 mi. If the passenger train travels 20 mi/h faster than the freight train, find the speed of each train.
rate for freight train: 92 mi/h; rate for passenger train: 112 mi/h

13. You can paint a fence twice as fast as your sister can. Working together, the two of you can paint a fence in 6 h. How many hours would it take each of you working alone?
you: 9 h; your sister: 18 h

Solve each equation. Check each solution.

14. $\frac{2}{x-3} - \frac{4}{x+3} = \frac{8}{x^2-9}$
$x = 5$

15. $\frac{3}{x+5} + \frac{2}{5-x} = \frac{-4}{x^2-25}$
$x = 21$

16. $\frac{3}{x^2-1} + \frac{4x}{x+1} = \frac{1.5}{x-1}$
$x = 0.375$

17. You are planning a school field trip to a local theater. It costs $60 to rent the bus. Each theater ticket costs $5.50.

a. Write a function $c(x)$ to represent the cost per student if x students sign up for the trip. $c(x) = 5.50 + \frac{60}{x}$

b. How many students must sign up if the cost is to be no more than $10 per student? **14 students**

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8-6 Standardized Test Prep

Solving Rational Equations

Gridded Response

For Exercises 1–8, what are the solutions of each rational equation? Enter your answer in the grid provided. If necessary, enter your answer as a fraction.

1. $\frac{3 - x}{6} = \frac{6 - x}{12}$

2. $\frac{2}{6x + 2} = \frac{x}{3x^2 + 11}$

3. $\frac{3}{2x - 4} = \frac{5}{3x + 7}$

4. $\frac{2}{x + 2} + \frac{5}{x - 2} = \frac{6}{x^2 - 4}$

5. $\frac{7}{x^2 - 5x} + \frac{2}{x} = \frac{3}{2x - 10}$

6. $\frac{1}{4 - 5x} = \frac{3}{x + 9}$

7. $\frac{7}{2} = \frac{7x}{8} - 4$

8. $4 + \frac{2y}{y - 5} = \frac{8}{y - 5}$

Answers

1. 0

2. 11

3. 41

4. 0

5. 6

6. 3/16

7. 60/7

8. 14/3

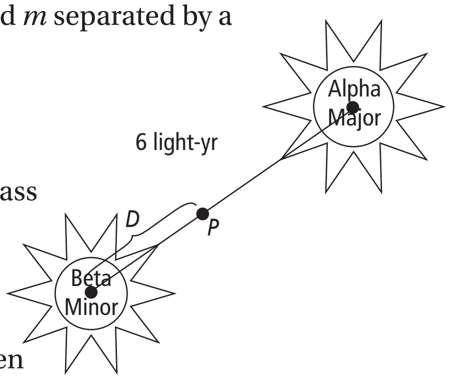
8-6 Enrichment

Solving Rational Equations

Gravitational Attraction

Many physical phenomena obey inverse-square laws. That is, the strength of the quantity is inversely proportional to the square of the distance from the source.

Isaac Newton was the first to discover that gravity obeys an inverse-square law. The gravitational force F between objects of masses M and m separated by a distance D is given by $F = \frac{GMm}{D^2}$, where G is a constant.



Suppose that two stars, Alpha Major and Beta Minor, are separated by a distance of 6 light-years. Alpha Major has four times the mass of Beta Minor. Let M represent the mass of Beta Minor. Suppose that an object, represented by point P, of mass m is placed between the two stars at a distance of D light-years from Beta Minor.

1. Write an expression for the gravitational force between this object and Beta Minor. $\frac{GMm}{D^2}$

2. Write an expression for the gravitational force between this object and Alpha Major. $\frac{4GMm}{(6 - D)^2}$

3. What is the distance of a neutral position of the object P with mass m from Beta Minor? At neutral position, both Beta Minor and Alpha Major exert equal force on point P. **2 light-yr**

A spaceship is stationary between a planet and its moon, experiencing an equal gravitational pull from each. When measurements are taken, it is determined that the craft is 300,000 km from the planet and 100,000 km from the moon.

4. What is the ratio of the mass of the planet to the mass of the moon? **9 : 1**

5. What would be the ratio of their masses if the distance of the spaceship from the planet was R times the distance of the spaceship to the moon? $R^2 : 1$

Once every 277 yr, the two moons of the planet Omega Minus line up in a straight line with the planet. The moons are equal in mass, and the inner moon is equidistant from the outer moon and from the planet. Measurements show that an object two thirds of the distance from the planet to the inner moon, and in the same line as all three, experiences an equal gravitational pull in both directions.

6. What is the ratio of the mass of the planet to the mass of one of its moons? **17 : 4**

Name ______________________ Class ____________ Date ____________

8-6 Reteaching

Solving Rational Equations

When one or both sides of a rational equation has a sum or difference, multiply each side of the equation by the LCD to eliminate the fractions.

Problem

What is the solution of the rational equation $\frac{6}{x} + \frac{x}{2} = 4$? Check the solutions.

$2x\left(\frac{6}{x}\right) + 2x\left(\frac{x}{2}\right) = 2x(4)$	Multiply each term on both sides by the LCD, $2x$.
$2\cancel{x}\left(\frac{6}{\cancel{x}}\right) + \cancel{2}x\left(\frac{x}{\cancel{2}}\right) = 2x(4)$	Divide out the common factors.
$12 + x^2 = 8x$	Simplify.
$x^2 - 8x + 12 = 0$	Write the equation in standard form.
$(x - 2)(x - 6) = 0$	Factor.
$x - 2 = 0$ or $x - 6 = 0$	Use the Zero-Product Property.
$x = 2$ or $x = 6$	Solve for x.

Check

$\frac{6}{x} + \frac{x}{2} \stackrel{?}{=} 4$	$\frac{6}{x} + \frac{x}{2} \stackrel{?}{=} 4$
$\frac{6}{2} + \frac{2}{2} \stackrel{?}{=} 4$	$\frac{6}{6} + \frac{6}{2} \stackrel{?}{=} 4$
$3 + 1 \stackrel{?}{=} 4$	$1 + 3 \stackrel{?}{=} 4$
$4 = 4$ ✓	$4 = 4$ ✓

The solutions are $x = 2$ and $x = 6$.

Exercises

Solve each equation. Check the solutions.

1. $\frac{10}{x+3} + \frac{10}{3} = 6$ $\frac{3}{4}$

2. $\frac{1}{x-3} = \frac{x-4}{x^2-27}$ $\frac{39}{7}$

3. $\frac{6}{x-1} + \frac{2x}{x-2} = 2$ $\frac{8}{5}$

4. $\frac{7}{3x-12} - \frac{1}{x-4} = \frac{2}{3}$ 6

5. $\frac{2x}{5} = \frac{x^2-5x}{5x}$ −5

6. $\frac{8(x-1)}{x^2-4} = \frac{4}{x-2}$ 4

7. $x + \frac{4}{x} = \frac{25}{6}$ $\frac{3}{2}, \frac{8}{3}$

8. $\frac{2}{x} + \frac{6}{x-1} = \frac{6}{x^2-x}$ no solution

9. $\frac{2}{x} + \frac{1}{x} = 3$ 1

10. $\frac{4}{x-1} = \frac{5}{x-1} + 2$ $\frac{1}{2}$

11. $\frac{1}{x} = \frac{5}{2x} + 3$ $-\frac{1}{2}$

12. $\frac{x+6}{5} = \frac{2x-4}{5} - 3$ 25

8-6 Reteaching (continued)

Solving Rational Equations

You often can use rational equations to model and solve problems involving rates.

Problem

Quinn can refinish hardwood floors four times as fast as his apprentice, Jack. They are refinishing 100 ft^2 of flooring. Working together, Quinn and Jack can finish the job in 3 h. How long would it take each of them working alone to refinish the floor?

Let x be Jack's work rate in ft^2/h. Quinn's work rate is four times faster, or $4x$.

square feet refinished per hour by Jack and Quinn together	=	square feet of floor they refinish together	÷	hours worked together
ft^2/h	=	ft^2	÷	h

$x + 4x = \frac{100}{3}$ Their work rates sum to 100 ft^2 in 3 h.

$3(x) + 3(4x) = 3\left(\frac{100}{3}\right)$ They work for 3 h. Refinished floor area = rate × time.

$15x = 100$ Simplify.

$x \approx 6.67$ Divide each side by 15.

Jack works at the rate of 6.67 ft^2/h. Quinn works at the rate of 26.67 ft^2/h.

Let j be the number of hours Jack takes to refinish the floor alone, and let q be the number of hours Quinn takes to refinish the floor alone.

$$6.67 = \frac{100}{j} \qquad 26.67 = \frac{100}{q}$$

$$j(6.67) = j\left(\frac{100}{j}\right) \qquad q(26.67) = q\left(\frac{100}{q}\right)$$

$$6.67j = 100 \qquad 26.67q = 100$$

$$j \approx 15 \qquad q \approx 3.75$$

Jack would take 15 h and Quinn would take 3.75 h to refinish the floor alone.

Exercises

13. An airplane flies from its home airport to a city and back in 5 h flying time. The plane travels the 720 mi to the city at 295 mi/h with no wind. How strong is the wind on the return flight? Is the wind a headwind or a tailwind? **about 14 mi/h; headwind**

14. Miguel can complete the decorations for a school dance in 5 days working alone. Nasim can do it alone in 3 days, and Denise can do it alone in 4 days. How long would it take the three students working together to decorate? **about 1.3 days**

Name ______________________________ Class ____________ Date ____________

Chapter 8 Quiz 1

Form G

Lessons 8-1 through 8-3

Do you know HOW?

When $x = 2$ and $y = 3$, $z = 42$. Write the function that models each of the following relationships.

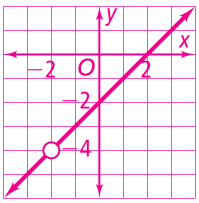
1. z varies inversely with x and directly with y $z = \frac{28y}{x}$
2. z varies jointly with x and y $z = 7xy$
3. z varies directly with x and inversely with the square of y $z = \frac{189x}{y^2}$

Write an equation for the translation of $y = \frac{4}{x}$ that has the given asymptotes.

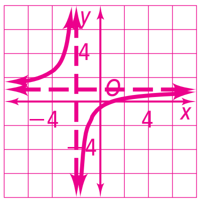
4. $x = 1$ and $y = 2$ $y = \frac{4}{x - 1} + 2$
5. $x = 0$ and $y = -1$ $y = \frac{4}{x} - 1$
6. $x = -2$ and $y = -5$ $y = \frac{4}{x + 2} - 5$
7. $x = -4$ and $y = 0$ $y = \frac{4}{x + 4}$

Sketch the graph of each rational function.

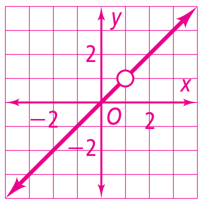
8. $y = \frac{x^2 - 4}{x + 2}$



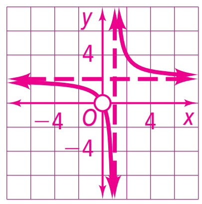
9. $y = \frac{x - 1}{x + 2}$



10. $y = \frac{x^2 - x}{x - 1}$



11. $y = \frac{2x^2}{x^2 - x}$



Do you UNDERSTAND?

12. **Open-Ended** Write a rational function that has a removable discontinuity in its graph at $x = 3$. **Check students' work.**

13. **Reasoning** How many inverse variation functions have (0, 0) as a solution? Explain. **None; division by zero is undefined.**

14. **Reasoning** (c, d) is a solution of the equation $y = \frac{a}{x}$. Name one solution of the equation $y = -\frac{a}{x}$. Justify your answer.
Answers may vary. Sample: $(c, -d)$; the functions are reflections of each other over the x-axis.

Name ______________________ Class ____________ Date ____________

Chapter 8 Quiz 2

Form G

Lessons 8-4 through 8-6

Do you know HOW?

Simplify each expression.

1. $\frac{8x^2 - 10x + 3}{6x^2 + 3x - 3}$ **$\frac{4x - 3}{3x + 3}$**

2. $\frac{x^2 - 4}{x^2 - 1} \cdot \frac{x + 1}{x^2 + 2x}$ **$\frac{x - 2}{x(x - 1)}$**

3. $\frac{7}{5y + 25} - \frac{4}{3y + 15}$ **$\frac{1}{15(y + 5)}$**

4. $\frac{x^2}{x^2 + 2x + 1} \div \frac{3x}{x^2 - 1}$ **$\frac{x(x - 1)}{3(x + 1)}$**

5. $\frac{2x + 4}{3x - 3} \cdot \frac{12x - 12}{x + 5}$ **$\frac{8(x + 2)}{x + 5}$**

6. $\frac{7}{2xy^2} + \frac{3}{8x^2y}$ **$\frac{28x + 3y}{8x^2y^2}$**

7. $\frac{x^2 - 16}{2x + 8} \div \frac{(x - 4)^2}{8x - 32}$ **4**

8. $5 - \frac{4x^2 - 5x + 1}{x^2 - x}$ **$\frac{x + 1}{x}$**

9. $\frac{\frac{1}{3x}}{\frac{5}{6y}}$ **$\frac{2y}{5x}$**

10. $\frac{\frac{2}{y} - 1}{\frac{3}{x} + 1}$ **$\frac{2x - xy}{3y + xy}$**

Solve each equation. Check each solution.

11. $\frac{3}{1 - x} = \frac{2}{1 + x}$ **$-\frac{1}{5}$**

12. $\frac{1}{2x + 2} = \frac{1}{x - 1}$ **-3**

13. $\frac{x + 3}{x^2 + 3x - 4} = \frac{x + 2}{x^2 - 16}$ **-5**

14. Becky and Kendra start a business painting fences. They can paint 200 ft^2 of fence in 40 min if they work together. If Kendra paints four times faster than Becky, how long would it take each of them to paint a 500-ft^2 fence working alone? **Kendra: 125 min, Becky: 500 min**

15. You bowl 3 games and get an average score of 120. How many additional consecutive games with scores of 145 would you need to bring your average up to 130? **2**

Do you UNDERSTAND?

16. **Error Analysis** A student claims that $\frac{x(x^2 - 9)}{x^2 - 2x - 3}$ is in simplest form. Is the student correct? Explain.
No; $x - 3$ can be divided out of both the numerator and the denominator.

17. **Writing** Explain how to simplify $\frac{1 + \frac{3}{x + 4}}{\frac{x - 1}{x^2 + 3x - 4}}$.

Rewrite 1 as $\frac{x + 4}{x + 4}$, add the terms in the numerator, rewrite division as multiplication of the numerator by the reciprocal of the denominator, factor $x^2 + 3x - 4$, and divide out common factors. The simplest form of the expression is $x + 7$.

Name ______________________ Class ____________ Date ____________

Chapter 8 Test

Form G

Do you know HOW?

Write a function that models each variation.

1. $x = -1$ when $y = 5$. y varies inversely as x. $y = -\frac{5}{x}$

2. $x = 3$ and $y = 12$ when $z = 2$. z varies directly with y and inversely with x. $z = \frac{0.5y}{x}$

Is the relationship between the values in each table a direct variation, an inverse variation, or neither? Write an equation to model any direct or inverse variation.

3.

x	-2	4	6
y	4	-8	-12

direct variation; $y = -2x$

4.

x	-2	-1	3
y	$-\frac{1}{2}$	-1	$\frac{1}{3}$

inverse variation; $y = \frac{1}{x}$

Write an equation for the translation of $y = \frac{2}{x}$ with the given asymptotes.

5. $x = 1, y = -1$ $y = \frac{2}{x - 1} - 1$

6. $x = 5, y = \frac{1}{2}$ $y = \frac{2}{x - 5} + \frac{1}{2}$

For each rational function, identify any holes or horizontal or vertical asymptotes of its graph.

7. $y = \frac{x}{x - 3}$

vertical asymptote, $x = 3$;
horizontal asymptote, $y = 1$

8. $y = \frac{-2(x - 8)}{8 - x}$

hole at $x = 8$; no horizontal or vertical asymptote

9. $y = \frac{x + 3}{(x + 2)(x + 3)}$

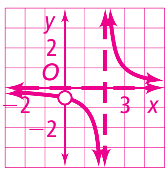
hole at $x = -3$; vertical asymptote, $x = -2$; horizontal asymptote, $y = 0$

10. $y = \frac{1}{x + 4} - 3$

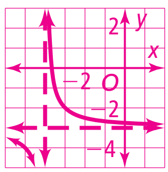
vertical asymptote, $x = -4$; horizontal asymptote, $y = -3$

Sketch the graph of each rational function.

11. $y = \frac{x}{x(x - 2)}$



12. $y = \frac{1}{x + 4} - 3$



Simplify each rational expression. State any restrictions on the variable.

13. $\frac{3x^2 - 12}{x^2 - x - 6}$ $\frac{3(x - 2)}{(x - 3)}$; $x \neq 3$ or -2

14. $\frac{2x^2 - x}{4x^2 - 4x + 1} \div \frac{x}{8x - 4}$ 4; $x \neq \frac{1}{2}$ or 0

Find the least common multiple of each pair of polynomials.

15. $x^2 - 16$ and $5x + 20$ $5(x - 4)(x + 4)$

16. $7(x - 2)(x + 5)$ and $2(x + 5)^2$ $14(x - 2)(x + 5)^2$

Simplify each sum or difference.

17. $\frac{2}{x + 5} + \frac{x}{x - 5}$ $\frac{x^2 + 7x - 10}{(x + 5)(x - 5)}$

18. $\frac{3x}{x^2 - 4} - \frac{1}{x^2}$ $\frac{3x^3 - x^2 + 4}{x^2(x^2 - 4)}$

Simplify each complex fraction.

19. $\dfrac{1 + \frac{2}{3}}{\frac{3}{4} - \frac{1}{3}}$ **4**

20. $\dfrac{1 + \frac{1}{x}}{5 - \frac{1}{y}}$ $\dfrac{xy + y}{5xy - x}$

Solve each equation. Check each solution.

21. $\frac{x}{3} + \frac{x}{2} = 10$ **12**

22. $\frac{y - 3}{5} = \frac{y + 1}{7}$ **13**

23. $\frac{x}{2} = 2x - 3$ **2**

24. $-\frac{x}{4} = \frac{2x}{3}$ **0**

25. $\frac{1}{x} - \frac{1}{6} = \frac{4}{3x^2}$ **2, 4**

26. $\frac{2x - 4}{x - 5} = 0$ **2**

27. Chad can paint a room in 2 h. Cassie can paint the room in 3 h. How long would it take them to paint the room working together? **1.2 h**

28. How many milliliters of 0.65% saline solution must be added to 100 mL of 3% saline solution to get a 0.7% solution? **4600 mL**

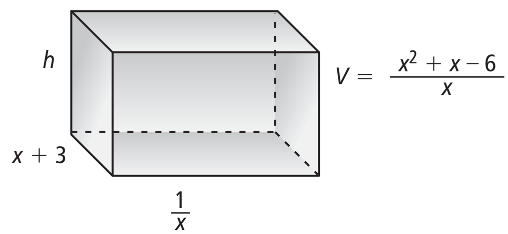
Do you UNDERSTAND?

29. Writing Explain what it means to simplify a rational expression.
The numerator and denominator have no common factors.

30. Reasoning Is $(3, y)$ on the graph of $y = \frac{3}{x - 3} + 3$? Justify your answer.
No; the graph of the equation has a vertical asymptote at $x = 3$.

31. Reasoning Is it possible to write a rational equation that has a graph with no vertical asymptotes? Explain.
Yes; answers may vary. Sample: $\frac{x}{x^2 + 1}$

32. Reasoning Write an expression in simplest form for the height of the rectangular prism shown at the right. **$x - 2$**



33. Writing Describe how the variables in the given equation are related. $y = \frac{w^2(x - 2)}{z}$
y varies directly with the square of w and the difference $x - 2$, and varies inversely with z.

Name ______________________ Class ____________ Date ____________

Chapter 8 Quiz 1

Form K

Lessons 8–1 through 8–3

Do you know HOW?

If $z = 15$ when $x = 4$ and $y = 5$, write a function that models the relationship.

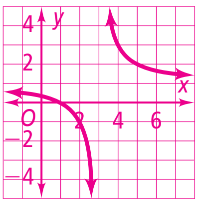
1. z varies directly with y and inversely with x. $z = \frac{12y}{x}$

2. z varies jointly with x and y. $z = \frac{3xy}{4}$

3. z varies inversely with the product of both x and y. $z = \frac{300}{xy}$

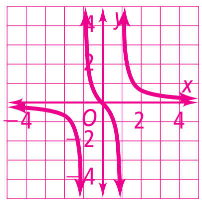
Sketch the graph of each rational function. Then identify the domain and range.

4. $y = \frac{x - 1}{x - 3}$



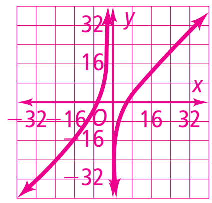
domain: all real numbers except $x = 3$; range: all real numbers except $y = 1$

5. $y = \frac{x}{x^2 - 1}$



domain: all real numbers except $x = 1$ and $x = -1$; range: all real numbers

6. $y = \frac{x^2 - 36}{x + 1}$



domain: all real numbers except $x = -1$; range: all real numbers

Do you UNDERSTAND?

7. **Writing** Explain how to find the asymptotes of $y = -\frac{3}{x - 2} + 11$.

Write the right side of the equation as a single term: $y = \frac{11x - 19}{x - 2}$. Because 2 is a zero of the denominator and is not a zero of the numerator, $x = 2$ is the vertical asymptote. The degree of the numerator is less than the degree of the denominator, so the horizontal asymptote is $y = \frac{11}{1} = 1$.

8. Your school is renting an indoor water park for a school outing. The water park costs $1500 for the day.

   a. How many students need to go to the water park so that each person pays $7 each? 215 students

   b. **Reasoning** The water park can hold up to 450 people. Does the park hold enough students so that each person pays only $3 each? Explain.
   No; to pay $3 each, 500 students would need to go to the park.

Name ______________________ Class ______________ Date ______________

Chapter 8 Quiz 2

Form K

Lessons 8–4 through 8–6

Do you know HOW?

Simplify each rational expression. State any restrictions on the variables.

1. $\frac{3x - 6}{5x - 20} \cdot \frac{x - 8}{5x - 10}$

$\frac{3(x - 8)}{25(x - 4)}$; $x \neq 4, 2$

2. $\frac{y^2 - 25}{y^2 - 16} \div \frac{2y + 10}{y^2 - 4y}$

$\frac{y(y - 5)}{2(y + 4)}$; $y \neq 4, 0, -4, -5$

3. $\frac{14x + 7}{4x - 6} \cdot \frac{8x - 12}{42x + 21}$

$\frac{2}{3}$; $x \neq -\frac{1}{2}, \frac{3}{2}$

4. $\frac{8}{3x^3y} + \frac{4}{9xy^3}$

$\frac{4x^2 + 24y^2}{9x^3y^3}$; $x \neq 0, y \neq 0$

5. $3x - \frac{x^2 - 5x}{x^2 - 2}$

$\frac{3x^3 - x^2 - x}{x^2 - 2}$; $x \neq \pm\sqrt{2}$

6. $\frac{5x}{2y + 4} - \frac{6}{y^2 + 2y}$

$\frac{5xy - 12}{2y(y + 2)}$; $y \neq 0, -2$

Solve each equation. Check your solutions.

7. $\frac{-4}{5(x + 2)} = \frac{3}{x + 2}$

no solution

8. $\frac{-2}{x^2 - 2} = \frac{2}{x - 4}$

$x = 2$ or -3

9. $\frac{1}{2x} - \frac{2}{5x} = \frac{1}{2}$

$x = \frac{1}{5}$

Do you UNDERSTAND?

10. A large snowplow can clear a parking lot in 4 h. A small snowplow needs more time to clear the lot. Working together, they can clear the lot in 3 h. How long would it take the small plow to clear the lot by itself? 12 h

11. Error Analysis Your classmate says that the simplified form of the complex fraction $\frac{\frac{1}{x} + 3}{4 + \frac{5}{y}}$ is $\frac{1 + 3x}{4y + 5}$. What mistake did he make? What is the correct answer?

He multiplied the numerator by x and the denominator by y instead of multiplying both by xy; $\frac{y + 3xy}{5x + 4xy}$

Name ______________________ Class ____________ Date ____________

Chapter 8 Test

Form K

Do you know HOW?

Write a function that models each variation.

1. $x = 5$ when $y = 15$, y varies inversely with x. $y = \frac{75}{x}$

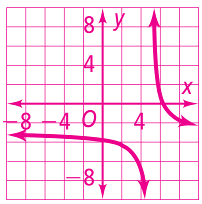
2. $z = 6$ when $x = 4$ and $y = 3$, z varies directly with y and inversely with x. $z = \frac{8y}{x}$

3. $z = 8$ when $x = 2$ and $y = 4$, z varies jointly with x and y. $z = xy$

Write and graph an equation of the translation of $y = \frac{4}{x}$ that has the given asymptotes.

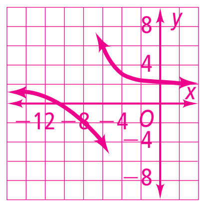
4. $x = 5;\ y = -3$

$y = \frac{4}{x - 5} - 3$



5. $x = -7;\ y = 2$

$y = \frac{4}{x + 7} + 2$



For each rational function, identify any holes or horizontal or vertical asymptotes of the graph.

6. $y = \frac{x^2 - 36}{x + 1}$

horizontal asymptote: none; vertical asymptote: $x = -1$

7. $y = \frac{5}{x - 2} + 1$

horizontal asymptote: $y = 1$; vertical asymptote: $x = 2$

8. $y = \frac{(x - 2)(x + 4)}{(x + 2)(x - 3)}$

horizontal asymptote: $y = 1$; vertical asymptotes: $x = -2$, $x = 3$

Simplify each rational expression. State any restrictions on the variable.

9. $\frac{x^2y - x^2}{x^3 - x^3y}$

$-\frac{1}{x}$; $x \neq 0$, $y \neq 1$

10. $\frac{4x}{5y} \cdot \frac{10y^4}{24x^3}$

$\frac{y^3}{3x^2}$; $x \neq 0$, $y \neq 0$

11. $\frac{x^2 + 2x - 15}{x - 3} \div \frac{x^2 - 4}{2}$

$\frac{2(x + 5)}{(x + 2)(x - 2)}$; $x \neq 3, 2, -2$

12. $\frac{y}{y - 1} - \frac{y}{y - 2}$

$\frac{-y}{(y - 1)(y - 2)}$; $y \neq 1, 2$

13. $\dfrac{\frac{x + y}{2x - y}}{\frac{x + y}{2x + y}}$

$\frac{2x + y}{2x - y}$; $x \neq \frac{y}{2}, -\frac{y}{2}, -y$

14. $\frac{x - 4}{x^2 - 2x - 8} - \frac{x - 2}{x^2 - 16}$

$\frac{-12}{(x + 4)(x - 4)(x + 2)}$; $x \neq -4, 4, -2$

Chapter 8 Test (continued)

Form K

Solve each equation. Check your solutions.

15. $\frac{9}{28} + \frac{3}{x + 2} = \frac{3}{4}$ **$x = 5$**

16. $x + \frac{x^2 - 5}{x^2 - 1} = \frac{x^2 + x + 2}{x + 1}$ **$x = 3$**

17. $\frac{x - 4}{x - 2} = \frac{x - 2}{x + 2} + \frac{1}{x - 2}$ **$x = 14$**

Do you UNDERSTAND?

18. A concrete supplier sells premixed concrete in 300-ft^3 truckloads. The area *A* that the concrete will cover is inversely proportional to the depth *d* of the concrete.

a. Write a model for the relationship between the area and the depth of a truckload of poured concrete. **$A = \frac{300}{d}$**

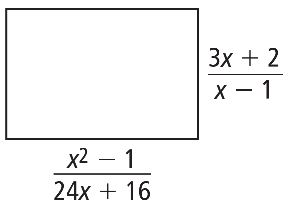
b. What area will the concrete cover if it is poured to a depth of 0.5 ft? A depth of 1 ft? A depth of 1.5 ft? **600 ft^2; 300 ft^2; 200 ft^2**

c. When the concrete is poured into a circular area, the depth of the concrete is inversely proportional to the square of the radius *r*. Write a model for this relationship. **$d = \frac{300}{r^2}$**

19. Writing Explain why zero cannot be in the domain of an inverse variation.
If zero were included in the domain, then the denominator of a rational expression would be equal to 0, which creates an undefined expression.

20. Error Analysis Your friend said that there is no horizontal asymptote for the rational expression $y = \frac{x + 5}{x^2 - 2x + 1}$. How do you know that this is incorrect? What is the correct horizontal asymptote?
The degree of the numerator is 1 and the degree of the denominator is 2, so the horizontal asymptote is $y = 0$.

21. Reasoning Write a simplified expression for the area of the rectangle at the right. **$\frac{x + 1}{8}$**



Name ______________________ Class __________ Date __________

Chapter 8 Performance Tasks

Task 1

a. Write a function that models an inverse variation situation.
b. Find the constant of the inverse variation.
c. Determine the dependent and independent variables.
d. Identify the domain and range.
e. Find the values of any asymptotes.
f. Graph the function, making sure to indicate any asymptotes.

[4] Check students' work. Student writes an appropriate function and correctly identifies constant of variation, independent and dependent variables, domain and range, and values of asymptotes. Graph is accurate. The only errors are minor computational or copying errors.

[3] Student writes an appropriate function. Uses appropriate methods to correctly identify some, but not all, of the following: constant of variation, independent and dependent variables, domain and range, and values of asymptotes. Graph is incomplete, but work presented is correct.

[2] Student writes a rational function that is not an inverse variation. Uses appropriate methods to correctly identify some, but not all, of the following: constant of variation, independent and dependent variables, domain and range, and values of asymptotes. Graph shows some understanding of the task.

[1] Student writes a rational function that is not an inverse variation. Student attempts to identify constant of variation, independent and dependent variables, domain and range, and values of the asymptotes, but work contains major errors. Graph is incorrect or missing.

[0] Student makes no attempt or no response is given.

Task 2

a. Write a function that models a combined variation situation with two independent variables. Let one of the variables have a direct variation with the dependent variable, and the other have an inverse variation.
b. Let the independent variables be equal to 3 and 6. Find the value of the dependent variable.
c. Let the dependent variable be 15, and one of the independent variables be 5. Find the value of the other independent variable.
d. Let the independent variable that varies directly with the dependent variable be 2. Does the other independent variable increase, decrease, or remain constant when the dependent variable increases in value? Now let the independent variable that varies inversely with the dependent variable be 4. Does the other independent variable increase, decrease, or remain constant when the dependent variable increases in value?
e. Let the dependent variable be 1. Does one of the independent variables increase, decrease, or remain constant when the other increases in value?

[4] Check students' work. Student writes an appropriate function and uses appropriate methods to find the value of variables and describe end behavior of the function. The only errors are minor computational or copying errors.

[3] Student writes an appropriate function and uses appropriate methods to identify variables, with several errors. Description of end behavior is partly correct.

[2] Student writes a rational function that is not an inverse variation. Correctly identifies variables, but does not describe end behavior.

[1] Student attempts a solution, but shows little understanding of problem.

[0] Student makes no attempt or no response is given.

Chapter 8 Performance Tasks (continued)

Task 3

To answer each question, use the function $t(r) = \frac{d}{r}$, where t is the time in hours, d is the distance in miles, and r is the rate in miles per hour.

a. Sydney drives 10 mi at a certain rate and then drives 20 mi at a rate 5 mi/h faster than the initial rate. Write expressions for the time along each part of the trip. Add these times to write an equation for the total time in terms of the initial rate, $t_{\text{Total}}(r)$.

b. Determine the reasonable domain and range and describe any discontinuities of $t_{\text{Total}}(r)$. Graph $t_{\text{Total}}(r)$ on your graphing calculator.

c. At what rate, to the nearest mi/h, must Sydney drive for the first 10 miles if the entire 30 mi must be covered in about 45 min? Find the answer using the graph and using algebraic methods.

d. How long will Sydney take to drive the entire 30 mi if the car's initial rate varies between 10 mi/h and 20 mi/h? Use the graph and algebraic methods to find the answer.

[4] Student uses appropriate methods to write correct expressions, $\frac{10}{r}$, $\frac{20}{r+5}$, equation, $t_{\text{Total}}(r) = \frac{30r + 50}{r(r+5)}$, identify domain $r > 0$ and range $t > 0$, and horizontal asymptote $t = 0$ (vertical asymptotes $r = 0$, $r = -5$ not within reasonable domain), and find that Sydney must drive about 37 mi/h for the first 10 mi to go 30 mi in 45 min, or will take between 1 h 18 min and 2 h 20 min if the car's rate varies between 10 mi/h and 20 mi/h.

[3] Student applies an appropriate solution strategy, but misunderstood part of the problem or ignored a condition in the problem.

[2] Student chooses an appropriate solution strategy, but applied it incorrectly or incompletely. Student shows some understanding of the problem.

[1] Student attempts a solution, but shows little understanding of the problem.

[0] Student makes no attempt or no response is given.

Name ______________________ Class ____________ Date ____________

Chapter 8 Cumulative Review

Multiple Choice

For Exercises 1–10, choose the correct letter.

1. Which matrix represents the system of equations $\begin{cases} 3a + c = -2 \\ -b + 4c = 0 \\ -6a + b = 8 \end{cases}$? C

(A) $\left[\begin{array}{cc|c} 3 & 1 & -2 \\ -1 & 4 & 0 \\ -6 & 1 & 8 \end{array}\right]$

(B) $\left[\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & -1 & 4 \\ -6 & 1 & 0 \end{array}\right]$

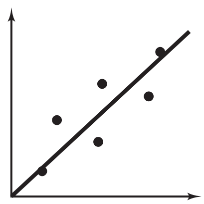
(C) $\left[\begin{array}{ccc|c} 3 & 0 & 1 & -2 \\ 0 & -1 & 4 & 0 \\ -6 & 1 & 0 & 8 \end{array}\right]$

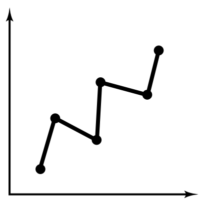
(D) $\left[\begin{array}{ccc|c} 3 & 1 & 0 & -2 \\ -1 & 4 & 0 & 0 \\ -6 & 1 & 0 & 8 \end{array}\right]$

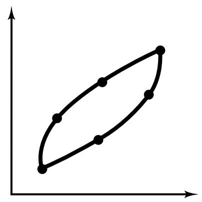
2. Which point lies on the graph of $2x + y - z = 3$? G

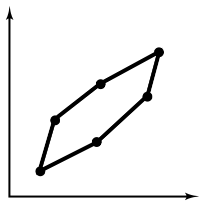
(F) (3, 0, −2) (G) (0, 1, −2) (H) (1, 0, 0) (I) (0, 0, 3)

3. Which of these shows a trend line for the plotted data? A

(A)


(C)


(B)


(D)


4. Which of these is a binomial quadratic? F

(F) $4x^2 + 6$ (G) $x^2 + 3x - 2$ (H) $2x^2 + 2x + 2$ (I) $x^3 - 2x^2$

5. What is the inverse of $y = 6x - 2$? C

(A) $y = \frac{1}{6}x - \frac{1}{2}$ (B) $y = 6x + 2$ (C) $y = \frac{1}{6}x + \frac{1}{3}$ (D) $y = \frac{1}{6}x + 2$

6. Which is a zero of the function $f(x) = x^2 - 36$? G

(F) 0 (G) −6 (H) −18 (I) 36

Chapter 8 Cumulative Review (continued)

7. Which number equals $\log_2 2 + (\log_2 20 - \log_2 5)$? **B**

(A) $\log_2 17$ (B) 3 (C) $\log_2 6$ (D) 4

8. Which matrix contains $a_{21} = 4$? **F**

(F) $\begin{bmatrix} -4 & -4 \\ 4 & -4 \end{bmatrix}$ (G) $\begin{bmatrix} -4 & -4 \\ -4 & 4 \end{bmatrix}$ (H) $\begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$ (I) $\begin{bmatrix} 4 & -4 \\ -4 & -4 \end{bmatrix}$

9. Which point cannot be on the line $y = 2x + 5$? **D**

(A) $(-4, -3)$ (B) $(3, 11)$ (C) $(1, 7)$ (D) $(0, 6)$

10. Which system has no solution? **G**

(F) $\begin{cases} y = 2x + 3 \\ y = 3x + 2 \end{cases}$ (G) $\begin{cases} y = 4x + 3 \\ y = 4x - 3 \end{cases}$ (H) $\begin{cases} y = \frac{1}{2}x \\ y = \frac{1}{20}x \end{cases}$ (I) $\begin{cases} y > 3 \\ x < 2 \end{cases}$

Short Response

11. The graph of a certain direct variation includes the point (2, 6). Write the equation of the direct variation. Show your work. **$y = kx$; $k = \frac{y}{x} = \frac{6}{2} = 3$; $y = 3x$**

12. Identify any vertical or horizontal asymptotes on the graph of $y = \frac{x + 3}{2x + 8}$. Justify your answer. **Vertical asymptote at $x = -4$, because division by zero is undefined; degree of numerator and denominator are equal, so horizontal asymptote at $y = \frac{a}{b} = \frac{1}{2}$.**

Extended Response

13. The height of a triangle is 9 mm less than the square of its base length. The height of a similar triangle is 6 mm more than twice the length of the base of the first triangle. Write an expression for the area of the second triangle in simplified form. State any restrictions on the variables.

[4] Student solves proportion $\frac{x}{x^2 - 9} = \frac{b_2}{2x + 6}$ to get $b_2 = \frac{2x}{x - 3}$, uses area formula to get $A_2 = \frac{1}{2}\left(\frac{2x}{x - 3}\right)(2x + 6) = \frac{2x(x + 3)}{x - 3}$, and notes $x > 3$ (a length cannot be < 0).

[3] Student uses appropriate strategies, but misunderstood part of the problem or ignored a condition in the problem.

[2] Student attempts to use appropriate strategies, but applies them incorrectly or incompletely.

[1] Student work contains significant errors and little evidence of strategies used.

[0] no answers given

Chapter 8 Project Teacher Notes: Under Pressure

About the Project

The Chapter Project gives students an opportunity to explore safety issues related to scuba diving. They use inverse proportions to find volumes of air in lungs and to find the sizes of tanks needed to hold enough air to dive to various depths.

Introducing the Project

- Ask students how they think a graph of the distance below the surface versus the volume of air in the lungs might look.
- Review the project with students and instruct students to make a list of questions they will need to answer to complete the project.

Activity 1: Graphing

Students make tables and graphs to find how the volume of air in their lungs varies with depth and pressure.

Activity 2: Writing

Students explain why they must exhale while ascending from driving.

Activity 3: Solving

Students use Boyle's Law to find how large a scuba diving tank needs to be.

Activity 4: Solving

Students use an inverse variation to find how long the air in a tank lasts at various depths.

Finishing the Project

You may wish to plan a project day on which students share their completed projects. Encourage students to explain their processes as well as their results. Ask students to review their project and update their folders.

- Ask students to review their methods for finding, recording, and solving formulas, and for making the tables and graphs used in the project.
- Ask groups to share their insights that resulted from completing the project, such as any shortcuts they found for solving formulas or making graphs.

Name ______________________ Class ____________ Date ____________

Chapter 8 Project: Under Pressure

Beginning the Chapter Project

Modern scuba-diving equipment allows divers to stay under water for long periods of time. You will use mathematics to explore safety issues related to scuba diving. Then, you will design a poster or brochure about scuba-diving safety.

Activities

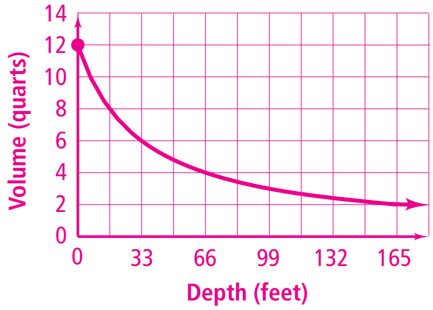
Activity 1: Graphing

Scuba divers must learn about pressure under water. At the water's surface, air exerts 1 atmosphere (atm) of pressure. Under water, the pressure increases. The pressure P (atm) varies with depth d (ft) according to the equation $P = \frac{d}{33} + 1$.

Boyle's law states that the volume V of air varies inversely with the pressure P. If you hold your breath, the volume of air in your lungs increases as you ascend. If you have 4 qt of air in your lungs at a depth of 66 ft ($P = 3$ atm), the air will expand to 6 qt when you reach 33 ft, where $P = 2$ atm.

- Using the data in the example above, make a table and graph to show how the volume of air in your lungs varies with depth.
- Make a table and graph to show how the volume of air in your lungs varies with pressure.

Depth (ft)	Volume (qt)
0	12
33	6
66	4
99	3
132	2.4
165	2



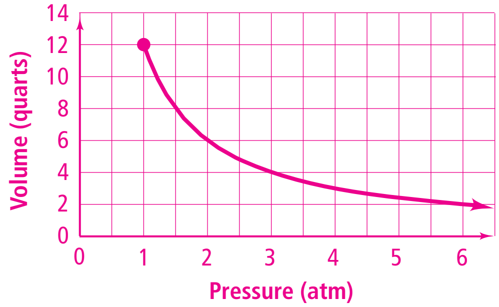
Activity 2: Writing

The volume of air in a diver's lungs could more than double as the diver resurfaces. This expansion can cause the membranes of the lungs to rupture.

- If you fill your lungs with 4 qt of air at a depth of 66 ft, how many quarts of air will you need to exhale during your ascent to still have 4 qt of air in your lungs when you reach the surface?
- Write an explanation of why beginning divers are told "Don't hold your breath!" Refer to your tables and graphs.

Pressure (atm)	Volume (qt)
1	12
2	6
3	4
4	3
5	2.4
6	2

8 qt; Answers may vary. Sample: Divers must exhale during ascent to prevent the expanding air in their lungs from damaging their lungs.



Chapter 8 Project: Under Pressure (continued)

Activity 3: Solving

A popular size of scuba-diving tank holds the amount of compressed air that would occupy 71.2 ft^3 at a normal surface pressure of 1 atm. The air in the tank is at a pressure of about 2250 lb/in.^{2}, so the tank itself can have a volume much less than 71.2 ft^3. How large does the tank need to be to hold 71.2 ft^3? (*Hint*: Use Boyle's Law: $PV = k$. Remember that 1 atm = 14.7 lb/in.^{2}.)

0.465 ft^3

Activity 4: Solving

The rate at which a scuba diver uses air in the tank depends on many factors, like the diver's age and lung capacity. Another important factor is the depth of the dive.

At greater depths, a diver uses the air in the tank more quickly. Assume that the amount of time the air will last is inversely proportional to the pressure at the depth of the dive.

- Suppose a tank has enough air to last 60 min at the surface. How long will it last at a depth of 99 ft? (The pressure is 4 atm, or 4 times as great.)
- Make a table showing how long the air will last at 0 ft, 20 ft, 33 ft, 40 ft, 50 ft, 66 ft and 99 ft. **15 min**

Depth (ft)	Pressure (atm)	Time (min)
0	1.00	60.0
20	1.61	37.3
33	2.00	30.0
40	2.21	27.1
50	2.51	23.9
66	3.00	20.0
99	4.00	15.0

Finishing the Project

Design a poster or brochure explaining what you learned about scuba-diving safety in this chapter. Use graphs, tables, and examples to support your conclusions.

Reflect and Revise

Work with a classmate to review your poster or brochure. Check that your graphs and examples are correct and that your explanations are clear. If necessary, refer to a book on scuba diving. Discuss your poster or brochure with an adult who works in the area of sports safety, such as a lifeguard, coach, physical education teacher, or recreation director. Ask for their suggestions for improvements.

Extending the Project

What other safety issues must scuba divers consider? Ask a scuba diver or refer to a book to find other things a scuba diver must consider to dive safely.

Name ______________________ Class ______________ Date ______________

Chapter 8 Project Manager: Under Pressure

Getting Started

Read the project. As you work on the project, you will need a calculator, materials on which you can record your calculations, and materials to make accurate and attractive graphs. Keep all of your work for the project in a folder.

Checklist

- ☐ Activity 1: graphing air volume vs. depth and pressure
- ☐ Activity 2: understanding breathing while diving
- ☐ Activity 3: determining tank size
- ☐ Activity 4: determining duration of air supply
- ☐ project display

Suggestions

- ☐ Make a table and a graph for each comparison.
- ☐ Explain when and why divers must exhale.
- ☐ Use Boyle's Law.
- ☐ First, find the pressure at each depth.
- ☐ What visual aids might help someone viewing your project for the first time to understand lung capacity and its importance when scuba diving? What physical conditions of a diver might make a dive more dangerous than it would be otherwise?

Scoring Rubric

4 Calculations are correct. Graphs are neat, accurate, and clearly show the relationship between the variables. Explanations are clear and complete. The poster or brochure is clear and neat.

3 Calculations are mostly correct. Graphs are neat and mostly accurate with minor errors. Explanations are not complete.

2 Calculations contain both minor and major errors. Graphs are not accurate. Explanations and the poster or brochure are not clear or are incomplete.

1 Major concepts are misunderstood. Project satisfies few of the requirements and shows poor organization and effort.

0 Major elements of the project are incomplete or missing.

Your Evaluation of Project Evaluate your work, based on the *Scoring Rubric*.

Teacher's Evaluation of Project