

Necessary and Sufficient Conditions for H_∞ Static Output-Feedback Control

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Necessary and sufficient conditions are presented for static output-feedback control of linear time-invariant systems using the H_∞ approach. Simplified conditions are derived which only require the solution of two coupled matrix design equations. It is shown that the static output-feedback H_∞ solution does not generally yield a well-defined saddle point for the zero-sum differential game; conditions are given under which it does. This paper also proposes a numerically efficient solution algorithm for the coupled design equations to determine the output-feedback gain. A major contribution is that an initial stabilizing gain is not needed. An F-16 normal acceleration design example is solved to illustrate the power of the proposed approach.

Nomenclature

A	=	system or plant matrix
B	=	control input matrix
C	=	output or measurement matrix
D	=	disturbance matrix
$d(t)$	=	disturbance
H	=	hamiltonian
$J(K, d)$	=	value functional
K	=	static output feedback gain matrix
Q	=	state weighting matrix
R	=	control weighting matrix
$u(t)$	=	control input
$x(t)$	=	internal state
$y(t)$	=	measured output
$z(t)$	=	performance output
γ	=	system L_2 gain

I. Introduction

THE static output-feedback problem is one of the most researched problems in systems and control theory. The use of output feedback allows flexibility and simplicity of implementation. Moreover, in practical applications full state measurements are not usually possible. The restricted-measurement static output-feedback (OPFB) problem is of extreme importance in practical controller design applications including flight control in Ref. 1, manufacturing robotics in Ref. 2, and elsewhere where it is desired that the controller have certain prespecified desirable structure, for example, unity gain outer tracking loop and feedback only from certain available sensors. A survey of OPFB design results is presented in Ref. 3. Finally, though many theoretical conditions have been offered for the existence of OPFB, there are few good solution algorithms. Most existing algorithms require the determination of an initial stabilizing gain, which can be extremely difficult.

It is well known that the OPFB optimal control solution can be prescribed in terms of three coupled matrix equations,⁴ namely, two

associated Riccati equations and a spectral radius coupling equation. A sequential numerical algorithm to solve these equations is presented in Ref. 5. OPFB stabilizability conditions that only require the solution of two coupled matrix equations are given in Refs. 6–8. Some recent linear matrix inequalities (LMI) approaches for OPFB design are presented in Refs. 9–11. These allow the design of OPFB controllers using numerically efficient software, for example, the MATLAB[®] LMI toolbox.¹² However several problems are still open. Most of the solution algorithms are hard to implement, are difficult to solve for higher-order systems, can impose numerical problems, and can have restricted solution procedures such as the initial stabilizing gain requirements.

H_∞ design has played an important role in the study and analysis of control theory since its original formulation in an input-output setting in Ref. 13. It is well known that, though conservative, they provide better response in the presence of disturbance than H_2 optimal techniques. State-space H_∞ solutions were rigorously derived for the linear time-invariant case that required solving several associated Riccati equations in Ref. 14. Later, more insight into the problem was given after the H_∞ linear control problem was posed as a zero-sum two-player differential game.¹⁵ A thorough treatment of H_∞ design is given in Ref. 16, which also considers the case of OPFB using *dynamic* feedback. An excellent treatment of H_2 and H_∞ is given in Ref. 17.

Static OPFB design, as opposed to dynamic output feedback with a regulator, is suitable for the design of aircraft controllers of prescribed structure. Recently H_∞ design has been considered for static OPFB; Hol and Scherer¹⁸ addressed the applicability of matrix-valued sum-of-squares (sos) techniques for the computations of LMI lower bounds. Prempain and Postlethwaite¹⁹ presented conditions for a static output loop shaping controller in terms of two coupled matrix inequalities.

In this paper, we show that the H_∞ approach can be used for static OPFB design to yield a simplified solution procedure that only requires the solution of one associated Riccati equation and a coupled gain matrix condition. This explains and illuminates the results in Ref. 7. That is, H_∞ design provides more straightforward design equations than optimal control, which requires solving three coupled equations. We have two objectives. First, we give necessary and sufficient conditions for OPFB with H_∞ design. Second, we suggest a less restrictive numerical solution algorithm with no initial stabilizing gain requirement. An F-16 design example is included.

II. Necessary and Sufficient Condition for H_∞ OPFB Control

Consider the linear time-invariant system of Fig. 1 with control input $u(t)$ output $y(t)$ and disturbance $d(t)$ given by

$$\dot{x} = Ax + Bu + Dd, \quad y = Cx \quad (1)$$

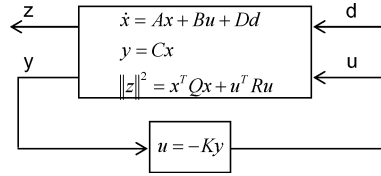
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Fig. 1 System description.



and a performance output $z(t)$ that satisfies

$$\|z(t)\|^2 = x^T Qx + u^T Ru, \quad y = Cx$$

for some positive matrices $Q \geq 0$ and $R > 0$. It is assumed that C has full row rank, a standard assumption to avoid redundant measurements.

By definition the pair (A, B) is said to be *stabilizable* if there exists a real matrix K such that $A - BK$ is (asymptotically) stable. The pair (A, C) is said to be *detectable* if there exists a real matrix L such that $A - LC$ is stable. System (1) is said to be *output feedback stabilizable* if there exists a real matrix K such that $A - BKC$ is stable. The system L_2 gain is said to be *bounded or attenuated* by γ if

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{\int_0^\infty \|d(t)\|^2 dt} = \frac{\int_0^\infty (x^T Qx + u^T Ru) dt}{\int_0^\infty (d^T d) dt} \leq \gamma^2 \quad (2)$$

Bounded L_2 Gain Design Problem:

Defining a constant output-feedback control as

$$u = -Ky = -KCx \quad (3)$$

it is desired to find a constant output-feedback gain K such that the system is stable and the L_2 gain is bounded by a prescribed value γ .

To achieve this, one can define the value functional

$$\begin{aligned} J(K, d) &= \int_0^\infty (x^T Qx + u^T Ru - \gamma^2 d^T d) dt \\ &= \int_0^\infty [x^T (Q + C^T K^T RKC)x - \gamma^2 d^T d] dt \end{aligned} \quad (4)$$

and the corresponding Hamiltonian is defined as

$$\begin{aligned} H(x, V_x, K, d) &= \frac{\partial V^T}{\partial x} [(A - BKC)x + Dd] \\ &\quad + x^T (Q + C^T K^T RKC)x - \gamma^2 d^T d \end{aligned} \quad (5)$$

with costate $\partial V/\partial x$. It is known that for linear systems the value functional $V(x)$ is quadratic and can be taken in the form $V = x^T Px > 0$ without loss of generality. We shall do so throughout the paper.

Two lemmas simplify the presentation of our main theorem 1, which solves this OPFB control problem. Lemma 1 is a mathematical description of Hamiltonian H , Eq. (5) at given predefined disturbance d^* , Eq. (6) and gain K^* , Eq. (7). It shows that if the gain K^* exists the Hamiltonian takes on a special form.

Lemma 1: For the disturbance defined as

$$d^*(t) = (1/\gamma^2)D^T Px \quad (6)$$

if there exists K^* satisfying

$$K^*C = R^{-1}(B^T P + L) \quad (7)$$

for some matrix L , then one can write

$$\begin{aligned} H(x, V_x, K^*, d^*) &= x^T [PA + A^T P + Q + (1/\gamma^2)PDD^T P \\ &\quad - PBR^{-1}B^T P + L^T R^{-1}L]x \end{aligned} \quad (8)$$

Remark: The meaning of d^* and K^* and the special Hamiltonian $H(x, V_x, K^*, d^*)$ will be discussed later. The existence of K^* satisfying Eq. (7) is addressed in theorem 1.

Proof: Introduce a quadratic form $V(x)$

$$V = x^T Px > 0 \quad (9)$$

Then $\partial V/\partial x = 2Px$, and substitution in Eq. (5) will give

$$\begin{aligned} H(x, V_x, K, d) &= 2x^T P[(A - BKC)x + Dd] \\ &\quad + x^T (Q + C^T K^T RKC)x - \gamma^2 d^T d \end{aligned} \quad (10)$$

Note that $H(x, V_x, K, d)$ is globally concave in d . To find a maximizing disturbance set, $0 = \partial H/\partial d = 2D^T Px - 2\gamma^2 d$. This defines the maximizing or worst-case disturbance (6). Substitute Eq. (6) into Eq. (10) to get

$$\begin{aligned} H(x, V_x, K, d^*) &= 2x^T P[(A - BKC)x + D(1/\gamma^2)D^T Px] \\ &\quad + x^T (Q + C^T K^T RKC)x - \gamma^2 [(1/\gamma^2)D^T Px]^T (1/\gamma^2)D^T Px \\ &= x^T [PA + A^T P + Q + (1/\gamma^2)PDD^T P - PBKC \\ &\quad - C^T K^T B^T P + C^T K^T RKC]x \end{aligned}$$

Completing the squares yields

$$\begin{aligned} H(x, V_x, K, d^*) &= x^T [PA + A^T P + Q + (1/\gamma^2)PDD^T P \\ &\quad - PBR^{-1}B^T P + (KC - R^{-1}B^T P)^T R(KC - R^{-1}B^T P)]x \end{aligned} \quad (11)$$

Substituting the gain defined by Eq. (7) into Eq. (11) yields Eq. (8)

$$\begin{aligned} H(x, V_x, K^*, d^*) &= x^T [PA + A^T P + Q + (1/\gamma^2)PDD^T P \\ &\quad - PBR^{-1}B^T P + (R^{-1}B^T P + R^{-1}L - R^{-1}B^T P)^T \\ &\quad \times R(R^{-1}B^T P + R^{-1}L - R^{-1}B^T P)]x \end{aligned}$$

or

$$\begin{aligned} H(x, V_x, K^*, d^*) &= x^T [PA + A^T P + Q + (1/\gamma^2)PDD^T P \\ &\quad - PBR^{-1}B^T P + L^T R^{-1}L]x \end{aligned} \quad \blacksquare$$

The next lemma expresses the Hamiltonian for any K and $d(t)$ in terms of the Hamiltonian for K^* and $d^*(t)$.

Lemma 2: Suppose there exists K^* so that lemma 1 holds, then for any $x(t)$, K , and $d(t)$, one can write

$$\begin{aligned} H(x, V_x, K, d) &= H(x, V_x, K^*, d^*) + x^T [L + R(K - K^*)C]^T \\ &\quad \times R^{-1}[L + R(K - K^*)C]x - x^T [L^T R^{-1}L]x - \gamma^2 \|d - d^*\|^2 \end{aligned} \quad (12)$$

for K^* satisfying Eq. (7) and d^* satisfying Eq. (6).

Proof: Now one has for any $x(t)$, K , $d(t)$, and a quadratic form $V(x)$ defined by Eq. (9)

$$\begin{aligned} H(x, V_x, K, d) &= 2x^T P[(A - BKC)x + Dd] \\ &\quad + x^T (Q + C^T K^T RKC)x - \gamma^2 d^T d \end{aligned} \quad (13)$$

whence, one can derive

$$\begin{aligned} H(x, V_x, K, d) &= x^T [PA + A^T P + Q + (1/\gamma^2)PDD^T P \\ &\quad - PBR^{-1}B^T P + L^T R^{-1}L]x + x^T [-PBKC \\ &\quad - C^T K^T B^T P + C^T K^T RKC - (1/\gamma^2)PDD^T P \\ &\quad + PBR^{-1}B^T P - L^T R^{-1}L]x + x^T 2PDd - \gamma^2 d^T d \end{aligned}$$

or

$$\begin{aligned} H(x, V_x, K, d) &= H(x, V_x, K^*, d^*) + x^T [-PB(KC - R^{-1}B^T P) \\ &\quad - C^T K^T (B^T P - RKC) - L^T R^{-1}L]x \\ &\quad + x^T [-(1/\gamma^2)PDD^T P]x + x^T 2PDd - \gamma^2 d^T d \end{aligned} \quad (14)$$

Substituting $R^{-1}(B^T P + L) = K^* C$, $R^{-1} B^T P = K^* C - R^{-1} L$, $B^T P = R K^* C - L$, and $P B = C^T (K^*)^T R - L^T$ into the first term in square brackets yields, after some manipulations

$$\begin{aligned} & x^T [-P B (K C - R^{-1} B^T P) - C^T K^T (B^T P - R K C) - L^T R^{-1} L] x \\ &= x^T [C^T (K - K^*)^T R (K - K^*) C] x + x^T [L^T R^{-1} L \\ &+ C^T (K - K^*)^T L + L^T (K - K^*) C - L^T R^{-1} L] x \end{aligned}$$

The result contains nonsquare terms. One must change these into square form and study the contribution in order to reach any conclusion; therefore, complete the square to see that

$$\begin{aligned} & x^T [L^T R^{-1} L + C^T (K - K^*)^T L + L^T (K - K^*) C - L^T R^{-1} L] x \\ &= x^T \{ [L^T + C^T (K - K^*)^T R] R^{-1} [L + R (K - K^*) C] \\ &- C^T (K - K^*)^T R R^{-1} R (K - K^*) C - L^T R^{-1} L \} x \end{aligned}$$

Therefore one has

$$\begin{aligned} & x^T [-P B (K C - R^{-1} B^T P) - C^T K^T (B^T P - R K C) - L^T R^{-1} L] x \\ &= x^T \{ [L^T + C^T (K - K^*)^T R] R^{-1} [L + R (K - K^*) C] \\ &- L^T R^{-1} L \} x \end{aligned} \quad (15)$$

Consider now the remaining three terms on the right-hand side of Eq. (14). One has $d^* = (1/\gamma^2) D^T P x$, so that

$$\begin{aligned} (d^*)^T &= (1/\gamma^2) x^T P D, & x^T P D &= \gamma^2 (d^*)^T \\ \text{and } \gamma^4 d^* (d^*)^T &= x^T (P D D^T P) x \end{aligned}$$

Therefore one can show

$$x^T [-(1/\gamma^2) P D D^T P] x + x^T P D d - \gamma^2 d^T d = -\gamma^2 \|d - d^*\|^2 \quad (16)$$

Substituting now Eqs. (16) and (15) into Eq. (14) yields Eq. (12). ■

Remarks:

1) According to the proof and the form of the Hamiltonian in Eq. (12), $d^*(t)$ given by Eq. (6) can be interpreted as a worst-case disturbance because the equation is negative definite in $\|d - d^*\|$.

2) The form (12) of the Hamiltonian does not allow the interpretation of K^* defined by Eq. (7) as a minimizing control. More shall be said about this subsequently.

The following main theorem shows necessary and sufficient conditions for output feedback stabilizability with prescribed attenuation γ .

Theorem 1—necessary and sufficient conditions for H - ∞ static OPFB control:

Assume that $Q \geq 0$ and (A, \sqrt{Q}) is detectable. Then the system defined by Eq. (1) is output-feedback stabilizable with L_2 gain bounded by γ , if and only if 1) (A, B) is stabilizable, and (A, C) is detectable and 2) there exist matrices K^* and L such that

$$K^* C = R^{-1} (B^T P + L) \quad (17)$$

where $P > 0$, $P^T = P$ is a solution of

$$P A + A^T P + Q + (1/\gamma^2) P D D^T P - P B R^{-1} B^T P + L^T R^{-1} L = 0 \quad (18)$$

Proof:

To prove *sufficiency* first, note that lemma 1 shows that $H(x, V_x, K^*, d^*) = 0$ if 2 holds. It is next required to show bounded L_2 gain if 2 holds. From lemma 1 and lemma 2, one has for any K , $x(t)$, and $d(t)$

$$\begin{aligned} H(x, V_x, K, d) &= x^T [L + R (K - K^*) C]^T R^{-1} [L + R (K - K^*) C] x \\ &- x^T [L^T R^{-1} L] x - \gamma^2 \|d - d^*\|^2 \end{aligned} \quad (19)$$

Note that one has, along the system trajectories, for $u = -K y = -K C x$

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x} \dot{x} = \frac{\partial V^T}{\partial x} (A x + B u + D d) \\ &= \frac{\partial V^T}{\partial x} [(A - B K C) x + D d] \end{aligned}$$

so that from Eq. (5)

$$H(x, V_x, K, d) = \frac{dV}{dt} + x^T (Q + C^T K^T R K C) x - \gamma^2 d^T d \quad (20)$$

With Eqs. (19) and (20)

$$\begin{aligned} & x^T [L + R (K - K^*) C]^T R^{-1} [L + R (K - K^*) C] x \\ &- x^T [L^T R^{-1} L] x - \gamma^2 \|d - d^*\|^2 \\ &= \frac{dV}{dt} + x^T (Q + C^T K^T R K C) x - \gamma^2 d^T d \end{aligned}$$

Selecting $K = K^*$, for all $d(t)$ and $x(t)$

$$\begin{aligned} \frac{dV}{dt} + x^T (Q + C^T K^T R K C) x - \gamma^2 d^T d \\ = -x^T [L^T R^{-1} L] x - \gamma^2 \|d - d^*\|^2 \leq 0 \end{aligned} \quad (21)$$

Integrating this equation yields

$$V[x(T)] - V[x(0)] + \int_0^T [x^T (Q + C^T K^T R K C) x - \gamma^2 d^T d] dt \leq 0 \quad (22)$$

Selecting $x(0) = 0$ and noting that nonnegativity implies $V[x(T)] \geq 0 \forall T$, one obtains

$$\int_0^T x^T (Q + C^T K^T R K C) x dt \leq \gamma^2 \int_0^T d^T d dt \quad (23)$$

for all $T > 0$, so that the L_2 gain is less than γ .

Finally, to prove the stability of the closed-loop system, letting $d(t) = 0$ in Eq. (21) one has

$$\frac{dV}{dt} \leq -x^T (Q + C^T K^T R K C) x \leq -x^T Q x \quad (24)$$

Now detectability of (A, \sqrt{Q}) shows that the system is locally asymptotically stable with Lyapunov function $V(x)$.

To prove *necessity*, suppose that there exists an output feedback gain K that stabilizes the system and satisfies L_2 gain $< \gamma$. It follows that $A_c \equiv A - B K C$ is stable. Because $A - B K C = A + \bar{L} C = A + B \bar{K}$, then 1 follows.

Consider the equation

$$A_c^T P + P A_c + (1/\gamma^2) P D D^T P + Q + C^T K^T R K C = 0 \quad (25)$$

From Knobloch et al.,¹⁶ theorem 2.3.1, closed-loop stability, and L_2 gain boundedness implies that Eq. (25) has a unique symmetric solution such that $P \geq 0$. Rearranging Eq. (25) and completing the solution will yield

$$\begin{aligned} P A + A^T P + Q + (1/\gamma^2) P D D^T P - P B R^{-1} B^T P \\ + (K C - R^{-1} B^T P)^T R (K C - R^{-1} B^T P) = 0 \end{aligned} \quad (26)$$

Equation (18) is obtained from Eq. (26) for the gain defined by Eq. (17) and 2 is verified. ■

Note that Eq. (25) is a Lyapunov equation referred to the output $z(t)$ because $\|z(t)\|^2 = x^T Q x + u^T R u$. Moreover, this theorem reveals the importance of the Hamiltonian $H(x, V_x, K^*, d^*)$ because the equation $H(x, V_x, K^*, d^*) = 0$ must hold for a stabilizing OPFB with bounded H - ∞ gain. Note further that if $C = I$, $L = 0$, this theorem reduces to known results for full state variable feedback.

III. Existence of Output-Feedback Game Theoretic Solution

The form of Eq. (12) does not allow the interpretation of (K^*, d^*) as a well-defined saddle point. The purpose of this section is to study when the two policies are in saddle point equilibrium for static output-feedback H_∞ . This means one has a Nash equilibrium in the game theoretic sense as discussed in Ref. 15, so that the H_∞ OPFB problem has a unique solution for the resulting L . In fact, this is the

case when theorem 1 is satisfied with $L = 0$, as we now show using notions from two-player, zero-sum differential game theory.^{15,16} The minimizing player controls $u(t)$, and the maximizing player controls $d(t)$.

Theorem 2—existence of well-defined game theory solution: (K^*, d^*) is a well-defined game theoretic saddle point corresponding to a zero-sum differential game if and only if L is such that

$$M \equiv L^T(K - K^*)C + C^T(K - K^*)^T L + C^T(K - K^*)^T R(K - K^*)C \geq 0 \quad (27)$$

when $K \neq K^*$.

Note that this is always true if $L = 0$.

Proof: Equation (12) becomes

$$\begin{aligned} H(x, V_x, K, d) &= H(x, V_x, K^*, d^*) + x^T [L + R(K - K^*)C]^T \\ &\quad \times R^{-1} [L + R(K - K^*)C] x - x^T [L^T R^{-1} L] x - \gamma^2 \|d - d^*\|^2 \\ H(x, V_x, K, d) &= H(x, V_x, K^*, d^*) + x^T [L + R(K - K^*)C]^T \\ &\quad \times R^{-1} [L + R(K - K^*)C] x - x^T [L^T R^{-1} L] x - \gamma^2 \|d - d^*\|^2 \\ &= H(x, V_x, K^*, d^*) + x^T L^T R^{-1} L x + x^T L^T (K - K^*) C x \\ &\quad + x^T [R(K - K^*)C]^T R^{-1} L x + x^T [R(K - K^*)C]^T \\ &\quad \times R^{-1} [R(K - K^*)C] x - x^T [L^T R^{-1} L] x - \gamma^2 \|d - d^*\|^2 \\ H(x, V_x, K, d) &= H(x, V_x, K^*, d^*) + x^T M x - \gamma^2 \|d - d^*\|^2 \quad (28) \end{aligned}$$

under the condition defined by Eq. (27), one has

$$H(x, V_x, K^*, d) \leq H(x, V_x, K^*, d^*) \leq H(x, V_x, K, d^*) \quad (29)$$

or

$$\frac{\partial^2 H}{\partial u^2} > 0, \quad \frac{\partial^2 H}{\partial d^2} < 0 \quad (30)$$

at (K^*, d^*) . It is known that a saddle point at the Hamiltonian implies a saddle point at the value function J when considering finite-horizon zero-sum games. For the infinite horizon case, the same strategies remain in saddle point equilibrium when sought among the class of stabilizing nonanticipative strategies.²⁰ Therefore, this implies that

$$\frac{\partial^2 J}{\partial u^2} > 0, \quad \frac{\partial^2 J}{\partial d^2} < 0 \quad (31)$$

which guarantees a game theoretic saddle point. ■

Remarks:

1) To complete the discussion in the remarks following lemma 2, note that theorem 2 allows the interpretation of K^* defined by Eq. (7), when $L = 0$, as a minimizing control in a game theoretic sense. It is important to understand that introducing L in theorem 1 provides the extra design freedom needed to provide *necessary* and *sufficient* conditions for the existence of the H - ∞ OPFB solution.

2) If $L \neq 0$, then there can exist a saddle point in some cases. However counterexamples are easy to find.

IV. Proposed Solution Algorithm

Most existing iterative algorithms for OPFB design require the determination of an initial stabilizing gain, which can be very difficult for practical aerospace systems. The following algorithm is proposed to solve the two coupled design equations in theorem 1. Note that *it does not require an initial stabilizing gain* because, in contrast to Kleinman's algorithm²¹ and the algorithm of Moerder and Calise,⁵ it uses a Riccati equation solution, not a Lyapunov equation, at each step.

1) Initialize:

Set $n = 0$, $L_0 = 0$, and select γ , Q , and R .

2) n th iteration:

Solve for P_n in

$$\begin{aligned} P_n A + A^T P_n + Q + (1/\gamma^2) P_n D D^T P_n - P_n B R^{-1} B^T P_n \\ + L_n^T R^{-1} L_n = 0 \end{aligned} \quad (32)$$

Evaluate gain and update L

$$K_{n+1} = R^{-1} (B^T P_n + L_n) C^T (C C^T)^{-1} \quad (33)$$

$$L_{n+1} = R K_{n+1} C - B^T P_n \quad (34)$$

If K_{n+1} and K_n are close enough to each other, go to 3, otherwise, set $n = n + 1$ and go to 2.

3) Terminate:

Set $K = K_{n+1}$. ■

Note that this algorithm uses well-developed techniques for solving Riccati equations available, for instance, in MATLAB. It generalizes the algorithm in Ref. 8 to the case of nonzero initial gain.

Lemma 3: If this algorithm converges, it provides the solution to Eqs. (17) and (18).

Proof: Clearly at convergence Eq. (18) holds for P_n . Note that substitution of Eq. (33) into Eq. (34) yields

$$L_{n+1} = R [R^{-1} (B^T P_n + L_n) C^T (C C^T)^{-1}] C - B^T P_n$$

Defining $C^+ = C^T (C C^T)^{-1}$ as the right inverse of C , one has

$$L_{n+1} = (B^T P_n + L_n) C^+ C - B^T P_n$$

$$L_{n+1} = -B^T P_n (I - C^+ C) + L_n C^+ C$$

At convergence $L_{n+1} = L_n \equiv L$, $P_n \equiv P$ so that

$$0 = L (I - C^+ C) + B^T P (I - C^+ C)$$

$$0 = (B^T P + L) (I - C^+ C)$$

$$B^T P + L = (B^T P + L) C^+ C \quad (35)$$

This guarantees that there exists a solution K^* to Eq. (17) given by $K = R^{-1} (B^T P + L) C^+$.

This algorithm has been applied to several aircraft design examples of reasonable complexity (e.g., the F-16 lateral regulator in example 8.1-1 of Ref. 4). It has excellent performance and converges quickly.

V. F-16 Normal Acceleration Regulator Design

In aircraft control design, it is very important to design feedback control regulators of prescribed structure for both stability augmentation systems and control augmentation systems (CAS).¹ Therefore, static OPFB design is required. This example shows the power of the proposed static H_∞ OPFB design technique because it is easy to include model dynamics, sensor processing dynamics, and actuator dynamics, but no additional dynamics (e.g., regulator) are needed.

The OPFB design algorithm presented is applied to the problem of designing an output-feedback normal acceleration regulator for the F-16 aircraft Ref. 1 (Sec. 5.4). The control system is shown in Fig. 2, where n_z is the normal acceleration, r is the reference input in g , and the control input $u(t)$ is the elevator actuator angle. To ensure zero steady-state error, an integrator has been added in the feed-forward path; this corresponds to the compensator dynamics. The integrator output is ε . The short-period approximation is used so that the aircraft states are pitch rate q and angle of attack α . Because alpha measurements are quite noisy, a low-pass filter with the cutoff frequency of 10 rad/s is used to provide filtered measurements α_F of the angle of attack. An additional state δ_ε is introduced by the elevator actuator.

The state vector is as follows: $x(1) = \alpha$: angle of attack; $x(2) = q$: pitch rate; $x(3) = \delta_\varepsilon$: elevator actuator; $x(4) = \alpha_F$: filtered measurement of angle of attack; and $x(5) = \varepsilon$: integral controller.

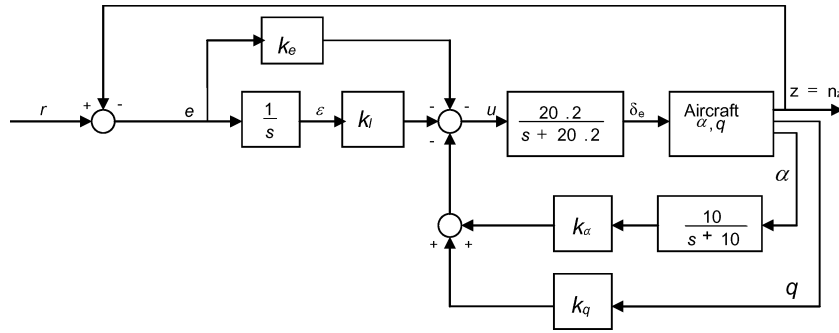


Fig. 2 G command system.

The measurement outputs are $y = [\alpha_F \ q \ e \ \varepsilon]^T$.

We use the short-period approximation to the F-16 dynamics linearized about the nominal flight condition described in Ref. 1, Table 3.6-3 (502 ft/s, level flight, dynamic pressure of 300 psf, $x_{cg} = 0.35\bar{c}$), and the dynamics are augmented to include the elevator actuator, angle-of-attack filter, and compensator dynamics. The result is

$$\dot{x} = Ax + Bu + Dd, \quad y = Cx \quad (36)$$

with

$$A = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 & 0 & 0 \\ 0.82225 & -1.07741 & -0.17555 & 0 & 0 \\ 0 & 0 & -20.2 & 0 & 0 \\ 10 & 0 & 0 & -10 & 0 \\ -16.26 & -0.9788 & 0.4852 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 20.2 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 57.2958 & 0 \\ 0 & 57.2958 & 0 & 0 & 0 \\ -16.26 & -0.9788 & 0.4852 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The factor of 57.2958 is added to convert angles from radians to degrees.

The control input is $u = -Ky = -[k_\alpha \ k_q \ k_e \ k_l]y$. It is required to select the output-feedback gains to yield stability with good closed-loop response. Note that k_α and k_q are feedback gains, while k_e and k_l are feed-forward gains. This approach allows the adjustment of both for the best bounded L_2 gain performance. The algorithm just presented was used to design an H_∞ pitch-rate regulator for a prescribed value of γ .

For the computation of the output feedback gain K , it is necessary to select Q and R . Using the algorithm just described for the given γ , Q , and R , the control gain K is easily found using MATLAB in a few seconds. If this gain is not suitable in terms of time responses and closed-loop poles, the elements of Q and R can be changed and the design repeated. After repeating the design several times, we selected the design matrices as

$$Q = \begin{bmatrix} 264 & 16 & 1 & 0 & 0 \\ 16 & 60 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{bmatrix}, \quad R = [0.1]$$

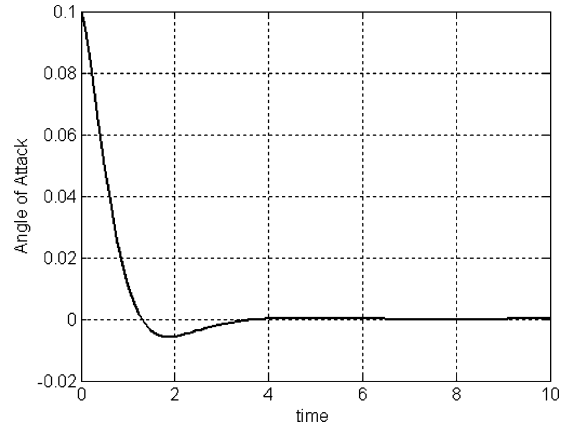


Fig. 3 Angle of attack.

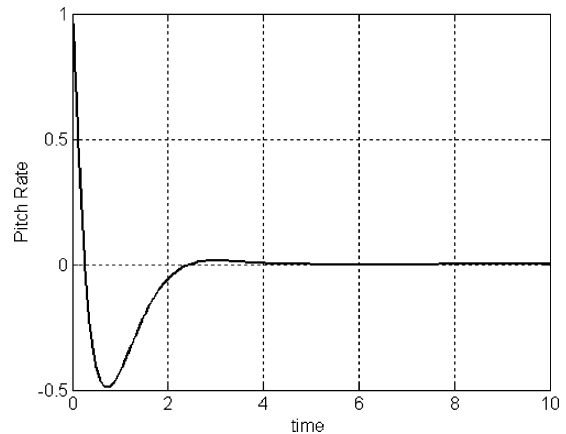


Fig. 4 Pitch rate.

which yields the feedback matrix

$$K = [0 \quad -0.1778 \quad 12.4336 \quad 31.7201]$$

The resulting closed-loop poles are at

$$s = -28.3061, -1.4974 \pm 1.2148i, -3.1809, -10$$

The resulting gains are applied to the system, and a unit step disturbance $d(t)$ is introduced in simulations to verify robustness of the design. The resulting time responses shown in Figs. 3 and 4 are very good. Note that, though we designed an H_∞ regulator, the structure of the static OPFB controller with the prescribed loops also guarantees good tracking.

The gain parameter γ defines the L_2 bound for a given disturbance. One can quickly perform the design using the preceding algorithm for a prescribed value of γ in a few seconds using MATLAB. If the algorithm converges, the parameter γ can be reduced. If γ is

taken too small, the algorithm will not converge because the ARE has no positive definite solution. This provides an efficient and fast trial-and-error method for determining the smallest allowable γ , for given Q and R design matrices, which solves the H_∞ problem. For this example, the H_∞ value of γ is found to be equal to 0.2, for which the preceding results were obtained.

VI. Conclusions

The problem of disturbance attenuation with stability using static output feedback for linear time-invariant systems has been studied. Necessary and sufficient conditions were developed, which yield two coupled matrix design equations to be solved for the OPFB gain. A computational algorithm to solve for the output-feedback gain that achieves a prespecified disturbance attenuation was developed. The algorithm requires no initial stabilizing gain, in contrast to other existing recursive OPFB solution algorithms. This procedure allows output-feedback control design with prespecified controller structures and guaranteed performance. A G-command control augmentation system CAS for the F-16 aircraft was designed using this algorithm.

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