A New Network Reduction Methodology for Congestion Study

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Abstract – **System planning on a large-scale electric power system is computationally challenging. Network reduction into a small system can significantly reduce the computational expense. The Ward equivalent technique is widely used for the reduction; however, it may not yield the same flow pattern as the original network. In this paper, a new methodology for network reduction is proposed and the results are compared with those from other methodologies.**

Index Terms – **Eigenvalue, eigenvector, DC power flow, factorization, transmission network reduction, power transfer distribution factor (PTDF), Ward reduction.**

I. NOMENCLATURE

B_{branch}	Branch impedance matrix
B_{bus}	Bus impedance matrix
C	Node-branch incidence matrix with cardinality of L-by-N
Fk	k^{th} flow group
Gk	k^{th} injection group
H	PTDF matrix with cardinality of L-by-N
I_w	Identity matrix with cardinality of w-by-w
L	Number of lines in the original network
Le	Number of inter-group lines in the original network
Li	Number of intra-group lines in the original network
N	Number of buses in the original network
P_f	Permutation matrix arranging flows (i.e., intra-group flow on top and inter-group flow on bottom of the flow vector)
P_{flow}	L-by-1 power flow vector in the original network
P_g	Permutation matrix ordering injection according to injection group
P_{inj}	N-by-1 net injection vector in the original network; generation – load
P_{ref}	N-by-N column permutation matrix placing slack bus on top
P_T	n-by-n column permutation matrix placing slack bus on top
diag(x)	Diagonal matrix with the diagonal elements of x
e_j	j^{th} unit vector
f	Power flow on the reduced network
g	Power injection vector in the reduced network
ℓ	Number of lines in the reduced network
m_K	Number of elements in a set K
n	Number of buses in the reduced network
x	L-by-1 reactance vector
y	L-by-1 inverse reactance vector
θ	N-by-1 voltage angle vector
Θ_{flow}	ℓ-by-Le matrix to sum flows
$\Theta_{injection}$	n-by-N matrix to sum bus injections

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II. INTRODUCTION

With the growing concern regarding climate change, the integration of renewable electric technologies into the transmission network has become increasingly important over the past decade. Recently, an efficient expansion-planning algorithm was developed to optimize both the transmission network and generation [1-5]. However, the power system planning for the integration on a large-scale power system is computationally challenging. By using small, equivalent networks, the computational requirements can be significantly reduced.

Network reduction is usually performed by computing impedances and by eliminating unnecessary elements [6-11]. This reduction usually results in a highly dense impedance matrix; therefore, using the reduced network may not significantly increase efficiency. Equivalent networks have been used for short circuit studies because they can reproduce the same voltages and currents of the remaining buses as the original systems do. However, the flows of the eliminated branches cannot be approximated in the reduced networks. Therefore, the usage of the reduced networks is limited in the power flow analysis (i.e., the flows using the reduced networks are significantly different from the flows from the original networks).

The reduced networks of the conventional equivalent techniques are dependent on the operation set point; therefore, the reduced networks at different operation set points on an identical network may be significantly different. Generation expansion planning is a process to find an optimal configuration of generation at various load profiles. Because the network is an input of the planning, it is necessary to have an interpretation of the network independent of the set points.

For planning, a highly nonlinear, full AC power flow would be the most accurate interpretation. Due to the high computational demands of AC, DC power flow is widely used because it is a linear approximation and captures most features of the AC power flow model. Because of its linearity, the sensitivity of power flows to the power injection (power transfer distribution factor, PTDF) does not depend on operation set points. There was an attempt to reduce network using zonal power exchange [11]. However, the method has the operation set point dependence and yields significant error in the flow profile for a different set point, even with the same congestion profile as the original case used for the reduction.

In this paper, a new method to reduce network is proposed, and the result is compared with those from the conventional methods. Furthermore, the new method is applied to the Western Electricity Coordinating Council (WECC) system for power flow study.

III. STRUCTURAL CHARACTERISTICS OF PTDF

H, the PTDF matrix with cardinality of L-by-N, is very useful in power flow studies because it relates power flows to the power injections that are control variables of optimal power flow. In a DC formulation, power injection and power flow are linearly related to the voltage angle θ:

$$\begin{aligned} P_{\text{inj}} &= B_{\text{bus}}\,\theta \\ P_{\text{flow}} &= B_{\text{branch}}\,\theta \end{aligned} \tag{1}$$

Since B_{bus} is a rank-deficient matrix, (1) can be solved for power flow only after selecting the reference bus for voltage angle and power injection.

Make P_{ref} a permutation matrix to re-organize P_{inj} so that the injection at the reference bus and those at the non-reference buses are located at the top and on the bottom of the injection vector, respectively. Using the first equation in (1), one can relate the voltage angle at the non-reference buses with that in the reference bus and the power injection at the non-reference buses. For example,

$$\begin{aligned} & P_{\text{ref}}\,P_{\text{inj}} = \begin{bmatrix} P_{\text{inj}}^{\text{ref}} \\ P_{\text{inj}}^{\text{non-ref}} \end{bmatrix} = P_{\text{ref}}\,B_{\text{bus}}\,P_{\text{ref}}^{T}\,P_{\text{ref}}\,\theta = P_{\text{ref}}\,B_{\text{bus}}\,P_{\text{ref}}^{T} \begin{bmatrix} \theta_{\text{inj}}^{\text{ref}} \\ \theta_{\text{inj}}^{\text{non-ref}} \end{bmatrix} \\ & \rightarrow P_{\text{inj}}^{\text{non-ref}} = \begin{pmatrix} 0 & I_{N-1} \end{pmatrix} P_{\text{ref}}\,P_{\text{inj}} \\ & \qquad = \begin{pmatrix} 0 & I_{N-1} \end{pmatrix} \begin{bmatrix} PBP_{11}^{1\times 1} & PBP_{12}^{1\times(N-1)} \\ PBP_{21}^{(N-1)\times 1} & PBP_{22}^{(N-1)\times(N-1)} \end{bmatrix} \begin{pmatrix} \theta_{\text{inj}}^{\text{ref}} \\ \theta_{\text{inj}}^{\text{non-ref}} \end{pmatrix} \\ & \qquad = PBP_{21}\,\theta_{\text{inj}}^{\text{ref}} + PBP_{22}\,\theta_{\text{inj}}^{\text{non-ref}} \\ & \rightarrow \theta_{\text{inj}}^{\text{non-ref}} = \left[PBP_{22}\right]^{-1} \left(P_{\text{inj}}^{\text{non-ref}} - PBP_{21}\,\theta_{\text{inj}}^{\text{ref}}\right) \\ & \textit{where } PBP = P_{\text{ref}}\,B_{\text{bus}}\,P_{\text{ref}}^{T}, \text{ and } P_{\text{ref}}^{T}\,P_{\text{ref}} = I_N \end{aligned} \tag{2}$$

Similarly, power flow can be expressed as:

$$\begin{aligned} P_{\text{flow}} &= B_{\text{branch}}\,\theta = B_{\text{branch}}\,P_{\text{ref}}^{T}\,P_{\text{ref}}\,\theta = B_{\text{branch}}\,P_{\text{ref}}^{T} \begin{bmatrix} \theta_{\text{inj}}^{\text{ref}} \\ \theta_{\text{inj}}^{\text{non-ref}} \end{bmatrix} \\ &= \begin{pmatrix} BP_{\text{branch}}^{\text{ref}} & BP_{\text{branch}}^{\text{non-ref}} \end{pmatrix} \begin{bmatrix} \theta_{\text{inj}}^{\text{ref}} \\ \theta_{\text{inj}}^{\text{non-ref}} \end{bmatrix} \\ &= BP_{\text{branch}}^{\text{ref}}\,\theta_{\text{inj}}^{\text{ref}} + BP_{\text{branch}}^{\text{non-ref}}\,\theta_{\text{inj}}^{\text{non-ref}} \\ &= BP_{\text{branch}}^{\text{ref}}\,\theta_{\text{inj}}^{\text{ref}} + BP_{\text{branch}}^{\text{non-ref}} \left[PBP_{22}\right]^{-1} \left(P_{\text{inj}}^{\text{non-ref}} - PBP_{21}\,\theta_{\text{inj}}^{\text{ref}}\right) \\ &= BP_{\text{branch}}^{\text{non-ref}} \left[PBP_{22}\right]^{-1} P_{\text{inj}}^{\text{non-ref}} + BP\,\theta_{\text{inj}}^{\text{ref}} \\ & \textit{where } BP_{\text{branch}} = B_{\text{branch}}\,P_{\text{ref}}^{T} \\ & \qquad BP = BP_{\text{branch}}^{\text{ref}} - BP_{\text{branch}}^{\text{non-ref}} \left[PBP_{22}\right]^{-1} PBP_{21} \end{aligned} \tag{3}$$

By setting the reference angle to zero, (3) gives the linear relationship between power flow and power injection:

$$\begin{aligned} & P_{\text{flow}} = BP_{\text{branch}}^{\text{non-ref}} \left[PBP_{22}\right]^{-1} P_{\text{inj}}^{\text{non-ref}} \\ & \rightarrow P_{\text{flow}} = H'\,P_{\text{inj}}^{\text{non-ref}} = \begin{pmatrix} 0 & H' \end{pmatrix} \begin{pmatrix} P_{inj}^{ref} \\ P_{inj}^{non-ref} \end{pmatrix} = H\,P_{\text{inj}} \end{aligned} \tag{4}$$

BP and the PBP matrix can easily be derived by using the node-branch incidence matrix C and the reactance x:

$$\begin{aligned} & BP'_{\text{branch}} = diag\left(1/x\right) C' \\ & PBP_{22} = C'^{T}\ diag\left(1/x\right) C' \\ & \textit{where } diag\left(1/x\right) = \begin{bmatrix} 1/x_1 & 0 & \cdots & 0 \\ 0 & 1/x_2 & & \\ \vdots & & \ddots & \\ 0 & & & 1/x_L \end{bmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{pmatrix} \end{aligned} \tag{5}$$

where C' is the C matrix with the column corresponding to the eliminated reference bus. Therefore, H' can be expressed in terms of C' and x:

$$\begin{aligned} & H' = \left[diag\left(1/x\right) C'\right] \left\{ C'^{T}\ diag\left(1/x\right) C' \right\}^{-1} \\ & \rightarrow H'\,C'^{T}\ diag\left(1/x\right) C' = diag\left(1/x\right) C' \end{aligned} \tag{6}$$

Because both H' and C' are rank (N – 1) matrices, the product $H'C'^{T}$ also has the rank of N – 1. Therefore, L – (N – 1) eigenvalues of $H'C'^{T}$ are zeros, and the corresponding eigenvectors span the null space of $H'C'^{T}$. Equation (6) implies that N – 1 eigenvalues of $H'C'^{T}$ are unity, and that the corresponding eigenvectors are the column vectors of diag(1/x) C'. Therefore, diag(1/x) C' spans the real space of $H'C'^{T}$. Consequently, the eigenvalue decomposition of $H'C'^{T}$ yields either 1) zero eigenvalue of which corresponding eigenvectors span the null space of $H'C'^{T}$, or 2) unity eigenvalue and the corresponding eigenvectors span the real space of $H'C'^{T}$.

To evaluate x from the PTDF matrix, one needs to take the eigenvalue decomposition of $H'C'^{T}$, select eigenvectors of which eigenvalues are unity, and then assign the set of eigenvectors V'. The eigenvectors are not uniquely defined because the eigenvalues are unity; therefore, it is not easy to calculate x directly from the PTDF matrix. Indeed, any linear combination of the eigenvectors can span the real space of $H'C'^{T}$. The physical interpretation of this property in the power system follows: A linear network attached to the meshed system does not affect the PTDF matrix, and in that case, the column vector in the space of V' is a unit vector. Because V' spans the same space as $H'C'^{T}$, the multiplication of the unit vector with $H'C'^{T}$ yields the unit vector itself. Suppose there exists a unit vector e satisfying the condition stated above, and that the unit vector is added in V'—let the matrix be V. Then the rank of V' and that of V (= [V' e]) should be the same. In other words, e is a linear combination of the column vector in V'. Consequently, some columns of V' can be replaced by e—let the matrix be V". Note that V" is more sparse than V'.

QR-factorization of the V matrix yields the real and null spaces spanned by the matrix:

$$V^{L\times(N-1+ne)} = Q\,R = \begin{bmatrix} Q_1^{L\times(N-1)} & Q_2^{L\times(L-N+1)} \end{bmatrix} \begin{bmatrix} R_1^{(N-1)\times(N-1+ne)} \\ 0 \end{bmatrix} \tag{7}$$

where ne is the number of such e vectors. Because the unit vectors are a linear combination of other column vectors in V, the first N – 1 columns of R_1 must be the same as R-factor of QR-factorization of V'.

Note that QR-factorization of V yields sparser Q_2 than V' does. Therefore, it is computationally more efficient to proceed with V than with V'. Because the null space of V, which is same as the null space of diag(1/x) C', is perpendicular to the real space of V, then:

$$Q_2^T\ diag(1/x)\,C' = 0$$
$$OR\quad Q_2^T\ diag(1/x)\,c_j = 0\ where\ C' = \begin{bmatrix} c_1 & c_2 & \cdots & c_{N-1}\end{bmatrix} \tag{8}$$

Simple algebra gives:

$$Q_2^T\ diag(1/x)\,c_j = 0 \leftrightarrow Q_2^T\ diag(c_j)\left(1/x\right) = 0$$
$$\rightarrow \Omega\left(1/x\right)^{L\times 1} = 0 \tag{9}$$
$$where\ \Omega = \begin{bmatrix} Q_2^T diag(c_1) \\ Q_2^T diag(c_2) \\ \vdots \\ Q_2^T diag(c_{N-1})\end{bmatrix}^{[(L-N+1)L]\times L}$$

Note that a trivial solution exists (i.e., 1/x = 0); however, the desired solution is non-trivial to satisfy (9). Because the uniform increase in the value of x is canceled out in evaluating H' (see (6)), there is an infinite number of sets of x to choose from to satisfy (9). Therefore, (9) is modified to:

$$\min_x \left|\Omega\left(1/x\right)\right|_k,\quad s.t.\left|\frac{1}{x}\right|_k \geq M(>0) \tag{10}$$

where M stands for a small positive number, which is a lower bound of k-norm of 1/x vector. For convenience, 2-norm was used for the optimization problem. A LaGrange function can be formed for the optimization problem:

$$L = y^T\Omega^T\Omega y + \lambda\left(M - y^T y\right)$$
$$\text{where } y = [y_i],\ and\ y_i = 1/x_i \tag{11}$$

An optimality condition says:

$$\frac{\partial L}{\partial y} = 2\Omega^T\Omega y - 2\lambda y = 0 \rightarrow \left(\Omega^T\Omega\right) y = \lambda y \tag{12}$$

Note that the change in M by one unit does not affect H (see (6)), but it does affect $|\Omega\, y|$ by λ. Therefore, $|\Omega\, y|$ is minimized for a given value of M at a small value of λ. For example, y is the eigenvector corresponding to the least eigenvalue in the absolute value of $\Omega^T\Omega$.

After H(x) is evaluated using (6) with a given value of x, the relative errors are calculated:

$$\text{error} = \frac{\|H - H(x)\|_2}{\|H\|_2} \tag{13}$$

The errors are typically in the range of the numerical error, 10^{-11}.

IV. REDUCED PTDF MATRIX, H_r

An ideal reduced PTDF matrix for the power flow study finds a sensitivity matrix of reduced flow to the reduced power injection. For defining the reduced flow and injection, the groups of buses that are aggregated must be defined. Let the injection groups (G1, G2,…, Gn); injection (P_{Gi}); the intra-group flow ($P_{\text{flow}}^{\text{int}}$); and the inter-group flow ($P_{\text{flow}}^{\text{ext}}$) be defined between groups. Then:

$$P_{\text{flow}} = P_{\text{f}}^T\ P_{\text{f}}\ P_{\text{flow}} = P_{\text{f}}^T\begin{bmatrix} P_{\text{flow}}^{\text{int}} \\ P_{\text{flow}}^{\text{ext}}\end{bmatrix}$$
$$P_{\text{inj}} = P_{\text{g}}^T\ P_{\text{g}}\ P_{\text{inj}} = P_{\text{g}}^T\begin{bmatrix} P_{G1} \\ \vdots \\ P_{Gn}\end{bmatrix} = P_{\text{g}}^T\ P_{\text{injection}} \tag{14}$$

where P_f and P_g are permutation matrices ordering rows according to flow and injection groups, respectively, and $P_{\text{injection}}$ is the power injection vector rearranged according to group. Then reduced flow and power injection give:

$$f = \left[\textstyle\sum_{\substack{k\in Gi\\ l\in Gj}} P_{\text{flow}}^{k\to l}\right] = \Theta_{\text{flow}}\begin{bmatrix}0^{\text{Le}\times\text{Li}} & I_{\text{Le}}\end{bmatrix} P_{\text{sign}}\ P_{\text{flow}}^{\text{ext}}$$
$$= \Theta_{\text{flow}}\begin{bmatrix}0^{\text{Le}\times\text{Li}} & I_{\text{Le}}\end{bmatrix} P_{\text{sign}}\ P_{\text{f}}\ P_{\text{flow}}$$
$$g = [g_{Gi}] = \left[\textstyle\sum_{k\in Gi} P_{\text{inj}}^{k}\right] = \Theta_{\text{injection}}\ P_{\text{injection}} = \Theta_{\text{injection}}\ P_g\ P_{\text{inj}} \tag{15}$$
$$where \begin{cases} \Theta_{\text{flow}}^{l\times Le} = \begin{bmatrix} 1_{F1}^T & & 0 \\ & \ddots & \\ 0 & & 1_{Fl}^T\end{bmatrix},\ 1_{Fk} = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}^{m_{Fk}\times 1} \\ \Theta_{\text{injection}}^{n\times N} = \begin{bmatrix} 1_{G1}^T & & 0 \\ & \ddots & \\ 0 & & 1_{Gn}^T\end{bmatrix},\ 1_{Gk} = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}^{m_{Gk}\times 1}\end{cases}$$

where P_{sign} is a diagonal matrix where an element is 1 if the corresponding flow is in the same direction as the inter-group flow; otherwise, the element is -1.

A reduced PTDF H_r is, by definition, a sensitivity matrix of the reduced flow to the reduced power injection. Therefore:

$$f = \Theta_{\text{flow}}\begin{bmatrix}0 & I\end{bmatrix} P_{\text{sign}}\ P_{\text{f}}\ P_{\text{flow}} = \Theta_{\text{flow}}\begin{bmatrix}0 & I\end{bmatrix} P_{\text{sign}}\ P_{\text{f}}\ H\ P_{\text{inj}}$$
$$= H_r\ g = H_r\ \Theta_{\text{injection}}\ P_{\text{g}}\ P_{\text{inj}} \tag{16}$$

Trying to find H_r to satisfy (16) for any power injection P_{inj}, leads to:

$$\Theta_{\text{flow}}\begin{bmatrix}0 & I\end{bmatrix} P_{\text{sign}}\ P_{\text{f}}\ H = H_r\ \Theta_{\text{injection}}\ P_{\text{g}} \tag{17}$$

Equation (17) is an over determined problem, therefore finding the solution is an error minimization process. One finds the solution for H_r is:

$$H_{\text{r}}^{l\times n} = \Theta_{\text{flow}}^{l\times \text{Le}}\ H_{\text{R}}^{\text{Le}\times \text{N}}\ \Theta_{\text{injection}}^T\ W_{\text{injection}}^{n\times n}$$
$$\text{where } H_R = \begin{bmatrix}0^{\text{Le}\times\text{Li}} & I_{\text{Le}}\end{bmatrix} P_{\text{sign}}^{L\times L}\ P_{\text{f}}^{L\times L}\ H^{L\times N}\left(P_g^{N\times N}\right)^T$$
$$W_{\text{injection}} = \left(\Theta_{\text{injection}}^{n\times N}\ \Theta_{\text{injection}}^T\right)^{-1}$$
$$= \begin{bmatrix}\frac{1}{m_{G1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{m_{Gn}}\end{bmatrix}^{n\times n} \tag{18}$$

Note that H_R is the row and column rearranged PTDF matrix according to the flow and the injection groups, and all intra-group flows are deleted.

The transmission network topology is determined once the network reduction is performed. With a given topology, a reduced node-branch incidence matrix C_r can be constructed:

$$C_{\mathrm{r}}^{l\times n} = W_{\mathrm{flow}}^{l\times l}\,\Theta_{\mathrm{flow}}^{l\times \mathrm{Le}}\,C_R\,\Theta_{\mathrm{injection}}^{T}$$
$$\text{where } C_R = \left[0^{\mathrm{Le}\times\mathrm{Li}}\quad I_{\mathrm{Le}}\right] P_{\mathrm{sign}}^{L\times L}\,P_{\mathrm{f}}^{L\times L}\,C^{L\times N}\,P_{\mathrm{g}}^{T} \tag{19}$$
$$W_{\mathrm{flow}} = \begin{bmatrix} \frac{1}{m_{F1}} & & & 0 \\ & \frac{1}{m_{F2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{m_{Fl}} \end{bmatrix}^{l\times l}$$

For a large system, L and N are very large numbers. For example, the WECC system contains approximately 18,000 branches and 16,000 buses. The calculation of H' requires the inversion of the matrix in the curly bracket in (6). Therefore, it would be computationally demanding to evaluate H because the calculation involves the inversion of an N-by-N matrix, requiring time and space of an order of $\vartheta(N^3)$. Using (6), one finds:

$$\begin{aligned} H' &= \left[diag(1/x)\,C'\right]\left[C'^{T}\,diag(1/x)\,C'\right]^{-1} I_{N-1} \\ \rightarrow h_k &= H' e_k = \left[diag(1/x)\,C'\right]\left\{C'^{T}\,diag(1/x)\,C'\right\}^{-1} e_k \end{aligned} \tag{20}$$

By performing a sparse LU-factorization of the matrix in the curly bracket with the minimum degree orderings [12] and saving L, U, and P matrices, each column of H' matrix (h_k) is evaluated. Then, columns in H_R are calculated by ignoring all of the intra-group flows. After H_R is evaluated, Θ_{flow} is multiplied. The multiplication is a summation of flows in the same flow group. Because each group is known in advance, the multiplication of $\Theta_{\mathrm{injection}}^{T}\,W_{\mathrm{injection}}$ is equivalent to updating an appropriate column of H_r. In this way, the reduced PTDF matrix can be calculated without a large storage space capacity. Fig. 1 illustrates the procedure to calculate H_r:

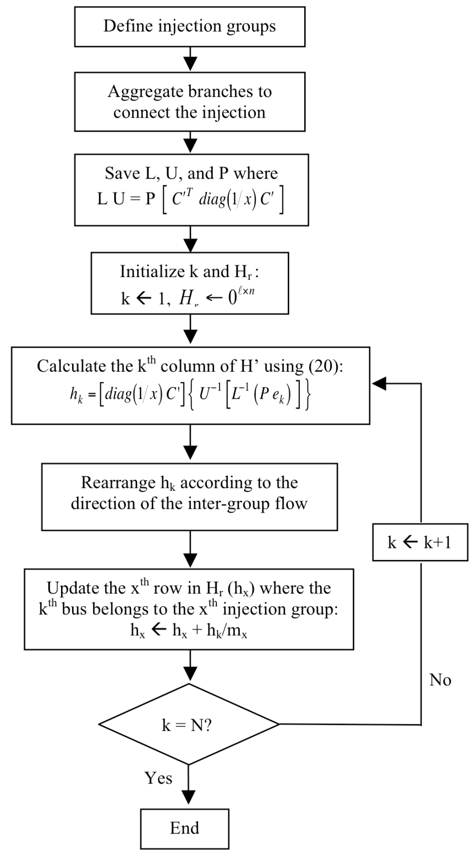

Fig. 1. Flow-chart to illustrate how to calculate H_r.

V. PROPERTIES OF H_r

For the reduced PTDF matrix, H_r would be a full-rank matrix, which does not have a slack bus. Consequently, H_r may not have the required structural properties discussed in IV. The reduced H_r and C_r are transformed to eliminate a column corresponding to a slack bus as follows:

$$C_T = C_r\,P_T \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{pmatrix} \rightarrow C'_T = C_r\,P_T \begin{pmatrix} 0 & \cdots & 0 \\ & & \\ & I & \\ & & \end{pmatrix}$$

$$\begin{aligned} H_T &= \left[H_r\,P_T - H_r\,P_T\,e_1\,(1 \ \cdots \ 1)\right] \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{pmatrix} \\ &= H_r\,P_T \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{pmatrix} \rightarrow H'_T = H_r\,P_T \begin{pmatrix} -1 & \cdots & -1 \\ & & \\ & I & \\ & & \end{pmatrix} \end{aligned} \tag{21}$$

The power balance equation implies that the product of C'^{T} and flow yields injection; therefore, multiplying flow with $H'C'^{T}$ results in the flow itself. Multiplying C'^{T} on both sides of (6) shows that $C'^{T}H'$ equals an identity matrix. Therefore, the multiplication of $H'C'^{T}$ and flow yields flow, and that of $C'^{T}H'$ and g is g. This also applies to the reduced

C and H. For example, the power balance equation for any injection g is written as:

$$\begin{pmatrix} -1^T g \\ g \end{pmatrix} = P_T^T C_r^T f = P_T^T C_r^T H_r P_T \begin{pmatrix} -1^T g \\ g \end{pmatrix}$$

$$\rightarrow P_T^T C_r^T H_r P_T = \begin{bmatrix} 0 & -1 & \cdots & -1 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{bmatrix} \tag{22}$$

Note that the power injected g is balanced by ejecting the same amount at the slack bus. Equations (21) and (22) yield:

$$\begin{aligned} C'^T_T H'_T &= \begin{pmatrix} 0 & \\ \vdots & I \\ 0 & \end{pmatrix} P_T^T C_r^T H_r P_T \begin{pmatrix} -1 & \cdots & -1 \\ & I & \end{pmatrix} \\ &= \begin{pmatrix} 0 & \\ \vdots & I \\ 0 & \end{pmatrix} \begin{bmatrix} 0 & -1 & \cdots & -1 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{bmatrix} \begin{pmatrix} -1 & \cdots & -1 \\ & I & \end{pmatrix} = I \end{aligned} \tag{23}$$

To see if the reduced PTDF matrix has the eigenvalue/eigenvector property as the original matrix, $H'_T C'^T_T$ is considered:

$$\begin{aligned} H'_T C'^T_T &= H_r P_T \begin{pmatrix} -1 & \cdots & -1 \\ & I & \end{pmatrix} \begin{pmatrix} 0 & \\ \vdots & I \\ 0 & \end{pmatrix} P_T^T C_r^T \\ &= H_r P_T P_T^T C_r^T = H_r C_r^T \\ &= \Theta_{\text{flow}} \begin{bmatrix} 0 & I \end{bmatrix} P_{\text{sign}} P_{\text{f}} H' \\ &\quad \times \left\{ P_{\text{g}}^T \Theta_{\text{injection}}^T W_{\text{injection}} \Theta_{\text{injection}} P_g \right\} \\ &\quad \times C'^T P_{\text{f}}^T P_{\text{sign}}^T \begin{bmatrix} 0 \\ I \end{bmatrix} \Theta_{\text{flow}}^T W_{\text{flow}} \end{aligned} \tag{24}$$

The quantity shown in the last row of (24) is the injection on the original network $P_{\text{injection}}$ that yields flow f at the reduced network. Multiplication with the second row rearranges the injection so that the power injection at any non-slack bus is compensated by the slack bus injection. The rearranged injection is multiplied with H' to give the feasible flow in the original network P_{flow}. Therefore, the multiplication of the rearranged injection with the first row yields a flow on the reduced network, equaling flow f. The resulting flow is in the space spanned by the feasible flow space, and the flow satisfies the nodal power balance equation. Therefore, $H_T C_T^T$ has the eigenvector and eigenvalue pair such that eigenvalues are unity. Note that the ranks of H_T and C_T^T are n – 1, as it is for $H_T C_T^T$. Therefore, $H_T C_T^T$ has n – 1 of eigenvectors of which the eigenvalue is unity. Because $H'C'^T$ has the eigenvalue property discussed in III and all other matrices in (24) only give a linear combination of the eigenvalue and eigenvectors, the property is preserved.

Because the size of H'_T and C'^T_T are ℓ–by-(n – 1), and the size of $H'_T C'^T_T$ is ℓ–by-ℓ, the eigenvalue decomposition of $H'_T C'^T_T$ gives (ℓ – n + 1) zero eigenvalue. For example,

$$H'_T C'^T_T = \begin{pmatrix} V & W \end{pmatrix} \begin{bmatrix} diag(\lambda) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} V & W \end{pmatrix}^{-1} \tag{25}$$

V and W span the real and the null space of $H'_T C'^T_T$, respectively. QR-factorization of W gives:

$$\begin{aligned} & W = \begin{pmatrix} Q_{WR} & Q_{WN} \end{pmatrix} \begin{pmatrix} R_W \\ 0 \end{pmatrix}, \\ & \text{and } Q_{WR}^T Q_{WR} = I \\ & H'_T C'^T_T Q_{WR} = 0 \leftrightarrow C'^T_T Q_{WR} = 0 \\ & Q_{WR}^T H'_T C'^T_T = 0 \leftrightarrow Q_{WR}^T H'_T = 0 \end{aligned} \tag{26}$$

Note that Q_{WR} span the null space of $H'_T C'^T_T$, and H'_T and C'^T_T are not zero matrices. Equations (23) and (26) yield:

$$\begin{aligned} & H'_T C'^T_T = \begin{pmatrix} H'_T & Q_{WR} \end{pmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} C'^T_T \\ Q_{WR}^T \end{pmatrix} \\ & \rightarrow H'_T C'^T_T \begin{pmatrix} H'_T & Q_{WR} \end{pmatrix} \\ & \quad = \begin{pmatrix} H'_T & Q_{WR} \end{pmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} C'^T_T \\ Q_{WR}^T \end{pmatrix} \begin{pmatrix} H'_T & Q_{WR} \end{pmatrix} \\ & \quad = \begin{pmatrix} H'_T & Q_{WR} \end{pmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{27}$$

$H'_T C'^T_T$ has the eigenvector and eigenvalue pair such that eigenvalue is either unity or zero. Because the non-zero eigenvalue is unity, any linear combination of the column vectors of H'_T can be the eigenvectors of $H'_T C'^T_T$. Note that the ranks of H'_T and C'^T_T are n – 1, as is $H'_T C'^T_T$. Therefore, $H'_T C'^T_T$ has n – 1 of eigenvectors of which eigenvalues are unity. That is:

$$H'_T C'^T_T F = F^{l \times (n-1)} I_{n-1}, \quad where \;\; F = [f] \tag{28}$$

Therefore, it is possible to evaluate the reactance of the lines in the reduced network from the procedure as described in III.

VI. NUMERICAL EXAMPLES

A. Simple Illustrative Example

To illustrate how this method works, the six-bus example in [11] is used. For the system, the reactance values are all 0.1j. The original and the reduced network are illustrated in Fig. 2.

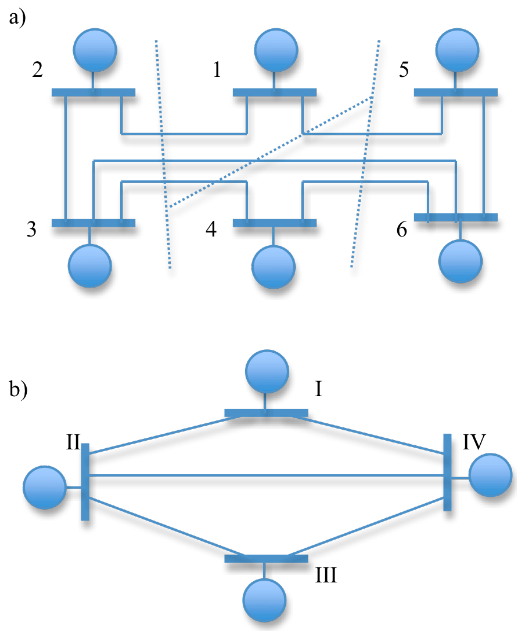

Fig. 2. a) Original 6-bus system; b) resulting system after the network reduction where {1} = I, {2, 3} = II, {4} = III, and {5, 6} = IV. Note that the dotted lines in a) show the group boundaries among I – IV.

The PTDF matrix of the original network can be evaluated using (6):

$$H' = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1\to2 \\ 1\to5 \\ 2\to3 \\ 3\to4 \\ 3\to6 \\ 4\to6 \\ 5\to6 \end{array} & \left[\begin{array}{cccccc} 0 & -0.786 & -0.571 & -0.500 & -0.214 & -0.429 \\ 0 & -0.214 & -0.429 & -0.500 & -0.786 & -0.571 \\ 0 & 0.214 & -0.571 & -0.500 & -0.214 & -0.429 \\ 0 & 0.071 & 0.143 & -0.500 & -0.071 & -0.143 \\ 0 & 0.143 & 0.286 & 0 & -0.143 & -0.286 \\ 0 & 0.071 & 0.143 & 0.500 & -0.071 & -0.143 \\ 0 & -0.214 & -0.429 & -0.500 & 0.214 & -0.571 \end{array}\right] \end{array} \quad (29)$$

As mentioned in the introduction, the network reductions in [6] and [11] are obtained based on the operation set point. Therefore, the PTDF on the reduced network depends on the dispatch. For example, a significantly different reduced PTDF might be obtained if the set point is different. From the reactance values and (6), it is possible to evaluate H:

$$H'_{[6]} = \begin{array}{cc} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{array}{c} I\to II \\ I\to IV \\ II\to III \\ II\to IV \\ III\to IV \end{array} & \left[\begin{array}{cccc} 0 & 0.952 & -0.333 & -0.191 \\ 0 & 0.119 & -0.167 & -0.238 \\ 0 & 0.786 & 0.500 & 0.429 \\ 0 & 0.095 & 0.667 & -0.191 \\ 0 & 0.214 & 0.500 & 0.571 \end{array}\right] \end{array} \quad (30)$$

$$H'_{[11]} = \begin{array}{cc} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{array}{c} I\to II \\ I\to IV \\ II\to III \\ II\to IV \\ III\to IV \end{array} & \left[\begin{array}{cccc} 0 & 0.0506 & -0.527 & -0.156 \\ 0 & 0.111 & -0.130 & -0.342 \\ 0 & 0.839 & 0.657 & 0.498 \\ 0 & 0.051 & 0.473 & -0.156 \\ 0 & 0.161 & 0.343 & 0.502 \end{array}\right] \end{array} \quad (31)$$

$$H'_{r} = \begin{array}{cc} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{array}{c} I\to II \\ I\to IV \\ II\to III \\ II\to IV \\ III\to IV \end{array} & \left[\begin{array}{cccc} 0 & -0.679 & -0.500 & -0.321 \\ 0 & -0.321 & -0.500 & -0.679 \\ 0 & 0.107 & -0.500 & -0.107 \\ 0 & 0.214 & 0 & -0.214 \\ 0 & 0.107 & 0.500 & -0.107 \end{array}\right] \end{array} \quad (32)$$

Equations (30) and (31) show clearly that the elements in both H'$_{[6]}$ and H'$_{[11]}$ have the same sign even though the values are different. However, the reduced PTDF shown in (32) indicates significantly different values and sign patterns. The set point used for reduction is not listed in [11], so it is not possible to compare the results at the same set point. To compare the results, the flow on the original network is calculated at the injection of [-5, 1, 1, 1, 1, 1]:

$$\mathrm{flow}\left(\begin{bmatrix} \mathrm{flow}_{1\to2} \\ \mathrm{flow}_{1\to5} \\ \mathrm{flow}_{2\to3} \\ \mathrm{flow}_{3\to4} \\ \mathrm{flow}_{3\to6} \\ \mathrm{flow}_{4\to6} \\ \mathrm{flow}_{5\to6} \end{bmatrix}\right) = H \begin{pmatrix} -5 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -2.5 \\ -2.5 \\ -1.5 \\ -0.5 \\ 0 \\ 0.5 \\ -1.5 \end{bmatrix} \quad (33)$$

The flow in (33) shows that power flows among groups I – IV are [-2.5, -2.5, -0.5, 0, 0.5], and it is clear that the power injection at each group is [-5; 2; 1; 2]. The flows are calculated using the PTDF matrices given in (30), (31), and (32), and the values for the flows are listed in Table I. To quantify the accuracy, the following performance index is introduced, and listed in Table I:

$$\mathrm{error} = \frac{\left|\mathrm{flow}_{\mathrm{original}} - \mathrm{flow}_{\mathrm{reduced}}\right|_2}{\left|\mathrm{flow}_{\mathrm{original}}\right|_2} \quad (34)$$

where original and reduced in subscript represent the original network and reduced network, respectively.

TABLE I
FLOWS CALCULATED BASED ON THE PTDF MATRICES IN (29) AND (30) USING THE INJECTION OF [-5; 2; 1; 2].

Flows	Actual Flow	Ref. [6]	Ref. [11]	This Study
I → II	-2.5	-0.524	-0.738	-2.5
I → IV	-2.5	-0.405	-0.593	-2.5
II → III	-0.5	2.929	3.331	-0.5
II → IV	0	0.476	0.262	0
III→ IV	0.5	2.071	1.669	0.5
error	-	132 %	133 %	0 %

It is evident that the errors in the flow are significant and the flow directions are wrong for some lines when the methods in [6] and [11] are used. Note that both the case in [11] and that from this example are non-congested case (i.e., the identical congestion pattern). Clearly, the method suggested here is more suitable for the power flow study.

B. WECC system

WECC is comprised of approximately 15,000 buses and 18,000 branches. Performing a DC OPF on the system took about 15 minutes using MATPOWER [13]. Finding an optimal expansion plan for the system described in [3] was not feasible due to the dimension of the problem. Therefore, it is necessary to reduce the system tradeoff between accuracy and computational efficiency.

The flow chart in Fig. 3 illustrates how the network reduction was performed, and Fig. 4 shows the result of the reduced network.

A 176-bus simplified equivalent of the WECC system [14] was used to choose the initial set of buses. In the aggregation process, some buses can be added to multiple nuclei. Equation (18) provides a heuristic approach to this problem (i.e., the error between both sides is kept small). If buses can be added to multiple nuclei, a nucleus was selected to minimize an error defined in (34). Create a new nucleus if doing so yields a significant decrease in error. Using the criterion, the flow error remains at an acceptable level. In this study, a 10% decrease in error was the criterion for the significant reduction in error. As a result, three additional buses were added to the original 176-bus system.

The state boundary condition is not necessary for the power system analysis, and might result in a large error. However, the condition was applied because it was useful for studying the state renewable portfolio. Due to the criterion, Bus 180 was added. As a result, the WECC system was reduced to 180 buses and 414 branches.

The "Ward" reduction was performed using the PowerWorld software. Because the reduction was manually performed, it is not possible to provide the execution time. However, it is worth mentioning that reducing 15,000 buses to 180 buses using the software is not the normal process, and multiple software crashes were observed. The result may also have the path dependency. The method in [11] was robust, but one must re-evaluate Ψ and F matrices if the operation set point changes. The most computationally expensive part is calculating the Ψ matrix, which takes about 8 hours using a 2.53 GHz computer. Constructing the F matrix also takes significant time and space. It is important to note that cardinality of F is ($\ell \times n$)-by-n (i.e., 74,520-by-180); therefore, constructing the F matrix takes significant time and space.

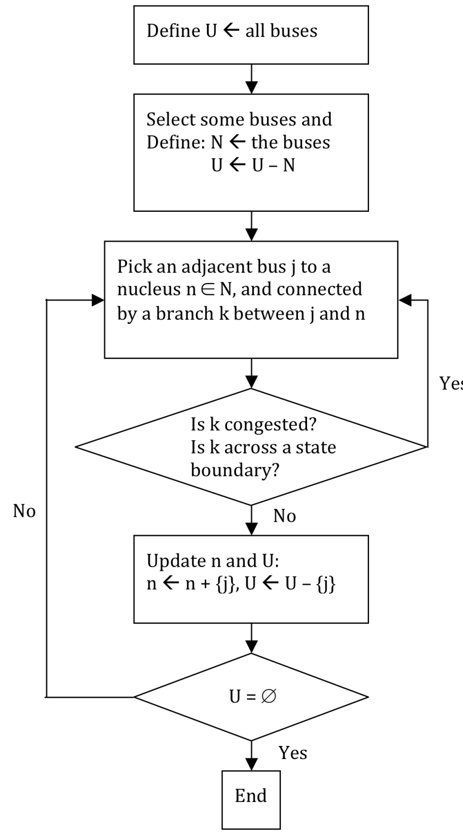

Fig. 3. Flow chart showing the bus aggregation procedure.

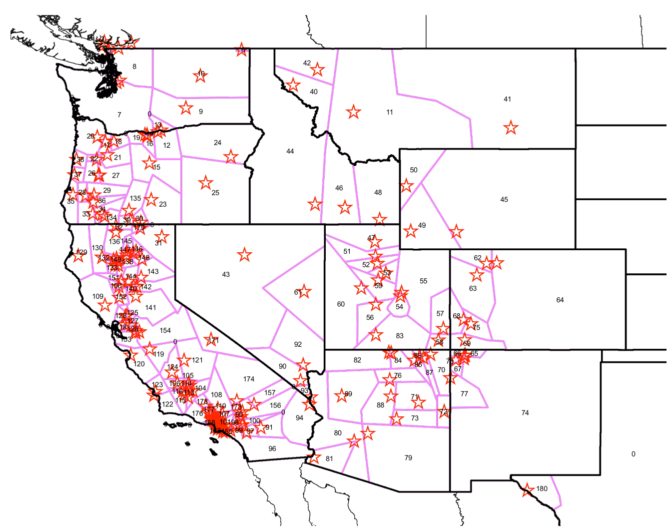

Fig. 4. An aggregate WECC 180-bus system.

The errors as defined in (34) were approximately 80% from [6] and 10% from [11] when the same set point was used as the selected load flow case. The errors were further increased to 180% and 150% in a low-load period, respectively. It is not surprising that the "Ward" reduction method yields a large error because the method is neither for a load flow study nor for such a significant size reduction (15,000 buses to 180 buses). The method in [11] yields a very small error at the same set point; however, a significant error is observed for a different condition because the method is based on the power flow of a specific case, which is affected by operation set point as well as the PTDF matrix. for a different set point.

The reduced WECC system was obtained within 8 hours using the method proposed in this study as described in Fig. 1. The error was at most 30% for both cases. Note that this method does not require a load flow case, and therefore, its result does not depend on the set point.

The method proposed in this study is based on the DC power flow approximation. Therefore, it does not address several issues such as voltage and reactive power. Consequently, it may perform poorly in cases where the DC model is not a good approximation to AC.

VII. CONCLUSIONS

Power systems are, in general, very large systems; therefore, precise power system optimization is practically infeasible. Several reduction methods were suggested, but their usage was limited due to the inaccuracy and the operation set point dependence. In this paper, a network reduction algorithm is proposed to reduce network using the PTDF matrix. The reduced PTDF matrix has the same structural properties as the original network's PTDF does. The method is tested on a simple system and performance compared with other methods observed in the references in terms of power flow. It yields more precise representation of the reduced network than the conventional methods. Another advantage is that the reduced network does not depend on the operation set point. As a result, it provides a concise and precise representation of the transmission network for a power flow study. Therefore, it can be used in a large system OPF, national corridor, and renewable portfolio studies, among other things.

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IX. BIOGRAPHY

HyungSeon Oh received his M.S. and Ph.D. in electrical and computer engineering from Cornell University in 2002 and 2005, respectively. He is an Engineer II at the National Renewable Energy Laboratory. His research interests include: power system planning, storage, artificial intelligence, energy system economics, computer simulation, and nonlinear dynamics.