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About two geometric inequalities

Michel Bataille

Abstract

Two geometric inequalities, Kooi's inequality and a companion, are surprisingly both derived from a single algebraic inequality. Comparison with inequalities of the same kind and applications are also offered.

1 Introduction

The main purpose of this note is to give a simple proof of the inequalities

$$\frac{2r(2R-r)(4R+r)}{R} \le s^2 \le \frac{R(4R+r)^2}{2(2R-r)}, \tag{1}$$

where s, r, and R are the semiperimeter, the inradius, and the circumradius of a triangle, respectiely.

The right inequality, presented without proof in section 5-7 of [2], is known as Kooi's inequality. Proofs of these inequalities have appeared recently: in [1], M. Lukarevski offers a short proof of Kooi's inequality, and the left inequality is the key to J. Zacharias's solution to a problem [4]. Each of these proofs rests upon a clever use of one of Blundon's strong (but complicated) geometric inequalities [2, p. 51]:

$$f(R,r) \le s^2 \le F(R,r),$$

where $f(R,r) = 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2rR}$ and $F(r,r) = 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2rR}$.

Here, in contrast, we deduce (1) from the following algebraic inequality:

If u,v,w are positive real numbers and

$$e_1 = u+v+w, \quad e_2 = uv+vw+wu, \quad e_3 = uvw,$$

then

$$4e_2(e_1e_2 - 3e_3) \leq e_1^2(e_1e_2 - e_3). \tag{2}$$

We first prove (1) assuming that (2) holds, and prove (2) in Section 3. In the final section, after some remarks about our inequalities, we present two situations where they prove useful.

2 Proof of the inequalities (1)

Let a,b,c be the sides of the triangle and A,B,C the angles opposite a,b,c, respectively. The only prerequisite of the proof is the following bunch of results about $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$:

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{4R+r}{s}, \quad \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r}{s}$$

and

$$\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1. \tag{3}$$

(These equalities are readily obtained using the basic formulas

$$\tan\frac{A}{2} = \frac{r}{s-a}, \quad \tan\frac{B}{2} = \frac{r}{s-b}, \quad \tan\frac{C}{2} = \frac{r}{s-c},$$

and $rs = \sqrt{s(s-a)(s-b)(s-c)}$ and $ab+bc+ca = s^2+r^2+4rR$.)

The proof consists in combining formulas (3) with inequality (2) in two ways.

First, we take $u = \tan\frac{A}{2}$, $v = \tan\frac{B}{2}$ and $w = \tan\frac{C}{2}$. Then,

$$e_1 = \frac{4R+r}{s}, \quad e_2 = 1, \quad e_3 = \frac{r}{s},$$

and (2) yields

$$4\left(\frac{4R+r}{s} - 3\cdot\frac{r}{s}\right) \leq \left(\frac{4R+r}{s}\right)^2 \left(\frac{4R+r}{s} - \frac{r}{s}\right).$$

Rearranging immediately leads to Kooi's inequality.

Second, we take $u = \tan\frac{B}{2}\tan\frac{C}{2}$, $v = \tan\frac{C}{2}\tan\frac{A}{2}$ and $w = \tan\frac{A}{2}\tan\frac{B}{2}$. This time, we have

$$e_1 = 1, \quad e_2 = \frac{r(r+4R)}{s^2}, \quad e_3 = \frac{r^2}{s^2},$$

and (2) gives

$$\frac{4r(r+4R)}{s^2} \cdot \left(\frac{r(r+4R)}{s^2} - \frac{3r^2}{s^2}\right) \leq \frac{r(r+4R)}{s^2} - \frac{r^2}{s^2}.$$

The left inequality of (1) follows.

3 Proof of inequality (2)

Due to homogeneity, we may suppose that $e_1 = u + v + w = 1$. Then, (2) can be written as

$$(4e_2 - 1)(e_2 - 3e_3) \leq 2e_3. \tag{4}$$

We observe that

$$e_2 - 3e_3 = e_2 e_1 - 3e_3 = uv(u+v) + vw(v+w) + wu(w+u) > 0,$$

hence (4) certainly holds if $4e_2 \leq 1$. From now on, we suppose that $4e_2 > 1$.

Because of the general inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ for $x, y, z \in \mathbb{R}$, we have $e_1^2 = u^2 + v^2 + w^2 + 2e_2 \geq 3e_2$, hence

$3e_2 \leq 1$. Following a known method (see [3], for example), we set $t = \sqrt{1 - 3e_2}$. Then, $0 \leq t < \frac{1}{2}$ and, since $e_2 = \frac{1-t^2}{3}$, (4) is transformed into

$$1 - 5t^2 + 4t^4 \leq 9e_3(3 - 4t^2). \tag{5}$$

Now, observing that the derivative $p'(x) = 3x^2 - 2x + e_2$ of the polynomial

$$p(x) = (x-u)(x-v)(x-w) = x^3 - x^2 + e_2 x - e_3$$

vanishes at $\frac{1-t}{3}$ and $\frac{1+t}{3}$, a quick study of the variations of $p(x)$ on the interval $(0, \infty)$ shows that we must have $p(\frac{1+t}{3}) \leq 0$. This provides the inequality $27e_3 \geq 1 - 3t^2 - 2t^3$.

Then, noticing that $3 - 4t^2 > 0$ (since $0 \leq t < \frac{1}{2}$), we see that (5) holds if $q(t) \geq 0$, where

$$q(t) = (3 - 4t^2)(1 - 3t^2 - 2t^3) - 3(1 - 5t^2 + 4t^4).$$

Since $q(t) = 2t^2(2t-1)^2(t+1) \geq 0$, the proof is complete.

4 Remarks and examples

4.1 Case of equality

Examining closely the above proof of (2), we see that equality holds if and only if $4e_2 > 1$ and $t = 0$. This occurs if and only if $e_2 = 1/3$, that is, $u = v = w$. It easily follows that either of the inequalities (1) is an equality if and only if the triangle is equilateral.

4.2 Comparison with other inequalities

Inequalities (1) are less strong than Blundon's inequalities (as pointed out in the introduction). However, they are stronger than the frequently used inequalities of Gerretsen [2, p. 50]:

$$r(16R - 5r) \leq s^2 \leq 4R^2 + 4rR + 3r^2. \tag{6}$$

For instance, a short calculation shows that

$$r(16R-5r) \le \frac{2r(2R-r)(4R+r)}{R}$$

rewrites as $r^2(R-2r) \ge 0$, which holds by Euler's well-known inequality $R \ge 2r$.

Similarly, the reader will prove that

$$\frac{R(4R+r)^2}{2(2R-r)} \le 4R^2+4rR+3r^2$$

holds.

Thus, inequalities (1) appear as being intermediate between Gerretsen's and Blundon's inequalities.

4.3 Applications

To see the inequalities (1) at work, we present two problems with solutions.

Problem 1. *Prove that*

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} \ge \frac{9}{s}.$$

Improve this inequality by showing that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} \ge \frac{2}{r}\sqrt{1-\frac{r}{2R}}. \tag{7}$$

Solution. First, from the harmonic mean-arithmetic mean inequality, we deduce that

$$\frac{1}{3}\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right) \ge \frac{3}{s-a+s-b+s-c} = \frac{3}{s}.$$

The first inequality follows.

Now, since

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}=\frac{ab+bc+ca-s^2}{(s-a)(s-b)(s-c)}=\frac{r+4R}{rs},$$

(7) is equivalent to $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$, Kooi's inequality, so that (7) holds.

Note that $\frac{2}{r}\sqrt{1-\frac{r}{2R}} \geq \frac{9}{s}$, which is equivalent to $s^2(4R-2r) \geq 81r^2R$, holds since, by (6) and $R \geq 2r$, we have

$$\begin{aligned} s^2(4R-2r)-81r^2R &\geq (16rR-5r^2)(4R-2r)-81r^2R \\ &= r(R-2r)(64R-5r) \geq 0. \end{aligned}$$ □

Problem 2. *Prove that*

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right) \leq \frac{1}{3}.$$

Improve this inequality by showing that

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right) \leq \frac{R}{2(2R-r)}. \quad (8)$$

Solution. First, let

$$\alpha=\tan\frac{B}{2}\tan\frac{C}{2}, \quad \beta=\tan\frac{C}{2}\tan\frac{A}{2}, \quad \gamma=\tan\frac{A}{2}\tan\frac{B}{2}.$$

Since $\alpha+\beta+\gamma=1$, we also have $3(\alpha\beta+\beta\gamma+\gamma\alpha) \leq 1$ (cf. Section 3). The inequality follows.

Next, using the results (3), we obtain that

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{r(4R+r)}{s^2}$$

and, therefore, (8) holds, as it can be written as $s^2 \geq \frac{2r(2R-r)(4R+r)}{R}$, the left inequality of (1).

From Euler's inequality $R \geq 2r$ we immediately deduce that $\frac{R}{2(2R-r)} \leq \frac{1}{3}$, hence (8) does improve the previous inequality. □

References

[1] Bottema, O. et al. *Geometric Inequalites*. Wolters-Noordhoff, 1968.

[2] Lukarevski, M. “A Simple Proof of Kooi’s Inequality”. *Mathematics Magazine* 93.3 (2020), p. 225.

[3] Pham Van Thuan and Le Vi. “A Useful Inequality Revisited”. *Crux Mathematicorum* 35.6 (2009), p. 164.

[4] Zacharias, J. “Solution to Problem 11686”. *The American Mathematical Monthly* 122.2 (2015), p. 176.

Michel Bataille
6 square des Boulots
76520 Franqueville-Saint-Pierre, France
michelbataille@wanadoo.fr

On a kind of Gaussian integral with the square on the left

S. Camosso

Abstract

In this short note we study the following integral:

$$\int_0^{+\infty} e^{-{}^2x}\,\mathrm{d}x,$$

involving the tetration function $f(x) = {}^2x = x^x$.

1 Introduction

Tetration is a mathematical operation defined for $x > 0$ as

$$^{\alpha}x = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}},$$

where $\alpha = 0, 1, 2, 3, \ldots$ is the level of the "power tower" (with $^0x = 1$). A nice article on the tetration operation is [1]. Infinite tetration $^{+\infty}x$ is treated in [5], [7] and [3]. The tetration is not an associative operation always evaluated from top to bottom. Tetration admits and inverse function know as the Lambert function W. This function satisfies the equation

$$W(x)e^{W(x)} = x.$$

The Lambert function can be used to solve various exponential and logarithmic equations of self-exponential nature. Details on the Lambert function and its relation with the tetration operation are in [2] and [4].

In this note the aim is the study of the integral

$$\int_0^{+\infty} e^{-{}^{\alpha}x}\,\mathrm{d}x, \tag{1}$$

with $\alpha = 2$. The integral can be decomposed into two integrals, one in the interval $[0,1]$ and the second in the interval $[1,+\infty)$. The first integral can be estimated using a result of [8]. The second integral can be estimated using inequalities.

2 Results

In this section we study the integral (1) for the case where $\alpha = 2$. In the first proposition presented here we give an explicit description in power series of the integral (1) for $x \in [0,1]$. The second and third propositions are, respectively, an upper and lower bound of the integral (1). The lower bound involves a special function $\mathrm{Ei}(\mathrm{x},\mathrm{y})$ called the " exponential integral function" defined in the last proposition.

Proposition 1. *The following holds:*

$$\int_0^1 e^{-x^x}\,\mathrm{d}x = 1 + \sum_{k=1}^{+\infty}\left\{\frac{(-1)^{k-1}}{k^k}\cdot\left[\sum_{j=1}^{+\infty}\frac{(-1)^j}{j!}j^{k-1}\right]\right\}.$$

Proof. The first step consists on considering the Taylor expansion of the integral

$$\int_0^1 e^{-x^x}\,\mathrm{d}x = \int_0^1\left(1 - x^x + \frac{x^{2x}}{2!} - \frac{x^{3x}}{3!} + \dots\right)\mathrm{d}x.$$

As a second step, we use the linearity of the integral

$$\int_0^1 e^{-x^x}\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x - \int_0^1 x^x\,\mathrm{d}x + \int_0^1 \frac{x^{2x}}{2!}\,\mathrm{d}x - \int_0^1 \frac{x^{3x}}{3!}\,\mathrm{d}x + \dots. \tag{2}$$

At this point we consider the remarkable formula of [8]:

$$\int_0^1 x^{c\cdot x^a}\,\mathrm{d}x = 1 - \frac{c}{(a+1)^2} + \frac{c^2}{(2a+1)^3} - \frac{c^3}{(3a+1)^4} + \ldots, \tag{3}$$

where a and c are constants.

In the last step, we apply (3) with $a = 1$ to each integral in (2):

$$\begin{aligned}&\int_0^1 e^{-x^x}\,\mathrm{d}x \\ &\quad = 1 - \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^k} + \frac{1}{2!}\sum_{k=1}^{+\infty}\frac{(-2)^{k-1}}{k^k} - \frac{1}{3!}\sum_{k=1}^{+\infty}\frac{(-3)^{k-1}}{k^k} + \cdots\end{aligned}$$

The result follows by recollecting the term $\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^k}$ after the first 1. □

Proposition 2. *The following inequality holds:*

$$\int_0^{+\infty} e^{-x^x}\,\mathrm{d}x \leq I_1 + e^{-1},$$

where $I_1 = \int_0^1 e^{-x^x}\,\mathrm{d}x$.

Proof. To prove the proposition we must estimate the integral $\int_1^{+\infty} e^{-x^x}\,\mathrm{d}x$. Observing that

$$e^{-x^x} \leq e^{-x}$$

for $x \geq 1$, then

$$\int_1^{+\infty} e^{-x^x}\,\mathrm{d}x \leq \int_1^{+\infty} e^{-x}\,\mathrm{d}x = \frac{1}{e}. \qquad \square$$

Proposition 3. *The following holds:*

$$\int_0^{+\infty} e^{-x^x}\,\mathrm{d}x \geq I_1 - \mathrm{Ei}(1, \mathrm{e}^{\mathrm{e}}) + \mathrm{Ei}(1, \mathrm{e}) + \mathrm{I}_2,$$

where

$$I_1 = \int_0^1 e^{-x^x}\,\mathrm{d}x, \qquad I_2 = \int_e^{+\infty} x^{-x^x}\,\mathrm{d}x$$

and

$$\mathrm{Ei}(\mathrm{x}, \mathrm{y}) = \int_1^{+\infty} \mathrm{e}^{-\mathrm{t}\mathrm{y}}\mathrm{t}^{-\mathrm{x}}\,\mathrm{d}\mathrm{t}.$$

Proof. To prove the proposition we must estimate the integrals $A = \int_1^e e^{-x^x}\,\mathrm{d}x$ and $B = \int_e^{+\infty} e^{-x^x}\,\mathrm{d}x$.

For the first integral A we observe that

$$x^x = e^{xW(x\ln(x))},$$

where W is the Lambert function. Now let us consider the following inequality:

$$W(x) \le \ln\left(\frac{x+y}{1+\ln y}\right),$$

for $y > 1/e$ and $x > -1/e$ (see [6]).

If $y = e$, we have that $W(x) \le \ln\left(\frac{x+e}{2}\right)$ and

$$x^x \le e^{x\ln\left(\frac{x\ln x+e}{2}\right)} \le e^x$$

for $x \in [1, e]$. Thus,

$$\int_1^e e^{-x^x}\,\mathrm{d}x \ge \int_1^e e^{-e^x}\,\mathrm{d}x = -\mathrm{Ei}(1, \mathrm{e}^{\mathrm{x}})\Big|_1^{\mathrm{e}},$$

where $\mathrm{Ei}(\mathrm{x}, \mathrm{y}) = \int_1^{+\infty} \mathrm{e}^{-\mathrm{t}\mathrm{y}}\mathrm{t}^{-\mathrm{x}}\,\mathrm{d}\mathrm{t}$.

For the second integral B we use the inequality

$$e^{-x^x} \ge x^{-x^x},$$

for $x \ge e$, and the result follows. □

Observation 1. *The integral B can be computed using approximation techniques. Another estimation for B may be given using the inequality $W(x) \le \ln x$ (see [6]) for $x \ge e$. In this case, the integral B can be estimated by the integral*

$$B \ge \int_e^{+\infty} \frac{1}{x^{x^x(\ln x)^{x-1}}}\,\mathrm{d}x.$$

3 Other related results

In this third section we find two results similar to Proposition 1. In what follows we find in series form the integral (1) on $[0, 1]$, respectively in the case where the exponent is replaced by $-(x^x)^x$ and ${}^2(-x)$. In both cases, the central role of the proof is played by the formula (3).

Proposition 4. *The following equality holds:*

$$\int_0^1 e^{-x^{x^2}}\,\mathrm{d}x = 1 + \sum_{k=0}^{+\infty}\left\{\frac{(-1)^k}{(2k+1)^{(k+1)}}\cdot\left[\sum_{j=0}^{+\infty}\frac{(-1)^{(j+1)}}{(j+1)!}(j+1)^k\right]\right\}. \tag{4}$$

Proof. The proof is similar to the proof of Proposition 1 using (3) with $a = 2$.

We start by the Taylor expansion of the integrand,

$$\int_0^1 e^{-x^{x^2}}\,\mathrm{d}x = \int_0^1\left(1 - x^{x^2} + \frac{x^{2x^2}}{2!} - \frac{x^{3x^2}}{3!} + \dots\right)\mathrm{d}x.$$

Now, by linearity, we find that

$$\int_0^1 e^{-x^{x^2}}\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x - \int_0^1 x^{x^2}\,\mathrm{d}x + \int_0^1 \frac{x^{2x^2}}{2!}\,\mathrm{d}x - \int_0^1 \frac{x^{3x^2}}{3!}\,\mathrm{d}x + \dots$$

By the formula (3) we have that

$$\begin{aligned}\int_0^1 e^{-x^{x^2}}\,\mathrm{d}x = 1 &- \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(2k-1)^k}\\ &+ \frac{1}{2!}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}\cdot 2^{k-1}}{(2k-1)^k}\\ &- \frac{1}{3!}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}\cdot 3^{k-1}}{(2k-1)^k} + \dots\end{aligned}$$

Recollecting the main term,

$$\int_0^1 e^{-x^{x^2}}\,\mathrm{d}x = 1 + \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(2k-1)^k}\left(-1 + \frac{2^{k-1}}{2!} - \frac{3^{k-1}}{3!}\dots\right).$$

This is exactly the formula (4) with translated indices. $\square$

Proposition 5. *We have that*

$$\int_0^1 e^{{}^2(-x)}\,\mathrm{d}x = 1 + \sum_{n=0}^{+\infty} \frac{1}{n^n}\left[\sum_{k=1}^{+\infty}(-1)^k\frac{k^n}{k!}\right].$$

Proof. As in Proposition 1, we consider the Taylor expansion of the integrand:

$$\int_0^1 e^{-x^{-x}}\,\mathrm{d}x = \int_0^1 \left(1 - x^{-x} + \frac{x^{-2x}}{2!} - \frac{x^{-3x}}{3!} + \dots\right)\mathrm{d}x.$$

We proceed calculating each integral using the formula (3) with $a = 1$ and now $c = -1$:

$$\begin{aligned}
\int_0^1 1\,\mathrm{d}x &= 1,\\
\int_0^1 x^{-x}\,\mathrm{d}x &= \sum_{n=0}^{+\infty}\frac{1}{n^n},\\
\int_0^1 x^{-2x}\,\mathrm{d}x &= \sum_{n=0}^{+\infty}\frac{2^n}{n^n},\\
&\vdots
\end{aligned}$$

Thus, we have that

$$\int_0^1 e^{-x^{-x}}\,\mathrm{d}x = 1 + \sum_{n=0}^{+\infty}\frac{1}{n^n}\cdot\left[-1 + \frac{2^n}{2!} - \frac{3^n}{3!} + \dots\right].$$

The result follows. □

4 Conclusion

The operation of tetration is the evolution that corresponds to the repeated exponentiation. It is related to Ackermann's function and can be viewed as a "new arithmetical operation" [4]. These operations are called hyper-operations. It is interesting to examine classical results in this new light. In this article we examined an analogue case to the classical Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi},$$

where we considered ${}^2x$ instead the classical square x^2 at exponent. In this case, it seems we do not have a simple closed expression such as $\sqrt{\pi}$ (here we cannot use the Gauss trick to pass to polar coordinates!). Apart from this, we have found an upper/lower bound for the integral and some interesting results in power series on the interval $[0, 1]$.

References

[1] Chun, J. H. *What is... tetration?* 2014. URL: https://math.osu.edu/sites/math.osu.edu/files/chun_tetration.pdf.

[2] Edwards, S. "Extension of Algebraic Solutions Using The Lambert W Function". *arXiv e-prints* (2019). arXiv: 1902.08910.

[3] Euler, L. "De formulis exponentialibus replicatis". *Acta Academiae Scientiarum Petropolitanae* 1 (1778), pp. 38–60.

[4] Euler, L. "De serie Lambertina Plurimisque eius insignibus proprietatibus". *Acta Academiae Scientiarum Petropolitanae* 2 (1783), pp. 29–51.

[5] Galidakis, I. N. "On an application of Lambert's W function to infinite exponentials". *Complex Variables, Theory Appl.* 49.11 (2004), pp. 759–780. ISSN: 0278-1077; 1563-5066/e. DOI: 10.1080/02781070412331298796.

[6] Hoorfar, A. and Hassani, M. "Inequalities on the Lambert W function and hyperpower function". *JIPAM, J. Inequal. Pure Appl. Math.* 9.2 (2008), Id/No 51, 5 p. ISSN: 1443-5756/e.

[7] Knoebel, R. A. "Exponentials reiterated". *Am. Math. Mon.* 88 (1981), pp. 235–252. ISSN: 0002-9890. DOI: 10.2307/2320546.

[8] Nahin, P. J. *Inside interesting integrals. A collection of sneaky tricks, sly substitutions, and numerous other stupendously clever, awesomely wicked, and devilishly seductive maneuvers for computing nearly 200 perplexing definite integrals from physics, engineering, and mathematics (plus 60 challenge problems with complete, detailed solutions).* Undergraduate Lecture Notes in Physics. New York: Springer, 2015, pp. xxiii + 412. ISBN: 978-1-4939-1276-6/pbk; 978-1-4939-1277-3/ebook. DOI: 10.1007/978-1-4939-1277-3.

[9] Rubtsov, C. A. and Romerio, G. F. *Ackermann's function and new arithmetical operations*. 2004. URL: http://www.rotarysaluzzo.it/Z_Vecchio_Sito/filePDF/Iperoperazioni%20(1).pdf.

Simone Camosso
Department of Mathematics,
I.I.S. G.Soleri-A.Bertoni (Saluzzo, Italia)
via dei Ronchi n.14, 12030 Envie, Italia
r.camosso@alice.it

Dense sets and Kroneker's theorem

Arkady M. Alt

1 Introduction

Among the problems that appear in mathematical olympiads, there are problems that by one way or another are related to the approximations of irrational numbers by rational ones. Such problems directly lead to theorems of the theory of Diophantine approximations, such as the Kronecker theorem and Dirichlet theorem, and to the concept of subset that is dense in a given set (a concept important for understanding the fundamental properties of real numbers). Thus, such problems, in addition to their competitive olympiad assignments, become a cognitive stimulus.

In this note we present a short introduction to the topic mentioned above with applications to olympiad problems.

2 Basic results

We begin stating some basic facts that will be used later on:

1. For every real number x and any integers m, n, it holds that $\{n\{mx\}\} = \{nmx\}$. Indeed, $\{n\{mx\}\} = \{n(mx - \lfloor mx \rfloor)\} = \{nmx\}$.
2. For every irrational τ and any integer n the number $\{n\tau\}$ is irrational. Suppose on the contrary that $\{n\tau\}$ is rational; then $\tau = \dfrac{\{n\tau\} + \lfloor n\tau \rfloor}{n} \in \mathbb{Q}$ (contradiction).

3. For any real $\alpha > 0$ there is a positive integer n such that $n\alpha > 1$. (Archimedes' Axiom)

Next, we state and prove some lemmas that will be used hereafter:

Lemma 1. *Each interval (α, β) with $\beta - \alpha > 1$ contains at least one integer number.*

Proof. Denote $n = \lfloor\alpha\rfloor + 1$. Then, from $\lfloor\alpha\rfloor \le \alpha < n$ and $\alpha + 1 < \beta$ it follows that $\alpha < n = \lfloor\alpha\rfloor + 1 \le \alpha + 1 < \beta$. □

Lemma 2. *Let τ be an irrational number such that $0 < \tau < 1$. Then, there exists a unique nonnegative integer k and an irrational ρ such that $k\tau + \rho = 1$ and $0 < \rho < \tau$.*

Proof. Let $k = \lfloor 1/\tau \rfloor$ and $\rho = \tau\{1/\tau\}$. Then, from $1/\tau = \lfloor 1/\tau\rfloor + \{1/\tau\}$ we immediately obtain $k\tau + \rho = 1$ and $0 < \rho < \tau$, where ρ is nonzero because τ is irrational and integer $k \ge 0$ on account that $0 < \tau < 1$. □

Lemma 3. *Let θ be an irrational number. Then, for any positive integer k, there exists a nonzero integer m such that $\{m\theta\} < 1/k$, where $|m| \le k$.*

Proof. First note that, since $\{m\{\theta\}\} = \{m\,\theta\}$, we can WLOG assume that $\theta \in (0, 1)$. Consider the numbers $x_i = \{i\theta\}$, where $1 \le i \le k + 1$. We claim that all of them are distinct. Indeed, suppose that $x_i = x_j$ for some $i \ne j$. Then, $\{i\theta\} = \{j\theta\}$ or $i\theta - \lfloor i\theta\rfloor = j\theta - \lfloor j\theta\rfloor$ and $\theta(i - j) = \lfloor i\theta\rfloor - \lfloor j\theta\rfloor$, from which we get that

$$\theta = \frac{\lfloor i\theta\rfloor - \lfloor j\theta\rfloor}{i - j} \in \mathbb{Q}.$$

This contradicts the irrationality of θ and the claim follows.

Since all these numbers are distinct then there are x_i and x_j such that $0 < x_i - x_j < 1/k$. In fact, assume the contrary and suppose that $|x_i - x_j| \ge 1/k$ for all $i \ne j$. Let $y_1 < y_2 < \ldots < y_{k+1}$ be all terms of the sequence $x_1, x_2, \ldots, x_{k+1}$ sorted in increasing order. Since by assumption $y_{i+1} - y_i \ge 1/k$ for all $1 \le i \le k$, then we obtain

$$\begin{aligned} y_{k+1} - y_1 &= (y_{k+1} - y_k) + (y_k - y_{k-1}) + \ldots + (y_2 - y_1) \\ &\geq k \cdot \frac{1}{k} > 1. \end{aligned}$$

But this contradicts the fact that $0 < y_1 < y_{k+1} < 1$. Since

$$\begin{aligned} x_i - x_j = \{i\theta\} - \{j\theta\} &= (i\theta - \lfloor i\theta \rfloor) - (j\theta - \lfloor j\theta \rfloor) \\ &= \theta(i - j) - \lfloor i\theta \rfloor + \lfloor j\theta \rfloor \end{aligned}$$

and $0 < x_i - x_j < 1/k$, then we obtain

$$x_i - x_j = \{\theta(i - j) - \lfloor i\theta \rfloor + \lfloor j\theta \rfloor\} = \{\theta(i - j)\}.$$

So, $\{m\theta\} < 1/k$ for $m = i - j$ and $|m| \leq k$ because $-k = 1 - (k+1) \leq i - j \leq (k+1) - 1 = k$. □

Remark. *Actually, it is not necessary to claim that $\theta \in (0,1)$. Indeed, by Lemma 3, for any irrational $\theta \in (0,1)$ the number $\{\theta\} \in (0,1)$ and there is an integer $m \neq 0$ such that $\{m\{\theta\}\} < 1/k$ and $\{m\{\theta\}\} = \{m\theta\}$.*

An immediate consequence of the preceding are the following corollaries.

Corollary 1. *Let θ be irrational and k be any positive integer. Then, there exists a positive integer m such that $\{m\theta\} < 1/k$.*

Proof. Suppose that the number m obtained in Lemma 3 is negative. Then, by Lemma 2, $1 = l \cdot \{m\theta\} + \theta_1$, where $l \in \mathbb{N}$ and $0 < \theta_1 < \{m\theta\}$. Hence, $\theta_1 = \{\theta_1\} = \{1 - l \cdot \{m\theta\}\} = \{-l \cdot m\theta + l\lfloor m\theta \rfloor\} = \{-l \cdot m\theta\} = \{m_1\theta\}$, where $m_1 = -lm > 0$, and since $\theta_1 < \{m\theta\} < 1/k$ we have now a positive m_1 such that $\{m_1\theta\} < 1/k$. □

Corollary 2 (Dirihlet's theorem). *Let θ be an irrational number and k be an arbitrary natural number. Then, there exist integers m and l such that*

$$|m\theta - l| < \frac{1}{k}$$

and $0 < m \leq k$.

Proof. By Lemma 3 we have $0 < m\theta - \lfloor m\theta \rfloor < 1/k \implies |m\theta - \lfloor m\theta \rfloor| < 1/k \iff ||m|\theta - \lfloor m\theta \rfloor \cdot sign(m)| < 1/k$. Let $l = \lfloor m\theta \rfloor \cdot sign(m)$, $m = |m|$. Then, we obtain $|m\theta - l| < 1/k$ where $0 < m \leq k$. □

Corollary 3. *For any irrational θ and any natural number k there is a rational $r = l/m$ such that $|\theta - r| < 1/mk$ and $0 < m \leq k$.*

Corollary 4. *Let θ be an irrational number and $\epsilon > 0$ a real number. Then, the following inequalities have infinitely many solutions:*

(a) $\{x \cdot \theta\} < \epsilon$, $x \in \mathbb{N}$.
(b) $\{x \cdot \theta - y\} < \epsilon$, $x \in \mathbb{N}$, $y \in \mathbb{Z}$.

Proof. (a) The inequality $\{x \cdot \theta\} < 1/k$, where $k \in \mathbb{N}$ and $1/k < \epsilon$, has at least one solution in $\mathbb{N}$ which is also a solution of $\{x \cdot \theta\} < \epsilon$. Suppose there is an $\epsilon > 0$ such that the set S of all natural solutions of $\{x \cdot \theta\} < \epsilon$ is finite. Then, $\delta = \min_{x \in S}\{x \cdot \theta\} > 0$ (because $\{x \cdot \theta\} = 0$ implies $\theta = \lfloor \{x \cdot \theta\} \rfloor / x \in \mathbb{Q}$) and for this δ the set $\{x \mid \{x \cdot \theta\}\} < \delta, x \in \mathbb{N}\}$ is the empty set. But this is a contradiction, because for any natural number k such that $1/k < \delta$, by Corollary 1, the inequality $\{x \cdot \theta\} < 1/k$ has a solution in $\mathbb{N}$.

(b) can be proved in a similar way. □

3 Kronecker theorem

We start recalling two definitions of a dense set.

- A proper subset A of the numerical set X is dense in X if for any real $\varepsilon > 0$ and any $x \in X$ there is $a \in A$ such that $|x - a| < \varepsilon$. (Approximation Form)
- If $X = (p, q)$ and $A \subsetneq (p, q)$ then it is easy to see that A is dense in (p, q) if for any subinterval $(\alpha, \beta) \subset (p, q)$ there is $a \in A$ such that $\alpha < a < \beta$. (Interval form)

If $A \subset \mathbb{R}$ is dense in $\mathbb{R}$, we say that A is everywhere dense.

Using the preceding definitions we state and prove the following.

Lemma 4. *If $A \subset \mathbb{R}$ is dense in $\mathbb{R}$ and τ is a nonzero real number, then $\tau + A$ and τA are dense in $\mathbb{R}$.*

Proof. Let $(\alpha, \beta) \in \mathbb{R}$. Then, for the interval $(\alpha - \tau, \beta - \tau)$ there is $a \in (\alpha - \tau, \beta - \tau) \iff a + \tau \in (\alpha, \beta)$, and in the case $\tau > 0$ for interval $(\alpha/\tau, \beta/\tau\ )$ there is $a \in (\alpha/\tau, \beta/\tau) \iff \tau a \in (\alpha, \beta)$. If $\tau < 0$, then for the interval $(\beta/\tau, \alpha/\tau)$ there is $a \in (\beta/\tau, \alpha/\tau) \iff \tau a \in (\alpha, \beta)$. □

Theorem 1 (Kronecker). *The following hold.*

(a) *For any irrational number θ, the set $\{\{n\theta\} \mid n \in \mathbb{N}\}$ is dense in $(0, 1)$.*
(b) *For any irrational number θ, the set $\{n\theta + m \mid n \in \mathbb{N}, m \in \mathbb{Z}\}$ is everywhere dense (dense in $\mathbb{R}$). That is, for any $a \in \mathbb{R}$ and $\varepsilon > 0$ there are $n \in \mathbb{N}$, $m \in \mathbb{Z}$ such that $|a - (n\theta + m)| < \varepsilon$.*

Proof. (a) Suppose that $\theta \in (0, 1)$. Then, we will prove that, for any $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$, there exists a natural number n such that $\alpha < \{n\theta\} < \beta$. By Corollary 1, there exists $m \in \mathbb{N}$ such that $\{m\theta\} < \beta - \alpha$. Let $\delta = \{m\theta\}$ and consider the sequence $\{0, \delta, 2\delta, \ldots, n\delta, \ldots\}$. Since $\beta - \alpha > \delta$, then $\beta/\delta - \alpha/\delta > 1$ and, by Lemma 1, there is $n \in \mathbb{N}$ such that

$$\frac{\alpha}{\delta} < n < \frac{\beta}{\delta},$$

from which it follows that $\alpha < n\delta < \beta$. Since $n\delta \in (0, 1)$, then $n\delta = \{n\delta\} = \{n\{m\theta\} = \{nm\theta\}$ and for $n := nm$ (Here, $:=$ is an assigning operator. That is, $n := nm$ means that the new value of n is the old value of n multiplied by m), we get $\alpha < \{n\theta\} < \beta$. Let now θ be any irrational number. Then, $\theta_1 = \theta - \lfloor\theta\rfloor$ is also irrational and, therefore, there exists $n \in \mathbb{N}$ such that $\alpha < \{n\theta_1\} < \beta$ or $\alpha < \{n\theta - n\lfloor\theta\rfloor\} < \beta$, from which it follows that $\alpha < \{n\theta\} < \beta$.

(b) First, we prove that, for any interval (α, β), there exist $m, n \in \mathbb{N}$ such that $\alpha < n\theta + m < \beta$. WLOG we may assume that $\beta - \alpha \leq 1$. Then, $(\{\alpha\}, \beta - \lfloor\alpha\rfloor) \subset [0, 1]$ and, by (a), there exists $n \in \mathbb{N}$ such that $\{\alpha\} < \{n\theta\} < \beta - \lfloor\alpha\rfloor$ or

$$\{\alpha\} < \{n\theta\} + \lfloor\alpha\rfloor < \beta \iff \{\alpha\} < n\theta - \lfloor n\theta\rfloor + \lfloor\alpha\rfloor < \beta.$$

Putting $m = \lfloor \alpha \rfloor - \lfloor n\theta \rfloor \in \mathbb{Z}$, then we get $\alpha < n\theta + m < \beta$. Let a be a real number. Then, for any $\epsilon > 0$, there are $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $a - \epsilon < n\theta + m < a + \epsilon$ or $|a - (n\theta + m)| < \epsilon$. □

Now we will give another constructive proof of Kronecker's theorem. The next two lemmas correspond to part (a) of the theorem. Furthermore, we also give an algorithm for finding n for any interval (α, β) and $\epsilon > 0$ depending on the definition of density (interval or approximation form).

Lemma 5. *For any irrational number $\tau \in (0, 1)$ there is a natural number $k \geq 2$ such that $\{k\tau\} < \tau/2$.*

Proof. For a given τ we have the representation $1 = k_0\tau + \tau_1$, where $k_0 \in \mathbb{N}$ and $0 < \tau_1 < \tau$. If $0 < \tau_1 < \tau/2$, then again (because τ_1 is irrational and $\tau_1 \in (0, 1)$) we have $1 = k_1\tau_1 + \tau_2$, where $k_1 \geq 2$ because $\tau_1 < 1/2$ and $0 < \tau_2 < \tau_1 < \tau/2$. Therefore, $\tau_2 = \{\tau_2\} = \{1 - k_1\tau_1\} = \{-k_1(1 - k_0\tau)\} = \{k\tau\} < \tau/2$, where $k = k_0k_1 \geq 2$. If $\tau/2 < \tau_1$, then from $\tau - \tau_1 = \{\tau - \tau_1\} = \{\tau - 1 + k_0\tau\} = \{(k_0 + 1)\tau\}$ it follows that $\{k\tau\} < \tau/2$, where $k = k_0 + 1 \geq 2$. □

Lemma 6. *Let $\theta \in (0, 1)$ be an irrational number. Then, there is a sequence of natural numbers $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\{n_k\theta\} < \theta/2^k$.*

Proof. By Lemma 5, there exists a natural number $k \geq 2$ such that $\{k\theta\} < \theta/2$. Let $n_1 = k$. Suppose that we already have $n_1 < n_2 < \ldots < n_i$ such that $\theta_j = \{n_j\theta\} < \theta/2^j$ for $j = 1, 2, \ldots i$. Applying Lemma 5 to the irrationals θ_i we obtain $\theta_{i+1} = \{k_i\theta_i\} < \theta_i/2$ for some natural $k_i \geq 2$. But $\theta_i < \theta/2^i$ and $\{k_i\theta_i\} = \{k_i\{n_i\theta\}\} = \{n_{i+1}\theta\} < \theta/2^{i+1}$, where $n_{i+1} = k_in_i > n_i$. □

Corollary 5. *Let $\theta \in (0, 1)$ and $\epsilon > 0$. Then, there exist infinitely many positive integers n such that $\{n\theta\} < \epsilon$. More precisely, there exists an increasing sequence of positive integers $\{n_k\}_{k\geq 1}$ such that $\epsilon > \{n_k\theta\}$ and $\{n_{k+1}\theta\} < \{n_k\theta\}/2$.*

Proof. For $\{n\theta\}$, there exists $m \geq 2$ such that

$$\{mn\theta\} = \{m\{n\theta\}\} < \frac{\{n\theta\}}{2} < \epsilon.$$

Then, for n_k we get the integer $n_{k+1} = mn_k > n_k$ for which $\{n_{k+1}\theta\} < \{n\theta\}/2$. $\square$

Corollary 6. *Let $\theta \in (0,1)$ be an irrational number. The set $\{\{n\theta\} \mid n \in \mathbb{N}\}$ is dense in $(0,1)$. Moreover, for each interval $(\alpha,\beta) \subset (0,1)$ there exist infinitely many positive integers x such that $\alpha < \{x\theta\} < \beta$.*

Proof. By the preceding results, there exists a positive integer m such that $\{m\theta\} < \beta - \alpha$. Then, the interval

$$\left(\frac{\alpha}{\{m\theta\}}, \frac{\beta}{\{m\theta\}}\right)$$

has length greater than 1 and contains a positive integer n. Namely,

$$\frac{\alpha}{\{m\theta\}} < n < \frac{\beta}{\{m\theta\}} \Leftrightarrow \alpha < n\{m\theta\} < \beta \Rightarrow \alpha < \{nm\theta\} < \beta,$$

because from $n\{m\theta\} \in (0,1)$ if follows that $n\{m\theta\} = \{n\{m\theta\}\} = \{nm\theta\}$. For example, we may choose $n = \lfloor \alpha/\{m\theta\} \rfloor + 1$. So, we have a positive integer $x = mn$ such that $\alpha < \{x\theta\} < \beta$ holds. By the preceding result, there always exists a positive integer $m' > m$ such that $\{m'\theta\} < \{m\theta\}/2$. Then, $n' = \lfloor \alpha/\{m'\theta\} \rfloor + 1 > n$. Actually, $n' \geq 2n - 1$ because

$$n' - 1 = \left\lfloor \frac{\alpha}{\{m'\theta\}} \right\rfloor \geq \left\lfloor \frac{2\alpha}{\{m\theta\}} \right\rfloor \geq 2\left\lfloor \frac{\alpha}{\{m\theta\}} \right\rfloor = 2(n-1).$$

Thus, we got another integer solution $x' = m'n' > x$ of $\alpha < \{x\theta\} < \beta$ and this process can be continued infinitely. So, starting with m and $n = \lfloor \alpha/\{m\theta\} \rfloor + 1$we may construct an increasing sequence of positive integers such that $\alpha < \{x\theta\} < \beta$, as desired. $\square$

For applications, it is often convenient to consider the following interval form of Kronecker's Theorem.

Corollary 7 (Kronecker). *If $\theta \in (0,1)$ is irrational, then for any interval $(\alpha,\beta) \subset \mathbb{R}$ there exist positive integers n, m such that $\alpha < n\theta - m < \beta$.*

Proof. Let $(\alpha, \beta) \subset \mathbb{R}$. WLOG we may assume that $\lfloor \alpha \rfloor = \lfloor \beta \rfloor$. Since $(\{\alpha\}, \{\beta\}) \in (0,1)$, then $\{\alpha\} < \{n\theta\} < \{\beta\}$ is satisfied for $n \in \mathbb{N}$ as big as we need. In particular, for $n \geq (\lfloor \alpha \rfloor + 1)/\theta$. Then,

$$n\theta \geq \lfloor \alpha \rfloor + 1 \Rightarrow \lfloor n\theta \rfloor \geq \lfloor \alpha \rfloor + 1 \Leftrightarrow \lfloor n\theta \rfloor - \lfloor \alpha \rfloor \geq 1.$$

Let us denote by $m = \lfloor n\theta \rfloor - \lfloor \alpha \rfloor$, then we have

$$\{\alpha\} < \{n\theta\} < \{\beta\} \Leftrightarrow \alpha - \lfloor \alpha \rfloor < n\theta - \lfloor n\theta \rfloor < \beta - \lfloor \beta \rfloor$$

or $\alpha < n\theta - (\lfloor n\theta \rfloor - \lfloor \alpha \rfloor) < \beta$, from which it follows that $\alpha < n\theta - m < \beta$. □

4 Some applications

Below, we apply the preceding results to solve some problems. We begin with the following.

Problem 1. *Prove that, for any positive integer M with k digits, there is a natural number n such that the first k digits of 2^n are precisely M.*

Solution. On account of the statement of the problem, we have to prove that there exists $m \in \mathbb{N} \cup \{0\}$ such that

$$M = \left\lfloor \frac{2^n}{10^m} \right\rfloor \iff M \leq \frac{2^n}{10^m} < M + 1$$

or

$$\log M \leq n \log 2 - m < \log(M+1).$$

Since M has k digits, then $\lfloor \log M \rfloor = k$. Let $\alpha = \{\log M\} = \log M - k$ and $\beta = \min\{1, \log(M+1) - k\}$, so $(\alpha, \beta) \subset (0,1)$. By the preceding, we know that there are infinitely many natural numbers such that $\{x \log 2\} < \beta - \alpha$. We choose $n > k/\log 2$ with $\{n \log 2\} < \beta - \alpha$. Then, the interval

$$\left(\frac{\alpha}{\{n \log 2\}}, \frac{\beta}{\{n \log 2\}} \right)$$

has length greater that 1 and, consequently, contains a natural number, say ℓ. So, we have $\alpha < \ell\{n \log 2\} < \beta$, or

$$\log M < n \log 2 - (\ell \lfloor n \log 2 \rfloor - k) < \beta + k \leq \log(M+1).$$

Putting $m := \ell \lfloor n \log 2 \rfloor - k$ and $n := \ell n$, we obtain

$$\log M \leq n \log 2 - m < \log(M+1),$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. □

Problem 2. *Prove that there exists an irrational number θ such that the set*

$$\{2^n \theta \mid n \in \mathbb{N}\}$$

is everywhere dense in $[0, 1)$.

Solution. First, we write the positive integers in the binary system and we get

$$\mathbb{N} = \{1, 10, 11, 100, 101, 110, 111, 1000, \ldots\}.$$

Let θ be the real number whose decimal figures are the natural numbers written in binary notation. That is,

$$\theta = 0.11011100101110111 1000\ldots.$$

This number is irrational because its binary representation contains zero segments of any length. This number also has the following interesting property: For each number $b = 0.\beta_1\beta_2\ldots\beta_k$, we can find a natural number which indicates the position in θ from where the digits of b start a θ segment of digits. Let $\ell(b)$ be the function that shows the least of starting positions of b. Thus, if $\theta = 0.\theta_1\theta_2\ldots\theta_m\ldots$, then

$$\{2^{\ell(b)}\theta\} = 0.\beta_1\beta_2\ldots\beta_k\theta_{\ell(b)+k+1}\ldots$$

Let $\alpha = 0.\alpha_1\alpha_2\ldots\alpha_i\ldots \in (0, 1)$ and let p be a positive integer. Then, for

$$b = 2^{-p}\lfloor 2^p \alpha \rfloor = 0.\alpha_1\alpha_2\ldots\alpha_p,$$

the numbers α and $\{2^{\ell(b)}\theta\}$ have the same first p digits $\alpha_1, \alpha_2, \ldots, \alpha_p$. Therefore,

$$|\alpha - \{2^{\ell(b)}\theta\}| = |0.\alpha_{p+1}\alpha_{p+2}\ldots - 0.\theta_{\ell(b)+p+1}\ldots| < 2^{-p},$$

and the proof is complete. □

Problem 3 (A. Ya. Dorogovtsev [2]). *Prove that the sets* $A = \{\sqrt{n} - \sqrt{m} \mid n, m \in \mathbb{N}\}$ *and* $B = \{\sqrt[3]{n} - \sqrt{m} \mid n, m \in \mathbb{N}\}$ *are everywhere dense.*

Solution. First, we will see that, for any real interval (a, b), there exist two positive integers n, m such that $a < \sqrt{n} - \sqrt{m} < b$. Let m be a positive integer such that $a + \sqrt{m} > 0$. Then,

$$a < \sqrt{n} - \sqrt{m} < b \Longleftrightarrow (a + \sqrt{m})^2 < n < (b + \sqrt{m})^2.$$

Now, we claim that $(b + \sqrt{m})^2 - (a + \sqrt{m})^2 > 1$. Indeed,

$$(b + \sqrt{m})^2 - (a + \sqrt{m})^2 > 1 \Longleftrightarrow \sqrt{m} > \frac{1 - a^2 + b^2}{2(b - a)}.$$

Thus, for any $m \in \mathbb{N}$ such that

$$\sqrt{m} > \max\left\{-a, \frac{1 - a^2 + b^2}{2(b - a)}\right\},$$

by Lemma 1, there exists $n \in \mathbb{N}$ such that $(a + \sqrt{m})^2 < n < (b + \sqrt{m})^2$, and the set A is dense everywhere.

To prove that B is everywhere dense, we have to see that, for any real interval (a, b), there exist two positive integers n, m such that $a < \sqrt[3]{n} - \sqrt{m} < b$. Let n be a positive integer such that $\sqrt[3]{n} > b$. Then,

$$a < \sqrt[3]{n} - \sqrt{m} < b \Longleftrightarrow (\sqrt[3]{n} - b)^2 < m < (\sqrt[3]{n} - a)^2.$$

Now, we claim that $(\sqrt[3]{n} - a)^2 - (\sqrt[3]{n} - b)^2 > 1$. Indeed,

$$(\sqrt[3]{n} - a)^2 - (\sqrt[3]{n} - b)^2 > 1 \Longleftrightarrow \sqrt[3]{n} > \frac{1 - a^2 + b^2}{2(b - a)}.$$

Thus, for any $n \in \mathbb{N}$ such that

$$\sqrt[3]{n} > \max\left\{b, \frac{1 - a^2 + b^2}{2(b - a)}\right\},$$

by Lemma 1, there exists $m \in \mathbb{N}$ such that $(\sqrt[3]{n} - b)^2 < m < (\sqrt[3]{n} - a)^2$, and the set B is dense everywhere. $\square$

Problem 4 (Yu. S. Ochan [3]). *Prove that the set* $\{\ln(r^2+1) \mid r \in \mathbb{Q}\}$ *is dense in* $[0,+\infty)$.

Solution. Let $(a,b) \subset [0,+\infty)$. We have that

$$a < \ln(x^2+1) < b \iff e^a - 1 < x^2 < e^b - 1$$
$$\iff \sqrt{e^a-1} < |x| < \sqrt{e^b-1}.$$

Then, on account of Archimede's Axiom there exists $n \in \mathbb{N}$ such that

$$n\left(\sqrt{e^b-1} - \sqrt{e^a-1}\right) > 1.$$

By Lemma 1, for this n the interval $(n\sqrt{e^a-1}, n\sqrt{e^b-1})$ contains a natural number m. That is, $n\sqrt{e^a-1} < m < n\sqrt{e^b-1}$ or, equivalently,

$$a < \ln\left(\left(\frac{m}{n}\right)^2 + 1\right) < b. \qquad \square$$

Problem 5 (V. I. Bernik et al. [1]).

(a) *Prove that the set* $\{\sin r \mid r \in \mathbb{Q}\}$ *is dense in* $[-1,1]$.
(b) *Prove that* $\{\{\log n\} \mid n \in \mathbb{N}\}$ *is dense in* $(0,1)$.

Solution. (a) Let $(a,b) \subset [-1,1]$ and let $\alpha = \arcsin a$ and $\beta = \arcsin b$. By Archimede's Axiom, there exists $n \in \mathbb{N}$ such that $n(\beta-\alpha) > 1$. Then, by Lemma 1, the interval $(n\alpha, n\beta)$ contains a natural number m. That is, $n\alpha < m < n\beta$ and $\alpha < m/n < \beta$. Since $f(x) = \sin x$ is increasing in $[-\pi/2, \pi/2]$ and $\alpha, \beta \in [-\pi/2, \pi/2]$, then we obtain

$$a = \sin\alpha < \sin\frac{m}{n} < \sin\beta = b,$$

and the set $\{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1,1]$. The preceding, jointly with the fact that $\mathbb{N} \subset \mathbb{Q}$, imply that $\{\sin r \mid r \in \mathbb{Q}\}$ is dense in $[-1,1]$.

An alternative proof of (a) can be given by using the following.

Proposition 1. *Let* f *be a continuous function on* $[a,b]$ *and suppose that* $f([a,b]) = [m,M]$. *If* $A \subset [a,b]$ *is dense in* $[a,b]$, *then* $f(A)$ *is dense in* $[m,M]$.

Proof. Let $q \in [m, M]$ and suppose that $f(p) = q$ for some $p \in [a, b]$. Then, for any $\epsilon > 0$ there is $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - q| < \epsilon$. Since A is dense in $[a, b]$, there is $c \in A$ such that $|c - p| < \delta$. Then, $|f(c) - q| < \epsilon$, and this means that $f(A)$ is dense in $[m, M]$. □

Applying the above proposition to the function $f(x) = \sin x$ we get that $\{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1, 1]$. Indeed,

$$\frac{n}{2\pi} = \left\lfloor \frac{n}{2\pi} \right\rfloor + \left\{ \frac{n}{2\pi} \right\} \Rightarrow \sin n = \sin\Big(\left\lfloor \frac{n}{2\pi} \right\rfloor + \left\{ \frac{n}{2\pi} \right\} \Big) = \sin\Big(2\pi \left\{ \frac{n}{2\pi} \right\} \Big).$$

Since $\left\{\frac{n}{2\pi}\right\}$ is dense in $[0, 1)$ then $2\pi\left\{\frac{n}{2\pi}\right\}$ is dense in $[0, 2\pi)$ and $\{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1, 1]$.

(b) Let $B = \{\{\log n\} \mid n = 2^m, m \in \mathbb{N}\} = \{m \log 2 \mid m \in \mathbb{N}\} \subset A$. Since $\log 2$ is irrational, then by Kronecker's Theorem B is dense in $(0, 1)$. This implies that A is dense in $(0, 1)$ because $B \subset A$. □

References

[1] Bernik, V. I., Zhuk, I. K., and Melnikov, O. V. *Collection of Olympiad problems in mathematics.* 1st ed. Belarus: Minsk, Publishing house "Narodnaya Asveta", 1980.

[2] Dorogovtsev, A. Y. *Mathematical analysis. Collection of problems.* 1st ed. Ukraine: Kiev, Publishing house "Vischa shkola", 1987.

[3] Ochan, Y. S. *Collection of problems and theorems on the theory of functions of a real variable.* 1st ed. Russia: Publishing house "Prosveschenie", 1983.

Arkady M. Alt
San Jose, California, USA.
arkady.alt@gmail.com

Some inequalities with radicals in triangle geometry

Marius Drăgan and Mihály Bencze

Abstract

In this paper we obtain lower and upper bounds for

$$\sqrt{P(a,b,c)} + \sqrt{P(b,c,a)} + \sqrt{P(c,a,b)},$$

where P is a positive rational function and a, b, c are the length of the sides of a triangle ABC.

1 Introduction

In the following for a triangle ABC we denote its sides by $a = |BC|$, $b = |AC|$, $c = |AB|$, by $s = \dfrac{a+b+c}{2}$ the semiperimeter, by R the circumscribed radius and by r the inscribed radius.

Recall the inequality $\sum \sin \frac{A}{2} \leq \sqrt{\frac{4R+r}{2R}}$ from [2], and its improvement $\sum \sin \frac{A}{2} \leq \frac{\sqrt{2}R+(3-2\sqrt{2})r}{R}$, given in [1].

Corollary 1 improves these inequalities and furthermore gives us the best inequality of the type

$$f(R,r) \leq \sum \sin \frac{A}{2} \leq g(R,r),$$

where $f(R,r)$ and $g(R,r)$ are two homogeneous functions.

We also have the inequality $\sum m_a \leq \frac{32R^2+7r^2}{8R-r}$ (see [5]), which represents a rational refinement of the well-known inequality

$$\sum m_a \leq 4R + r.$$

Corollary 7 improves this inequality and furthermore gives us the best inequality of type $f(R,r) \leq \sum m_a \leq g(R,r)$, where $f(R,r)$ and $g(R,r)$ are two homogeneous functions.

2 Main results

Theorem 1 (Blundon). *In any triangle ABC we have $s_1 \leq s \leq s_2$, where*

$$s_1 = \sqrt{2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}},$$
$$s_2 = \sqrt{2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}}.$$

We obtain equality in the case of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ with sides of lengths

$$a_1 = \frac{2r(R+r-d)}{\sqrt{(R-d)^2 - r^2}}, \qquad b_1 = c_1 = \frac{(R+r-d)(R-d)}{\sqrt{(R-d)^2 - r^2}},$$
$$a_2 = \frac{2r(R+r+d)}{\sqrt{(R+d)^2 - r^2}}, \qquad b_2 = c_2 = \frac{(R+r+d)(R+d)}{\sqrt{(R+d)^2 - r^2}},$$

where $d = |OI| = \sqrt{R^2 - 2Rr}$.

In the following we will use the results contained in the next lemma:

Lemma 1. *In any triangle ABC the following identities are true*

1) $\sum \frac{s-a}{a} = \frac{s^2+r^2-8Rr}{4Rr}$.
2) $\sum \frac{(s-b)(s-c)}{bc} = \frac{2R-r}{2R}$.
3) $\sum \frac{(s-b)(s-c)}{a} = \frac{r}{R}\frac{s^2+(4R+r)^2}{4s}$.
4) $\sum \frac{s(s-a)}{(s-b)(s-c)} = \frac{s^2-2\left(r^2+4Rr\right)}{r^2}$.

5) $\sum \frac{s^2}{(s-a)^2} = \frac{(4R+r)^2-2s^2}{r^2}$.
6) $\sum \frac{(s-b)(s-c)}{s(s-a)} = \frac{(4R+r)^2-2s^2}{s^2}$.
7) $\sum \frac{(s-a)^2}{s^2} = \frac{s^2-2r^2-8Rr}{s^2}$.
8) $\sum \frac{bc}{(s-b)(s-c)} = \frac{s^2+r^2-8Rr}{r^2}$.
9) $\sum \frac{a}{s-a} = \frac{4R-2r}{r}$.
10) $\sum \frac{bc}{s(s-a)} = \frac{s^2+(4R+r)^2}{s^2}$.
11) $\sum \frac{a}{(s-b)(s-c)} = \frac{8R+2r}{sr}$.
12) $\sum m_a^2 m_b^2 = \frac{9}{16}\left[\left(s^2+r^2+4Rr\right)^2-16Rrs^2\right]$.
13) $(4m_a m_b m_c)^2 = s^6+\left(33r^2-12Rr\right)s^4-\left(33r^4+60R^2r^2-120r^3\right)s^2-\left(r^2+4Rr\right)^3$.

Proof. We prove each item.

1) $\sum \frac{s-a}{a} = \frac{s\sum bc-3abc}{abc} = \frac{s\left(s^2+r^2+4Rr\right)-12sRr}{4Rrs} = \frac{s^2+r^2-8Rr}{4Rr}$.

2) $\sum \frac{(s-b)(s-c)}{bc} = \frac{s^2\sum a-2s\sum ab+3abc}{abc} = \frac{2R-r}{2R}$.

3) $\sum \frac{(s-b)(s-c)}{a} = \frac{s^2\sum bc-2s^2\sum bc+3abcs+\left(\sum bc\right)^2-4abc}{abc} = \frac{s^2r^2+\left(r^2+4Rr\right)^2}{4Rrs}$.

4) $\sum \frac{s(s-a)}{(s-b)(s-c)} = \frac{s\sum(s-a)^2}{\prod(s-a)} = \frac{s^2\left(3s^2-4s^2+2s^2-2r^2-8Rr\right)}{s^2r^2} = \frac{s^2-2r^2-8Rr}{r^2}$.

5) $\sum \frac{s^2}{(s-a)^2} = \frac{\left\{\left[\sum(s-b)(s-c)\right]^2-2\prod(s-a)s\right\}s^2}{\left(\prod(s-a)\right)^2} = \frac{\left(4Rr+r^2\right)^2-2s^2r^2}{r^4}$.

6) $\sum \frac{(s-b)(s-c)}{s(s-a)} = \frac{1}{s}\cdot\frac{\left[\sum(s-b)(s-c)\right]^2-2\prod(s-a)s}{\prod(s-a)} = \frac{(4R+r)^2-2s^2}{s^2}$.

7) $\sum \frac{(s-a)^2}{s^2} = \frac{3s^2-4s^2+2s^2-2r^2-8Rr}{s^2r^2} = \frac{s^2-2r^2-8Rr}{s^2}$.

8) $\sum \frac{bc}{(s-b)(s-c)} = \frac{\left(s\sum bc-3abc\right)s}{S^2} = \frac{s^2+r^2-8Rr}{r^2}$.

9) $\sum \frac{a}{s-a} = \frac{\left(2s^3-2s\sum ab+3abc\right)s}{s^2r^2} = \frac{4R-2r}{r}$.

10) $\sum \frac{bc}{s(s-a)} = \frac{\sum bc(s-b)(s-c)}{s^2r^2} = \frac{s^2r^2+\left(4Rr+r^2\right)^2}{s^2r^2}$.

11) $\sum \frac{a}{(s-b)(s-c)} = \frac{\sum a(s-a)}{\prod(s-a)} = \frac{2s^2-2s^2+2r^2+8Rr}{sr^2} = \frac{8R+2r}{sr}$.

12) $\sum m_a^2 m_b^2 = \frac{9}{16}\sum a^2b^2 = \frac{9}{16}\left[\left(s^2+r^2+4Rr\right)^2-16Rrs^2\right]$.

13) We denote $S_2 = a^2+b^2+c^2$, $s^2 = t$ and $r^2+4Rr = \alpha$. Then,

we have

$$
\begin{aligned}
&64m_a^2m_b^2m_c^2 = \prod\left[2\left(b^2+c^2\right)-a^2\right] = \prod\left(2s_2-3a^2\right)\\
&= 8s_2^3 - 12s_2^3 + +9\sum a^2b^22s_2 - 27a^2b^2c^2\\
&= -4s_2^3 + 18s_2\left[\left(\sum ab\right)^2 - 4abcs\right] - 27\cdot 16\cdot R^2r^2s^2\\
&\quad -32(t-\alpha)^3 + 36(t-\alpha)\cdot\left[(t-\alpha)^2 - 16Rrs\right] - 432R^2r^2s^2\\
&= 4\left\{-8(t-\alpha)^3 + 9(t-\alpha)\left[(t+\alpha)^2 - 16Rrs - 108R^2r^2s^2\right]\right\}\\
&= 4\left[-8t^3 + 24t^2\alpha - 24t\alpha^2 + 8\alpha^3\right.\\
&\qquad \left. +(9t-9\alpha)\left(t^2+\alpha^2+2\alpha t - 16Rrt\right) - 108R^2r^2s^2\right]\\
&= 4(-8t^3 + 24t^2\alpha - 24t\alpha^2 + 8\alpha^3 + 9t^3 + 9\alpha^2t + 18\alpha t^2\\
&\quad - 144Rrt^2 - 9\alpha r^2 - 9\alpha^3 - 18\alpha^2t + 144\alpha Rrt - 108R^2r^2s^2)\\
&= 4[t^3 + (24\alpha + 18\alpha - 144Rr - 9\alpha)t^2\\
&\quad + (-24\alpha^2 + 9\alpha^2 - 18\alpha^2 + 144\alpha Rr - 108R^2r^2)t + 8\alpha^3 - 9\alpha^3]\\
&= 4[t^3 + (33\alpha - 144Rr)t^2 + (-33\alpha^2 + 144\alpha Rr - 108R^2r^2)t - \alpha^3]\\
&= 4[t^3 + \left(33r^2 - 12Rr\right)t^2 + (-33r^4 - 60R^2r^2 - 120Rr^3)t - \alpha^3].
\end{aligned}
$$

This concludes the proof. □

We consider the functions $f, g, h, u\colon [s_1, s_2] \to \mathbb{R}$ defined by

$$
\begin{aligned}
&f(s) = P(a,b,c) + P(b,c,a) + P(c,a,b),\\
&g(s) = P(a,b,c)P(b,c,a) + P(a,b,c)P(c,a,b) + P(b,c,a)P(c,a,b),\\
&h(s) = \sqrt{P(a,b,c)P(b,c,a)P(c,a,b)},\\
&u(s) = \sqrt{P(a,b,c)} + \sqrt{P(b,c,a)} + \sqrt{P(c,a,b)},
\end{aligned}
$$

where $P(a,b,c)$, $P(b,c,a)$, $P(c,a,b)$ are three rational functions.

Theorem 2. *Let $f, g, h\colon [s_1, s_2] \to \mathbb{R}$ be as defined above, so they are increasing, continuous and differentiable on $[s_1, s_2]$. If there exists a unique function $u\colon [s_1, s_2] \to \mathbb{R}$ which is continuous and differentiable on (s_1, s_2), and such that*

$$
\left(u^2(s) - f(s)\right)^2 = 4(g(s) + 2h(s)u(s)), \tag{1}
$$

then u is an increasing function on $[s_1, s_2]$.

Proof. We have to prove that u' is positive on (s_1, s_2). After we differentiate (1) we obtain

$$\begin{aligned}&(u^2(s) - f(s))(2u(s)u'(s) - f'(s))\\ &= 2(g'(s) + 2h'(s)u(s) + 2h(s)u'(s)),\end{aligned}$$

or

$$\begin{aligned}&u'(s)\big(2u^3(s) - 2f(s)u(s) - 4h(s)\big)\\ &= 2(g'(s) + 2h'(s)u(s)) + f'(s)\big(u^2(s) - f(s)\big).\end{aligned} \tag{2}$$

However,

$$u^2(s) = f(s) + 2\sum\sqrt{P(a,b,c)P(b,c,a)} \geq f(s) + 6\sqrt[3]{h^2(s)},$$

so

$$u^2(s) - f(s) \geq 6\sqrt[3]{h^2(s)} > 0$$

and we have

$$2(g'(s) + 2h'(s)u(s)) + f'(s)\big(u^2(s) - f(s)\big) > 0. \tag{3}$$

From (2) and (3) it follows that, to prove that $u'(s) > 0$ for all $s \in [s_1, s_2]$, it will suffice to show that

$$2u^3(s) - 2f(s)u(s) - 4h(p) > 0. \tag{4}$$

But

$$2u(s)\big(u^2(s) - f(s)\big) - 4h(s) > 12\sqrt[3]{h^2(s)}u(s) - 4h(s). \tag{5}$$

To prove (4) from (5), it suffices to show that

$$u^2(s) > \frac{1}{9}\sqrt[3]{h^2(s)}.$$

This inequality is true, as we have

$$u^2(s) > 6\sqrt[3]{h^2(s)} > \frac{1}{9}\sqrt[3]{h^2(s)}. \qquad \square$$

In the following, we consider the relation (1), which is equivalent to

$$u^4 - 2f(s)u^2 - 8h(s)u + f^2(s) - 4g(s) = 0.$$

As $u^2 \geq f(s)$, we can define the function $F\colon \left[\sqrt{f(s)}, +\infty\right) \to \mathbb{R}$ given by the formula

$$F(u) = u^4 - 2f(s)u^2 - 8h(s)u + f^2(s) - 4g(s).$$

Theorem 3. *There is a unique continuous function $u\colon [s_1, s_2] \to \mathbb{R}$ which is differentiable on the interval (s_1, s_2) such that*

$$F(u(s)) = 0, \quad \forall\, s \in [s_1, s_2].$$

Proof. We have

$$F'(u) = 4u^3 - 4f(s)u - 8h(s) = 4u\big(u^2 - f(s)\big) - 8h(s),$$

and so F is a strictly increasing function on $\left[\sqrt{f(s)}, +\infty\right)$ and, as $F'\left(\sqrt{f(s)}\right) = -8h(s) < 0$ and $\lim\limits_{u\to\infty} F'(u) = +\infty$, it follows that F' has a unique root on $\left[\sqrt{f(s)}, +\infty\right)$. Let u_0 be this root. As

$$F\left(\sqrt{f(s)}\right) = -8h(s)\sqrt{f(s)} - 4g(s) < 0 \ \text{ and } \ \lim_{u\to\infty} F(u) = +\infty,$$

if follows from the Rolle sequence that the equation has a unique solution, either in the interval $\left[\sqrt{f(s)}, u_0\right]$ if $F(u_0) > 0$ or in the interval $(u_0, +\infty)$ if $F(u_0) < 0$.

On account of the implicit function theorem, it results that there exists a unique continuous function $u\colon [s_1, s_2] \to \mathbb{R}$ which is differentiable on (s_1, s_2). □

Corollary 1. *In any triangle ABC the following inequalities hold:*

$$\frac{r}{R-d} + \sqrt{\frac{R-d}{R}} \leq \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \leq \frac{r}{R+d} + \sqrt{\frac{R+d}{R}}.$$

Proof. We have

$$\sum \sin\frac{A}{2} = \sum \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

We have according to Lemma 1 that

$$\begin{aligned}
f(s) &= \sum \frac{(s-b)(s-c)}{bc} = \frac{2R-r}{2R}, \\
g(s) &= \left(\sum \frac{s-a}{a}\right) \frac{(s-a)(s-b)(s-c)}{abc} \\
&= \frac{s^2+r^2-8Rr}{4Rr} \cdot \frac{r}{4R} = \frac{s^2+r^2-8Rr}{16R^2}, \\
h(s) &= \frac{(s-a)(s-b)(s-c)}{abc} = \frac{r}{4R}.
\end{aligned}$$

It results that f, g and h are increasing functions. From Theorems 2 and 3, it follows that there exists a differentiable and increasing function on (s_1, s_2), $u(s) = \sum \sin \frac{A}{2}$, such that $u(s_1) \leq u(s) \leq u(s_2)$, where

$$u(s_1) = \frac{r}{R-d} + \sqrt{\frac{R-d}{R}} \text{ and } u(s_2) = \frac{r}{R+d} + \sqrt{\frac{R+d}{R}}. \quad \square$$

Corollary 2. *In any triangle ABC the following inequalities hold:*

$$\begin{aligned}
&\sqrt{\frac{R+r-d}{R-d}}\left(\sqrt{\frac{R-r-d}{R-d}}+\sqrt{2}\right) \\
&\quad \leq \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \\
&\quad\quad \leq \sqrt{\frac{R+r+d}{R+d}}\left(\sqrt{\frac{R-r+d}{R+d}}+\sqrt{2}\right). \quad (6)
\end{aligned}$$

Equality holds on the left side of (6) *for the triangle $A_1B_1C_1$, and on the right side for the triangle $A_2B_2C_2$.*

Proof. By Lemma 1, we have $\sum \cos \frac{A}{2} = \sum \sqrt{\frac{s(s-a)}{bc}}$. The functions

$$\begin{aligned}
f(s) &= s\sum \frac{s-a}{bc} = \frac{s}{4Rrs}\sum a(s-a) = \frac{4R+r}{2R},\\
g(s) &= \frac{s^2}{abc}\sum \frac{(s-a)(s-b)}{c}\\
&= \frac{s^2}{4Rrs}\cdot \frac{s^2r^2+(r^2+4Rr)^2}{4Rrs} = \frac{s^2+(4R+r)^2}{16R^2},\\
h(s) &= \frac{s^3(s-a)(s-b)(s-c)}{a^2b^2c^2} = \frac{s^2}{16R^2}
\end{aligned}$$

are strictly increasing on $[s_1, s_2]$.

From Theorems 2 and 3, it follows that there exists a unique function $u\colon [s_1, s_2] \to R$ which is differentiable and increasing on (s_1, s_2), $u(s) = \sum \sin \frac{A}{2}$, such that $u(s_1) \le u(s) \le u(s_2)$. We have

$$\begin{aligned}
u(s_1) &= \sqrt{\frac{R+r-d}{R-d}}\left(\sqrt{\frac{R-r-d}{R-d}}+\sqrt{2}\right),\\
u(s_2) &= \sqrt{\frac{R+r+d}{R+d}}\left(\sqrt{\frac{R-r+d}{R+d}}+\sqrt{2}\right).
\end{aligned}$$ □

Corollary 3. *In any triangle ABC, the following inequalities hold:*

$$\begin{aligned}
\frac{R-d}{r} + 4\sqrt{\frac{R-d}{2(R-r-2d)}} &\le \frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}\\
&\le \frac{R+d}{r} + 4\sqrt{\frac{R+d}{2(R+r-2d)}}.
\end{aligned}$$

Proof. We have $\sum \frac{1}{\sin \frac{A}{2}} = \sum \sqrt{\frac{bc}{(s-b)(s-c)}}$. From Lemma 1 we have

$$\begin{aligned}
f(s) &= \sum \frac{bc}{(s-b)(s-c)} = \frac{s^2+r^2-8Rr}{r^2},\\
g(s) &= \sum \frac{bc}{(s-b)(s-c)} \cdot \frac{ab}{(s-a)(s-b)}\\
&= \frac{abc}{\prod(s-a)} \sum \frac{b}{s-b} = \frac{4R(4R-2r)}{r^2},\\
h(s) &= \sqrt{\left(\frac{abc}{\prod(s-a)}\right)^2} = \frac{4R}{r}.
\end{aligned}$$

It results that f, g and h are three increasing functions on the interval $[s_1, s_2]$, so we are under the conditions of Theorem 2. From Theorems 2 and 3 it follows that $u(s_1) \le u(s) \le u(s_2)$, where $u(s_1)$ and $u(s_2)$ appear in the inequality from the statement. □

Corollary 4. *In any triangle ABC, the following inequalities hold:*

$$\begin{aligned}
&\sqrt{\frac{R+d}{R+r+d}}\left(\sqrt{\frac{R+r}{R-r+d}}+2\sqrt{2}\right)\\
&\le \sum \frac{1}{\cos \frac{A}{2}} \le \sqrt{\frac{R-d}{R+r-d}}\left(\sqrt{\frac{R-d}{R-r-d}}2+2\sqrt{2}\right).
\end{aligned}$$

Proof. We have $\sum \frac{1}{\cos \frac{A}{2}} = \sqrt{\frac{bc}{s(s-a)}}$. From Lemma 1, it follows that

$$\begin{aligned}
f(s) &= \sum \frac{bc}{s(s-a)} = \frac{s^2+(4R+r)^2}{s^2},\\
g(s) &= \frac{abc}{s^2} \sum \frac{a}{(s-b)(s-c)} = \frac{4R(8R+2r)}{s^2},\\
h(s) &= \sqrt{\frac{(abc)^2}{s^2 \prod(s-a)}} = \frac{4R}{s}.
\end{aligned}$$

So we have $f'(s) < 0$, $g'(s) < 0$ and $h'(s) < 0$.

By Theorems 2 and 3, there exists a unique decreasing function on $[s_1, s_2]$. So we have $u(s_2) \le u(s) \le u(s_1)$. After we calculate $u(s_1)$ and $u(s_2)$, we obtain the inequality from the statement. □

Corollary 5. *In any triangle ABC, the following inequalities hold:*

$$\frac{2R-r+2d}{\sqrt{(R+d)^2-r^2}} \le tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2} \le \frac{2R-r-2d}{\sqrt{(R-d)^2-r^2}}.$$

Proof. We have $\sum \frac{1}{tg\frac{A}{2}} = \sum \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, and from Lemma 1 it follows that

$$\begin{aligned}
f(s) &= \sum \frac{(s-b)(s-c)}{s(s-a)} = \frac{(4R+r)^2-2s^2}{s^2},\\
g(s) &= \sum \frac{(s-a)(s-b)(s-c)}{s(s-c)s(s-b)} = \frac{s^2-2r^2-8Rr}{s^2},\\
h(s) &= \frac{r}{s},
\end{aligned}$$

with the derivatives

$$f'(s) = -\frac{2(4R+r)^2}{s^3},\quad g'(s) = \frac{4r(4R+r)}{s^3},\quad h'(s) = -\frac{r}{s^2}.$$

As $f'(s)<0$, $g'(s)>0$ and $h'(s)<0$ for $s\in[s_1,s_2]$, we cannot apply Theorem 2. We will prove that

$$f'(s)\big(u^2(s)-f(s)\big)+2g'(s)+4h'(s)u(s)<0. \tag{7}$$

But

$$u^2(s)-f(s)\ge 6\sqrt[3]{h^2(s)} \text{ or } f'(s)\big(u^2(s)-f(s)\big)\le 6f'(s)\sqrt[3]{h^2(s)},$$

or

$$\begin{aligned}
&f'(s)\big(u^2(s)-f(s)\big)+2g'(s)+4h'(s)u(s)\\
&<6f'(s)\sqrt[3]{h^2(s)}+2g'(s)+4h'(s)u(s).
\end{aligned} \tag{8}$$

As $h'(s)<0$, it follows from (8) that, to prove (7), it suffices to show that

$$\begin{aligned}
6f'(s)\sqrt[3]{h^2(s)}+2g'(s)<0 &\iff g'(s)<-3f'(s)\sqrt[3]{h^2(s)}\\
&\iff \frac{2r}{3(4R+r)}<\sqrt[3]{\frac{r^2}{s^2}}\\
&\iff s^2\le\frac{27}{8}\cdot\frac{(4R+r)^3}{r}.
\end{aligned} \tag{9}$$

But from the well-known Gerretsen inequality, we have

$$s^2 \leq 4R^2 + 4Rr + 3r^2. \tag{10}$$

From (10) it follows that, to prove (9), it suffices to show that

$$4R^2 + 4Rr + 3r^2 \leq \frac{27}{8} \cdot \frac{(4R+r)^3}{r}. \tag{11}$$

Denoting $x = \frac{R}{r}$, the inequality (11) is equivalent with

$$32x^2 + 32x + 24 \leq 27(4x+1)^3,$$

which is true.

We have proved that u is decreasing. So $u(s_2) \leq u(s) \leq u(s_1)$. Calculating $u(s_1)$ and $u(s_2)$ we obtain the stated inequality. □

Corollary 6. *In any triangle ABC the following inequalities hold:*

$$\begin{aligned}\sqrt{\frac{R+r-d}{R-r-d}} \cdot \frac{R+r-d}{r} &\leq ctg\frac{A}{2} + ctg\frac{B}{2} + ctg\frac{C}{2} \\ &\leq \sqrt{\frac{R+r+d}{R-r+d}} \cdot \frac{R+r+d}{r}.\end{aligned}$$

Proof. We have $\sum ctg\frac{A}{2} = \sum \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ and, by Lemma 1,

$$\begin{aligned} f(s) &= \sum \frac{s(s-a)}{(s-b)(s-c)} = \frac{s^2 - 2(r^2+4Rr)}{r^2}, \\ g(s) &= \sum \frac{s^2}{(s-a)^2} = \frac{(4R+r)^2 - 2s^2}{r^2}, \\ h(s) &= \frac{s}{r}. \end{aligned}$$

As $f'(s) > 0$, $g'(s) < 0$ and $h'(s) > 0$ for $s \in (s_1, s_2)$, we cannot apply Theorem 2. If we consider the equality (2) from Theorem 2, we obtain

$$\begin{aligned} &u'(s)\big(2u^3(s) - 2f(s)u(s) - 4h(s)\big) \\ &= 2(g'(s) + 2h'(s)u(s)) + f'(s)\big(u^2(s) - f(s)\big). \end{aligned} \tag{12}$$

From (4) we have that

$$2u^3(s) - 2f(s)u(s) - 4h(s) > 0. \tag{13}$$

It will be sufficient to prove that

$$2(g'(s) + 2h'(s)u(s)) + f'(s)u^2(s) - f'(s)f(s) > 0, \tag{14}$$

But $u^2(s) - f(s) \geq 6\sqrt[3]{h^2(s)}$. It results that

$$\begin{aligned} &f'(s)\big(u^2(s) - f(s)\big) + 2g'(s) + 2h'(s)u(s) \\ &\geq 6f'(s)\sqrt[3]{h^2(s)} + 2g'(s) + 4h'(s)u(s). \end{aligned} \tag{15}$$

We will also prove that

$$6f'(s)\sqrt[3]{h^2(s)} + 2g'(s) > 0 \text{ or } 3f'(s)\sqrt[3]{h^2(s)} > -g'(s), \tag{16}$$

or

$$3 \cdot \frac{2s}{r^2} \cdot \sqrt[3]{\frac{s^2}{r^2}} > \frac{4s}{r^2} \text{ or } s^2 > \frac{8}{27}r^2.$$

This inequality follows from

$$s^2 \geq 16Rr - 5r^2 > \frac{8}{27}r^2.$$

From (15), (16) and as $h'(s) > 0$ for all $s \in [s_1, s_2]$, it follows that the inequality (14) holds. From (12), (13) and (14) it follows that u is an increasing function. After we calculate $u(s_1)$ and $u(s_2)$, we obtain the inequality from the statement. □

Corollary 7. *In any triangle ABC, the following inequalities hold:*

$$\begin{aligned} &\sqrt{\frac{R+r-d}{R-r-d}}\left[\sqrt{8r^2 + (R-d)^2} + \sqrt{(R-d)^2 - r^2}\right] \\ &\leq m_a + m_b + m_c \leq \sqrt{\frac{R+r+d}{R-r+d}}\left[\sqrt{8r^2+(R+d)^2}+\sqrt{(R+d)^2-r^2}\right]. \end{aligned}$$

Proof. As $\sum m_a = \sum \sqrt{\frac{2(b^2+c^2)-a^2}{4}}$, from Lemma 1 we have

$$f(s) = \sum \frac{2(b^2 + c^2) - a^2}{4} = \sum m_a^2 = \frac{3}{2}\big(s^2 - r^2 - 4Rr\big),$$

which is an increasing function, and

$$g(s) = \sum m_a^2 m_b^2 = \frac{9}{16}\Big[\big(s^2 + r^2 + 4Rr\big)^2 - 16Rrs^2\Big].$$

We will prove that g is an increasing function. Indeed,

$$g'(s) = \frac{9}{4}s\big(s^2 + r^2 - 4Rr\big) > 0$$

as $s^2 \geq 16Rr - 5r^2 > 4Rr - r^2$. Then,

$$h(s) = m_a^2 m_b^2 m_c^2 = \frac{1}{4}\Big[s^6 + \big(33r^2 - 12Rr\big)s^4 \\ - \big(33r^4 + 60R^2r^2 + 120Rr^3\big)s^2 - \big(r^2 + 4Rr\big)^3\Big].$$

We next prove that h is an increasing function. Denote $s^2 = t$, $x = 33r^2 - 12Rr$, $y = 33r^4 + 60R^2r^2 + 120Rr^3$, $z = (r^2 + 4Rr)^3$.

We have

$$h\big(\sqrt{t}\big) = \frac{1}{4}\big[t\big(t^2 + xt - y\big) - z\big].$$

Let be $v\colon (0, +\infty) \to \mathbb{R}$, $v(t) = t^2 + xt - y$. We will prove that v is an increasing function:

$$v'(t) = 2t + x > 0 \iff 2s^2 > 12Rr - 33r^2.$$

But $s^2 \geq 16Rr - 5r^2$. It will be sufficient to prove that

$$32Rr - 10r^2 > 12Rr - 33r^2 \text{ or } 20Rr > -23r^2.$$

From Theorems 2 and 3, it follows that there exists a function u which is differentiable and increasing on (s_1, s_2). So we have

$$u(s_1) \leq u(s) \leq u(s_2).$$

If we calculate $u(s_1)$ and $u(s_2)$, we obtain

$$\begin{aligned}
u(s_1) &= \sum \sqrt{\frac{2\left(a_1^2+b_1^2\right)-c_1^2}{4}} \\
&= \sqrt{\frac{R+r-d}{R-r-d}}\left[\sqrt{8r^2+(R-d)^2}+\sqrt{(R-d)^2-r^2}\right], \\
u(s_2) &= \sum \sqrt{\frac{2\left(a_2^2+b_2^2\right)-c_2^2}{4}} \\
&= \sqrt{\frac{R+r+d}{R-r+d}}\left[\sqrt{8r^2+(R+d)^2}+\sqrt{(R+d)^2-r^2}\right]. \qquad \square
\end{aligned}$$

References

[1] Bencze, M. and Dragan, M. “A new refinement of inequality $\sum \sin \frac{A}{2} \leq \sqrt{\frac{4R+r}{2R}}$”. *Scientia Magna* 8.4 (2011), pp. 1–10.

[2] Bencze, M. and Shanhe, W. “Anextension of inequality $\sum \sin \frac{A}{2} \leq \frac{3}{2}$”. *Octogon Mathematical Magazine* 17.2 (2009), pp. 697–699.

[3] Blundon, W. “Inequalities associated with a triangle”. *Canadian Math. Bull* 8 (1965).

[4] Blundon, W. “Problem 1935”. *The Amer. Math. Monthly* 73 (1966), p. 1122.

[5] Dragan, M. “Estimation of the sum of medians in a triangle”. *Rev. Arhimede* 1-2 (2011), pp. 32–34.

[6] Dragan, M., Maftei, I., and Radulescu, S. *Inegalitati matematice (extinderi si generalizari)*. EDP, 2012.

[7] Makovski, A. “Solution to the Problem E1935”. *The Amer. Math. Monthly* 78 (1968), p. 404.

[8] Mitrinovic, D., Pecaric, J., and Volonec, V. *Recent Advances in Geometric inequalities*. Dordrecht: Springer, 1989.

Marius Drăgan
61311 bd. Timişoara Nr. 35, Bl. 0D6, Sc. E, et. 7, Ap. 176, Sect. 6, Bucureşti, Romania
marius.dragan2005@yahoo.com

Mihály Bencze
Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Braşov, Romania
benczemihaly@yahoo.com

Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

October 31, 2021

Elementary Problems

E–89. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In an urn we have balls of 20 different colors. Balls are drawn at random from the urn and placed on a table. What is the minimum number of balls that must be drawn from the urn to ensure that, among them, there are at least 102 that share the same color?

E–90. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all functions $f\colon \mathbb{N}^* \to \mathbb{N}^*$ that for all $x, y \in \mathbb{N}^*$ satisfy

$$\frac{x(f(x)+y)}{yf(x)+1} + \frac{y(f(y)+x)}{xf(y)+1} = \frac{[f(x+y)]^2}{xy+1}.$$

E–91. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Each 1×1 square of a 7×211 rectangle is painted either black or white. Prove that it is possible to choose four rows and four columns of the rectangle so that the sixteen 1×1 squares into which they intersect are painted with the same color.

E–92. *Proposed by Michel Bataille, Rouen, France.* Let m, n be positive integers such that $n \geq m$. Prove that $m+1$ divides $(mn-1) \cdot \binom{n}{m}$.

E–93. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle where a, b, c are the lengths of its sides. Prove that its incenter lies at the point

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c},\ \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

E–94. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all nonnegative integers n and p such that $n^2 + 9n + 1 = 11^p$.

Easy–Medium Problems

EM–89. *Proposed by Michel Bataille, Rouen, France.* Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

$$2(a^2 + b^2 + c^2) \geq a + b + c + 3.$$

EM–90. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_n = 18^n + 20^n$. Find the remainder of a_{722} when divided by 722.

EM–91. *Proposed by Ghiţescu Cosmina, student 8^{th} grade, Bucureşti, Romania.* Let ABC be a triangle with the sides $a > b > c$. If R is the circumradius of $\triangle ABC$, r_a, r_b, r_c are the rays of the excircles, p is the semiperimeter and $a^2 + b^2 + c^2 = 8R^2$, then prove that

$$\frac{3\sqrt{3}}{\sqrt{2}+1}(\sin A + \sin B + \sin C) < \frac{p}{r_a + r_b + r_c - 4R}.$$

EM–92. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Find all the solutions in the set of positive integers of the equation

$$\frac{x^4 + y^3}{x^2 + y} = x + y.$$

EM–93. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let ABC be an isosceles triangle with $AB = BC$. Let M be an interior point such that $\angle AMC = 2 \cdot \angle ABC$. Let K be a point on AM such that $\angle BKM = \angle ABC$. Prove that $4\,KM \cdot MC \leq BK^2$.

EM–94. *Proposed by Toma–Ioan Dumitrescu, Bucharest, Romania.* Let a, b, c be positive reals such that $a+b+c=5$. Prove that

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2+b^2+c^2}{a+b+c}\geq 10.$$

Medium–Hard Problems

MH–89. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let F_n be the n-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{k=1}^{n} \frac{F_k^3}{F_{k+1}} \geq \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \sqrt{\frac{F_k^5}{F_{k+1}}} \right)^2.$$

(Here, the subscripts are taken modulo n).

MH–90. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle. Let N be the center of the nine-point circle of $\triangle ABC$ and $\angle\alpha = \angle BAC$. If AN is parallel to BC, prove that

$$\cos\alpha = \frac{-bc}{2R^2}.$$

MH–91. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* A triplet $u, v, w \in \mathbb{R}[x, y, z]$ is called Barcelonian if there exist polynomials $A, B, C \in \mathbb{R}[x, y, z]$ such that

$$u^{2021}A + v^{2021}B + w^{2021}C = 2021.$$

(a) Is the triplet of polynomials $u = x + 2y + z$, $v = y + 2z + 3$, $w = x + y + z$ Barcelonian?
(b) Is the triplet of polynomials $u = x + 2y + z$, $v = y + 2z + 3$, $w = x + y - z + 4$ Barcelonian?

MH–92. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let A and B be two symmetric $n \times n$ matrices with real entries and such that $A(A - B) = B$. Show that

$$\mathrm{Tr}(B^3(B - A)) \geq \mathrm{Tr}(A^3(B - A)),$$

where Tr stands for the trace.

MH–93. *Proposed by Daniel Vilardell Regué, CFIS, Barcelona-Tech, Barcelona, Spain.* Let F_n be the n-th Fibbonacci number. Prove that for all odd n the following equality holds:

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2} + k}{2k}.$$

MH–94. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be an acute triangle, with $\measuredangle(BAC) = 45°$. Outside the triangle take the points X, Y, Z so that AB and AC are hypotenuses in the isosceles right triangles ABX and AYC and the quadrilateral $ABZC$ is a parallelogram. If $CX \cap BY = \{W\}$, then show that AW and AZ are isogonal lines.

Advanced Problems

A–89. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral

$$\int_0^\infty \frac{\sqrt{x}\ln x}{x^2+x+1}\,\mathrm{d}x.$$

A–90. *Proposed by Michel Bataille, Rouen, France.* For $n \in \mathbb{N}$, let $P_n = \prod_{k=1}^{n}(2k-1)$. Find

$$\ell = \lim_{n\to\infty} \frac{\sqrt[n]{P_n}}{n} \qquad \text{and} \qquad \lim_{n\to\infty}\left(\sqrt[n]{P_n} - n\ell\right).$$

A–91. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be an acute, scalene triangle with orthocenter H and centroid G. Let A_1, B_1, C_1 be the feet of perpendiculars from H on the sidelines BC, CA, AB, respectively. Let G_1 be the centroid of $\triangle A_1B_1C_1$. If HG is parallel to BC, then prove that HG_1 is parallel to B_1C_1.

A–92. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer, and consider the set of 2^{n^2} matrices whose coefficients are either 0 or 1. Let α_n be the proportion of those matrices whose determinant is an odd number. Prove that $\alpha = \lim_{n\to\infty} \alpha_n$ exists and show that

$$\frac{1}{4} < \alpha < \frac{1}{2\sqrt{e}}.$$

A–93. *Proposed by José Pérez Cano, CFIS, BarcelonaTech, Barcelona, Spain.* Let $p > 2$ be a prime number such that $p \equiv 2 \pmod 3$ and let a, b, c, d be positive integers. If p divides $ac - 3bd$ and $ad + bc$, prove that either p divides a and b or p divides c and d.

A–94. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let f be a real function having second derivative in the interval $[x_1, x_2]$ and $f'(x_1) = f'(x_2) = 0$. Prove that there exists at least one point $c \in (x_1, x_2)$ such that

$$\frac{1}{4}\,|f''(c)| \geq \frac{|f(x_2) - f(x_1)|}{(x_2 - x_1)^2}.$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Exponent of primes: a closer look

Navid Safaei

Abstract

In this mathlesson we show how adopting the p-adic valuation would provide a more subtle and much deeper account to solve number theory problems.

1 Introduction

In recent years, many instructors have found that, for teaching elementary number theory, adopting approaches concerning the exponent of primes, that is, p-adic valuation offers rich opportunities and vocabularies to solve many challenging problems. Since these approaches examine the problems under the study with the utmost precision and concession, in many instances, they play the role of the decisive idea.

The rest of the article is divided into four parts: firstly, we provide basic notations and facts, secondly, we start with the introductory problems, thirdly, we continue with intermediate problems that simultaneously need the implementation of other approaches, fourthly, we provide some advanced problems on this topic, and finally, we end this article with concluding remarks.

2 Basic notations and facts

We all know that the ring of integers and the field of rational numbers are Unique Factorization Domains (UFDs). This gives a great and however less attended account for solving number theory problems. Let r be a rational number, we can write $r = p_1^{\alpha_1} \dots p_t^{\alpha_t}$, where $p_1 < \dots < p_t$ are distinct prime numbers and $\alpha_1, \dots, \alpha_t$ are non-zero integers. By $v_{p_i}(r)$ we mean[1] the exponent of p_i in the prime decomposition of r. Therefore, $v_{p_i}(r) = \alpha_i$, $i = 1, \dots, t$, and for each prime $p \notin \{p_1, \dots, p_t\}$, $v_p(r) = 0$. For example, for $r = 175/108 = 2^{-2} \cdot 3^{-3} \cdot 5^2 \cdot 7$, $v_2(r) = -2$, $v_3(r) = -3$, $v_5(r) = 2$, $v_7(r) = 1$. Moreover, $v_{11}(r) = v_{13}(r) = \dots = 0$.

Indeed, $v_p(r)$ is a function from rational numbers to integers, that is, $v_p \colon \mathbb{Q} \to \mathbb{R}$. There are some simple yet interesting properties concerning this new function. You can find some of their preliminary properties in Proposition 1. The proofs of all of them are so easy, but the reader can find good materials about them in the works of Andreescu and Dospinescu [1, 2], and Andreescu, Mushkarov, and Dospinescu [3].

Proposition 1. *The function $v_p(\cdot)$ satisfies the following properties, for every prime number p:*

i) $v_p(ab) = v_p(a) + v_p(b)$.
ii) $v_p(a/b) = v_p(a) - v_p(b)$.
iii) If a and b are two integers such that a divides b, then, for each prime number p, $v_p(a) \leq v_p(b)$.
iv) If a and b are two integers such that a does not divide b, then there is a prime number p such that $v_p(a) > v_p(b)$.
v) If r is a rational number such that there is a positive integer $n > 1$ and a rational number s such that $r = s^n$, then, for each prime number p, $v_p(r)$ is divisible by n.
vi) $v_p(\gcd(a,b)) = \min\{v_p(a), v_p(b)\}$ *and* $v_p(\mathrm{lcm}(a,b)) = \max\{v_p(a), v_p(b)\}$.
vii) If $\gcd(a,b,c) = 1$, then, for each prime p, $v_p(a)v_p(b)v_p(c) = 0$.

[1] The abbreviation "v" comes from the term "valuation". This term is rooted in the "p-adic" number fields. See, for example, [4].

Thus far, it seems that the function $v_p(\cdot)$ has properties in common with the logarithm function. But it also has an important property for the addition of integers.

Proposition 2. *Let p be a prime number and c, d be integers such that $\gcd(p, cd) = 1$. Then,*

$$v_p(ac + bd) \geq \min\{v_p(a), v_p(b)\}.$$

Proof. Writing $a = p^{v_p(a)}x$, $b = p^{v_p(b)}y$, $\gcd(p, xy) = 1$. It follows that

$$ac + bd = cxp^{v_p(a)} + dyp^{v_p(b)}.$$

Without loss of generality, assume that $v_p(a) \geq v_p(b)$ then $ac + bd = p^{v_p(a)}\big(cx + dyp^{v_p(b)-v_p(a)}\big)$. Whence, if $v_p(a) > v_p(b)$ then $v_p(ac + bd) = v_p(a) = \min\{v_p(a), v_p(b)\}$. Further, if $v_p(a) = v_p(b)$ then $ac + bd = p^{v_p(a)}(cx + dy)$, whence, there is a possibility that $cx + dy$ is divisible by p. Whence, $v_p(ac + bd) \geq v_p(a) = \min\{v_p(a), v_p(b)\}$. This completes our proof. □

Remark 1. *For example, we have $v_2(32) = 5$ and $v_2(96) = 5$, but $v_2(5 \cdot 96 + 32) = v_2(16 \cdot 32) = 9 > \min\{v_p(32), v_p(96)\}$. The reader must be more careful in the case $v_p(a) = v_p(b)$, otherwise, it may lead to a somewhat unsatisfactory result.*

3 Introductory problems

The basic properties of the function $v_p(\cdot)$ were spelled out in the last section. Now, we are ready to work on some introductory problems.

Problem 1. *Vlatka[2] envisaged natural numbers a, b, c and intends to find natural numbers x, y, z such that*

$$a = \operatorname{lcm}(x, y), \quad b = \operatorname{lcm}(x, z), \quad c = \operatorname{lcm}(z, y).$$

It turned out that x, y, z are unique. Vlatka told Patricija[3] about that and only told her a, b. Prove that Patricija can find c.

[2] A Croatian name.
[3] A Croatian name.

Solution. Let p be a prime dividing xyz, $v_p(x) \geq v_p(y) \geq v_p(z)$. If $v_p(y) > 0$, then we can change $v_p(z)$ in the range from 0 to $v_p(y)$ and a, b, c would not change. Hence, $v_p(y) = v_p(z) = 0$. That is, x, y, z are pairwise coprime. Hence, $a = xy$, $b = xz$, $c = yz$. If Patricija has a, b then $c = yz = \operatorname{lcm}(a,b)/\gcd(a,b)$. □

For the next problem, we need a basic consideration, that is, if for example $a^x = b^y$ for some positive integers a, b and x, y, then the set of primes dividing a, b are the same. Although this is very fundamental, it will lead to some interesting results.

Problem 2. *Let a, b be distinct positive integers such that $a^{9a} = b^{2b}$. Find the minimum possible values of $b + 2a$.*

Solution. It is clear that $a < b$. Therefore, $9a > 2b$. Moreover, the prime divisors of a, b are the same. Examining the exponent of p in both sides, we find that

$$9a \cdot v_p(a) = 2b \cdot v_p(b).$$

Then for each p, $v_p(a) < v_p(b)$. Thus, a divides b. Writing $b = ac$ for some positive integer $c > 1$, then

$$a^{9a} = (ac)^{2ac},$$

Hence, $a^9 = (ac)^{2c}$, that is, $a^{9-2c} = c^{2c}$. Therefore, $2c < 9$ yields $c = 2, 3, 4$.

If $c = 2$, then $a^5 = 2^4 = 16$, and this leads us to a contradiction. If $c = 3$, then $a^3 = 3^6$, thus $a = 9$, $b = 27$. If $c = 4$, then $a = 4^8 = 2^{16}$ and $b = 2^{18}$. Hence, the answer is 45. □

Problem 3 (Belarussian TST, 2018). *Let a, b, c be positive integers such that $a^b \cdot b^c = c^a$. Then,*

i) Prove that any prime divisor of a, divides b as well.
ii) Solve the equation under the condition $b \geq a$.

Solution. i) Assume for a contradiction that there is a prime p that divides a but does not divide b. Let $v_p(a) = \alpha$, $a = p^\alpha a_0$, $v_p(c) = \gamma$, $a = p^\gamma c_0$. Then, $v_p\big(a^b \cdot b^c\big) = v_p\big(a^b\big) = b\alpha$, and

$v_p(c^a) = a \cdot \gamma$. Yielding to the fact that $p^\alpha a_0 \gamma = b\alpha$. Since $p^\alpha > \alpha$, we find that α could not divide p^α. Thus, $\gcd(p, b) > 1$.

ii) Note that $c^a = a^b \cdot b^c > a^b \geq a^a$, hence $c > a$. Moreover,

$$c^a = a^b \cdot b^c \geq a^a \cdot a^c > a^a \cdot a^a = (a^2)^a.$$

Therefore, $c > a^2$. Suppose now $c \geq a^m$, $m \geq 2$. Then,

$$c^a = a^b \cdot b^c \geq a^a \cdot a^{a^m}, \text{ thus, } c \geq a^{1+a^{m-1}} \geq a^{m+1}.$$

Thus, c would be greater than every exponent of a. This leads us to the contradiction. □

Remark 2. *There are infinitely many a, b, c satisfying the above equation. That is, $a = n^n$, $b = n^{n-1}$, $c = n^n$.*

Problem 4. *Let x, y be positive integers such that $(x - 2y)(1-2y)$ divides $x^2 - 4y + 1$. Prove that $|x - 2y|$ is a perfect square.*

Solution. Let $a = x - 2y \neq 0$ and $b = 2y - 1$. Then, $x^2 - 4y + 1 = a^2 + b^2 + 2ab + 2a$. Then, ab divides $a^2 + b^2 + 2a$. It is easy to find that a is odd. Note that, for any odd prime p,

$$v_p(ab) = v_p(a) + v_p(b) \leq v_p\big(a^2 + b^2 + 2a\big).$$

Note that

$$\begin{aligned} v_p\big(a^2 + b^2 + 2a\big) &\geq \min\{2v_p(a), 2v_p(b), v_p(a)\} \\ &= \min\{2v_p(b), v_p(a)\}. \end{aligned}$$

Now, if $v_p(a) \neq 2v_p(b)$, then $v_p(a^2 + b^2 + 2a) = \min\{2v_p(b), v_p(a)\}$. That is,

$$v_p(a) + v_p(b) \leq \min\{2v_p(b), v_p(a)\}.$$

Now, if $\min\{2v_p(b), v_p(a)\} = 2v_p(b)$, then $2v_p(b) < v_p(a) \leq v_p(a) + v_p(b)$. This is impossible. If now $\min\{2v_p(b), v_p(a)\} = v_p(a)$, then $v_p(a) < 2v_p(b)$, hence $v_p(b) > 0$ and, thus, $v_p(a) < v_p(a) + v_p(b)$. This is again impossible. Hence, $v_p(a) = 2v_p(b)$. This implies that for any odd prime number $v_p(a)$ is even. Further, since a is odd, $v_2(a) = 0$. Then, a is a perfect square. □

Remark 3. *One can write* $x^2 - 4y + 1 = z(x - 2y)(1 - 2y)$, *for some positive integer* z. *Therefore,* $(2y - x)(2yz + 2y + x - z) = (2y - 1)^2$. *Working on* $\gcd(2y - x, 2yz + 2y + x - z)$, *we find that* $2y - x$ *is a perfect square.*

Remark 4. *This problem is indeed the reformulation of some old problems in elementary number theory. Here is one example:* Let a, b be positive integers such that

$$\frac{(a+b)^2 + 4a}{ab}$$

is an integer. Assume that b is odd. Prove that a is a perfect square *(Greece 2018).*

In the next problems, we shall use the divisibility implications of the function $v_p(\cdot)$.

Problem 5. *Find all positive integers* x, y *such that* x^2+y^2 *divides* $x^3 - 2y$ *and* $y^3 - 2x$.

Solution. Let $v_p(x) = m$, $v_p(y) = n$. Assume $m \geq n \geq 0$. Then, $v_p(x^3 - 2y)$, $v_p(y^3 - 2x) \geq 2n$. That is, $v_p(2y) \geq 2n$. Since $v_p(2y) \leq 1 + n$, we find that $n = 0$ or $n = 1$, $p = 2$. Further,

$$x\big(x^2 + y^2\big) - \big(x^3 - 2y\big) = y(xy + 2).$$

Hence, $x^2 + y^2$ divides $y(xy + 2)$. If $n = 0$, then gcd(x, y)$= 1$ and $x^2 + y^2$ divides $xy + 2$. Therefore,

$$x^2 + y^2 - xy = (x - y)^2 + xy \leq 2.$$

If $n = 1$, then $\gcd(x, y) = 2$, $x = 2a$, $y = 2b$. Hence, $a^2 + b^2$ divides $2a^3 - b$ and $2b^3 - a$. That is, $a^2 + b^2$ divides $2a(a^2 + b^2) - (2a^3 - b) = b(2ab + 1)$. Since $\gcd(a, b) = 1$, then $a^2 + b^2$ divides $2ab + 1$. Hence, $(a - b)^2 \leq 1$. Thus, the answer is $(x, y) = (2k, 2k + 2), (2k + 2, 2k)$, for some positive integer k. $\square$

Problem 6. *Find all integers* (x, y) *such that* x^2+y^2 *divides* x^3+y *and* $y^3 + x$.

Solution. Let $v_p(\gcd(x,y)) = r$. Then, p^{2r} must divide $x^3 + y$ and $y^3 + x$. However, either $v_p(x^3 + y)$ or $v_p(y^3 + x)$ is r. Thus, $\gcd(x,y) = 1$. Hence, $x^2 + y^2$ divides $x^3 + y - x(x^2 + y^2) = y(xy - 1)$. That is, $xy - 1$ is divisible by $x^2 + y^2$. But $|xy - 1| \leq 1 + |xy| \leq x^2 + y^2$. □

In some cases, we firstly need a reformulation and then using the exponent function. Please note the following example.

Problem 7. *Find all prime numbers p such that there are positive integers x and y such that*

$$\frac{1}{x} + \frac{1}{y} - \frac{1}{x+y} = \frac{1}{p}.$$

Solution. We may rewrite the original equation as $p(x^2 + xy + y^2) = xy(x + y)$. If $p > 2$ and at least one of x, y is odd, then the left side is even while the right side is odd. That is, $x = 2^a z$, $y = 2^b t$, $\gcd(2, zt) = 1$. Then, if $a > b$ we have $v_2(x^2 + xy + y^2) = 2a$, $v_2(xy(x + y)) = 2a + b$. This is impossible. Hence, $a = b$. Then, $v_2(x^2 + xy + y^2) = 2a$. But $v_2(xy(x + y)) \geq 2a + b$, which is a contradiction.

If $p = 2$, then $x^2(x - 2) + y^2(y - 2) = 2xy$. If $x, y > 2$, assume $x \geq y$. Let $y \geq 3$. Then,

$$x^2(x - 2) + y^2(y - 2) \geq x^2 + y^2 \geq 2xy.$$

Then, $x = y = 3$. If $y = 1, 2$ we arrive at a contradiction. □

In the next problem, we need a more detailed look at $v_p(ac + bd)$. Before this look, we cannot advance a decisive argument. Another family of problems that could easily be solved through the exponents of primes is problems about the sum of some rational numbers that eventually become an integer.

Problem 8 (Austrian Olympiad, 2018). *Let a, b, c be integers such that $\frac{ac}{b} + \frac{bc}{a} + \frac{ba}{c}$ is integer. Prove that $\frac{ac}{b}$, $\frac{bc}{a}$ and $\frac{ba}{c}$ are all integers.*

Solution 1. Assume for a contradiction that at least one of $\frac{ac}{b}$, $\frac{bc}{a}$, $\frac{ba}{c}$ is not integer. Then, there is a prime p that divides abc and whose exponent in the denominator is higher than its exponent in the numerator. Assume then $v_p(a) \leq v_p(b) \leq v_p(c)$. Then, it follows that p can only be problematic in the case of $\frac{ba}{c}$, that is, $v_p(c) > v_p(b) + v_p(a)$. Now, rewrite $\frac{ac}{b} + \frac{bc}{a} + \frac{ba}{c} = \frac{a^2b^2+b^2c^2+c^2a^2}{abc}$. Note

$$\begin{aligned}(v_p(a^2b^2), v_p(a^2c^2), v_p(c^2b^2)) = (&2v_p(a) + 2v_p(b),\\ &2v_p(a) + 2v_p(c),\\ &2v_p(c) + 2v_p(b)).\end{aligned}$$

According to our assumption,

$$2v_p(c) + 2v_p(b) \geq 2v_p(a) + 2v_p(c) > 2v_p(a) + 2v_p(b).$$

Hence, $v_p(a^2b^2 + b^2c^2 + c^2a^2) = 2v_p(a) + 2v_p(b)$. On the other hand, $v_p(abc) = v_p(a) + v_p(b) + v_p(c)$.

Since $\frac{ac}{b} + \frac{bc}{a} + \frac{ba}{c}$ is an integer, it follows that

$$\begin{aligned}v_p\big(a^2b^2 + b^2c^2 + c^2a^2\big) &= 2v_p(a) + 2v_p(b)\\ &\geq v_p(abc) = v_p(a) + v_p(b) + v_p(c).\end{aligned}$$

It follows that

$$v_p(a) + v_p(b) \geq v_p(c).$$

We have thus reached a contradiction. □

Solution 2. Assume $x = \frac{ba}{c}$, $y = \frac{bc}{a}$, $z = \frac{ac}{b}$, $p = x + y + z$. Then, $q = xy + yz + zx = a^2 + b^2 + c^2$, $r = xyz = abc$ are both integers. Hence, the polynomial $(t - x)(t - y)(t - z) = t^3 - pt^2 + qt - r$ has integer coefficients and rational roots. Since it is monic, its roots are integers. □

Remark 5. *Here is a similar problem, that you can solve as a simple exercise:* Let a, b, c, d be non-zero integers such that $\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}$, $(\frac{a}{b} + \frac{b}{a})(\frac{c}{d} + \frac{d}{c})$ are integers. Prove that $|a| = |b| = |c| = |d|$.

Problem 9 (Czech and Slovak Math Olympiad, 2018). *Let a, b, c be positive integers, such that they are sides of a non-degenerate triangle and $\gcd(a, b, c) = 1$. It is known that $\frac{a^2+b^2-c^2}{a+b-c}$, $\frac{b^2+c^2-a^2}{b+c-a}$ and $\frac{c^2+a^2-b^2}{c+a-b}$ are all integers. Prove that $(a+b-c)(b+c-a)(c+a-b)$ is either a square or twice a square.*

Solution. Let $z = a+b-c$, $x = b+c-a$ and $y = c+a-b$. Hence, $a = \frac{y+z}{2}$, $b = \frac{x+z}{2}$ and $c = \frac{y+x}{2}$. Then,

$$a^2 + b^2 - c^2 = \frac{1}{2}(z(x+y+z) - xy).$$

Hence, z must divide xy, x must divide zy, and y must divide xz. If $v_2(xyz)$ is even, then we can put it away. On the other hand, if it is odd, then write it as 2 times a square. Therefore, we can again put it away. Consider an odd prime p that divides xyz. Note that, if p divides all of x, y, z, then it must divide a, b and c, which is impossible. Assume now $v_p(x) \leq v_p(y) \leq v_p(z)$. It follows that $v_p(x) = 0$. It follows from the fact z divides xy that

$$v_p(z) \leq v_p(y).$$

Combining this with $v_p(y) \leq v_p(z)$, it follows that $v_p(y) = v_p(z)$. This shows that $v_p(xyz)$ is even. Hence, xyz is either a square or twice a square. □

Problem 10 (Mongolian Olympiad, 2019). *Let x, y, z be positive integers such that $A = \frac{xy^2}{z} + \frac{y^3z^4}{x} + \frac{z^5x^6}{y}$ is an integer. Prove that each summand in the preceding sum is an integer.*

Solution. Assume, for example, that $\frac{xy^2}{z}$ is not an integer. Then, there is a prime p dividing z such that $v_p(z) > v_p(xy^2)$. Writing $y = p^{v_p(y)}b$, $z = p^{v_p(z)}c$ and $x = p^{v_p(x)}a$, we have $\gcd(abc, p) = 1$. We find that

$$v_p(z) > v_p(x) + 2v_p(y).$$

Now, consider the sum $ab\left(\frac{y^3z^4}{x} + \frac{z^5x^6}{y}\right) = \frac{y^3z^4ab}{x} + \frac{z^5x^6ab}{y} = \frac{y^3z^4b}{p^{v_p(x)}} + \frac{z^5x^6a}{p^{v_p(y)}}$. By the above assumption, we find that $ab\left(\frac{y^3z^4}{x} + \frac{z^5x^6}{y}\right)$ is an integer. Therefore, $abA - ab\left(\frac{y^3z^4}{x} + \frac{z^5x^6}{y}\right)$ would be an integer.

Thus, $\frac{xy^2ab}{z}$ is an integer. Hence,

$$v_p(z) \leq v_p(x) + 2v_p(y).$$

We have thus reached a contradiction. $\square$

The next problems can show the far-reaching implications of this approach. To put it much more simply, it can solve some advanced problems like the next ones.

Problem 11 (adapted after a Serbian TST, 2005). *Find all integers a, b, c such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$.*

Solution. Rewrite it as follows: $\frac{a^2c+b^2a+c^2b}{abc} = 3$. We can assume that $\gcd(a, b, c) = 1$. Then, if p is a prime that divides abc, it must divide exactly two of them. Then,

$$v_p\big(a^2c\big) = 2v_p(a), \quad v_p\big(c^2b\big) = v_p(b), \quad v_p\big(b^2a\big) = 2v_p(b) + v_p(a).$$

Hence, $\min\{2v_p(a), v_p(b), 2v_p(b) + v_p(a)\} = 2v_p(a)$ or $v_p(b)$. In either case, it is less than $v_p(a)+v_p(b) = v_p(abc)$. Hence, $2v_p(a) = v_p(b)$. By the same argument, we can find that

$$a = A^2B, \qquad b = B^2C, \qquad c = C^2A,$$

where $\gcd(A, B) = \gcd(A, C) = \gcd(B, C) = 1$. Hence, we find that $A^3 + B^3 + C^3 = 3ABC$. Then,

$$A + B + C = 0.$$

That is, $a = A^2B$, $b = -B^2(A + B)$ and $c = (A + B)^2A$. $\square$

Problem 12 (Tournament Of Towns, early '80s). *If a, b, c are integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$, $\frac{a}{c} + \frac{b}{a} + \frac{c}{b}$ are both integers, prove that*

$$|a| = |b| = |c|.$$

Solution. By the same argument we find that $a = A^2B$, $b = B^2C$, $c = C^2A$ and

$$a = B^2A, \qquad b = C^2B, \qquad c = A^2C.$$

Hence $A = B = C = 1, -1$. That is,

$$|a| = |b| = |c|. \qquad \square$$

Problem 13. *Let a, b, c be integers such that $a^2c+c^2b+b^2a=0$. Find all possible values of $a^2b+b^2c+c^2a$.*

Solution. We can assume that $\gcd(a,b,c)=1$. Then, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{a^2c+c^2b+b^2a}{abc}=0$. Let p be a prime dividing abc. We can assume that p does not divide c, hence p only divides a and b. Writing

$$a=p^{v_p(a)}x, \qquad b=p^{v_p(b)}y, \qquad \text{we have} \quad \gcd(p,xy)=1.$$

Now, we shall use the following observation.

If $X+Y+Z=0$, $v_p(X)=\alpha$, $v_p(Y)=\beta$ and $v_p(Z)=\gamma$, then the smallest two of α, β, γ are equal.

Then, $v_p(a^2c)=2v_p(a)$, $v_p(c^2b)=v_p(b)$, $v_p(b^2a)=2v_p(b)+v_p(a)$. According to our lemma, either $v_p(a)=2v_p(b)$ or $v_p(b)=2v_p(a)$. Continuing this way, we find that there are positive integers A, B, C such that $a=A^2B$, $b=B^2C$, $c=C^2A$. Hence,

$$a^2c+c^2b+b^2a=(ABC)^2\big(A^3+B^3+C^3\big).$$

Thus, $A^3+B^3+C^3=0$. Hence, by Fermat's last theorem, at least one of A, B, C is zero. Hence, two of a, b, c become zero. Thus, $a^2b+b^2c+c^2a=0$. □

Problem 14 (Navid Safaei, Mathematical Reflections). *Let x, y, z be positive integers such that $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=6$ and $\gcd(z,x)=1$. Find the maximal value of $x+y+z$.*

Solution. Since $\gcd(z,x)=1$, we deduce that $\gcd(x,y,z)=1$. Thus, for each prime number p there is at least one of x, y, z that p does not divide. Assume that p is an arbitrary prime that divides xyz, let $x=p^{v_p(x)}a, y=p^{v_p(y)}b$ where $\gcd(p,ab)=1$. Further, assume that $\gcd(p,z)=1$. Now, rewrite the left side of our equation as

$$\frac{x^2z+z^2y+y^2x}{xyz}.$$

Note that $v_p(xyz)=v_p(x)+v_p(y)$. Thus, we should have

$$v_p\big(x^2z+z^2y+y^2x\big)\geq v_p(x)+v_p(y).$$

Note that $v_p(x^2z) = 2v_p(x)$, $v_p(z^2y) = v_p(y)$, and $v_p(y^2x) = 2v_p(y) + v_p(x)$. It is easy to deduce that, if $2v_p(x) \neq v_p(y)$, then $v_p(x^2z + z^2y + y^2x) = v_p(y)$ or $2v_p(x)$. In the former case, the above inequality clearly fails; in the later case, we should have $v_p(y) > 2v_p(x)$, therefore,

$$v_p\big(x^2z + z^2y + y^2x\big) = 2v_p(x) < v_p(y) < v_p(x) + v_p(y).$$

The inequality fails again. Hence, $v_p(y) = 2v_p(x)$. By the same argument, for each prime p that divides y and z we have $2v_p(y) = v_p(z)$, and $2v_p(z) = v_p(x)$ for each prime p that divides x and z. Hence, there are positive integers A, B, C such that $x = A^2B$, $y = B^2C$ and $z = C^2A$. This implies that

$$\frac{x^2z + z^2y + y^2x}{xyz} = \frac{A^3 + B^3 + C^3}{ABC} = 6.$$

Since $\gcd(x, z) = 1$, we find that $A = 1$, that is, $x = B$, $y = B^2C$, $z = C^2$. That is,

$$B^3 + C^3 + 1 = 6BC.$$

Thus, $B^3 + C^3 + 8 - 6BC = 7$. Hence,

$$(B + C + 2)((B - C)^2 + (B - 2)^2 + \big(C - 2)^2\big) = 14.$$

That is, $B+C+2 = 7$, $(B-C)^2+(B-2)^2+(C-2)^2 = 2 = 1+1+0$, which yields

$$(B, C) = (2, 3), (3, 2).$$

Hence, $(x, y, z) = (2, 12, 9), (3, 18, 4)$. That is, $x + y + z = 23, 25$. The answer is 25. $\square$

4 Intermediate problems

We before long arrived at some problems that simultaneously need good knowledge about exponent of primes and knowledge about other things in number theory. After solving these problems, the reader finds that this approach has precisely become a pillar in solving elementary number theory problems.

Problem 15 (Mongolian Olympiad, 2019). *Let m, n be positive integers such that $B = n^3 + 4mn^2 + 3m^2n + 4m^3 - 1$ divides $A = m^4 + 3m^2n^2 + n$. Show that B is divisible by the fourth power of a prime.*

Solution. Let $B = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$. It suffices to show that there is an index i such that $\alpha_i \geq 4$. Assume to the contrary that $\alpha_i < 4$ for all i. Then, $B \mid C^3$, where C is the product of all primes dividing B. Observe that $A + nB = (m+n)^4$. Thus, $C \mid (m+n)^4$. Therefore, $C \mid m+n$, implying that

$$C^3 \mid (m+n)^3.$$

This yields

$$B = n^3 + 4mn^2 + 3m^2n + 4m^3 - 1?(m+n)^3.$$

We have thus reached a contradiction. □

Remark 6. *A similar problem was proposed for the Canadian Mathematical Olympiad 2019.* Let a, b be positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that the latter expression is divisible by the cube of an integer greater than 1.

Problem 16 (Japanese problem). *Find all positive integers $x, y, z \leq 500$ with $\gcd(x, y, z) = 1$ such that $\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^3}{x}$ is an integer.*

Solution. Let $y = p^{v_p(y)}b$, $z = p^{v_p(z)}c$ and $x = p^{v_p(x)}a$, so $\gcd(abc, p) = 1$. Rewrite $\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$ in the following form:

$$\frac{p^{3v_p(x)+v_p(z)}a^4c + \dots}{p^{v_p(x)+v_p(y)+v_p(z)}abc}.$$

Since $\gcd(x, y, z) = 1$, assume that $v_p(z) = 0$. Then, the above expression could be written as

$$\frac{p^{3v_p(x)}a^4c + p^{3v_p(y)+v_p(x)}b^4a + p^{v_p(y)}c^4b}{p^{v_p(x)+v_p(y)}abc}.$$

Now, if $v_p(y) < 3v_p(x)$, then

$$v_p\big(p^{3v_p(x)}a^4c + p^{3v_p(y)+v_p(x)}b^4a + p^{v_p(y)}c^4b\big) = v_p(y).$$

This leads us to a contradiction. If $v_p(y) > 3v_p(x)$, then

$$v_p\big(p^{3v_p(x)}a^4c + p^{3v_p(y)+v_p(x)}b^4a + p^{v_p(y)}c^4b\big) = 3v_p(x).$$

This is impossible and hence we arrived at a contradiction. Hence, $v_p(y) = 3v_p(x)$. Pursing this way, we find that there are positive integers A, B, C with $\gcd(A,B,C) = 1$ such that $x = C^3A$, $y = A^3B$, $z = B^3A$. That is,

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^3}{x} = \frac{A^7 + B^7 + C^7}{ABC}.$$

Since $C^3A, A^3B, B^3A \leq 500$, it follows that $\mathrm{Max}\{A,B,C\} \leq 7$. The rest is handed over to the reader. $\square$

Problem 17. *Let m, n, k be positive integers such that n is odd, prove that the following number is not an integer:*

$$\frac{1}{m} + \frac{1}{m+n} + \ldots + \frac{1}{m+kn}.$$

Solution. Let $2^t \leq k < 2^{t+1}$, where t is a non-negative integer. Further, if $s \geq t+1$ is an integer. We prove that there is unique i, $0 \leq i \leq k$, such that $m + in$ is divisible by 2^s. That is, assume that such a number does not exist. Then, consider the numbers $m+n, m+3n, \ldots, m+(2^t-1)n$. There is exactly one term that is divisible by 2^t. If there is a positive integer r such that $2^{t+1}-1 > r > 2^t-1$ and $m+rn$ is divisible by 2^t, assume $m+ln$ is divisible by 2^t, for some l, $1 \leq l \leq 2^t-1$. If $m+ln = 2^t(2b+1)$, then from $m+rn \equiv m+ln \pmod{2^t}$ we find that $r-l$ is divisible by 2^t. That is, $r = l + 2^ta < 2^{t+1}$. Hence, $a = 1$ and $r - l = 2^t$. Therefore,

$$\begin{aligned} m + rn &= m + \big(l + 2^t\big)n = m + ln + 2^tn \\ &= 2^t(2b+1) + 2^tan = 2^t(2b+n+1). \end{aligned}$$

Hence, $m + rn$ is divisible by 2^{t+1}. But this is impossible. Thus, between the denominators the term with the highest exponent of 2 is unique. Hence, after taking the least common denominators, the exponent of 2 in the denominator is at least one more than the exponent of 2 in the numerator. This implies that the expression is not an integer. $\square$

Problem 18 (APMO, 2017). *We call a rational number r* powerful *if r can be expressed in the form $\frac{n^k}{m}$ for some relatively prime positive integers n, m and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose that there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful.*

Solution. Let $(a, b, c) = \left(\frac{d}{g}, \frac{e}{h}, \frac{f}{i}\right)$, where $\gcd(d, g) = \gcd(e, h) = \gcd(f, i) = 1$. Moreover,

$$a^x + b^y + c^z = \frac{d^x h^y i^z + g^x e^y i^z + g^x h^y f^z}{g^x h^y i^z}.$$

Hence, g^x divides $d^x h^y i^z$ and since $\gcd(d, g) = 1$, it follows that g^x divides $h^y i^z$. Similarly, we find that $h^y | g^x i^z$. Since $def = ghi$, we find that if p is a prime factor of f. Since $\gcd(f, i) = 1$, then p divides gh. Without loss of generality, assume that p divides g. Since g^x divides $h^y i^z$ it follows that p divides h. Therefore, if p divides f, g, h, then p does not divide d, e, i. Hence, $v_p(f) = v_p(g) + v_p(h)$. Further, combining the fact that g^x divides $h^y i^z$ with the fact that p does not divide i, we find $xv_p(g) \leq yv_p(h)$. Further, since $h^y \mid g^x i^z$, we find that $yv_p(h) \leq xv_p(g)$. Hence,

$$xv_p(g) = yv_p(h).$$

This implies that $yv_p(f) = yv_p(h) + yv_p(g) = (x + y)v_p(g)$. Therefore,

$$\frac{v_p(f)}{v_p(g)} = \frac{x + y}{y} = \frac{\frac{x+y}{\gcd(x,y)}}{\frac{y}{\gcd(x,y)}}.$$

Hence, $v_p(f)$ is divisible by $k = \frac{x+y}{\gcd(x,y)} > 1$, for each prime p that divides f. Hence, $f = m^k$ for some positive integer m. That is, $c = \frac{m^k}{i}$ is a powerful number. The same holds true for a, b. □

Before trying the next problem, we need a bit more knowledge about the $v_p(\cdot)$. That is, we already knew $v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$. The last formula has a closed form as follows, that is, $v_p(n!) = \frac{n - s_p(n)}{p-1}$, where $s_p(n)$ is the sum of the digits of n in base p. The latter formula can easily be proven by writing $n = (\overline{a_d \dots . a_0})_p$, where

$a_d \neq 0$, $0 \leq a_i \leq p-1$, $i = 0, \ldots, d$. That is, considering the *base-p* representation of n. Let us see what bearings this fact has.

Problem 19 (China, South Eastern Math Olympiad, 2019). *Given a positive integer m, let $A_l = (4l+1)(4l+2)\ldots(4(5^m+1)l)$. Prove that there are infinitely many l such that $v_5(A_l) = l \cdot 5^m$. Find the least value of l such that the above condition holds.*

Solution. Note that

$$\begin{aligned} v_5(A_l) &= v_5\left(\frac{(4(5^m+1)l)!}{(4l)!}\right) \\ &= v_5((4(5^m+1)l)!) - v_5((4l)!) \\ &= \frac{(4(5^m+1)l) - S_5((4(5^m+1)l))}{4} - \frac{4l - S_5(4l)}{4} \\ &= l \cdot 5^m - \frac{S_5((4(5^m+1)l)) - S_5(4l)}{4}. \end{aligned}$$

Let us denote $4l$ by n. Then, we must prove that there are infinitely many n such that $S_5(n) = S_5((5^m+1)n)$. If n has less than $m+1$ digits in base 5, then

$$\begin{aligned} S_5((5^m+1)n) = S_5(n \cdot 5^m + n) &= S_5(n) + S_5(n \cdot 5^m) \\ &= 2S_5(n) \neq S_5(n). \end{aligned}$$

If n has $m+1$ digits, then we shall prove $n = 5^{m+1} - 1$ satisfies the problem condition. Let us write $n = (a_m \ldots a_0)_5$, where $0 \leq a_i \leq 4$, $i = 0, \ldots, m$, $a_m \neq 0$. Now, assume that the number $n \cdot 5^m + n$ has t times 5 to 1. It is easy to find that $t \leq m+1$. Thus,

$$S_5(n \cdot 5^m + n) = S_5(n) + S_5(n \cdot 5^m) - 4t = 2S_5(n) - 4t.$$

This implies $S_5(n) = 4t$. Now, if $a_0 + a_m \geq 5$, $a_1 = \ldots = a_{t-1} = 4$, for $t < m+1$, then

$$S_5(n) \geq a_0 + a_m + 4(t-1) > 4t.$$

Thus, for this case we should have $t = m+1$, $a_0 + a_m \geq 5$, $a_0 > 0$, $a_1 = \cdots = a_{t-1} = 4$. That is, $n = 5^{m+1} - 1$. Thus, $l = \frac{5^{m+1}-1}{4}$.

Now, we prove that if l satisfies the statement of the problem, then $5^a l$ also satisfies the statement of our problem. This holds since

$$S_5(4(5^m+1)\cdot 5^a l) = S_5((4(5^m+1)l)) = S_5(4l) = S_5(4l\cdot 5^a),$$

and we are indeed done. $\square$

5 Advanced problems

We are now in the route of the last part of our journey. As you have seen, this approach provides a wonderful toolbox to solve elementary number theory problems. In this part, we need to be more creative and think out of the box. For the first problem, we only need to consider some powers of primes.

Problem 20. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that, for any two positive integers a and b, $f(ab)$ divides $\max\{f(a), b\}$. Prove that there are infinitely many positive integers n such that $f(n) = 1$.*

Solution. Assume that $f(1) < p < q$ are two prime numbers. Then we shall inductively prove that, for all $r \geq 1$, $f(p^r)$ divides p. For $r = 1$, $f(p)$ divides $\max\{f(1), p\} = p$. Hence, either $f(p) = 1$ or $f(p) = p$. Assume that the statement holds true for all positive integers less than or equal to r. Now, $f(p^{r+1})$ divides $\max\{f(p^r), p\}$. Since $f(p^r)$ divides p, we are done.

Now, take a positive integer s, $p^s > q$. Setting $n = qp^s$, then $f(n)$ divides $\max\{p^s, f(q)\}$ and $\max\{f(p^s), q\}$. Note that $p^s > q \geq f(q)$ and $q > p \geq f(p^s)$. Thus,

$$\max\{p^s, f(q)\} = p^s, \qquad \max\{f(p^s), q\} = q.$$

Hence, $f(n)$ simultaneously divides p^s and q, so $f(n) = 1$. $\square$

The next problem defines an interesting function.

Problem 21 (Navid Safaei). *Let $p \geq 5$ be a prime number. Find all functions $f\colon \mathbb{Z} \to \mathbb{N}$ such that*

i) *For each* $a_i \neq 0, 1$, $i = 1, \ldots, p-2$, *such that* p *does not divide* $a_i - 1$,
$$f(a_1) + \ldots + f(a_{p-2}) = f(a_1 \cdots a_{p-2});$$
ii) *If* $a \equiv b(\bmod p)$, *then* $f(a) = f(b)$;
iii) *For each positive integer* n, *there is an integer* l *such that* $f(l) = n$.

Solution. Putting $a_i = -1$, then $(p-2)f(-1) = f(-1)$ implies that $f(-1) = 0$. Putting $a_i = 1 + i$, $i = 1, \ldots, p-2$, then $f(2) + \ldots + f(p-1) = f((p-1)!) = f(-1) = 0$. This implies $f(2) = \ldots = f(p-1) = 0$. Plugging $a_1 = 4$, $a_2 = \ldots = a_{p-2} = 2$, then[4]
$$0 = f\left(2^{p-1}\right) = f(1).$$
Therefore, if $\gcd(a, p) = 1$, then $f(a) = 0$. Assume now p divides n. Letting $a_1 = a$, $a_2 = b$, $ab \equiv -1(\bmod p)$, $a_3 = a_4 = p$, $a_5 = \ldots = -1$ yields $2f(p) = f(p^2)$. Analogously, assume p does not divide $r-1$ and $\gcd(p, r) = 1$. Then, setting $a_1 = r$, $a_2 = p^{k-1}$, $a_3 = p$, $a_5 = \ldots = -1$, we have $f\left(rp^k\right) = f(p) + f\left(p^{k-1}\right)$. Now, through induction on k, we find that $f\left(rp^k\right) = kf(p)$. Now assume that p divides $r-1$, that is, $r = 1+mp$; plugging $a_1 = (1+mp)p^k$, $a_2 = (mp-1)p^k$, $a_3 = a$, $a_4 = b$, $ab \equiv -1(\bmod p)$, $a_5 = \ldots = -1$ yields
$$\begin{aligned} f\left((1+mp)p^k\right) + f\left((1+mp)p^k\right) &= f\left((1+mp)p^k\right) + kf(p) \\ &= f\left((m^2p^2-1)p^{2k}\right) = 2kf(p). \end{aligned}$$
Thus, $f\left((1+mp)p^k\right) = kf(p)$. That is, we find that $f(n) = f(p)v_p(n)$. According to the condition (iii), $f(p) = 1$ and $f(n) = v_p(n)$. □

The next problem needs good combinatorial insight, too.

Problem 22 (Russian problem). *Does there exist some sequence* $a_1, \ldots, a_n, \ldots$ *of natural numbers such that, for each* k, $2^k a_1 \cdots a_k$ *is divisible by* a_{k+1}^k?

[4]Through these lines, we used Wilson's theorem and Fermat's little theorem. I think the reader knows many things about them. Otherwise, see [3].

Solution. If $p \geq 3$ divides a_1 and $v_p(a_1) = n$, then $v_p(a_2) \leq n$, by the same argument

$$2v_p(a_3) \leq v_p(a_1) + v_p(a_2) \leq 2n.$$

Hence, $v_p(a_3) \leq n$. That is, $v_p(a_2), \ldots$ are all at most n.

Hence, each a_k is the product of an integer no more than a_1 multiplied by $2^{f(k)}$. If the sequence $f(1), f(2), \ldots$ is bounded, that is, $f(i) \leq M$ for some M, then for each i, $a_i \leq 2^M a_1$. That is, the sequence $a_1, \ldots, a_n, \ldots$ would be bounded; therefore, some terms would appear infinitely many times in the sequence; impossible.

So there are infinitely many positive integer k such that $f(k+1) > f(i)$ for all $i \leq k$, that is, otherwise, the sequence $f(1), f(2), \ldots$ would be bounded.

Then, a_{k+1}^k is divisible by $2^{kf(k+1)}$. Hence,

$$\begin{aligned} kf(k+1) \leq v_2\big(2^k a_1 \ldots a_k\big) &\leq k + v_2(a_1) + \ldots + v_2(a_k) \\ &\leq k + kf(k+1) - k = kf(k+1). \end{aligned}$$

Hence, the equality case occurs. Thus,

$$f(1) = f(2) = \ldots = f(k) = f(k+1) - 1$$

for infinitely many k.

Thus, for infinitely many k, $f(k) = 1 + f(1)$, that is, $a_k \leq 2a_1$ for infinitely many k. This shows that there is no such sequence. □

The next two problems define two new functions of primes and products of primes. We presented both of them here to prepare you to think recursively, which we need in the last problem.

Problem 23 (Chinese Olympiad, 2020). *Let $P(n)$ be the largest prime divisor of n. Let $a_1 > 1$, $a_{n+1} = a_n + P(a_n)$. Prove that there is at least one perfect square in the sequence.*

Solution 1. Since $P(a_n)$ divides the right side, we find that $P(a_{n+1}) \geq P(a_n)$. We show that the sequence $P(a_n)$ is unbounded. Otherwise, for all $n \geq N$, $P(a_n) = p$. Hence, $a_N = pt$,

$a_{N+i} = p(t+i)$, $i = 0, \ldots$. Setting $i = q - t$, where $q > p$ is a prime, we find that $P(a_{N+q-t}) = q > p$. Let

$$I = \{i \geq 2 : P(a_i) > P(a_{i-1})\}.$$

We find that I is infinite. Order the elements of I by $i_1 = 1 < i_2 < \ldots$. For all $j \geq 1$, $P\big(a_{i_j}\big) = p_j$ and for all $j \geq 2$, $P\big(a_{i_{j-1}}\big) = p_{j-1}$, $a_{i_j} = a_{i_{j-1}} + p_{j-1}$. Then, $a_{i_j} = p_{j-1}p_j m_j$, for some positive integer m_j. Now, we prove that m_j is a monotonically decreasing sequence.

Suppose $j \geq 2$, $i_j \leq k < i_{j+1}$. Then, $P(a_k) = p_j$. Hence,

$$a_k = a_{i_j} + p_j(k - i_j) = p_j(p_{j-1}m_j + k - i_j).$$

Since $p_{j-1}m_j, p_{j-1}m_j+1, \ldots, p_{j-1}m_j+i_{j+1}-i_j-1$ are not divisible by p_{j+1}, we find that

$$i_{j+1} - i_j < p_{j+1}.$$

Hence,

$$m_{j+1} = \frac{a_{i_j+1}}{p_j p_{j+1}} = \frac{a_{i_j} + (i_{j+1} - i_j)p_j}{p_j p_{j+1}} < 1 + m_j.$$

Hence, $m_{j+1} \leq m_j$.

Then, after some point $j \geq j_0$, $m_j = m$. We prove that $m = 1$. If $m \geq 2$, take a prime $q > mp_{j_0}$. Hence, there is $j \geq j_0$ such that $mp_j < q < mp_{j+1}$. Let $i_j \leq k \leq i_{j+1}$, then $a_k = a_{i_j} + (k - i_j)p_j = p_j(p_{j-1}m + k - i_j)$. Further, $a_{1+i_j} = p_j p_{j+1} m$, and the factors of the numbers $p_{j-1}m, p_{j-1}m + 1, \ldots, p_{j+1}m - 1$ do not exceed p_j. But, on the other hand,

$$q \in \{p_{j-1}m, p_{j-1}m + 1, \ldots, p_{j+1}m - 1\}, \quad q > mp_j > p_j,$$

a contradiction, resulting $m = 1$.

Thus, for all $j \geq j_0$, $a_{1+i_j} = p_j p_{j+1}$. Hence,

$$a_{i_j+p_j-p_{j-1}} = p_j p_{j-1} + (p_j - p_{j-1})p_j = p_j^2. \qquad \square$$

Solution 2. Let $p_n = P(a_n)$, $a_n = p_n b_n$. If there is a positive integer n such that $p_n \geq b_n$, then

$$a_{n+p_n-b_n} = a_n + p_n(p_n - b_n) = p_n^2.$$

Thus, assume that $p_n < b_n$ for each n. Now, we shall prove that $b_{n+1} - b_n \leq 1$. That is, since $a_{n+1} = p_n b_n + p_n$, then p_n divides a_{n+1}. Hence, $p_{n+1} \geq p_n$. Therefore, $b_{n+1} = \frac{p_n(1+b_n)}{p_{n+1}} \leq 1 + b_n$. We are done. Now, since a_n is unbounded and $b_n \geq \sqrt{a_n}$, it follows that b_n is also unbounded.

Now, let $p > b_1$ be a prime number. There is a positive integer m such that $b_m > p > b_1$. Since $b_{n+1} - b_n \leq 1$, there is a positive integer k satisfying $b_k = p$, yielding to $a_n = p_k b_k = pp_k$. Then, $p = b_k > p_k$. The latter conclusion contradicts the definition of p_k. We thus far find that there is a positive integer n satisfying $p_n \geq b_n$. This completes our proof. □

Problem 24 (Mongolian Olympiad, 2001). *Let* $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, *define* $\mathrm{rad}(n) = p_1 \dots p_k$, *define* $a_{n+1} = a_n + \mathrm{rad}(a_n)$. *Prove for all* N *the sequence contains* N *consecutive terms of an arithmetic progression.*

Solution. We can prove by induction that $a_n \leq 2^{n-1} a_1$. Let us define $b_n = \mathrm{rad}(a_n)$. It is clear that $b_1 \mid b_2 \mid \dots$ We shall prove that, for each M, there is a row of M equal terms in the sequence b_n. Assume for a contradiction that, for each n, $b_{n+M} > b_n$. That is,

$$b_{2nM} = b_M \frac{b_{2M}}{b_M} \cdots \frac{b_{2nM}}{b_{(2n-1)M}}.$$

We have $2n$ factors that are pair-wise relatively prime and greater than 1. Hence, it would be greater than $(2n)! > n^n$. Thus,

$$2^{2nM} a_1 > a_{2nM} \geq b_{2nM} > n^n.$$

If we set $n = (4a_1)^M$, the above inequality would be absurd! We have thus reached a contradiction. □

Now, we finally arrive at the last problem; a problem that needs several attacks and importantly needs deep insight about the condition of the functions defined.

Problem 25 (China, South Eastern Math Olympiad, 2020). *For a positive integer $n > 1$, define the set S_n as*

$$\mathrm{S}_n = \{p^r : p \mid n, r \in \mathbb{N}, p^r \mid n, r \equiv v_p(n) (\mathrm{mod} 2)\}.$$

Let $f(n)$ be the sum of all elements of S_n, for $n > 1$, and $f(1) = 1$. Let m be a given positive integer and, for all $n > m$, define the sequence $a_1, \ldots$ satisfying

$$a_{n+1} = \max\{f(a_n), f(1 + a_{n-1}), \ldots, f(m + a_{n-m})\}.$$

(i) Prove that there are a, b, $0 < a < 1$, such that, for any positive integer with at least 2 distinct prime divisors, $f(n) < an + b$.
(ii) Prove that the sequence a_n is bounded.
(iii) Prove that there are positive integers N, T such that, for all $n \geq N$, $a_{n+T} = a_n$.

Solution. (i) Let $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$, $p_1 < \ldots < p_t$. Then,

$$f(n) = \sum_{i=1}^{t} \sum_{j=0}^{\lfloor \frac{\alpha_i}{2} \rfloor} p_i^{\alpha_i - 2j}.$$

Thus, $f(n) \leq \sum_{i=1}^{t} p_i^{\alpha_i} \frac{1}{1 - \frac{1}{p_i^2}}$. Now, we shall prove that $f(n) \leq \frac{2}{3}(n + 4)$. Note that $f(n) \leq \sum_{i=1}^{t} p_i^{\alpha_i} \frac{1}{1 - \frac{1}{p_i^2}} \leq \frac{4}{3} \sum_{i=1}^{t} p_i^{\alpha_i}$. It suffices to prove that $\frac{2}{3}\left(2 \sum_{i=1}^{t} p_i^{\alpha_i} - \prod_{i=1}^{t} p_i^{\alpha_i}\right) \leq \frac{8}{3}$. It is easy to find that, for all $t \geq 2$,

$$\frac{2}{3}\left(2 \sum_{i=1}^{t} p_i^{\alpha_i} - \prod_{i=1}^{t} p_i^{\alpha_i}\right) \leq \frac{2}{3}\left(4t - 2^t\right).$$

For $t = 2, 3$, the right side is indeed $\frac{8}{3}$, and for all $t > 3$ the right side is negative. We are done. Now let $M_n = \max\{a_1, \ldots, a_n\}$. It is easy to verify that M_n is an increasing sequence. Then, for all $n \geq 2m + 8$, if there is an index $j \in \{1, 2, \ldots, m\}$ such that $n+j$ has at least two distinct prime divisors, it follows that $f(n+j) \leq \frac{2}{3}(n+m+4) \leq n$.

(ii) Assume that the sequence has no upper bound. Then, there is a positive integer N such that, for all $n > N$, $M_n > 2m+8$. Thus, there are $n_1 < n_2 < \dots$ such that all of them satisfy $M_{n_{i+1}} > M_{n_i}$. Therefore, for each $j = 1, \dots$ there must be a prime number p_j and a positive integer α_j such that $T_j = M_{1+n_j} = f\big(p_j^{\alpha_j}\big)$. Let us define $A_j = \frac{1}{1-\frac{1}{p_j^2}}$. Then, $f\big(p_j^{\alpha_j}\big) \leq A_j p_j^{\alpha_j} \leq A_j(m + T_{j-1})$. Hence, writing the above inequality for $p_{l+1}, \dots, p_{l+s}$, we can prove that there is a constant $C < 2$ such that $T_{l+s} \leq CT_l + Csm$. We can show $C = 3^{\frac{7}{12}} < 2$. Note that, for primes $q_1, \dots, q_t$ and $\gamma = \prod_{i=1}^{t} \frac{1}{1-\frac{1}{q_i^2}}$, we have[5]

$$\begin{aligned}
\ln\gamma &= \sum_{i=1}^{t} \ln\left(1 + \frac{1}{q_i^2 - 1}\right) \leq \sum_{i=1}^{t} \frac{1}{q_i^2 - 1} \\
&\leq \frac{1}{2^2 - 1} + \sum_{q \geq 3} \frac{1}{q^2 - 1} = \frac{1}{3} + \frac{1}{2} \sum_{q \geq 3} \left(\frac{1}{q-1} - \frac{1}{q+1}\right) = \frac{7}{12}.
\end{aligned}$$

Now, writing $T_{j+s} \leq A_{j+s}(m + T_{j+s-1})$ for $j = 1, \dots, l$, then

$$T_{l+s} \leq T_l \prod_{j=1}^{m} A_{j+s} + m \sum_{r=1}^{s} \prod_{j=1}^{r} A_{l+s+1-j}.$$

Note that $\prod_{j=1}^{r} A_{l+s+1-j} \leq \prod_{j=1}^{m} A_{j+s} \leq C$.

Hence, $T_{l+s} \leq CT_l + Csm$. Now, take $j \geq N+1$. Then, if $p_{j+1}, \dots$ are distinct, there would be a positive integer s such that $T_{j+s} \geq 2T_j$. Further, we can characterize the desired s, as well. That is,

$$T_{j+s} \leq CT_j + Csm \leq \frac{C}{2} T_{j+s} + Csm.$$

Hence, it suffices to have $s \geq \frac{1-\frac{C}{2}}{Cm} T_{j+s} = DT_{j+s}$. Further, since $p_{j+1}, \dots, p_{j+s}$ are all distinct and are less than T_{j+s}, there are at

[5]Here we have used the well-known inequality $\ln(1+x) \leq x$, which can be easily proven by using $f(x) = x - \ln(1+x)$, then considering $f'(x)$ and examining whether the function is strictly increasing.

least DT_{j+s} prime numbers less than T_{j+s}. On the other hand, for all large enough T_{j+s}, the number of primes less than or equal to T_{j+s} is close to $\frac{T_{j+s}}{\ln T_{j+s}}$. Hence,

$$DT_{j+s} < \frac{T_{j+s}}{\ln T_{j+s}},$$

which is impossible.

Now, if there are infinitely many identical terms in the sequence $p_{N+1}, \dots$, one can set $p_j = p_{j+s}$ and $p_{j+1}, p_{j+2}, \dots, p_{j+s}$ are pairwise distinct. Then, there are infinitely many pairs (j, s). Therefore,

$$T_{j+s} = f\left(p_{j+s}^{\alpha_{j+s}}\right) > T_j = f\left(p_j^{\alpha_j}\right).$$

This yields $\alpha_{j+s} \geq 1+\alpha_j$. Therefore, $T_{j+s} = f\left(p_{j+s}^{\alpha_{j+s}}\right) \geq p_j f\left(p_j^{\alpha_j}\right) \geq 2T_j$. Thus, $s \geq DT_{j+s}$, which implies that the prime numbers that do not exceed T_{j+s} are at least DT_{j+s}. Since there are infinitely many (j, s), for all large enough T_{j+s}, the number of primes less than or equal to T_{j+s} is close to $\frac{T_{j+s}}{\ln T_{j+s}}$. Therefore,

$$DT_{j+s} < \frac{T_{j+s}}{\ln T_{j+s}},$$

a contradiction. Hence, the sequence is bounded!

(iii) Since the sequence is bounded, then there is a positive integer B such that $a_n < B$. Hence, consider $(a_k, a_{k+1}, \dots, a_{k+m})$. Then, according to the pigeon-hole principle, there are c, d such that

$$(a_c, a_{c+1}, \dots, a_{c+m}) = (a_{c+d}, a_{c+d+1}, \dots, a_{c+d+m}).$$

It follows that $a_{c+m+1} = a_{c+d+m+1}$. That is, for all $n \geq N$,

$$(a_n, a_{n+1}, \dots, a_{n+m}) = (a_{n+d}, a_{n+d+1}, \dots, a_{n+d+m}). \qquad \square$$

6 Concluding remarks

Through this article, we developed a rich toolbox that was based on a single fact, i.e., the unique factorization. At the very beginning, we provided the reader with the properties of the function $v_p(\cdot)$. Then, we started with some easy problems to stabilize what the reader has learned. We then continued to sweep out the implications of our approach through the middle of Section 3 and Section 4. Importantly, the reader has seen several times that considering $v_p(\cdot)$ was indeed the decisive idea,

The cardinal reason behind this approach is to transform the global condition, i.e., an equation or a statement about integers, to a local condition, i.e., checking the exponent of a prime in both sides of an equation or in the numerator and denominator. Then, we need to split what has been yielded for several primes. Finally, after developing an inequality or applying an external fact from number theory, we can finish the problem.

References

[1] Andreescu, T. and Dospinescu, G. *Problems From The Book*. 2nd ed. USA: XYZ-Press, 2012. ISBN: 0979926904.

[2] Andreescu, T. and Dospinescu, G. *Straight From The Book*. 1st ed. USA: XYZ-Press, 2012. ISBN: 0979926939.

[3] Andreescu, T., Mushkarov, O., and Dospinescu, G. *Number Theory: Concepts and Problems*. 1st ed. USA: XYZ-Press, 2017. ISBN: 09885622.

[4] Cohen, H. *Number Theory, Volume 1: Tools and Diophantine Equations*. 1st ed. Springer, 2007. ISBN: 978-0-387-49922-2.

Navid Safaei,
Sharif University,
Iran.
navid-safaei@gsme.sharif.edu

Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Problems and solutions from the 35th edition of the Iberoamerican Mathematical Olympiad

O. Rivero Salgado

1 Introduction

The 35th edition of the Iberoamerican Mathematical Olympiad took place in November 2020 in Peru. Unfortunately, and due to the Covid pandemic, the 84 participants could not gather together and the competition followed an online format. It was developed in two consecutive days, and contestants had to solve 3 problems each day in a maximum time of four hours and a half. Each problem was graded with an integer mark between 0 and 7 points, so the maximum possible score was 42 points, achieved this year by two students. According to the usual standards, at most half of the students can get a medal, and then these are awarded in the proportion 1:2:3 for gold, silver and bronze, respectively.

The Spanish team made a very good performance, achieving two silver medals (Pau Cantos and Javier Nistal) and two bronze medals (Leonardo Costa and Ignacio Císcar). The delegation was completed by María Gaspar, as the chief of the delegation, Óscar Rivero, as the deputy leader, and Marc Felipe, as an observer.

We present now the problems of the competition, and include the solutions given to them by our team. In all the cases, the solutions follow the ideas presented by the contestants, but we have done some little modifications to ease the exposition.

2 Problems and solutions

We now present the problems that appeared in the paper and their solutions.

Problem 1. Let ABC be an acute scalene triangle such that $AB < AC$. The midpoints of sides AB and AC are M and N, respectively. Let P and Q be points on the line MN such that $\angle CBP = \angle ACB$, and $\angle QCB = \angle CBA$. The circumscribed circle of triangle ABP intersects line AC at D (with $D \neq A$), and the circumscribed circle of triangle AQC intersects line AB at E (with $E \neq A$). Show that lines BC, DP, and EQ are concurrent.

Solution by Javier Nistal. As usual, let $\alpha = \angle BAC$, $\beta = \angle CBA$, and $\gamma = \angle ACB$. Let A' be the intersection of lines BP and CQ. Then,

$$\angle A'BC = \angle CBP = \gamma, \quad \angle A'CB = \angle BCQ = \beta,$$

and therefore

$$\angle BA'C = 180^\circ - \angle A'BC - \angle A'CB = \alpha.$$

We conclude that $ABCA'$ is cyclic.

By the construction, $AECQ$ and $ADBP$ are also cyclic, so

$$\angle ABA' = \angle ACA' = \angle ACQ = \angle AEQ.$$

Since A, B and E are collinear, this means that BA' is parallel to EQ.

Observe that triangles ABC and $A'CB$ are symmetric, and the points M and N correspond to Q and P, respectively. This means that $BP = A'P$ and $CQ = A'Q$. Then, Thales' theorem implies that EQ passes through the midpoint of BC.

We finally see that DP also passes through that point:

$$\angle ACA' = \angle ABA' = \angle ABP = \angle ADP.$$

Since C, A and D are collinear, DP and $A'C$ are parallel. And since P is the midpoint of $A'B$, using again Thales' theorem we get that DP passes through the midpoint of BC.

Therefore, all three lines BC, DP and EQ are concurrent at the midpoint of BC.

Problem 2. Let T_n stand for the least natural number such that $1 + 2 + \ldots + T_n$ is divisible by n. Find all natural numbers n such that $T_n \geq n$.

Solution by Ignacio Císcar. We begin by observing that for all powers of two $T_n \geq n$. Indeed, let $n = 2^k$ and assume that n divides $\frac{T_n(T_n+1)}{2}$. Since $\gcd(T_n, T_n + 1) = 1$, this means that either 2^{k+1} divides T_n or that 2^{k+1} divides $T_n + 1$. In particular, $T_n \geq 2^k = n$.

The case where n is odd and strictly greater than 1 is almost immediate. If $n = 2m + 1$, with $m \geq 1$, then $2m + 1$ divides $\frac{2m(2m+1)}{2}$ and $T_n \leq n - 1$.

We finally consider the case where $n = (2m+1)2^k$, where m, $k \geq 1$. Let i stand for the inverse of 2^{k+1} modulo $2m + 1$, taken in such a way that $0 < i < 2m + 1$. If $0 < i < m + 1$, since $i \cdot 2^{k+1} - 1$ is a multiple of $2m + 1$, it happens that

$$n = (2m + 1)2^k \text{ divides } \sum_{j=1}^{i \cdot 2^{k+1}-1} j = (i \cdot 2^{k+1} - 1) \cdot i \cdot 2^k.$$

In particular, $T_n \leq i \cdot 2^{k+1} - 1 \leq 2m \cdot 2^k < (2m + 1) \cdot 2^k = n$. Similarly, if $m < i < 2m + 1$, let $i' = 2m + 1 - i$, which clearly satisfies that $0 < i' \leq m$. Then,

$$\begin{aligned} i' \cdot 2^{k+1} &\equiv (2m + 1 - i) \cdot 2^{k+1} \\ &\equiv (2m + 1)2^{k+1} - i \cdot 2^{k+1} \equiv -1 \pmod{2m + 1}. \end{aligned}$$

This means that $i' \cdot 2^{k+1} + 1$ is a multiple of $2m + 1$. Then,

$$\begin{aligned} n = (2m + 1)2^k \text{ divides } \sum_{j=1}^{i' \cdot 2^{k+1}} j &= \frac{(i' \cdot 2^{k+1}) \cdot (i' \cdot 2^{k+1} + 1)}{2} \\ &= i' \cdot 2^k \cdot (i' \cdot 2^{k+1} + 1). \end{aligned}$$

Again, we can easily prove that $T_n < n$.

We conclude that $T_n \geq n$ if and only if n is a power of 2.

Problem 3. Let $n \geq 2$ be an integer. A sequence $\alpha = (a_1, a_2, \ldots, a_n)$ of n integers is called *limeña* if

$$\gcd\{a_i - a_j \text{ such that } a_i > a_j \text{ and } 1 \leq i, j \leq n\} = 1.$$

One operation consists on choosing two elements a_k and a_ℓ from a sequence, with $k \neq \ell$, and replacing a_ℓ by $a'_\ell = 2a_k - a_\ell$.

Show that, given a collection of $2^n - 1$ *limeña* sequences, each one formed by n integers, there are two of them, say β and γ, such that it is possible to transform β into γ through a finite number of operations.

Remark. If all the elements of a sequence are equal, then that sequence is not *limeña*.

Solution by Pau Cantos. Observe that a sequence is not *limeña* if and only if there exist positive integer numbers (a, d), with $0 \leq a < d$, such that $a_i \equiv a$ modulo d for all $i = 1, \ldots, n$. We can define an equivalence relation between sequences of n terms as follows: two sequences β and γ are said to be equivalent if we can transform β into γ using the operation described in the statement (as many times as needed). It is rather easy to check that this is indeed an equivalence relation:

(i) *Transitivity:* if α is equivalent to β, and β is equivalent to γ, we can clearly compose both sequence of operations in order to pass from α to γ.
(ii) *Symmetry:* let (ℓ, k) stand for the operation which replaces a_ℓ by $a'_\ell = 2a_k - a_\ell$. Observe that this is an involution, since $2a_k - a'_\ell = 2a_k - (2a_k - a_\ell) = a_\ell$ and in particular this means that it is its own inverse and we can revert the operation.
(iii) *Reflexivity:* tautological.

Further, the operation transforms *limeña* sequences into *limeña* sequences. Indeed, if α is not *limeña*, there exists a pair of integers

(a, d) such that $a_i \equiv a$ modulo d; then, $a'_\ell = 2a_k - a_\ell \equiv a$ modulo d. Therefore, in the new sequence all the terms are congruent with a modulo d. Similarly, if α is *limeña* and transforms under the operation into a non-*limeña* sequence, applying the operation again we get a non-*limeña* sequence, contradicting the fact that the operation is an involution.

We will be done if we can show that there are at most $2^n - 2$ equivalence classes of *limeña* sequences. To see this, we begin by noting that the operation clearly preserves the parity of all the terms of the sequence. This means that each binary vector $(b_1, \ldots, b_n)$ must contain at least one of these equivalence classes; observe that the vectors $(0, \ldots, 0)$ and $(1, \ldots, 1)$ do not correspond to *limeña* sequences. Hence, we will prove that whenever α and β are *limeña* sequences with $\alpha \equiv \beta$ modulo 2, then α can be transformed into β by successive iterations of the operation. In particular, we will prove that α can be transformed in the corresponding binary vector consisting of 0's and 1's.

Let $S(\alpha) = \sum_{i=1}^{n} |a_i|$. Note that among those sequences congruent with α modulo 2, the one which minimizes $S(\alpha)$ is precisely the one formed by 0's and ± 1's. But if $b \neq (1, \ldots, 1)$, there would exist an index k such that $a_k = 0$, and applying the operation (ℓ, k) we can replace a_ℓ by $-a_\ell$. Then, it is enough with proving that given a sequence α whose terms are not all equal to $\{0, \pm 1\}$ we can make a sequence of transformations passing from α to α', with $S(\alpha') < S(\alpha)$. If there are two indexes (k, ℓ) such that either $0 < a_k < a_\ell$ or $a_\ell < a_k < 0$, we can conclude by applying the operation (ℓ, k): in the former case, $-a_\ell < 2a_k - a_\ell < a_\ell$, and in the latter, $a_\ell < 2a_k - a_\ell < -a_\ell$. If this were not the case, all the positive (resp. negative) terms would be equal to a given value $x > 0$ (resp. $y < 0$). If there were no positive values (resp. no negative values), the *limeña* condition would mean that $y = -1$ (resp. $x = 1$), and we would be done. If there were both positive and negative terms, $\gcd(x, y) = 1$. In this case, if there were terms equal to zero, we can change signs and proceed as before reducing the value of $S(\alpha)$; if not, all the differences would be divisible by $x - y \geq 1 - (-1) = 2$, which is a contradiction with the *limeña* condition. This finishes the problem.

Problem 4. Show that there exists a set $\mathcal{C}$ of 2020 distinct, positive integers that satisfies simultaneously the following properties:

- When one computes the greatest common divisor of each pair of elements of $\mathcal{C}$, one gets a list of numbers that are all distinct.
- When one computes the least common multiple of each pair of elements of $\mathcal{C}$, one gets a list of numbers that are all distinct.

Solution by Leonardo Costa. Choose 4040 different prime numbers

$$\{p_1, p_2, \ldots, p_{2020}, q_1, q_2, \ldots, q_{2020}\}.$$

Let $N = q_1 q_2 \cdots q_{2020}$, and consider the following set of 2020 elements:

$$\left\{p_1 \cdot \frac{N}{q_1}, p_2 \cdot \frac{N}{q_2}, \ldots p_{2020} \cdot \frac{N}{q_{2020}}\right\}.$$

It clearly holds that all the elements are distinct. Now, observe that

$$\gcd\left(p_i \cdot \frac{N}{q_i}, p_j \cdot \frac{N}{q_j}\right) = \frac{N}{q_i q_j}.$$

Similarly,

$$\operatorname{lcm}\left(p_i \cdot \frac{N}{q_i}, p_j \cdot \frac{N}{q_j}\right) = p_i p_j N.$$

This proves that our set of 2020 numbers satisfies the given conditions, since $q_i q_j = q_{i'} q_{j'}$ if and only if $\{i, j\} = \{i', j'\}$, and $p_i p_j = p_{i'} p_{j'}$ if and only if $\{i, j\} = \{i', j'\}$.

Problem 5. Determine all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x-y)) + yf(x) = x + y + f(x^2),$$

for any real numbers x, y.

Solution by Ignacio Císcar. We begin by observing that the function $f(x) = x + 1$ satisfies the given equation. Indeed,

$$\begin{aligned} f(xf(x-y)) + yf(x) &= f(x(x-y+1)) + y(x+1) \\ &= x(x-y+1) + 1 + y(x+1) + 1 \\ &= x + y + x^2 + 1 = x + y + f(x^2). \end{aligned}$$

We will prove now that this is the unique solution.

First claim. $f(0) = 1$ and $f(1) = 2$.

Set $x = 0$ and $y = 1$ in the given equation. This directly gives $f(0) = 1$. Similarly, setting $x = y = 1$, we get that

$$f(f(0)) + f(1) = 2 + f(1),$$

which implies that $f(1) = 2$.

Second claim. $f(f(t)) = t + 2$ for all t.

In the given equation, replace $x = 1$ and $y = 1 - t$. Then, using the previous claim,

$$f(f(t)) + 2(1 - t) = 2 - t + 2,$$

and from here the conclusion immediately follows.

In particular, this claim trivially implies that f is injective.

Third claim. $f(f(x) - 2) = x$ for all x.

To prove this result, observe that

$$f(x) = (-2 + f(x)) + 2 = f(f(f(x) - 2)).$$

Using the injectivity of f, the conclusion follows.

We can finally prove the result, replacing $y = x + 2 - f(x)$. Then,

$$f(xf(f(x) - 2)) + (x + 2 - f(x))f(x) = 2x + 2 - f(x) + f(x^2).$$

Using the previous claim, this implies that

$$f(x)^2 + (-3 - x)f(x) + 2x + 2 = (f(x) - 2)(f(x) - x - 1).$$

However, due to the injectivity of f together with the first claim, $f(x) = 2$ happens if and only if $x = 1$. This means that the only possible candidate is $f(x) = x + 1$, and we have already checked that this function works.

Problem 6. Let ABC be an acute, scalene triangle. Let H be the orthocenter and O be the circumcenter of triangle ABC, and let P be a point interior to the segment HO. The circle with center P and radius PA intersects the line AB and AC again at S and S, respectively. Denote by Q the symmetric point of P with respect to the perpendicular bisector of BC. Prove that points P, Q, R and S lie on the same circle.

Solution by Javier Nistal. Let A' be the symmetric point of A with respect to OH. Then, A' is the intersection of the circumscribed circles to both ARS and ABC. This means that it is the center of the spiral similarity carrying BOC to RPS.

Let E be the circumcenter of BOC, and θ the angle $\angle A'BR = \angle A'OP = \angle A'CS$. If E' stands for the circumcenter of RPS, we will establish that it is over the perpendicular bisector of BC, which is precisely EO. For that purpose, it is enough with proving that $\angle A'EO = \theta$. Let H' be the symmetric point of H with respect to BC, which belongs to the circumcircle of ABC. Then,

$$\angle A'H'A = \angle A'CA = \angle A'CS = \theta.$$

Moreover, $H'A$ is parallel to EO. It only remains to prove that A', H' and E are collinear, since this directly gives that $\angle A'EO = \angle A'H'H = \theta$, thus finishing the proof.

To see the claimed colinearity, we observe that $A'HOH'$ is cyclic, since $\angle A'H'H = \theta = \angle A'OH$. Let O' be the symmetric points of O with respect to BC. Since the perpendicular bisector of HH' is BC, the point O' also belongs to that circle. Let F be the intersection point of the radical axis $A'H'$ with the perpendicular bisector of BC. Then, the power of F with respect to the circles $ABH'C$ and $A'HOH'$ are equal, and

$$FO^2 - OB^2 = FO \cdot FO' = FO \cdot (FO - OO').$$

In particular, $OO' \cdot FO = OB^2$. This means that the triangles $OO'B$ and OBF, which share the angle $\angle BOO'$, have proportional sides and therefore are similar. This proves that OBF is isosceles, and consequently $F = E$.

Óscar Rivero Salgado
Department of Mathematics
BarcelonaTech
Barcelona, Spain
riverosalgado@gmail.com

Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Elementary Problems

E–83. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a_1, a_2, a_3, a_4, a_5, a_6$ be six distinct positive integers. Show that

$$\prod_{1\le i<j\le 6}(a_i - a_j)$$

is a multiple of 320.

Solution 1 by Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany. We first note that at least one of the differences is a multiple of 5. Indeed, consider the six given integers modulo 5; since we have six integers, by the pigeonhole principle we must have that at least two of the integers are equal $(\bmod\ 5)$. Then, the difference of these two is a multiple of 5.

Now consider the six integers and divide them into two groups, those which are even and those which are odd. Any difference of two odd integers is even, and hence a multiple of 2; similarly, the difference of any two even numbers is also a multiple of two. Assume that a of the six numbers are odd; then, the number of pairs whose difference is even is $b := \binom{a}{2} + \binom{6-a}{2}$, and we can easily check that $b \geq 6$ (with equality if and only if $a = 3$).

Combining the two remarks (at least one of the differences is a multiple of 5, and at least 6 of the differences are multiples of 2) yields the result.

Solution 2 by Michel Bataille, Rouen, France. Since $320 = 5 \times 2^6$, we have to show that the product $P = \prod_{1 \leq i < j \leq 6} (a_i - a_j)$ has a factor $a_i - a_j$ divisible by 5 and six even factors.

For $1 \leq i < j \leq 6$, let $d_{i,j} = a_i - a_j$ and consider the list $d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}$. If one of the numbers of this list is a multiple of 5, so is P. Otherwise, each is congruent to one of the numbers $1, 2, 3, 4$ modulo 5. Since there are 5 numbers and four possible residues, two of the numbers are congruent to the same number, say $d_{1,i} \equiv d_{1,j} \pmod 5$ with $2 \leq i < j \leq 6$. But then, we have $d_{i,j} = d_{1,j} - d_{1,i} \equiv 0 \pmod 5$ and P is a multiple of 5.

We keep considering the same list. If at least three numbers of the list are even, say $d_{1,i}, d_{1,j}, d_{1,k}$ where $2 \leq i < j < k \leq 6$, then in addition to these three even factors, the factors $d_{i,j} = d_{1,j} - d_{1,i}, d_{i,k}, d_{j,k}$ are also even. Altogether, we have six even factors.

If exactly two numbers of the list, say $d_{1,i}, d_{1,j}$ are even, then the other numbers $d_{1,k}, d_{1,\ell}, d_{1,m}$ (with $k < \ell < m$) of the list are odd. It follows that the six factors $d_{1,i}, d_{1,j}, d_{i,j}, d_{k,\ell}, d_{k,m}, d_{\ell,m}$ are even.

Similarly, if exactly one, say $d_{1,i}$, is even, then $d_{1,j}, d_{1,k}, d_{1,\ell}, d_{1,m}$ with $j, k, \ell, m \neq i$, $j < k < \ell < m$ are odd and again, we obtain six even factors namely, $d_{1,i}, d_{j,k}, d_{j,\ell}, d_{j,m}, d_{k,m}, d_{\ell,m}$.

Lastly, if the $d_{1,i}$'s are all odd, then $d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{3,4}, d_{3,5}$ are even.

Solution 3 by the proposer. First, we observe that the difference of two even numbers is even and the same occurs if they are odd.

Now, we consider pairs of the form $(\boldsymbol{Even}, \boldsymbol{Odd})$ and we have for the given integers the following possible situations:

$$(6,0),(5,1),(4,2),(3,3),(2,4),(1,5),(0,6).$$

For the first case, we have that all the 15 factors in the product are even and the factor 2^{15} divides the product. For the remaining pairs, the powers of two that appear are $2^{10}, 2^7, 2^6, 2^7, 2^{10}, 2^{15}$, respectively. So, the number is a multiple of $2^6 = 64$. Now, we claim that it is a multiple of 5. Indeed, Dividing an integer by 5 the possible remainders are $0,1,2,3,4$. Since we have six integers $a_1, a_2, a_3, a_4, a_5, a_6$ and five possible remainders by the PHP two of them must be equal and the difference of these number is a multiple of five, as claimed. This completes the proof.

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–84. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase New York, USA.* The following function is constant on the interval I:

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}.$$

Determine the interval I and the constant value of f on I.

Solution 1 by Michel Bataille, Rouen, France. Note that f is defined on $[1, \infty)$: if $x \geq 1$, then $\sqrt{x-1}$ exists and $2\sqrt{x-1} \leq x$ (since $x^2 \geq 4x - 4$).

For $x \geq 1$, $f(x) \geq 0$ and

$$\begin{aligned}(f(x))^2 &= x + 2\sqrt{x-1} + x - 2\sqrt{x-1} + 2\sqrt{x^2 - 4(x-1)} \\ &= 2x + 2|x-2|.\end{aligned}$$

It follows that, if $x \geq 2$, then $f(x)^2 = 4(x-1)$, while if $x \leq 2$, $(f(x))^2 = 2x + 4 - 2x = 4$. Thus, f is constant on the interval $I = [1,2]$ and $f(x) = 2$ for $x \in I$.

Solution 2 by the proposer. First note that the function is not real-valued unless $x \geq 1$. Then, we have

$$[f(x)]^2 = (x + 2\sqrt{x-1}) + (x - 2\sqrt{x-1}) + 2\sqrt{x^2 - 4(x-1)},$$
$$[f(x)]^2 = 2x + 2\sqrt{(x-2)^2} = 2x + 2|x-2|.$$

If $x > 2$, $2x + 2|x-2| = 2x + 2(x-2) = 4x - 4$, which is not constant. However, if $x \in [1,2]$, then $2x + 2|x-2| = 2x + 2(-(x-2)) = 4$ and $f(x) = 2$. (By definition, $f(x)$ is positive.)

Therefore, the interval is $I = [1,2]$ and $f(x) \equiv 2$ on I.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. Note

$$(\sqrt{x-1} + 1)^2 = x - 1 + 2\sqrt{x-1} + 1 = x + 2\sqrt{x-1}$$

and

$$(\sqrt{x-1} - 1)^2 = x - 1 - 2\sqrt{x-1} + 1 = x - 2\sqrt{x-1},$$

so

$$f(x) = \left|\sqrt{x-1} + 1\right| + \left|\sqrt{x-1} - 1\right|,$$

with domain $x \geq 1$. Now, $\sqrt{x-1} + 1 > 0$ for all $x \geq 1$, while $\sqrt{x-1} - 1 \leq 0$ for $1 \leq x \leq 2$ and is greater than 0 for $x > 2$. Thus,

$$f(x) = \begin{cases} 2, & 1 \leq x \leq 2, \\ 2\sqrt{x-1}, & x > 2. \end{cases}$$

The interval I is $1 \leq x \leq 2$, and the constant value of f on I is 2.

Also solved by *Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany; Rovsen Pirgulyev, Sumgait city, Azerbaijan; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA; Daniel Văcaru, Pitești, Romania; Todor Zaharinov, Sofia, Bulgaria, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–85. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be distinct positive numbers such that

$$\frac{\ln a}{b^2 - c^2} = \frac{\ln b}{c^2 - a^2} = \frac{\ln c}{a^2 - b^2}.$$

Compute the value of $a^{a^2} b^{b^2} c^{c^2}$.

Solution 1 by Michel Bataille, Rouen, France. Let $X = a^{a^2} b^{b^2} c^{c^2}$ and let r be the common value of the ratios $\frac{\ln a}{b^2-c^2}, \frac{\ln b}{c^2-a^2}, \frac{\ln c}{a^2-b^2}$. Then, we have

$$\begin{aligned}\ln X &= a^2 \ln a + b^2 \ln b + c^2 \ln c \\ &= a^2 \cdot r(b^2 - c^2) + b^2 \cdot r(c^2 - a^2) + c^2 \cdot r(a^2 - b^2) \\ &= r[a^2 b^2 - a^2 c^2 + b^2 c^2 - b^2 a^2 + c^2 a^2 - c^2 b^2) = 0.\end{aligned}$$

It follows that $X = 1$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Let $V = a^{a^2} b^{b^2} c^{c^2}$. Then,

$$\begin{aligned}\ln V &= a^2 \ln a + b^2 \ln b + c^2 \ln c \\ &= a^2 \left(\frac{(b^2 - c^2) \ln c}{a^2 - b^2} \right) + b^2 \left(\frac{(c^2 - a^2) \ln c}{a^2 - b^2} \right) + c^2 \ln c \\ &= \left(\frac{b^2 c^2 - a^2 c^2}{a^2 - b^2} + c^2 \right) \ln c = 0,\end{aligned}$$

which implies that $V = 1$.

Solution 3 by the proposer. Let $t \in \mathbb{R}$ be a real number such that

$$\frac{\ln a}{b^2 - c^2} = \frac{\ln b}{c^2 - a^2} = \frac{\ln c}{a^2 - b^2} = t.$$

Then, we have

$$\begin{aligned}\ln a &= t(b^2 - c^2), \\ \ln b &= t(c^2 - a^2), \\ \ln c &= t(a^2 - b^2),\end{aligned}$$

and

$$\begin{aligned} a^2 \ln a &= ta^2(b^2 - c^2), \\ b^2 \ln b &= tb^2(c^2 - a^2), \\ c^2 \ln c &= tc^2(a^2 - b^2). \end{aligned}$$

Adding up the preceding expressions, yields

$$a^2 \ln a + b^2 \ln b + c^2 \ln c = t(a^2b^2 - a^2c^2 + b^2c^2 - b^2a^2 + c^2a^2 - c^2b^2) = 0,$$

from which we get

$$\ln\left(a^{a^2} b^{b^2} c^{c^2}\right) = 0 \qquad \text{and} \qquad a^{a^2} b^{b^2} c^{c^2} = 1.$$

Also solved by *Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Rovsen Pirgulyev, Sumgait city, Azerbaijan; Daniel Văcaru, Pitești, Romania, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–86. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Show that no prime of the form $4n + 27$ can be written as the sum of two squares.

Solution 1 by Michel Bataille, Rouen, France. Let p be a prime of the form $4n + 27$ and assume that $p = x^2 + y^2$ for some integers x, y. Since p is odd, x, y cannot be both even or both odd. Say, for instance, that x is even and y is odd. Then $x^2 \equiv 0 \pmod 4$ and $y^2 \equiv 1 \pmod 4$ and it follows that $p = x^2 + y^2 \equiv 1 \pmod 4$, a contradiction since $4n + 27 = 4(n + 6) + 3 \equiv 3 \pmod 4$. Thus, p cannot be written as the sum of two squares.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. In fact, no number of the form $4n+27$, where n is a positive integer, can be written as the sum of two squares. Any number of the form $4n + 27$ is congruent to 3 modulo 4. However, a perfect square is congruent to either 0 or 1 modulo 4, so the sum of two perfect squares will be congruent to either 0, 1, or 2 modulo 4; that is, a sum of two perfect squares can never be congruent to 3 modulo 4. Thus, no number of the form $4n + 27$, where n is a positive integer, can be written as the sum of two squares.

Solution by the proposer. If p is a prime of the form $4n + 27$, then $p \equiv 3 \pmod 4$. To prove that it is not possible to express p as the sum of two squares, we argue by contradiction. So, suppose that $p = a^2 + b^2$ for some integers a and b. Since p is odd, then one of the squares is odd, say a^2, and the other is even, say b^2. Then a is odd and b is even, as can be easily checked. If $a = 2i + 1$ and $b = 2j$ for some integers i, j, then

$$p^2 = a^2 + b^2 = (2i + 1)^2 + (2j)^2 = 4(i^2 + j^2 + i) + 1 \equiv 1 \pmod 4.$$

This contradicts the fact that $p \equiv 3 \pmod 4$, and the proof is complete.

Also solved by *Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany; Daniel Văcaru, Pitești, Romania, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

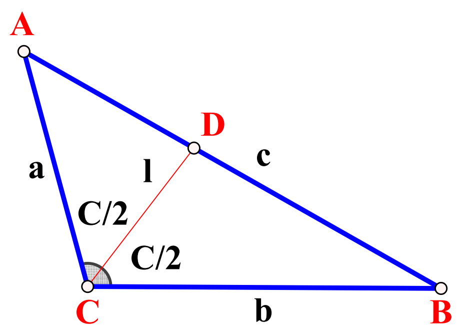
E–87. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let ℓ_a, ℓ_b, ℓ_c be the length of the bisectors of the angles of triangle ABC. If a, b, c are the length of its sides, then prove that

$$\frac{\ell_a}{\cos A/2} + \frac{\ell_b}{\cos B/2} + \frac{\ell_c}{\cos C/2} \leq \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

Solution 1 by the proposer. To prove the statement we insert the term $a + b + c$ and we prove that

$$\frac{\ell_a}{\cos A/2} + \frac{\ell_b}{\cos B/2} + \frac{\ell_c}{\cos C/2} \leq a + b + c \leq \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

To prove the LHS inequality, we compute the area of triangle ABC in two ways. Indeed, we have that $[ABC] = [ADC] + [BDC]$.



Scheme for solving problem E-87.

Since $[ADC] = \frac{a\ell_c}{2}\sin C/2$, $[BDC] = \frac{b\ell_c}{2}\sin C/2$, and $[ABC] = \frac{ab}{2}\sin C$, then

$$\frac{a\ell_c}{2}\sin C/2 + \frac{b\ell_c}{2}\sin C/2 = \frac{ab}{2}\sin C,$$

from which it follows that $\ell_c = \frac{2ab}{a+b}\cos C/2$ and $\frac{\ell_c}{\cos C/2} = \frac{2ab}{a+b}$. Likewise, we get $\frac{\ell_a}{\cos A/2} = \frac{2bc}{b+c}$ and $\frac{\ell_b}{\cos B/2} = \frac{2ca}{c+a}$. Adding up the preceding expressions and taking into account the HM-AM inequality yields

$$\begin{aligned}\frac{\ell_a}{\cos A/2} + \frac{\ell_b}{\cos B/2} + \frac{\ell_c}{\cos C/2} &= \frac{2ab}{a+b} + \frac{2bc}{b+c} + \frac{2ca}{c+a}\\ &\le \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2}\\ &= a+b+c.\end{aligned}$$

Equality holds when $a = b = c$. That is when $\triangle ABC$ is equilateral.

To prove the RHS inequality, we have

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \ge a+b+c$$

or, equivalently,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{a+b+c}{abc}=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}.$$

Putting $x=1/a, y=1/b$ and $z=1/c$ in the well-known inequality $x^2+y^2+z^2\geq xy+yz+zx$, we get

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}.$$

Equality holds when $a=b=c$.

Solution 2 by Michel Bataille, Rouen, France. Twice the area of ABC is $bc\sin A=2bc\sin A/2\cos A/2$ as well as $b\ell_a\sin A/2+c\ell_a\sin A/2$. We deduce that $\frac{\ell_a}{\cos A/2}=\frac{2bc}{b+c}$ and it follows that the problem amounts to showing that

$$\frac{2bc}{b+c}+\frac{2ca}{c+a}+\frac{2ab}{a+b}\leq\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}. \tag{1}$$

Now, $2bc=\frac{(b+c)^2-(b-c)^2}{2}\leq\frac{(b+c)^2}{2}$, hence the left-hand side L of (1) satisfies

$$L\leq\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}=a+b+c$$

and it suffices to prove that

$$a+b+c\leq\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}. \tag{2}$$

We are done because, using that $\frac{x}{y}+\frac{y}{x}\geq 2\sqrt{\frac{x}{y}\frac{y}{x}}=2$ for positive x,y, the inequalities

$$\frac{ab}{c}+\frac{ca}{b}=a\left(\frac{b}{c}+\frac{c}{b}\right)\geq 2a,$$
$$\frac{bc}{a}+\frac{ca}{b}=c\left(\frac{b}{a}+\frac{a}{b}\right)\geq 2c,$$
$$\frac{ab}{c}+\frac{bc}{a}=b\left(\frac{a}{c}+\frac{c}{a}\right)\geq 2b$$

give by addition

$$2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)\geq 2(a+b+c),$$

and therefore (2) holds.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. With

$$\ell_a = \frac{2bc}{b+c}\cos\frac{A}{2}, \quad \ell_b = \frac{2ac}{a+c}\cos\frac{B}{2}, \quad \text{and} \quad \ell_c = \frac{2ab}{a+b}\cos\frac{C}{2},$$

it follows that

$$\frac{\ell_a}{\cos A/2} + \frac{\ell_b}{\cos B/2} + \frac{\ell_c}{\cos C/2} = \frac{2bc}{b+c} + \frac{2ac}{a+c} + \frac{2ab}{a+b}.$$

By the arithmetic mean - harmonic mean inequality,

$$\frac{2bc}{b+c} \le \frac{b+c}{2}.$$

Similarly,

$$\frac{2ac}{a+c} \le \frac{a+c}{2} \quad \text{and} \quad \frac{2ab}{a+b} \le \frac{a+b}{2},$$

so

$$\frac{\ell_a}{\cos A/2} + \frac{\ell_b}{\cos B/2} + \frac{\ell_c}{\cos C/2} \le a+b+c.$$

It therefore suffices to show that

$$a+b+c \le \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

By the arithmetic mean - geometric mean inequality,

$$\begin{aligned} \frac{1}{2}\left(\frac{ab}{c} + \frac{ca}{b}\right) &\ge \sqrt{a^2} = a, \\ \frac{1}{2}\left(\frac{ab}{c} + \frac{bc}{a}\right) &\ge \sqrt{b^2} = b, \text{ and} \\ \frac{1}{2}\left(\frac{bc}{a} + \frac{ca}{b}\right) &\ge \sqrt{c^2} = c. \end{aligned}$$

Adding these last three inequalities yields

$$a+b+c \le \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

Also solved by *Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; Rovsen Pirgulyev, Sumgait city, Azerbaijan; Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA; Daniel Văcaru, Pitești, Romania; Titu Zvonaru, Comanesti, Romania (three solutions), and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–88. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let AB be the diameter of a circle Γ. Points C and D lie on either side of AB, so that $AD = AC$. Knowing that $E \in BD$, $F \in BC$ so that $DF \perp AE$, $AF \cap EC = \{X\}$ and Y is the middle point of AE, show that XYD is an isosceles triangle.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. In Figure 1a), $\angle ADB$ is a right angle, since it is inscribed in a semicircle. Hence,

$$\angle ADE = \angle ADB = 90^\circ.$$

Let us assume for a moment that $AF \perp CE$, a fact that we shall establish at the end of this solution. It follows that AE is the common hypotenuse of the two right-angled triangles EXA and ADE. But we are told that Y is the midpoint of AE, so

$$XY = (\text{in } \triangle EXA) = \frac{1}{2}AE = (\text{in } \triangle ADE) = DY.$$

Hence $\triangle XDY$ is an isosceles triangle, as was to be shown.

We now complete the argument by proving:

Lemma. *Let BCD an isosceles triangle, with $BC = BD$; let A be the point on the circumcircle of $\triangle BCD$ diametrically opposite to B. In the interior of sides BC, BD, any points E, F, respectively, are selected. Then,*

$$AF \perp CE \iff AE \perp DF.$$

Proof. (See Figure 1b)) We consider a rectangular coordinate system with the unit of measurement the same along both axes.

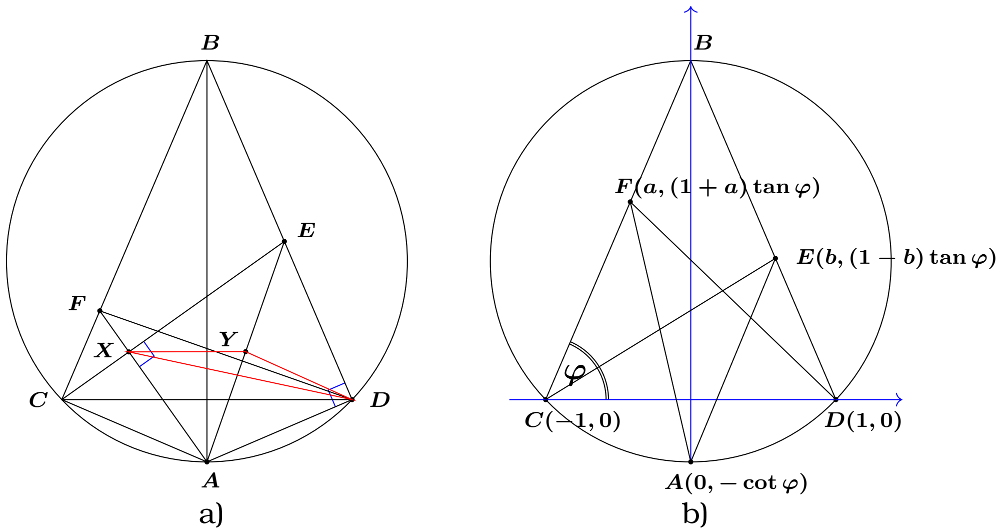


Figure 1: Schemes for solution 1 of Problem E–88.

Suppose that the angles at the base of $\triangle BCD$ are each φ. We place B at $(-1, 0)$ and D at $(1, 0)$.

The coordinates of A are $(0, -\cot\varphi)$, the coordinates of F are $(a, (1+a)\tan\varphi)$, and those of E are $(b, (1-b)\tan\varphi)$, where $-1 < a < 0$ and $0 < b < 1$.

The slopes of AE, DF, AF, CE are therefore given by

$$m_{AE} = \frac{(1-b)\tan\varphi + \cot\varphi}{b}, \qquad m_{DF} = \frac{(1+a)\tan\varphi}{a-1},$$
$$m_{AF} = \frac{(1+a)\tan\varphi + \cot\varphi}{a}, \qquad m_{CE} = \frac{(1-b)\tan\varphi}{1+b},$$

from which we obtain

$$\begin{aligned} b(1-a)(m_{AF}\cdot m_{CE}+1) &= 1+a-b+ab+(1+a)(1-b)\tan^2\varphi \\ &= a(1+b)(m_{AE}\cdot m_{DF}+1), \end{aligned}$$

where the factors $b(1-a)$ and $a(1+b)$ are non-zero.

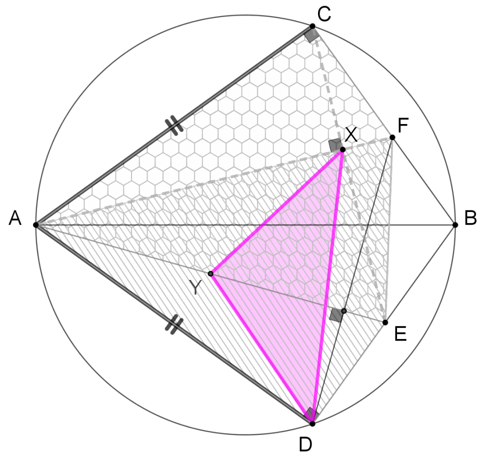
We conclude that

$$m_{AF}\cdot m_{CE}+1=0 \iff m_{AE}\cdot m_{DF}+1=0,$$

equivalent to the desired equivalence. □

Solution 2 by the proposer. We begin with the following lemma.

Lemma. *In a quadrilateral $ABCD$, $AC \perp BD$ if and only if $AB^2 + CD^2 = AD^2 + BC^2$.*



Scheme for solution 2 of Problem E–88.

In the quadrilateral $ADEF$, $DF \perp AE$, so

$$AD^2 + EF^2 = DE^2 + AF^2. \tag{1}$$

In $AEFC$, if $CE \perp AF \Rightarrow$, then

$$AE^2 + FC^2 = AC^2 + EF^2. \tag{2}$$

Now,

$$\begin{cases} \triangle\ AFC,\ \measuredangle\widehat{ACF} = 90^\circ \Rightarrow FC^2 = AF^2 - AC^2, \\ \triangle\ ADE,\ \measuredangle\widehat{ADE} = 90^\circ \Rightarrow AE^2 = AD^2 + DE^2, \end{cases}$$

so (2) becomes

$$AD^2 + DE^2 + AF^2 - AC^2 = AC^2 + EF^2.$$

But $AD = AC$, hence $AD^2 = AC^2$, so

$$DE^2 + AF^2 = AD^2 + EF^2 \Rightarrow CE \perp AF. \tag{3}$$

Finally,

$$\begin{cases} DY \text{ median in } \triangle ADE \Rightarrow DY = \frac{AE}{2}, \\ XY \text{ median in } \triangle AXE \Rightarrow XY = \frac{AE}{2}, \end{cases}$$

so $DY = XY$ and, therefore, YDX is an isosceles triangle.

Also solved by *Michel Bataille, Rouen, France.*

Easy–Medium Problems

EM–83. *Proposed by Oriol Baeza Guasch, CFIS, BarcelonaTech, Terrassa, Spain.* [Correction] Let Γ_1 and Γ_2 be two non-intersecting, non-overlapping circles, whose common external tangents meet at O. One of the internal common tangents meets Γ_1 at R, and Γ_2 at P. Denote by S the intersection of ray OP with Γ_1 closest to P, and denote by Q the intersection of ray OR with Γ_2 closest to R. Prove that PR, QS and the line joining the centers of Γ_1 and Γ_2 concur.

Solution 1 by the proposer. First of all, denote by C_1, C_2 the centers of Γ_1, Γ_2, respectively. Also let I be the point where the common internal tangents of Γ_1 and Γ_2 meet.

Next, let's show that $PQRS$ is cyclic. Indeed, let rays OSP, ORQ meet Γ_2 by second time at S' and R', respectively. Since O is the external homothety center we have the similarity $\triangle OSR \sim \triangle OS'R'$. Hence, a simple angle chasing shows

$$\angle QPS = 180^\circ - \angle QPS' = \angle S'R'Q = \angle SRO = 180^\circ - SRQ.$$

Thus, $(PQRS)$ is cyclic.

Now, denote by $J = (QSO) \cap (PRO)$ the second point of intersection, different from O. Also, let $I \neq J$ be the second intersection of circles $(QJR) \cap (PJS)$. Then, by concurrence of radical axes, J, I, O are collinear. To end the problem, let us show $PR \cap QS = I$.

Indeed $\angle QRI = 180^\circ - \angle QJI = 180^\circ - \angle QJO = 180^\circ - \angle QSO = \angle QSP = \angle QRP$, hence R, I, P are collinear. One can show analogously that Q, I, S are collinear, and we are done.

Solution 2 by the proposer. Once we have shown that $(PQRS)$ is cyclic, consider ψ the inversion with center O and ratio $k = \sqrt{OS \cdot OP}$ (also equal to $\sqrt{OR \cdot OQ}$ because of power of a point). Clearly, it swaps $P \leftrightarrow S$ and $Q \leftrightarrow R$, hence

$$\overline{QS} \leftrightarrow (PRO), \qquad \overline{PR} \leftrightarrow (QSO).$$

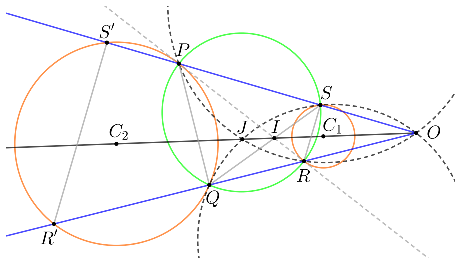


Figure 2: Scheme for solving Problem EM–83.

Let us denote by J the second intersection of circles (PRO) and (QSO), which lies on OI by concurrency of radical axis. Then, it is enough to prove that $OI \cdot OJ = k^2 = OR \cdot OQ$, because that will imply

$$I = \psi(J) = \psi((QSO) \cap (PRO)) = PR \cap QS.$$

Finally, notice that $OI \cdot OJ = OR \cdot OQ$ is equivalent to showing that $(QRIJ)$ is cyclic, which is true by Reim's theorem, and we are done.

Also solved by *Michel Bataille, Rouen, France.*

EM–84. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let a, b, c be positive real numbers. Prove that

$$\frac{a^9}{bc} + \frac{b^9}{ca} + \frac{c^9}{ab} \geq a^2b^2c^2\,(a + b + c).$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. The given inequality is equivalent to

$$a^{10} + b^{10} + c^{10} \geq a^3b^3c^3(a + b + c) = a^4b^3c^3 + a^3b^4c^3 + a^3b^3c^4.$$

We now present two proofs of this equivalent inequality.

Proof 1. By the arithmetic mean - geometric mean inequality,

$$\frac{4a^{10}+3b^{10}+3c^{10}}{10} \geq \sqrt[10]{a^{40}b^{30}c^{30}} = a^4b^3c^3,$$
$$\frac{3a^{10}+4b^{10}+3c^{10}}{10} \geq \sqrt[10]{a^{30}b^{40}c^{30}} = a^3b^4c^3,$$

and

$$\frac{3a^{10}+3b^{10}+4c^{10}}{10} \geq \sqrt[10]{a^{30}b^{30}c^{40}} = a^3b^3c^4.$$

Adding these last three inequalities yields

$$a^{10}+b^{10}+c^{10} \geq a^4b^3c^3 + a^3b^4c^3 + a^3b^3c^4. \qquad \square$$

Proof 2. The sequence $(10, 0, 0)$ majorizes the sequence $(4, 3, 3)$, so, by Muirhead's inequality,

$$a^{10}+b^{10}+c^{10} \geq a^4b^3c^3 + a^3b^4c^3 + a^3b^3c^4. \qquad \square$$

Solution 2 by the proposer. Assuming $a \geq b \geq c$, we have $a^8 \geq b^8 \geq c^8$ and $\frac{a}{bc} \geq \frac{b}{ac} \geq \frac{c}{ab} \iff a^2 \geq b^2 \geq c^2$.

Applying the rearrangement inequality yields

$$\begin{aligned}\frac{a^9}{bc}+\frac{b^9}{ac}+\frac{c^9}{ab} &= a^8\frac{a}{bc}+b^8\frac{b}{ac}+c^8\frac{c}{ab} \geq a^8\frac{b}{ac}+b^8\frac{c}{ab}+c^8\frac{a}{bc}\\ &= a^7\frac{b}{c}+b^7\frac{c}{a}+c^7\frac{a}{b} = a^6\frac{ab}{c}+b^6\frac{bc}{a}+c^6\frac{ac}{b}.\end{aligned}$$

Since $a \geq b \geq c \Rightarrow a^6 \geq b^6 \geq c^6$ and $\frac{ab}{c} \geq \frac{bc}{a} \geq \frac{ac}{b}$,

$$\begin{aligned}a^6\frac{ab}{c}+b^6\frac{bc}{a}+c^6\frac{ab}{c} &\geq a^6\frac{bc}{a}+b^6\frac{ac}{b}+c^6\frac{ab}{c}\\ &= a^5bc+b^5ac+c^5ab = abc\big(a^4+b^4+c^4\big)\\ &\geq abc\big(a^2b^2+a^2c^2+b^2c^2\big)\\ &\geq abc(ab\cdot ac+ab\cdot bc+ac\cdot bc)\\ &= a^2b^2c^2(a+b+c).\end{aligned}$$

Finally,

$$\frac{a^9}{bc}+\frac{b^9}{ac}+\frac{c^9}{ab}\geq a^2b^2c^2(a+b+c).$$

Equality is reached for $a=b=c \Rightarrow a^7+a^7+a^7=a^6\cdot 3a \Rightarrow 3a^7=3a^7$.

Also solved by *Michel Bataille, Rouen, France; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA, and Titu Zvonaru, Comanesti, Romania (two solutions).*

EM–85. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase New York, USA.* A group of people joined a parade. If they formed the same three lines, then there was one person left (not included in the line). If they formed four lines, there were two people left. If they formed five lines, there were three people left. What is the minimum number of people in the group?

Solution by the proposer. This problem asks for the minimum solution of the system

$$\begin{aligned} x &\equiv 1 \pmod{3},\\ x &\equiv 2 \pmod{4},\\ x &\equiv 3 \pmod{5}. \end{aligned}$$

The Chinese Remainder Theorem (CRT) guarantees a unique solution modulo $M=3\cdot 4\cdot 5=60$. Following the standard proof of the CRT (noting that the moduli are pairwise relatively prime), we consider the congruences $(M/3)x_1\equiv 1 \pmod{3}$, $(M/4)x_2\equiv 1 \pmod{4}$, and $(M/5)x_3\equiv 1 \pmod{5}$, or $20x_1\equiv 1 \pmod{3}$, $15x_2\equiv 1 \pmod{4}$, and $12x_3\equiv 1 \pmod{5}$. The unique solutions of the last set of congruences are easily found: 2, 3, and 3, respectively. Then, the CRT implies that the unique solution of our original system is given by

$$\begin{aligned} x &\equiv (1)(20)x_1+(2)(15)x_2+(3)(12)x_3 \pmod{60}\\ &\equiv (2)(20)+(30)(3)+(36)(3) \pmod{60}\\ &\equiv 238 \mod 60 \equiv 58 \pmod{60}. \end{aligned}$$

This says that the general solution of the original system is $x = 58 + 60t$, so 58 is the minimum number of people in the group.

Also solved by *Michel Bataille, Rouen, France, and Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA.*

EM–86. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $y = ax^2 - bx - c$, $y = bx^2 - cx - a$ and $y = cx^2 - ax - b$ be the equations of three parabolas where a, b, c are non-zero real numbers. If their graphics have a common point, then prove that $a = b = c$.

Solution by the proposer. Let us denote $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$ and $P_3(x) = cx^2 - ax - b$, respectively. Suppose there exists a point $(u, v) \in \mathbb{R}^2$ such that $P_1(u) = P_2(u) = P_3(u) = v$. Then, we have

$$\begin{aligned} au^2 - bu - c &= v, \\ bu^2 - cu - a &= v, \\ cu^2 - au - b &= v. \end{aligned}$$

Multiplying the above relations respectively by $b - c$, $c - a$ and $a - b$, we get

$$\begin{aligned} (b - c)(au^2 - bu - c) &= v(b - c), \\ (c - a)(bu^2 - cu - a) &= v(c - a), \\ (a - b)(cu^2 - au - b) &= v(a - b), \end{aligned}$$

and adding up the preceding relations yields

$$\begin{aligned} &(b - c)(au^2 - bu - c) + (c - a)(bu^2 - cu - a) \\ &+ (a - b)(cu^2 - au - b) - v(b - c) - v(c - a) - v(a - b) = 0 \end{aligned}$$

or

$$(ab + bc + ca - a^2 - b^2 - c^2)(u - 1) = 0.$$

Thus, either $ab + bc + ca - a^2 - b^2 - c^2 = 0$ or $u = 1$. In the first case,

$$0 = ab + bc + ca - a^2 - b^2 - c^2 = \frac{1}{2}\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right]$$

shows that $a = b = c$. If $u = 1$, then we obtain $a - b - c = b - c - a = c - a - b$, and once again we obtain $a = b = c$.

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Daniel Văcaru, Piteşti, Romania, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

EM–87. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase New York, USA.* For any positive integer N, prove that

$$\frac{\pi^2}{6} - \frac{1}{N} < \sum_{n=1}^{N} \frac{1}{n^2} < \frac{\pi^2}{6} - \frac{1}{N+1}.$$

Solution 1 by the proposer. It is well known that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. (This is $\zeta(2)$, where ζ denotes Riemann's zeta function.)

For any positive integer N, we see that

$$\begin{aligned}\frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=N+1}^{\infty} \frac{1}{n^2} &< \sum_{n=N+1}^{\infty} \frac{1}{n(n-1)} \\ &= \lim_{M\to\infty} \sum_{n=N+1}^{M} \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \lim_{M\to\infty} \left(\frac{1}{N} - \frac{1}{M}\right) = \frac{1}{N}. \qquad (1)\end{aligned}$$

Similarly, we determine that

$$\begin{aligned}\sum_{n=N+1}^{\infty} \frac{1}{n^2} > \sum_{n=N+1}^{\infty} \frac{1}{n(n+1)} &= \lim_{M\to\infty} \sum_{n=N+1}^{M} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \lim_{M\to\infty} \left(\frac{1}{N+1} - \frac{1}{M+1}\right) = \frac{1}{N+1}. \qquad (2)\end{aligned}$$

Combining (1) and (2), we obtain the desired result.

Solution 2 by Michel Bataille, Rouen, France. Since $\frac{\pi^2}{6} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, the inequality to be proved can be written as

$$\frac{1}{N+1} < \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{1}{N}.$$

Let $f_n(x) = \frac{1}{x^2} - \frac{1}{n^2}$. On the interval $[n-1, n]$ where $n \geq 2$, f_n is continuous and satisfies $f_n(x) \geq 0$ and $f_n(x) > 0$ if $x \neq n$. It follows that $\int_{n-1}^{n} f_n(x)\,\mathrm{d}x > 0$, that is,

$$\int_{n-1}^{n} \frac{\mathrm{d}x}{x^2} > \frac{1}{n^2}.$$

Now, suppose that $a_n \leq b_n$ for all integers $n \geq 1$ and that the series $\sum_{n\geq 1} a_n$, $\sum_{n\geq 1} b_n$ are convergent. If $a_m < b_m$ for some positive integer m, then $\sum\limits_{n=1}^{\infty} a_n < \sum\limits_{n=1}^{\infty} b_n$. (Indeed, this follows from $\sum\limits_{n=m+1}^{\infty} a_n \leq \sum\limits_{n=m+1}^{\infty} b_n$ and $\sum\limits_{n=1}^{m} a_n < \sum\limits_{n=1}^{m} b_n$.)

We deduce that

$$\frac{1}{N} = \int_{N}^{\infty} \frac{\mathrm{d}x}{x^2} = \sum_{n=N+1}^{\infty} \int_{n-1}^{n} \frac{\mathrm{d}x}{x^2} > \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

Similarly, considering f_n on the interval $[n, n+1]$, we obtain $\int_{n}^{n+1} \frac{\mathrm{d}x}{x^2} < \frac{1}{n^2}$ and

$$\frac{1}{N+1} = \int_{N+1}^{\infty} \frac{\mathrm{d}x}{x^2} = \sum_{n=N+1}^{\infty} \int_{n}^{n+1} \frac{\mathrm{d}x}{x^2} < \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA.*

EM–88. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_1, x_2, x_3, x_4 be the roots of the equation $x^4 - x^3 + 5x^2 + 2x + 1 = 0$. Find an equation of fourth degree with roots t_1, t_2, t_3, t_4 such that $t_1 = (x_2 + x_3 + x_4)\, x_2x_3x_4$, $t_2 = (x_3 + x_4 + x_1)\, x_3x_4x_1$, $t_3 = (x_4 + x_1 + x_2)\, x_4x_1x_2$ and $t_4 = (x_1 + x_2 + x_3)\, x_1x_2x_3$, respectively.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. By Viete's formulas,

$$\begin{aligned}
x_1+x_2+x_3+x_4 &= 1,\\
x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 &= 5,\\
x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4 &= -2,\\
x_1x_2x_3x_4 &= 1.
\end{aligned}$$

From here, it follows that

$$\begin{aligned}
&t_1 = (x_2+x_3+x_4)x_2x_3x_4 = (1-x_1)x_2x_3x_4 = x_2x_3x_4-1,\\
&t_2 = (1-x_2)x_1x_3x_4 = x_1x_3x_4-1,\\
&t_3 = (1-x_3)x_1x_2x_4 = x_1x_2x_4-1, \text{ and}\\
&t_4 = (1-x_4)x_1x_2x_3 = x_1x_2x_3-1.
\end{aligned}$$

Thus,

$$\sum_{j=1}^{4} t_j = x_2x_3x_4+x_1x_3x_4+x_1x_2x_4+x_1x_2x_3-4 = -6,$$

and

$$\begin{aligned}
\prod_{j=1}^{4} t_j &= (1-x_1)(1-x_2)(1-x_3)(1-x_4)x_1^3x_2^3x_3^3x_4^3\\
&= 1-\sum_{j=1}^{4}x_j+\sum_{1\le j<k\le 4}x_jx_k-\sum_{1\le j<k<\ell\le 4}x_jx_kx_\ell+x_1x_2x_3x_4\\
&= 1-1+5-(-2)+1 = 8.
\end{aligned}$$

Next,

$$\begin{aligned}
t_1t_2t_3 &= x_1^2x_2^2x_3^2x_4^3-(x_1x_2x_3^2x_4^2+x_1x_2^2x_3x_4^2+x_1^2x_2x_3x_4^2)\\
&\quad+(x_2x_3x_4+x_1x_3x_4+x_1x_2x_4)-1\\
&= x_4-(x_1x_4+x_2x_4+x_3x_4)\\
&\quad+(x_2x_3x_4+x_1x_3x_4+x_1x_2x_4)-1,\\
t_1t_2t_4 &= x_3-(x_1x_3+x_2x_3+x_3x_4)\\
&\quad+(x_2x_3x_4+x_1x_3x_4+x_1x_2x_3)-1,\\
t_1t_3t_4 &= x_2-(x_1x_2+x_2x_3+x_2x_4)\\
&\quad+(x_2x_3x_4+x_1x_2x_4+x_1x_2x_3)-1,\\
t_2t_3t_4 &= x_1-(x_1x_2+x_1x_3+x_1x_4)\\
&\quad+(x_1x_3x_4+x_1x_2x_4+x_1x_2x_3)-1,
\end{aligned}$$

and

$$\sum_{1\le j<k<\ell\le 4} t_j t_k t_\ell = \sum_{j=1}^{4} x_j - 2\sum_{1\le j<k\le 4} x_j x_k + 3\sum_{1\le j<j<\ell\le 4} x_j x_k x_\ell$$
$$= 1 - 2(5) + 3(-2) - 4 = -19.$$

Finally,

$$\begin{aligned} t_1t_2 &= x_1x_2x_3^2x_4^2 - x_2x_3x_4 - x_1x_3x_4 + 1 \\ &= x_3x_4 - x_2x_3x_4 - x_1x_3x_4 + 1, \\ t_1t_3 &= x_2x_4 - x_2x_3x_4 - x_1x_2x_4 + 1, \\ t_1t_4 &= x_2x_3 - x_2x_3x_4 - x_1x_2x_3 + 1, \\ t_2t_3 &= x_1x_4 - x_1x_3x_4 - x_1x_2x_4 + 1, \\ t_2t_4 &= x_1x_3 - x_1x_3x_4 - x_1x_2x_3 + 1, \\ t_3t_4 &= x_1x_2 - x_1x_2x_4 - x_1x_2x_3 + 1, \end{aligned}$$

and

$$\sum_{1\le j<k\le 4} t_j t_k = \sum_{1\le j<k\le 4} x_j x_k - 3\sum_{1\le j<k<\ell\le 4} x_j x_k x_\ell + 6$$
$$= 5 - 3(-2) + 6 = 17.$$

Therefore, t_1, t_2, t_3, t_4 are roots of the equation

$$t^4 + 6t^3 + 17t^2 + 19t + 8 = 0.$$

Solution 2 by the proposer. First, we observe that all the roots of the given equation are nonzero. We have that

$$t_1 = (x_2 + x_3 + x_4)\, x_2x_3x_4 = (x_1 + x_2 + x_3 + x_4 - x_1)\cdot \frac{x_1x_2x_3x_4}{x_1}.$$

Since on account of Viète formulae $x_1 + x_2 + x_3 + x_4 = 1$ and $x_1x_2x_3x_4 = 1$, then $t_1 = \dfrac{1-x_1}{x_1} = \dfrac{1}{x_1} - 1$ from which $x_1 = \dfrac{1}{1+t_1}$. Likewise, $x_2 = \dfrac{1}{1+t_2}$, $x_3 = \dfrac{1}{1+t_3}$ and $x_4 = \dfrac{1}{1+t_4}$. Carrying out in the given equation the change of variables $x = \dfrac{1}{1+t}$ we get

$$\frac{1}{(1+t)^4} - \frac{1}{(1+t)^3} + \frac{5}{(1+t)^2} + \frac{2}{1+t} + 1 = 0,$$

from which it follows that $t^4 + 6t^3 + 17t^2 + 19t + 8 = 0$.

Solution 3 by Todor Zaharinov, Sofia, Bulgaria. Let

$$t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 0 \tag{1}$$

be an equation of fourth degree with roots t_1, t_2, t_3, t_4. By Vieta's formulas,

$$\begin{aligned} -a_3 &= t_1 + t_2 + t_3 + t_4, \\ a_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4, \\ -a_1 &= t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4, \\ a_0 &= t_1t_2t_3t_4. \end{aligned}$$

From the equation

$$x^4 - x^3 + 5x^2 + 2x + 1 = 0$$

and Vieta's formulas, it follows that

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 + x_4, \\ 5 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ -2 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \\ 1 &= x_1x_2x_3x_4. \end{aligned}$$

Observe that

$$\begin{aligned} t_1 &= (x_2 + x_3 + x_4)x_2x_3x_4 = (1 - x_1)x_2x_3x_4 \\ &= x_2x_3x_4 - x_1x_2x_3x_4 = x_2x_3x_4 - 1 = \frac{1}{x_1} - 1. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} t_2 &= x_1x_3x_4 - 1 = \frac{1}{x_2} - 1, \\ t_3 &= x_1x_2x_4 - 1 = \frac{1}{x_3} - 1, \\ t_4 &= x_1x_2x_3 - 1 = \frac{1}{x_4} - 1. \end{aligned}$$

Now,

$$\begin{aligned}
-a_3 &= t_1 + t_2 + t_3 + t_4\\
&= (x_2x_3x_4 - 1)+(x_1x_3x_4 - 1)+(x_1x_2x_4 - 1)+(x_1x_2x_3 - 1)\\
&= -2 - 4 = -6, \quad \text{so} \quad a_3 = 6;\\
a_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4\\
&= \frac{1}{x_1x_2x_3x_4} \cdot ((x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)\\
&\quad - 3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) + 6(x_1x_2x_3x_4))\\
&= 5 - 3 \cdot (-2) + 6 = 17;\\
-a_1 &= t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4\\
&= \frac{1}{x_1x_2x_3x_4}.((x_1 + x_2 + x_3 + x_4)\\
&\quad - 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)\\
&\quad + 3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) - 4(x_1x_2x_3x_4))\\
&= 1 - 2 \cdot 5 + 3 \cdot (-2) - 4 = -19, \quad \text{or} \quad a_1 = 19;\\
a_0 &= t_1t_2t_3t_4 =\\
&= \frac{1}{x_1x_2x_3x_4} \cdot (1 - (x_1 + x_2 + x_3 + x_4)\\
&\quad + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)\\
&\quad - (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) + (x_1x_2x_3x_4))\\
&= 1 - 1 + 5 - (-2) + 1 = 8.
\end{aligned}$$

It follows that (1) becomes

$$t^4 + 6t^3 + 17t^2 + 19t + 8 = 0.$$

Also solved by *Michel Bataille, Rouen, France; Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA, and Daniel Văcaru, Pitești, Romania.*

Medium–Hard Problems

MH–83. *Proposed by Dorin Mărghidanu, Colegiul National "A.I. Cuza", Corabia, Romania.* Let a, b, c, d be the lengths of the sides of a quadrilateral. Prove that

$$\sum_{\text{cyclic}} \sqrt{\frac{b+c+d-a}{a}} > 4.$$

Solution 1 by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Since a, b, c, d are the lengths of the sides of a quadrilateral, then $x = \sqrt{b+c+d-a}$, $y = \sqrt{c+d+a-b}$, $z = \sqrt{d+a+b-c}$, $t = \sqrt{a+b+c-d}$ are positive reals. Squaring them, we get $x^2 = b+c+d-a$, $y^2 = c+d+a-b$, $z^2 = d+a+b-c$, $t^2 = a+b+c-d$, and

$$a = \frac{y^2+z^2+t^2-x^2}{4}, \qquad b = \frac{z^2+t^2+x^2-y^2}{4},$$
$$c = \frac{t^2+x^2+y^2-z^2}{4}, \qquad d = \frac{x^2+y^2+z^2-t^2}{4}.$$

On account of the preceding and the AM-GM inequality, we have

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{\frac{b+c+d-a}{a}} &= 2\sum_{\text{cyclic}} \frac{x}{\sqrt{y^2+z^2+t^2-x^2}} \\
&= 2\sum_{\text{cyclic}} \frac{x^2}{x\sqrt{y^2+z^2+t^2-x^2}} \\
&\geq 4\sum_{\text{cyclic}} \frac{x^2}{x^2+(y^2+z^2+t^2-x^2)} \\
&= 4\sum_{\text{cyclic}} \frac{x^2}{y^2+z^2+t^2} \\
&> 4\sum_{\text{cyclic}} \frac{x^2}{x^2+y^2+z^2+t^2} = 4.
\end{aligned}$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA. We have

$$\begin{aligned}\sum_{\text{cyclic}} \sqrt{\frac{b+c+d-a}{a}} &= \sum_{\text{cyclic}} \frac{b+c+d-a}{\sqrt{a(b+c+d-a)}} \\ &\geq 2\sum_{\text{cyclic}} \frac{b+c+d-a}{b+c+d} \\ &= 2\sum_{\text{cyclic}} \left(1-\frac{a}{b+c+d}\right) \\ &= 2\left(4-\sum_{\text{cyclic}} \frac{a}{b+c+d}\right) > 4,\end{aligned}$$

where the inequality holds by the AM-GM inequality. Now, the last inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{a}{b+c+d} < 2.$$

We note that $b+c+d > (a+b+c+d)/2$, or $a/(b+c+d) < 2a/(a+b+c+d)$ follows from $b+c+d > a$. Therefore,

$$\sum_{\text{cyclic}} \frac{a}{b+c+d} < \frac{2}{a+b+c+d} \sum_{\text{cyclic}} a = 2,$$

as required.

Also solved by *Daniel Văcaru, Pitești, Romania.*

MH–84. *Proposed by Nicolae Papacu, Slobozia, Romania..* [Correction] Let z_1, z_2, z_3 be three distinct nonzero complex numbers and let $z = z_1 + z_2 + z_3$.

1. If $|z - z_k| = |z_k|$ for $1 \leq k \leq 3$, then prove that $z_1 + z_2 + z_3 = 0$.
2. Let p be an integer that it is not a multiple of 3. If $|z_1| = |z_2| = |z_3|$, then prove that $z_1^p + z_2^p + z_3^p = 0$.

Solution by the proposer. (1) For $1 \leq k \leq 3$, we have $|z - z_k| = |z_k| \iff (z - z_k)(\overline{z} - \overline{z}_k) = z_k\overline{z}_k$ or $z\overline{z} = z\overline{z}_k + z_k\overline{z}$. Adding up these expressions yields

$$3\,z\overline{z} = z\sum_{k=1}^{3}\overline{z}_k + \overline{z}\sum_{k=1}^{3} z_k \iff |z| = z\overline{z} = 0 \Rightarrow z = 0.$$

(2) Let $u = z_2/z_1$ and $v = z_3/z_1$. Since $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, then $|u| = |v| = 1$ and $1 + u + v = 0$. Let $u = \cos\alpha + i\sin\alpha$ and $v = \cos\beta + i\sin\beta$ with $\alpha, \beta \in [0, 2\pi)$. From $1 + u + v = 0$, we get

$$\begin{aligned} 1 + \cos\alpha + \cos\beta &= 0, \\ \sin\alpha + \sin\beta &= 0. \end{aligned}$$

We have $1 = \sin^2\alpha + \cos^2\beta = \sin^2\alpha + (1+\cos\alpha)^2$ or $1+2+2\cos\alpha$, from which we get $\cos\alpha = -1/2$. Likewise, we obtain $\cos\beta = -1/2$. Then, $\alpha, \beta \in \{2\pi/3, 4\pi/3\}$. Let $\alpha = 2\pi/3$ and $\beta = 4\pi/3$, then $u = \epsilon$ and $v = \epsilon^2$, where $\epsilon = \cos(2\pi/3) + i\sin(2\pi/3)$. Then, $\epsilon^3 = 1$ and $1 + \epsilon + \epsilon^2 = 0$.

Thus, $z_1^p + z_2^p + z_3^p = z_1^p(1 + \epsilon^p + \epsilon^{2p})$. We distinguish two cases:

- If $p \equiv 1 \pmod 3$, then $\epsilon^p = \epsilon$, $\epsilon^{2p} = \epsilon^2$ and

$$z_1^p + z_2^p + z_3^p = z_1^p(1 + \epsilon^p + \epsilon^{2p}) = z_1^p(1 + \epsilon + \epsilon^2) = 0.$$

- If $p \equiv 2 \pmod 3$, then $\epsilon^p = \epsilon^2$, $\epsilon^{2p} = \epsilon^4 = \epsilon$ and

$$z_1^p + z_2^p + z_3^p = z_1^p(1 + \epsilon^p + \epsilon^{2p}) = z_1^p(1 + \epsilon^2 + \epsilon) = 0.$$

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

MH–85. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all triples (x, y, z) of nonnegative integers such that $x^3 + 4y^3 = 2z^3$.

Solution 1 by Todor Zaharinov, Sofia, Bulgaria. It is readily checked that $(0, 0, 0)$ is a solution.

First assume that $x, y, z > 0$.

Assume that (x, y, z) is a solution in positive integers. Clearly x is divisible by 2, so $x = 2x_1$ for some positive integer x_1. Then,

$$8x_1^3 + 4y^3 = 2z^3, \text{ hence } 4x_1^3 + 2y^3 = z^3.$$

Now $z = 2z_1$ for some positive integer z_1, and $2x_1^3 + y^3 = 4z_1^3$.

Finally, $y = 2y_1$ for some positive integer y_1, and we find

$$x_1^3 + 4y_1^3 = 2z_1^3.$$

Thus, if (x, y, z) is a solution of the equation

$$x^3 + 4y^3 = 2z^3 \tag{1}$$

in positive integers, then so is $(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$.

Repeating this argument we find that for every positive solution there is a smaller solution in positive integers: but this is nonsense, thus there is no solution in positive integers.

Let $x = 0$, $y > 0$, $z > 0$. Assume that $(0, y, z)$ is a solution. Then, (1) is $2y^3 = z^3$, $z = 2z_1$, $y^3 = 4z_1^3$, $y = 2y_1$, $2y_1^3 = z_1^3$ and if $(0, y, z)$ is a solution, then so is $(0, \frac{y}{2}, \frac{z}{2})$. Repeating the arguments above, we see that there is no solution for $x = 0$.

Similarly we see that no solution and for $y = 0$ or $z = 0$.

Thus, $(0, 0, 0)$ is the only solution.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA. We see that $(x, y, z) = (0, 0, 0)$ is a triple satisfying the given Diophantine equation. Now we will use the method of infinite descent to show that this is the *only* solution in nonnegative integers.

If there is a solution (x, y, z) in positive integers, the Well-Ordering Principle implies that there is a solution (x, y, z) with x *minimal*. Writing the equation as $x^3 = 2z^3 - 4y^3 = 2(z^3 - 2y^3)$, we see that x^3 is even, implying that x must be even. Letting $x = 2m$, $m \in \mathbb{Z}^+$, we see that $(2m)^3 = 2(z^3 - 2y^3)$, or $z^3 = 2(2m^3 + y^3)$, so that z

is even. If we let $z = 2n$, $n \in \mathbb{Z}^+$, we can write $y^3 = 2(2n^3 - m^3)$ and conclude that $y = 2u$ for some positive integer u.

Finally, we have $(2u)^3 = 2(2n^3 - m^3)$, or $m^3 = 2n^3 - 4u^3$, so that (m, u, n) is a solution of our original Diophantine equation. But this is a contradiction since $m < x$, and we assumed that x was a minimal value. Therefore, $x^3 + 4y^3 = 2z^3$ has only the solution $(0, 0, 0)$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Solution 3 by the proposer. The triple $(0, 0, 0)$ trivially satisfies the equations and it is a solution. Triples of the form $(0, x, y)$ are not solution because, if $x = 0$, then $4y^3 = 2z^3$ or $y^3 = (1/2)z^3$ and $(y/z)^3 = 1/2$ has no solution in integers. Likewise, triples of the form $(x, 0, z)$ and $(x, y, 0)$ are not solutions. Suppose that (x_1, y_1, z_1) is a solution of the given equation, where x_1, y_1, z_1 are positive integers. From $x_1^3 + 4y_1^3 = 2z_1^3$, it follows that x_1 is even, say $x_1 = 2x_2$ with x_2 a positive integer smaller that x_1. Plugging that in the last equation, we get $8x_2^3 + 4y_1^3 = 2z_1^3$, which simplifies to $4x_2^3 + 2y_1^3 = z_1^3$. By a similar argument, $z_1 = 2z_2$. Plugging that in, we get $4x_2^3 + 2y_1^3 = 8z_2^3$, which simplifies to $2x_2^3 + y_1^3 = 4z_2^3$, and $y_1 = 2y_2$. Plugging these in and simplifying gives us $x_2^3 + 4y_2^3 = 2z_2^3$. This way, we obtain a new solution (x_2, y_2, z_2) to our equation with $x_1 > x_2$, $y_1 > y_2$, $z_1 > z_2$. But by repeating the procedure, we can construct infinitely many solutions (x_i, y_i, z_i), where $x_1 > x_2 > \ldots > x_i > \ldots$ which is impossible by Fermat Method of Infinite Descent. So, the given equation has no solutions in positive integers and the only solution in nonnegative integers is $(0, 0, 0)$.

Also solved by *Michel Bataille, Rouen, France; Daniel Văcaru, Pitești, Romania, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

MH–86. *Proposed by Mihály Bencze, Brașov, Romania.* Let $F, G\colon \mathbb{R} \to \mathbb{R}$ be the primitives of the functions $f, g\colon \mathbb{R} \to \mathbb{R}$, respectively. Determine $f(x)$ and $g(x)$ if for all $x \in \mathbb{R}$ it holds that $f(x) + G(x) = x$ and $F(x) + g(x) = -x$.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca,

Spain. According to the given equations, by differentiation, we obtain

$$f'(x) + g(x) = 1, \qquad f(x) + g'(x) = -1.$$

Adding these two equations, we get

$$(f(x) + g(x))' + (f(x) + g(x)) = 0,$$

and subtracting the second from the first,

$$(f(x) - g(x))' - (f(x) - g(x)) = 2.$$

We solve these two first order linear differential equations, yielding

$$f(x) + g(x) = Pe^{-x}, \qquad f(x) - g(x) = Qe^{x} - 2, \tag{1}$$

respectively, where P and Q are constants.

Solving (1) simultaneously, we find

$$f(x) = Ae^{-x} + Be^{x} - 1, \qquad g(x) = Ae^{-x} - Be^{x} + 1,$$

where $A = \frac{1}{2}P$ and $B = \frac{1}{2}Q$.

Solution 2 by Michel Bataille, Rouen, France. The functions F, G are differentiable and $F'(x) = f(x)$, $G'(x) = g(x)$ for all $x \in \mathbb{R}$. Suppose that the conditions of the statement hold. Then, from $f(x) + G(x) = x$ for all $x \in \mathbb{R}$, we deduce that $f(x) = x - G(x)$, hence f is differentiable as the sum of two differentiable functions and $f'(x) = 1 - g(x)$. Similarly, g is differentiable and $g'(x) = -1 - f(x)$. Now let $s(x) = f(x) + g(x)$ and $d(x) = f(x) - g(x)$. Since $f'(x)+g(x) = 1$, $g'(x)+f(x) = -1$, we have $s'(x)+s(x) = 0$ and $d'(x) - d(x) = 2$. It first follows that $s(x) = ae^{-x}$ and $d(x) = -2 + be^{x}$ for some real constants a, b and then that

$$f(x) = \frac{1}{2}\big(-2 + ae^{-x} + be^{x}\big), \qquad g(x) = \frac{1}{2}\big(2 + ae^{-x} - be^{x}\big). \tag{2}$$

Conversely, if f anf g are defined by (2), then F, G defined by $F(x) = -1 - x + \frac{be^{x} - ae^{-x}}{2}$ and $G(x) = 1 + x - \frac{be^{x} + ae^{-x}}{2}$ for all $x \in \mathbb{R}$ are primitives of f and g, respectively, and $f(x) + G(x) = x$, $F(x) + g(x) = -x$ hold for all $x \in \mathbb{R}$.

Thus, the solutions are given by (2), a, b being arbitrary real constants.

Also solved by *Daniel Văcaru, Pitești, Romania; the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia, and the proposer.*

MH–87. *Proposed by Oriol Baeza Guasch, CFIS, BarcelonaTech, Terrassa, Spain.* Let ABC be a triangle, and let D, E be the contact points of the incircle and the A-excircle on side BC, respectively. Let F be the intersection of ray AD with the A-excircle, closest to D; and let G be the intersection of ray AE with the incircle, closest to E. Prove that FG, side BC and the bisector of A concur.

Solution 1 by the proposer.

Claim 1. *$DEFG$ is cyclic, with center M the midpoint of BC.*

Proof. Let rays AD, AG intersect by second time the A-excircle at $D' \neq F$, $G' \neq E$. Then, $ADG \sim AD'G'$ and so a simple angle chasing using the A-excircle gives

$$\begin{aligned} \angle ADG = \angle AD'G' = \angle FD'G' = 180^\circ - \angle FEG' = \angle FEG \\ \implies \angle FDG = 180^\circ - \angle ADG = 180^\circ - \angle FEG, \end{aligned}$$

so $(DEFG)$ is cyclic.

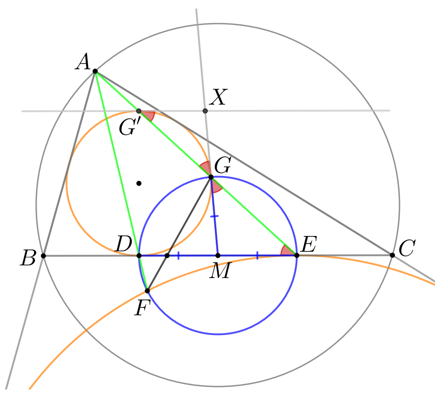


Figure 3: Scheme for solution 1 of Problem MH–87 (proof of Claim 1).

To find the center, let M be the point where the tangent to the incircle by G meets BC. By power of a point, $MD = MG$.

Let us also show that $MG = ME$. Indeed, recall that the other intersection point of ray AE with the incircle, say point G', is diametrically opossite to D. Hence, the tangent by G' is a line parallel to BC.

With that in mind, and putting X the point where the tangents to the incircle by G' and by G meet (that is, the pole of GG'), we have a simple angle chasing

$$\angle MGE = \angle G'GX = \angle GG'X = \angle GEM.$$

Hence, M is the circumcentre of (DEG), thus also of $(DEFG)$. Since M is on BC and it is well known that D, E are isotomic conjugates, we conclude that M is the midpoint of BC. □

Now we introduce an auxiliary point, that is, $K := DG \cap EF$. Also, denote by $N := FG \cap DE$, and we will show that N belongs to AI. For that, we will show that AI and AN are both perpendicular to MK.

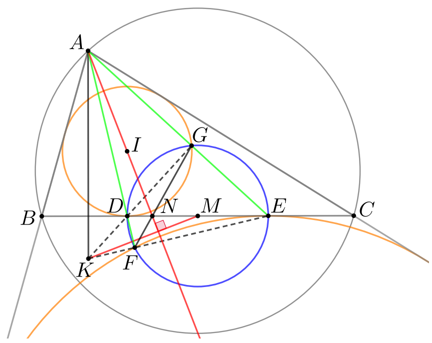


Figure 4: Scheme for solution 1 of Problem MH–87.

Claim 2. $AN \perp MK$.

Proof. This follows from construction as AK is the polar of N, and since M is the center of $(DEFG)$ we get $AN \perp MK$. □

Claim 3. $AI \perp MK$.

Proof. Let r be the radical axis of the incircle and the A-excircle. By concurrence of the radical axis with these circles together with $(DEFG)$ we have that $K = DG \cap EF$ belongs to r.

On the other hand, since $MD = ME$, that also means M belongs to r. Hence, $r = K \vee M$ and given that the radical axis is perpendicular to the line joining the centers (that is, the bisector of A), we conclude that $AI \perp KM$. $\square$

So A, I, N are collinear, meaning AI, FG, BC concur, and we are finished.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let a, b, c denote the sides of $\triangle ABC$ in the usual order and let s be its semiperimeter.

Let the ray AD meets the $A-$ excircle again at F' and let the ray AE meets the incircle again at G' .

Since the length of the tangents from vertex A to the incircle are $s - a$ and the tangents from A to the excircle beyond a are length s, the circle with center A and radius $\sqrt{s(s-a)}$ invert the incircle and the $A-$ excircle into themselves, so that

$$AD \cdot AF = s(s-a) = AG \cdot AE. \tag{1}$$

By the power of a point theorem,

$$AF \cdot AF' = s^2, \quad AG \cdot AG' = (s-a)^2. \tag{2}$$

Remembering that $BD = s - b = CE$, it follows that $DE = a - 2(s-b) = b - c$ (which we are assuming to be positive; otherwise interchange B and C). Again by the power of a point theorem,

$$EG \cdot EG' = (b-c)^2 = DF \cdot DF'.$$

From (1) and (2), then,

$$\frac{AD}{AF'} = \frac{s-a}{s} = \frac{AG'}{AE}.$$

Hence,

$$G'E = AE - AG' = AE - \frac{s-a}{s}AE = \frac{a}{s}AE$$

and

$$DF' = AF' - AD = \frac{s}{s-a}AD - AD = \frac{a}{s-a}AD.$$

Let $\{X\} = FG \cap BC$. Applying Menelaus's theorem to the triad of points FXG on the sides of triangle ADE, we obtain

$$\frac{AF}{FD} \cdot \frac{DX}{XE} \cdot \frac{EG}{GA} = 1.$$

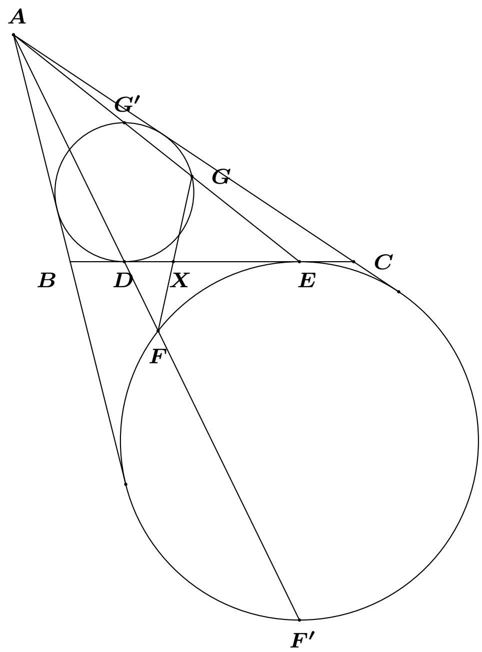


Figure 5: Scheme for solution 2 of Problem MH–87.

Using the above, $AD \cdot AF = s(s-a)$, $DF \cdot DF' = (b-c)^2$, and $\frac{DF'}{AD} = \frac{a}{s-a}$, we get

$$\frac{AF}{FD} = \frac{as}{(b-c)^2}.$$

Likewise, using that $AG \cdot AE = s(s-a)$, $EG \cdot EG' = (b-c)^2$, and $\dfrac{G'E}{AE} = \dfrac{a}{s}$ we obtain

$$\frac{EG}{GA} = \frac{(b-c)^2}{a(s-a)}.$$

Therefore,

$$\frac{as}{(b-c)^2} \cdot \frac{DX}{XE} \cdot \frac{(b-c)^2}{a(s-a)} = 1,$$

yielding

$$\frac{DX}{XE} = \frac{s-a}{s}.$$

Adding 1 to each side gives

$$\frac{DX + XE}{XE} = \frac{2s-a}{s}.$$

Since $DX + XE = DE = b - c$ and $2s = a + b + c$, we have

$$XE = \frac{(b-c)s}{b+c}$$

and

$$XC = XE + EC = \frac{(b-c)s}{b+c} + (s-b) = \frac{ab}{b+c},$$

making AX the angle bisector of $\angle CAB$ (accordingly the reciprocal of the internal angle bisector theorem applied to $\triangle ABC$ at A) and we are done.

Also solved by *Michel Bataille, Rouen, France, and Titu Zvonaru, Comanesti, Romania.*

MH–88. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The 2020-th prime number is 17573, and the next prime is 17579. Show that there exists a sequence of 2020^{2020} consecutive positive integers that contains exactly 2020 prime numbers.

Solution by the proposer. It is well-known and easy to see that there exists a sequence of N consecutive positive integers that are all composite. Indeed, the sequence

$$(N+1)!+2,(N+1)!+3,\ldots,(N+1)!+N,(N+1)!+(N+1)$$

consists of N consecutive composite numbers, as claimed.

Let $N = 2020^{2020}$. Since $N > 17579$, there are more than 2020 primes in the sequence $1, 2, \ldots, N$. To solve the problem, let

$$a, a+1, a+2, \ldots, a+N-1$$

be a sequence of N consecutive positive integers with no prime. We carry out repeatedly the following operation to the numbers in the preceding sequence: delete the far-right number $a+N-1$, and append to the far-left the number $a-1$. The resulting sequence

$$a-1, a, a+1, \ldots, a+N-2$$

has at most one prime. We may repeat this operation until we reach the sequence $1, 2, \ldots, N$, which has more than 2020 primes. Performing such operation either keeps, increases by one, or decreases by one the number of primes of the previous sequence. Since the starting sequence has no prime at all, while the last sequence has more than 2020 primes, there exists a sequence (after applying the operation a convenient number of times) that contains exactly 2020 primes.

Advanced Problems

A–83. *Proposed by Henry Ricardo, Westchester Area Math Circle, NY, USA.* For any nonnegative integer n, define

$$I_n = \int_0^{\pi/2} \frac{\sin^2 nt}{\sin t}\,\mathrm{d}t.$$

Find $\lim\limits_{n\to\infty}(2I_n - \ln n)$.

Solution 1 by Michel Bataille, Rouen, France. The required limit is $\gamma + 2\ln 2$, where γ denotes the Euler constant.

Let t be such that $\sin t \neq 0$. For all positive integer n, the following relation (R_n) holds:

$$\frac{\sin^2 nt}{\sin t} = \sum_{k=1}^{n} \sin[(2k-1)t].$$

This follows from

$$\begin{aligned}\sum_{k=1}^{n} (\sin t)\sin[(2k-1)t] &= \frac{1}{2}\sum_{k=1}^{n}(\cos[(2k-2)t] - \cos(2kt))\\ &= \frac{1}{2}(1-\cos(2nt)) = \sin^2(nt).\end{aligned}$$

The relation (R_n) first shows that $\lim\limits_{t\to 0^+} \frac{\sin^2 nt}{\sin t} = 0$, hence the integral exists, and then that

$$I_n = \sum_{k=1}^{n}\left[\frac{-\cos[(2k-1)t]}{2k-1}\right]_0^{\pi/2} = \sum_{k=1}^{n}\frac{1}{2k-1} = H_{2n} - \frac{H_n}{2},$$

where $H_m = \sum\limits_{k=1}^{m}\frac{1}{k}$ denotes the mth harmonic number.

From the known $H_n = \ln n + \gamma + o(1)$ as $n \to \infty$, we deduce that

$$\begin{aligned}2I_n &= 2H_{2n} - H_n = 2(\ln(2n) + \gamma + o(1)) - (\ln n + \gamma + o(1))\\ &= \ln n + 2\ln 2 + \gamma + o(1)\end{aligned}$$

and the announced result follows.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Using the identity

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$

it follows that

$$\begin{aligned}\sin t \sum_{j=1}^{n} \sin(2j-1)t &= \frac{1}{2}\sum_{j=1}^{n}(\cos(2j-2)t - \cos 2jt)\\ &= \frac{1}{2}(1-\cos 2nt) = \sin^2 nt.\end{aligned}$$

Thus,

$$\frac{\sin^2 nt}{\sin t} = \sum_{j=1}^{n} \sin(2j-1)t,$$

and

$$\begin{aligned}I_n &= \int_0^{\pi/2} \frac{\sin^2 nt}{\sin t}\,\mathrm{d}t = \sum_{j=1}^{n}\int_0^{\pi/2} \sin(2j-1)t\,\mathrm{d}t\\ &= \sum_{j=1}^{n}\left(-\frac{1}{2j-1}\cos(2j-1)t\right)\Bigg|_0^{\pi/2}\\ &= \sum_{j=1}^{n}\frac{1}{2j-1} = \sum_{j=1}^{2n}\frac{1}{j} - \frac{1}{2}\sum_{j=1}^{n}\frac{1}{j} = H_{2n} - \frac{1}{2}H_n, ,\end{aligned}$$

where H_n denotes the nth harmonic number. As $n \to \infty$,

$$H_n \sim \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

where γ is the Euler-Mascheroni constant, so

$$I_n \sim \ln 2 + \frac{1}{2}\ln n + \frac{1}{2}\gamma + O\left(\frac{1}{n^2}\right).$$

Finally,

$$\begin{aligned}\lim_{n\to\infty}(2I_n - \ln n) &= \lim_{n\to\infty}\left(2\ln 2 + \ln n + \gamma - \ln n + O\left(\frac{1}{n^2}\right)\right)\\ &= 2\ln 2 + \gamma.\end{aligned}$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. We show that the limit as $n \to \infty$ equals $\gamma + 2\ln 2$, where γ denotes the Euler-Mascheroni constant.

It is easy to see that $I_0 = 0$ and $I_1 = 1$. Now for $n \geq 1$ we have

$$I_n = \int_0^{\pi/2} \frac{\sin^2 nt}{\sin t}\,\mathrm{d}t = \int_0^{\pi/2} \frac{1 - \cos 2nt}{2\sin t}\,\mathrm{d}t,$$

so

$$\begin{aligned} I_n - I_{n-1} &= \int_0^{\pi/2} \frac{\cos 2(n-1)t - \cos 2nt}{2\sin t}\,\mathrm{d}t \\ &= \int_0^{\pi/2} \frac{2\sin t \cdot \sin(2n-1)t}{2\sin t}\,\mathrm{d}t = \int_0^{\pi/2} \sin(2n-1)t\,\mathrm{d}t \\ &= \frac{1}{2n-1}. \end{aligned}$$

Then,

$$\begin{aligned} I_n &= \sum_{k=1}^{n}(I_k - I_{k-1}) = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right) \\ &= H_{2n} - \frac{1}{2}H_n, \end{aligned}$$

where H_n denotes the harmonic number $\sum_{k=1}^{n} 1/k$.

Finally,

$$\begin{aligned} 2I_n - \ln n &= 2H_{2n} - H_n - \ln n \\ &= 2(H_{2n} - \ln 2n) - (H_n - \ln n) + 2\ln 2, \end{aligned}$$

which tends to $2\gamma - \gamma + 2\ln 2 = \gamma + 2\ln 2$ as $n \to \infty$.

Also solved by *Moti Levy, Rehovot, Israel; Daniel Văcaru, Pitești, Romania; the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia, and the proposer.*

A–84. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $A \in M_2(\mathbb{R})$ be a matrix with $\det A = 3$ and

$$\det\bigl(A^4 + 5A^2 + 9I_2\bigr) = 81.$$

Show that at least one of the matrices $A + I_2$ or $A - I_2$ is invertible.

Solution 1 by Michel Bataille, Rouen, France. For the purpose of a contradiction, assume that neither $A + I_2$ nor $A - I_2$ is invertible. Then, 1 and -1 are eigenvalues of A, hence are the eigenvalues of A. It follows that $\det A = -1$ (the product of the eigenvalues), contradicting the hypothesis $\det A = 3$. Thus, $A + I_2$ or $A - I_2$ must be invertible.

Solution 2 by Daniel Văcaru, Pitești, Romania. We know that $\det(A + x\mathrm{I}_2) = \det A + \mathrm{tr}A \cdot x + x^2$. Then, we have

$$\begin{aligned}
\det^2(A+\mathrm{I}_2)+\det^2(A-\mathrm{I}_2) &= (\det A+\mathrm{tr}A+1)^2+(\det A-\mathrm{tr}A+1)^2 \\
&= (4 - \mathrm{tr}A)^2 + (4 + \mathrm{tr}A)^2 \\
&= 32 + 2\mathrm{tr}^2 A > 0.
\end{aligned}$$

It follows that at least one of $\det(A + \mathrm{I}_2)$ and $\det(A - \mathrm{I}_2)$ is $\neq 0$. That proves the statement.

Solution 3 by Moti Levy, Rehovot, Israel. If $A \in M_2(\mathbb{R})$, the following holds:

$$\begin{aligned}
\det(A + I_2) &= 1 + \det(A) + \mathrm{tr}(A), \\
\det(A - I_2) &= 1 + \det(A) - \mathrm{tr}(A).
\end{aligned}$$

In our case,

$$\begin{aligned}
\det(A + I_2) &= 4 + \mathrm{tr}(A), \\
\det(A - I_2) &= 4 - \mathrm{tr}(A).
\end{aligned} \tag{1}$$

It follows that, if $\mathrm{tr}(A) \neq -4$, then $A + I_2$ is invertible; if $\mathrm{tr}(A) \neq 4$, then $A - I_2$ is invertible.

So let us find the trace of A.

If $A, B \in M_2(\mathbb{R})$ then following three identities hold:

$$\det(A+B) = \det(A) + \det(B) + \operatorname{tr}\bigl(\det(A)A^{-1}B\bigr), \tag{2}$$

$$\operatorname{tr}\bigl(A^{-1}\bigr) = \frac{\operatorname{tr}(A)}{\det(A)}, \tag{3}$$

$$\operatorname{tr}\bigl(A^{-2}\bigr) = \left(\frac{\operatorname{tr}(A)}{\det(A)}\right)^2 - \frac{2}{\det(A)}. \tag{4}$$

Observe

$$\begin{aligned}\det\bigl(A^4 + 5A^2 + 9I_2\bigr) &= \det\bigl(\bigl(A^2 + A + 3I_2\bigr)\bigl(A^2 - A + 3I_2\bigr)\bigr)\\ &= \det\bigl(A^2 + A + 3I_2\bigr)\det\bigl(A^2 - A + 3I_2\bigr).\end{aligned} \tag{5}$$

Using (2), (3), and (4) we express $\det(A^2 + A + 3I_2)$ in terms of $\operatorname{tr}(A)$:

$$\begin{aligned}&\det\bigl(A^2 + A + 3I_2\bigr)\\ &= \det\bigl(A^2\bigr) + \det(A + 3I_2) + \operatorname{tr}\bigl(\det\bigl(A^2\bigr)A^{-2}(A + 3I_2)\bigr)\\ &= 9 + \det(A + 3I_2) + 9\operatorname{tr}\bigl(A^{-1}\bigr) + 27\operatorname{tr}\bigl(A^{-2}\bigr)\\ &= 9 + 3 + 9 + 3\operatorname{tr}(A) + 9\operatorname{tr}\bigl(A^{-1}\bigr) + 27\operatorname{tr}\bigl(A^{-2}\bigr)\\ &= 21 + 3\operatorname{tr}(A) + 9\operatorname{tr}\bigl(A^{-1}\bigr) + 27\operatorname{tr}\bigl(A^{-2}\bigr)\\ &= 21 + 3\operatorname{tr}(A) + 9\frac{\operatorname{tr}(A)}{3} + 27\left(\frac{1}{9}(\operatorname{tr}(A))^2 - \frac{2}{3}\right)\\ &= 3(\operatorname{tr}(A) + 1)^2.\end{aligned} \tag{6}$$

Again, using (2), (3), and (4) we express $\det(A^2 - A + 3I_2)$ in terms of $\operatorname{tr}(A)$:

$$\begin{aligned}&\det\bigl(A^2 - A + 3I_2\bigr)\\ &= \det\bigl(A^2\bigr) + \det(-A + 3I_2) + \operatorname{tr}\bigl(\det\bigl(A^2\bigr)A^{-2}(-A + 3I_2)\bigr)\\ &= 9 + \det(-A + 3I_2) - 9\operatorname{tr}\bigl(A^{-1}\bigr) + 27\operatorname{tr}\bigl(A^{-2}\bigr)\\ &= 9 + 3 + 9 - 3\operatorname{tr}(A) - 9\operatorname{tr}\bigl(A^{-1}\bigr) + 27\operatorname{tr}\bigl(A^{-2}\bigr)\\ &= 21 - 3\operatorname{tr}(A) - 9\operatorname{tr}\bigl(A^{-1}\bigr) + 27\operatorname{tr}\bigl(A^{-2}\bigr)\\ &= 21 - 3\operatorname{tr}(A) - 9\frac{\operatorname{tr}(A)}{3} + 27\left(\frac{1}{9}(\operatorname{tr}(A))^2 - \frac{2}{3}\right)\\ &= 3(\operatorname{tr}(A) - 1)^2.\end{aligned} \tag{7}$$

By (6), (7) and (5),

$$\det\left(A^4 + 5A^2 + 9I_2\right) = 9(\mathrm{tr}(A)+1)^2(\mathrm{tr}(A)-1)^2 = 81.$$

The real solutions for $\mathrm{tr}(A)$ are

$$\mathrm{tr}(A) = 2 \quad \text{or} \quad \mathrm{tr}(A) = -2.$$

We conclude that both $A + I_2$ and $A - I_2$ are invertible.

Also solved by *Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA; the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia, and the proposer.*

A–85. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate

$$\int_{-1}^{1} \frac{\arccos x}{\sqrt{3x^4 + 2x^2 + 3}}\,\mathrm{d}x.$$

Solution 1 by Michel Bataille, Rouen, France. Let I denote the integral to be evaluated. The change of variables $x = -u$ and the relation $\arccos(-u) = \pi - \arccos u$ give

$$I = \int_{-1}^{1} \frac{\pi - \arccos u}{\sqrt{3u^4 + 2u^2 + 3}}\,\mathrm{d}u = 2\pi \cdot J - I, \tag{1}$$

where

$$J = \int_{0}^{1} \frac{\mathrm{d}u}{\sqrt{3u^4 + 2u^2 + 3}}.$$

We calculate J with the change of variables $u = \tan t$. Observing that

$$3u^4 + 2u^2 + 3 = 3(u^2+1)^2 - 4u^2 = \frac{3}{\cos^4 t} - \frac{4\sin^2 t}{\cos^2 t} = \frac{3 - \sin^2 2t}{\cos^4 t}$$

and that $\mathrm{d}u = \frac{\mathrm{d}t}{\cos^2 t}$, we obtain

$$J = \frac{1}{\sqrt{3}} \int_{0}^{\pi/4} \frac{\mathrm{d}t}{\sqrt{1 - \frac{1}{3}\sin^2 2t}} = \frac{1}{2\sqrt{3}} \int_{0}^{\pi/2} \frac{\mathrm{d}v}{\sqrt{1 - \frac{1}{3}\sin^2 v}}.$$

Thus,

$$J = \frac{1}{2\sqrt{3}} K(1/\sqrt{3}),$$

where for $0 \le k < 1$, $K(k)$ denotes the elliptic integral (of the first kind) $\int_0^{\pi/2} \frac{\mathrm{d}v}{\sqrt{1-k^2\sin^2 v}}$.

Returning to (1), we conclude that

$$I = \pi J = \frac{\pi}{2\sqrt{3}} K(1/\sqrt{3}).$$

Solution 2 by Moti Levy, Rehovot, Israel. Set

$$\begin{aligned} I &:= \int_{-1}^{1} \frac{\arccos(x)}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x \\ &= \int_{-1}^{1} \frac{\arccos(x) - \frac{\pi}{2}}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x + \frac{\pi}{2}\int_{-1}^{1} \frac{1}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x. \end{aligned}$$

The first definite integral is zero since the integrand is an odd function. Hence,

$$I = \pi \int_0^1 \frac{1}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x.$$

A quote from the classic book of Richard Courant, *Differential and Integral Calculus*:

"Attempts to express general integrals such as $\int \frac{\mathrm{d}x}{\sqrt{a_0+a_1x+\ldots+a_nx^n}}$, $\int \sqrt{a_0+a_1x+\ldots+a_nx^n}\,\mathrm{d}x$ or $\int \frac{e^x}{x}\,\mathrm{d}x$ in terms of elementary functions have always ended in failure; and in the nineteenth century it was finally proved that it is actually impossible to carry out these integrations in terms of elementary functions... The first and most important example which leads us beyond the region of elementary functions is given by *elliptic integrals*."

By changing the integration variable $t = x^2$,

$$I = \frac{\pi}{2\sqrt{3}} \int_0^1 \left(\frac{1}{\sqrt{t}\sqrt{t^2+\frac{2}{3}t+1}} \right) \mathrm{d}t.$$

The elliptic integral of the first kind is defined as

$$F(\varphi,k) := \int_0^{\varphi} \frac{\mathrm{d}u}{\sqrt{1-k^2\sin^2(u)}}.$$

By changing the integration variable $u = 2\arctan\sqrt{z}$, $z = \tan^2\left(\frac{u}{2}\right)$, we obtain

$$F(\varphi,k) = \int_0^{\tan^2\left(\frac{\varphi}{2}\right)} \frac{1}{\sqrt{z}\sqrt{(z^2+2(1-2k^2)z+1)}}\,\mathrm{d}z.$$

(The idea for this specific change of variable came from Gradshteyn and Ryzhik, entry 3.138, 5).

Setting $\varphi = \frac{\pi}{2}$ and $k = \frac{1}{\sqrt{3}}$, we get

$$\mathrm{K}\left(\frac{1}{\sqrt{3}}\right) := F\left(\frac{\pi}{2}, \frac{1}{\sqrt{3}}\right) = \int_0^1 \frac{1}{\sqrt{z}\sqrt{\left(z^2+\frac{2}{3}z+1\right)}}\,\mathrm{d}z.$$

We conclude that

$$I = \frac{\pi}{2\sqrt{3}}\mathrm{K}\left(\frac{1}{\sqrt{3}}\right) \cong 1.57249,$$

where $\mathrm{K}(k)$ is the complete elliptic integral of the first kind.

Solution 3 by the proposer. We will use the notation

$$A = \int_{-1}^{1} \frac{\arccos x}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x.$$

We also consider the integral

$$B = \int_{-1}^{1} \frac{\arcsin x}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x.$$

We have

$$A+B = \int_{-1}^{1} \frac{\arccos x + \arcsin x}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x.$$

Because the function under the integral sign is even, we have

$$\int_{-1}^{1} \frac{1}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x = 2\int_0^1 \frac{1}{\sqrt{3x^4+2x^2+3}}\,\mathrm{d}x.$$

We are going to calculate the integral

$$C = \int_0^1 \frac{1}{\sqrt{3x^4 + 2x^2 + 3}}\,\mathrm{d}x.$$

We will show that the integral C can be expressed using the complete elliptic integral of the first kind, which is defined by the relationship

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}}\,\mathrm{d}\theta,\ k \in (-1, 1).$$

With the substitution

$$t = \tan\frac{\theta}{2} \implies \sin\theta = \frac{2t}{1+t^2};\quad \mathrm{d}\theta = \frac{2}{1+t^2}\,\mathrm{d}t,$$

we obtain

$$K(k) = \int_0^1 \frac{1}{\sqrt{1 - k^2\frac{4t^2}{(1+t^2)^2}}}\frac{2}{1+t^2}\,\mathrm{d}t = 2\int_0^1 \frac{1}{\sqrt{t^4 + (2-4k^2)t^2 + 1}}\,\mathrm{d}t.$$

We also have

$$C = \frac{1}{\sqrt{3}} \int_0^1 \frac{1}{\sqrt{x^4 + \frac{2}{3}x^2 + 1}}\,\mathrm{d}x.$$

We put the condition

$$2 - 4k^2 = \frac{2}{3} \implies k = \frac{1}{\sqrt{3}} \quad \text{(we put the condition } k > 0\text{).}$$

We obtain $C = \frac{1}{\sqrt{3}}\frac{1}{2}K\left(\frac{1}{\sqrt{3}}\right)$.

The integral B is 0, because the function under the integral sign is odd. Thus, we have $A = \dfrac{\pi}{2}2C$.

We obtained the value of the integral required in the problem statement:

$$A = \frac{\pi}{2\sqrt{3}}K\left(\frac{1}{\sqrt{3}}\right).$$

A–86. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Suppose V is a finite inner-product space and $v, w \in V$. Let $T\colon V \to V$ be a linear transformation defined by $Tu = \langle u, v\rangle w$. Find a formula for the trace of T.

Solution 1 by Michel Bataille, Rouen, France. The trace of T is $\langle w, v\rangle$. This is clearly true if $w = 0$. If $w \neq 0$, we can find vectors in V such that $(w, e_2, \ldots, e_n)$ is a basis of V (where $n = \dim V$). The columns of the matrix of T in this basis then are

$$\begin{pmatrix} \langle w, v\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \langle e_2, v\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \langle e_n, v\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The trace of T can be read on this matrix: it is $\langle w, v\rangle = \langle w, v\rangle + 0 + \cdots + 0$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V, and let $v = \sum_i a_i v_i$, $w = \sum_i b_i v_i$. Then,

$$\begin{aligned}
Tv_1 &= \langle v_1, v\rangle w = \langle v_1, \sum_i a_i v_i\rangle w = a_1\langle v_1, v_1\rangle w = a_1 w \\
&= a_1 b_1 v_1 + a_1 b_2 v_2 + \ldots = a_1 b_n v_n, \\
Tv_2 &= \langle v_2, v\rangle w = \langle v_2, \sum_i a_i v_i\rangle w = a_2\langle v_2, v_2\rangle w = a_2 w \\
&= a_2 b_1 v_1 + a_2 b_2 v_2 + \ldots = a_2 b_n v_n, \\
&\vdots \\
Tv_n &= \langle v_n, v\rangle w = \langle v_n, \sum_i a_i v_i\rangle w = a_n\langle v_n, v_n\rangle w = a_n w \\
&= a_n b_1 v_1 + a_n b_2 v_2 + \ldots = a_n b_n v_n.
\end{aligned}$$

Since the trace of a linear operator is independent of the basis, we see that

$$\text{Trace}(T) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \langle v, w\rangle.$$

Solution 3 by the proposer. First suppose that $v \neq 0$. Extend $(v/||v||)$ to an orthonormal basis $\left(\frac{v}{||v||}, e_2, \ldots, e_n\right)$ of V. Note that, for $2 \leq j \leq n$, we have

$$Te_j =< e_j, v > w = 0w = 0$$

(because $< e_j, v >= 0$). The trace of T equals the sum of the diagonal entries in the matrix of T with respect to the basis $\left(\frac{v}{||v||}, e_2, \ldots, e_n\right)$, as is well-known. Thus,

$$\begin{aligned}
\operatorname{tr} T &= \left\langle T\frac{v}{||v||}, \frac{v}{||v||}\right\rangle + \left\langle Te_2, e_2\right\rangle + \ldots + \left\langle Te_n, e_n\right\rangle \\
&= \left\langle\left\langle\frac{v}{||v||}, v\right\rangle w, \frac{v}{||v||}\right\rangle = \left\langle\frac{||v||^2}{||v||} w, \frac{v}{||v||}\right\rangle \\
&= \langle w, v\rangle.
\end{aligned}$$

If $v = 0$, then $T = 0$ and so $\operatorname{tr} T = 0 = \langle w, v\rangle$. Thus we have the formula

$$\operatorname{tr} T = \langle w, v\rangle$$

regardless of whether or not $v = 0$.

Also solved by *Moti Levy, Rehovot, Israel, and the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

A–87. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq k \geq 0$ be integers. If $\gamma = \{z \in \mathbb{C} : |z| = 1\}$, then prove that

$$\frac{1}{2^{n+1}\pi i} \sum_{k=0}^{n} \oint_\gamma \frac{(1+z)^n}{z^{k+1}} \,\mathrm{d}z$$

is an integer and determine its value.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. The problem follows by Cauchy's differentiation formula

$$f^{k)}(a) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{k+1}}$$

where, in our case, $f(z) = (1+z)^n$ is infinitely differentiable. Since $f(z) = \sum_{k=0}^{n} \binom{n}{k} z^k$, then, by Taylor expansion, $\binom{n}{k} = \frac{f^{k)}(0)}{k!}$ and therefore the given expression is

$$\frac{1}{2^{n+1}\pi i} \sum_{k=0}^{n} \oint_{\gamma} \frac{(1+z)^n}{z^{k+1}}\, \mathrm{d}z = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} = 1.$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. If $\gamma = \{z \in \mathbb{C} : |z| = 1\}$, then $z = e^{i\theta}$ and $\mathrm{d}z = ie^{i\theta}\, \mathrm{d}\theta$, so that

$$\begin{aligned}
\oint_{\gamma} \frac{(1+z)^n}{z^{k+1}}\, \mathrm{d}z &= \int_0^{2\pi} \frac{(1+e^{i\theta})^n}{e^{i(k+1)\theta}} \cdot ie^{i\theta}\, \mathrm{d}\theta \\
&= i \int_0^{2\pi} \sum_{j=0}^{n} \binom{n}{j} e^{i(j-k)\theta}\, \mathrm{d}\theta \\
&= i \sum_{j=0}^{n} \binom{n}{j} \int_0^{2\pi} e^{i(j-k)\theta}\, \mathrm{d}\theta = i \binom{n}{k} 2\pi
\end{aligned}$$

since the last integral is 0 for $j \neq k$ and 2π for $j = k$.

Therefore,

$$\frac{1}{2^{n+1}\pi i} \sum_{k=0}^{n} \oint_{\gamma} \frac{(1+z)^n}{z^{k+1}}\, \mathrm{d}z = \frac{1}{2^{n+1}\pi i} \cdot 2\pi i \cdot \sum_{k=0}^{n} \binom{n}{k} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} = 1.$$

Comment by Henry Ricardo. The article “Summation of Series by the Residue Theorem,” *Math. Mag.* **44** (1971), 24-26 provides simple examples of the residue theorem applied to the summation of real series involving binomial coefficients.

Also solved by *Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Moti Levy, Rehovot, Israel; the SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia, and the proposer.*

A–88. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $(A, +, \circ)$ be a commutative ring. If I and J are two ideals of A, then prove that

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$

and compute $\sqrt{4320\mathbb{Z}} \cap \sqrt{3024\mathbb{Z}}$.

Solution 1 by Michel Bataille, Rouen, France. For $x \in A$ and $n \in \mathbb{N}$, let x^n be defined by $x^1 = x$ and the recursion $x^n = x \circ x^{n-1}$ for $n \geq 2$. Recall that the set $\sqrt{I}$ of all $x \in A$ such that $x^n \in I$ for some positive integer n is an ideal containing I.

Let $x \in \sqrt{I \cap J}$. Then, $x^n \in I \cap J$ for some positive integer n, hence $x^n \in I$ and $x^n \in J$ and therefore $x \in \sqrt{I}$ and $x \in \sqrt{J}$. Thus, $\sqrt{I \cap J} \subset \sqrt{I} \cap \sqrt{J}$. Conversely, if $x \in \sqrt{I} \cap \sqrt{J}$, then $x^n \in I$ for some positive integer n and $x^m \in J$ for some positive integer m. It follows that $x^{n+m} = x^m \circ x^n \in I$ (since $x^n \in I$ and $x^m \in A$) and similarly $x^{n+m} = x^n \circ x^m \in J$. As a result, $x^{m+n} \in I \cap J$ and $x \in \sqrt{I \cap J}$. This shows that $\sqrt{I} \cap \sqrt{J} \subset \sqrt{I \cap J}$. Finally, we have $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Note that, by an immediate induction, if $I_1, I_2, \ldots, I_s$ are ideals of A $(s \geq 2)$, then $\sqrt{I_1 \cap I_2 \cap \ldots \cap I_s} = \sqrt{I_1} \cap \sqrt{I_2} \cap \ldots \cap \sqrt{I_s}$.

Now, let p be a prime and let m be a positive integer. Then, $\sqrt{p^m\mathbb{Z}}$ is an ideal $k\mathbb{Z}$ where k is a positive divisor of p^m (since $p^m\mathbb{Z} \subset k\mathbb{Z}$). We have $k \neq 1$ (since $1^n = 1 \notin p^m\mathbb{Z}$ for any positive integer n), hence $k \in \{p, p^2, \ldots, p^m\}$. In any event, we have $\sqrt{p^m\mathbb{Z}} = k\mathbb{Z} \subset p\mathbb{Z}$. Conversely, if $x \in p\mathbb{Z}$, then $x^m \in p^m\mathbb{Z}$ and so $x \in \sqrt{p^m\mathbb{Z}}$. In conclusion, $\sqrt{p^m\mathbb{Z}} = p\mathbb{Z}$.

Now, we have $\sqrt{4320\mathbb{Z}} = \sqrt{(2^5 \times 3^3 \times 5)\mathbb{Z}} = \sqrt{(2^5\mathbb{Z} \cap 3^3\mathbb{Z} \cap 5\mathbb{Z})}$, hence

$$\sqrt{4320\mathbb{Z}} = \sqrt{2^5\mathbb{Z}} \cap \sqrt{3^3\mathbb{Z}} \cap \sqrt{5\mathbb{Z}} = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}.$$

Since $3024 = 2^4 \times 3^3 \times 7$, we similarly obtain

$$\sqrt{3024\mathbb{Z}} = (2 \times 3 \times 7)\mathbb{Z} = 42\mathbb{Z}$$

and we conclude that

$$\sqrt{4320\mathbb{Z}} \cap \sqrt{3024\mathbb{Z}} = (6 \times 5)\mathbb{Z} \cap (6 \times 7)\mathbb{Z} = (6 \times 5 \times 7)\mathbb{Z} = 210\mathbb{Z}.$$

Solution 2 by the proposer. First, we claim that, if $I \subset J$, then $\sqrt{I} \subset \sqrt{J}$. Indeed, if $x \in \sqrt{I}$, then there exists n such that $x^n \in I \subset J$ and $x \in \sqrt{J}$, from which it follows that $\sqrt{I} \subset \sqrt{J}$.

Since $I \cap J \subset I$ and $I \cap J \subset J$, then $\sqrt{I \cap J} \subset \sqrt{I}$ and $\sqrt{I \cap J} \subset \sqrt{J}$, from which it follows that $\sqrt{I \cap J} \subset \sqrt{I} \cap \sqrt{J}$. Now, suppose $x \in \sqrt{I} \cap \sqrt{J}$. Then, there exist two integers m, n such that $x^n \in I$ and $x^m \in J$. On account of the definition of an ideal, we have $x^n \cdot x^m$ belongs to I and to J. So, $x^n \cdot x^m = x^{n+m} \in I \cap J$. Hence, x is an element of $\sqrt{I \cap J}$ and $\sqrt{I} \cap \sqrt{J} \subset \sqrt{I \cap J}$. From the preceding, we get $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$, as desired.

In $\mathbb{Z}$, we have that $\sqrt{4320\mathbb{Z}}$ is the set of integer x such that there exist a power x^n which is a multiple of 4320. Since $4320 = 2^5 \cdot 3^3 \cdot 5$ then a power of x say x^n will be divisible by these factors if and only if x is divisible by $2 \cdot 3 \cdot 5 = 30$, and one have that $\sqrt{4320\mathbb{Z}} = 30\mathbb{Z}$. Likewise, since $3024 = 2^4 \cdot 3^3 \cdot 7$, then we have that $\sqrt{3024\mathbb{Z}} = 42\mathbb{Z}$. Finally, we get

$$\sqrt{4320\mathbb{Z} \cap 3024\mathbb{Z}} = \sqrt{4320\mathbb{Z}} \cap \sqrt{3024\mathbb{Z}} = 30\mathbb{Z} \cap 42\mathbb{Z} = 210\mathbb{Z}.$$

Remark. Note that, $\sqrt{4320\mathbb{Z} \cap 3024\mathbb{Z}} = \sqrt{30240\mathbb{Z}} = 210\mathbb{Z}$ because $30240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$.

Also solved by *Moti Levy, Rehovot, Israel; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and Daniel Văcaru, Pitești, Romania.*

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