ENGR 313—CIRCUITS AND INSTRUMENTATION

Node-Voltage Method Activity

INTRODUCTION

Combining resistive elements into equivalent resistances and using Kirchhoff's voltage and current laws are powerful tools for circuit analysis. For more complicated circuits containing multiple power supplies, dependent power supplies, and multiply connected resistors, a more systematic analysis approach is required. In this activity, you will learn the node-voltage method of circuit analysis. This method uses a system of independent equations based on Kirchhoff's current law to analyze the circuit.

Work in groups of up to three to complete the following exercises. Each group must turn in *one* completed worksheet.

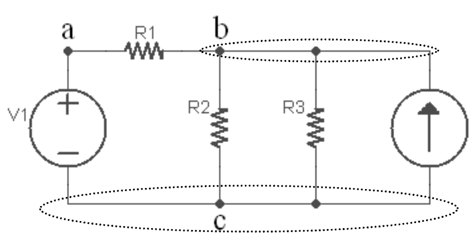
LEARNING OBJECTIVES

- Identify nodes, essential nodes, branches, and essential branches in a circuit
- Determine the number of independent simultaneous equations required to analyze a circuit using the node-voltage method.
- Use the node-voltage method to analyze and solve circuits.

EXERCISE 1—TERMINOLOGY

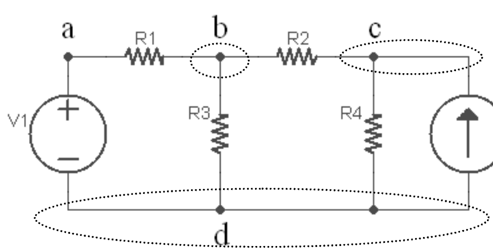
Examine the following three examples.

Example 1



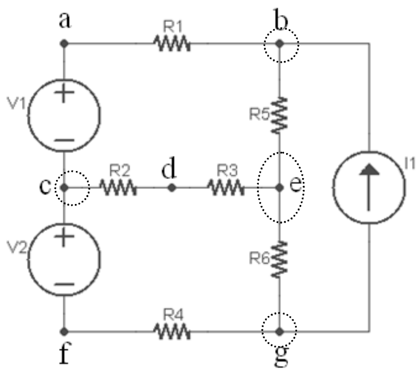
- The *nodes* of this circuit are: a, b, c
  - Note that four circuit elements are connected to node b — R_1, R_2, R_3, and I_1 (the dashed ovals are intended to illustrate a single node — they will be removed in future examples.)
  - Note that four elements are connected to node c — V_1, R_2, R_3, and I_1
- The *essential nodes* of this circuit are: b, c
- The *branches* of this circuit are: R_1, R_2, R_3, V_1, I_1

Example 2



- The *nodes* of this circuit are: a, b, c, d
  - Node b connects three circuit elements
  - Node c connects three circuit elements
  - Node d connects four circuit elements
- The *essential nodes* of this circuit are: b, c, d
- The *branches* of this circuit are: R_1, R_2, R_3, R_4, V_1, I_1
- The *essential branches* of this circuit are: $R_1 \leftrightarrow V_1$, R_2, R_3, R_4, I_1
- Two (2) *node-voltage equations* are required for this circuit.

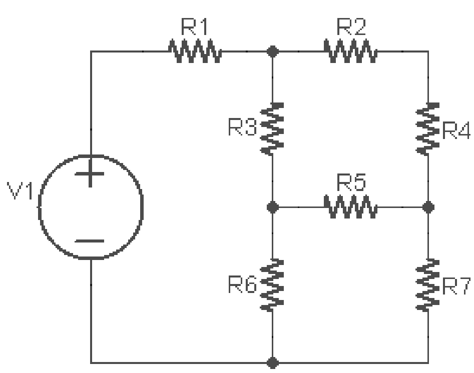
Example 3



- The *nodes* of this circuit are: a, b, c, d, e, f, g
  - Node b connects three circuit elements
  - Node c connects three circuit elements
  - Node e connects three circuit elements
  - Node g connects three circuit elements
- The *essential nodes* of this circuit are: b, c, e, g
- The *branches* of this circuit are: R_1, R_2, R_3, R_4, R_5, R_6, V_1, V_2, I_1
- The *essential branches* of this circuit are: $R_1 \leftrightarrow V_1$, R_5, I_1, $R_2 \leftrightarrow R_3$, $V_2 \leftrightarrow R_4$, R_6
- Three (3) *node-voltage equations* are required to solve this circuit.

EXERCISE 1 QUESTIONS

1. Based on the examples above, define the following circuit terminology :
   - Node—
   - Essential node—
   - Branch—
   - Essential Branch—
2. What is the *minimum* number of circuit elements that are connected to an essential node?__________
3. What is the relationship between the number of required node-voltage equations and the number of essential nodes, n_e, in the circuit?
4. For the following circuit, identify the nodes, essential nodes, branches, essential branches, and the number of node-voltage equations required for full analysis of the circuit.



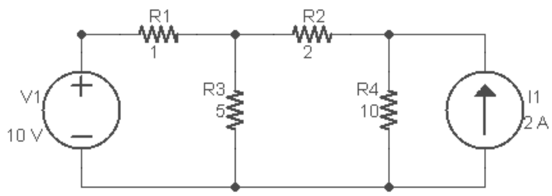
- Nodes:
- Essential Nodes:
- Branches:
- Essential Branches:
- Number of node-voltage equations:

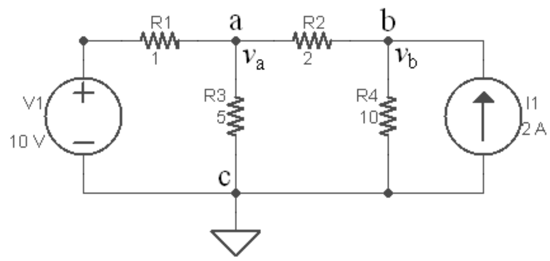
EXERCISE 2—USING THE NODE-VOLTAGE METHOD

Examine the following two examples:

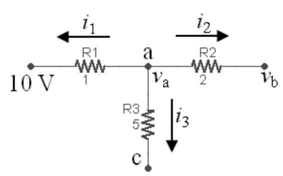
Example 4

Determine the current through R_4, the 10 Ω resistor, using the node-voltage method.





- This circuit has three essential nodes; therefore, two node-voltage equations are required.
- Label the essential nodes as a, b, and c.
- The essential branches are $R_1 \leftrightarrow V_1$, R_2, R_3, R_4, and I_1
- Arbitrarily choose an essential node as a reference node. Here c will be the *reference* node and designate with a signal ground symbol. All node voltages will be referenced to this point.
- Define the voltage, with respect to the reference node, at the essential nodes a and b as v_a and v_b, respectively.
- Use Kirchoff's current law to write conservation equations for each node, in terms of node voltages.



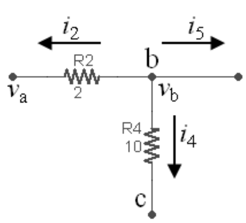
- Isolate node a and apply Kirchhoff's current law to this node:

$$i_1 + i_2 + i_3 = 0$$

- Use Ohm's law to compute the current through each branch in terms of node voltages:

$$\frac{v_a - V_1}{R_1} + \frac{v_a - v_b}{R_2} + \frac{v_a - v_c}{R_3} = \frac{v_a - 10\text{ V}}{1\,\Omega} + \frac{v_a - v_b}{2\,\Omega} + \frac{v_a - 0\text{ V}}{5\,\Omega} = 0\text{ A}$$

- Using algebra, the first node-voltage equation is: $17v_a - 5v_b = 100\text{ V}$



- Next, isolate node b and apply Kirchhoff's current law to this node:

$$i_2 + i_4 + i_5 = 0$$

- Using Ohm's law and recognizing that i_5 = -2A because of the current provided by supply I_1:

$$\frac{v_b - v_a}{R_2} + \frac{v_b - v_c}{R_4} - I_1 = \frac{v_b - v_a}{2\,\Omega} + \frac{v_b - 0\text{ V}}{10\,\Omega} - 2\text{ A} = 0\text{ A}$$

- Multiplying by 10 Ω and rearranging gives the second node-voltage equation:

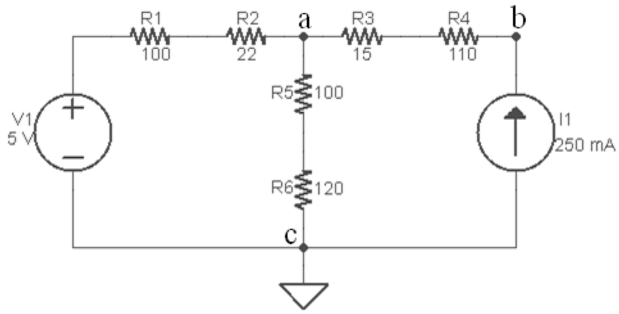
$$-5v_a + 6v_b = 20\text{ V}$$

- Solving the two simultaneous node-voltage equations (using substitution, matrices, and/or a calculator) gives: v_a = 9.09 V and v_b = 10.9 V.
- Knowing v_b and using Ohm's law, the current through R_4 can be computed to be 1.09 A.

$$i_4 = \frac{v_b}{R_4} = \frac{10.9\text{ V}}{10\,\Omega} = 1.09\text{ A}$$

Example 5

Determine the voltage at node b with respect to the signal ground of the following circuit.



- This circuit has two essential nodes (a and c); therefore, one node-voltage equation is required.
- The essential branches are $R_1 \leftrightarrow R_2 \leftrightarrow V_1$, $R_3 \leftrightarrow R_4 \leftrightarrow I_1$, and $R_5 \leftrightarrow R_6$.
- Choose node c to be the *reference* node.

- Let i_1 be the current away from node a through the essential branch $R_1 \leftrightarrow R_2 \leftrightarrow V_1$. Similarly, let i_2 and i_3 represent the current away from node a through the essential branches $R_3 \leftrightarrow R_4 \leftrightarrow I_1$ and $R_5 \leftrightarrow R_6$, respectively. Applying Kirchhoff's current law to essential node a gives:

$$i_1 + i_2 + i_3 = 0$$

$$\frac{v_a - V_1}{R_1 + R_2} - I_1 + \frac{v_a - v_c}{R_5 + R_6} = \frac{v_a - 5\text{ V}}{100\ \Omega + 22\ \Omega} - 0.25\text{ A} + \frac{v_a - 0\text{ V}}{100\ \Omega + 120\ \Omega} = 0\text{ A}$$

- Rearranging and solving for v_a gives $v_a = 22.8$ V
- With the voltage at node a known, and knowing that 250mA of current flows from node b to node a, we can use Ohm's law to compute the potential difference in the branch between node a and node b.

$$v_b - v_a = v_b - 22.8\text{ V} = i_2(R_3 + R_4) = (0.25\text{ A})(15\ \Omega + 110\ \Omega)$$

- Solving, we find that $v_b = 54.05$ V = 54.1 V with respect to the signal ground (essential node c).

EXERCISE 2 QUESTIONS

6. When applying Kirchhoff's law to the essential nodes, what assumption is made about the direction of the current flow?

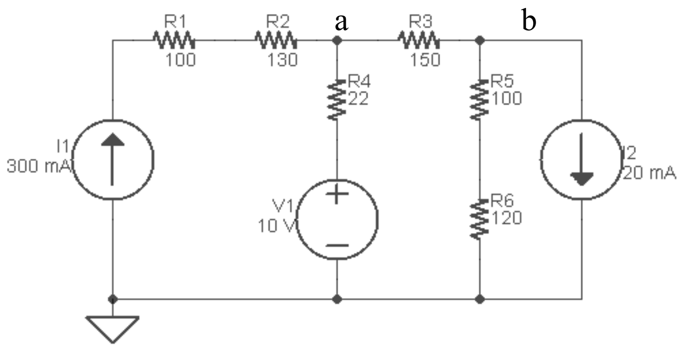
7. What is the assumed potential of the *arbitrarily selected* reference node?

8. How is the current in each essential branch represented and/or calculated?

9. Why were the currents from the current supplies in the examples written as negative values when developing the node voltage equations?

EXERCISE 2 QUESTIONS, CONTINUED

10. Determine the node voltage equations for the following circuit and solve for the voltages at nodes a and b of the circuit, with respect to the signal ground shown. Clearly show all of your steps and use additional paper if necessary.



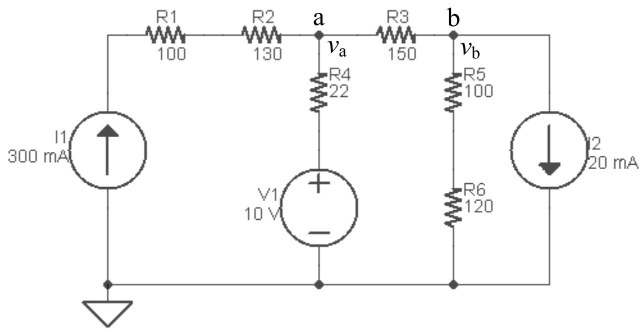
Answers:
$v_a = 15.4$ V
$v_b = 7.39$ V

Exercise 3—Special Cases for the Node-Voltage Method

Examine the following four examples:

Example 6

This circuit, from Question 10, has three essential nodes and requires the two independent node-voltage equations shown to completely specify the circuit.

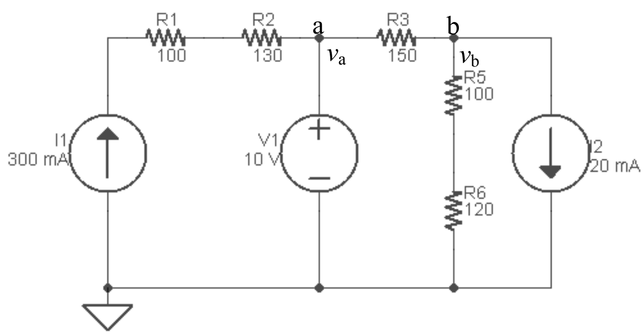


$$-0.3\text{ A}+\frac{v_a-10\text{ V}}{22\,\Omega}+\frac{v_a-v_b}{150\,\Omega}=0\text{ A}$$

$$\frac{v_b-v_a}{150\,\Omega}+\frac{v_b-0\text{ V}}{100\,\Omega+120\,\Omega}+0.02\text{ A}=0\text{ A}$$

Example 7

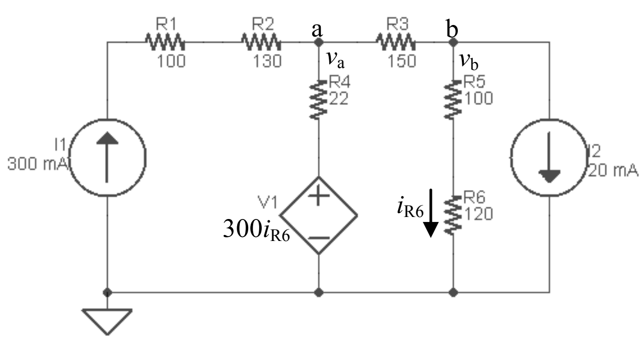
This circuit has three essential nodes. By recognizing that $v_a = 10$ V, only one node-voltage equation is needed.



$$\frac{v_b-10\text{ V}}{150\,\Omega}+\frac{v_b-0\text{ V}}{100\,\Omega+120\,\Omega}+0.02\text{ A}=0\text{ A}$$

Example 8

The *dependent* voltage supply, V_1, in the circuit below produces a voltage output that is 300 times the current through resistor R_6 (i.e., a gain of 300 V/A). This circuit has three essential nodes and requires two node-voltage equations and an additional *constraint* equation.



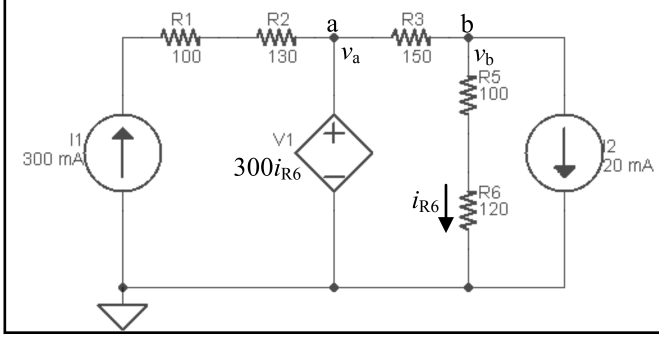
$$-0.3\text{ A}+\frac{v_a-300\,i_{R6}}{22\,\Omega}+\frac{v_a-v_b}{150\,\Omega}=0\text{ A}$$

$$\frac{v_b-v_a}{150\,\Omega}+\frac{v_b-0\text{ V}}{100\,\Omega+120\,\Omega}+0.02\text{ A}=0\text{ A}$$

$$i_{R6}=\frac{v_b-0\text{ V}}{100\,\Omega+120\,\Omega}$$

Example 9

The dependent voltage supply, V_1, in the circuit below produces a voltage output that is 300 times the current through resistor R_6. This circuit has three essential nodes. Recognizing that the voltage at node a with respect to the reference node is the voltage produced by the dependent voltage supply, $v_a = 300 i_{R6}$, one node-voltage equation and one *constraint* equation is required.



$$\frac{v_b - 300 i_{R6}}{150\ \Omega} + \frac{v_b - 0\ \text{V}}{100\ \Omega + 120\ \Omega} + 0.02\ \text{A} = 0\ \text{A}$$

$$i_{R6} = \frac{v_b - 0\ \text{V}}{100\ \Omega + 120\ \Omega}$$

EXERCISE 3 QUESTIONS

11. How does the number of required equations change when an essential node is connected to the reference node with a branch containing only a voltage supply (dependent or independent)? Why?

12. How does the number of required equations change when a dependent source is included in the circuit? Why?

Note that this is true when a dependent voltage or current source (not shown in the examples) is added to the circuit.

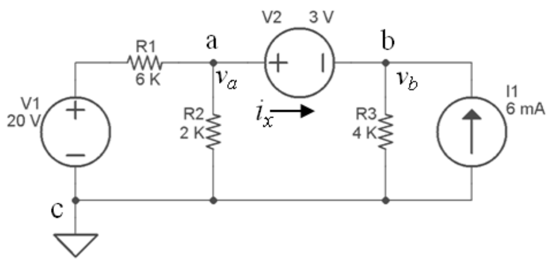
13. What is a *constraint* equation? How is it determined?

EXERCISE 4—SUPERNODES

Examine the following two examples:

Example 10—Supernode with Independent Voltage Supply

When an ideal voltage supply directly connects two nodes, neither of which are the zero-voltage reference node, a *supernode* exists. Nodes a and b in the following circuit form a *supernode*. Based on the schematic, the supply V_2 creates a 3 V potential between nodes a and b; however, we do not know the potential of node a or b with respect to the reference node, c.



To solve for the unknown node voltages in this circuit, begin by applying Kirchhoff's current law at node a.

$$i_{R1} + i_{R2} + i_{V2} = 0$$

Using Ohm's Law, the current through R_1 and R_2 can be expressed in terms of the unknown node voltage at node a. The voltage supply does not have a resistance, but current will leave (or enter) the node through the voltage supply. If we *arbitrarily assume* that current *leaves* node a through the voltage supply, and we call this unknown current i_x, we can rewrite the current balance at node a as

$$\frac{v_a - 20\text{ V}}{6{,}000\ \Omega} + \frac{v_a - 0\text{ V}}{2{,}000\ \Omega} + i_x = 0$$

Next, applying Kirchhoff's Current Law to node b gives

$$i_{V2} + i_{R3} + i_{I1} = 0$$

The unknown current through the voltage supply from node a, i_x, must enter (or leave) node b. Based on the assumption that the unknown current *exited* node a, we must assume that it *enters* node b. The current from the ideal current supply, I_1, is also directed towards the node. Therefore, the current balance at node b can be rewritten as

$$-i_x + \frac{v_b - 0\text{ V}}{4{,}000\ \Omega} - 0.006\text{ A} = 0$$

The current balance equations for nodes a and b both contain the unknown current, i_x, and can be combined to eliminate this as a variable, providing the *supernode equation*

$$\frac{v_a - 20\text{ V}}{6{,}000\ \Omega} + \frac{v_a - 0\text{ V}}{2{,}000\ \Omega} + \frac{v_b - 0\text{ V}}{4{,}000\ \Omega} = 0.006\text{ A}$$

Unfortunately, there is now one equation for the two unknown variables, v_a and v_b. To solve this circuit network, we need an additional independent equation. Luckily, we know from the schematic that the voltage supply V_2 creates a 3 V potential between nodes a and b. This can be written as a *constraint* equation, which provides the second equation necessary to solve for the unknown voltages at the essential nodes. Observing the polarity of the voltage supply (node a will be at a higher potential, since the + symbol is connected to this node), the constraint equation is written as

$$v_a - v_b = 3\text{ V}$$

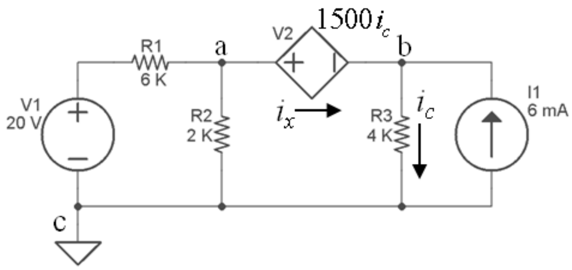
Solving the system of 2 equations for two unknowns gives

$$v_a = 11\text{ V}$$
$$v_b = 8\text{ V}$$

Example 11—Supernode with Dependent Voltage Supply

In this example, V_2 is a dependent voltage supply that creates a potential that is proportional to the current, i_C, through resistor R_3. This voltage supply provides a voltage of 1,500 V for every ampere of current through R_3 (i.e., 1,500 V/A). Because V_2 joins node a and b, neither of which are the reference node, this connection forms a supernode. As in Example 10, a current balance can be performed on nodes a and b, resulting in the supernode equation of

$$\frac{v_a - 20\text{ V}}{6{,}000\ \Omega} + \frac{v_a - 0\text{ V}}{2{,}000\ \Omega} + \frac{v_b - 0\text{ V}}{4{,}000\ \Omega} = 0.006\text{ A}$$



Again, the creation of this supernode equation reduces the number of independent equations available to completely solve the system. A constraint equation that defines the potential developed between nodes a and b must be developed. Based on the orientation of the voltage supply, node a must be at a higher potential than node b, and the potential difference must $1{,}500 i_C$.

Therefore, a constraint equation is $v_a - v_b = 1{,}500 i_C$

Unfortunately, the value of i_C is not known, so yet another constraint equation is necessary to solve this system. This constraint equation must relate the controlling current of the dependent supply, i_C, to the unknown node voltages. Luckily, we can apply Ohm's law to determine this constraint equation. Noting that the designated direction (given in the diagram) of i_C is *away* from node b, the potential at node b must be higher than the potential at node c, which has been selected as the 0 V reference node. Therefore, the second constraint equation becomes

$$i_C = \frac{v_b - 0V}{4{,}000\,\Omega}$$

Now, there are three equations to solve for the three unknown variables—v_a, v_b, and i_C. Solving the system of equations gives

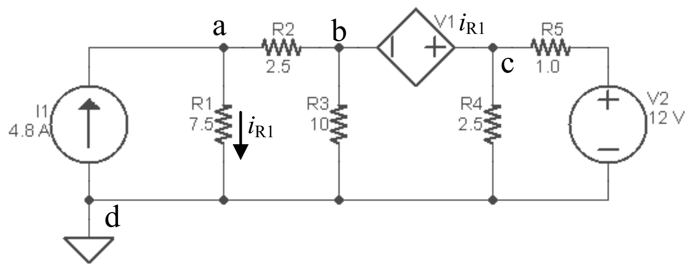
$$v_a = 11\text{ V}$$
$$v_b = 8\text{ V}$$
$$i_C = 0.002\text{ A} = 2\text{ mA}$$

EXERCISE 4 QUESTIONS

14. What is a *supernode*? When does it occur? How does it change the number of equations necessary to completely solve the system?

15. Outline the process for solving a circuit containing a supernode.

16. In the circuit below, the output of the dependent voltage supply is proportional to the current through resistor R_1 with a gain of 1 V/A. How many independent equations are required to analyze this circuit? Write the node-voltage and constraint equations and solve for the unknown node voltages. Clearly show your work and all steps involved. Use additional paper if necessary.



Answers:
$v_a = 15$ V
$v_b = 8$ V
$v_c = 10$ V
$i_{R1} = 2$ A