NON-EXAMPLES AND PROOF BY CONTRADICTION

Samuele Antonini

Department of Mathematics - University of Pisa, Italy

Researches in Mathematics Education about proof by contradiction revealed some difficulties of the students but also that this kind of argumentation comes spontaneously in certain situations. In this paper we shall show some processes that might lead the student to produce a proof by contradiction. In particular, we shall point out a deep link between a certain kind of examples (which we call non-examples) generated during the stage of conjecture production, and the structure of the argumentation with which the conjecture is justified.

INTRODUCTION

There are few researches in Mathematics Education which have specifically dealt with proof by contradiction. An analysis of these studies points out that the proof by contradiction, from cognitive and didactical points of view, seems to have the form of a paradox.

First, it is well known that proving by contradiction is a complex activity for the students of various scholastic levels. See, for example, Bernardi (2002), Antonini (2001), Reid (1998), Epp (1998), Thompson (1996), Barbin (1988), Leron (1985).

On the other side, some studies describe proof by contradiction as an argumentation that students spontaneously produce. For example, Freudenthal says:

"The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously '... if it were not so, it would happen that...' " (Freudenthal, 1973, p. 629).

Freudenthal concludes that "*Before the indirect proof is exhibited, it should have been experienced by the pupil*" (Freudenthal, 1973, p.629). Also Thompson writes:

"If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure."(Thompson, 1996, p.480).

Following Freudenthal, we take as working definition of *indirect argumentation* an argumentation like "*…if it were not so, it would happen that…*". A more articulate definition has been developed in [Antonini, 2003] but the one just given is enough for the topics treated in this paper.

Assuming that students spontaneously produce indirect argumentations, it is very interesting to study favourable conditions that can help the generation of such argumentations. The goal of this paper is to investigate the processes that lead to the construction of an indirect argumentation. In particular, we shall focus on some factors that favour the rise of indirect argumentation.

THEORETICAL FRAMEWORK

A suitable theoretical framework is the Cognitive Unity which has been introduced and developed in [Garuti et al.,1996a,1996b,1998; Mariotti et al.,1997; Pedemonte,2002]. The framework of Cognitive Unity is based on the following:

"during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices; during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ("arguments") produced during the construction of the statements according to a logical chain." (Garuti et al. 1996b, p.113).

Cognitive Unity regards the relations between processes of argumentation and of proof construction. In particular, Pedemonte (2002) analyses and compares the structure of the argumentation and of the proof produced by students. In this theoretical framework, we basically pose two questions. First, which are the factors that can favour the production of an indirect argumentation. Then, which factors can induce or block the transition from this argumentation to a proof by contradiction. In this paper we focus on the first question.

We begin by remarking that the analysis of the processes of the construction of the conjecture has shown how frequent it is for the students to produce one or more examples during the solving of an open-ended mathematical problem. We divide these examples in two classes: the ones that verify the conditions given by the problem and the ones that do not. In particular, a student facing a task like "*given an hypothesis A what can you deduce?*" can produce examples which verify A and examples which do not verify A. The latter will be called *non-examples*.

Non-examples are particularly interesting because, with the construction of a non-example, students seem to produce indirect argumentations like "*…if it were not so, it would happen that…*". Ones a non-example has been generated, the student seems to be led to a formulation of a conjecture in negative terms, like "*given A, it is not possibile for B to be true*".

Moreover, sometimes the students links the argumentation of the conjecture with the argumentation given to justify the fact that the example generated is a non-example: "*it is not possible for B to be true because, otherwise, A would not be true (as seen in the non-example)*".

The hypothesis we formulate is the following:

In task like "given A what can you deduce?" the conjecture can be produced via the analysis of a non-example. The argumentation that justifies the fact that the generated example is a non-example can be re-elaborated and become part of the argumentation of the conjecture. In this case, the argumentation takes an indirect form.

METHODOLOGY

The research that we are going to explain here is part of a wider work on proof by contradiction which is the subject of [Antonini, 2003]. To develop the research for this paper we interviewed 9 pairs of students: 7 pairs from the last three years of secondary school (grade 11 to 13) and 2 pairs from the second year of the degree course in biology at the University. The students have been given an open-ended problem and they have been asked to formulate a conjecture and to prove it. We decided to interviewed pairs of students to favour the rise of argumentative processes. This kind of methodology has already been successfully used by other researchers (for example see Pedemonte, 2002).

The choice of the problem for the interviews has been guided by the hypothesis made on the relations between the production of a non-examples and the construction of an indirect argumentation. The problem is the following:

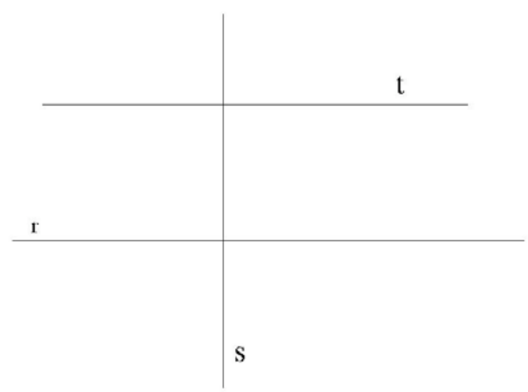
Two lines r and s on a plane have the following property: each line t intersecting r, intersect s too. Is there anything you can say about the reciprocal position of r and s? Why?

Let be A the property "*each line t intersecting r, intersect s too*". The problem asks to determine the reciprocal position of two lines verifying A. Hence the text is of the form: "*given A, what can you deduce?*". Moreover, since the position of the two lines is not known, indeed it is what has to be found out, the students have no initial configuration to refer to and that they can use to start the study of the problem. We think that this lack of a starting point can lead to the production of examples and to the analysis of various cases. In particular we expect that the students will consider two cases: two parallel lines and two crossing lines. Property A is verified only in the first case. Indeed, when r and s intersect each other, it is possible to find a line intersecting r and not intersecting s (just take any line parallel to s). We expect the students to produce non-examples, i.e. examples not verifying A, and, moreover, we expect them to propose an indirect argumentation for the conjecture formulated in this way.

Valerio and Cristina: non-examples and indirect argumentation

In what follows we give some excerpts from an interview in which it is evident how the production of a non-example determined the indirect structure of the argumentation. The two students, Valerio and Cristina (grade 13) are very well considered by their teacher.

18. V,C: .....
19. V: Oh, they [r and s] cannot be perpendicular.
20. C: No, that's not possible.
21. V: They cannot be perpendicular because otherwise it [the line t] could be parallel to one of the two and do not intersect the other one (he makes a drawing, see figure 1)



F

Figure 1

Valerio produce a non-example made by two perpendicular lines. The conjecture is formulated in negative terms ("*r and s cannot be perpendicular*") and it is justified indirectly: "*they cannot be perpendicular because otherwise*" A would not be true. Later, during the interview, Valerio generates a second non-example which recalls the first one.

31. V: Well, [the line t] cannot be parallel to any of the two lines because, if we have two crossing lines, even if they are not perpendicular (he makes a drawing, see figure 2), if [t] is parallel to one of the two, it intersect just one of them.
32. C: Yes, it's the same situation of the two perpendicular lines.

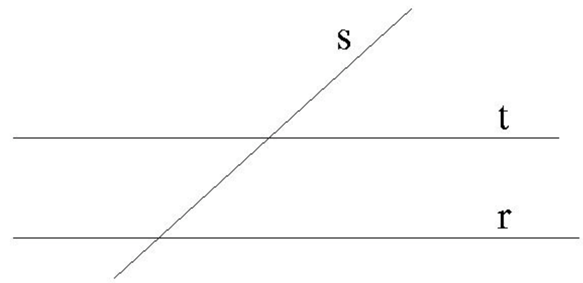


Figure 2

The second non-example is made by two crossing lines which do not verify the property A. The argumentation is indirect: the two lines do not intersect each other because, "*if we have two crossing lines*" they do not verify A. The second non-example is a generalisation of the first one: the condition of being perpendicular lines is generalised to the condition of being crossing lines. It is interesting to notice that the two students are aware of the fact that the second non-example is a generalisation of the first one. Indeed, Valerio says "*even if they are not perpendicular*" and Cristina adds "*Yes, it's the same situation of the two perpendicular lines*". Moreover, the argumentations of the fact that they are non-examples have the same structure, and finally, the argumentation of the conjecture is an indirect argumentation. Later on Valerio and Cristina go on as follows:

40. V: Oh yes, then they [r and s] definitely have to be parallel.
41. C: Parallel.
42. I: Why?
43. V: Because, they would never intersect each other if they are parallel.
44. C: Because…
45. V: They would never intersect each other and then there could not be a situation like this (he points at figure 2), in which, since they [r and s] cross, the line t is parallel to r or to s and then it [t] does not intersect any of them.
46. C: The line…
47. V: Am I making myself clear? If they [r and s] are not parallel there will be always a point in which they intersect, there can always be a situation in which there is a parallel to one line only, which [the parallel] then intersect just one line.

The conjecture is now expressed in affirmative terms: *the two lines are parallel*. The argumentation of the conjecture recalls the argumentation of the fact the ones produced are non-examples. Hence such argumentation is indirect: *if they are not parallel, it exists, like in the two non-examples, a line which intersects only one of the given lines r and s*. Later on, during the proof production stage, Valerio says:

87. Hence, it is enough to prove that, if we menage to find a line parallel to one of the two which does not cross both, we have proved that they [r and s] definitely have to be parallel… by contradiction, yes.

The argumentation is still the same. We remark two important aspects. First, we are dealing with a case of *cognitive unity* between argumentation and proof construction: Valerio affirms that to prove the conjecture we need to do exactly what we have done before during the construction of the non-example. Second, what Valerio says expresses a continuity between the production of the non-example and the proof which is going to be by contradiction: the argumentation is based on the second non-example, which is a generalisation of the first one.

DISCUSSION AND CONCLUSIONS

The presented protocol is very enlightening.

It is confirmed that indirect argumentation can arise spontaneously: in the protocol, we can observe many indirect argumentation spontaneously produced. Moreover, the hypothesis we formulated on the production of non-examples and on the role they can have in the production of indirect argumentation seems to be confirmed.

Valerio and Cristina start by noticing that if r and s are perpendicular there is a line t intersecting r but not intersecting s. Hence the configuration of the two perpendicular lines is a non-example. The generalisation of this configuration leads to the following conjecture: if the lines r and s intersect each other then the property A is not verify. The same argumentation that justifies the fact that the lines could not be perpendicular has been applied to the case of the crossing lines, leading to an indirect argumentation: the two lines are parallel because *if it were not so, it would exist* a line t intersecting r but not intersecting s. The conjecture (r and s are parallel) is true because *if it were not so, we would be* in a situation like the non-example.

We have interviewed eighteen students, almost all of them produced non-examples, but sometimes the generation of non-examples did not lead to an indirect argumentation. This happened when the students, instead of focusing on the analysis of a non-example (as Valerio does), produced a set of situations in which each example and non-example remain isolated.

We believe that the process followed by Valerio to produce a conjecture is a special case of one of the processes of statements production like "*if A then B*" described in [Boero et al. 1999]; in particular the one classified as PGC 4 and described as follows:

"the regularity found in a particular generated case can put into action "expansive" research of a "general rule" whose particular starting case was an example; during research, new cases can be generated"(Boero et al., 1999).

It seems that the other students we have interviewed produced the conjecture according to the model classified as PGC3 in [Boero et al., 1999]. This model is described as a "*synthesis and generalisation process starting with an exploration of a meaningful sample of conveniently generated examples*". In our opinion, the models suggested in [Boero et al., 1999] can be further analysed and their classification can be refined according to our distinction between example and non-example. A study in this direction requires a deeper research that we consider of great interest from both the cognitive and the didactical point of view.

As regards the transition from the argumentation to the proof, the case of Valerio can be considered similar to the transition from example to proof through what Balacheff (1987) calls *generic example*. As the *generic example* of Balacheff, we can formulate the hypothesis that the *non-example*, in some cases, can be developed to a certain level of generality (becoming a *generic non-example*), so that the argumentation of the fact that the example is a non-example is used to support the conjecture: this argumentation takes an indirect structure.

The non-example one is not the only process that can lead to a proof by contradiction. Further researches are needed to point out some others. The non-example process seems to be linked to a certain kind of task ("*given A what can you deduce*?"). It is reasonable to think that different kinds of task can give rise to different processes and that the kind of task given to the student is one of the fundamental variables for the production of argumentation with different structures.

The educational implications of this research are very interesting.

The studies of Cognitive Unity have been showing that the processes of conjecture production are extremely important for the student in order to construct the proof. From the didactical point of view, a meaningful approach to theorems seems to be the one of producing them entirely, starting from the production of the conjecture, going through the argumentation and, finally, reaching the proof. In this theoretical framework, we think that a way to help the student overcome the difficulties of proof by contradiction can be to favour argumentations that can become proofs by contradiction.

The relevance of this research relies on the fact that it offers to the teacher tools to implement didactical activities in order to introduce pupils to proof by contradiction.

Indeed, we have pointed out a tool which can lead students to indirect argumentation. From the didactical point of view, is very important guiding the students to the awareness of the structure of their argumentations, in order to offer them the opportunity to construct a proof by contradiction starting from the indirect argumentation produced.

References

Antonini,S.(2001), Negation in mathematics: obstacles emerging from an exploratory study, *Proceedings of the 25th PME Conference, Utrecht, The Netherlands*, vol. 2 pp. 49-56.

Antonini,S.(2003), *Dimostrare per assurdo: analisi cognitiva in una prospettiva didattica*, Tesi di Dottorato, Dipartimento di Matematica, Università di Pisa (submitted to referees).

Balacheff,N.(1987), Processus de preuve et situations de validation, *Educational Studies in Mathematics*, vol. 18 n.2 pp. 147-76.

Barbin,E.(1988), La démonstration mathématique: significations épistémologiques et questions didactiques, *Bulletin APMEP* 366, pp. 591-620.

Bernardi,C.(2002), Ricerche in Didattica della Matematica e in Matematiche Elementari, *Bollettino Unione Matematica Italiana*, Serie VIII, Vol. V-A pp. 193-213.

Boero,P.,Garuti,R.&Lemut,E.(1999), About the Generation of Conditionality of Statements and its Links with Proving, in *Proceedings of th 23th PME Conference, Haifa,* vol. 2 pp. 137-144.

Epp,S.S.(1998), A Unified Framework for Proof and Disproof, *The Mathematics Teacher,* v. 91(8) pp. 708-713.

Freudenthal,H.(1973), *Mathematics as an educational task*, Reidel Publishing Company: Dordrecht, Holland.

Garuti,R.,Boero,P.,Mariotti,M.A.(1996a), Some dynamic mental processes underlying producing and proving conjectures, in *Proceedings of th 20th PME Conference, Valencia,* vol. 2 pp. 121-128.

Garuti,R.,Boero,P.,Lemut,E.&Mariotti,M.A.(1996b), Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems, in *Proceedings of th 20th PME Conference, Valencia,* vol. 2 pp. 113-120.

Garuti,R.,Boero,P.,Lemut,E.(1998), Cognitive Unity of Theorems and Difficulties of Proof, in *Proceedings of th 22th PME Conference, Stellenbosch, South Africa,* vol. 2 pp. 345-352.

Leron,U.(1985), A Direct approach to indirect proofs, *Educational Studies in Mathematics* vol. 16 (3) pp. 321-325.

Mariotti,M.A., Bartolini Bussi,M., Boero, P.,Ferri,F., Garuti,R.(1997), Approaching geometry theorems in contexts: from history and epistemology to cognition, *Proceedings of the 21st PME Conference, Lathi, Finland* vol.1 pp. 180-195.

Pedemonte,B.(2002), *Etude didactique et cognitive des rapports de l'argumentation et de la démonstration dans l'apprentissage des mathématiques*, Thèse, Université Joseph Fourier, Grenoble.

Reid,D.&Dobbin,J.(1998), Why is proof by contradiction difficult?, *Proceedings of th 22th PME Conference, Stellenbosch, South Africa* vol. 4 pp. 41-48.

Thompson,D.R.(1996), Learning and Teaching Indirect Proof, *The Mathematics Teacher* v. 89(6) pp. 474-82.

Page Intentionally Left Blank