Nonlinear Control
Lecture 8: Nonlinear Control System Design

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Nonlinear Control Problems

- **Objective of Control design:** given a physical system to be controlled and specifications of its desired behavior, construct a feedback control law to make the closed-loop system display the desired behavior.
- **Control problems:**
  1. Stabilization (regulation): stabilizing the state of the closed-loop system around an Equ. point, like: temperature control, altitude control of aircraft, position control of robot manipulator.
  2. Tracking (Servo): makes the system output tracks a given time-varying trajectory such as aircraft fly along a specified path, a robot manipulator draw straight lines.
  3. Disturbance rejection or attenuation: rejected undesired signals such as noise
  4. Various combination of the three above

Stabilization Problems

- **Asymptotic Stabilization Problem**: *Given a nonlinear dynamic system:*

$$\dot{x} = f(x, u, t)$$

find a control law, u, s.t. starting from anywhere in region $\Omega \rightsquigarrow x \longrightarrow 0$ as $t \longrightarrow \infty$.

- *If the objective is to drive the state to some nonzero set-point x_d, it can be simply transformed into a zero-point regulation problem $x - x_d$ as the state.*
  - Static control law: the control law depends on the measurement signal directly, such as proportional controller.
  - Dynamic control law: the control law depends on the measurement through a differential Eq, such as lag-lead controller

Feedback Control

- State feedback: for system $\dot{x} = f(t, x, u)$
- Output feedback for the system

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned}$$

  - The measurement of some states is not available.
  - an observer may be required

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- Static control law:
  - $u = \gamma(t, x)$
- Dynamic control law:
  - $u = \gamma(t, x, z)$
  - z is the solution of a dynamical system driven by x: $\dot{z} = g(t, x, z)$
  - The origin to be stabilize is $x = 0, z = 0$

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- For nonlinear systems
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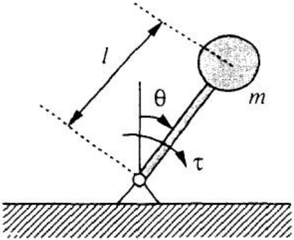
  - If g.a.s is achieved, FB provides global stabilization
  - If FB control does not achieve global stabilization, but can be designed s.t. any given compact set (no matter how large) can be included in the RoA, FB achieves semiglobal stabilization

Example

- Consider the system $\dot{x} = x^2 + u$
- Linearize at the origin $\rightsquigarrow \dot{x} = u$
- Stabilize by $u = -kx,\ k > 0$
- $\therefore$ the closed loop system $\dot{x} = -kx + x^2$
- RoA is $x < k$
- It is regionally stabilized
- Given any compact set $B_r = \{|x| \leq r\}$, we can choose $k > r$
- $\therefore$ FB achieves semiglobal stabilization.
  - Once k is fixed and the controller is implemented, for $x_0 < k$ a.s. is guaranteed
- Global stabilization is achieved by FB: $u = -x^2 - kx$

Example: Stabilization of a Pendulum

- Consider the dynamics of the pendulum:
$$J\ddot{\theta} - mgl\sin\theta = \tau$$
- Objective: take the pendulum from a large initial angel ($\theta = 60^o$) to the vertical up position
- A choice of stabilizer:
a feedback part for stability (PD)+
a feedforward part for gravity compensation:
$$\tau = -k_d\dot{\theta} - k_p\theta - mgl\sin\theta$$
k_d and k_p are pos. constants.
- $\therefore$ globally stable closed-loop dynamics:prove it
$$J\ddot{\theta} + k_d\dot{\theta} + k_p\theta = 0$$
- In this example feedback (FB) and feedforward (FF) control actions modify the plant into desirable form.



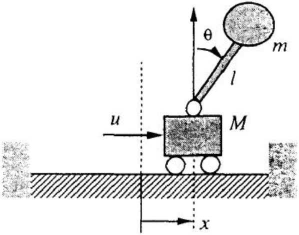
Example: Stabilization of an Inverted Pendulum with Cart

- Consider the dynamics of the inverted pendulum shown in Fig.:

$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = u$$

$$m\ddot{x}\cos\theta + ml\ddot{\theta} - ml\dot{x}\dot{\theta}\sin\theta + mg\sin\theta = 0$$

  mass of the cart is not negligible



- Objective: Bring the inverted pendulum from vertical-down at the middle of the lateral track to the vertical-up at the same lateral point.
- It is not simply possible since degree of freedom is two, # inputs is one (under actuated).

Tracking Problems

- **Asymptotic Tracking Problem**: *Given a nonlinear dynamics:*
$$\begin{aligned}\dot{x} &= f(x,u,t)\\ y &= h(x,u,t)\end{aligned}$$
and a desired output, y_d, find a control law for the input u s.t. starting from any initial state in region Ω, the tracking error $y(t) - y_d(t)$ goes to zero, while whole state x remain bounded.
- A practical point: Sometimes x can just be remained *reasonably bounded*, i.e., bounded within the range of system model validity.
- Perfect tracking: proper initial states imply zero tracking error for all time: $y(t) \equiv y_d(t) \quad \forall t \geq 0$; in asymptotic/ exponential tracking perfect tracking is achieved asymptotically/ exponentially
- Assumption throughout the rest of the lectures:
  - y_d and its derivatives up to a sufficiently high order (generally equal to the system's order) are cont. and bounded.
  - y_d and its derivatives available for on-line control computation
  - y_d is planned ahead

- Sometimes derivatives of the desired output are not available.
- A reference model is applied to provide the required derivative signals
- **Example**: For tracking control of the antenna of a radar, only the position of the aircraft $y_a(t)$ is available at a given time instant (it is too noisy to be differentiated numerically).
- desired position, velocity and acceleration to be tracked is obtained by

$$\ddot{y}_d + k_1\dot{y}_d + k_2 y_d = k_2 y_a(t) \tag{1}$$

  k_1 and k_2 are pos. constants
- $\therefore$ following the aircraft is translated to the problem of tracking the output y_d of the reference model
- The reference model serves as
  - providing the desired output of the tracking system in response to the aircraft position
  - generating the derivatives of the desired output for tracker design.
- (1) Should be fast for y_d to closely approximate y_a

Tracking Problem

- Perfect tracking and asymptotic tracking is not achievable for non-minimum phase systems.
- **Example:** Consider $\ddot{y} + 2\dot{y} + 2y = -\dot{u} + u$.
- It is non-minimum phase since it has zero at $s = 1$.
- Assume the perfect tracking is achieved.
- $\therefore\ \dot{u} - u = -(\ddot{y}_d + 2\dot{y}_d + 2y_d) \Rightarrow u = -\frac{s^2+2s+2}{s-1} y_d$
- Perfect tracking is achieved by infinite control input.
- $\therefore$ Only bounded-error tracking with small tracking error is achievable for desired traj.
- Perfect tracking controller is inverting the plant dynamics

Tracking Problem in Presence of Disturbance

- **Asymptotic disturbance rejection:** *Given a nonlinear dynamics:*

$$\begin{aligned} \dot{x} &= f(x,u,w,t) \\ y &= h(x,u,w,t) \end{aligned}$$

and a desired output, y_d, find a control law for the input u s.t. starting from any initial state in region Ω, the tracking error $y(t) - y_d(t)$ goes to zero, while whole state x remain bounded.

  - When the exogenous signals y_d and w are generated by a known model, asymptotic output tracking and disturbance rejection can be achieved by including such model in the FB controller.

- For T.V disturbance $w(t)$, achieving asymptotic disturbance rejection may not be feasible. $\rightsquigarrow$ look for **disturbance attenuation**:
  - achieve u.u.b of the tracking error with a prescribed tolerance: $\|e(t)\| < \epsilon, \ \forall t > T, \ \epsilon$ is a prespecified (small) positive number.
  - **OR** consider attenuating the closed-loop input-output map from the disturbance input w to the tracking error $e = y - y_d$
    - e.g. considering w as an L_2 signal, goal is min the L_2 gain of the closed-loop I/O map from w to e

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- **For tracking problem one can design:**
  - Static/Dynamic state FB controller
  - Static/Dynamic output FB controller
- Tracking may achieve locally, regionally, semiglobally, or globally:
  - These phrases refer not only to the size of the initial state, but to the size of the exogenous signals y_d, w
  - Local tracking means tracking is achieved for sufficiently small initial states and sufficiently small exogenous signals
  - Global tracking means tracking is achieved for any initial state and any y_d, w

Relation between Stabilization and Tracking Problems

- Tracking problems are more difficult to solve than stabilization problems
- In tracking problems the controller should
  - not only keep the whole state stabilized
  - but also drive the system output toward the desired output
- However, for tracking problem of the plant:

$$\ddot{y} + f(\dot{y}, y, u) = 0$$

  - $e(t) = y(t) - y_d(t)$ goes to zero
  - It is equivalent to the asymptotic stabilization of the system

$$\ddot{e} + f(\dot{e}, e, u, y_d, \dot{y}_d, \ddot{y}_d) = 0 \tag{2}$$

    with states e and $\dot{e}$
- $\therefore$ tracking problem is solved if we can design a stabilizer for the non-autonomous dynamics (2)
- On the other hand, stabilization problems can be considered as a special case of tracking problem with desired trajectory being a constant.

Specify the Desired Behavior

- In Linear control, the desired behavior is specified in
  - time domain: rise time, overshoot and settling time for responding to a step command
  - frequency domain: the regions in which the loop transfer function must lie at low and high frequencies
- So in linear control the quantitative specifications of the closed-loop system is defined, the a controller is synthesized to meet the specifications
- For nonlinear systems the system specification of nonlinear systems is less obvious since
  - response of the nonlinear system to one command does not reflect the response to an other command
  - a frequency description is not possible
- $\therefore$ In nonlinear control systems some qualitative specifications of the desired behavior is considered.

- **Some desired qualitative specifications of nonlinear system:**
  - Stability must be guaranteed for the nominal model, either in local or global sense. In local sense, the region of stability and convergence are of interest.
    - stability of nonlinear systems depends on initial conditions and only temporary disturbances may be translated as initial conditions
  - Robustness is the sensitivity effect which are not considered in the design like persistent disturbance, measurement noise, unmodeled dynamics, etc.
  - Accuracy and Speed of response for some typical motion trajectories in the region of operation. For instance, sometimes appropriate control is desired to guarantee consistent tracking accuracy independent of the desired traj.
  - Cost of a control which is determined by # and type of actuators, sensors, design complexity.
- The mentioned qualitative specifications are not achievable in a unified design.
- A good control can be obtained based on effective trade-offs of them.

Nonlinear Control Problems

- A Procedure of designing control
    1. Specify the desired behavior and select actuators and sensors
    2. model the physical plant by a set of differential Eqs
    3. design a control law
    4. analyze and simulate the resulting control system
    5. implement the control system in hardware
- Experience and creativity are important factor in designing the control
- Sometimes, addition or relocation of actuators and sensors may make control of the system easier.
- **Modeling Nonlinear Systems**
- Modeling is constructing a mathematical description (usually as a set of differential Eqs.) for the physical system to be controlled.

Modeling Nonlinear Systems

- Two points in modeling:
  1. To obtain tractable yet accurate model, good understanding of system dynamics and control tasks requires.
     - Note: more accurate models are not always better. They may require unnecessarily complex control design and more computations.
     - Keep essential effects and discard insignificant effects in operating range of interest.
  2. In modeling not only the nominal model for the physical system should be obtained, but also some characterization of the model uncertainties should be provided for using in robust control, adaptive design or simulation.
- Model uncertainties: difference between the model and real physical system
  - parametric uncertainties: uncertainties in parameters
- **Example:** model of controlled mass: $m\ddot{x} = u$
  - Uncertainty in m is parametric uncertainty
  - neglected motor dynamics, measurement noise, and sensor dynamics are non-parametric uncertainties.
  - Parametric uncertainties are easier to characterize; $2 \leq m \leq 5$

Feedback and FeedForward

- Feedback (FB) plays a fundamental role in stabilizing the linear as well as nonlinear control systems
- Feedforward (FF) in nonlinear control is much more important than linear control
- FF is used to
  - cancel the effect of known disturbances
  - provide anticipate actions in tracking tasks
- for FF a model of the plant (even not very accurate) is required.
- Many tracking controllers can be written in the form: $u =$ FF+ FB
  - FF: to provide necessary input to follow the specified motion traj and canceling the effect of known disturbances
  - FB to stabilize the tracking error dynamics.

Example

- Consider a minimum-phase system

$$A(s)y = B(s)u \tag{3}$$

where $A(s) = a_0 + a_1 s + ... + a_{n-1}s^{n-1} + s^n,\ B(s) = b_0 + b_1 s + ... + b_m s^m$

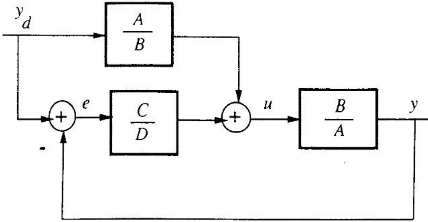
- Objective: make the output $y(t)$ follow a time-varying traj $y_d(t)$
  1. To achieve $y = y_d$, input should have a FF term of $\frac{A(s)}{B(s)}$:

$$u = v + \frac{A(s)}{B(s)} y_d \tag{4}$$

     - Substitute (4) to (3): $A(s)e = B(s)v$, where $e(t) = y(t) - y_d(t)$
  2. Use FB to stabilize the system:
     - $v = \frac{C(s)}{D(s)} e \rightsquigarrow$ closed loop system $(AD + BC)e = 0$.
     - Choose D and C to poles in desired places
- $\therefore\ u = \frac{A}{B} y_d + \frac{C}{D} e$
- $e(t)$ is zero if initial conditions $y^{(i)}(0) = y_d^{(i)}(0),\ i = 1, ..., r$,otherwise exponentially converges to zero

Example Cont'd

- If some derivatives of y_d are not available, one can simply omit them from FF $\rightsquigarrow$ only bounded tracking error is guaranteed,
- This method is not applicable for non-min phase systems.
  - low freq. components of desired traj in FF,$\rightsquigarrow$ good tracking in freq lower than the LHP zeros of plant
  - By defining FF term as $\frac{A}{B_1}y_d \rightsquigarrow e(t) = \frac{D}{AD-BC}[\frac{B}{B_1} - 1]Ay_d$
  - If B_1 eliminates the RHP zeros of $B \rightsquigarrow$ good tracking for desired traj with frequencies lower than the RHP zeroes (but we may not have internal stability)



Linear Tracking Control system

Importance of Physical Properties

- In nonlinear control design, explanation of the physical properties may make the control of complex nonlinear plants simple;
- **Example:** Adaptive control of robot manipulator was long recognized to be far of reach.
- Because robot's dynamics is highly nonlinear and has multiple inputs
- Using the two physical facts:
  - pos. def. of inertia matrix
  - possibility of linearly parameterizing robot dynamics

  yields adaptive control with global stability and desirable tacking convergence.

Available Methods for Nonlinear Control

- There is no general method for designing nonlinear control
- Some alternative and complementary techniques to particular classed of control problem are listed below:
  - **Trail-and Error**: The idea is using analysis tools such as phase-plane methods, Lyapunov analysis , etc, to guide searching a controller which can be justified by analysis and simulations.
    - This method fails for complex systems
  - **Feedback Linearization:** transforms original system models into equivalent models of simpler form (like fully or partially linear)
    - Then a powerful linear design technique completes the control design
    - This method is applicable for input-state linearizable and minimum phase systems
    - It requires full state measurement
    - It does not guarantee robustness in presence of parameter uncertainties or disturbances.
    - It can be used as model-simplifying for robust or adaptive controllers

Available Methods for Nonlinear Control

- **Robust Control** is deigned based on consideration of nominal model as well as some characterization of the model uncertainties
  - An example of robust controls is sliding mode control
  - They generally require state measurements.
  - In robust control design tries to meet the control objective for any model in the "ball of uncertainty."
- **Adaptive Control** deals with uncertain systems or time-varying systems.
  - They are mainly applied for systems with known dynamics but unknown constant or slowly-varying parameters.
  - They parameterizes the uncertainty in terms of certain unknown parameters and use feedback to learn these parameters on-line , during the operation of the system.
  - In a more elaborate adaptive scheme, the controller might be learning certain unknown nonlinear functions, rather than just learning some unknown parameters.

Available Methods for Nonlinear Control

- **Gain Scheduling** Employs the well developed linear control methodology to the control of nonlinear systems.
  - A number of operating points which cover the range of the system operation is selected.
  - Then, at each of these points, the designer makes a linear TV approximation to the plant dynamics and designs a linear controller for each linearized plant.
  - Between operating points, the parameters of the compensators are interpolated, (scheduled), resulting in a global compensator.
  - It is simple and practical for several applications.
- The main problems of gain scheduling:
  - provides limited theoretical guarantees of stability in nonlinear operation
  - The system should satisfy some conditions:
    - the scheduling variables should change slowly
    - The scheduling variables should capture the plant's nonlinearities".
  - Due to the necessity of computing many linear controllers, this method involves lots of computations.