Nonlinear Dynamics, Chaos and Strange Attractors with Applications

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PRESENTATION OVERVIEW

- Some milestones in strange attractor theory history
- Notation, definitions, preliminaries and some basic results on strange attractors
- Some new theorems on *radial strange attractors* and *multihorseshoe attractors*
- Applications in population, granular flow, walking droplet and reaction-diffusion dynamics
- Concluding remarks and work-in-progress

1 Milestones in Strange Attractor Research

- **1963**. Focused strange attractor research begins as a result of Lorenz's investigation of his simplified ODE model of his chaotic atmospheric equations (Fig. 1)
- **1976**. Hénon introduces his planar map as an approximate model of a Poincaré section of the Lorenz equations (Fig. 2)
- **1978**. Lozi devises a simplified piecewise linear analog of the Hénon map (Fig. 3)
- **1980**. M. Misiurewicz proves that the Lozi map has a chaotic strange attractor for certain parameters (*Annals of NYAS*)
- **1991**. M. Benedicks & L. Carleson prove the Hénon map has a chaotic strange attractor (*Annals of Math.*)
- **2001-2008**. Q. Wang & L-S. Young extend and generalize the work of Benedicks & Carleson in their *rank-one theory* (*Commun. Math. Phys. – Annals of Math*.)

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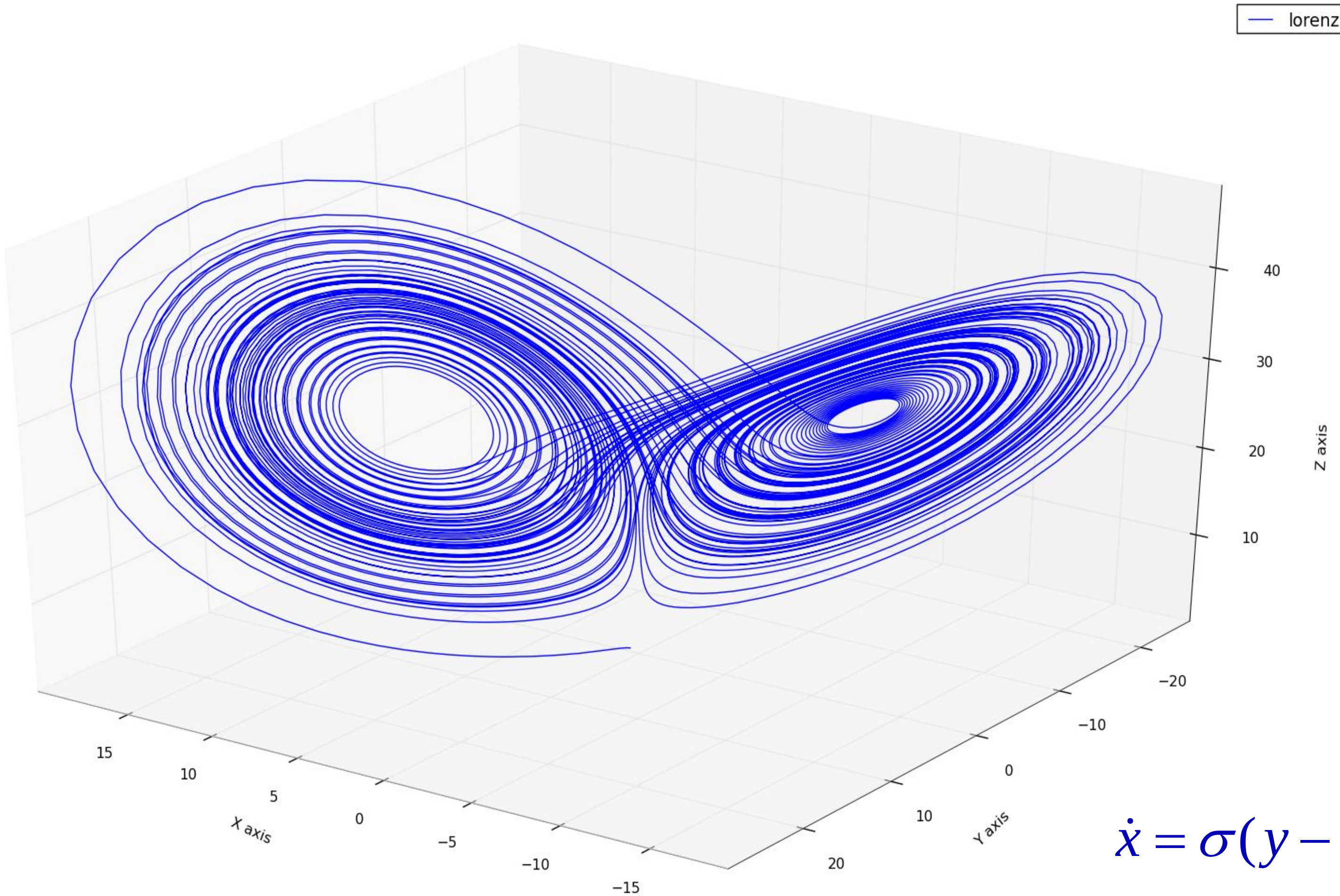

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - (8/3)y$$

Fig.1. The Lorenz attractor for σ = 10, r = 28 and b = 8/3.

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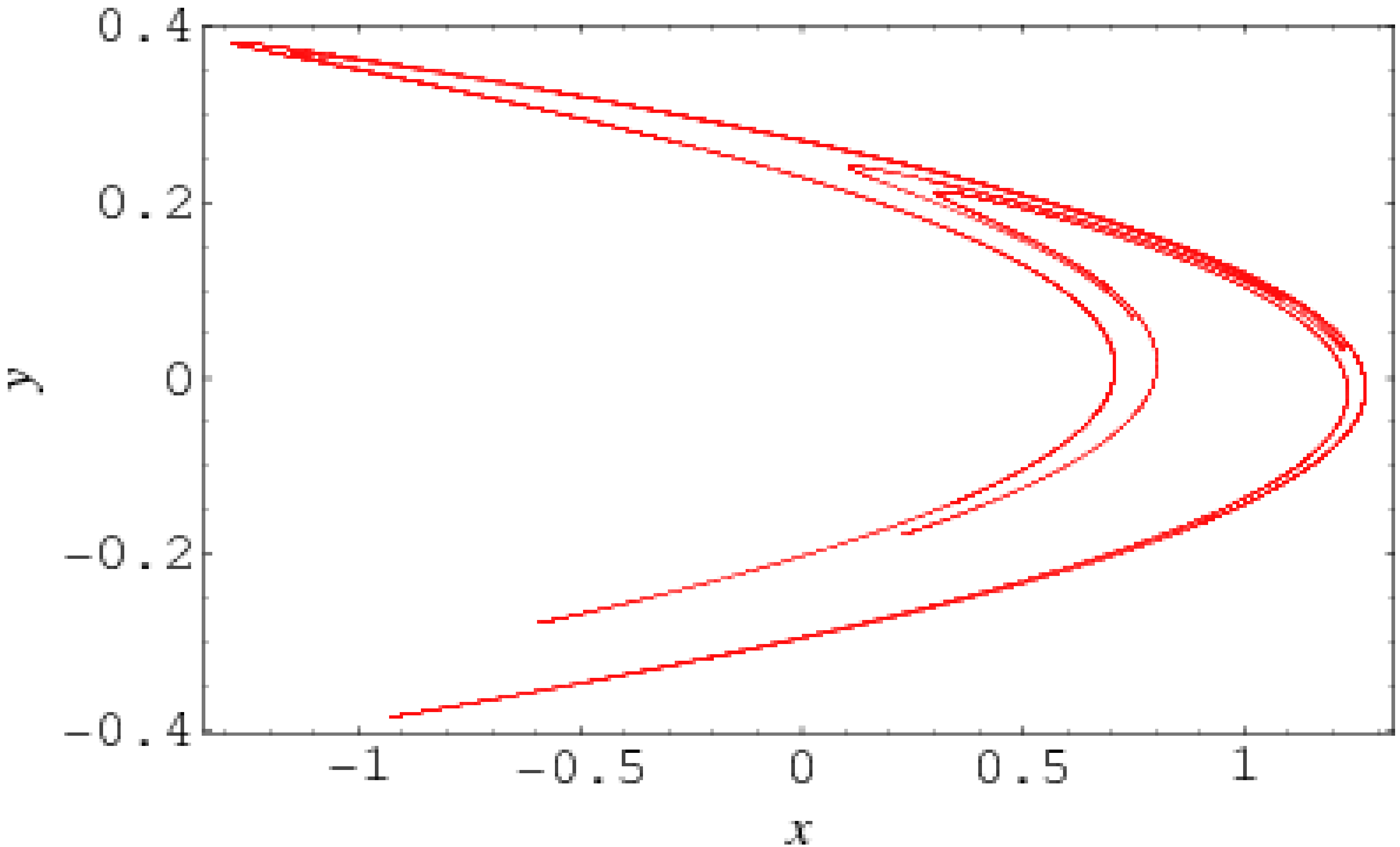

Fig.2. The Hénon attractor for

$$H(x,y) := (1 - ax^2 + y, bx), \quad a = 1.4,\ b = 0.3.$$

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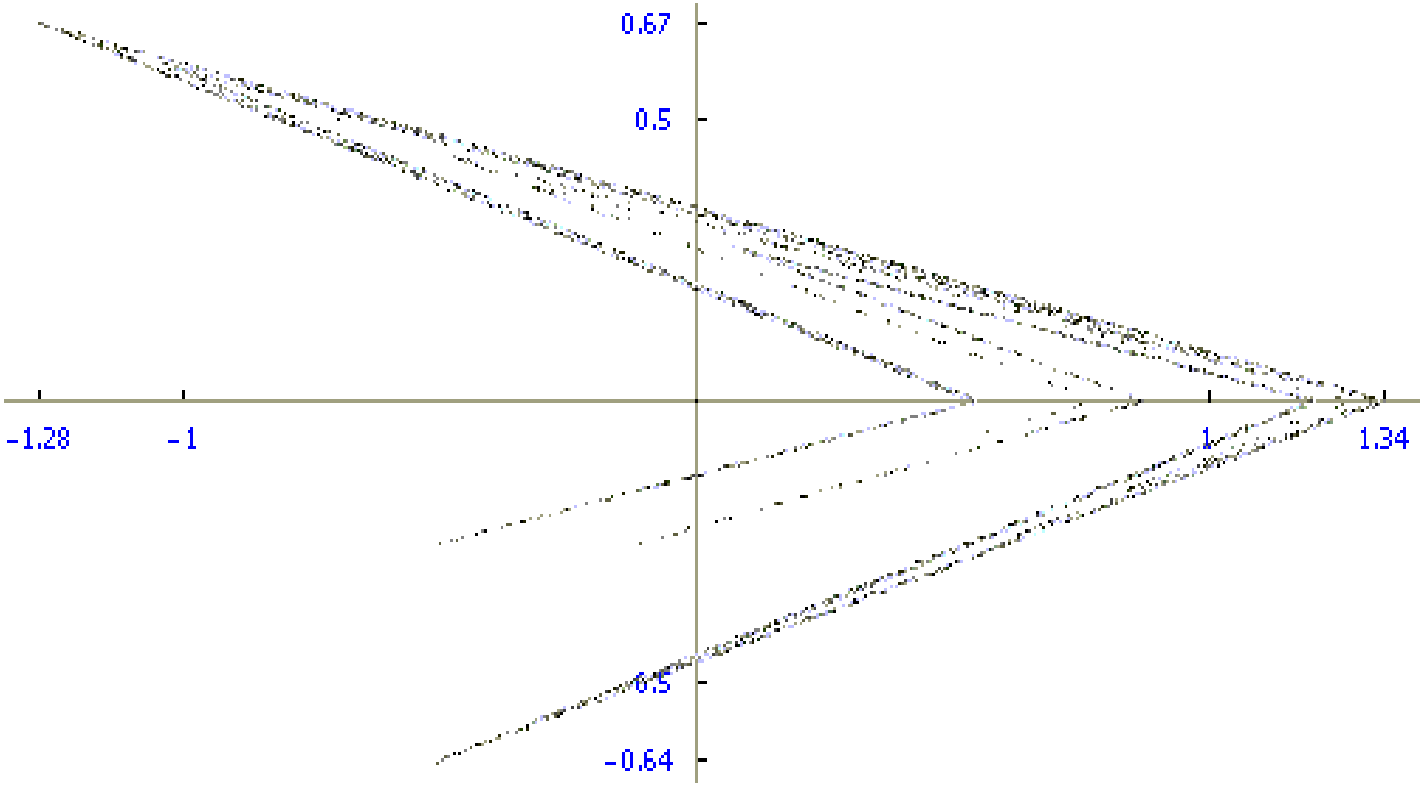

Fig.3 .The Lozi attractor for

$$L(x, y) := (1 - a|x| + y, bx), \quad a = 1.7, \mathrm{b} = 0.5$$

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2 Preliminaries and Basic Results

We focus on continuous maps (sometimes with additional smoothness) and the associated discrete (semi-) dynamical system of their iterates of the form

(1) $$f:\mathbb{R}^m \to \mathbb{R}^m, \quad f^n := f^{n-1}\circ f:\mathbb{R}^m \to \mathbb{R}^m.$$

Definition 1

If A is a (positively) invariant subset of $\mathbb{R}^m$, then

$$f:A\to A$$

is ***chaotic*** if it is (i) *topologically transitive*, i.e. U and V open in $A \Rightarrow f^k(U)\cap V \neq \varnothing$ for some $k\in\mathbb{N}$. (ii) The set of periodic points, Per(*f*), is dense in A. ($\Rightarrow$ sensitive dependence (Banks *et al.*))

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Definition 2

A is an *attracting set* for the map (1) if :

(AS1) It is nonempty, closed and (positively) invariant;

(AS2) There is an open set U containing A such that

$$x \in U \Rightarrow d(f^n(x), A) \to 0 \text{ as } n \to \infty.$$

Definition 3

An attracting set is a *semichaotic attracting set* if: (SCAS1) it is compact; and (SCAS2) the map is differentiable almost everywhere on a nonempty invariant subset A_* on which it is sensitively dependent on initial conditions as

$$n^{-1} \log \left\| (f^n)'(x) \right\| = n^{-1} \sum_{k=1}^{n} \log \left\| f'(f^{k-1}(x)) \right\| \geq l > 0 \ \forall (x,n) \in A_* \times \mathbb{N}.$$

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Definition 4

An attracting set A for (1) is a *chaotic attracting* set if: (CAS1) it is an attracting set; and (CAS2) there is a nonempty closed invariant subset, A_*, of A such that the restriction $f_{|A_*} : A_* \to A_*$ is chaotic.

Definition 5

An attracting set A for (1) is an ***attractor*** if it is minimal with respect to properties (AS1) and (AS2).

Definition 6

A is a ***strange attractor*** for (1) if: (SA1) it is an attractor; and (SA2) it is *fractal*, with a noninteger fractal (Hausdorff) dimension.

Definition 6

A is a ***chaotic strange attractor*** for (1) if: (CSA1) it is a strange attractor; and (CSA2) it is a chaotic attracting set.

Definition 7

A is a ***semichaotic strange attractor*** for (1) if: (SCSA1) it is a strange attractor; and (SCSA2) it is a semichaotic attracting set.

We note here that there are strange attractors that are not chaotic (see e.g. Grebogi *et al*., *Physica D* **13** (1984).

2.1 Basic dynamical properties of special maps

The following types of maps are quite ubiquitous when it comes to modeling – especially in population dynamics.

Definition 8

The map (1) is ***asymptotically zero*** **(*AZ*)** if

$$\left| f(x) \right| \to 0 \ as \ \left| x \right| \to \infty.$$

We shall also consider the following special AZ maps.

Definition 9

The map (1) is ***eventually zero*** **(*EZ*)** if

$$\exists R \text{ such that } f^{-1}(\{x \in \mathbb{R}^m : \left| x \right| \geq R > 0\}) = \{0\}.$$

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Lemma 1

If (1) is an AZ map, so that $|f| \le M$ $(M > 0)$ *on* $\mathbb{R}^m$*, then* f *and all of its iterates* f^n *have their fixed points in the compact ball* $B_M(0) := \{x \in \mathbb{R}^m : |x| \le M\}$*; in fact, they are contained in the globally contracting set*

(2) $$A := \bigcap_{n=1}^{\infty} f^n(B_M(0)) \subset B_M(0).$$

Proof Sketch. For every $R \ge M$,

$$f^n(B_R(0)) \subset B_M(0) \subset B_R(0)\ \forall R \ge M, n \in \mathbb{N},$$

so the first part follows from Brouwer's fixed point theorem, while (2) is a consequence of the definitions. □

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Lemma 2

Let f, M *and* A *be as in Lemma* 1, *and suppose* f *satisfies the the additional properties*: (i) $f(0) = 0$; *and* (ii) $|f(x)| < |x|$ *when* $0 < |x| \le R_M := \max\{|x| : |f(x)| = M\}$. *Then* $\{0\}$ *is a global attractor for* f.

Proof Sketch. By hypothesis, it suffices to consider the case for which the initial point and none of its iterates are equal to zero. Then $|x_k| := |f^k(x_0)|$ is strictly decreasing and must have limit zero in view of (ii). □

3 Radial Strange Attractors

Here we prove two theorems on strange attractors that have distinctive radial characteristics (cf. Figs. 5, 6, 7 and 12).

3.1 Attractors for EZ maps expanding at 0

Our first theorem has a rather lengthy list of hypotheses, but as we shall see they can readily be distilled to fairly simple criteria that are easily checked for applications.

Theorem 1

Let $f:\mathbb{R}^m \to \mathbb{R}^m$ *be a continuous EZ map, with* M *and* R_M *as in Lemma* 2, *satisfying the following additional properties:*

(i) $f^{-1}(\{0\}) = \{0\} \cup Z$, *where* $Z := \{x \in \mathbb{R}^m : |x| \geq \zeta(x/|x|) > 0\}$,

$\zeta : \mathbb{S}^{m-1} \to \mathbb{R}$ *is a* C^1 *function satisfying* $R_M < \zeta(u) < M$ *for all* $u \in \mathbb{S}^{m-1} := \{u \in \mathbb{R}^m : |u| = 1\}$ – *the unit* $(m-1)$ – *sphere.*

(ii) $S_* := f^{-1}(Z) = \{x \in \mathbb{R}^m : 0 < \alpha(x/|x|) \leq |x| \leq \beta(x/|x|)\}$, *where*

$\alpha, \beta : \mathbb{S}^{m-1} \to \mathbb{R}$ *are* C^1, *positive and* $0 < \beta(u) - \alpha(u) < \zeta(u)$ *on* $\mathbb{S}^{m-1}$.

(iii) $f \in C^1(D := \{x \in \mathbb{R}^m : 0 < |x| < \zeta(x/|x|)\})$ *and* $f'(x)$ *is invertible*

on $D \setminus \{x \in \mathbb{R}^m : 0 < \alpha(x/|x|) \leq |x| \leq \beta(x/|x|)\} = D \setminus S_*$.

(iv) *The radial derivative* $\partial_r |f|(x) := \langle \nabla |f|(x), x/|x| \rangle$, *when it exists, is such that* $\exists\, \lambda, \mu$ *with* $\mathfrak{M}/\mathfrak{m} < \lambda < \mu$ *for which* $\lambda \leq \partial_r |f|(x) \leq \mu \;\forall x \in \{x \in \mathbb{R}^m : 0 < |x| < \alpha(x/|x|)\}$ *and* $-\mu \leq \partial_r |f|(x) \leq -\lambda \;\forall x \in \{x \in \mathbb{R}^m : \beta(x/|x|) < |x| < \zeta(x/|x|)\}$. *Here* $\mathfrak{m} = \min\{\alpha(u) : u \in \mathbb{S}^{m-1}\}$ *and* $\mathfrak{M} = \max\{\beta(u) : u \in \mathbb{S}^{m-1}\}$.

Then

(3) $$\Lambda := \bar{D} \setminus \bigcup_{n=1}^{\infty} f^{-n}(S_*^o)$$

is a compact semichaotic strange globally attracting set of $m-$ *dimensional Lebesgue measure zero, where* $\bar{E}$ *and* E^o *denote the closure and interior, respectively, of a set* E.

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Proof Sketch. It follows from the hypotheses that S_*^o is (homeomorphic to) an open (*m*-1)-spherical shell enclosing 0 and

$$\bar{D} \setminus f^{-1}(S_*^o) = \Sigma_0 \vee \Sigma_1,$$

where Σ_0 is a closed *m*-ball and Σ_1 is a closed (*m*-1)-spherical shell. Whence, we obtain the disjoint union of a closed *m*-ball and three closed (*m*-1)-spherical shells

$$\bar{D} \setminus f^{-1}(S_*^o) \cup f^{-2}(S_*^o) = \Sigma_{00} \vee \Sigma_{01} \vee \Sigma_{10} \vee \Sigma_{11}.$$

If this is continued, we see it is just the inductive construction of a Cantor set, so

$$\Lambda := \bar{D} \setminus \bigcup_{n=1}^{\infty} f^{-n}(S_*^o) = \bigvee_{s \in 2^{\mathbb{N}}} \Sigma_s, \quad (2^{\mathbb{N}} := \{s : s : \mathbb{N} \to \{0,1\}\}).$$

This implies that that this set is homeomorphic to the fractal 'cone' pinched at the origin; namely,

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(4) $$\Lambda \cong (\mathbb{S}^{m-1} \times C) / (\mathbb{S}^{m-1} \times 0),$$

where C is the standard two-component Cantor set on the unit interval [0,1]. Hence, (4) is fractal. For sensitive dependence, we compute that for all x in D

$$n^{-1} \log \| (f^n)'(x) \| = n^{-1} \sum_{k=1}^{n} \log \| f'(f^{k-1}(x)) \|$$

$$\geq n^{-1} \sum_{k=1}^{n} \log | \partial_r | f | (f^{k-1}(x)) | \geq \log \lambda > 0$$

$$\Rightarrow \liminf_n n^{-1} \log \| (f^n)'(x) \| > 0,$$

so sensitive dependence on initial conditions is established and the proof sketch is complete. □

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The proof of Theorem can actually be used to obtain the following estimate for the fractal dimension of (4):

(5) $m-1+(\log 2/\log(1+\mu)) \le \dim_H(\Lambda) \le m-1+(\log 2/\log(1+\lambda))$.

Using standard constructions from symbolic dynamics, we get the following result from (the proof of) Theorem 1.

Corollary 1.1

The hypotheses of Theorem 1 imply that f on Λ is conjugate to

$$\hat{f}:(\mathbb{S}^{m-1}\times 2^{\mathbb{N}})/(\mathbb{S}^{m-1}\times 0)\to(\mathbb{S}^{m-1}\times 2^{\mathbb{N}})/(\mathbb{S}^{m-1}\times 0),$$

where

$$\hat{f}(x,s):=(\nu(x,s),\sigma(s)),$$

σ is the shift map, $\sigma(s)=\sigma(.a_1a_2a_3\ldots):=.a_2a_3a_4\ldots$, and ν is continuous.

To improve Theorem 1 so as to obtain a chaotic strange attractor, we need further assumptions on the map that imply topological transitivity and the density of Per(*f*), such as

Corollary 1.2

Suppose that in addition to the hypotheses of Theorem 1: *There exists a set of* C^1 *curves* $\{\gamma_1, \ldots, \gamma_k\}$ *such that:*

(*a*) *Each* γ_i *begins at* 0 *and ends at a distinct point of* $\partial \bar{D}$.

(*b*) *The curves are all transverse to* ∂S_* *and all its preimages.*

(*c*) $f(\gamma_j) = \gamma_{j+1}, 1 \leq j \leq k-1$, *and* $f(\gamma_k) = \gamma_1$.

(*d*) $E := \{\gamma_1, \ldots, \gamma_k\}$ *is* ***conically attracting*** *in that there is a conical open set* W (*pinched at* 0) *such that* $x \in W \Rightarrow d(f^n(x), E) \to 0$.

Then the following set is a chaotic strange global attractor:

(6) $$\mathfrak{A} := \Lambda \cap E.$$

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Proof Sketch. It follows from the hypotheses that v in Corollary 1.2 restricted to $\mathfrak{A}$ is periodic in x. Consequently, the periodic density and transitivity properties of the shift map imply the same holds true for $\hat{f}$. □

3.2 Attractors for AZ maps contracting at 0

There are analogs of Theorem 1, its corollaries and formulas for AZ maps contracting at the origin, which, for example, are common in discrete dynamical models of ecological phenomena associated with what are known as *climax species*. The proofs of all these results can be obtained by straightforward modification of the proofs in 3.2, so we shall just state our main theorem.

Theorem 2

Let $f:\mathbb{R}^m \to \mathbb{R}^m$ *be a* C^1 *AZ map, with* M *and* R_M *as in Lemma* 2*, satisfying the following additional properties:*

(i) $f^{-1}(\{0\})=\{0\}, \| f'(0) \| < 1$ *and* $\{0\}$ *has a basin of attraction*

$$B(0):=\{x\in\mathbb{R}^m : 0 \le |x| \le \alpha_0(x/|x|)\},$$

where $\alpha_0 : \mathbb{S}^{m-1} \to \mathbb{R}$ *is a positive* C^1 *function and* $f^{-1}(B(0))$ *is a semi-infinite* $(m-1)$*-spherical shell of the form*

$$Z:=\{x\in\mathbb{R}^m : \zeta(x/|x|) < |x|\},$$

where $\zeta : \mathbb{S}^{m-1} \to \mathbb{R}$ *is a* C^1 *function such that* $R_M < \zeta(u) < M$ *on* $\mathbb{S}^{m-1}$.

(ii) $S_* := f^{-1}(\bar{Z})$ *is an* $(m-1)$*-spherical shell of the form*

$$S_* = \{x\in\mathbb{R}^m : 0 < \alpha(x/|x|) \le |x| \le \beta(x/|x|)\},$$

where $\alpha, \beta : \mathbb{S}^{m-1} \to \mathbb{R}$ *are* C^1, >0 *and* $0 < \beta(u) - \alpha(u) < \zeta(u)$ *on* $\mathbb{S}^{m-1}$.

(iii) $f'(x)$ *is invertible on* $D \setminus \{x\in\mathbb{R}^m : 0 < \alpha(x/|x|) \le |x| \le \beta(x/|x|)\}$.

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(iv) *The radial derivative* $\partial_r |f|(x) := \langle \nabla |f|(x), x/|x| \rangle$, *when it exists, is such that* $\exists\, \lambda, \mu$ *with* $\mathfrak{M}/\mathfrak{m} < \lambda < \mu$ *for which* $\lambda \leq \partial_r |f|(x) \leq \mu \; \forall x \in \{x \in \mathbb{R}^m : 0 < |x| \leq \alpha(x/|x|)\}$ *and* $-\mu \leq \partial_r |f|(x) \leq -\lambda \; \forall x \in \{x \in \mathbb{R}^m : \beta(x/|x|) < |x| < \zeta(x/|x|)\}$. *Here* $\mathfrak{m} = \min\{\alpha(u) : u \in \mathbb{S}^{m-1}\}$ *and* $\mathfrak{M} = \max\{\beta(u) : u \in \mathbb{S}^{m-1}\}$.

Then

(7)
$$\Gamma := \{0\} \vee \Gamma_C,$$

where

$$\Gamma_C = \bar{D} \setminus \bigcup_{n=1}^{\infty} f^{-n}(S_*^o)$$

is a compact semichaotic strange minimal globally attracting set of $m-$*dimensional Lebesgue measure zero.*

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4 Multihorseshoe Strange Attractors

The lynchpin of the results for these kinds of attractors is the definition of an ***attracting 1×(m-1)-horseshoe at p for f***, which is simple, but lengthy, to describe, so we just do it with a picture; namely Fig. 4.

Moreover, we shall also sketch the proofs of our main theorems using the same picture. This definition and proofs by picture turn out to be a simple, yet very effective way of sketching the rather detailed descriptions needed for a precise definition and rigorous proofs – indeed, a picture turns out to be literally worth several hundred words.

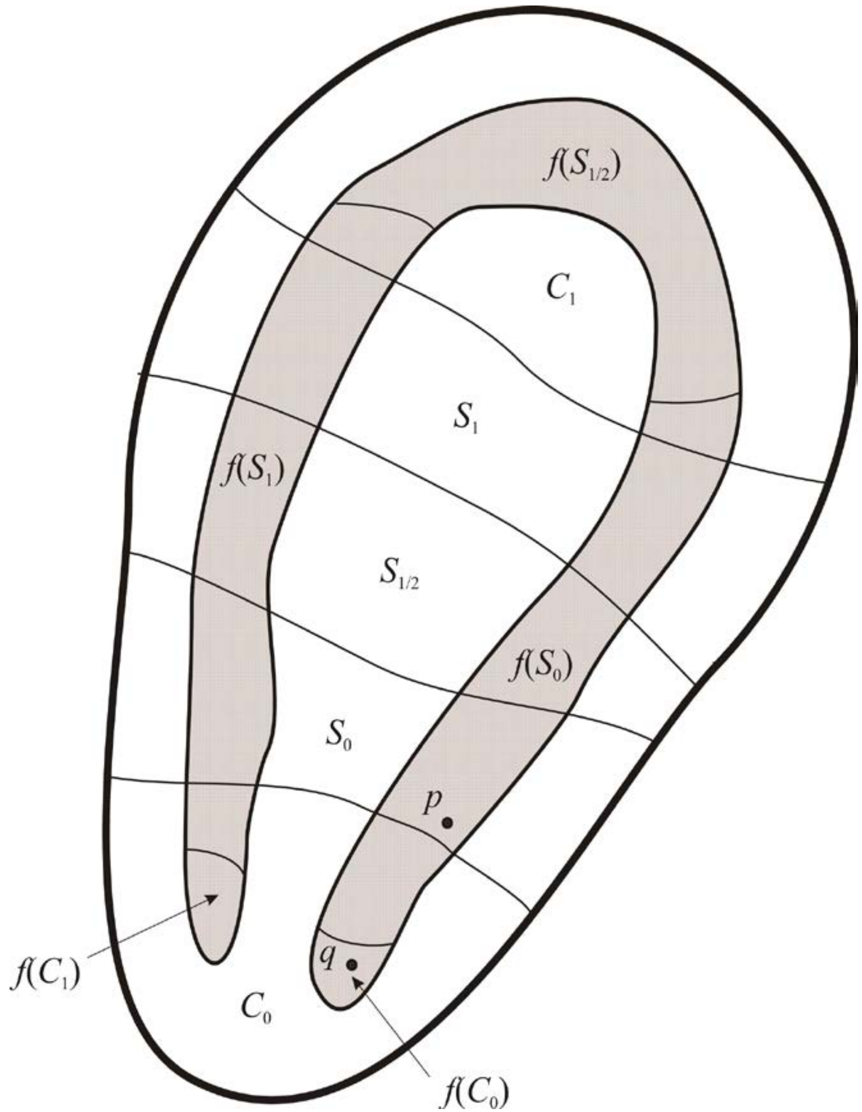

Fig.4. An attracting horseshoe

Theorem 3

Let $f : E \to E$ *be a* C^1 *map of a connected open subset of* $\mathbb{R}^m$.
If f *has an attracting* $1 \times (m-1)$*-horseshoe* H *at* $p \in E$, *then*

(8) $$\mathfrak{A} := \bigcap_{n=1}^{\infty} f^n(H) = \overline{W^u(p)}$$

is a chaotic strange attractor of f *with basin of attraction containing* H *and* $\mathfrak{A}$ *is homeomorphic to the quotient space*

$$(K \times [0,1]) / (K \times \{0,1\}),$$

where K *is a two-component Cantor space.*

Theorem 4

Let $f: E \to E$ *be a* C^1 *map of a connected open subset of* $\mathbb{R}^m$ *that has a* $k-$*cycle of distinct points (with* $k>1$*) and* f^k *has an attracting* $1\times(m-1)-$*horseshoe* H *at one of the points* p *in the cycle, then*

(9) $$\mathfrak{A} := \overline{W^u(p)} \cup f\left(\overline{W^u(p)}\right) \cup \cdots \cup f^{k-1}\left(\overline{W^u(p)}\right)$$

is a chaotic strange attractor of f *with basin of attraction containing*

$$H \cup f(H) \cup \cdots \cup f^{k-1}(H).$$

Proof Sketches. The proofs of Theorems 3 and 4 follow directly from Fig. 4. □

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5 Examples and Simulations

We now illustrate our results by applying them to planar AZ maps that have proven quite effective in modeling ecological population dynamics. The dynamics in each case is illustrated by direct simulations of the iterates.

5.1 Radial chaotic strange attractors

Consider the map $f:\mathbb{R}^2 \to \mathbb{R}^2$, defined as

(10) $$f(x,y) = f(x,y;a) := ae^{-x^2-y^2}\left(x\cos(2\pi\theta) - y\sin(2\pi\theta),\, x\sin(2\pi\theta) + y\cos(2\pi\theta)\right),$$

which is just a rotated version of a standard discrete dynamical systems model for competing pioneer species.

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We shall consider three cases for (10) corresponding to the quasiperiodic rotations associated with the following cases:

(i) $\theta = (1 + \sqrt{5})/2$, the golden mean;

(ii) $\theta = \mathrm{e}$, the base of the natural logarithm;

(iii) $\theta = 1/\sqrt{11}$.

In each case we vary a from $a = 2.7$ to $a = 6$ in increments of $\Delta a = 0.3$. The simulations for (i), (ii) and (iii) are shown in Figs. 5, 6 and 7, respectively.

5.2 Multihorseshoe chaotic strange attractors

Here we consider the dynamical systems model for an ecological pair comprised of a pioneer and climax species given by a planar AZ map $f:\mathbb{R}^2 \to \mathbb{R}^2,$ of the form

(11) $$f(x,y) = f(x,y;a,b) := (xe^{a-0.8x-0.2y}, y(0.2x+0.8y)e^{b-0.2x-0.8y}),$$

In Fig. 8, we see that this map has an attractor comprised of a 6-cycle when a = 2.4 and b = 2.5, so the dynamics is quite regular. As a and b are increased there is a global bifurcation so that when $a = b = 3$, there is a very interesting chaotic strange multihorseshoe attractor as shown in Fig. 9.

Another chaotic strange multihorseshoe attractor, this time for the planar map (representing pioneer and climax species).

(12) $$f(x,y) := (xe^{3-0.8x}, y(0.2x+0.8y)e^{3-0.2x-0.8y}),$$

is illustrated in Fig. 10., and another such attractor for the planar map

(13) $$f(x,y) := (xe^{3-0.5(x+y)}, 0.5y(x+y)e^{3-0.5(x+y)}),$$

is illustrated in Fig. 11.

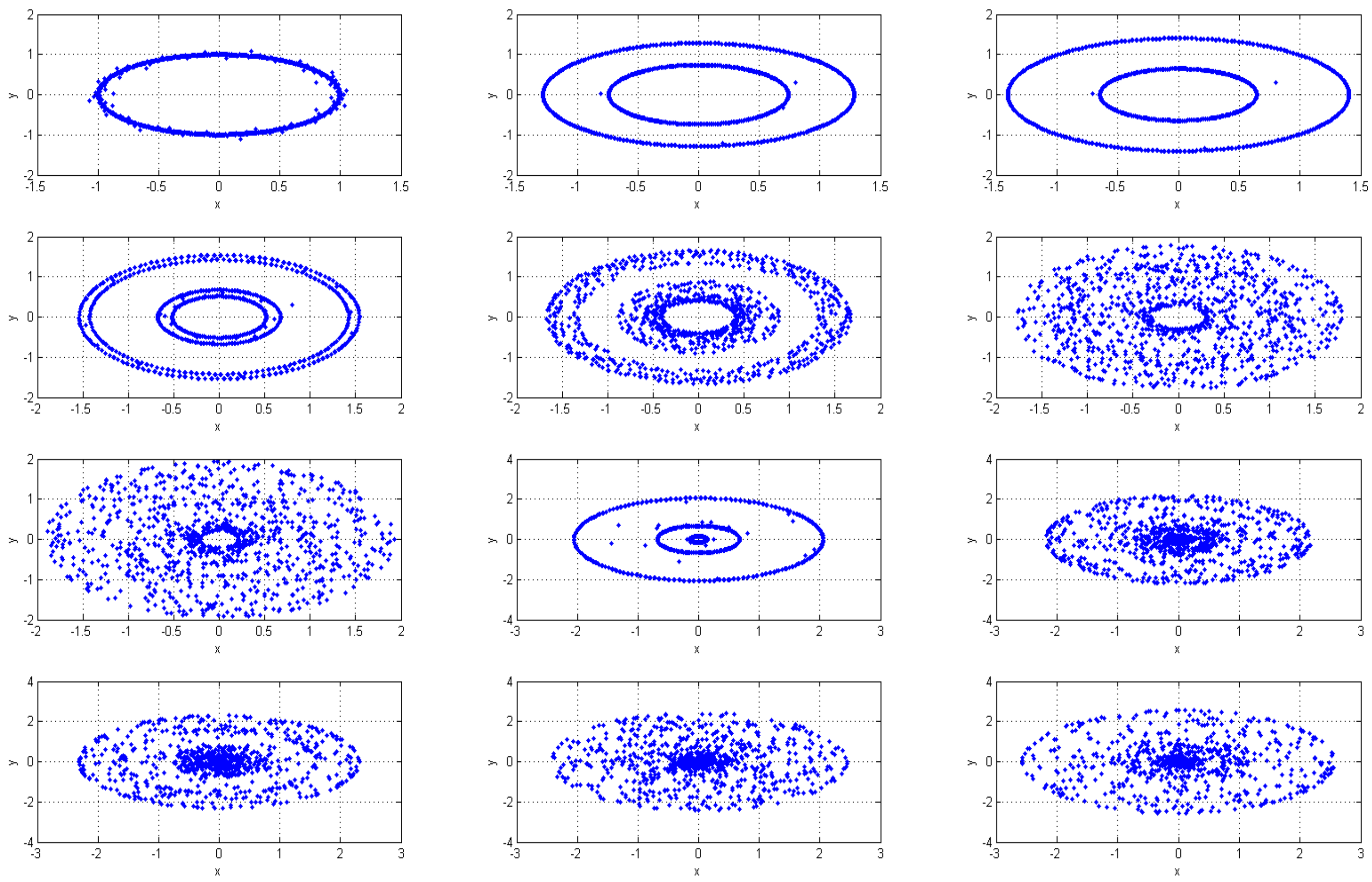

Fig. 5. Case (i): θ = □, $a = 2.7, 3.0, \ldots, 6.0$, left to right then down

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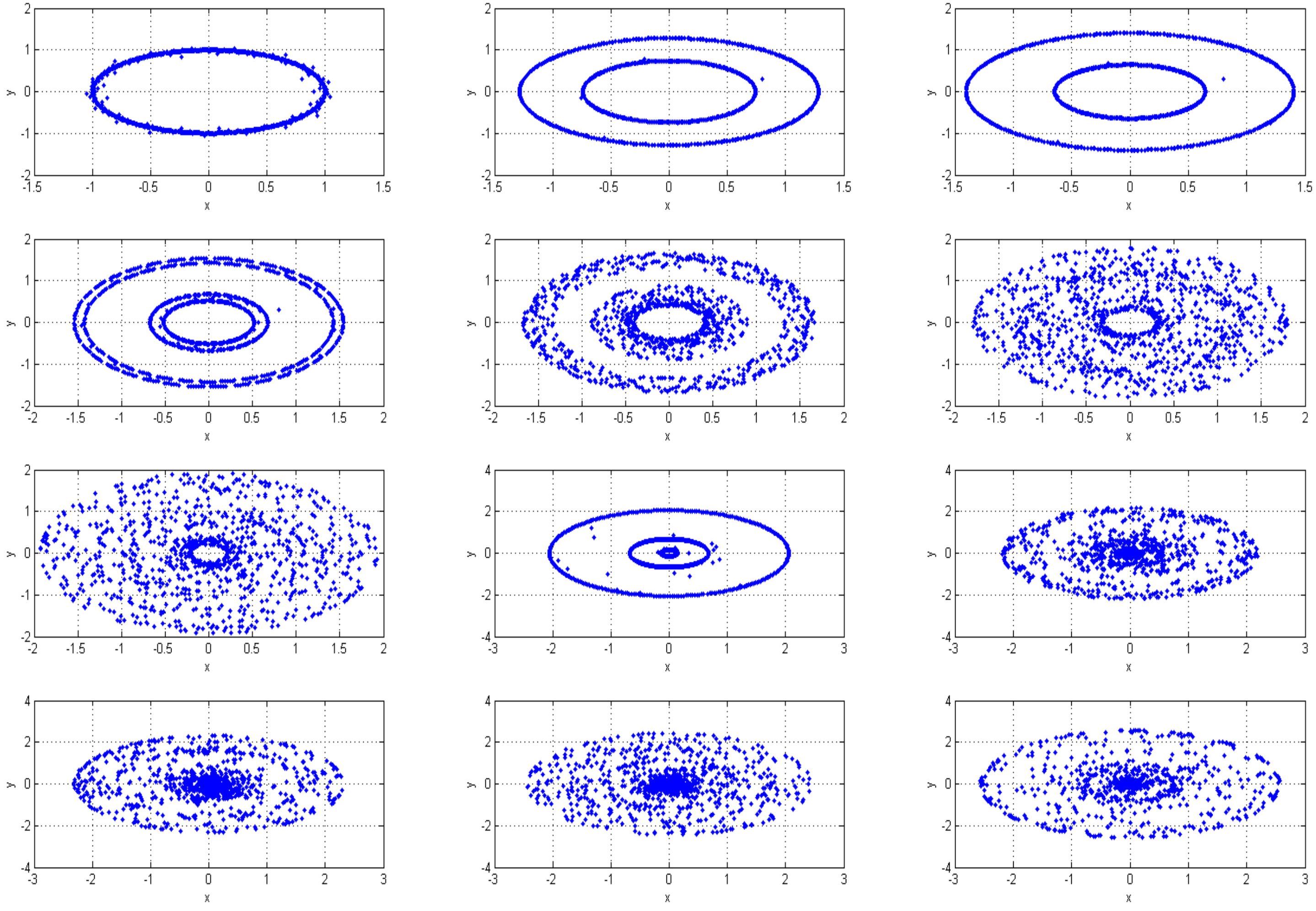

Fig. 6. Case (ii): $\theta = \mathrm{e}$, $a = 2.7, 3.0, \ldots, 6.0$, left to right then down

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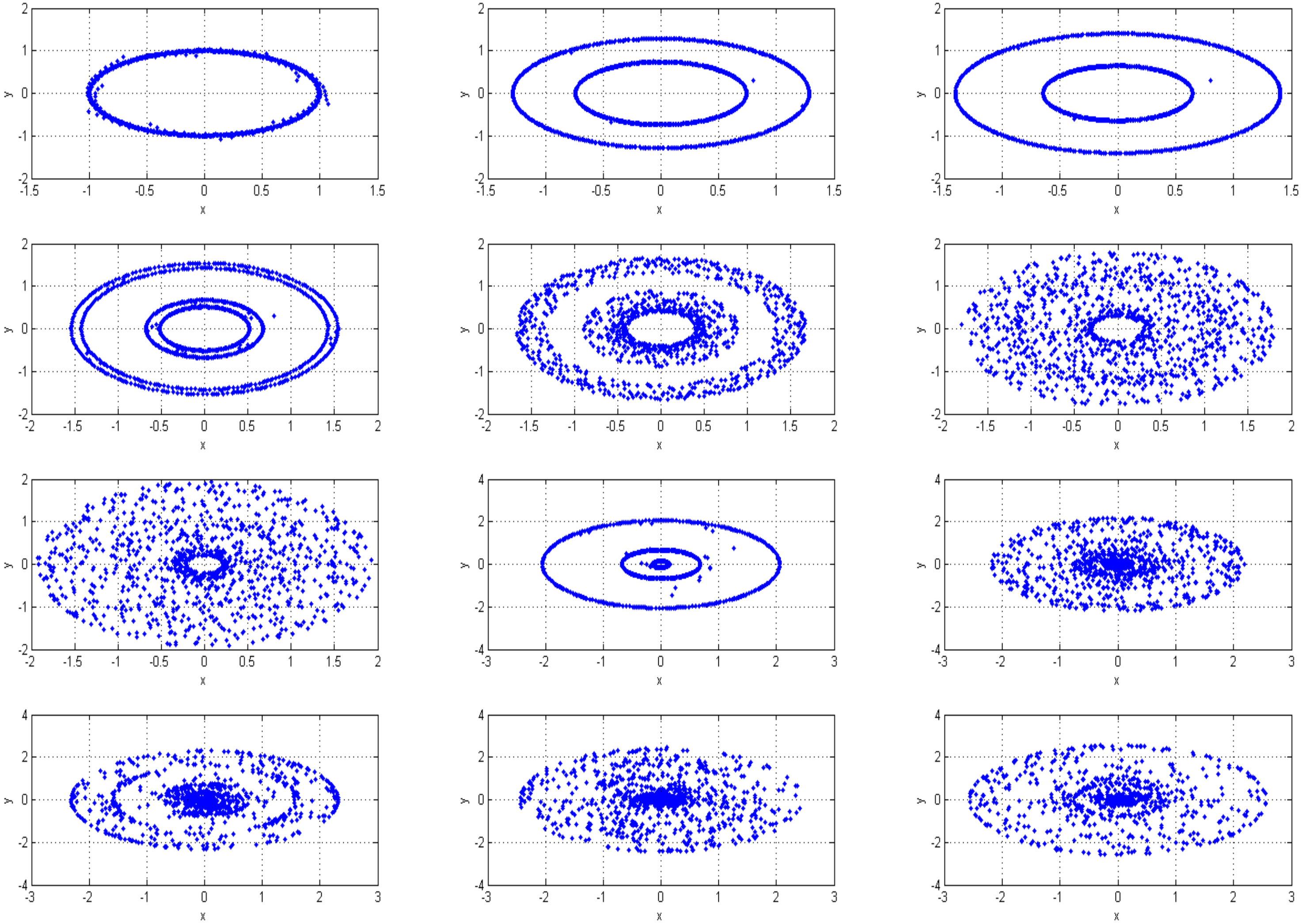

Fig. 7. Case (iii): $\theta = 1/\sqrt{11}$, a = 2.7, 3.0, …, 6.0, left to right then down

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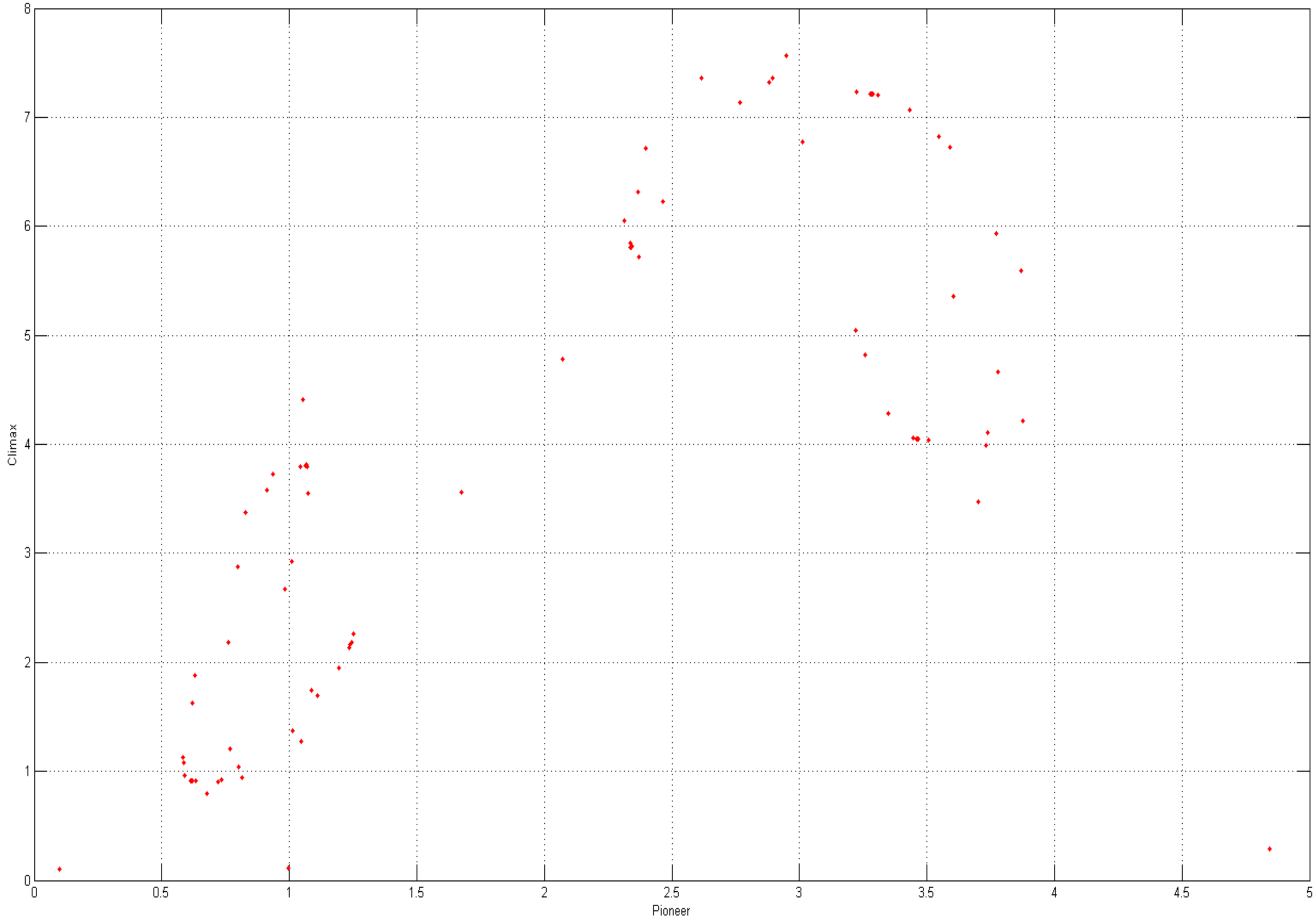

Fig.8. The iterates of map (11) for a = 2.4, b = 2.5.

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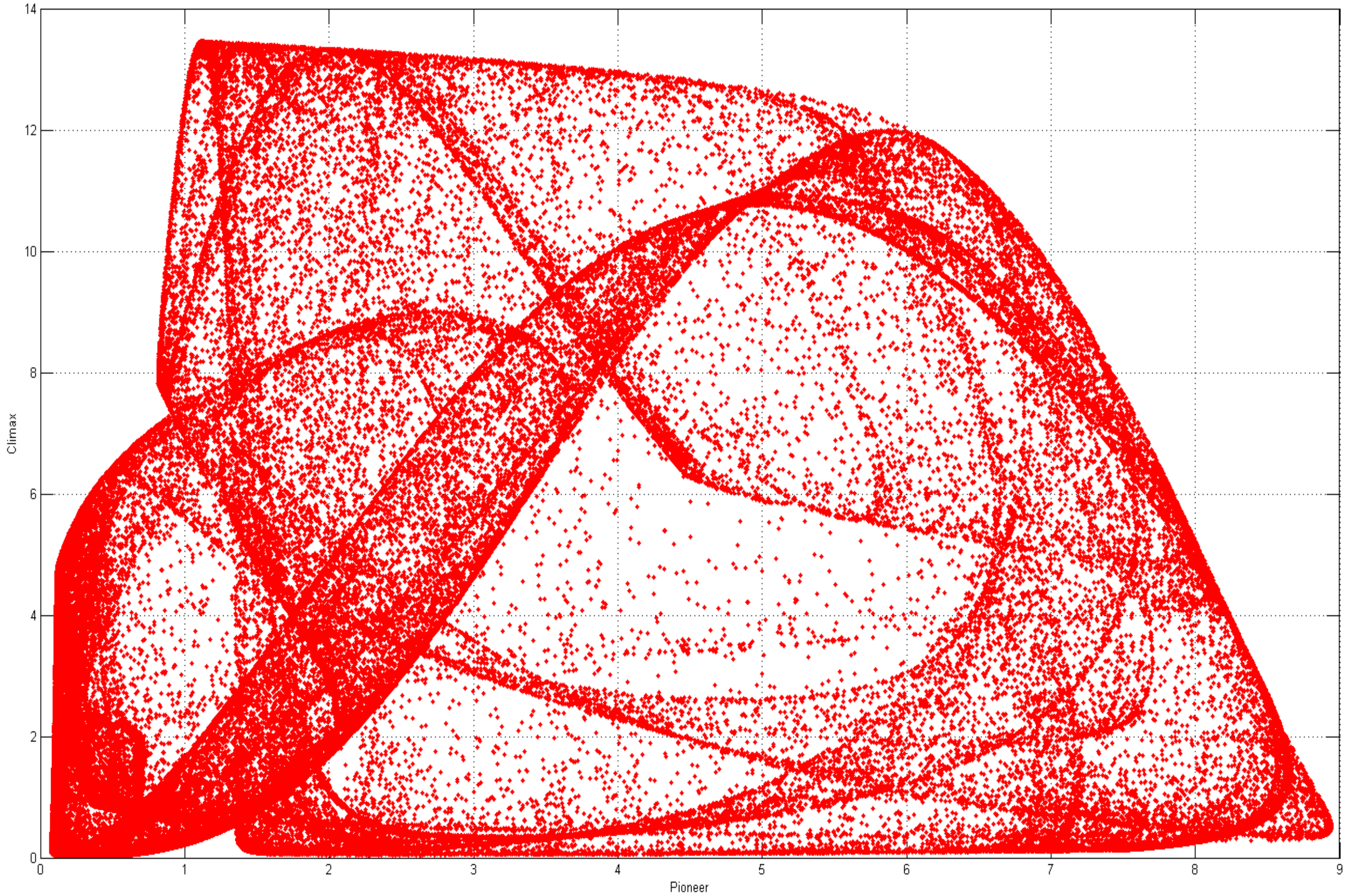

Fig.9. The dynamics of map (11) for $a = b = 3$

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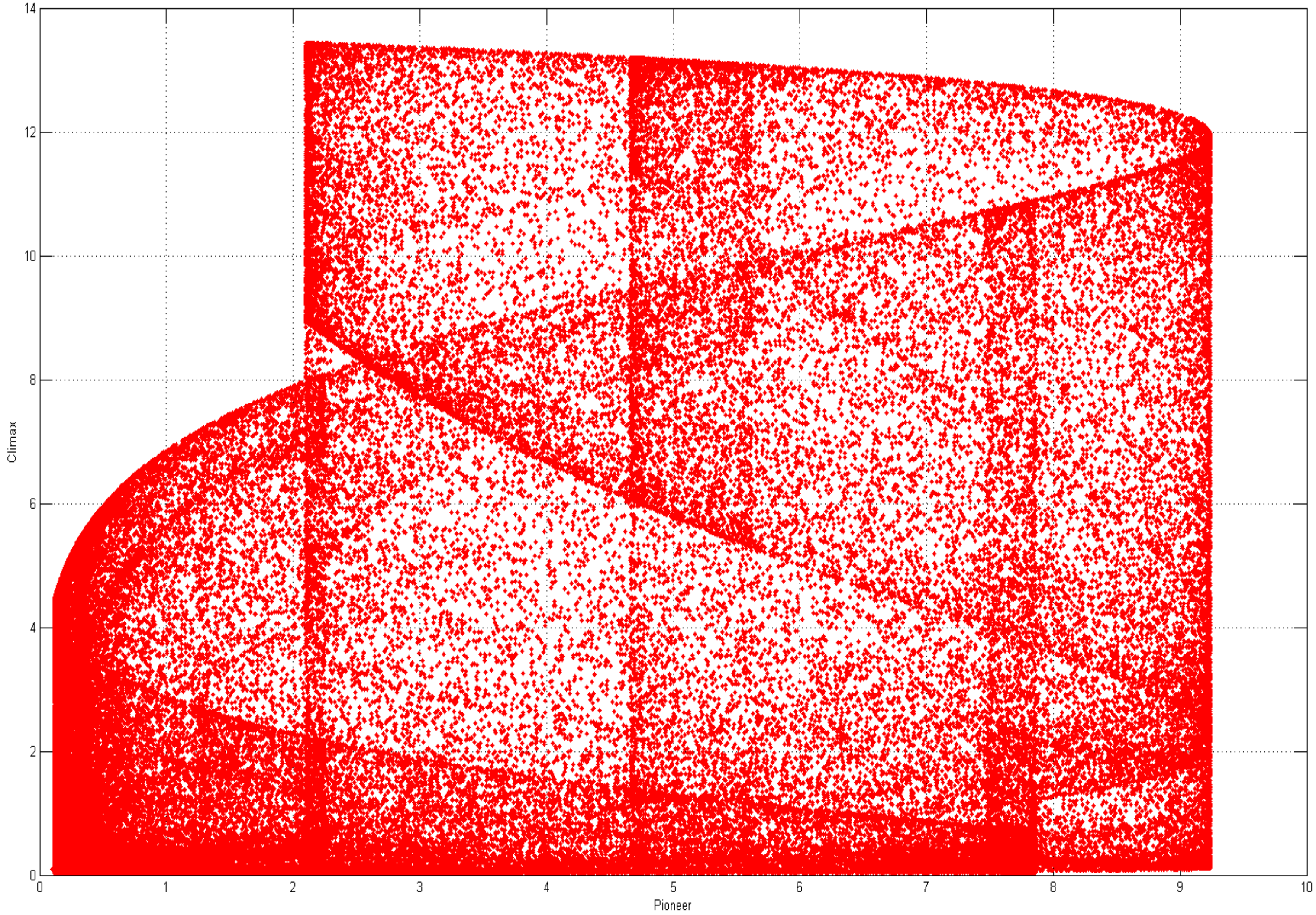

Fig.10. The attractor for map (12)

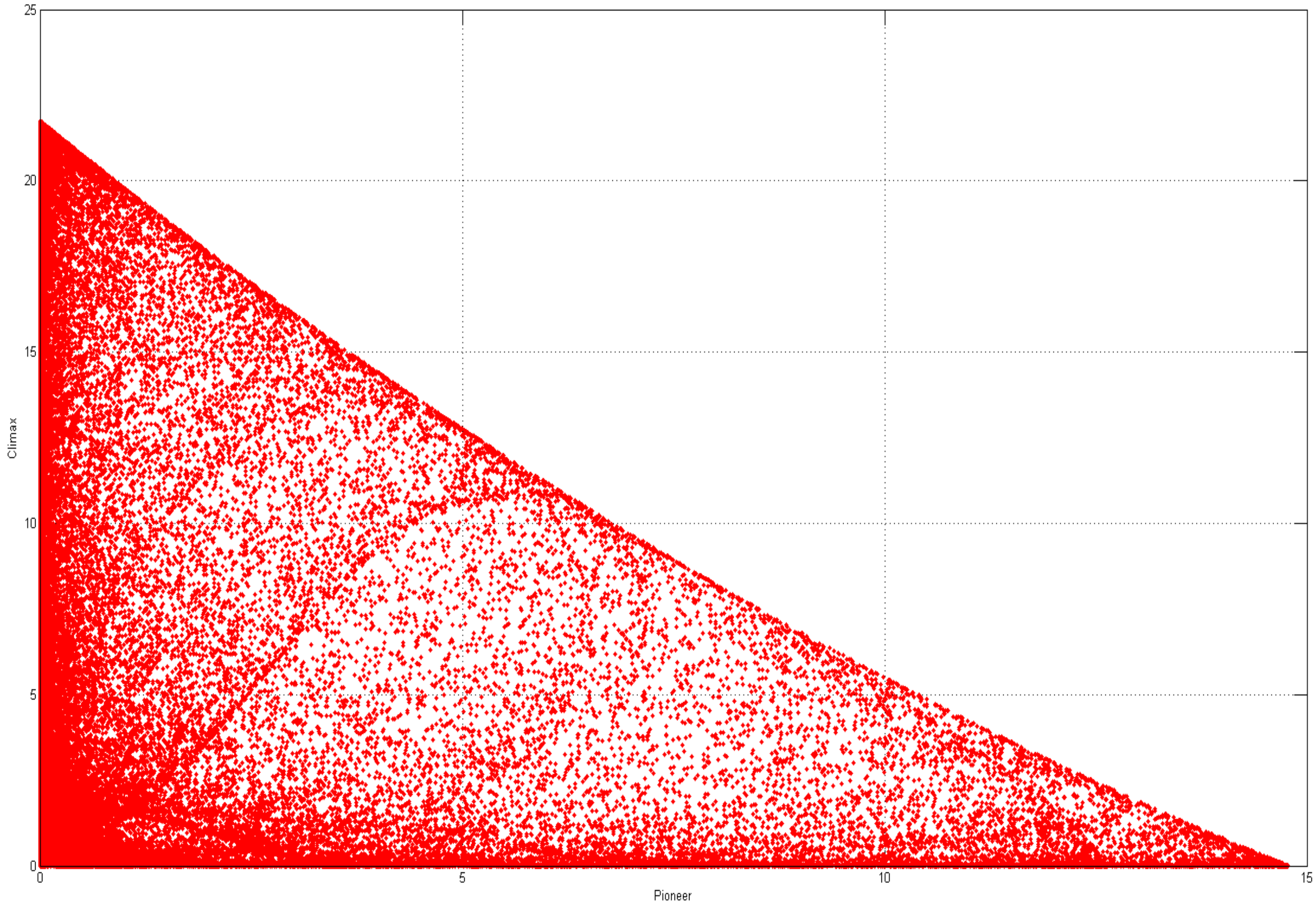

Fig.11. The multihorseshoe attractor for the map (13).

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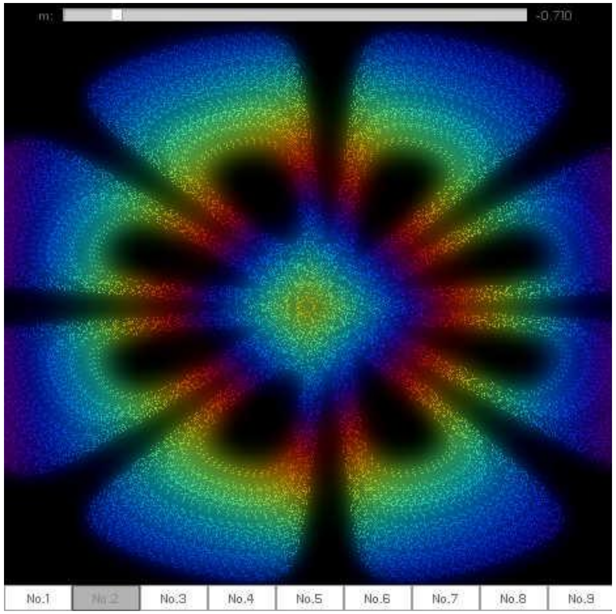
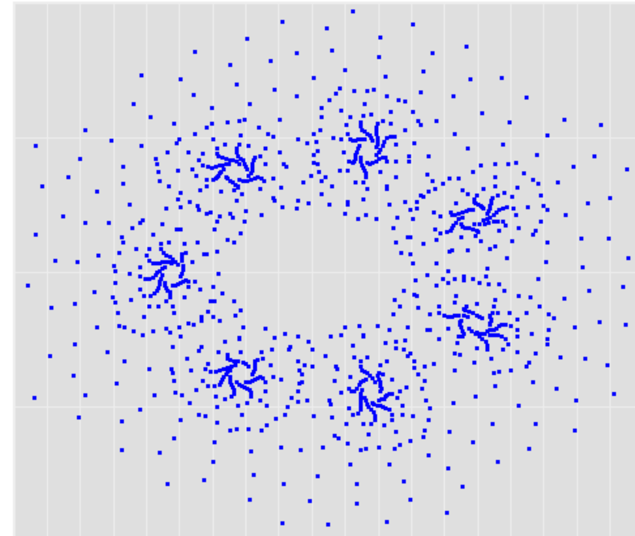
6 More Applications

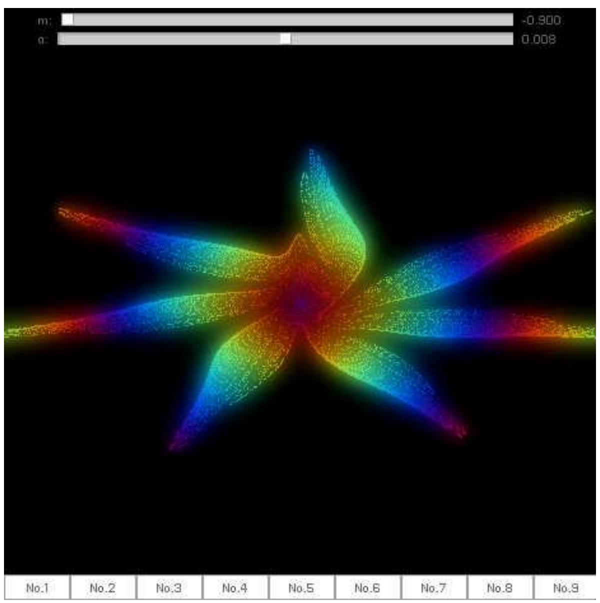
We have applied and are applying our methods to several other areas that feature well-developed dynamical systems models ranging from the finite-dimensional discrete to the infinite-dimensional continuous.

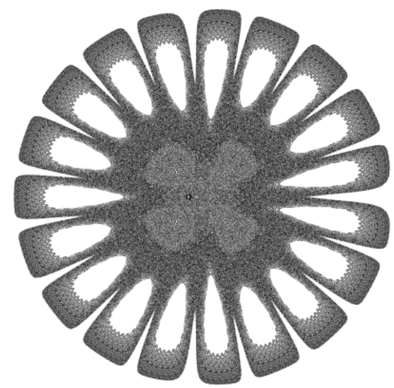
6.1 Gumowski--Mira strange attractors

These attractors have been found in simulations of simplified discrete dynamical systems models for certain quantum phenomena (see Fig. 12). Notice the radial "symmetry". We are in the process of generalizing our radial attractor theory to prove that the simulations are as they appear – chaotic strange attractors.









$$x_{n+1} = f(x_n) + By_n, \;\; y_{n+1} = f(x_{n+1}) - x_n,$$

$$f(x) := Ax + 2(1-A)x^2(1+x^2)^{-2}$$

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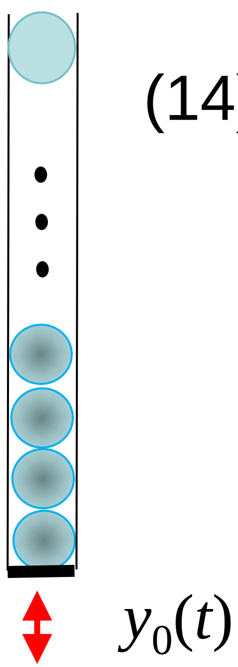
Fig.12. Gumowski—Mira attractors

6.2 Tapped Granular Columns

The motion of the column of particles shown below can be approximated by the iterates of a variant of the planar standard map. Figure 12 is a simulation of the dynamics.

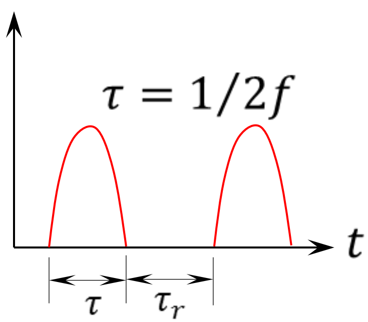


(14)

$$\Phi : \mathbf{S}^1 \times \mathbb{R} \to \mathbf{S}^1 \times \mathbb{R},$$

$$\Phi(\theta, v) := \left([\![\theta + v]\!], ev + \gamma W\left([\![\theta + v]\!]\right)\right),$$

$$[\![\theta + v]\!] := \theta + v \bmod \omega T$$



$$\gamma := \frac{2a\omega^2(1+e)}{g_*} \quad (g_* := g / N).$$

$$W(s) := \begin{cases} \cos s, & 0 \le s \le \pi \\ 0, & \pi \le s \le \omega T. \end{cases}$$

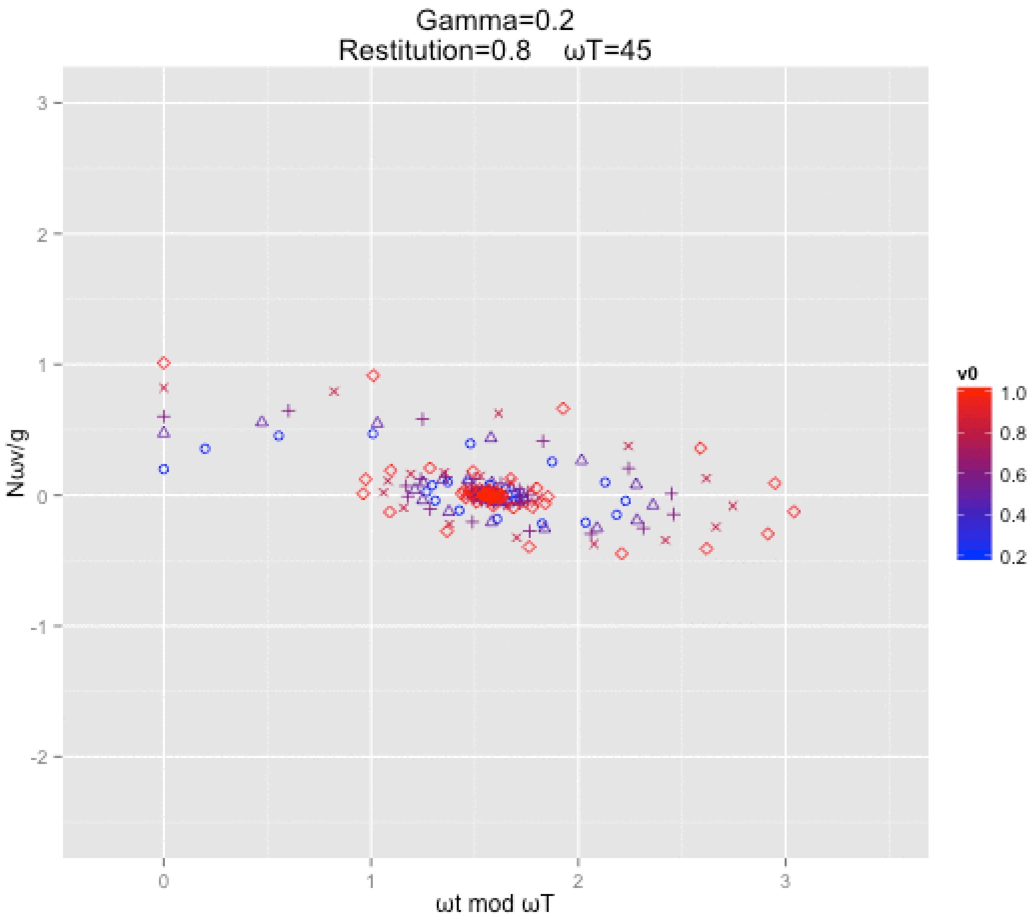


Fig.13. Attractor for tapped column model

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6.3 Walking droplet dynamics

Gilet developed a rather successful planar discrete dynamical system model for walking droplet motion:

(15)
$$
\begin{aligned}
w_{n+1} &= \mu\left[w_n + \Psi(x_n)\right] \\
x_{n+1} &= x_n - C w_n \Psi(x_n), \quad (0 \le \mu, C \le 1)
\end{aligned}
$$

where Ψ is a typical eigenmode such as

$$\Psi := \frac{1}{\sqrt{\pi}}\left(\cos\beta \sin 3x + \sin\beta \sin 5x\right)$$

We have proved that (15) has a Neimark—Sacker bifurcation. Moreover, as a parameter is increased there is a novel type of bifurcation producing a strange attractor – with a proof in the works (see Figs. 14 - 16).

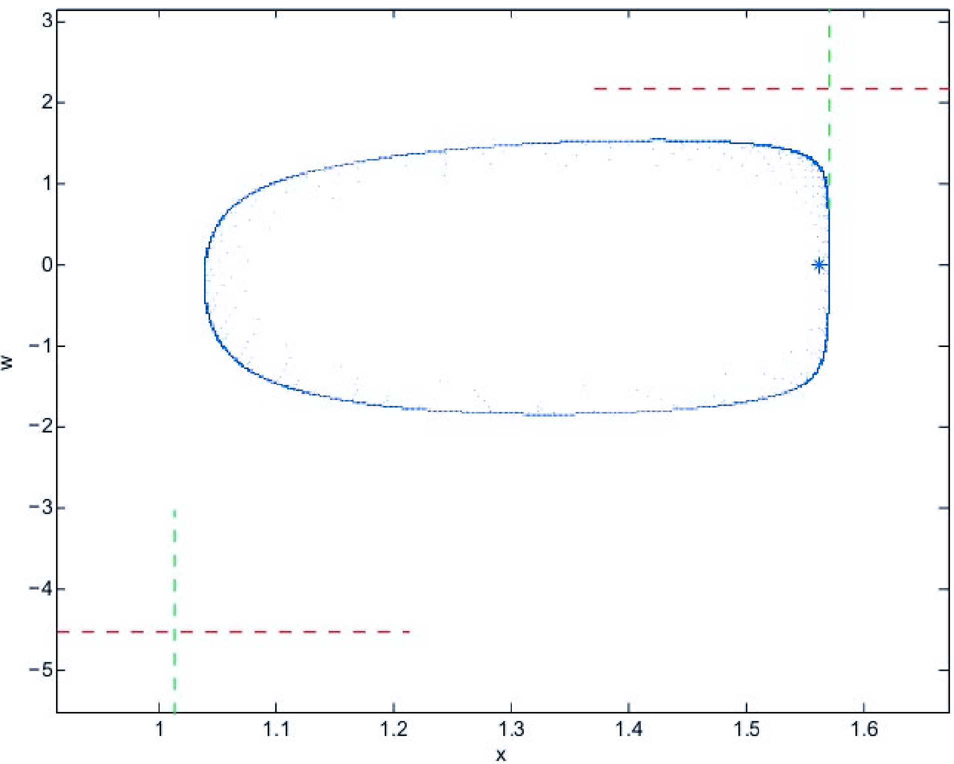

Fig.14. Invariant “circle” for $\mu = 0.913$

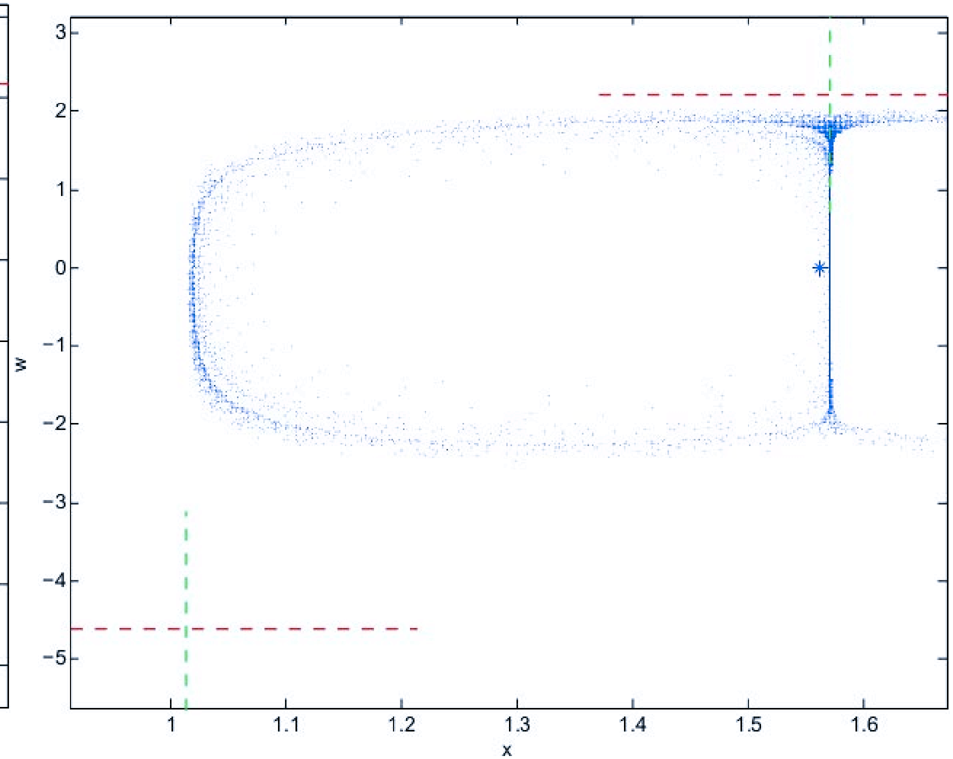

Fig.15. Chaos for $\mu = 0.915$

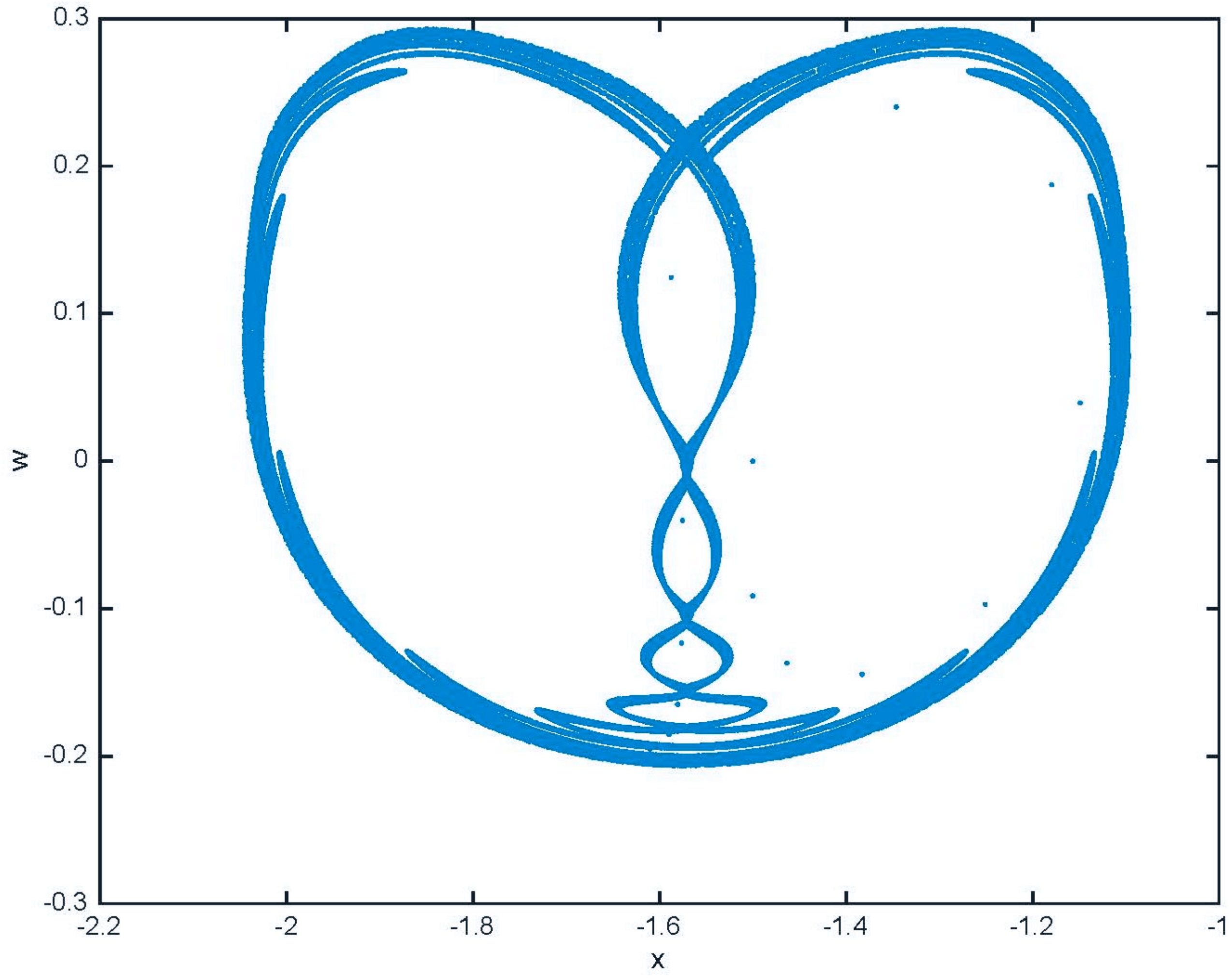

Fig.16. Strange attractor for $\mu \simeq 0.92$

6.4 Chaotic reaction-diffusion dynamics

We are also studying chaotic regimes for reaction-diffusion equations of the type

$$u_t - \Delta u + g\left(t, x, u\right) = 0$$

$$u(0, x) = u_0(x) \in W^{1,2}\left(\Omega\right)$$

$$u_{|[0,T)\times\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, $\partial\Omega \in Lip$ and g is

$$g(t, x, u) := a\left\|u\right\|_2^\rho + h(t, x), \text{ or } := a(t, x)\left|u\right|^\rho + h(t, x).$$

For certain initial conditions, related to the spectrum of the Laplacian, spatio-temporal chaos occurs on an attractor.

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CONCLUSIONS & BEYOND

- ♦ We have proved new theorems for radial strange attractors and introduced and proved results for multihorseshoe chaotic strange attractors
- ♦ The new results have been applied to problems in ecological, granular flow and reaction-diffusion dynamics
- The radial approach has many generalizations, and may be applicable to Gumowski--Mira attractors (Fig. 12)
- We plan to show that the multihorseshoe approach can be generalized in many ways that yield higher dimensional attractors that are definitely not rank-one.
- New applications will be found for our theorems, e.g. a much simpler proof for the Hénon attractor.
- We plan to construct SRB measures for our attractors.