Nonlinear Impulse of Ocean Waves on Floating Bodies

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Abstract

A new formulation is presented of the nonlinear loads exerted on floating bodies by steep irregular surface waves. The forces and moments are expressed in terms of the time derivative of the fluid impulse which circumvents the time consuming computation of the temporal and spatial derivatives in Bernoulli's equation. The nonlinear hydrostatoc force on a floating body is shown to point vertically upwards and the nonlinear Froude-Krylov force and moment are derived as the time derivative of an impulse that involves the time derivative of a simple integral of the ambient velocity potential over the time dependent body wetted surface. The nonlinear radiation and diffraction forces and moments are expressed as time derivatives of two impulses, a body impulse and a free impulse that represents higher order wave loads acting along body waterline. Applications discussed include the nonlinear seakeeping of ships and offshore platforms and the extreme wave loads and responses of offshore wind turbines.

1. Introduction

The momentum conservation principle been widely used in fluid mechanics for the evaluation of steady and unsteady forces and moments acting on bodies in lifting and non-lifting flows. Examples include the derivation of expressions for the unsteady potential flow forces on a body undergoing an arbitrary motion in an infinite fluid and of the induced drag on a three-dimensional lifting surface in terms of the free vorticity downstream on the Treftz plane. Distinct advantages of such expressions for the forces is that they circumvent the computation of the hydrodynamic pressure over the body boundary from Bernoulli's equation which is computationally demanding, they are exact, easy to implement and offer valuable insights of the fluid flow physics responsible for the force and moments exerted on the body.

The momentum conservation principle has found little use for the derivation of expressions for the nonlinear loads exerted on floating bodies by steep irregular waves. The Bernoulli equation is used instead to obtain the hydrodynamic pressure over the instantaneous body wetted surface which upon integration leads to the force and moment acting on the body. A drawback of this approach is that the computation of the partial time derivative and spatial gradients of the velocity potential over the instantaneous body wetted surface is computationally challenging leading to small time steps, fine meshing and large computational efforts. The proper application of the momentum conservation principle for such problems would circumvent these computational shortcomings of the Bernoulli equation and lead to expressions for the time dependent force and moment that do not require the evaluation of the time or space derivatives on the body boundary. A classical example is the expression for the exact force and moment acting on a body in an infinite potential flow in terms of the time derivative of the flow impulse which is a simple integral of the velocity potential over the body boundary.

The concept of the impulse in fluid flow dates back to Kelvin (1868). It arises from the insight that an inviscid flow field governed by the Euler equations may be started from rest if an impulse is applied instantaneously in the fluid domain in the form a local push over the body boundary or a twist in parts of the fluid domain where vorticity is shed. This topic is discussed in Lamb (1932) where the impulsive form of Euler's equations are derived and expressions are obtained for the forces and moments acting on bodies as time derivatives of impulses in potential and vortical flows. In the absence of vorticity the potential flow around a rigid body in a domain of infinite extent may be started from rest by imparting an impulse on its boundary equal to the velocity potential multiplied by the unit normal vector. In lifting flows with shed wakes the impulse also includes an integral over the domain of free vorticity in the fluid domain. Its time derivative provides expressions for the lifting forces and moments generalizations of the Kutta-Joukowski theorem [Lighthill (1986)]. The time derivative of the impulse provides the forces and moments exerted on the rigid body in terms of the time invariant added mass tensor leading to the generalized equations of motion of a rigid body in an ideal fluid [Newman (1977)].

The present article derives expressions for the impulses in the nonlinear interaction of steep and irregular surface waves with floating bodies the time domain. The force and moment acting on the body follow at time derivatives of the impulses which involve simple integrals of the velocity potentials over the body boundary and the nonlinear ambient wave profile assumed known a priori. The ambient waves are assumed irregular of large amplitude and modeled by potential flow theory using a perturbation or a nonlinear method. A floating body interacting with the ambient waves is allowed to undergo large amplitude motions and the radiation and diffraction wave disturbances induced by the body are also governed by potential flow theory. Expressions are obtained of components of the impulse, including the Froude-Krylov (FK) Impulse and the impulse due to the Radiation-Diffraction (RD) wave disturbances. The nonlinear wave force acting on the body is shown to be the sum of a nonlinear hydrostatic force which points vertically upwards at all times and of the time derivative of the Froude-Krylov and the Radiation-Diffraction Impulses which involve simple integrals of the velocity potentials over the instantaneous body boundary. Expressions are also derived for the corresponding moment impulses. The use of the Bernoulli's equation is circumvented mitigating the time consuming computation of time and space derivatives of the velocity potential known to be an impediment of the efficiency of nonlinear wave body computations.

Viscous effects are neglected. In certain applications they are important and may be accounted for as additive forces nonlinear functions of the ambient flow kinematics as in Morison's equation. The application of the new expressions for the nonlinear wave forces is discussed for the evaluation of the large amplitude motion and nonlinear loads of ships, large volume offshore platforms and small volume offshore wind turbines either bottom mounted or supported by floaters.

2. Boundary Value Problem and Nonlinear Forces

Figure 2.1 illustrates a body floating on a free surface interacting with a nonlinear ambient wave which is assumed irregular. The reference coordinate system (X,Y,Z) is fixed in space with its origin located on the calm water surface and the positive Z-axis pointing upwards. The free surface elevation due to the ambient wave is denoted by the dashed line and the free surface elevation that includes the radiation and diffraction wave disturbances by the solid line.

Denote the total velocity potential by φ(X,Y,Z,t), the ambient wave potential by φ_I and the disturbance potential by φ_D. They are all subject to the Laplace equation in the fluid domain

(2.1)
$$\varphi = \varphi_I + \varphi_D$$
$$\frac{\partial^2 \varphi_{I,D}}{\partial X^2} + \frac{\partial^2 \varphi_{I,D}}{\partial Y^2} + \frac{\partial^2 \varphi_{I,D}}{\partial Z^2} = 0$$

On the body boundary the normal velocity of the total potential is equal to the normal velocity of the body boundary

(2.2)
$$\frac{\partial \varphi}{\partial n} = U_n, \ \ on \ \ S_B(t)$$

On the nonlinear free surface $Z = \zeta_T(Y,t)$ the total velocity potential satisfies the nonlinear free-surface condition

$$\left(\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial Z} + 2\nabla\varphi \cdot \nabla \frac{\partial \varphi}{\partial t} + \tfrac{1}{2}\nabla\varphi \cdot \nabla\left(\nabla\varphi \cdot \nabla\varphi\right) \right)_{Z=\zeta_T(X,Y,t)} = 0, \tag{2.3}$$
$$\zeta_T(X,Y,t) = -\frac{1}{g}\left(\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}\nabla\varphi \cdot \nabla\varphi \right)_{Z=\zeta_T(X,Y,t)}$$

The ambient wave potential by φ_I also satisfies (2.3) on the ambient wave free surface $Z = \zeta_I(X,Y,t)$

$$\left(\frac{\partial^2 \varphi_I}{\partial t^2} + g\frac{\partial \varphi_I}{\partial Z} + 2\nabla\varphi_I \cdot \nabla \frac{\partial \varphi_I}{\partial t} + \tfrac{1}{2}\nabla\varphi_I \cdot \nabla\left(\nabla\varphi_I \cdot \nabla\varphi_I\right) \right)_{Z=\zeta_I(X,Y,t)} = 0, \tag{2.4}$$
$$\zeta_I(X,Y,t) = -\frac{1}{g}\left(\frac{\partial \varphi_I}{\partial t} + \tfrac{1}{2}\nabla\varphi_I \cdot \nabla\varphi_I \right)_{Z=\zeta_I(X,Y,t)}$$

It is hereafter assumed that the nonlinear ambient wave velocity potential and free surface elevation are known. They may be determined by a perturbation method or by a nonlinear numerical algorithm. In many applications second order theory has been found to be adequate for the description of unidirectional or spread seas. In the remainder of the article the ambient waves are assumed to be irregular and of finite amplitude.

The definition of hydrodynamic force and moment acting on the body follows from the integration of the hydrodynamic pressure obtained form Bernoulli's equation over the instantaneous body wetted surface $S_B(t)$

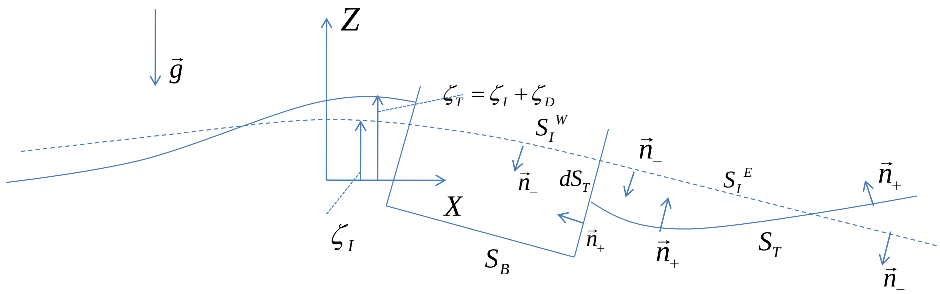

Figure 2.1

(2.5)
$$\vec{F}(t) = -\rho \int_{S_B(t)} \left(\frac{\partial \varphi}{\partial t} + \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi + gZ \right) \vec{n} ds$$
$$\vec{M}(t) = -\rho \int_{S_B(t)} \left(\frac{\partial \varphi}{\partial t} + \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi + gZ \right) \left[\vec{X} \times \vec{n} \right] ds$$

At a large distance from the body the ambient wave and radiation-diffraction wave elevations are assumed to vanish. The decay of the ambient wave disturbance at infinity is physically equivalent to the transition to the calm water region outside a seastate of finite spatial extent.

Denote by V(t) the fluid volume enclosed by the nonlinear free surface $Z = \zeta_T(Y,t)$ the body wetted surface S_B(t) and a control surface at infinity. By virtue of the transport theorem

(2.6)
$$\frac{d}{dt} \iint_{V(t)} \nabla \varphi \, dv = \iint_{V(t)} \frac{\partial}{\partial t} \nabla \varphi \, dv + \int_{S(t)} \nabla \varphi (\vec{U} \cdot \vec{n}) \, ds$$

The control surface S(t) encloses a potential flow which consists of the ambient wave and of the body wave disturbance. The unit normal vector points out of the fluid domain. The asymptotic decay of the radiation-diffraction disturbance at large radial distances is dipole like and leads to the vanishing of all surface integrals over a control surface at infinity. Therefore reference to that surface is hereafter omitted. In (2.6) $U_n = \vec{U} \cdot \vec{n} = \frac{\partial \varphi}{\partial n}$ is the normal velocity of the flow of the over the free surface and the body wetted surface and is equal to the normal velocity of the respective surfaces.

Applying Gauss's theorem in the left hand side and the first term in the right hand side of (2.6), and exchanging the time derivative with the gradient under the integral sign

$$\text{(2.7)} \qquad \frac{d}{dt} \int_{S(t)} \varphi \vec{n} ds = \int_{S(t)} \left[\frac{\partial \varphi}{\partial t} \vec{n} + \nabla \varphi \, \frac{\partial \varphi}{\partial n} \right] ds$$

Adding the quadratic term in Bernoulli's equation on both sides of (2.7) we obtain after a simple rearrangement of terms

$$\text{(2.8)} \qquad \int_{S(t)} \left[\frac{\partial \varphi}{\partial t} + \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi \right] \vec{n} \, ds = \frac{d}{dt} \int_{S(t)} \varphi \vec{n} \, ds - \int_{S(t)} \left[\nabla \varphi \, \frac{\partial \varphi}{\partial n} - \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi \vec{n} \right] ds$$

The following identity holds for any velocity potential over a closed surface S(t)

$$\text{(2.9)} \qquad \int_{S(t)} \left[\nabla \varphi \frac{\partial \varphi}{\partial n} - \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi \vec{n} \right] ds = 0$$

Combining (2.8) and (2.9) and adding the hydrostatic term on both sides of (2.8) we obtain

$$\text{(2.10)} \qquad \int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi + g \, Z \right] \vec{n} \, ds = \frac{d}{dt} \int_{S_B(t)+S_T(t)} \varphi \vec{n} \, ds + g \int_{S_B(t)+S_T(t)} Z \, \vec{n} \, ds$$

Expression (2.10) is valid for any nonlinear free surface flow around a floating body assuming that all disturbances vanish sufficiently fast at infinity. The surface $S_B(t)$ denotes the body instantaneous wetted surface and $S_T(t)$ the nonlinear free surface exterior to the body over which the hydrodynamic pressure vanishes. The left hand side of (2.10) multiplied by $-\rho$ is the total nonlinear hydrodynamic force acting on the floating body. The right-hand side involves no spatial or time derivatives under the integral sign and is the starting point for the derivation of the impulses and the new expressions for the forces below.

The corresponding expression for the moment follows from the application of the following identity over a closed surface S(t)

$$\int_{S(t)} \vec{X} \times \left[\nabla\varphi \frac{\partial \varphi}{\partial n} - \tfrac{1}{2} \nabla\varphi \cdot \nabla\varphi \, \vec{n} \right] ds = 0 \tag{2.11}$$

The resulting expression for the moment follows a derivation analogous to that used to obtain (2.10)

$$\begin{aligned} &\int_{S_B(t)} \left[\frac{d\varphi}{dt} + \tfrac{1}{2} \nabla\varphi \cdot \nabla\varphi + g\, Z \right] (\vec{X} \times \vec{n})\, ds = \\ &\qquad = \frac{d}{dt} \int_{S_B(t)+S_T(t)} \varphi (\vec{X} \times \vec{n})\, ds + g \int_{S_B(t)+S_T(t)} Z\, (\vec{X} \times \vec{n})\, ds \end{aligned} \tag{2.12}$$

If the body is allowed to shrink to a point with vanishing displacement the left-hand side of (2.10) reduces to

$$\begin{aligned} &0 = \frac{d}{dt} \left[\int_{S_I(t)} \varphi_I \, \vec{n}\, ds \right] + g \int_{S_I(t)} Z\, \vec{n}\, ds \\ &g \int_{S_I(t)} Z\, \vec{n}\, ds = g \int_{S_I(t)+S_{Z=0}} Z\, \vec{n}\, ds = g\vec{k} \iint_{\upsilon(t)} d\upsilon = g\vec{k} V(t) \end{aligned} \tag{2.13}$$

In (2.13) φ_I is the velocity potential and $S_I(t)$ the free surface of the ambient wave in the absence of a floating body. Therefore (2.13) represents an integral form of the nonlinear free surface condition satisfied by an irregular ambient wave. It states the balance between the time derivative of the impulse applied over the free surface and the restoring effect of the the ambient wave free surface elevation. Expression (2.13) is an alternative form of the classical nonlinear free surface condition (2.4) as an integral balance between inertial and restoring effects for ambient wave disturbances that vanish at infinity. A new mathematical representation of ocean waves that allows their decay at infinity is presented by Sclavounos (2011).

The second term in the right hand side of (2.13) is shown to be a vector pointing in the vertical direction by virtue of Gauss' theorem applied over the volume enclosed by the free surface and the Z=0 plane. The horizontal components of (2.13) lead to

$$
\begin{aligned}
&0 = \frac{d}{dt}\left[\int_{S_I(t)} \varphi_I \, n_X \, ds \right] = \frac{d}{dt}\left[\int_{S_I(t)} \varphi_I \, n_Y \, ds \right] \\
&\int_{S_I(t)} \varphi_I \, n_X \, ds = C_1, \quad \int_{S_I(t)} \varphi_I \, n_Y \, ds = C_2
\end{aligned}
\tag{2.14}
$$

The conservation law (2.14) states that the horizontal impulses of free ambient wave disturbances are constant and time invariant as they are not counteracted by restoring effects. Any nonlinear representation of ambient waves subject to (2.4) and the Laplace equation also obeys (2.13) and (2.14).

3. Nonlinear Impulses

Taking the difference between (2.10) and (2.13) and multiplying with both sides with $-\rho$ we obtain for the nonlinear force acting on the floating body

$$
\begin{aligned}
\vec{F}_B(t) &= -\rho \int_{S_B(t)} \left[\frac{\partial \varphi}{\partial t} + \tfrac{1}{2} \nabla \varphi \cdot \nabla \varphi + g\,Z \right] \vec{n}_+ \, ds = \\
&= -\rho g \left[\int_{S_B(t)} Z\,\vec{n}_+ \, ds + \int_{S^W{}_I(t)} Z\,\vec{n}_- \, ds \right] \\
&\quad - \rho \frac{d}{dt} \left[\int_{S_B(t)} \varphi\,\vec{n}_+ \, ds + \int_{S^W{}_I(t)} \varphi_I\,\vec{n}_- \, ds \right] \\
&\quad - \rho \frac{d}{dt} \left[\int_{S_T(t)} \varphi\,\vec{n}_+ \, ds + \int_{S^E{}_I(t)} \varphi_I\,\vec{n}_- \, ds \right] - \rho g \left[\int_{S_T(t)} Z\,\vec{n}_+ \, ds + \int_{S^E{}_I(t)} Z\,\vec{n}_- ds \right]
\end{aligned}
$$

(3.1)

In (3.1) the ambient wave free surface interior to the body is denoted by S_I^W and its surface exterior to the body is denoted by S_I^E. [Figure 2.1]. Moreover the notation is adopted that unit normal vectors pointing out of the fluid domain are denoted by $\vec{n}_+$ and unit normal vectors pointing into the fluid domain by $\vec{n}_-$. This will facilitate the application of Gauss' theorem in the derivation that follows.

Denote by $S_B^W(t)$ the body wetted surface defined as the intersection of its boundary with the ambient wave profile. It follows that $S_B(t)= S_B^W(t)+dS(t)$, where dS(t) is the differential surface on around the waterline bounded by the exact free surface elevation and the free surface elevation of the ambient wave. Introducing this notation in (3.1) and re-grouping terms we obtain

$$\vec{F}_B(t) = -\rho \int_{S_B(t)} \left[\frac{d\varphi}{dt} + \tfrac{1}{2}\nabla\varphi\cdot\nabla\varphi + g\,Z \right] \vec{n}_+ \, ds =$$

$$= -\rho g \left[\int_{S^W{}_B(t)} Z\,\vec{n}_+\,ds + \int_{S^W{}_I(t)} Z\,\vec{n}_-\,ds \right]$$

$$-\rho \frac{d}{dt} \left[\int_{S^W{}_B(t)} \varphi\,\vec{n}_+\,ds + \int_{S^W{}_I(t)} \varphi_I\,\vec{n}_-\,ds \right]$$

$$-\rho \frac{d}{dt} \left[\int_{S_T(t)+dS(t)} \varphi\,\vec{n}_+\,ds + \int_{S^E{}_I(t)} \varphi_I\,\vec{n}_-\,ds \right] - \rho g \left[\int_{S_T(t)+dS(t)} Z\,\vec{n}_+\,ds + \int_{S^E{}_I(t)} Z\,\vec{n}_-\,ds \right]$$

(3.2)

The total velocity potential $\varphi = \varphi_I + \varphi_D$ in (3.2) is the sum of the ambient wave and disturbance potentials. Substituting in (3.2) and re-grouping terms we obtain

$$\vec{F}_B(t) = -\rho \int_{S_B(t)} \left[\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}\nabla\varphi\cdot\nabla\varphi + g\,Z \right] \vec{n}_+ \, ds =$$

$$= -\rho g \left[\int_{S^W{}_B(t)} Z\,\vec{n}_+\,ds + \int_{S^W{}_I(t)} Z\,\vec{n}_-\,ds \right]$$

$$-\rho \frac{d}{dt} \left[\int_{S^W{}_B(t)} \varphi_I\,\vec{n}_+\,ds + \int_{S^W{}_I(t)} \varphi_I\,\vec{n}_-\,ds \right]$$

$$-\rho \frac{d}{dt} \left[\int_{S^W{}_B(t)} \varphi_D\,\vec{n}_+\,ds \right]$$

$$-\rho \frac{d}{dt} \left[\int_{S_T(t)+dS(t)} \varphi_D\,\vec{n}_+\,ds \int_{S^E{}_I(t)} \varphi_D\,\vec{n}_-\,ds + \right] + \rho \frac{d}{dt} \left[\int_{S^E{}_I(t)} \varphi_D\,\vec{n}_-\,ds \right]$$

$$-\rho \frac{d}{dt} \left[\int_{S_T(t)+dS(t)} \varphi_I\,\vec{n}_+\,ds + \int_{S^E{}_I(t)} \varphi_I\,\vec{n}_-\,ds \right] - \rho g \left[\int_{S_T(t)+dS(t)} Z\,\vec{n}_+\,ds + \int_{S^E{}_I(t)} Z\,\vec{n}_-\,ds \right]$$

(3.3)

In the fourth term on the right hand side of (3.3) the integral of the disturbance potential over the ambient wave surface was added and subtracted. The reason is that the surfaces $S_T(t)+dS(t)+S^E{}_I(t)$ define a closed differential volume [see Figure 2.1] over which Gauss' theorem will be applied to further reduce the integrals in the last two lines of the right-hand side of (3.3). It is also noted that the unit normal vectors over this closed differential surface have been defined so that they point either always outside or inside the enclosed volume leading to a positive or negative sign in the application of Gauss' theorem, respectively.

Expression (3.3) provides an exact expression for the nonlinear force acting on the body. It contains five terms in its right-hand side. Each is discussed separately next.

Nonlinear Buoyancy Force and Moment

The first term in the right hand side of (3.4) defines the nonlinear buoyancy force. Applying Gauss' theorem and noting that with the unit normal vectors point inside the body boundary, we obtain

(3.4)
$$\begin{aligned}\vec{F}_{B,1}(t) &= -\rho g\left[\int_{S^W{}_B(t)} Z\,\vec{n}_+\,ds + \int_{S^W{}_I(t)} Z\,\vec{n}_-\,ds\right]\\ &= \rho g\,\nabla_W(t)\,\vec{k}\end{aligned}$$

In (3.4) $\vec{k}$ is the unit vector pointing in the positive Z-direction and $\nabla_W(t)$ is the volume enclosed by the body wetted surface and the ambient wave surface interior to the body. Therefore the nonlinear hydrostatics force given by (3.4) acts on a closed volume and always points upwards. In the classical definition of the nonlinear body force obtained by integration of the hydrodynamic pressure using Bernoulli's equation in (2.5) the nonlinear hydrostatic force is coupled with the flow dynamics and does not necessary point upwards.

The nonlinear hydrostatic moment follows from (2.10) and a derivation that parallels that for the force

$$\vec{M}_{B,1}(t) = -\rho g \left[\int_{S^W{}_B(t)} Z\,(\vec{X} \times \vec{n}_+)ds + \int_{S^W{}_I(t)} Z(\vec{X} \times \vec{n}_-)\,ds \right] \tag{3.5}$$

It follows from (3.4) and (3.5) that the nonlinear buoyancy force points upwards and passes through the geometrical center of the volume $\nabla_W(t)$. These expressions extend the classical Archimedian hydrostatics in calm water via the introduction of the ambient wave free surface as the dynamic waterplane area of the floating body.

Froude-Krylov Impulse and Force and Moment

The second term in the right hand side of (3.3) leads to the definition of the Froude-Krylov Impulse and the force defined as its time derivative

$$\begin{aligned}
\vec{F}_{B,2}(t) &= -\rho \frac{d}{dt}\left[\int_{S^W{}_B(t)} \varphi_I\, \vec{n}_+\, ds + \int_{S^W{}_I(t)} \varphi_I\, \vec{n}_-\, ds \right] \\
\vec{I}_{F-K} &= -\rho \int_{S^W{}_B(t)} \varphi_I\, \vec{n}_+ ds - \rho \int_{S^W{}_I(t)} \varphi_I\, \vec{n}_-\, ds \\
&= -\rho \iint_{V_W(t)} \nabla\varphi_I\, dv
\end{aligned} \tag{3.6}$$

The surface integrations in (3.6) are again carried out over the intersection of the body boundary and the ambient wave profile which is assumed known. An additional integration is carried out over the ambient wave free surface interior to the body. An application of Gauss's theorem provides an alternative definition of the Froude-Krylov impulse as the integral of the ambient wave velocity vector over the volume internal to the body surface and its dynamic waterplane area. The evaluation of the new Froude-Krylov forces and moments requires knowledge of just the velocity potential of the ambient wave over the body boundary and not its partial time derivative or its spatial gradients.

In the limit of waves of small steepness the integrals over the interior waterplane area in (3.4) and (3.6) cancel out by virtue of the linearized dynamic free surface condition satisfied by the ambient wave on the Z=0 plane. Yet this cancellation does not occur in the nonlinear case.

The Froude-Krylov moment and its impulse take the form

$$
\begin{aligned}
&\vec{M}_{B,2}(t) = \frac{d\vec{L}_{F-K}}{dt} \\
&\vec{L}_{F-K} = -\rho \int_{S^{W}{}_{B}(t)} \varphi_I \,(\vec{X} \times \vec{n}_+) ds - \rho \int_{S^{W}{}_{I}(t)} \varphi_I \,(\vec{X} \times \vec{n}_-)\, ds
\end{aligned}
\tag{3.7}
$$

Radiation and Diffraction Body Impulse, Force and Moment

The third term in the right hand side of (3.3) leads to the body Radiation-Diffraction disturbance Impulse and respective force and moments

$$
\begin{aligned}
&\vec{F}_{B,3} = \frac{d\vec{I}_{B-D,I}}{dt} \\
&\vec{I}_{B-D,I} = -\rho \int_{S^{W}{}_{B}(t)} \varphi_D \, \vec{n}_+ \, ds \\
&\vec{M}_{B,3} = \frac{d\vec{L}_{B-D,I}}{dt} \\
&\vec{L}_{B-D,I} = -\rho \int_{S^{W}{}_{B}(t)} \varphi_D \,(\vec{X} \times \vec{n}_+)\, ds
\end{aligned}
\tag{3.8}
$$

The integrations in (3.8) are carried out over the body wetted surface defined by its intersection with the ambient wave profile. Again the evaluation of the forces and moments requires just the radiation and diffraction velocity potential over the body boundary and not their partial time derivative or spatial gradients.

Radiation and Diffraction Free Surface Impulse, Force and Moment

The fourth and fifth terms in the right hand side of (3.4) have been reduced to the form

$$\vec{F}_{B,5} = -\rho \frac{d}{dt}\left[\int_{S_T(t)+dS(t)} \varphi_D \, \vec{n}_+ \, ds + \int_{S^E{}_I} \varphi_D \, \vec{n}_- \, ds \right] + \rho \frac{d}{dt}\left[\int_{S^E{}_I} \varphi_D \, \vec{n}_- \, ds \right]$$
$$- \rho \frac{d}{dt}\left[\int_{S_T(t)+dS(t)} \varphi_I \, \vec{n}_+ \, ds + \int_{S^E{}_I(t)} \varphi_I \, \vec{n}_- \, ds \right] - \rho g \left[\int_{S_T(t)+dS(t)} Z \, \vec{n}_+ \, ds + \int_{S^E{}_I(t)} Z \, \vec{n}_- \, ds \right]$$

(3.9)

The integration in the first, third and fourth terms in the right hand side is over a closed differential surface bounded by the nonlinear free surface, the ambient wave surface and the differential surface around the body waterline. The unit normal vector over this closed surface points outside the enclosed volume when the total wave elevation is larger than the ambient wave elevation and points inside the differential volume otherwise. The first, third and fourth terms may be reduced by an application of Gauss' theorem over the volume bounded by the closed differential surface defined above. Denoting by $\zeta_D = \zeta_T - \zeta_I$ the radiation-diffraction disturbance wave elevation assumed to be a signed quantity the unit normal vectors over the differential surface point outside the differential volume when $\zeta_D > 0$ and inside the volume when $\zeta_D < 0$. Grouping terms and applying Gauss' theorem in (3.10) we obtain

$$\vec{F}_{B,5} = -\rho \frac{d}{dt}\left[\int_{S^E{}_I} \varphi_D \, \vec{n}_+ \, ds \right]$$
$$- \rho \frac{d}{dt}\left[\int_{S_T(t)+dS(t)} (\varphi_I + \varphi_D) \, \vec{n}_+ \, ds + \int_{S^E{}_I(t)} (\varphi_I + \varphi_D) \, \vec{n}_- \, ds \right]$$

(3.10)

$$- \rho g \left[\int_{S_T(t)+dS(t)} Z \, \vec{n}_+ \, ds + \int_{S^E{}_I(t)} Z \, \vec{n}_- \, ds \right]$$
$$= -\rho \frac{d}{dt}\left[\int_{S^E{}_I(t)} \varphi_D \, \vec{n}_+ \, ds \right]$$
$$- \rho \frac{d}{dt}\left[\iint_{\upsilon(t)} \left(\nabla \varphi_I + \nabla \varphi_D \right) dv \right] \operatorname{sgn}(\zeta_D) - \rho g \vec{k} \, \upsilon(t) \operatorname{sgn}(\zeta_D)$$

The second and third terms in the right-hand side of the second equality of (3.10) may be expressed as integrals over the ambient wave free surface exterior to the body by a Taylor's series expansion

$$
\begin{aligned}
\vec{F}_{B,5} = \frac{d\vec{I}_{B,5}}{dt} &= -\rho \frac{d}{dt} \int_{S^E{}_I(t)} \varphi_D \, \vec{n}_+ \, ds - \rho g \vec{k} \int_{S^E{}_I(\tau)} \zeta_D \, ds \\
&\quad - \rho \frac{d}{dt} \int_{S^E{}_I(t)} \left[\zeta_D \nabla(\varphi_I + \varphi_D) + \tfrac{1}{2} \zeta^2{}_D \frac{\partial}{\partial Z} \nabla(\varphi_I + \varphi_D) + \ldots \right] ds
\end{aligned}
$$

(3.11)

$$
\begin{aligned}
\vec{I}_{B,F-S} &= -\rho \int_{S^E{}_I(t)} \varphi_D \, \vec{n}_+ \, ds - \rho g \vec{k} \int_t d\tau \int_{S^E{}_I(\tau)} \zeta_D \, ds \\
&\quad - \rho \int_{S^E{}_I(t)} \left[\zeta_D \nabla(\varphi_I + \varphi_D) + \tfrac{1}{2} \zeta^2{}_D \frac{\partial}{\partial Z} \nabla(\varphi_I + \varphi_D) + \ldots \right] ds
\end{aligned}
$$

Expression (3.11) provides the impulse imparted upon the body by the radiation-diffraction disturbances via the ambient wave free surface.

The corresponding expression for the moment follows in an analogous manner

(3.12)

$$
\begin{aligned}
\vec{M}_{B,5} &= -\rho \frac{d}{dt} \left[\int_{S^E{}_I} \varphi_D (\vec{X} \times \vec{n}_+) \, ds \right] \\
&\quad - \rho \frac{d}{dt} \left[\int_{S_T(t)+dS(t)} (\varphi_I + \varphi_D)(\vec{X} \times \vec{n}_+) \, ds + \int_{S^E{}_I(t)} (\varphi_I + \varphi_D)(\vec{X} \times \vec{n}_-) \, ds \right] \\
&\quad - \rho g \left[\int_{S_T(t)+dS(t)} Z (\vec{X} \times \vec{n}_+) \, ds + \int_{S^E{}_I(t)} Z (\vec{X} \times \vec{n}_+) \, ds \right]
\end{aligned}
$$

Applying again Gauss' theorem the second and third terms in the right-hand side of (3.12) may be converted into an integral over the differential volume between the exact and the ambient wave surfaces

$$\vec{M}_{B,5} = -\rho \frac{d}{dt}\left[\int_{S^E{}_I(t)} \varphi_D \, (\vec{X} \times \vec{n}_+) ds \right]$$

(3.13)

$$-\rho \frac{d}{dt}\left[\iint_{\upsilon(t)} \vec{X} \times \left(\nabla \varphi_I + \nabla \varphi_D \right) dv \right] \mathrm{sgn}(\zeta_D) - \rho g \iint_{\upsilon(t)} \vec{X} \times \vec{k} \, dv \ \mathrm{sgn}(\zeta_D)$$

The further reduction of (3.13) into a form analogous to (3.11) is omitted for brevity. Summarizing, the nonlinear fluid force and moment acting on a body floating in an ambient irregular wave of large amplitude has been derived as the sum of a nonlinear buoyancy force pointing upwards and the time derivative of a sequence of impulses. The Froude-Krylov nonlinear impulse involves an integral of the ambient wave velocity potential over the instantaneous body wetted surface and the interior waterplane area defined by the ambient wave elevation. The body Radiation-Diffraction (RD) nonlinear impulse involves an integral of the RD velocity potentials over the body wetted surface. The free-surface RD nonlinear impulse involves integrals of the RD disturbances over ambient wave free surface. It is shown in the next section that it is small relative to the nonlinear Froude-Krylov and body RD forces may be neglected in a number of applications.

4. Free Surface Impulse

All expressions for the nonlinear impulses and forces derived above involve integrals of the velocity potentials over the instantaneous body wetted surface defined as its intersection with the ambient wave profile. For fixed bodies this wetted surface is known in terms of the ambient wave profile which is known a priori. For bodies free to oscillate it follows as the geometric intersection of the instantaneous body position as obtained from the time marching of its equations of motion and the ambient wave profile. This geometric may be simply carried out during each time step of the simulation of the body using a standard Computer Aided Design program.

Nonlinear ambient waves are assumed to have an amplitude A=O(1) comparable to the body dimension d, either the diameter of a leg of an offshore platform or the draft of a section of a ship cases where d~10-20m. The ambient wave steepness kA= kA= 2π A/ λ is assumed to be small and of O(δ) where δ ~ 0.1. For fixed offshore structures the magnitude of the body wave disturbance $(\varphi_D, \zeta_D) = O(kd) = O(2\pi d / \lambda) = O(\varepsilon)$. The case of the interaction of a breaking wave interaction with a structure requires a fully nonlinear treatment which is beyond the scope of the present article. For a ship floating in steep waves encountered in design seastates typically have a length larger than teh ship length therefore they are long compared to the ship draft and beam. Therefore the body wave disturbance is again of the $(\varphi_D, \zeta_D) = O(kd) = O(2\pi d / \lambda) = O(\varepsilon)$ where d is characteristic transverse dimension of the ship. Typically ε < 0.1 in design seastates. These order of magnitude estimates justify the linearization of the body wave disturbance about the ambient wave profile which assumed to have a large amplitude.

Invoking the fully nonlinear free surface condition (2.3) and linearizing the body disturbance about the ambient wave profile we obtain the dynamic surface condition for (φ_D, ζ_D)

$$\left[\frac{\partial \varphi_D}{\partial t} + \nabla\varphi_D \cdot \nabla\varphi_I\right] + \zeta_D \left\{ g + \frac{\partial}{\partial Z}\left[\frac{\partial \varphi_I}{\partial t} + \tfrac{1}{2}\nabla\varphi_I \cdot \nabla\varphi_I\right]\right\} = 0, \;\; Z = \zeta_I(X,Y,t)$$

(4.1) $$\frac{\partial \varphi_D}{\partial t} + g\zeta_D = -\left\{\nabla\varphi_D \cdot \nabla\varphi_I - \zeta_D \frac{\partial}{\partial Z}\left[\frac{\partial \varphi_I}{\partial t} + \tfrac{1}{2}\nabla\varphi_I \cdot \nabla\varphi_I\right]\right\} = O(\varepsilon\delta), \;\; Z = \zeta_I(X,Y,t)$$

The kinematic free surface condition takes the form

$$\frac{\partial \zeta_D}{\partial t} - \frac{\partial \varphi_D}{\partial Z} + \nabla\varphi_I \cdot \nabla\zeta_D + \nabla\zeta_I \cdot \nabla\varphi_D - \zeta_D\left[\frac{\partial^2 \varphi_I}{\partial Z^2} - \nabla\zeta_I \cdot \nabla\left(\frac{\partial \varphi_I}{\partial Z}\right)\right] = 0, \;\; Z = \zeta_I(X,Y,t)$$

(4.2)

The disturbance potential satisfies the Laplace equation in the fluid domain and on the body wetted surface is subject to the condition

(4.3) $$\frac{\partial \varphi_D}{\partial n} = U_n - \frac{\partial \varphi_I}{\partial n}, \;\; on \;\; S_B(t)$$

For a fixed body $U_n = 0$ and the disturbance potential reduces to the diffraction potential.

Vertical Free Surface Impulse

It follows from (3.12) that the vertical component of the force and free surface impulse takes the form

(4.4)
$$F_{Z-B,5} = -\rho \frac{d}{dt} \int_{S^E{}_I(t)} \varphi_D \, n_Z \, ds - \rho g \int_{S^E{}_I(\tau)} \zeta_D \, ds$$
$$- \rho \frac{d}{dt} \int_{S^E{}_I(t)} \left[\zeta_D \frac{\partial}{\partial Z}(\varphi_I + \varphi_D) + \tfrac{1}{2} \zeta^2{}_D \frac{\partial^2}{\partial Z^2} \nabla(\varphi_I + \varphi_D) + ... \right] ds$$

The unit normal vector on the ambient wave free is given by the expression

(4.5)
$$\vec{n} = \frac{\nabla(Z - \zeta_I(X,Y,t))}{\left|\nabla(Z - \zeta_I(X,Y,t))\right|} = \frac{-\vec{i}\,\zeta_{IX} - \vec{j}\zeta_{IY} + \vec{k}}{\sqrt{1 + \zeta^2{}_{IX} + \zeta^2{}_{IY}}} = \left(-\vec{i}\,\zeta_{IX} - \vec{j}\zeta_{IY} + \vec{k}\right)\left[1 + O(\delta^2)\right]$$

Substituting its vertical component in (4.6) and with errors of $O(\delta^2, \varepsilon^2)$ we obtain

(4.6)
$$F_{Z-B,5} = -\rho \frac{d}{dt} \int_{S^E{}_I(t)} \left(\varphi_D + \zeta_D \frac{\partial \varphi_I}{\partial Z} \right) ds - \rho g \int_{S^E{}_I(\tau)} \zeta_D \, ds$$

The integrals in (4.8) may be projected on the Z=0 using ds=dXdY $\left[1 + O(\delta^2)\right]$

$$F_{Z-B,5} = -\rho \frac{d}{dt} \int_{S_I^E} \left(\varphi_D + \zeta_D \frac{\partial \varphi_I}{\partial Z} \right)(X,Y,\zeta_I,t)\, ds - \rho g \int_{S_I^E} \zeta_D(X,Y,t)\, ds$$
$$= -\rho \int_{Z=\zeta_I} \left[\frac{\partial}{\partial t} + \frac{\partial \zeta_I}{\partial t} \frac{\partial}{\partial Z} \right] \left(\varphi_D + \zeta_D \frac{\partial \varphi_I}{\partial Z} \right) ds - \rho g \int_{Z=\zeta} \zeta_D(X,Y,t)\, ds$$
$$= -\rho \int_{Z=\zeta_I} \left[\frac{\partial \varphi_D}{\partial t} + g\zeta_D + \frac{\partial \zeta_I}{\partial t} \frac{\partial \varphi_D}{\partial Z} + \zeta_D \frac{\partial^2 \varphi_I}{\partial Z \partial t} + \zeta_D \frac{\partial \zeta_I}{\partial t} \frac{\partial^2 \varphi_D}{\partial Z^2} \right] ds = O(\varepsilon\delta)$$

(4.7)

The terms in the last equation of (4.6) are seen to be of $O(\varepsilon\delta)$ by virtue of the second equation in (4.1) and the order of magnitudes of the body wave disturbance and the ambient wave gradients.

By comparison, the nonlinear buoyancy force (3.5) for a surface piercing body is of the order of the wave amplitude hence of $O(1)$ and the same applies for the vertical component of nonlinear Froude-Krylov force (3.7) The body impulse force (3.9) is of the order of the disturbance potential hence of $O(\varepsilon)$. Consequently the vertical component of the free surface potential being of order $\varepsilon\delta$ may be neglected. Therefore the contribution of the vertical impulse and corresponding force may be neglected in most applications.

Horizontal Free Surface Impulse

Invoking (4.5) in (3.12) the free surface impulse and force in the X-direction, assumed to coincide with the direction of propagation of the ambient wave, takes the form

(4.8)

$$\begin{aligned} F_{Z-B,5} &= \rho \frac{d}{dt} \int_{S^E{}_I(t)} \left[\varphi_D \frac{\partial \zeta_I}{\partial X} - \zeta_D \frac{\partial}{\partial X}\left(\varphi_I + \varphi_D\right) \right] ds \\ &\quad - \tfrac{1}{2} \rho \frac{d}{dt} \int_{S^E{}_I(t)} \zeta^2{}_D \frac{\partial}{\partial Z} \frac{\partial}{\partial X} (\varphi_I + \varphi_D)\, ds + O(\delta^2) \\ &= \rho \frac{d}{dt} \int_{S^E{}_I(t)} \left[\varphi_D \frac{\partial \zeta_I}{\partial X} - \zeta_D \frac{\partial}{\partial X}\left(\varphi_I + \varphi_D\right) \right] ds + O(\delta^2, \delta\varepsilon^2) \end{aligned}$$

The first two terms term in the right hand side of (4.10) are seen to be of $O(\varepsilon\delta)$ and the last term of $O(\varepsilon^2)$. The nonlinear hydrostatic force in the horizontal direction is in the present formulation identically zero. The horizontal component of the nonlinear Froude-Krylov force (3.7) is of $O(\delta)$ and the force due to the body impulse given by (3.9) is of $O(\varepsilon)$. This order of magnitude analysis suggests that the contribution of may be neglected. The physics of the wave force represented by the horizontal free surface impulse (4.8) is further discussed in the next section.

5. Discussion and Applications

Expressions have been derived for the ocean wave impulse and the nonlinear loads on floating bodies find application in wide range of problems of in naval hydrodynamics and ocean engineering. They are discussed next

"Impulse Dispersion Relation" for Nonlinear Irregular Ocean Waves

Expression (2.13) states an integral dynamic equilibrium between the inertia and restoring effects of an irregular nonlinear wave train. Its derivation used both the nonlinear dynamic free surface condition of zero pressure and the kinematic free surface condition that the normal flow velocity equals the normal velocity of the free surface. It may therefore be regarded as a nonlinear impulse dispersion relation that constrains the irregular wave profiles that are admissible representations of ocean waves. Assuming a space and time evolution of a large amplitude free surface elevation a potential flow problem follows in the fluid domain the solution of which leads to a velocity potential on the presumed free surface which must satisfy (3.13). Therefore not all presumed ambient wave free surface elevations are admissible under the nonlinear free surface conditions. Equation (3.13) is an integral physical constrain and may be viewed as a generalized dispersion relation and by virtue of the fact that it represents an integral balance between inertia effects, the time derivative of the impulse, and restoring effects, the integral of the free surface elevation.

Nonlinear Hydrostatics and Impulse Froude-Krylov Force

A principal result of the present paper is the derivation of the coupled nonlinear buoyancy and Froude-Krylov force and moment in nonlinear irregular ambient waves given by expressions (3.5)-(3.8). It was shown that by introducing the ambient wave free surface elevation as a dynamic waterplane area for surface piercing bodies, the nonlinear buoyancy force always points upwards. The direct use of Bernoulli's equation which is often used in applications keeps the hydrostatic effects strongly coupled with the dynamic effects. Moreover the expressions of the Froude-Krylov component have the simple form of the time derivative of integrals of just the ambient wave velocity potential over the body wetted surface and the dynamic waterplane area. Gradients of the ambient wave velocity potential are absent and the force takes the form of the time derivatives of an impulse which may also be expressed as an integral of the ambient wave velocity inside the body.

The computation of the Froude-Krylov force thus reduces to a simple task if the ambient wave velocity potential is known. Moreover, in a number of seakeeping problems for ships and other surface piercing offshore structures the vertical component of the Froude-Krylov force and moment are known to dominate the force components due to the radiation and diffraction components. Therefore the accurate computation of the nonlinear Froude-Krylov force is essential in wave body interactions.

Nonlinear Seakeeping of Ships

The expressions derived in the present article for the wave loads on general floating bodies may be readily applied to ship like geometries. Slender body approximations may be invoked to simplify three dimensional integrals of the ambient wave and body velocity potentials over sections of the ship hull along its length. The section wetted surface remains time dependent and all dynamic forces follow as time derivatives of sectional impulses relative to the reference frame. By virtue of its simplicity the nonlinear buoyancy and Froude-Krylov force may be evaluated in their exact three-dimensional form in order to account as accurately as possible for three-dimensional effects near the ship ends..

Generalized Morison's Equation

The combination of the Froude-Krylov force and the body impulse force given by expression (3.9) applied to cylindrical members encountered in offshore engineering offer a generalization of the inertia terms in Morison's equation. Viscous effects may be added as in the classical Morison formulation. The present formulation allows for the complete account of nonlinear hydrostatic and Froude-Krylov effects in both the vertical and horizontal forces and moments and for a time dependent body wetted surface in steep irregular waves. The integral of the body disturbance potential in (3.9) over slender cylindrical members with diameters small compared to the ambient wavelength may be approximated in terms of the sectional added-mass coefficient of each member providing the fluid inertia force completing the formulation of a generalized Morison equation.

This generalization of Morison's equation leads to explicit expressions for the nonlinear forces on a wide range of geometries encountered in connection with offshore platforms and offshore wind turbines. The consistent, complete and explicit account of all horizontal and vertical forces acting on slender cylindrical members and pontoons of such structures is particularly useful in practice..

Horizontal Drift Forces

The horizontal free surface impulse contributes all the horizontal drift force acting on a floating body. This follows from the observation that the Froude-Krylov and body disturbance forces have been expressed as time derivatives of the corresponding impulses which are stationary time dependent quantities in irregular waves. The time average of the time derivative of any stationary signal of sufficiently long duration vanishes. This follows easily by an integration by parts. Therefore the only remaining force component that may account for the horizontal drift force is the horizontal impulse given in its exact form by (3.12). If the ambient wave interaction starts from rest the free surface area over which the integrations in (3.12) are carried out grow in time and this must be accounted for properly when computing the mean value of the force signal. Otherwise it follows from the analysis of Section 3 that the free surface impulse and force are quadratic functions of the ambient wave and body disturbances and therefore their time average is nonzero.

Ringing Loads on Offshore Structures and Wind Turbines

Offshore structures and wind turbines fixed on the ocean floor or constrained by vertical tethers as in the case of Tension leg Platforms have resonant flexural frequencies of the order of a few seconds which may be excited by ambient waves. This subject has been studied extensively by the oil industry and "ringing" events have been measured when a steep wave impinges upon the offshore structure. "Ringing" may also occur in offshore wind turbines and is a topic of particular interest since the lowest natural frequency of the tower is around 3 seconds and higher tower natural frequencies overlap with the natural frequencies of the blades. Excessive ringing may therefore lead to fatigue and structural failure.

The flexural natural frequencies of offshore structures and wind turbines fall well to right of the modal frequency of design seastates which is 12-15 seconds. Therefore the excitation of ringing events must arise primarily from nonlinearities in the wave loads. Such nonlinearities originate from the ambient wave kinematics the nonlinear form of the buoyancy and Froude-Krylov force, the body impulse force and viscous forces that may be modeled in a Morison like manner. The quadratic wave force and moment due to the horizontal free surface impulse, approximated in (4.8) for ambient waves of large amplitude and small steepness contributes a nonlinear wave load which must be compared to the other dominant nonlinear loads at the high flexural frequencies of the structure. The power spectral density of these nonlinear force components needs to be determined in design seastates. The force (4.8) may be evaluated numerically or estimated analytically and its power spectral density at the natural frequencies compared with that of the other force components. It should be pointed out that the accuracy in the estimation of the force spectral density at high frequencies must be balanced with an equally accurate estimate of the damping of the structure at resonance which is often of viscous origin and hard to estimate.

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