

Nonlinear oscillations of gas bubbles in liquids: steady-state solutions

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The nonlinear oscillations of a spherical gas bubble in an incompressible, viscous liquid subject to the action of a sound field are investigated by means of an asymptotic method. Approximate analytical solutions for the steady-state oscillations are presented for the fundamental mode, for the first and second subharmonics, and for the first and second harmonics to second order in the expansion. These results are compared with some numerical ones and a very good agreement is found.

Subject Classification: 35.10; 25.60.

INTRODUCTION

Because of the highly nonlinear nature of the governing equations, the oscillations of a gas bubble in a liquid present a difficult mathematical problem. Many of the existing studies are based either on linearized analyses¹⁻⁵ (for a recent review see Ref. 5) or on numerical computations.⁶⁻¹² A first analytical attempt at the full nonlinear problem was made in 1956 by G uth who, neglecting damping and surface tension, gave approximate results for the resonant and subharmonic oscillations.¹³ More recently, Eller and Flynn¹⁴ obtained an expression for the subharmonic threshold in an investigation motivated by the experimentally observed subharmonic signal from an acoustically cavitating liquid.¹⁵⁻¹⁷

In the present study, the steady-state, nonlinear oscillations are analyzed by means of an asymptotic expansion. The bubble is assumed to remain spherical and to be immersed in an incompressible, viscous liquid subject to steady sinusoidal ambient pressure oscillations. The effects of surface tension and of viscous damping are included; thermal and acoustic damping can be accounted for by the suitable redefinition of the viscosity parameter. Analytical, second-order results for the steady-state oscillations are presented for the fundamental mode, as well as for the first and second subharmonics and for the first and second harmonics. The response curves are compared with others obtained by numerical integration of the Rayleigh equation¹⁸ and a very good agreement is found.

I. STATEMENT OF THE PROBLEM

We study the nonlinear oscillations of a gas bubble assuming that the motion is spherically symmetric and that the effects of compressibility may be neglected. Under these hypotheses, Rayleigh's equation of motion for the bubble wall can be written as¹⁸

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 = \frac{1}{\rho} \left[p_i - p(t) - \frac{2\sigma}{R} - \frac{4\mu}{R} \frac{dR}{dt} \right], \quad (1)$$

where R is the instantaneous radius of the bubble, p_i is its internal pressure, and $p(t)$, the ambient pressure, oscillates with angular frequency Ω about its average value p_∞ :

$$p(t) = p_\infty (1 - \eta \cos \Omega t) .$$

This expression is a close approximation to the pres-

sure distribution in a liquid subject to a sound field the wavelength of which is large compared with the bubble radius. The nature of the liquid is described by its density ρ , surface tension σ , and viscosity μ . It will be assumed that the amount of vapor present in the bubble is negligible, and that p_i can be written in the form

$$p_i = p_0 (R_0 / R)^{3\gamma} , \quad (2)$$

where γ is a polytropic exponent and p_0 the internal pressure corresponding to the equilibrium radius R_0 .

It should be noted that a relation like Eq. 2 is acceptable only insofar as thermal damping can be neglected, since this effect produces a phase difference between pressure and volume variations.^{3,4} As can be deduced from Fig. 1 of Ref. 4, in water this will be the case for bubbles smaller than about 10^{-4} cm. Even if Eq. 2 is not strictly applicable to bubbles of larger radii, the results obtained for these will nevertheless be approximately valid if referred to the average, steady-state behavior of the bubbles, rather than to the instantaneous one.³ For such relatively large bubbles, however, thermal dissipation should be taken into account. To the same approximation of the computations presented below, this can be easily obtained by letting the parameter μ in Eq. 1 be the sum of the viscosity of the liquid plus a "thermal viscosity" defined according to the expressions given by Devin² or Chapman and Plesset⁴ for the thermal logarithmic decrement. As an example we give here the expression that can be deduced from Eq. 27 of the latter authors:

$$\mu_{\text{thermal}} = \frac{1}{4} \rho \Omega R_0^2 \frac{\text{Im}(G)}{\text{Re}(G) - 2\sigma/R_0} ,$$

where G is a function defined in Eq. 24 of Ref. 4. In a similar way, the acoustic damping can be introduced through an effective "acoustic viscosity" given by^{2,4}:

$$\mu_{\text{acoustic}} = \frac{1}{4} \frac{\rho \Omega^2 R_0^3}{c} ,$$

where c is the velocity of sound in the liquid.

The equilibrium radius of the bubble is determined by $p_0 - p_\infty = 2\sigma/R_0$. The physical mechanisms that may alter the value of R_0 , such as rectified diffusion,^{19,20} will be disregarded. From Table I of Ref. 19, it can be seen that the time scale for this process is by many orders of magnitude larger than the period of oscillation of a bubble. The only effect of rectified diffusion will

therefore be an exceedingly slow parametric variation of the value of R_0 in the equations presented below. This conclusion is also substantiated by the finds of Ref. 20, where it is shown that the growth by rectified diffusion can essentially be decoupled from the oscillation of the bubble.

II. PRELIMINARY CONSIDERATIONS

Since we are interested in the oscillations of the bubble about its equilibrium radius R_0 , in Eq. 1 we let

$$R = R_0(1 + x) , \tag{3}$$

and perform a power series expansion in x . If the following change in the time scale is made:

$$\tau = (\rho_0 / \rho)^{1/2} t R_0^{-1} ,$$

and terms of fourth and higher order are neglected,²¹ we obtain

$$\ddot{x} + \omega_0^2 x = \xi \cos \omega \tau + \left[-\frac{3}{2} \dot{x}^2 + \alpha_1 x^2 - \xi x \cos \omega \tau - 2b\dot{x} \right] + \left[\frac{3}{2} \dot{x}^2 x - \alpha_2 x^3 + \xi x^2 \cos \omega \tau + 4b\dot{x}x \right] , \tag{4}$$

where dots denote differentiation with respect to the dimensionless time τ and the following definitions have been used:

$$\begin{aligned} \alpha_1 &= \frac{3}{2} \gamma (\gamma + 1) - 2w , \\ \alpha_2 &= \frac{1}{2} \gamma (9\gamma^2 + 18\gamma + 11) - 3w , \\ w &= 2\sigma / R_0 \rho_0 , \\ b &= 2\mu / R_0 (\rho \rho_0)^{1/2} , \\ \xi &= (1 - w) \eta , \\ \omega &= R_0 \Omega (\rho / \rho_0)^{1/2} , \\ \omega_0^2 &= 3\gamma - w . \end{aligned}$$

When the expression for ω_0 , the resonant frequency of the bubble, is converted back into dimensional form, it reads

$$\Omega_0^2 = (3\gamma \rho_0 - 2\sigma R_0^{-1}) / \rho R_0^2 ,$$

a well-known result.^{1,4} It should also be observed that in Eq. 4 the combination $\xi = (1 - w)\eta$ enters, rather than η alone; ξ appears therefore as an "effective pressure amplitude" for the oscillations.

In the following, the approximate equation, Eq. 4, is studied. Only the results are presented here; for more details and the computational aspects, see Ref. 22 and Sec. VII below.

To third order, Eq. 4 possesses two harmonic and two subharmonic resonances, respectively, for $\omega \approx \omega_0 / 2$, $\omega \approx \omega_0 / 3$ and $\omega \approx 2\omega_0$, $\omega \approx 3\omega_0$. The solution $x(\tau)$ is found to have a similar form in all four of these frequency regions; namely,

$$\begin{aligned} x(\tau) &= C \cos \theta + \xi [(\omega^2 - \omega_0^2) + 4b^2 \omega^2]^{-1/2} \cos(\omega \tau + \delta) \\ &+ (c_1 + \lambda_1 c_3 \cos 2\omega \tau) \xi^2 + (c_5 + c_2 \cos 2\theta) C^2 \\ &+ [c_4 \cos(\omega \tau + \theta) + \lambda_2 c_0 \cos(\omega \tau - \theta)] \xi C . \end{aligned} \tag{5}$$

In this equation,

$$\delta = \tan^{-1} [2b\omega / (\omega^2 - \omega_0^2)] ,$$

$$\theta = n\omega\tau + \varphi ,$$

and the c_i 's, $0 \leq i \leq 5$, are functions of ω given by Eqs. A1-A7 of the Appendix A. Expressions for the quantities C and φ , the amplitude and phase of the resonant component, are given in the following sections for each resonance region. The other symbols, λ_1 , λ_2 , n , are explained in Table I. The solution in the intermediate regions (i.e., far from resonances) is given by Eq. 5 with $C = 0$.

III. SUBHARMONICS

In the first subharmonic region $\omega \approx 2\omega_0$, the amplitude C can have one of the following values:

$$\begin{aligned} C(\omega) &= 0 , \\ C(\omega) &= \left[\frac{\omega_0^2 - (1/4)\omega^2 + \xi^2 g_1(\omega) \pm \Delta(\omega)}{g_0(\omega)} \right]^{1/2} , \end{aligned} \tag{6}$$

where

$$\Delta(\omega) = \{ \xi^2 \beta_1^2(\omega) - \omega^2 b^2 \}^{1/2} ,$$

and the quantities β_1 , g_0 , g_1 are functions of ω given in Eqs. A8, A10, A11 of the Appendix A. The phase φ is undetermined for $C = 0$, while in the other two cases its value is given by

$$\begin{aligned} \sin 2\varphi &= b \frac{\omega \beta_1 \pm g_1 \Delta}{\xi \beta_1^2} , \\ \cos 2\varphi &= \frac{\pm \beta_1 \Delta - b^2 \omega g_1}{\xi \beta_1^2} . \end{aligned}$$

The function $C(\omega)$ is shown in Fig. 1 for a representative case. A stability analysis shows that, of the two branches of Eq. 6, the one corresponding to the lower sign is unstable. The value $C = 0$ is stable everywhere except in the frequency interval comprised between ω_+ , the starting points of the two branches and determined by the equations²²

$$\omega_0^2 - (1/4)\omega_+^2 + \xi^2 g_1(\omega_+) \pm \Delta(\omega_+) = 0 . \tag{7}$$

Therefore, there exists an interval of frequencies where the solution $x(\tau)$ may or may not exhibit a subharmonic component, depending on the initial conditions of the oscillations (Figs. 1 and 2). If ω_+ is real, this interval is located at the left of ω_+ on the frequency axis (Fig. 1).

The requirement that the quantity $\Delta(\omega)$ appearing in the above expressions be real, determines the (frequency-dependent) threshold ξ_c for the subharmonic excitation as

TABLE I. Values of the constants appearing in Eq. 5 in the different frequency regions.

	n	λ_1	λ_2
First subharmonic region	1/2	1	0
Second subharmonic region	1/3	1	1
First harmonic region	2	0	1
Second harmonic region	3	1	1
Intermediate regions	—	1	—

$$\xi_t = \frac{\omega b}{\beta_1(\omega)} \quad (8)$$

If surface tension is unimportant, this equation is similar (although in a somewhat more manageable mathematical form) to a result obtained by Eller and Flynn.¹⁴ There is, however, an important difference. Eller and Flynn's expression was derived by an investigation of the onset of instability for a solution of the Rayleigh equation similar to Eq. 5 with $C=0$. Because of its indirect nature, this technique cannot handle correctly a case like that shown in Fig. 2, where the $C=0$ solution is stable in the entire frequency range, but a subharmonic component may nevertheless set in if appropriate initial conditions are chosen for the motion.

If in Eq. 8 we set $\omega = 2\omega_0$, we get, in dimensional variables,

$$\xi_t = (1-w) \left(\frac{\rho_{max}}{\rho_{av}} \right)_t = \frac{(3\gamma-w)^{3/2}}{9\gamma^2-w} \frac{24\mu}{R_0(\rho p_0)^{1/2}}.$$

The linear increase with viscosity predicted by this relation appears to have been experimentally verified.¹⁷

For the second subharmonic resonance $\omega \approx 3\omega_0$, one gets similar results. Again, $C=0$ is a possibility for the amplitude, which can also, however, have nonvanishing values, the (positive) solutions of

$$g_0^2 C^4 + \{2g_0[\frac{1}{9}\omega^2 - \omega_0^2 - \xi^2(g_1 - g_2)] - \xi^2 g_6^2\} C^2 + \frac{4}{9}\omega^2 b^2 + [\frac{1}{9}\omega^2 - \omega_0^2 - \xi^2(g_1 - g_2)]^2 = 0,$$

where g_2, g_6 are functions of ω given by Eqs. A12 and A16 of the Appendix A. The condition that the discriminant of this quadratic equation be positive, determines a threshold condition analogous to Eq. 8. Of the two branches of the curve $C(\omega)$, the lower one turns out to

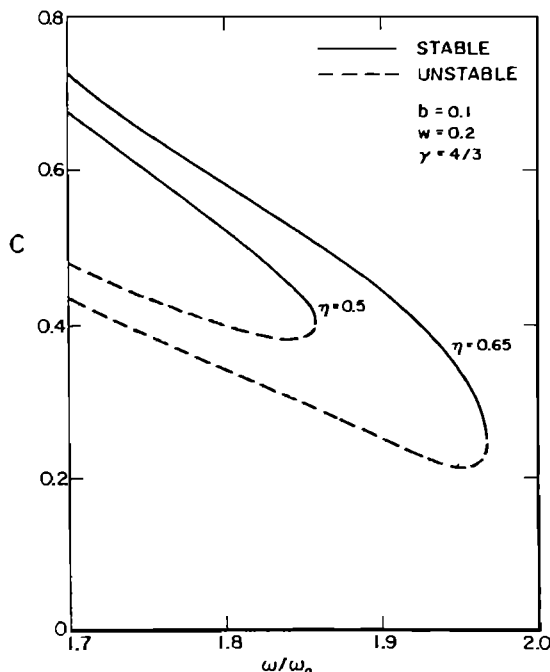


FIG. 2. Amplitude of the steady-state subharmonic component as a function of the ratio between the impressed frequency ω and the natural frequency of the bubble ω_0 .

be unstable, just as in the preceding case. The phase φ is determined by

$$\sin 3\varphi = -\frac{2\omega b}{3g_6 \xi C},$$

$$\cos 3\varphi = -\frac{(1/9)\omega^2 - \omega_0^2 - \xi^2(g_1 - g_2) + g_0 C^2}{g_6 \xi C}.$$

There appears to be experimental evidence for the occurrence of this mode.¹⁵

IV. HARMONICS

For the first harmonic resonance $\omega \approx (1/2)\omega_0$, the amplitude C and phase φ are determined by

$$g_0^2 C^6 + 2g_0 Q_2 C^4 + (Q_2^2 + 16b^2 \omega^2) C^2 - \xi^4 \beta_2^2 = 0,$$

$$\sin \varphi = \frac{bC}{\xi^2} \frac{g_3(Q_2 + g_0 C^2) - 4\omega \beta_2}{\beta_2^2}, \quad (9)$$

$$\cos \varphi = -\frac{C}{\xi^2} \frac{\beta_2(Q_2 + g_0 C^2)}{\beta_2^2},$$

where

$$Q_2 = 4\omega^2 - \omega_0^2 - \xi^2(g_1 - g_2),$$

and β_2, g_3 are functions of ω given by Eqs. A9 and A13 of the Appendix A.

Because of the characteristic difference in the coupling with the subharmonic case, Eq. 9 is cubic, rather than quadratic, in C^2 . This has the consequence that $C=0$ is no longer a possibility, and that (as is physically clear) there is no threshold for the harmonic oscillations.

It is easily verified that Eq. 9 has either one or—for high enough ξ or low enough b —three real, positive solutions. In the latter case, the peak is bent over to the

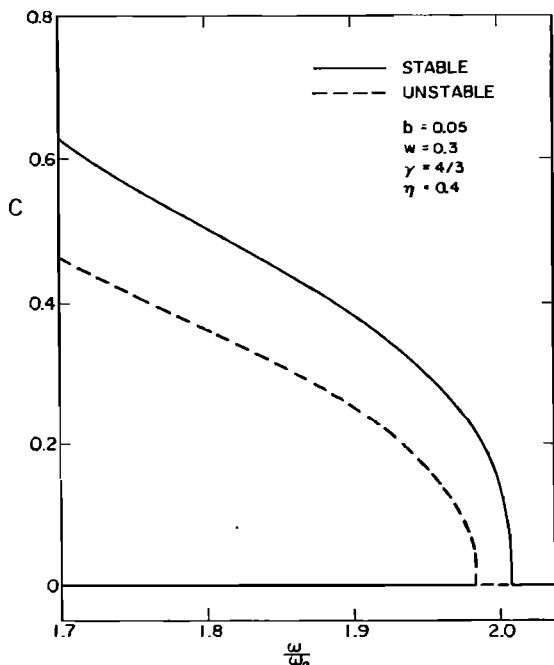


FIG. 1. Amplitude of the steady-state subharmonic component as a function of the ratio between the impressed frequency ω and the natural frequency of the bubble ω_0 .

left, with an unstable section comprised between the points of vertical tangency.²² The situation is qualitatively similar to the one depicted in Fig. 3. In the frequency interval where the function $C(\omega)$ is two-valued, the initial conditions determine the steady-state value of the amplitude.

In the second harmonic region $\omega \approx (1/3)\omega_0$ we have

$$g_0^2 C^6 + 2Q_3 g_0 C^4 + (Q_3^2 + 36\omega^2 b^2) C^2 - g_5^2 \xi^6 = 0, \quad (10)$$

$$\sin\varphi = -\frac{6wbC}{g_5 \xi^3},$$

$$\cos\varphi = -C \frac{Q_3 + g_0 C^2}{g_5 \xi^3},$$

with

$$Q_3 = 9\omega^2 - \omega_0^2 - \xi^2(g_1 - g_2)$$

and the function $g_5(\omega)$ given by Eq. A15 of the Appendix A. The possibilities for the solutions of Eq. 10 are as for the first harmonic. Two typical examples of the curves $C(\omega)$ obtained in this case are shown in Fig. 3.

As ω gets far from the resonance region, the values of C determined by Eqs. 9 and 10 get exceedingly small, thus justifying the last remark at the end of Sec. II on the solution in the intermediate (nonresonant) regions. In the case of the subharmonics, C equals 0 exactly far from resonance.

V. RESONANCE

Near the main resonance at $\omega \approx \omega_0$, the approximate solution of Eq. 4 is, in place of Eq. 5,

$$\begin{aligned} x(\tau) = & C \cos(\omega\tau + \varphi) + \frac{1}{2} [(4\omega^2 - \omega_0^2)^{-1} \\ & \times \cos(2\omega\tau - \varphi) - \omega_0^{-2} \cos\varphi] \xi C \\ & + \frac{1}{2} \{ [\alpha_1 - \frac{3}{2}\omega^2] \omega_0^{-2} - [\alpha_1 + \frac{3}{2}\omega^2] (4\omega^2 - \omega_0^2)^{-1} \\ & \times \cos(2\omega\tau + 2\varphi) \} C^2. \end{aligned}$$

The amplitude C and phase φ are solutions of the following system:

$$2wbC + \xi(1 - d_3 C^2) \sin\varphi + \frac{1}{4} \xi^2 C \omega_0^{-2} \sin 2\varphi = 0,$$

$$C[\omega^2 - \omega_0^2 + \frac{1}{2} \xi^2 (2\omega^2 - \omega_0^2) \omega_0^{-2} (4\omega^2 - \omega_0^2)^{-1}]$$

$$- d_1 C^3 + \xi(1 - d_2 C^2) \cos\varphi + \frac{1}{4} \xi^2 C \omega_0^{-2} \cos 2\varphi = 0,$$

and can be obtained with the aid of a digital computer. The quantities d_i 's, $1 \leq i \leq 3$, are functions of ω given by Eqs. A17–A19 of the Appendix A.

From the qualitative point of view, the situation here is identical with the one described in the previous section and shown in Fig. 3. Obviously, the quantitative difference lies mainly in the height of the response.

VI. COMPARISON WITH NUMERICAL RESULTS

Lauterborn has performed an extensive series of computations of the response–frequency relation of an oscillating bubble, integrating numerically an equation practically equal to Eq. 1 above. Figures 4–9 present a

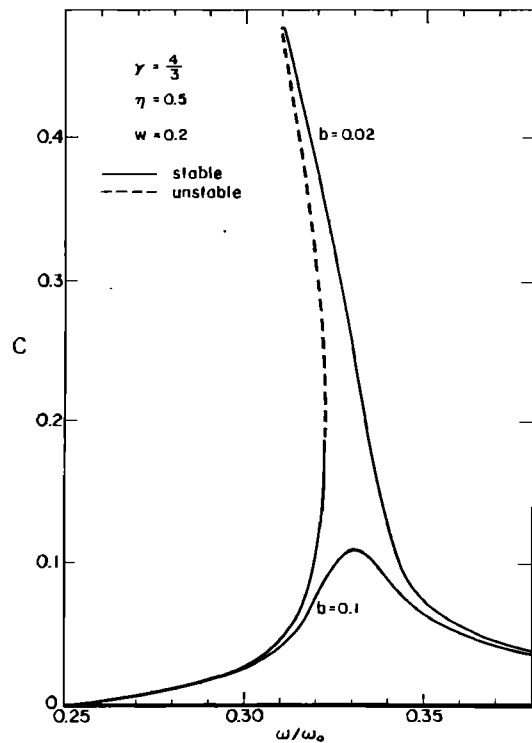


FIG. 3. Amplitude of the steady-state, second harmonic component as a function of the ratio between the impressed frequency and the natural frequency of the bubble. The parameter b , labeling the curves, is the damping parameter.

comparison of Lauterborn's results¹² with the analytical ones in a few cases. The figures are plots of $x_m = (R_{\max} - R_0)/R_0$ versus ω/ω_0 , where R_{\max} is the maximum value of the radius during a steady-state oscillation at frequency ω . The cases considered are the following:

$$\text{case 1: } b=0.128, \quad w=0.592;$$

$$\text{case 2: } b=0.019, \quad w=0.127;$$

for several values of the pressure amplitude η . If the liquid is water at 20 °C, the two cases would correspond to a bubble of radius 10^{-4} cm and 10^{-3} cm, respectively; for both the polytropic exponent is $\gamma=1.33$, which is the value used by Lauterborn.

Figures 4 and 5 refer to case 1 for $\eta=0.5$. Because of the large value of w , however, the effective pressure amplitude ξ is much lower, $\xi=0.204$. The agreement is seen to be extremely good, even near resonance where x_m is large (Fig. 4). In the following figure, corresponding to a higher value of ξ (Fig. 6, case 2, $\eta=0.3$, $\xi=0.262$), the agreement is not as good, but still very satisfactory. Figure 7 refers to case 1 for a large value of the effective pressure amplitude, $\xi=0.367$. The progressive worsening of the agreement between numerical and analytical results is evident, but nevertheless the fact that the latter still are not completely off at such a high value of the perturbation parameter should be noted.

Figures 8 and 9 present a detailed comparison in the region of the first subharmonic. Figure 9 (case 2, $\eta=0.7$, $\xi=0.611$) is included to show the accuracy of Eq. 7 for the interval of instability of the purely har-

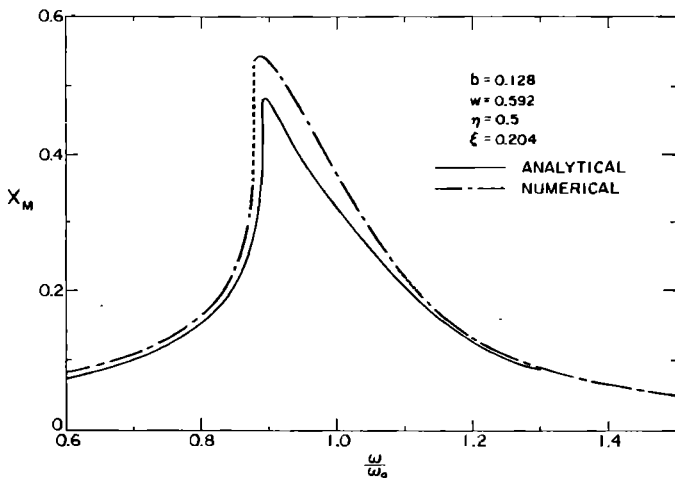


FIG. 4. Comparison between the numerical and analytical results in the frequency region of the main resonance, $\omega \approx \omega_0$ (case 1).

monic $C=0$ solution even for such a large value of ξ .

In Figures 4, 6, and 7 the dotted vertical segments on the curves of Lauterborn's results reflect the discontinuities that appear in the numerical solution as it passes from the left (lower) branch to the right (higher) one at the points at which the former becomes unstable; these are the points at which the tangent to the curve $x_m(\omega/\omega_0)$ becomes vertical (cf. Fig. 3 and Sec. IV). Both the stable and the unstable branches are indicated for the analytical curves. The stable ones continue above the corresponding numerical results. The reason for this is that Lauterborn has apparently integrated in the direction of increasing values of ω/ω_0 , so that he was unable to compute the response curve in the region in which it has two stable branches. For the same reason his results do not show the typical hysteresis of nonlinear oscillations.²³ In the case of the subharmonic regions, the point of vertical tangency is not recognizable as such for the numerical curve, but is very evident in the analytical one (Figs. 8 and 9).

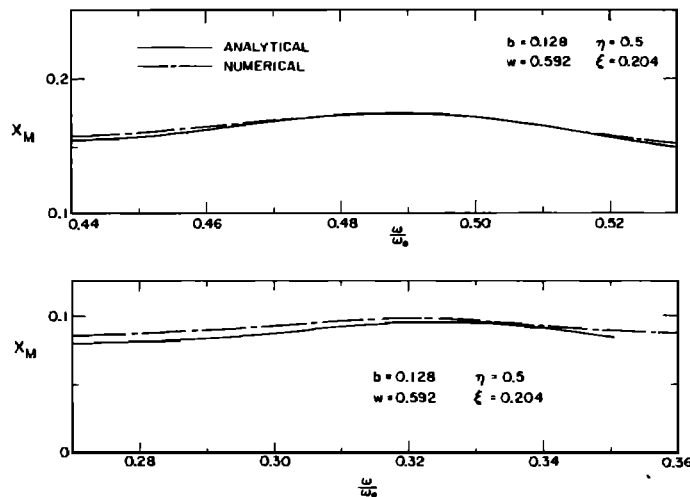


FIG. 5. Comparison between the numerical and analytical results in the frequency regions of the first (upper) and second (lower) harmonics, $\omega \approx (1/2) \omega_0$ and $\omega \approx (1/3) \omega_0$, respectively (case 1).

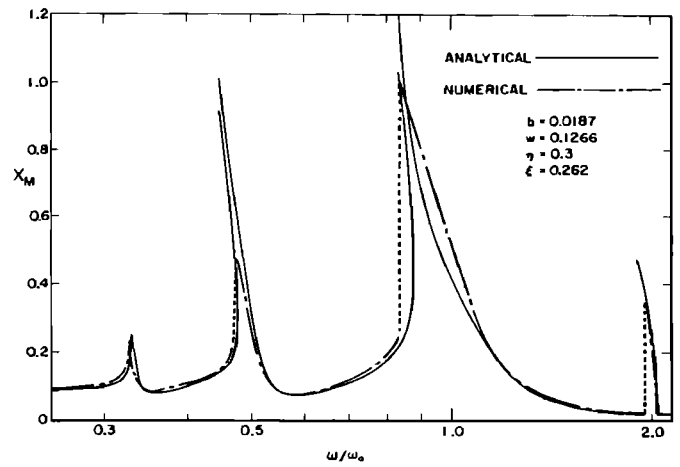


FIG. 6. Comparison between the numerical and analytical results for a moderately high value of the perturbation parameter (case 2).

As a test of the threshold equation, Eq. 8, we mention that, for case 1, Lauterborn obtains $1.5 < \eta_t < 1.6$ and, for case 2, $\eta_t \approx 0.12$. The analytical results are $\eta_t = 1.54$ and $\eta_t = 0.123$, respectively.²⁴

Finally, Fig. 10 presents two examples of radius-versus-time curves for subharmonic oscillations (case 2, $\xi = 0.262$, $\omega/\omega_0 = 1.955$). Both curves have been computed from Eq. 5, curve *a*, with $C(\omega)$ given by Eq. 6, curve *b*, with $C=0$.

VII. NOTE ON THE ASYMPTOTIC METHOD

The steady-state solutions presented in this study have been derived from the transient solutions computed with the aid of the Bogolyubov-Krylov asymptotic method.²⁵ This method is very well known, so that a very brief outline will be sufficient here.

Consider a weakly nonlinear differential equation of the form

$$\ddot{X} + 2\epsilon B \dot{X} + \omega_0^2 X = P \cos \omega t + \epsilon h_1(X, \dot{X}; \omega t) + \epsilon^2 h_2(X, \dot{X}; \omega t) + \dots$$

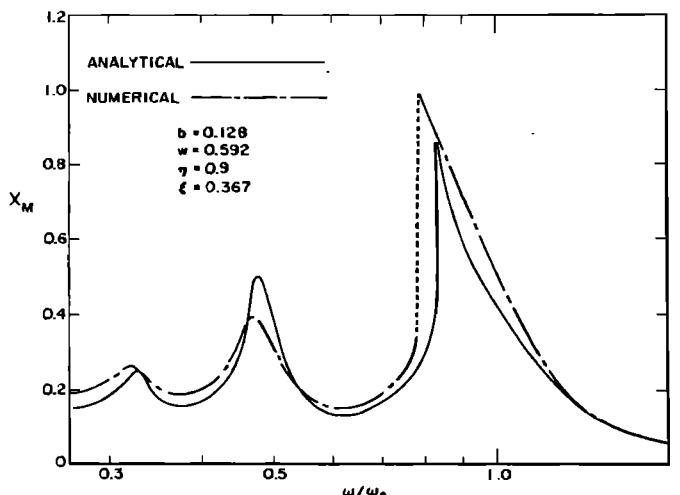


FIG. 7. Comparison between the numerical and analytical results for a high value of the perturbation parameter (case 1).

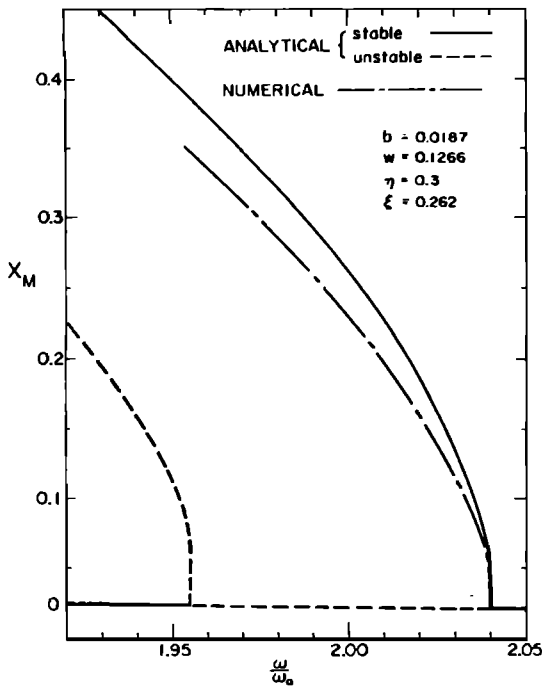


FIG. 8. Comparison between the numerical and analytical results in the subharmonic region. This figure is an enlargement of the subharmonic peak of Fig. 6 (case 2).

where $h_i(X, \dot{X}; \omega t)$ are functions consisting of linear combinations of terms of the form $X^n \dot{X}^m$, with coefficients periodic of period 2π in the variable ωt . We suppose for the moment that the driving frequency ω is far from the resonant frequency ω_0 , but that there is a rational number n such that $n\omega \sim \omega_0$. We seek a solution of the differential equation in the form of the asymptotic expansion

$$X = P[(\omega_0^2 - \omega^2)^2 + 4\epsilon^2 B^2 \omega^2]^{-1/2} \cos(\omega t + \delta) + A(t) \cos \theta + \epsilon X_1 + \epsilon^2 X_2 + \dots,$$

with $\theta = n\omega t + \varphi(t)$, $\delta = \tan^{-1}[2\epsilon B\omega/(\omega^2 - \omega_0^2)]$. If this expression is substituted into the differential equation, an expansion in powers of ϵ carried out, and the coefficients of like powers equated, a hierarchy of equations is obtained:

$$-n\omega \dot{A} \sin \theta - n\omega A \dot{\varphi} \cos \theta + \frac{d}{dt} [\dot{A} \cos \theta - A \dot{\varphi} \sin \theta] = (\nu^2 \omega^2 - \omega_0^2) A \cos \theta,$$

$$\ddot{X}_1 + \omega_0^2 X_1 = F_1(A, \varphi; r_1 \omega t, s_1 \theta) + f_1(A, \varphi) \cos \theta + g_1(A, \varphi) \sin \theta,$$

$$\ddot{X}_2 + \omega_0^2 X_2 = F_2(A, \varphi; r_2 \omega t, s_2 \theta) + f_2(A, \varphi) \cos \theta + g_2(A, \varphi) \sin \theta,$$

and so on, where r_i, s_i are rational numbers different from n and 1, respectively. Note first that this system is underdetermined as far as A and φ are concerned, so that any other relation between them can be imposed. It is convenient to choose

$$\dot{A} \cos \theta - A \dot{\varphi} \sin \theta = 0,$$

so that the first equation remains of first order. Now, since by hypothesis $n\omega \sim \omega_0$, the terms proportional to $\cos \theta$ and $\sin \theta$ in the RHS of the equations for X_i will give

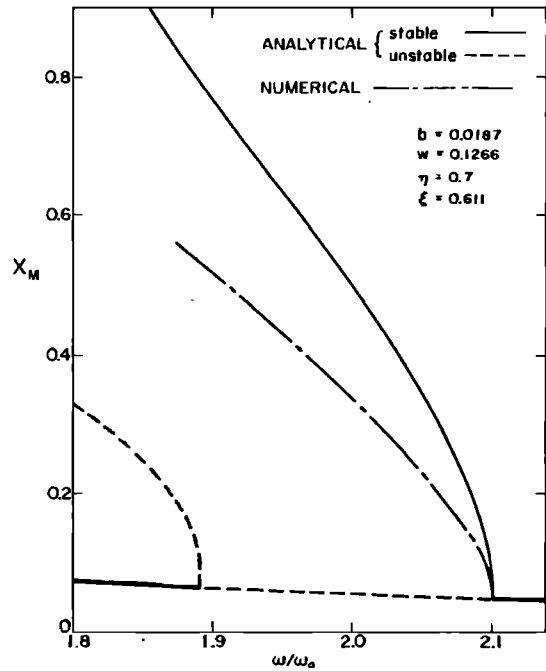


FIG. 9. Comparison between the numerical and analytical results in the subharmonic region for an extremely high value of the perturbation parameter (case 2).

rise to very large amplitudes, so that the validity of the asymptotic expansion breaks down. To avoid this difficulty, the functions $A(t)$ and $\varphi(t)$ must be defined in such a way that at any time the term $A(t) \cos \theta$ contains the entire component of frequency $n\omega$ present in X . To obtain this, all terms proportional to $\cos \theta$ and $\sin \theta$ must be subtracted from the RHS of the equations for X_i and put back into the first equation. One then obtains the following systems:

$$-n\omega \dot{A} \sin \theta - n\omega A \dot{\varphi} \cos \theta = (\nu^2 \omega^2 - \omega_0^2) A \cos \theta + (\epsilon f_1 + \epsilon^2 f_2 + \dots) \times \cos \theta + (\epsilon g_1 + \epsilon^2 g_2 + \dots) \sin \theta,$$

$$n\omega \dot{A} \cos \theta - n\omega A \dot{\varphi} \sin \theta = 0,$$

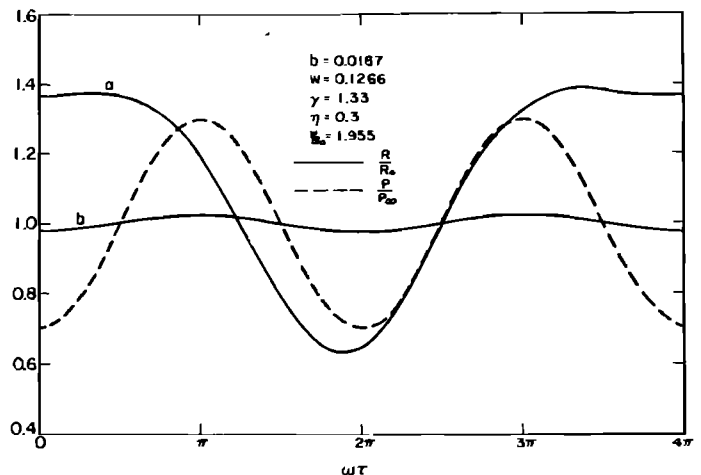


FIG. 10. Steady-state oscillations in the subharmonic region. Curve a, subharmonic oscillations; curve b, purely harmonic, $C=0$, oscillations. The pressure amplitude curve (---) is drawn to indicate the phase relationships.

$$\ddot{X}_1 + \omega_0^2 X_1 = F_1(A, \varphi; r_1 \omega t, s_1 \theta) ,$$

$$\ddot{X}_2 + \omega_0^2 X_2 = F_2(A, \varphi; X_1; r_1 \omega t, s_1 \theta) ,$$

and so on.

The first two equations are independent of the remaining ones and can be solved separately. In order to facilitate this, one makes use of the fact that A and φ are slowly varying functions of time, i.e., that $\dot{A} = o(1)$, $\dot{\varphi} = o(1)$, so that they can be considered constant over a time length $T = 2\pi/n\omega$. The two equations can then be combined and averaged over T to obtain

$$-2n\omega A \dot{\varphi} = (n^2 \omega^2 - \omega_0^2)A + \epsilon f_1(A, \varphi) + \epsilon^2 f_2(A, \varphi) + \dots ,$$

$$-2n\omega \dot{A} = \epsilon g_1(A, \varphi) + \epsilon^2 g_2(A, \varphi) + \dots .$$

The asymptotic values of A and φ are obtained by setting $\dot{A} = 0$, $\dot{\varphi} = 0$, and solving the remaining system of ordinary (not differential) equations. After this step, the equations for X_1, X_2, \dots can be solved and the asymptotic expansion of the steady-state solution determined. Equation 4 of the present study can be reduced to the form considered here by letting $x = \epsilon X$, $b = \epsilon B$, and $\xi = \epsilon P$, where ϵ is an artificial parameter selected in such a way that X is of order 1. The final results are, however, independent of the particular choice of the artificial parameter, as is obvious.

In the case $\omega \sim \omega_0$, the first term in the asymptotic expansion must be omitted, but otherwise the above procedure goes through unchanged. However, in this case \dot{A} and $\dot{\varphi}$ are approximately of order P , so that it is necessary to limit oneself to lower values of the excitation amplitude to get accurate results.

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APPENDIX A

The form of the various functions of ω which appear in the text are given below. The auxiliary quantity D is defined as:

$$D = (\omega_0^2 - \omega^2)^{-1} .$$

$$[\omega_0^2 - (n-1)^2 \omega^2] c_0(\omega) = D(\alpha_1 - \frac{3}{2} n \omega^2) - \frac{1}{2} , \tag{A1}$$

$$\omega_0^2 c_1(\omega) = \frac{1}{2} D [D(\alpha_1 - \frac{3}{2} \omega^2) - 1] , \tag{A2}$$

$$(\omega_0^2 - 4n^2 \omega^2) c_2(\omega) = \frac{1}{2} (\alpha_1 + \frac{3}{2} n^2 \omega^2) , \tag{A3}$$

$$(\omega_0^2 - 4\omega^2) c_3(\omega) = \frac{1}{2} D [D(\alpha_1 + \frac{3}{2} \omega^2) - 1] , \tag{A4}$$

$$[\omega_0^2 - (n+1)^2 \omega^2] c_4(\omega) = D(\alpha_1 + \frac{3}{2} n \omega^2) - \frac{1}{2} , \tag{A5}$$

$$\omega_0^2 c_5(\omega) = \frac{1}{2} (\alpha_1 - \frac{3}{2} n^2 \omega^2) , \tag{A6}$$

$$c_6(\omega) = 2\omega D^2 . \tag{A7}$$

$$\beta_1(\omega) = \frac{1}{2} - D(\alpha_1 - \frac{3}{2} n \omega^2) \tag{A8}$$

$$\beta_2(\omega) = \frac{1}{2} D [D(\alpha_1 + \frac{3}{2} \omega^2) - 1] . \tag{A9}$$

$$g_0(\omega) = \alpha_1 (2c_5 + c_2) + \frac{3}{2} n^2 \omega^2 (\frac{1}{4} - 2c_2) - \frac{3}{4} \alpha_2 , \tag{A10}$$

$$g_1(\omega) = -2\alpha_1 c_1 + c_4 \{ \frac{1}{2} - D[\alpha_1 - \frac{3}{2} (n+1) \omega^2] \} + \frac{3}{2} (\alpha_2 - \frac{1}{2} \omega^2) D^2 - D , \tag{A11}$$

$$g_2(\omega) = c_0 \{ D[\alpha_1 + \frac{3}{2} (n-1) \omega^2] - \frac{1}{2} \} , \tag{A12}$$

$$g_3(\omega) = c_6 [D(\alpha_1 + \frac{3}{2} \omega^2) - \frac{1}{2}] - 2D^2 \omega , \tag{A13}$$

$$g_4(\omega) = c_6 (\alpha_1 - \frac{3}{2} n \omega^2) + 2(n-1)\omega D , \tag{A14}$$

$$g_5(\omega) = c_3 [D(\alpha_1 + 3\omega^2) - \frac{1}{2}] + \frac{1}{4} [1 - D(\alpha_2 + \frac{3}{2} \omega^2)] D^2 , \tag{A15}$$

$$g_6(\omega) = c_0 [\alpha_1 + \frac{3}{2} n(n-1)\omega^2] + c_2 [D(\alpha_1 - 3n\omega^2) - \frac{1}{2}] + \frac{1}{4} + \frac{3}{4} [n(1 - \frac{1}{2}n)\omega^2 - \alpha_2] D . \tag{A16}$$

$$d_1(\omega) = \frac{3}{4} \alpha_2 - \alpha_1 [\omega_0^{-2} (\alpha_1 - \frac{3}{2} \omega^2) + \frac{1}{2} (\alpha_1 + \frac{3}{2} \omega^2) (\omega_0^2 - 4\omega^2)^{-1}] - \frac{3}{2} \omega^2 [\frac{1}{4} - (\alpha_1 + \frac{3}{2} \omega^2) (\omega_0^2 - 4\omega^2)^{-1}] , \tag{A17}$$

$$d_2(\omega) = \frac{3}{4} (\alpha_1 - \frac{3}{2} \omega^2) (\omega_0^2 - 4\omega^2)^{-1} + \frac{3}{2} \omega_0^{-2} (\alpha_1 - \frac{1}{2} \omega^2) - \frac{3}{4} , \tag{A18}$$

$$d_3(\omega) = \frac{1}{4} (\alpha_1 - \frac{15}{2} \omega^2) (\omega_0^2 - 4\omega^2)^{-1} + \frac{1}{2} \omega_0^{-2} (\alpha_1 - \frac{3}{2} \omega^2) - \frac{1}{4} . \tag{A19}$$

Finally, we give approximate expressions for the functions appearing in the subharmonic region. Letting $\omega = 2\omega_0$ (with $\omega_0^2 = 3\gamma - \omega$) in Eqs. A8, A10, and A11, one gets

$$\beta_1(2\omega_0) = \frac{1}{6} \omega_0^{-2} (9\gamma^2 - \omega) ,$$

$$g_0(2\omega_0) = \frac{1}{8} \omega_0^{-2} [\gamma^2 (54\gamma^2 - 27\gamma - 18)$$

$$+ w\gamma(27\gamma^2 - 21\gamma + 18) - \frac{7}{3} w^2] ,$$

$$g_1(2\omega_0) = \frac{1}{2} \omega_0^{-6} [\gamma^2 (\frac{9}{16} \gamma^2 + 9\gamma + 1) - \gamma w (\frac{3}{2} \gamma^2 + \frac{11}{8} \gamma + \frac{11}{8}) + \frac{85}{144} w^2] .$$

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