North Carolina Math 3

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Custom
Program Overview

WALCH®
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Introduction to the Program

Introduction

The *North Carolina Math 3 Custom Teacher Resource* is a complete set of materials developed around the North Carolina Standard Course of Study (NCSCOS) for Mathematics. Topics are built around accessible core curricula, ensuring that the *North Carolina Math 3 Custom Teacher Resource* is useful for striving students and diverse classrooms.

This program realizes the benefits of exploratory and investigative learning and employs a variety of instructional models to meet the learning needs of students with a range of abilities.

The *North Carolina Math 3 Custom Teacher Resource* includes components that support problem-based learning, instruct and coach as needed, provide practice, and assess students' skills. Instructional tools and strategies are embedded throughout.

The program includes:

- More than 150 hours of lessons
- Essential Questions for each instructional topic
- Vocabulary
- Instruction and Guided Practice
- Problem-based Tasks and Coaching questions
- Step-by-step graphing calculator instructions for the TI-Nspire and the TI-83/84
- Station activities to promote collaborative learning and problem-solving skills

Purpose of Materials

The *North Carolina Math 3 Custom Teacher Resource* has been organized to coordinate with the North Carolina Math 3 content map and specifications from the NCSCOS. Each lesson includes activities that offer opportunities for exploration and investigation. These activities incorporate concept and skill development and guided practice, then move on to the application of new skills and concepts in problem-solving situations. Throughout the lessons and activities, problems are contextualized to enhance rigor and relevance.

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This program includes all the topics addressed in the North Carolina Math 3 content map. These include:

- Functions and Their Inverses
- Exponential and Logarithmic Functions
- Polynomial Functions
- Modeling with Geometry
- Reasoning with Geometry with Circles
- Rational Functions
- Trigonometric Functions
- Statistics

The eight Standards for Mathematical Practice are infused throughout:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Structure of the Teacher Resource

The *North Carolina Math 3 Custom Teacher Resource* materials are completely reproducible. The Program Overview is the first section. This section helps you to navigate the materials, offers a collection of research-based Instructional Strategies along with their literacy connections and implementation suggestions, and shows the correlation between the NCSCOS for Mathematics and the district-specfic content map and course requirements.

The remaining materials focus on content, knowledge, and application of the eight units in the North Carolina Math 3 custom program: Functions and Their Inverses, Exponential and Logarithmic Functions, Polynomial Functions, Modeling with Geometry, Reasoning with Geometry with Circles, Rational Functions, Trigonometric Functions, and Statistics. The units in this program are

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designed to be flexible so that you can mix and match activities as the needs of your students and your instructional style dictate.

The Station Activities correspond to the content in the units and provide students with the opportunity to apply concepts and skills, while you have a chance to circulate, observe, speak to individuals and small groups, and informally assess and plan.

Each unit includes a mid-unit assessment and an end-of-unit assessment. These enable you to gauge how well students have understood the material as you move from lesson to lesson and to differentiate as appropriate.

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Correspondence to Standards for Mathematical Practice

How Do Walch Integrated Mathematics Resources Address the Standards for Mathematical Practice?

Walch's mathematics courses employ a problem-based model of instruction that supports and reinforces the eight Standards for Mathematical Practice. Although the following table focuses on Problem-Based Tasks, Walch's full programs also include hundreds of additional problems in warm-ups and practices. The Implementation Guides for selected PBTs highlight SMPs to focus on during implementation and discussion.

Standards for Mathematical Practice		Relevant Attributes of Walch Integrated Math Resources
1	Make sense of problems and persevere in solving them.	Each lesson is built around a Problem-Based Task (PBT) that requires students to "make sense of problems and persevere in solving them."
2	Reason abstractly and quantitatively.	Each PBT uses a meaningful real-world context that requires students to reason both abstractly about the situation/relationships and quantitatively about the values representing the elements and relationships.
3	Construct viable arguments and critique the reasoning of others.	Since the PBT provides opportunities for multiple problem-solving approaches and varied solutions, students are required to construct viable arguments to support their approach and answer. This, in turn, provides other students the opportunity to analyze and critique their classmates' reasoning.
4	Model with mathematics.	Each PBT represents a real-world situation and requires students to model it with mathematics.
5	Use appropriate tools strategically.	PBTs require students to make choices about using appropriate tools, such as calculators, spreadsheets, graph paper, manipulatives, protractors, and compasses. The tasks do not prescribe specific tools, but instead provide opportunities for their use.
6	Attend to precision.	The real-world contexts of the PBTs require students to be precise in their solutions, both in the ways that the solutions are stated, labeled, and explained, and in the degree of precision necessary given the context (e.g., tripling chili for a crowd vs. machining a part for an airplane engine).
7	Look for and make use of structure.	The PBTs present students with complicated scenarios that must be analyzed to discern patterns and significant mathematical features.
8	Look for and express regularity in repeated reasoning.	PBTs require multiple steps, providing opportunities for students to note repeated calculations, monitor their process, and continually evaluate reasonableness of intermediate results before arriving at a solution.

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Correspondence to NCTM *Principles to Actions* Teaching Practices

How Do Walch Integrated Mathematics Resources Address the NCTM *Principles to Actions* Mathematics Teaching Practices?

Walch's mathematics programs were designed by experienced educators and curriculum developers, informed by best-practice research, and refined through an iterative process of implementation and feedback. Together with professional development, these materials support and sustain good teaching practices.

NCTM Mathematics Teaching Practices	Relevant Attributes of Walch Integrated Resources
Establish mathematics goals to focus learning.	Each lesson in Walch's programs addresses specified standards which can be used as goals to focus learning. Essential Questions offer further focus.
Implement tasks that promote reasoning and problem solving.	Each lesson in Walch's programs is built around a Problem-Based Task (PBT), set in a meaningful real-world context and designed to promote reasoning and problem solving. The courses include dozens of PBTs as well as warm-up and practice problems.
Use and connect mathematical representations.	Walch's Integrated programs make frequent use of, and connections among and between, equations, tables, and graphs. PBTs often require students to use and connect two or more of these representations, and the representations are modeled through guided practice.
Facilitate meaningful mathematical discourse.	Several features of the programs support mathematical discourse, including warm-up debriefs with connections to upcoming lessons, implementation guides and optional coaching questions for the PBTs, and discussion guides for Station Activities. Explanations of PBT solutions are another opportunity for discourse. Please note: Mathematical discourse is an important topic for professional development, in conjunction with implementation of these materials.
Pose purposeful questions.	The implementation guides, coaching questions and discussion guides provide samples of purposeful questions. Note that this is another important topic for professional development.
Build procedural fluency from conceptual understanding.	The programs develop conceptual understanding through modeling, guided practice, and application, and then provide additional opportunities to practice and develop fluency.
Support productive struggle in learning mathematics.	The PBTs require "productive struggle;" implementation guides include suggestions for facilitation and monitoring, and coaching questions provide an option for additional support as appropriate, allowing students to proceed through the task and ensuring that the struggle remains productive rather than too frustrating.
Elicit and use evidence of student thinking.	Various discussions and PBTs require students to display their thinking. Implementation guides offer specific prompts and suggestions for eliciting and responding to student thinking. Professional development supports teachers in using that evidence to respond in instructionally appropriate ways.

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Unit Structure

All of the instructional units have common features. Each unit begins with a list of all the standards addressed in the lessons; Essential Questions; vocabulary (titled "Words to Know"); a list of recommended websites to be used as additional resources, and one or more conceptual activities.

Each lesson begins with a warm-up, followed by a list of identified prerequisite skills that students need to have mastered in order to be successful with the new material in the upcoming lesson. This is followed by an introduction, key concepts, common errors/misconceptions, guided practice examples, a problem-based task with coaching questions and sample responses, a closure activity, and practice. Each unit includes a Mid-Unit Assessment and an End-of-Unit Assessment to evaluate students' learning.

All of the components are described below and on the following pages for your reference.

North Carolina Standard Course of Study for the Unit

All standards that are addressed in the entire unit are listed.

Essential Questions

These are intended to guide students' thinking as they proceed through the unit. By the end of each unit, students should be able to respond to the questions.

Words to Know

A list of vocabulary terms that appear in the unit are provided as background information for instruction or to review key concepts that are addressed in the lesson. Each term is followed by a numerical reference to the lesson(s) in which the term is defined.

Recommended Resources

This is a list of websites that can be used as additional resources. Some websites are games; others provide additional examples and/or explanations. (*Note*: Links will be monitored and repaired or replaced as necessary.) Each Recommended Resource is also accessible through Walch's cloud-based Curriculum Engine Learning Object Repository as a separate learning object that can be assigned to students.

Conceptual Activities

Conceptual understanding serves as the foundation on which to build deeper understanding of mathematics. In an effort to build conceptual understanding of mathematical ideas and to provide more than procedural fluency and application, links to interactive open education and Desmos resources are included. (*Note*: These website links will be monitored and repaired or replaced as necessary.) These and many other open educational resources (OERs) are also accessible through the Learning Object Repository as separate objects that can be assigned to students.

Warm-Up

Each warm-up takes approximately 5 minutes and addresses either prerequisite and critical-thinking skills or previously taught math concepts.

Warm-Up Debrief

Each debrief provides the answers to the warm-up questions, and offers suggestions for situations in which students might have difficulties. A section titled Connection to the Lesson is also included in the debrief to help answer students' questions about the relevance of the particular warm-up activity to the upcoming instruction. Warm-Ups with debriefs are also provided in PowerPoint presentations.

Identified Prerequisite Skills

This list cites the skills necessary to be successful with the new material.

Introduction

This brief paragraph gives a description of the concepts about to be presented and often contains some Words to Know.

Key Concepts

Provided in bulleted form, this instruction highlights the important ideas and/or processes for meeting the standard.

Graphing Calculator Directions

Step-by-step instructions for using a TI-Nspire and a TI-83/84 are provided whenever graphing calculators are referenced.

Common Errors/Misconceptions

This is a list of the common errors students make when applying Key Concepts. This list suggests what to watch for when students arrive at an incorrect answer or are struggling with solving the problems.

Scaffolded Practice (Printable Practice)

This set of 10 printable practice problems provides introductory level skill practice for the lesson. This practice set can be used during instruction time.

Guided Practice

This section provides step-by-step examples of applying the Key Concepts. The three to five examples are intended to aid during initial instruction, but are also for individuals needing additional instruction and/or for use during review and test preparation.

Enhanced Instructional PowerPoint (Presentation)

Each lesson includes an instructional PowerPoint presentation with the following components: Warm-Up, Key Concepts, and Guided Practice. Selected Guided Practice examples include GeoGebra applets. These instructional PowerPoints are downloadable and editable.

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Problem-Based Task

This activity can serve as the centerpiece of a problem-based lesson, or it can be used to walk students through the application of the standard, prior to traditional instruction or at the end of instruction. The task makes use of critical-thinking skills.

Optional Problem-Based Task Coaching Questions with Sample Responses

These questions scaffold the task and guide students to solving the problem(s) presented in the task. They should be used at the discretion of the teacher for students requiring additional support. The Coaching Questions are followed by answers and suggested appropriate responses to the coaching questions. In some cases answers may vary, but a sample answer is given for each question.

Recommended Closure Activity

Students are given the opportunity to synthesize and reflect on the lesson through a journal entry or discussion of one or more of the Essential Questions.

Problem-Based Task Implementation Guide

This instructional overview, found with selected Problem-Based Tasks in each unit, highlights connections between the task and the lesson's key concepts and SMPs. The Implementation Guide also offers suggestions for facilitating and monitoring, and provides alternative solutions.

Printable Practice (Sets A and B) and Interactive Practice (Set A)

Each lesson includes two sets of practice problems to support students' achievement of the learning objectives. They can be used in any combination of teacher-led instruction, cooperative learning, or independent application of knowledge. Each Practice A is also available as an interactive Learnosity activity with Technology-Enhanced Items.

Answer Key

Answers for all of the Warm-Ups and practice problems are provided at the end of each unit.

Station Activities

Each unit includes a collection of station-based activities to provide students with opportunities to practice, reinforce, and apply mathematical skills and concepts. The debriefing discussions after each set of activities provide an important opportunity to help students reflect on their experiences and synthesize their thinking.

Mid-Unit and End-of-Unit Assessments

A mid-unit assessment and an end-of-unit assessment offer multiple-choice questions and extended-response questions that incorporate critical thinking and writing components. These can be used to document the extent to which students grasped the concepts and skills of each unit.

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Standards Correlations

Each lesson in this program was written specifically to address the North Carolina Standard Course of Study (NCSCOS) for Mathematics. Each unit lists the standards covered in all the lessons, and each lesson lists the standards addressed in that particular lesson. In this section, you'll find a comprehensive list mapping the lessons to the NCSCOS.

As you use this program, you will come across a star symbol (★) included with the standards for some of the lessons and activities. This symbol is explained below.

Symbol: ★

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

From http://www.walch.com/CCSS/00003

NORTH CAROLINA MATH 3 STANDARDS CORRELATIONS

Unit 1: Functions and Their Inverses			
Lesson	**Title**	**Standard(s)**	**Pages**
1.1	Comparing Properties of Functions Given in Different Forms	F–IF.9, F–LE.3	U1-1
1.2	Graphing Radical Functions	F–IF.7★	U1-34
1.3	Creating Absolute Value Equations and Inequalities in One Variable	A–CED.1★	U1-76
1.4	Absolute Value and Step Functions	F–IF.7★	U1-102
1.5	Creating and Graphing Absolute Value Equations and Inequalities with Two Variables	A–CED.2★, A–CED.3★, A–REI.11★	U1-142
1.6	Piecewise Functions	F–IF.7★	U1-178
1.7	Operating on Functions	F–BF.1b★	U1-210
1.8	Finding Inverse Functions	F–BF.4a	U1-229
1.9	Finding Inverse Functions in Various Forms	F–BF.4c, F–IF.9	U1-249
1.10	Determining Inverses of Quadratic Functions	F–BF.4c	U1-288

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Unit 2: Exponential and Logarithmic Functions			
Lesson	**Title**	**Standard(s)**	**Pages**
2.1	Creating and Interpreting Exponential Functions	A–CED.1★, A–SSE.1a★	U2-1
2.2	Translating Exponential Functions	F–BF.3	U2-21
2.3	Logarithmic Functions as Inverses	F–BF.4a, F–BF.4c	U2-43
2.4	Graphing Logarithmic Functions	F–LE.4★	U2-67
2.5	Solving Exponential Equations	F–LE.4★	U2-96
2.6	Creating and Solving Exponential Equations from Word Problems	A–CED.1★	U2-123
2.7	Writing Exponential Expressions in Equivalent Forms	A–SSE.3★	U2-147
2.8	Linear, Exponential, and Quadratic Functions	A–CED.2★, F–IF.9	U2-177

Unit 3: Polynomial Functions			
Lesson	**Title**	**Standard(s)**	**Pages**
3.1	Introduction to Polynomial Functions	A–SSE.1a★	U3-1
3.2	Graphing Quadratic and Cubic Functions	F–IF.7★	U3-17
3.3	Optimization of Volume	G–MG.1★	U3-51
3.4	Describing End Behavior and Turns	F–IF.7★, N–CN.9	U3-81
3.5	The Remainder Theorem	A–APR.2	U3-104
3.6	Zeros of Polynomial Functions	A–APR.3	U3-127
3.7	Building Polynomial Functions	F–BF.1a★	U3-157
3.8	End Behaviors of Functions	F–LE.3★, F–IF.9	U3-185

NORTH CAROLINA MATH 3 STANDARDS CORRELATIONS

Unit 4: Modeling with Geometry			
Lesson	**Title**	**Standard(s)**	**Pages**
4.1	Proving Theorems About Triangles	G–CO.10	U4-1
4.2	Proving Properties of Parallelograms	G–CO.11	U4-2
4.3	Proving Properties of Special Quadrilaterals	G–CO.11	U4-38
4.4	Two-Dimensional Cross Sections of Three-Dimensional Objects	G–GMD.4	U4-80
4.5	Volumes of Cylinders, Pyramids, Cones, and Spheres	G–GMD.3★	U4-109
4.6	Density	G–MG.1★	U4-136
4.7	Design	G–MG.1★	U4-162
4.8	Proving Centers of Triangles	G–CO.10	U4-198

Unit 5: Reasoning with Geometry with Circles			
Lesson	**Title**	**Standard(s)**	**Pages**
5.1	Deriving the Equation of a Circle	G–GPE.1, G–CO.14	U5-1
5.2	Similar Circles and Central and Inscribed Angles	G–C.2	U5-40
5.3	Chord Central Angles Conjecture	G–C.2	U5-75
5.4	Defining Radians	G–C.5	U5-96
5.5	Deriving the Formula for the Area of a Sector	G–C.5	U5-119
5.6	Properties of Tangents of a Circle	G–C.2, G–CO.14	U5-134
5.7	Inscribed Angles, Secants, Tangents, and Chords	G–C.2	U5-168

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Unit 6: Rational Functions			
Lesson	**Title**	**Standard(s)**	**Pages**
6.1	Graphing Rational Equations	A–CED.2★	U6-1
6.2	Graphing Rational Functions and Identifying Key Features	A–SSE.1a★, F–IF.4★, F–IF.7★	U6-45
6.3	Structures of Rational Expressions	A–APR.6	U6-78
6.4	Multiplying Rational Expressions	A–APR.7b	U6-104
6.5	Dividing Rational Expressions	A–APR.7b	U6-124
6.6	Adding and Subtracting Rational Expressions	A–SSE.2, A–APR.7a	U6-146
6.7	Solving Rational Equations	A–REI.2	U6-172
6.8	Creating Rational Equations	A–CED.1★	U6-203

Unit 7: Trigonometric Functions			
Lesson	**Title**	**Standard(s)**	**Pages**
7.1	Radians	F–IF.1, F–TF.1	U7-1
7.2	Special Angles in the Unit Circle	F–TF.2	U7-29
7.3	Periodic Phenomena and Amplitude, Frequency, and Midline	F–TF.5★	U7-64
7.4	Using Trigonometric Functions to Model Periodic Phenomena	F–TF.5★	U7-93

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Unit 8: Statistics			
Lesson	**Title**	**Standard(s)**	**Pages**
8.1	Identifying Surveys, Experiments, and Observational Studies	S–IC.3★	U8-1
8.2	Other Methods of Random Sampling	S–IC.3★	U8-30
8.3	Differences Between Populations and Samples	S–IC.3★	U8-65
8.4	Designing Surveys, Experiments, and Observational Studies	S–IC.3★	U8-103
8.5	Designing and Simulating Treatments	S–IC.5★	U8-125
8.6	Estimating Sample Proportions	S–IC.4★	U8-148
8.7	Estimating Sample Means	S–IC.4★	U8-179
8.8	Reading Reports	S–IC.6★	U8-203

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

Unit 1

- Desmos. "Absolute Value Inequalities on the Number Line."

  http://www.walch.com/ca/01048

  In this activity, students explore inequalities involving absolute value and make connections among multiple representations (including algebraic expressions, verbal statements, number line graphs, and solution sets).

- Desmos. "Domain and Range Introduction."

  http://www.walch.com/ca/01049

  In this activity, students practice finding the domain and range of piecewise functions. Students begin with an informal exploration of domain and range using a graph, and build up to representing the domain and range of piecewise functions using inequalities.

- Desmos. "Polygraph: Absolute Value."

  http://www.walch.com/ca/01050

  This activity is designed to spark vocabulary-rich conversations about transformations of the absolute value parent function. Key vocabulary that may appear in student questions includes: translation, shift, slide, dilation, stretch, horizontal, vertical, and reflect.

- Desmos. "Polygraph: Piecewise Functions."

  http://www.walch.com/ca/01052

  This activity is designed to spark vocabulary-rich conversations about piecewise functions. Key vocabulary that may appear in student questions includes: piecewise, continuous, and interval.

- Desmos. "Polygraph: Twelve Functions."

  http://www.walch.com/ca/01053

  This activity is designed to spark vocabulary-rich conversations about various functions. Key vocabulary that may appear in student questions includes: linear, quadratic, exponential, cubic, absolute value, rational, radical, sinusoid, and step.

Unit 2

- Desmos. "Card Sort: Exponentials."

  http://www.walch.com/ca/01044

  In this activity, students practice what they've learned about exponential functions by matching equations to properties of the graphs the functions will produce. Students will then use their knowledge of transforming exponential functions to pair equations with graphs.

- Desmos. "Polygraph: Exponential & Logarithmic Functions."

  http://www.walch.com/ca/01045

  This activity is designed to spark vocabulary-rich conversations about exponential and logarithmic functions. Key vocabulary that may appear in student questions includes: exponential, asymptote, logarithmic, and quadrant.

- Desmos. "Polygraph: Exponential Functions."

  http://www.walch.com/ca/01046

  This activity is designed to spark vocabulary-rich conversations about exponential functions. Key vocabulary that may appear in student questions includes: increasing, decreasing, asymptote, quadrant, and axis.

- Desmos. "Writing Rules: Linear, Quadratic, and Exponential."

  http://www.walch.com/ca/01047

  In this activity, students have an opportunity to deepen their understanding of linear, quadratic, and exponential functions by making connections between their tables, graphs, and equations.

Unit 3

- Desmos. "Constructing Polynomials."

  http://www.walch.com/ca/01054

  In this activity, students will consider properties of polynomial functions such as end behavior, leading terms, and properties of roots. They will explore connections between those properties and the factored forms of the equations of the polynomials.

- Desmos. "Polygraph: Polynomial Functions."

  http://www.walch.com/ca/01055

  This activity is designed to spark vocabulary-rich conversations about polynomial functions. Key vocabulary that may appear in student questions includes: degree, roots, end behavior, limit, quadrant, axis, increasing, decreasing, maximum, minimum, extrema, concave up, and concave down.

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- Desmos. "Polynomial Equation Challenges."

  http://www.walch.com/ca/01056

  In this activity, students will create polynomial equations (of degree 2, 3, and 4) to match given zeros and points. Students will explore how the factored form of the equations relates to the zeros and the order of those zeros.

Unit 5

- Desmos. "Circle Patterns."

  http://www.walch.com/ca/01039

  In this activity, students notice similarities and differences in a set of circles. They then use this information to practice writing equations of circles that extend a given pattern or match a given set of conditions.

- Desmos. "Equations of Circles."

  http://www.walch.com/ca/01040

  In this activity, students write equations of circles with different given information. The activity involves writing equations in both standard and general form.

- Desmos. "Sector Area."

  http://www.walch.com/ca/01041

  In this proportional reasoning activity, students explore the relationship between circle area, sector area, and sector angle.

Unit 6

- Desmos. "Marbleslides: Rationals."

  http://www.walch.com/ca/01057

  In this activity, students will transform rational functions to send marbles through stars.

- Desmos. "Polygraph: Rational Functions."

  http://www.walch.com/ca/01058

  This activity is designed to spark vocabulary-rich conversations about rational functions. Key vocabulary that may appear in student questions includes: asymptote, vertical, horizontal, quadrant, axis, increasing, decreasing, discontinuity, and hole.

- Mathematics Vision Project. "Rational Functions."

  http://www.walch.com/ca/01043

  This site provides multiple tasks on the topic of rational functions. Each task is designed to either develop or solidify a student's understanding of the topic covered, and includes whole-class, small-group, and homework activities.

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Unit 7

- Desmos. "Burning Daylight."

  http://www.walch.com/ca/01059

  In this activity, students use sinusoids to model daylight data for two U.S. cities. They predict which city has more total daylight during a given year, and then use their model to calculate an answer to that question.

- Desmos. "Graphing the Sine Function Using Amplitude, Period, and Vertical Translation."

  http://www.walch.com/ca/01060

  Students will build a visual understanding of amplitude, period, and phase shift in this introduction to trigonometric graphing. They will use this understanding to find models for given graphs of the sine function.

- Desmos. "Marbleslides: Periodics."

  http://www.walch.com/ca/01061

  In this activity, students will transform periodic functions to send marbles through stars.

- Desmos. "Polygraph: Sinusoids."

  http://www.walch.com/ca/01062

  This activity is designed to spark vocabulary-rich conversations about sinusoids. Key vocabulary that may appear in student questions includes: amplitude, periods, maximum, minimum, and shift.

- Desmos. "Polygraph: Sinusoids with Vertical Transformations."

  http://www.walch.com/ca/01063

  This activity is designed to spark vocabulary-rich conversations about vertical transformations of sinusoids. Key vocabulary that may appear in student questions includes: translation, dilation, amplitude, midline, and sinusoidal axis.

- Desmos. "Trigonometric Graphing: Introduction to Amplitude and Vertical Shift."

  http://www.walch.com/ca/01064

  In this activity, students will informally explore range, midline, and amplitude of trigonometric functions. They'll use what they learn about the relationships to write equations of sine and cosine graphs.

Unit 8

- Inside Mathematics. "Sorting Functions."

  http://www.walch.com/ca/01042

  In this activity, students are presented with multiple tasks challenging them to use their knowledge of equations to match tables, verbal descriptions, and graphs to equations.

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Station Activities Guide

Introduction

Each unit includes a collection of station-based activities to provide students with opportunities to practice and apply the mathematical skills and concepts they are learning. You may use these activities in addition to the instructional lessons, or, especially if the pre-test or other formative assessment results suggest it, instead of direct instruction in areas where students have the basic concepts but need practice. The debriefing discussions after each set of activities provide an important opportunity to help students reflect on their experiences and synthesize their thinking. Debriefing also provides an additional opportunity for ongoing, informal assessment to guide instructional planning.

Implementation Guide

The following guidelines will help you prepare for and use the activity sets in this section.

Setting Up the Stations

Each activity set consists of four or five stations. Set up each station at a desk, or at several desks pushed together, with enough chairs for a small group of students. Place a card with the number of the station on the desk. Each station should also contain the materials specified in the teacher's notes, and a stack of student activity sheets (one copy per student). Place the required materials (as listed) at each station.

When a group of students arrives at a station, each student should take one of the activity sheets to record the group's work. Although students should work together to develop one set of answers for the entire group, each student should record the answers on his or her own activity sheet. This helps keep students engaged in the activity and gives each student a record of the activity for future reference.

Forming Groups of Students

All activity sets consist of four or five stations. You might divide the class into four or five groups by having students count off from 1 to 4 or 5. If you have a large class and want to have students working in small groups, you might set up two identical sets of stations, labeled A and B. In this way, the class can be divided into eight groups, with each group of students rotating through the "A" stations or "B" stations.

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Assigning Roles to Students

Students often work most productively in groups when each student has an assigned role. You may want to assign roles to students when they are assigned to groups and change the roles occasionally. Some possible roles are as follows:

- Reader—reads the steps of the activity aloud
- Facilitator—makes sure that each student in the group has a chance to speak and pose questions; also makes sure that each student agrees on each answer before it is written down
- Materials Manager—handles the materials at the station and makes sure the materials are put back in place at the end of the activity
- Timekeeper—tracks the group's progress to ensure that the activity is completed in the allotted time
- Spokesperson—speaks for the group during the debriefing session after the activities

Timing the Activities

The activities in this section are designed to take approximately 10 minutes per station. Therefore, you might plan on having groups change stations every 10 minutes, with a two-minute interval for moving from one station to the next. It is helpful to give students a "5-minute warning" before it is time to change stations.

Since each activity set consists of four or five stations, the above time frame means that it will take about 50 to 60 minutes for groups to work through all stations.

Guidelines for Students

Before starting the first activity set, you may want to review the following "ground rules" with students. You might also post the rules in the classroom.

- All students in a group should agree on each answer before it is written down. If there is a disagreement within the group, discuss it with one another.
- You can ask your teacher a question only if everyone in the group has the same question.
- If you finish early, work together to write problems of your own that are similar to the ones on the activity sheet.
- Leave the station exactly as you found it. All materials should be in the same place and in the same condition as when you arrived.

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Debriefing the Activities

After each group has rotated through every station, bring students together for a brief class discussion. At this time, you might have the groups' spokespersons pose any questions they had about the activities. Before responding, ask if students in other groups encountered the same difficulty or if they have a response to the question. The class discussion is also a good time to reinforce the essential ideas of the activities. The questions that are provided in the teacher's notes for each activity set can serve as a guide to initiating this type of discussion.

You may want to collect the student activity sheets before beginning the class discussion. However, it can be beneficial to collect the sheets afterward so that students can refer to them during the discussion. This also gives students a chance to revisit and refine their work based on the debriefing session. If you run out of time to hold class discussions, you might want to have students journal about their experiences and follow up with a class discussion the next day.

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Digital Enhancements Guide

Introduction

With this program, you have access to the following digital components, described here with guidelines and suggestions for implementation.

Digital Instruction PowerPoints (Presentations)

These optional versions of the Warm-Ups, Warm-Up Debriefs, Introductions, Key Concepts, and Guided Practices for each lesson run on PowerPoint. (*Please note*: Computers may render PowerPoint images differently. For best viewing and display, use a PowerPoint Viewer and adjust your settings to optimize images and text.)

Each PowerPoint begins with the lesson's Warm-Up and is followed by the Warm-Up Debrief, which reveals the answers to the Warm-Up questions.

In the notes section of the last Warm-Up slide, you will find the "Connections to the Lesson," which describes concepts students will glean or skills they will need in the upcoming lesson. The "Connections" help transition from the Warm-Up to instruction.

GeoGebra Applets (Interactive Practice Problems)

One or two interactive GeoGebra applets are provided for most lessons. The applets model the mathematics in the Guided Practice examples for these lessons. Links to these applets are also embedded within the Instructional PowerPoints. With an Internet connection, simply click on the "Play" button slide that follows selected examples.

Once you've accessed the GeoGebra applet, please adjust your view to maximize the image. Each applet illustrates the specific problem addressed in the Guided Practice example. The applets allow you to walk through the solution by visually demonstrating the steps, such as defining points and drawing lines. Variable components of the applets (usually fill-in boxes or sliders) allow you to substitute different values in order to explore the mathematics. For example, "What happens to the line when we increase the amount of time?" or "What if we cut the number of students in half?" This experimentation and discussion supports development of conceptual understanding.

GeoGebra for PC/MAC

GeoGebra is not required for using the applets, but can be downloaded for free for further exploration at the following link:

http://www.geogebra.org/cms/en/download

GeoGebra Applet Troubleshooting

If you are experiencing any difficulty in using the applets in your browser, please visit the following link for our troubleshooting document.

http://www.walch.com/applethelp

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Curriculum Engine Item Bank

Walch's Curriculum Engine comes loaded with thousands of curated learning objects that can be used to build formative and summative assessments as well as practice worksheets. District leaders and teachers can search for items by standard and create assessments or worksheets in minutes using the three-step assessment builder.

For more information about the Curriculum Engine Item Bank, or for additional support, please contact Customer Service at (800) 341-6094 or customerservice@walch.com.

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Standards for Mathematical Practice Implementation Guide

Introduction

The eight Standards for Mathematical Practice describe features of lesson design, teaching pedagogy, and student actions that will lead to a true conceptual understanding of the mathematics standards. Walch's lessons, practice problems, and Problem-Based Tasks lend themselves to teaching through this framework. When the Walch resources are combined with high-level questioning and engaging teacher decisions in the classroom, it will lead to high-level math instruction and student achievement.

Here is a brief description of the SMPs and how they can be applied in the classroom:

SMP 1: Make sense of problems and persevere in solving them.

Students will read, interpret, and understand complicated mathematical and real-world problems, and they will be willing to try multiple methods with the ultimate goal of determining the correct answer. Strategies such as annotation and student discourse can lead to improvement on this standard. Presenting students with higher-level problems is essential to ensuring students achieve maximum understanding. Teacher prompts that can enhance this standard include:

- What is the problem asking you to solve?
- What are some (other) strategies you could use to solve this problem?
- Compare your answer with a classmate's answer. Who is correct? Why?

SMP 2: Reason abstractly and quantitatively.

Mathematical reasoning with numbers and variables is essential to understanding the connections among the standards. Students must be able to discover and formalize general rules using numbers and variables, and apply them to determine numerical quantities in other situations. Teacher prompts that can enhance this standard include:

- Substitute realistic numbers into the situation.
- What operation/strategy would you use?
- Will your strategy work for any number?
- For which categories of numbers (negative integers, all real numbers, etc.) will your strategy work?

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SMP 3: Construct viable arguments and critique the reasoning of others.

Many students are most concerned with the “what” aspects of mathematics, i.e. “what” do we do or “what” is the answer. However, math educators must develop the “why” of mathematics. Students must learn to question algorithms, challenge answers, and justify their reasoning in order to truly understand the concepts behind their answers. Teacher prompts that can enhance this standard include:

- How did you determine your answer?
- Why did you choose that strategy?
- Defend your answer based on a real-world situation.

SMP 4: Model with mathematics.

An important goal of mathematics instruction is for students to be able to apply mathematics to the world around them. Students should be able to link a real problem to a mathematical concept, identify quantities that are modeled well with mathematics, and use mathematics to find a solution. Emphasizing this standard will help students represent and interpret information using physical, visual, and abstract models. Encourage students to use any or all of their learning experiences to gain a deep and flexible understanding of mathematics. Teacher prompts that can enhance this standard include:

- Can you represent this situation with a visual model?
- How will it help you solve the problem?
- What information is needed to solve this problem?
- Is there another way to solve this problem?
- While working to solve this problem, what do you notice/wonder?

SMP 5: Use appropriate tools strategically.

There are many available tools suitable for mathematics, such as calculators, manipulatives, formulas, rulers, computers, and developed mathematical strategies. Choosing and using the correct tool to work through a problem is an important skill for mathematicians. Teacher prompts that can enhance this standard include:

- Can you graph this equation in the calculator to see a relationship?
- What formula or strategy might help you determine the answer to this question?
- How can you represent the situation using handheld tools (rulers, protractors, etc.) to determine an answer?

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SMP 6: Attend to precision.

When using mathematics to solve problems, an answer can be considered correct only if it is sufficiently precise and accurate for the situation to which it pertains. When applying mathematics, it is vital to clearly define the question, the reasoning, the answer, and the explanation. Vocabulary, units, numerical responses, and pictures must be represented precisely in questions and answers to ensure that the mathematical solutions represent the true answer to a question. Teacher prompts that can enhance this standard include:

- What does your answer represent in a real-world context?
- Is your answer reasonable based on your initial estimate?
- What units of measure help describe your numerical answer?

SMP 7: Look for and make use of structure.

Structure, whether geometric, algebraic, statistical, or numerical, is an important aspect of mathematical reasoning that students often overlook. Teachers often explicitly refer to geometric and other visual structures as explanations of mathematical concepts, but algebraic and numerical structures can often be just as important in analyzing and interpreting mathematical situations. These structures yield clues as to the meaning of expressions, equations, graphs, and other representations. As students interpret these structures, they will gain a greater understanding of the mathematical concepts. Teacher prompts that can enhance this standard include:

- What do the characteristics of the graph tell us about the situation?
- What do each of the variables and numbers in the equation/formula represent?
- How are these situations the same and different based on their representations?

SMP 8: Look for and express regularity in repeated reasoning.

Just as patterns appear in real life, patterns appear throughout the subject of mathematics. Recognizing and applying these patterns, and applying the reasoning contained within, is one of the most important skills teachers can instill in their students. Rather than teaching isolated algorithms to determine answers, have students discover relationships, create their own algorithms, and apply the reasoning to other situations. These skills can be applied throughout their education and will enrich their lives after high school. Teacher prompts that can enhance this standard include:

- What relationship do you notice in the graph/table/numbers?
- Why did you choose to use this process to solve this word problem/equation?
- How can you apply this process in other situations?

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Instructional Strategies

Ensuring Access for All Students

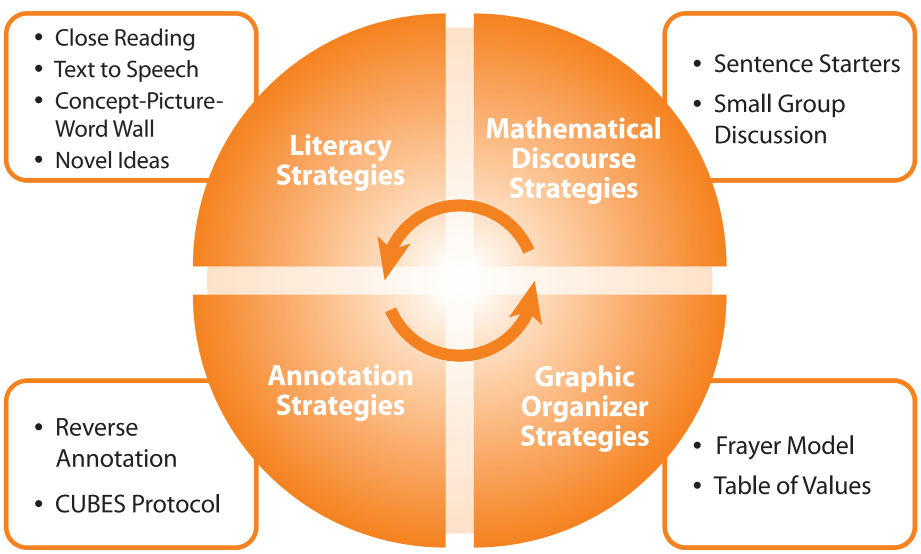
Introduction

The increased focus on literacy in math instruction can help some students navigate mathematical contexts, but for struggling readers, it can further complicate calculations. English language learners struggle to master difficult mathematical concepts while simultaneously processing a new language. Students with learning and behavioral disabilities struggle with the math concepts in their own contexts. This is where teachers and the strategies they select for their classrooms become essential.

The strategies presented here can help all students succeed in math, literacy, school, and, ultimately, in life. These instructional strategies provide teachers with a wide range of instructional support to aid English as a Second Language (ESL) students, students with disabilities (SWD), and struggling readers. These strategies provide support for the Mathematics Standards and the Standards of Mathematical Practice (SMP), English Language Development (ELD) Standards, English Language Arts Standards, and WIDA English Language Development Standards.

Within each lesson throughout this course, you will find suggested instructional strategies. These instructional strategies are research-based strategies and best practices that work well for all students.

The instructional strategies detailed here fall into four main categories: Literacy, Mathematical Discourse, Annotation, and Graphic Organizers. These strategies provide teachers with research-based strategies to address the needs of all students.



Source

- WIDA: https://www.wida.us/standards/eld.aspx

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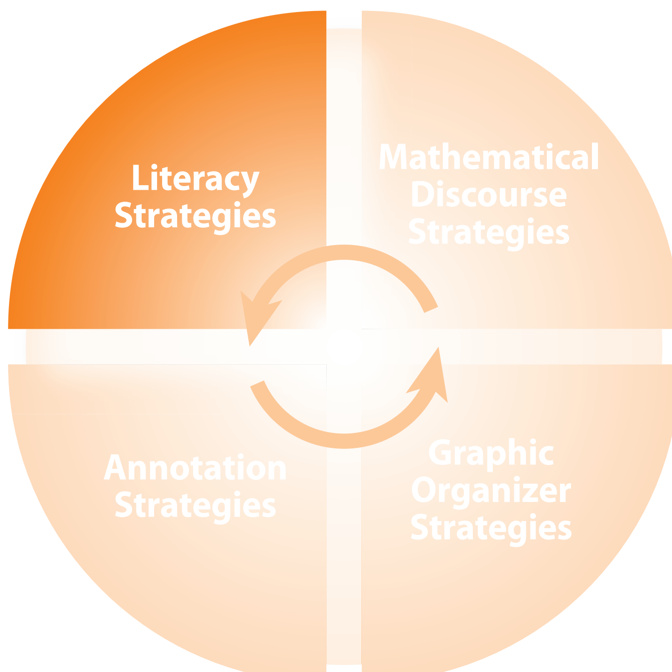
Understanding the Language of Mathematics: Literacy

Mathematics has its own language consisting of words, notations, formulas, and visuals. In education, the language of mathematics is often regarded solely in the context of word problems and articles. This neglects the vocabulary and other mathematical representations students must be able to interpret. The strategies presented here help students navigate the language of mathematics so that they can understand text and feel confident speaking in and listening to mathematical discussions. For students with disabilities, the stress on repetition and different representations in this approach is essential to their ability to grasp the math concepts. For ESL students, repetition and different representations can strip out some of the English language barriers to understanding the language of mathematics, as well as provide multiple means of accessing the content. Literacy strategies include Close Reading, Text-to-Speech, Concept-Picture-Word Walls, and Novel Ideas.



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Literacy Strategies

Close Reading with Guiding Questions

What is Close Reading with Guiding Questions?

Close Reading with Guiding Questions is a process that allows students to preview mathematical reading and problems by answering questions related to the text in advance and reviewing their responses during and/or after reading. Multiple reading protocols can be used in conjunction with guiding questions to enhance their effectiveness.

How do you implement Close Reading with Guiding Questions in the classroom?

When utilizing a textbook, task, or article in a math class, literacy struggles are often a strong barrier to entry into the mathematical ideas. Asking students to answer accessible questions before and/or as they read can lead them to the key information.

Prior to implementation, the teacher should determine the most important information students need to obtain from a text, whether it is a math problem to solve, a task to complete, or an informational lesson or article to read. Then, the teacher should come up with some questions to guide students before they read. These questions can:

- assess and relate prior knowledge
- define key vocabulary words
- discuss non-mathematical concepts in the text

The teacher should also prepare some questions to guide students as they read. These questions can:

- point out key concepts within the text
- relate the text and concepts to future learning
- assist students in identifying key facts in the text
- highlight the importance of text features (graphics, headings, etc.) in the text

To ensure the questions are accessible for students and to encourage reflection and debate after reading, many of these questions should be designed as either "True/False" or "Always True/Sometimes True/Never True." Students can represent their reasoning for their answer in writing, numbers, or graphic/pictorial representations. Students should complete the guiding questions and reading individually, with discussion to follow.

After students complete the reading, they should be given some time to individually evaluate their initial answers. Then, in partners or in groups, they can discuss their answers and come to final conclusions that will help them find the important information initially identified by the teacher. After deciphering the text through close reading, students will be able to complete the given activity.

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When would I use Close Reading with Guiding Questions in the classroom?

Close Reading with Guiding Questions can be used for any activity in which literacy could be a barrier to learning or demonstrating mastery of mathematical concepts. The number of questions and length of the discussions can be altered based on the length, importance, and difficulty of the text and concept. As students become more accustomed to mathematical literacy, the text complexity can be increased, but the adherence to close reading strategies must be maintained to ensure students can access the mathematical concepts. The length of time spent on the literacy aspect can be shortened as students become more skilled, but the questioning and discussions must occur to ensure students are properly interpreting the text in the mathematical context.

How can I use Close Reading with Guiding Questions with students needing additional support?

For struggling readers, including ESLs, Close Reading with Guiding Questions can help make an intimidating lesson, word problem, or task much more accessible. Questions focusing more on Tier 2 and Tier 3 vocabulary, text features, and real-world concepts can help struggling readers relate to the text and learn how to decipher the text in context. Discussions around the questions will help students grasp the math concepts.

Allowing struggling readers to explain their answers using words, numbers, or graphics/pictures ensures that they can express their opinion and rationale despite a potential lack of vocabulary. Through these representations and the ensuing discussion, students will begin to learn the necessary vocabulary to be successful.

What other standards does Close Reading with Guiding Questions address?

Standards of Mathematical Practice:

- SSMP.1
- SSMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.9
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4
- ELA–LITERACY.RST.9–10.7

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Literacy Strategies

Text-to-Speech Technology

What is Text-to-Speech Technology?

Text-to-Speech Technology is an adaptive technology that reads text aloud from a text source for students. It is usually accessed through an application or program on a computer, smartphone, or tablet. Some new programs utilize Mathematical Markup Language (MathML) to read mathematical notation in a common, understandable manner for students. Many programs also highlight the words and notation on the screen as the audio plays, which helps students relate the written representation to the words they hear. The use of Text-to-Speech Technology allows students who struggle with literacy to hear the words and notation and access the text in a different way.

How do you implement Text-to-Speech Technology?

A classroom community focused on everyone's learning and a growth mindset is the first step in implementing Text-to-Speech Technology. One of the main barriers to implementation is encouraging students to use the program. Once they do, they will realize how the audio can help them understand the difficult mathematical texts and interpret the math content within them. After students realize the benefits of Text-to-Speech Technology, it can become part of the regular routine for group and independent work.

The use of headphones can be very important for effective use of Text-to-Speech Technology. Students can use the technology to listen to lessons and texts at their own pace. Extra noise from other students working or other students listening at different paces can confuse students attempting to use Text-to-Speech Technology, and headphones can help mitigate these distractions. Many teachers are nervous about the potential disruption headphones can cause in class. However, well-managed use of headphones can help students successfully utilize the technology to learn.

When would I use Text-to-Speech Technology in the classroom?

Text-to-Speech Technology can be used at any time throughout the year, and if the program speaks in MathML, it can be used with any lesson. Without MathML, effective use could be limited to word problems without unusual notation. For example, if x^2 is read as "x-two" instead of "x-squared" or "x to the second power," that could confuse students more.

During a lesson or small group discussion, Text-to-Speech Technology could detract from students' ability to listen, question, and process information. However, during warm-ups, independent work, or assessments, Text-to-Speech Technology can help students process the information and access the activity. It can become a routine for students to automatically listen to the question, problem, or directions first, and then attempt the activity.

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How can I use Text-to-Speech Technology with students needing additional support?

Text-to-Speech Technology is an important adaptation and accommodation for struggling readers. Students who have read-aloud accommodations sometimes don't receive them because they are either embarrassed to accept them or because of staffing restrictions. These students can use Text-to-Speech Technology to supplement their math instruction by having text automatically read to them in a manner in which they can process it.

Additionally, for ESL students, hearing the English mathematical language, especially referring to mathematical representations and notation, can help put English words to the ideas they see. Some Text-to-Speech Technology can translate written and mathematical text into other languages, so students can hear the text in their natural language and see the English highlighted on the screen as they hear it. In this way, students are learning English vocabulary as well as learning the mathematical content in a language they can understand.

What other standards does Text-to-Speech Technology address?

Standards of Mathematical Practice:

- SMP.1
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.9
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4
- ELA–LITERACY.RST.9–10.7

Source

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Literacy Strategies

Concept-Picture-Word Wall

more> than

What is a Concept-Picture-Word Wall?

A Concept-Picture-Word Wall is a classroom display, often a bulletin board or a set of posters, that exposes students to important vocabulary words they will use in math class.

Posting vocabulary words in class helps reinforce the words students will see in textbooks, videos, websites, and test questions on math concepts. These Tier 3 vocabulary words are often not used in everyday language, and the exposure to the words visually through Concept-Picture-Word Walls can help students connect them to the math content.

How do you implement Concept-Picture-Word Walls in the classroom?

Just seeing the vocabulary on a Concept-Picture-Word Wall by itself will help students; more importantly, referring to the words as the teacher uses them in class helps students connect the visual to the application. A simple gesture to the wall makes a very explicit reference to the word as it is used and allows students to connect the unfamiliar word to its meaning in context. Additionally, students can be taught to refer to the wall as they use the words in class, and they can be asked to make sure they say at least 3 words from the wall during each class period in small-group discourse or as answers to whole-class questions. The comfort gained from using these Tier 3 words will help students to use appropriate math vocabulary while solving problems and will help students connect concepts more explicitly.

Postings on the Concept-Picture-Word Wall can be arranged strategically to connect concepts, units of study, or groups of words where appropriate. Having three sections of the Concept-Picture-Word Wall—for example, an "In the Future" section, a "Live in the Present" section, and a "Remember the Past" section—can help students see and remember the vocabulary throughout the entire course. Even without regular use of some words, just seeing the words before a unit can help instill a familiarity with the vocabulary. Leaving the words on the Concept-Picture-Word Wall after a unit is taught can help students connect "old" concepts to the current lesson and ensure that students still have access to the vocabulary.

When would I use Concept-Picture-Word Walls in the classroom?

Concept-Picture-Word Walls can be used for the entire year. The actual words might have to change, or at least be moved to different areas of the Concept-Picture-Word wall. The more exposure students have to the words, the more familiar and comfortable they will become. The constant exposure to the math context is beneficial for students throughout the entire course, especially for words with multiple meanings (bias, tangent, etc.) that could exist as Tier 2 words in everyday conversation but are Tier 3 words in the math classroom.

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How can I use Concept-Picture-Word Walls with students needing additional support?

For all students learning mathematics, knowing and using the math vocabulary is often a major barrier. This is a problem especially for ESL students, who are learning the English language along with math content. If teachers try to simplify the words too much for students, it does them a disservice as they seek out information from other teachers, textbooks, and online sources that use the proper vocabulary. Most tests, especially state tests, will expect students to have knowledge of the Tier 3, math-specific vocabulary. The more students see these words, the more familiarity they will have when they apply them.

Concept-Picture-Word Walls can also be written in multiple languages. Especially for students who are on-grade-level in their native language, a multi-lingual Concept-Picture-Word Wall can help students connect the content they already know in another language to the English vocabulary necessary for success on English-language math activities and tests.

This website can help you get started on an English-Spanish Concept-Picture-Word Wall: http://math2.org/math/spanish/eng-spa.htm

What other standards do Concept-Picture-Word Walls address?

Standards of Mathematical Practice:

- SMP.1
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.9
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4
- ELA–LITERACY.RST.9–10.7

Source

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Literacy Strategies

Novel Ideas

What is Novel Ideas?

Novel Ideas is a classroom activity that explores students' understanding of important Tier 2 vocabulary words they will use in math class. Instead of asking students to look up vocabulary words in the dictionary, Novel Ideas allows students to have conversations with their peers about vocabulary words in class. This reinforces the mathematical vocabulary students will see in textbooks, videos, websites, and test questions. These Tier 2 vocabulary words are often used in everyday language, but have specific meaning in mathematics. Exposure to the words through Novel Ideas can help students connect them to the math content.

How do you implement Novel Ideas in the classroom?

While building a rich representation of math content words and connecting the words to other words and concepts has inherent merit, it is more important to consider that pre-teaching the words before they are used in class helps students connect to the application. The understanding gained from discussing these Tier 2 words will help students apply them in a mathematical context to solve problems and connect concepts.

Here is a step-by-step process for implementing Novel Ideas:

1. Students separate into groups of four.
2. Students copy the teacher generated prompt/sentence starters and number their papers 1–8.
3. One student offers an idea, another echoes it, and all write it down.
4. After three minutes, students draw a line under the last item in the list.
5. All students stand, and the teacher calls one student from a group to read the group's list.
6. The student starts by reading the prompt/sentence starters, "We think a _______ called _______ may be about ... ," and then adds whatever ideas the team has agreed on.
7. The rest of the class must pay attention because after the first group has presented all their ideas, the teacher asks them to sit down and calls on a student from another team to add that team's "novel ideas only." Ideas that have already been presented cannot be repeated.
8. As teams complete their turns and sit down, each seated student should record novel ideas from other groups below the line that marks the end of his or her team's ideas.

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When would I use Novel Ideas in the classroom?

Novel Ideas can be used for the entire year. The more students are exposed to mathematical vocabulary, the more familiar and comfortable they become, leading to increased usage of these math terms in their conversation and writing. Using math vocabulary in context is beneficial for students throughout the entire course, especially for words with multiple meanings (bias, tangent, etc.) that could exist as Tier 2 words in everyday conversation but are Tier 3 words in the math classroom.

How can I use Novel Ideas with students needing additional support?

Most tests, especially state tests, will expect students to have knowledge of the Tier 3, math-specific vocabulary. The more students use these words in conversation, the more familiarity they will have when they apply them. Understanding Tier 2 words also helps students avoid misconceptions in mathematics. Twice a week before the start of a lesson, allow students to use sentence starters in small groups that include all students. Prepare the sentence starter "When I hear the word _______, I think about _______" to share out with whole class. This will allow students who know the vocabulary words to share their knowledge, and will allow other students to hear the meaning of the vocabulary words. This strategy is particularly helpful for ESL students.

What other standards does Novel Ideas address?

Standards of Mathematical Practice:

- SMP.1
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.9
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4
- ELA–LITERACY.RST.9–10.7

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Novel Ideas Sentence Starters

Slope

- When I hear the word climb, I think about ...
- When I hear the word steep, I think about ...

Volume

- When I hear the word filling, I think about ...

Equations

- When I hear the word balance, I think about ...
- When I hear the word equal, I think about ...

Graphing

- When I hear the word grid, I think about ...
- When I hear the word graph, I think about ...

Scatter Plots

- When I hear the word scattered, I think about ...

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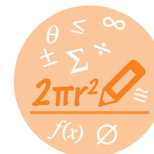
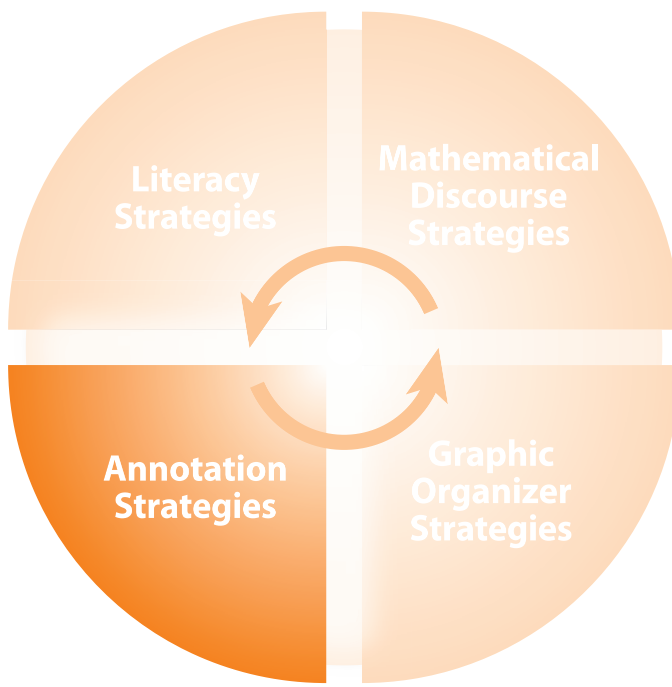
Understanding Mathematical Content: Annotation



Understanding mathematical content is an extremely important skill, both in the math classroom and in life. When students read word problems, articles, charts, graphs, equations, tables, or other forms of mathematical text, they must be able to decode and extract meaning from the text. Annotation can help. The strategies presented here help students identify and focus on key characteristics and facts from various forms of text while ignoring the non-essential information. For students with disabilities, many of whom struggle with the distractions inherent in many high-school level texts, making notes and drawing pictures to explain a problem can help them focus. ESL students will be pointed to certain Tier 3 vocabulary words and determine which Tier 2 vocabulary words they must learn to be proficient in math class and in the English language. Annotation strategies include Reverse Annotation and CUBES protocol.



Annotation Strategies

Reverse Annotation Protocol

What is Reverse Annotation?

Reverse Annotation is a strategy that asks students to identify and write down key information from math problems. This is especially helpful for problems given on a computer or tablet, where students can't annotate directly on the problem. A template is given at the end of this section.

How do you implement Reverse Annotation in the classroom?

Many annotation strategies ask students to write, underline, or mark directly on the text of a problem. While those forms of annotation are also beneficial, they are not always possible with technology. Whether the problem is given on paper or using technology, having students write the answers to these questions will ensure that they are thinking strategically and specifically about the strategies and information needed to solve the problem.

The three questions at the top of the Reverse Annotation template are the key to understanding mathematical problems. For every problem given in class, ask students:

1. What is the problem asking us to solve?
2. What key words tell us the mathematical steps we need to perform?
3. What information in the problem can help us figure it out?

After answering the initial questions, students should make a guess, or estimate, of what they think the answer will be. This helps grow their number sense, and provides an initial, reasonable solution to guide their work. Students can then use the strategies they selected to solve the problem and evaluate their solution using the questions at the bottom of the template.

When students first begin to use Reverse Annotation, the teacher should walk them through the steps individually to ensure they can accurately identify the question, key words, and important information. Teachers can also lead students through the estimation process, making a game out of which student has the closest estimate.

Work through each step individually for several "easy" problems first, so that difficult math doesn't interfere with the process. Increase the problem difficulty incrementally as students begin to master the process. This may seem like a long process at first, but the ultimate result is worth the time investment.

When would I use Reverse Annotation in the classroom?

Reverse Annotation can be used to solve any math problem, and is especially helpful for word problems. When Reverse Annotation is initially implemented, the steps should be discussed in detail. As students become accustomed to Reverse Annotation and begin thinking about problems in this manner automatically, the individual steps become less important and can be scaffolded out to

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improve efficiency. Students should reach the point where they immediately ask themselves the three initial questions when they first see a problem. However, the teacher should ensure that students are truly evaluating all the key information before routine discussions of the individual steps are removed.

How can I use Reverse Annotation with students needing additional support?

Annotation strategies can help students identify key information, even when certain vocabulary words are not known. As teachers introduce the content-specific Tier 3 vocabulary to their classes, annotation strategies such as reverse annotation can help students use these words to apply appropriate strategies while problem solving. Answering the three initial questions can help students organize the key facts and vocabulary, and the identification of key information can simplify the problem. This strategy is especially beneficial for ESL students.

Using reverse annotation with graphic organizers benefits ESL students by removing a lot of the confusing wording and allowing them to focus on the important pieces of a problem. When using Reverse Annotation, all students, including ESL students, will begin to think about problem solving in a way that encourages them to use the appropriate information to find a solution.

What other standards does the Reverse Annotation Protocol address?

Standards of Mathematical Practice:

- SMP.1
- SMP.2
- SMP.5
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.SL.9–10.3
- ELA–LITERACY.SL.9–10.2
- ELA–LITERACY.RST.9–10.4

Source

- Alliance for Excellent Education. "Six Key Strategies for Teachers of English Language Learners." https://uteach.utexas.edu/sites/default/files/files/SixKeyStrategiesELL.pdf

Reverse Annotation Template

Name: ______________________________ **Problem/Assignment:** ______________________________

Analyze the Problem

What is the problem asking us to solve?	
What key words will tell us the mathematical steps we need to perform?	
What information in the problem can help us figure it out?	

Initial estimate of solution:

Work Space

Remember to box in your solution!

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Name: ______________________________ Problem/Assignment: ______________________________

Check It Over

How close was your estimate?	
Does your answer make sense? Is it reasonable? How do you know?	
Did you perform the calculations correctly?	
What does your answer mean in context?	

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Annotation Strategies

CUBES Protocol

What is the annotation strategy CUBES?

CUBES is an annotation strategy in which students use different written designs to highlight the key aspects of word problems. It can help them choose the correct mathematical strategy to solve the problem accurately.

How do you implement CUBES in the classroom?

The steps for CUBES are:

1. **C**: **C**ircle all the key numbers.
2. **U**: **U**nderline the question.
3. **B**: **B**ox in the key words that will determine the operation(s) necessary and write the mathematical symbol for the operation(s).
4. **E**: **E**valuate the information given to determine the strategy needed. **E**liminate any unnecessary information.
5. **S**: **S**olve the problem, **s**how your work, and check your answer.

As students learn to use CUBES, walk them through the steps individually to ensure they can accurately identify the key numbers, question, key words, unnecessary information, and strategy. Work through each step individually for several “easy” problems first, so that difficult math doesn’t interfere with the process. Increase the problem difficulty incrementally as students begin to master the process. This may seem like a long process at first, but the ultimate result is worth the time investment.

A graphic organizer can help students master the process, especially when problems are given on a computer or tablet where students can’t always annotate directly on the problem. Students can write down the key numbers and circle them, write down the question and underline it, and so on. This will encourage students to truly think about the different pieces of the problem they are identifying, and how these pieces will guide the strategy and affect the solution.

When would I use CUBES in the classroom?

CUBES can be used to solve any math problem, and is especially helpful for word problems. When CUBES is initially implemented, the steps should be discussed in detail. As students become accustomed to using CUBES and begin thinking about problems in this manner automatically, the individual steps become less important and can be scaffolded out to improve efficiency. However, the teacher should ensure that students are truly evaluating all the key information before routine discussions of the individual steps are removed.

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How can I use CUBES with students needing additional support?

Design features can help students identify key words and features, even when certain vocabulary words are not known. As teachers introduce the content-specific Tier 3 vocabulary to their classes, annotation strategies such as CUBES can help students use these words to apply appropriate strategies while problem solving. Using circles, underlines, and boxes can help students organize the key facts and vocabulary, and the elimination of unnecessary information can simplify the problem. This strategy is especially beneficial for ESL students.

Combining CUBES with graphic organizers also benefits ESL students by removing a lot of the confusing wording and allowing them to focus on the important facts of a problem. When using CUBES with a graphic organizer, all students, including ESL students, will begin to think about problem solving in a way that helps encourage them to use the appropriate information to find a solution.

What other standards does the CUBES Protocol address?

Standards of Mathematical Practice:

- SMP.1
- SMP.2
- SMP.5
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.SL.9–10.3
- ELA–LITERACY.SL.9–10.2
- ELA–LITERACY.RST.9–10.4

Source

- Margaret Tibbett. “Comparing the effectiveness of two verbal problem solving strategies: Solve It! and CUBES.”

  https://rdw.rowan.edu/cgi/viewcontent.cgi?article=2633&context=etd

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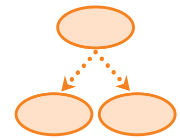
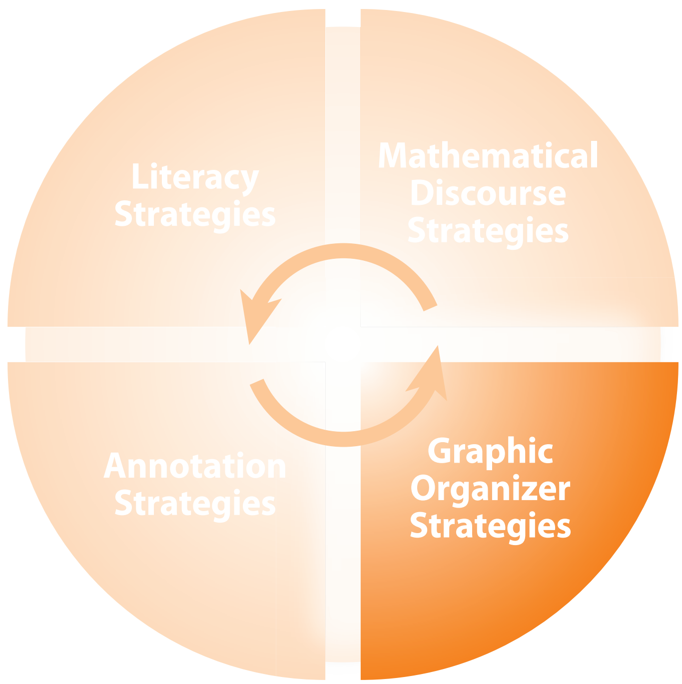
Organizing Mathematical Content: Graphic Organizers



Organizing mathematical content is a crucial skill for problem solving, exploring other possible methods for finding solutions, and managing math content. All students need strategies for organizing content to build conceptual understanding. For students with disabilities, visual representations and graphic organizers can help them clarify their thoughts and focus on the math. ESL students also benefit from visual representations and graphic organizers. Organizing mathematical knowledge with visuals can help ESL students navigate math content while learning the language. Graphic organizers include Frayer Models and Tables of Values.



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Graphic Organizers

Frayer Models

What is a Frayer Model?

A Frayer Model is a graphic organizer that can help students understand new vocabulary words and concepts by exploring their characteristics. A Frayer model lists the definition of a word or concept, describes some key facts, and gives examples and non-examples. Examples and non-examples can come from a mathematical or real-world context.

How do you implement Frayer Models in the classroom?

Students can learn to create Frayer Models the first week of school, and the process can be used throughout the year each time students experience a new word or concept.

While it is important for teachers to give students precise mathematical definitions with appropriate content vocabulary, it is maybe more important for students to understand the application of mathematical words and concepts in their own context. As students learn new information, small group discussions and think-pair-share activities are great ways for students to formulate their own definitions, review the characteristics and facts they have learned, and discuss examples and non-examples.

Discussions of the examples and non-examples can help lead to the mathematical definition. For example, if students use a Frayer Model to define a quadratic function, they would notice that all examples have a highest exponent of 2, and all non-examples would not have a highest exponent of 2. All examples would have parabolic graphs, and all non-examples would have other graphs. Through these comparisons, students will understand the definition of quadratics using different representations, and they will be able to apply it in different contexts.

When would I use Frayer Models in the classroom?

Frayer Models can be used at different points during instruction. They are appropriate as introductions to new concepts, summaries to ensure understanding of new concepts, or as note-organizers throughout the lesson for students to fill in as they learn new concepts. At first, students might need help figuring out how to list and differentiate between the definition, facts and characteristics, examples, and non-examples. As students adapt to the process, they will be able to categorize information on their own or in small groups. As they compare newer Frayer Models to previous models, they will also be able to see how concepts build upon each other.

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How can I use Frayer Models with students needing additional support?

Frayer Models can be a point of reference for students as they progress throughout the year. As students determine their own definitions for math-specific words and concepts, and use the examples and non-examples to determine the key facts, they will be able to put them in their own context and apply them to solve complicated problems. As math concepts build upon each other both within a unit and throughout the year, the use of Frayer Models to remind students of their initial definitions of words or concepts can help solidify their understanding. Using Frayer Models as part of a Word Wall or Concept Wall, or having a consistent notebook process to reference past Frayer models, can help consistently reinforce learning.

What other standards do Frayer Models address?

Standards of Mathematical Practice:

- SMP.1
- SMP.2
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.1
- ELA–LITERACY.SL.9–10.1
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4

Source

- Deborah K. Reed. "Building Vocabulary and Conceptual Knowledge Using the Frayer Model." https://iris.peabody.vanderbilt.edu/module/sec-rdng/cresource/q2/p07/

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Frayer Model

Definition	Characteristics
WORD	
Examples from Life	Non-Examples

Graphic Organizers

Tables of Values

What is a Table of Values?

A Table of Values is an organized way to list numbers that represent different categories of values. These values can be represented as ordered pairs, graphs, word problems, or lists. Tables can help students see and compare values in a different way.

How do you implement Tables of Values in the classroom?

Tables can be used throughout the year to support various mathematical standards. Some standards mention tables specifically, and in others, tables can be an effective support to help students organize and understand the meaning and application of values.

Tables can be set up with numerical values in rows or columns. The key to understanding the values lies in the headings. The headings must be specific enough to show students the meaning and/or application of the numerical values, but not so wordy that they interfere with the clarity of the numbers in the table. For example:

x (year)	*y* (population in millions)
1960	219
1970	230
1980	258
1990	312
2000	342

Mean (statistical average)	50	45
Median (middle value)	52	43
Quartile 1 (median of the lower 50%)	40	38
Quartile 3 (median of the upper 50%)	72	80
Range (difference of max and min values)	80	61
Interquartile Range (difference of quartiles)	32	42
Standard Deviation (measure of spread of data)	7.24	10.23

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When would I use Tables of Values in the classroom?

Various mathematical topics can be represented by tables. For example:

- An (x, y) table of values to represent coordinates on a graph or independent and dependent variables for a given context
- A table to represent coefficients and/or constants in an equation
- A table to show different statistical measures when comparing sets of data
- A table to compare output values for the same input given different functions

Each time numbers or values are being listed, compared, or graphed, a table can help students differentiate between the values. Tables are easy to create, and students can be encouraged to create them as another representation to clarify and compare numbers for nearly any topic.

How can I use Tables of Values with students needing additional support?

Tables of Values can help students focus on numerical values and their meaning in context without distraction. They clarify what each number represents, what numbers can be compared, and what ordered pairs can be graphed to give a visual representation. Additionally, headings can be used to either highlight the relevant facts from a context or to describe mathematical vocabulary.

In general, graphic organizers benefit students by removing much of the confusing wording and focusing on the important facts and numbers of a problem.

What other standards do Tables of Values address?

Standards of Mathematical Practice:

- SMP.1
- SMP.2
- SMP.6

WIDA English Language Development Standards:

- ELD Standard 3

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.1
- ELA–LITERACY.SL.9–10.1
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4

Source

- Alliance for Excellent Education. "Six Key Strategies for Teachers of English Language Learners." https://uteach.utexas.edu/sites/default/files/files/SixKeyStrategiesELL.pdf

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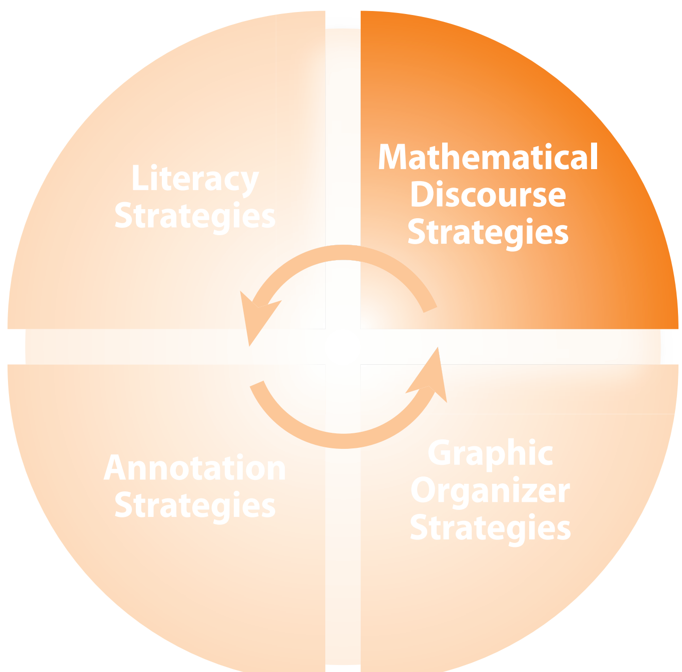
Communicating Mathematical Content: Mathematical Discourse

Reading, writing, speaking, and listening are all important ways to learn and express information, but the last two ways are often slighted in the math classroom. The mathematical discourse strategies presented here promote speaking and listening in a math-focused literacy context. Working these strategies into the daily routine of a classroom can help students become comfortable speaking and listening in a mathematical context, which will help them become comfortable with the mathematical content. Routines and structures are essential to support students with disabilities, as they often benefit from following a routine. This can lead to developing capability in their mathematical skills. These strategies also remove the barrier to entry for many ESL students, as structure and routine can help them focus on the math content rather than English language deficiencies. Mathematical Discourse strategies include Sentence Starters and Small Group Discussion.



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Mathematical Discourse Strategies

Sentence Starters

What is a Sentence Starter?

A Sentence Starter is a common phrase or mathematical sentence frame that can help students begin and sustain academic conversations around mathematical content. It helps guide students through the discussion and bring out pertinent ideas that can lead to greater understanding.

How do you implement Sentence Starters in the classroom?

Many people view math class as a place to calculate solutions to math problems. However, to ensure the conceptual understanding and proper application of a math concept, students need to be able to explain the concepts and reasoning behind a solution to a problem. As many students are not accustomed to having academic conversations about math, sentence starters can help begin and continue these conversations in a productive manner.

There are two main types of sentence starters for mathematical discussions: discourse starters and math starters. For example, a poster with these or other sentence starters can be displayed from the beginning of the year, and the expectation can be set that any answer to a question or comment in a discussion should be framed using one of these starters. As students become accustomed to framing mathematical conversations in this way, they can expand on the given sentence starters and create some of their own. They will begin to realize how these statements ensure that their conversations revolve around math, enhance understanding of the concept, and force them not only to state, but also to explain their thinking. They will gain confidence from the ability to engage, as the first step has already been taken for them.

When would I use Sentence Starters in the classroom?

Sentence Starters can be used throughout the entire school year with any concept. However, they are most important to use at the beginning of the school year to build a mathematical community in the classroom centered on a comfort with mathematical discourse. Especially at the beginning of the year, students should be encouraged to use these sentence starters for every math statement. Appropriate settings include during small group discussion, while responding to whole class questions, and when writing explanations for problem solutions.

Modifications can be introduced so that students must use certain mathematical vocabulary within the sentences, or must use certain sentence starters at different points in conversations or for different conversation types and situations. However the starters are implemented, it is important for students to realize that these are intended to enhance and focus their conversations, not limit them.

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How can I use Sentence Starters with students needing additional support?

Often, students are reluctant to talk about math concepts because they either lack confidence in their knowledge, are afraid to be "wrong," or don't know how to start or continue the conversation. Sentence starters can help students overcome this reluctance. The non-threatening, easy-to-interpret sentence starters remove the barrier to entry for students who don't know how to engage, and the respectful, mathematical focus promoted by sentence starters can help build confidence and provide a structure so that students will not fear being wrong.

For ESL students specifically, sentence starters can provide the English language support to help students engage with and discuss the math. The support of sentence structure removes language barriers to entry for students who don't fully understand English sentence structure.

Discourse Starters	Math Starters
I agree/disagree with ... because ... I understand/don't understand ... First/Next/Finally I ... because ... I noticed that ... I wonder ...	My answer was ... because ... The next step is ... because ... I used (insert formula/equation/concept) because ... My answer is right/reasonable because ...

What other standards do Sentence Starters address?

WIDA English Language Development Standards

- ELD Standard 3

Standards of Mathematical Practice:

- SMP.1
- SMP.3
- SMP.6

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.1
- ELA–LITERACY.SL.9–10.1
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4

Source

- AVID. "Sentence Starters."
  https://sweetwaterschools.instructure.com/files/29100523/download?download_frd=1&verifier=CBvje9CPNKUe6IkN4TPBJDuXmZY3464aTTK1Fk2r math sentence starters research

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Mathematical Discourse Strategies

Small Group Discussion

What is Small Group Discussion?

Small Group Discussion is a structured way for students to verbalize their mathematical thinking in a comfortable setting to solve a problem, build conceptual understanding, or summarize a concept.

How do you implement Small Group Discussion?

Small Group Discussion in math class depends on a trusting relationship between the teacher and the students. From there, students can build trusting relationships among themselves. Once this trust has been built, students will feel free to explore mathematical topics in groups, take risks, and engage in a productive struggle toward understanding or a solution.

Once these relationships have been established, certain structures should be established for Small Group Discussion to be effective. Discussion norms can be set by the class to ensure discussions are respectful and productive, and discussions should have predetermined time limits. The group composition is also important and should be based on instructional measures. For different activities, homogeneous groups, heterogeneous groups, or groups based on specific data by standard could be appropriate. Students should always be aware that the groups were chosen to maximize their learning.

Another structure that can be effective for Small Group Discussion is assigning group roles. These roles can include group leader, note taker, timekeeper, resource manager, culture keeper, or other roles determined to be appropriate for the classroom context. During the discussion, assigning each student a letter within the group (A, B, C, D, etc.) can help structure the discussion. Different roles can specify certain time limits for talk, which sentence starters to use, or other structured aspects of the discussion.

When implementing a Small Group Discussion, the question or task should inspire students to think in different ways about a concept. Through the structured format of the discussion, students will compare their ideas and arrive at an answer or explanation of the concept. Within the trusting framework of the class and group, students can focus on the common goal of the discussion and develop their thinking around the math concept. These rich discussions will enhance their understanding.

When would I use Small Group Discussion in the classroom?

Small Group Discussion can be used for nearly any topic, and it can be used at a variety of times in the classroom. The questions and tasks may need to change depending on when it is used. Opening activities for lessons can be Small Group Discussions where students explore properties of new math concepts or review/build upon their prior learning. Turn and talks throughout the lesson can be structured as Small Group Discussions if a consistent framework is in place. At the end of class, a Small Group Discussion can be used to come to a common understanding about an essential question from the lesson.

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Depending on when the Small Group Discussion is used in class, and what the goal of the discussion is, the discussion reporting may vary. For a warm-up, each group might be asked to share their thinking. For a guided practice, recording answers on chart paper and a gallery walk could be appropriate. For a closing activity, individual written responses to a question could be appropriate.

How can I use Small Group Discussion with students needing additional support?

As discussed in other Mathematical Discourse strategies, struggling students are reluctant to talk about math concepts because they lack confidence in their knowledge and don't always have the needed vocabulary in their toolbox. Structured discussions with effective grouping can help students through these barriers. After a trusting and respectful classroom environment has been established, struggling students often feel more comfortable sharing their ideas with just a few classmates rather than the whole class. Additionally, adding structure can help students engage by providing the expectation that they participate in the process.

The intentional grouping of students can also help them succeed using Small Group Discussion. At times, heterogeneous groups could be appropriate so that stronger students can help struggling students, and at other times, homogeneous groups could be appropriate so the teacher can work with an entire group of struggling students. ESL students can be grouped with other students with the same dominant language to help remove the language barrier from the conversation.

What other standards does Small Group Discussion address?

WIDA English Language Development Standards:

- ELD Standard 3

Standards of Mathematical Practice:

- SMP.1
- SMP.3
- SMP.6

Language Arts Standards:

- ELA–LITERACY.WHST.9–10.4
- ELA–LITERACY.WHST.9–10.1
- ELA–LITERACY.SL.9–10.1
- ELA–LITERACY.SL.9–10.4
- ELA–LITERACY.RST.9–10.3
- ELA–LITERACY.RST.9–10.4

Source

- Jessie C. Store. "Developing Mathematical Practices: Small Group Discussions."
  https://kb.osu.edu/dspace/bitstream/handle/1811/78055/OJSM_69_Spring2014_12.pdf

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PROGRAM OVERVIEW

Graphic Organizers

Overview

Graphic organizers can be a versatile tool in your classroom. Organizers offer an easy, straightforward way to visually present a wide range of material. Research suggests that graphic organizers support learning in the classroom for all levels of learners. Gifted students, students on grade level, and students with learning difficulties all benefit from the use of graphic organizers. They reduce the cognitive demand on students by helping them access information quickly and easily. Using graphic organizers, learners can understand content more clearly and can take concise notes. Ultimately, learners find it easier to retain and apply what they've learned.

Graphic organizers help foster higher-level thinking skills. They help students identify main ideas and details in their reading. They make it easier for students to see patterns such as cause and effect, comparing and contrasting, and chronological order. Organizers also help students master critical-thinking skills by asking them to recall, evaluate, synthesize, analyze, and apply what they've learned. Research suggests that graphic organizers contribute to better test scores because they help students understand relationships between key ideas, and enable them to be more focused as they study.

Types of Graphic Organizers

There are four main purposes for using graphic organizers in mathematics and a variety of tools within each category:

Purpose 1: Organizing, Categorizing, and Classifying	**Purpose 2: Problem Solving**	**Purpose 3: Understanding Mathematical Information**	**Purpose 4: Communicating Mathematical Information**
Tables Flowcharts Webs Venn Diagrams	Number Lines Geometric Drawings Factor Trees Attribute Tables Cause and Effect Maps Coordinate Plane Probability Trees	Frayer Model Semantic Map/ Concept Map Compare-and-Contrast Diagram	Line Graphs Bar Charts

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Tables

A table is simply a grid with rows and columns. Tables are useful because information stored in a table is easy to find—much easier than the same information embedded in text.

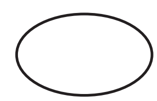
Usually, a table has a row (horizontal) for each item being listed. The columns (vertical) provide places for details about the listed items—the things they have in common. The places where the rows and columns meet are called cells. In each cell, we write information that fits both the topic of the row (the thing being listed) and the topic of the column (the aspect being examined). To create a table, we make rows and columns to fit the number of items and attributes.

Flowcharts

Flowcharts are graphic organizers that show the steps in a process. Flowcharts can be very simple—just a series of boxes with one step in each box. However, there is also a more formal type of flowchart. These flowcharts use special symbols to show different things, such as starting and stopping points, or points where decisions must be made. These symbols make flowcharts especially useful for showing complicated processes.

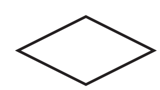
Each step in a flowchart is written in a box. The boxes are connected by arrows to show the sequence of steps. The boxes aren't all rectangular; different shapes are used to indicate different actions. The shapes and symbols are a kind of visual shorthand. Whenever a certain symbol is used, it always has the same meaning.

- Circles and ovals show starting and stopping points. They often contain the words start or stop. The "start" circle or oval has no arrows in and one arrow out. The "stop" circle or oval has one arrow in and no arrows out.



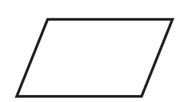
- Arrows show the direction in which the process is moving.

- Diamonds show points where a decision must be made or a question must be answered. The question can usually be answered either "yes" or "no."



- Rectangles and squares show steps where a process or an operation takes place.

- Parallelograms show input or output, such as writing or printing a result or solution.



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Webs

Webs are graphic organizers that help take notes, identify important ideas, and show relationships between and among pieces of information. In a web, the main idea is written in the center circle. Details are recorded in other circles with lines to connect related topics. Circles or lines can be added or deleted as necessary.

Number Lines

In its simplest form, a number line is any line that uses equally spaced marks to show numbers. Number lines are used to visualize equalities and inequalities, positive and negative numbers, and measurements of all kinds. They can "map" math problems, especially ones that involve negative numbers or distances.

Geometric Drawings

A geometric drawing is a representation on paper (or some other surface) of a geometric figure. The geometric drawings we make can never be as perfect as the geometric figures they represent, but as long as they are reasonably accurate, they can help us visualize the figures. In fact, it's often impossible to solve a geometry problem without making a drawing.

Factor Trees

There are several ways to find factors. One that helps to visually keep track of all the factors is called a factor tree. This is a diagram with a tree-like shape. It uses "branches" to show the factors of a number.

All whole numbers other than 1 can be written as the product of factors. A prime number is a number that has only two factors, itself and 1. An example of a prime number is 13. Its only factors are 13 and 1. A composite number is a number that has more than two factors. An example of a composite number is 6. Its factors include 6, 3, 2, and 1. Prime factors are factors that are also prime numbers. The greatest common factor (GCF) of two numbers is the largest number that is a factor of both numbers.

Coordinate Plane

This is the plane determined by a horizontal number line, called the x-axis, and a vertical number line, called the y-axis, intersecting at a point called the origin. A coordinate plane can be used to illustrate locations and relationships using ordered pairs of numbers.

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Venn Diagrams

A set is a list of objects in no particular order. Items in a set can be numbers, but they can also be letters or words. Venn diagrams are a visual way of showing how sets of things can include one another, overlap, or be distinct from one another.

Venn diagrams are often used to compare and contrast things. But they are also a useful tool to sort and classify information. You can use Venn diagrams to take notes on material that shows relationships between things or ideas. You can also use them to solve certain types of word problems. When a word problem names two or three different categories and asks you how many items fall into each category, a Venn diagram can be a useful problem-solving tool.

A Venn diagram begins with a rectangle representing the universal set. Then each set in the problem is represented by a circle. Circles can be separate, overlapping, or one within another. When two circles overlap, it means that the two sets intersect. Some members of one set are also members of the other set.

Venn Diagrams AND **Compare-and-Contrast Diagrams**

The Venn diagram is an organizing device for planning comparisons and contrasts. A completed Venn diagram helps students categorize and organize similarities and differences, and provides a blueprint for a comparison-and-contrast exercise. The compare-and-contrast diagram provides a structure to identify or list similarities and differences between two objects.

Attribute Tables

To solve logic problems, you need a way to keep track of the subjects and which attributes they have or don't have. An attribute table can help. This is a table with a row for each subject in the problem, and a column for each attribute. The rows and columns meet to form cells. Because the attributes in logic problems are usually exclusive, you can use Xs or check marks (✔) to show which attribute belongs to which subject.

Cause and Effect Maps

Cause and effect maps help you work through information to make sense of it. Write each cause in the oval. Write all its effects in the boxes. Add or delete ovals and boxes as needed.

Frayer Model

The Frayer Model is a word categorization activity that helps learners to develop their understanding of concepts. Using this model, students provide a definition, list characteristics, and provide examples and non-examples of the concept.

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Semantic Map

A semantic word map allows students to conceptually explore their knowledge of a new term or concept by mapping it with other related words, concepts, or phrases that are similar in meaning. Semantic maps portray the schematic relations that compose a concept. It assumes that there are multiple relations between a concept and the knowledge that is associated with the concept.

Line Graphs

Line graphs are often used to show how things change over time. They clearly show trends in data and can let you make predictions about future trends, too. Line graphs use two number lines, one horizontal and one vertical. The horizontal number line is called the *x*-axis. The vertical line is called the *y*-axis. The *x*-axis often shows the passage of time. The *y*-axis often shows a quantity of some kind, such as height, speed, cost, and so forth.

Bar Charts

Bar charts are useful when you want to compare things or to show how one thing changes over time. They are a good way to show overall trends. Bar charts use horizontal or vertical bars to represent data. Longer bars represent higher values. Different colors can be used to show different variables. When you look at a bar chart, it's easy to see which element has the greatest value—the one with the longest bar.

Bar charts have an *x*-axis (horizontal) and a *y*-axis (vertical). If the graph is being used to show how something changes over time, the *x*-axis has numbers for the time period. If the graph is being used to compare things, the *x*-axis shows which things are being compared. The *y*-axis has numbers that show how much of each thing there is.

Probability Trees

When we have probability problems with many possible outcomes, or events that depend on one another, probability trees can help. Probability trees show all the possible outcomes of an event. Whenever a problem calls for figuring out how many possible outcomes there are, and the probability that any one of them will happen, a probability tree can be useful.

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Table

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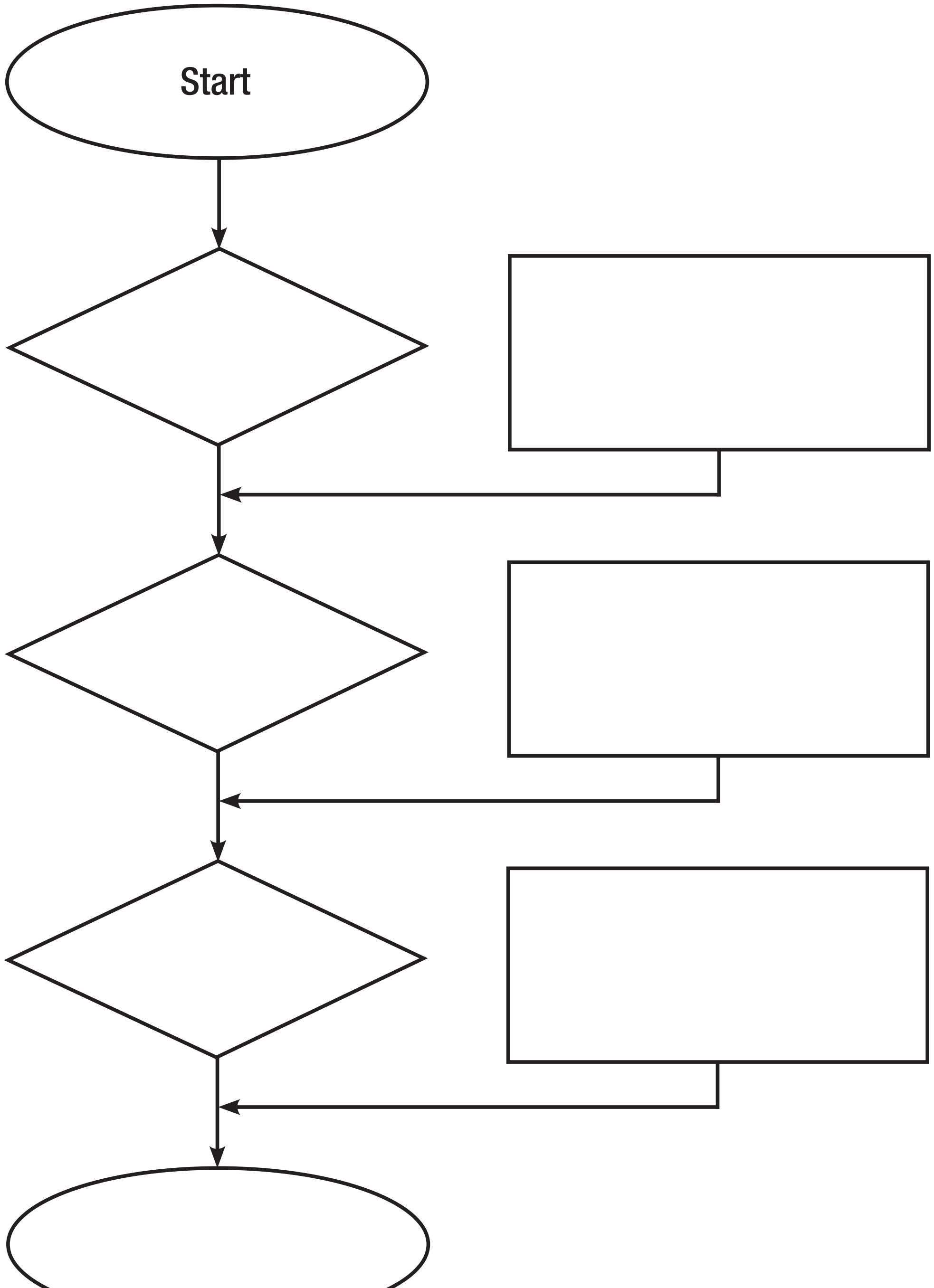
Flowchart



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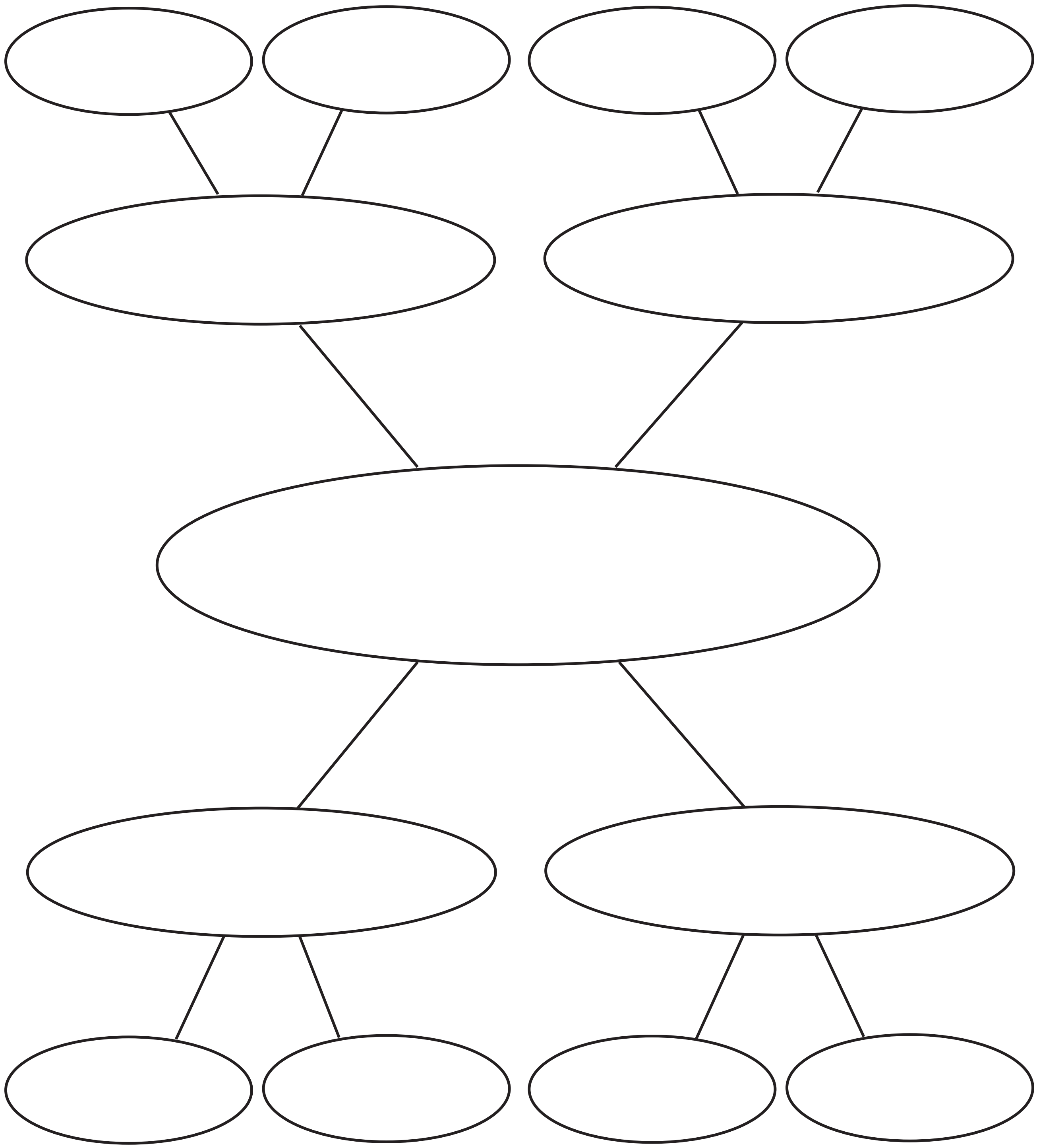
Web



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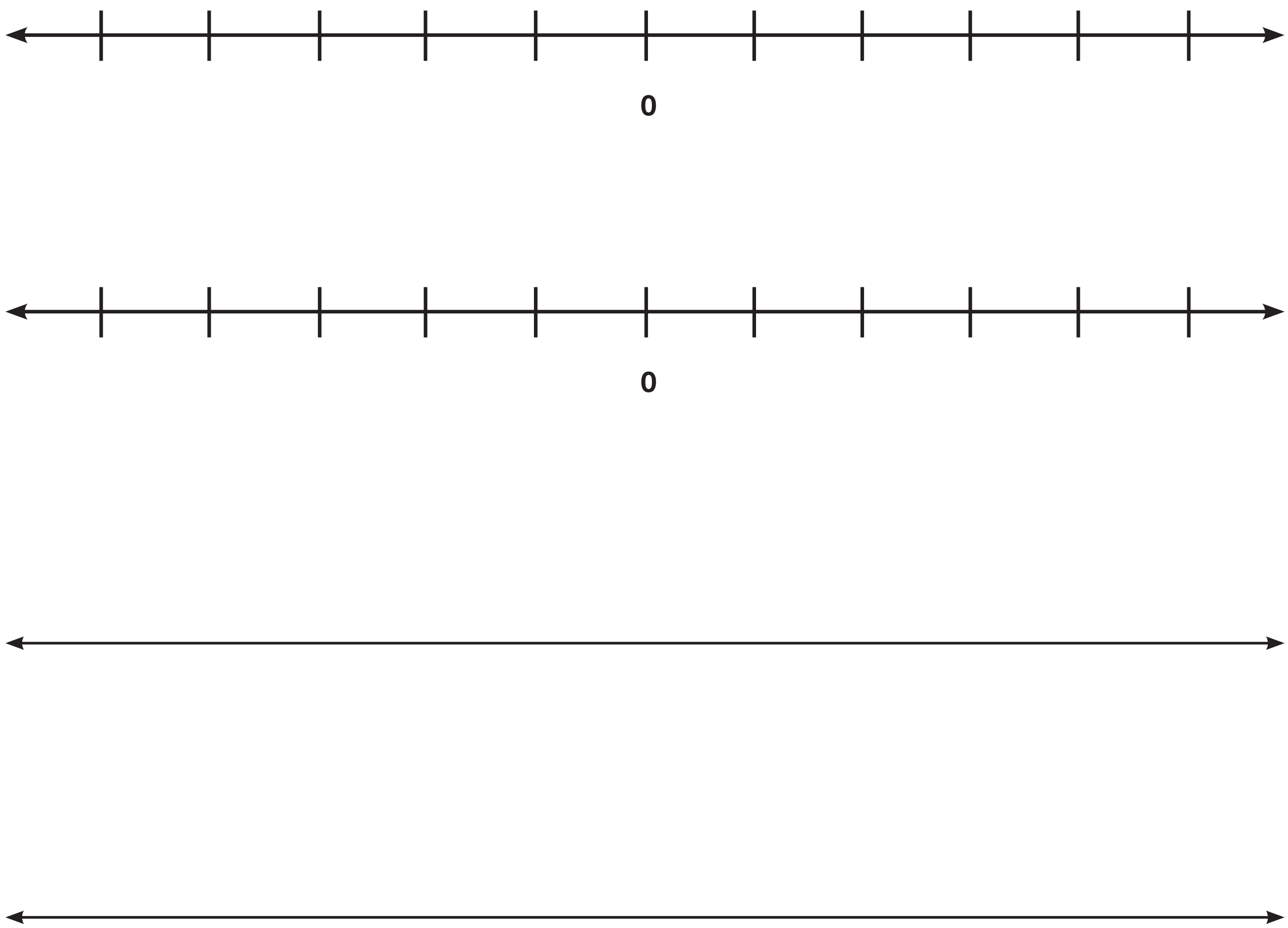
Number Line



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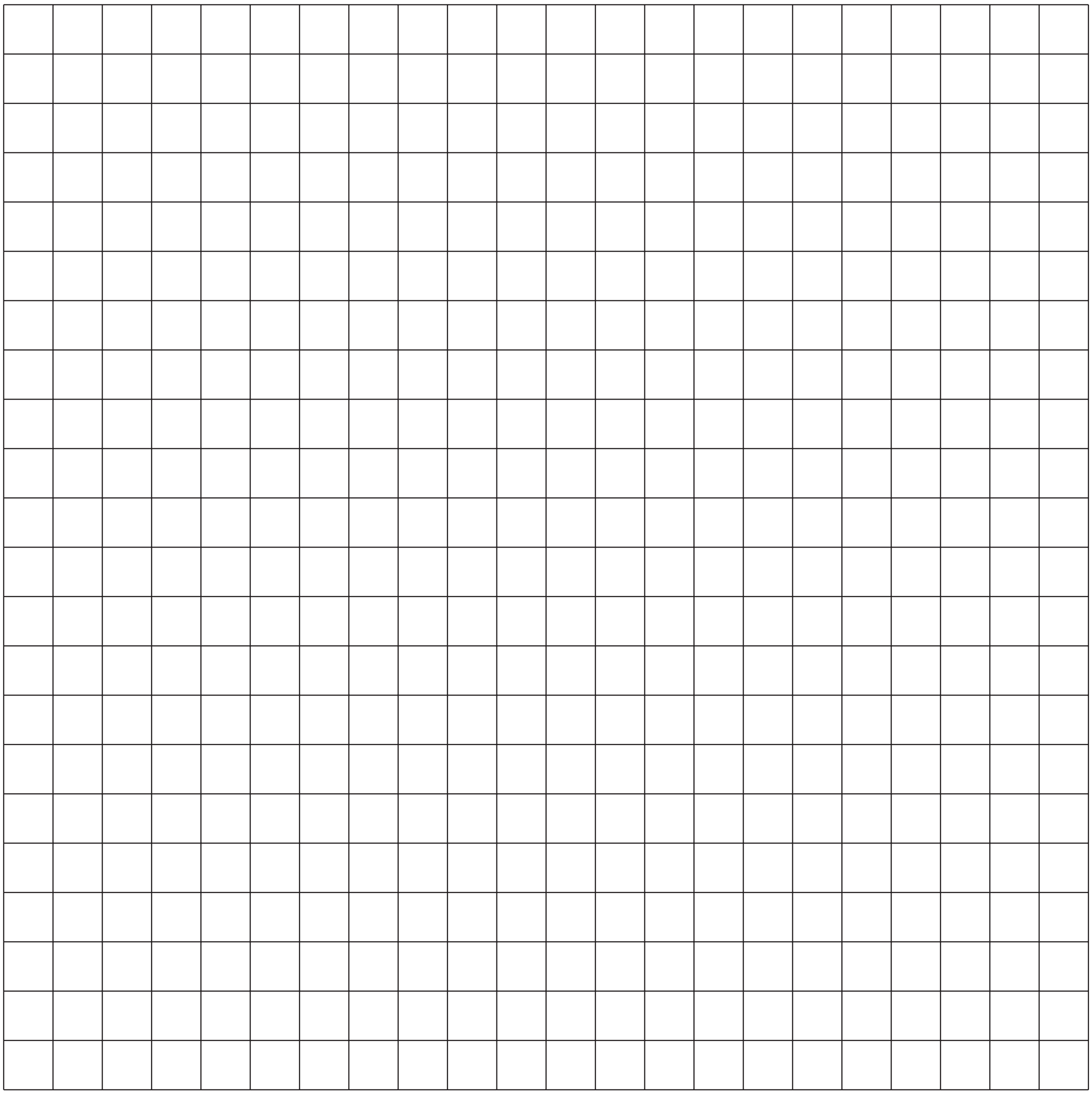
Geometric Drawing



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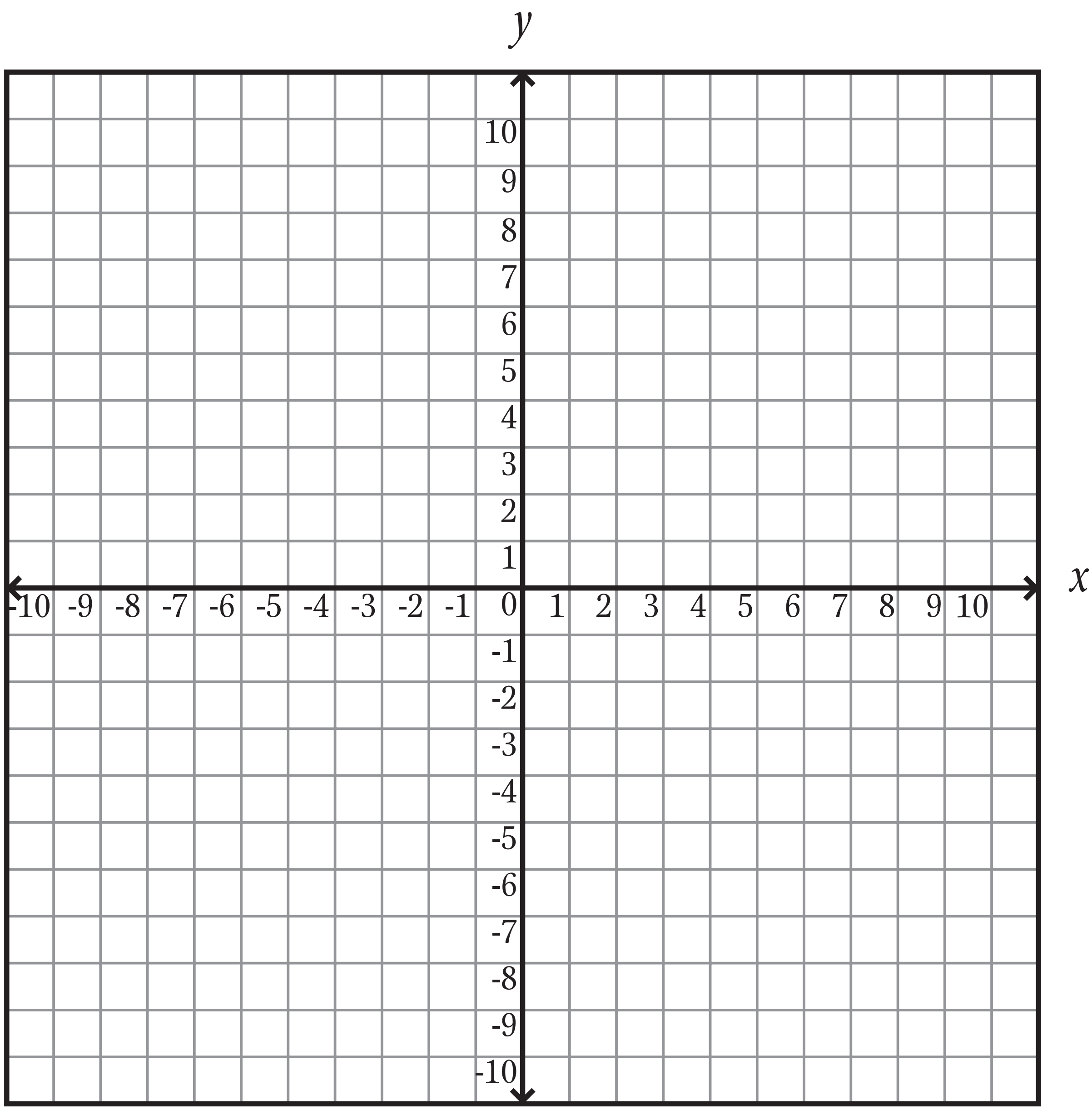
Coordinate Plane



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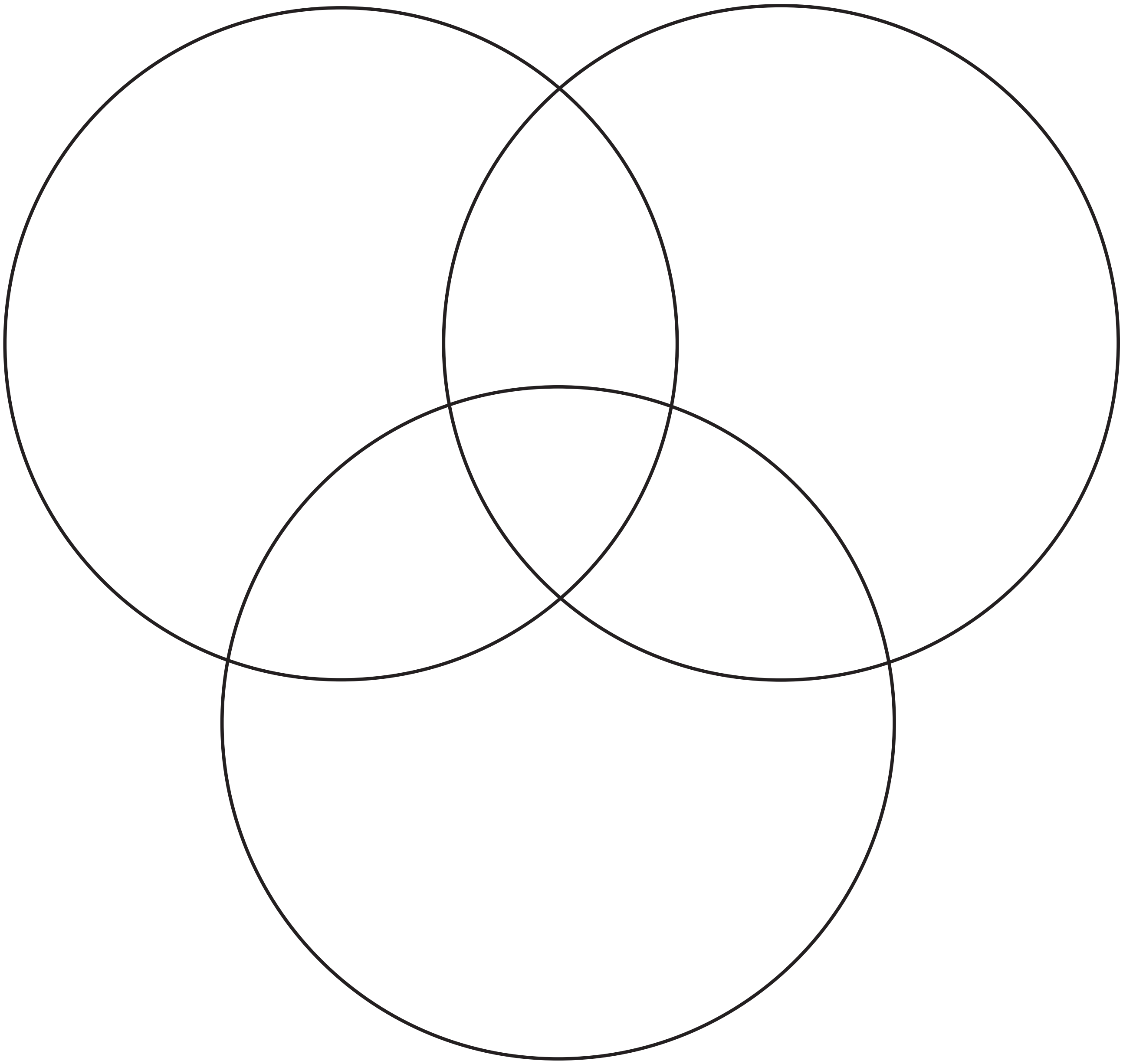
Venn Diagram



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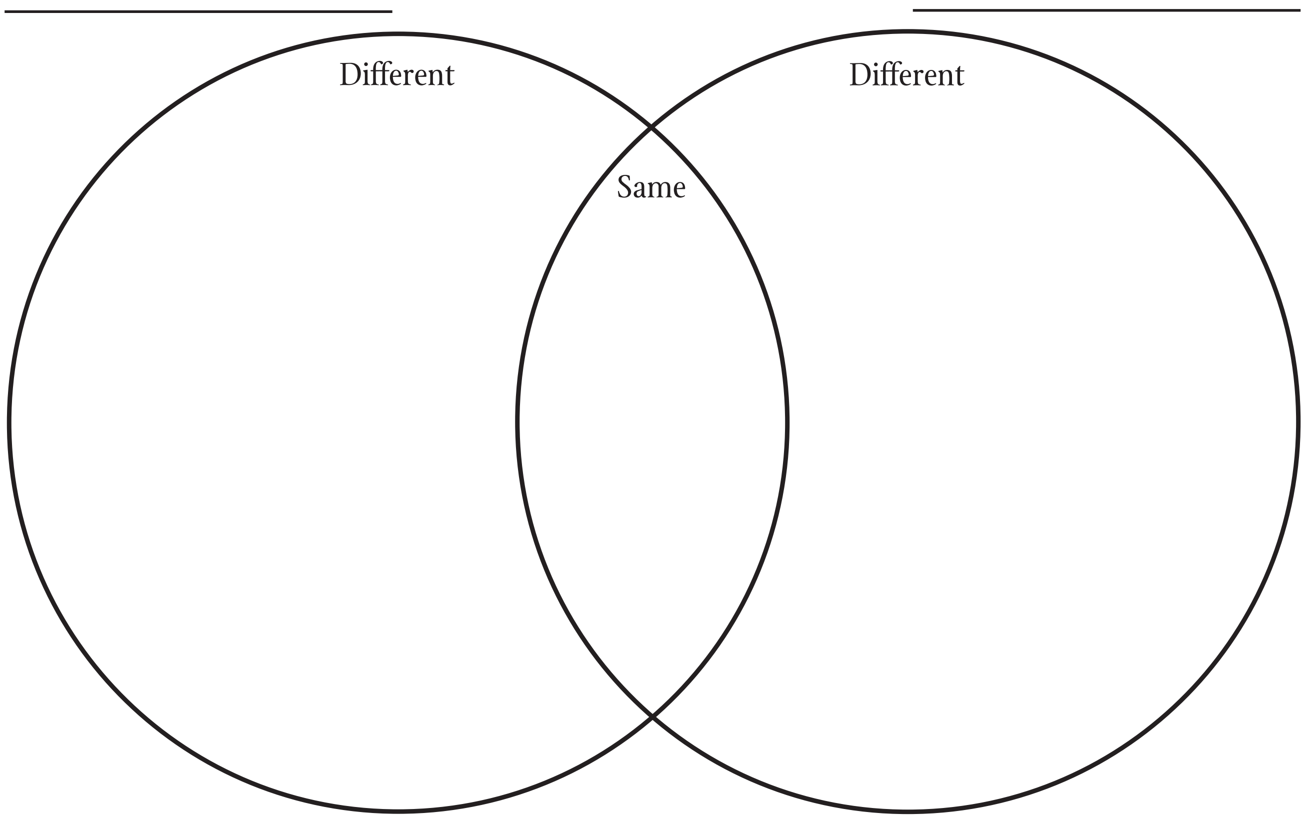
Venn Diagram



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Compare-and-Contrast Diagram

Item 1 ______________________________ Item 2 ______________________________

How Alike?

How Different?

With Regard To

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Attribute Table

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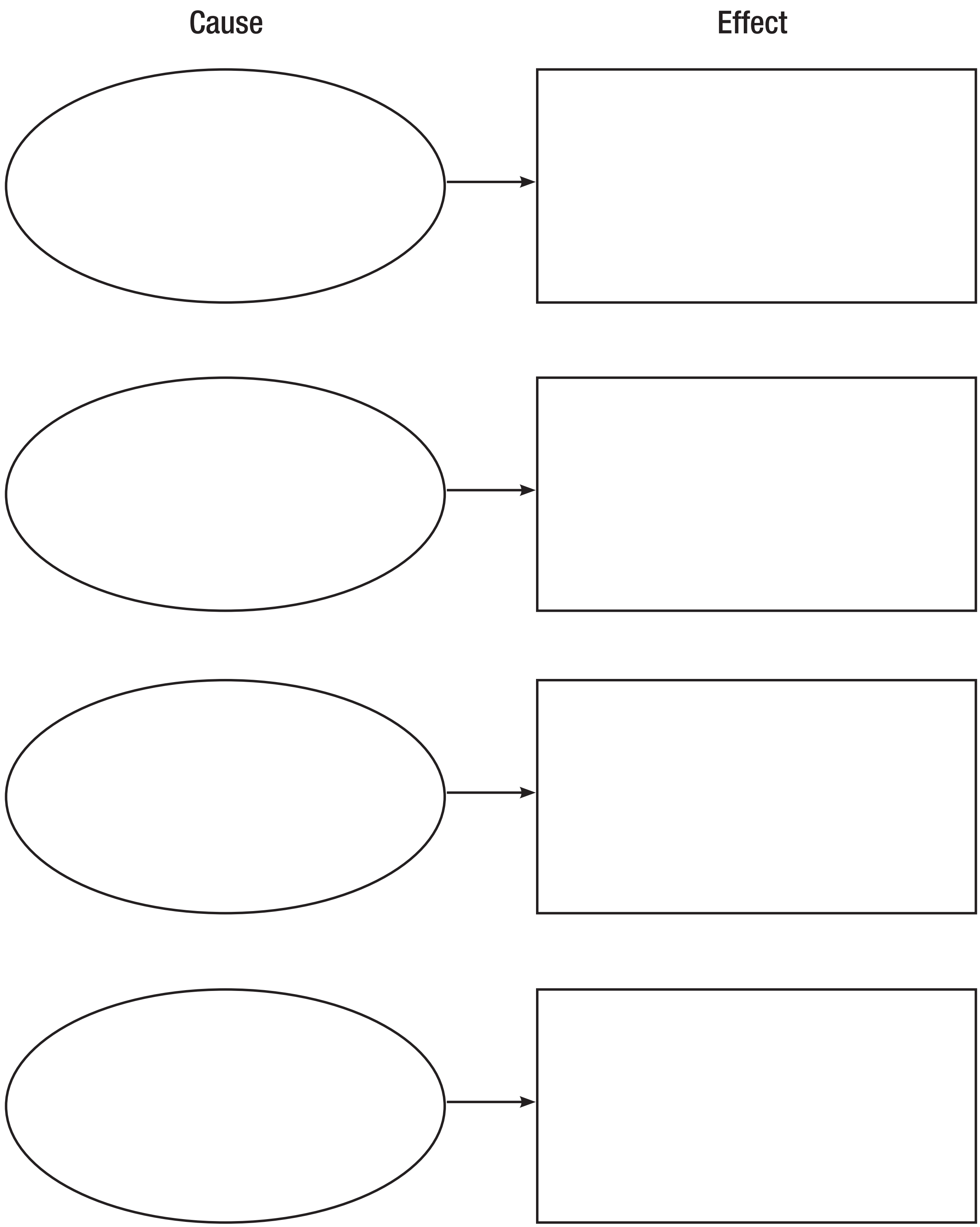
Cause and Effect Map



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Frayer Model

Definition	Characteristics
WORD	
Examples from Life	Non-Examples

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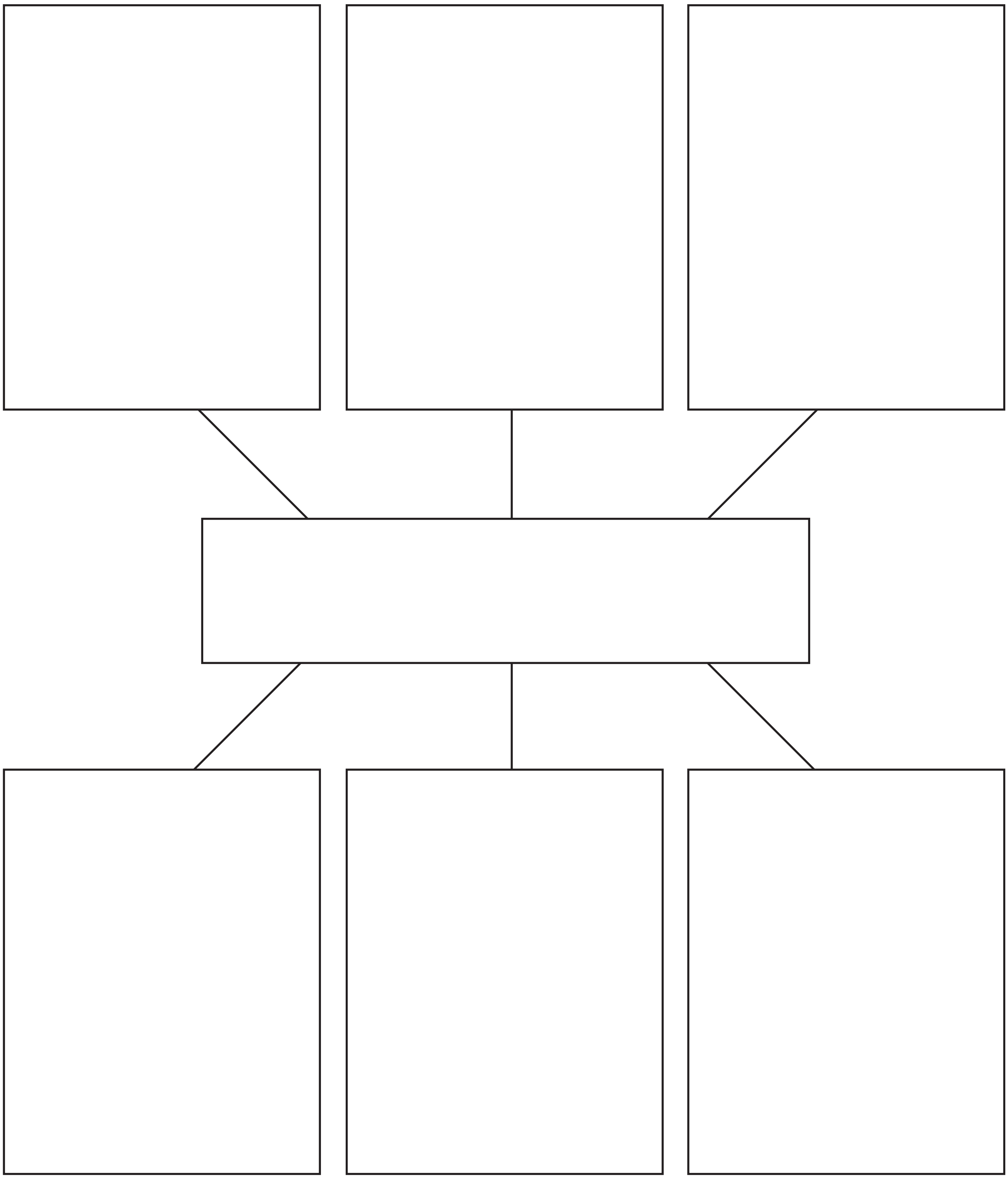
Semantic Map/Concept Map



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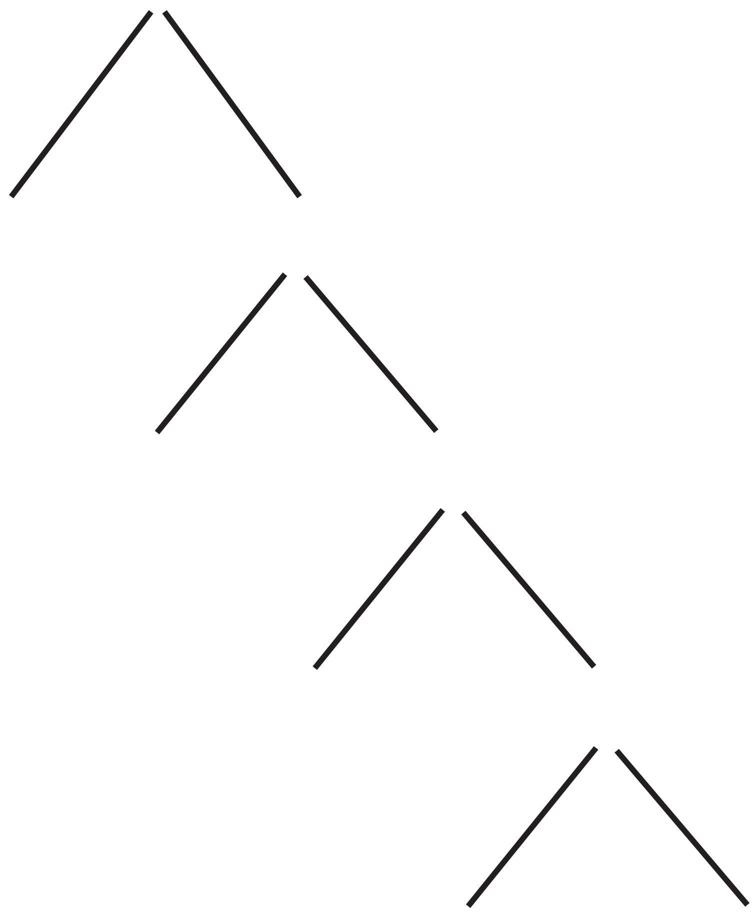
Factor Tree



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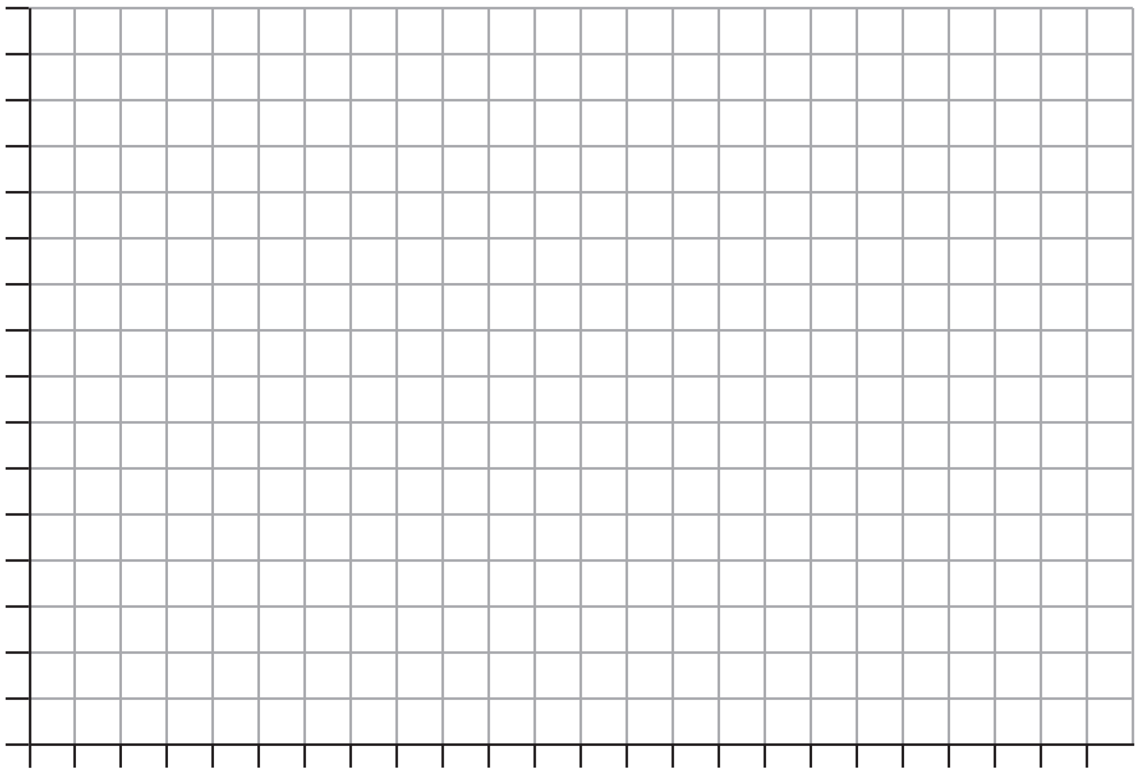
Line Graph

Graph title ______________________

Axis title ______________________



Axis title ____________________

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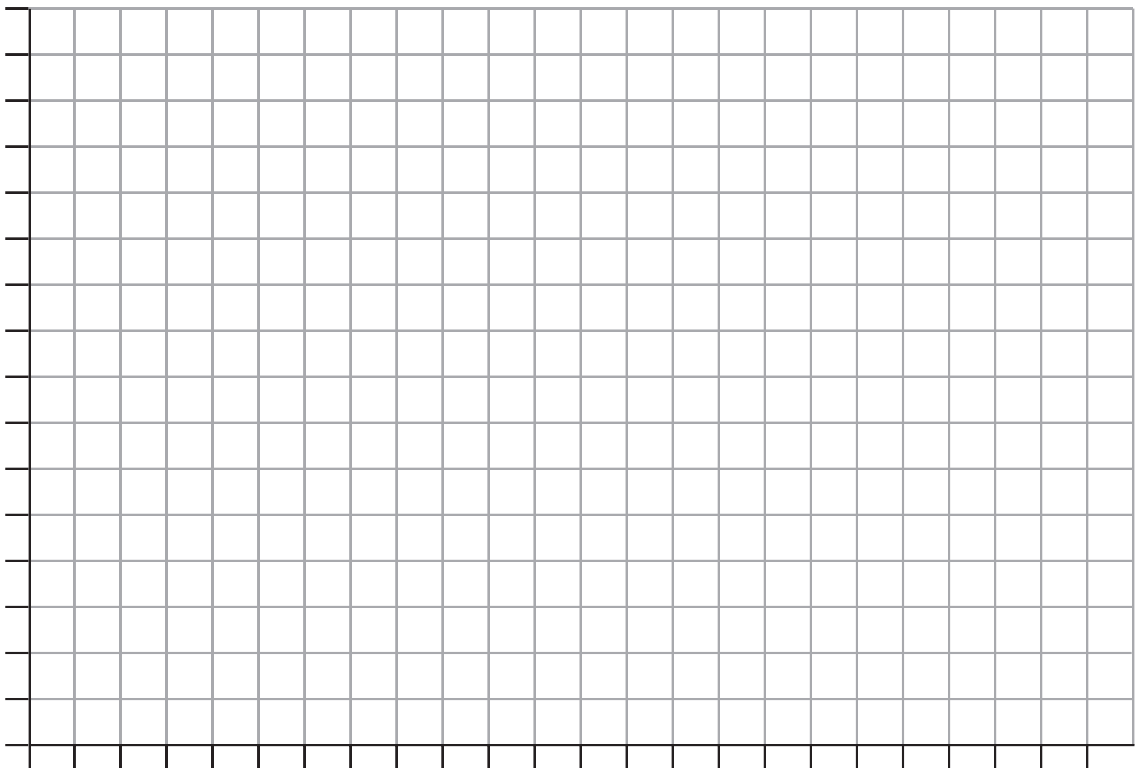
Bar Chart/Histogram

Graph title ______________________

Axis title ______________________



KEY

- ☐
- ☐
- ☐
- ☐

Axis title ______________________

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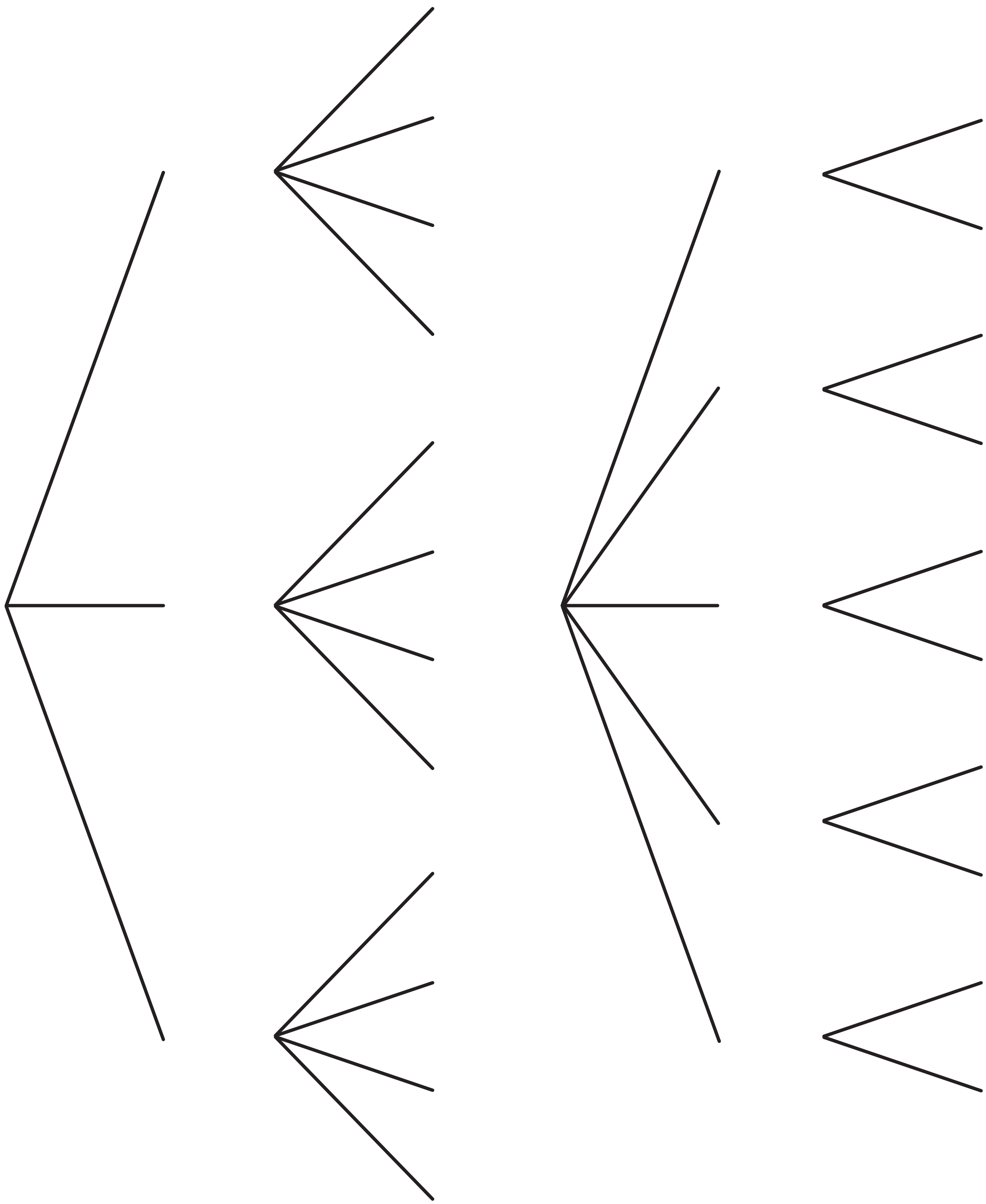
Probability Trees



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Formulas

ALGEBRA

Symbols	
$\approx$	Approximately equal to
$\neq$	Is not equal to
$\lvert a \rvert$	Absolute value of a
$\sqrt{a}$	Square root of a
∞	Infinity
[	Inclusive on the lower bound
]	Inclusive on the upper bound
(	Non-inclusive on the lower bound
)	Non-inclusive on the upper bound
Σ	Sigma
Δ	Delta

Linear Equations	
$m=\dfrac{y_2-y_1}{x_2-x_1}$	Slope
$y=mx+b$	Slope-intercept form
$ax+by=c$	General form
$y-y_1=m(x-x_1)$	Point-slope form

General	
(x, y)	Ordered pair
$(x, 0)$	x-intercept
$(0, y)$	y-intercept

Exponential Equations	
$A=P\left(1+\dfrac{r}{n}\right)^{nt}$	Compounded interest formula
Compounded...	n (number of times per year)
Yearly/annually	1
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Exponential Functions	
$1+r$	Growth factor
$1-r$	Decay factor
$f(t)=a(1+r)^t$	Exponential growth function
$f(t)=a(1-r)^t$	Exponential decay function
$f(x)=ab^x$	Exponential function in general form
$y=ab^{\frac{x}{t}}$	Exponential equation

Binomial Theorem

$$\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\bullet a^{n-k}b^k=1a^nb^0+\frac{n}{1}a^{n-1}b^1+\frac{n(n-1)}{1\bullet 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\bullet 2\bullet 3}a^{n-3}b^3+\cdots+1a^0b^n$$

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Functions	
$f(x)$	Function notation, "f of x"
$f^{-1}(x)$	Inverse function notation
$f(x) = mx + b$	Linear function
$f(x) = b^x + k$	Exponential function
$f(x) = ax^2 + b^x + c$	Quadratic function
$(f + g)(x) = f(x) + g(x)$	Addition
$(f - g)(x) = f(x) - g(x)$	Subtraction
$(f \bullet g)(x) = f(x) \bullet g(x)$	Multiplication
$(f \circ g)(x) = f(g(x))$	Composition
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Division
$\frac{f(b) - f(a)}{b - a}$	Average rate of change
$r = \frac{\Delta f(x)}{\Delta g(x)}$	Concise rate of change
$f(-x) = -f(x)$	Odd function
$f(-x) = f(x)$	Even function
$f(x) = \lfloor x \rfloor$	Floor/greatest integer function
$f(x) = \lceil x \rceil$	Ceiling/least integer function
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$	Polynomial function
$f(x) = a\sqrt[3]{(x - h)} + k$	Cube root function
$f(x) = a\sqrt[n]{(x - h)} + k$	Radical function
$f(x) = a\lvert x - h \rvert + k$	Absolute value function
$f(x) = \frac{p(x)}{q(x)};\quad q(x) \neq 0$	Rational function
$\log_a bx = c$	Logarithmic function

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Formulas

Quadratic Functions and Equations

$x=\frac{-b}{2a}$	Axis of symmetry
$x=\frac{p+q}{2}$	Axis of symmetry using the midpoint of the x-intercepts
$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$	Vertex
$f(x) = ax^2 + bx + c$	General form
$f(x) = a(x - h)^2 + k$	Vertex form
$f(x) = a(x - p)(x - q)$	Factored/intercept form
$b^2 - 4ac$	Discriminant
$x^2+bx+\left(\frac{b}{2}\right)^2$	Perfect square trinomial
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Quadratic formula

Properties of Exponents

Property	General rule
Zero Exponent	$a^0 = 1$
Negative Exponent	$b^{-\frac{m}{n}}=\frac{1}{b^{\frac{m}{n}}}$
Product of Powers	$a^m \bullet a^n = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n}=a^{m-n}$
Power of a Power	$\left(b^m\right)^n = b^{mn}$
Power of a Product	$\left(bc\right)^n = b^n c^n$
Power of a Quotient	$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$

Multiplication of Complex Conjugates

$(a + bi)(a - bi) = a^2 + b^2$

Common Polynomial Identities

$(a + b)^2 = a^2 + 2ab + b^2$	Square of Sums
$(a - b)^2 = a^2 - 2ab + b^2$	Square of Differences
$a^2 - b^2 = (a + b)(a - b)$	Difference of Two Squares
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Sum of Two Cubes
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	Difference of Two Cubes

Radicals to Rational Exponents

$\sqrt[n]{a}=a^{\frac{1}{n}}$

$\sqrt[n]{x^m}=x^{\frac{m}{n}}$

Imaginary Numbers

$i=\sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Logarithmic Functions

e	Base of a natural logarithm
$\log_a b=\frac{\log b}{\log a}$	Change of base formula
$\frac{2\pi}{b}$	Period
$-\frac{b}{a}$	Phase shift

Properties of Radicals

$\sqrt{ab}=\sqrt{a}\bullet\sqrt{b}$

$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

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Series and Sequences	
$r=\dfrac{a_n}{a_{n-1}}$	Common ratio
$a_n = a_1 \bullet r^{n-1}$	Explicit formula for a geometric sequence
$\sum_{k=1}^{n} a_1 r^{k-1}$	Finite geometric series
$\sum_{k=1}^{\infty} a_1 r^{k-1}$	Infinite geometric series
$a_n = a_{n-1} \bullet r$	Recursive formula for a geometric sequence
$S_n=\dfrac{a_1(1-r^n)}{1-r}$	Sum formula for a finite geometric series
$S_n=\dfrac{a_1}{1-r}$	Sum formula for an infinite geometric series
$P=\sum_{k=1}^{n} A\left(\dfrac{1}{1+i}\right)^{k-1}$	Amortization loan formula

Properties of Logarithms	
Product property	$\log_a (x \bullet y) = \log_a x + \log_b y$
Quotient property	$\log_a\left(\dfrac{x}{y}\right)=\log_a x-\log_a y$
Power property	$\log_a x^y = y \bullet \log_a x$

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STATISTICS AND DATA ANALYSIS

Symbols	
$\varnothing$	Empty/null set
$\cap$	Intersection, "and"
$\cup$	Union, "or"
$\subset$	Subset
$\overline{A}$	Complement of Set A
!	Factorial
${}_{n}C_{r}$	Combination
${}_{n}P_{r}$	Permutation
μ	Population mean
$\overline{x}$	Sample mean
σ	Standard deviation of a population
s	Standard deviation of a sample
$\hat{p}$	Sample proportion
SEM	Standard error of the mean
SEP	Standard error of the proportion
MOE	Margin of error
CI	Confidence interval
df	Degrees of freedom

Empirical Rule/68–95–99.7 Rule
$\mu \pm 1\sigma \approx 68\%$
$\mu \pm 2\sigma \approx 95\%$
$\mu \pm 3\sigma \approx 99.7\%$

Common Critical Values

Confidence level	99%	98%	96%	95%	90%	80%	50%
Critical value (z_c)	2.58	2.33	2.05	1.96	1.645	1.28	0.6745

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Formulas	
$\mu=\frac{x_1+x_2+\cdots+x_n}{n}$	Mean of a population
$\overline{x}=\frac{x_1+x_2+\cdots+x_n}{n}$	Mean of a sample
$\sigma=\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$	Standard deviation of a population
$s=\sqrt{\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2}{n}}$	Standard deviation of a sample
$z=\frac{x-\mu}{\sigma}$	*z*-score
$\hat{p}=\frac{p}{n}$	Sample proportion
$\text{SEM}=\frac{s}{\sqrt{n}}$	Standard error of the mean
$\text{SEP}=\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$	Standard error of the proportion
$\text{MOE}=\pm z_c\frac{s}{\sqrt{n}}$	Margin of error of a sample mean
$\text{MOE}=\pm z_c\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$	Margin of error for a sample proportion
$\text{CI}=\hat{p}\pm z_c\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$	Confidence interval for a sample population with proportion $\hat{p}$
$\text{CI}=\overline{x}\pm z_c\frac{s}{\sqrt{n}}$	Confidence interval for a sample population with mean $\overline{x}$
$t=\frac{\overline{x}_1-\overline{x}_2}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}}$	*t*-value for two sets of sample data
$t=\frac{\overline{x}-\mu_0}{\frac{s}{\sqrt{n}}}$	*t*-value for sample data and population
$df=\frac{n_1-1+n_2-1}{2}$	Degrees of freedom

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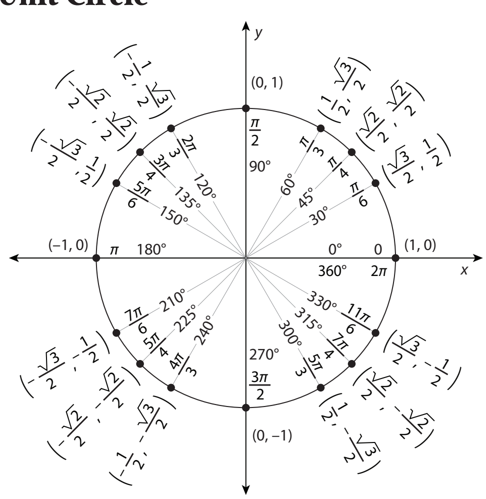
Rules and Equations	
$P(E)=\dfrac{\#\text{ of outcomes in }E}{\#\text{ of outcomes in sample space}}$	Probability of event E
$P(A\cup B)=P(A)+P(B)-P(A\cap B)$	Addition rule
$P(\overline{A})=1-P(A)$	Complement rule
$P(B\|A)=\dfrac{P(A\cap B)}{P(A)}$	Conditional probability
$E(X)=p_1P(X_1)+p_2P(X_2)+p_3P(X_3)$	Expected value
$P(A\cap B)=P(A)\bullet P(B\|A)$	Multiplication rule
$P(A\cap B)=P(A)\bullet P(B)$	Multiplication rule if A and B are independent
${}_nC_r=\dfrac{n!}{(n-r)!r!}$	Combination
${}_nP_r=\dfrac{n!}{(n-r)!}$	Permutation
$n!=n\bullet(n-1)\bullet(n-2)\bullet\cdots\bullet 1$	Factorial
$P=\begin{pmatrix} n \\ x \end{pmatrix}p^xq^{n-x}$	Binomial probability distribution

GEOMETRY

Unit Circle



Symbols	
$\overset{\frown}{ABC}$	Major arc length
$\overset{\frown}{AB}$	Minor arc length
$\angle$	Angle
$\odot$	Circle
$\cong$	Congruent
$\overleftrightarrow{PQ}$	Line
$\overline{PQ}$	Line segment
$\overrightarrow{PQ}$	Ray
$\parallel$	Parallel
$\perp$	Perpendicular
$\bullet$	Point
$\triangle$	Triangle
$\square$	Parallelogram
A'	Prime
$^\circ$	Degrees
θ	Theta
ϕ	Phi
π	Pi
ρ	Rho

Trigonometric Ratios		
$\sin\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta=\dfrac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}$
$\csc\theta=\dfrac{\text{hypotenuse}}{\text{opposite}}$	$\sec\theta=\dfrac{\text{hypotenuse}}{\text{adjacent}}$	$\cot\theta=\dfrac{\text{adjacent}}{\text{opposite}}$

Volume	
$V=lwh$	Rectangular prism
$V=Bh$	Prism
$V=\frac{1}{3}\pi r^2h$	Cone
$V=\frac{1}{3}Bh$	Pyramid
$V=\pi r^2h$	Cylinder
$V=\frac{4}{3}\pi r^3$	Sphere

Pythagorean Theorem
$a^2+b^2=c^2$

Trigonometric Identities
$\sin\theta=\cos(90^\circ-\theta)$
$\cos\theta=\sin(90^\circ-\theta)$
$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$
$\csc\theta=\dfrac{1}{\sin\theta}$
$\sec\theta=\dfrac{1}{\cos\theta}$
$\cot\theta=\dfrac{1}{\tan\theta}$
$\cot\theta=\dfrac{\cos\theta}{\sin\theta}$
$\sin^2\theta+\cos^2\theta=1$

Area	
$A=lw$	Rectangle
$A=\frac{1}{2}bh$	Triangle
$A=\frac{1}{2}ab\sin C$	Triangle with unknown height
$A=\pi r^2$	Circle
$A=\frac{1}{2}(b_1+b_2)h$	Trapezoid

Distance Formula
$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Pi Defined
$\pi=\dfrac{\text{circumference}}{\text{diameter}}=\dfrac{\text{circumference}}{2\bullet\text{radius}}$

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Circumference of a Circle	
$C=2\pi r$	Circumference given the radius
$C=\pi d$	Circumference given the diameter

Inverse Trigonometric Functions
Arcsin $\theta = \sin^{-1}\theta$
Arccos $\theta = \cos^{-1}\theta$
Arctan $\theta = \tan^{-1}\theta$

Converting Between Degrees and Radians
$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180}$

Arc Length	
$s=\theta r$	Arc length (θ in radians)

Laws of Sines and Cosines	
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Law of Sines
$c^2 = a^2 + b^2 - 2ab \cos C$	Law of Cosines

Density	
$\text{Density}_{\text{Area}} = \frac{\text{mass or quantity}}{\text{number of } \textit{square} \text{ units}}$ or $\rho_A = \frac{m}{A}$	Area density
$\text{Density}_{\text{Volume}} = \frac{\text{mass or quantity}}{\text{number of } \textit{cubic} \text{ units}}$ or $\rho = \frac{m}{V}$	Volume density

MEASUREMENTS

Length
Metric
1 kilometer (km) = 1000 meters (m)
1 meter (m) = 100 centimeters (cm)
1 centimeter (cm) = 10 millimeters (mm)
Customary
1 mile (mi) = 1760 yards (yd)
1 mile (mi) = 5280 feet (ft)
1 yard (yd) = 3 feet (ft)
1 foot (ft) = 12 inches (in)

Volume and Capacity
Metric
1 liter (L) = 1000 milliliters (mL)
Customary
1 gallon (gal) = 4 quarts (qt)
1 quart (qt) = 2 pints (pt)
1 pint (pt) = 2 cups (c)
1 cup (c) = 8 fluid ounces (fl oz)

Weight and Mass
Metric
1 kilogram (kg) = 1000 grams (g)
1 gram (g) = 1000 milligrams (mg)
1 metric ton (MT) = 1000 kilograms
Customary
1 ton (T) = 2000 pounds (lb)
1 pound (lb) = 16 ounces (oz)

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English	Unit/Lesson	Español
A		
absolute value a number's distance from 0 on a number line; the positive value of a quantity	1.3 1.4	**valor absoluto** distancia de un número a partir del 0 en una recta numérica; valor positivo de una cantidad
absolute value function a function of the form $f(x)=\|ax+b\|+c$, where *x* is the independent variable and *a*, *b*, and *c* are real numbers	1.3 1.4	**función de valor absoluto** función de la forma $f(x)=\|ax+b\|+c$, donde *x* es la variable independiente *a*, *b* y *c* son números reales
amplitude the coefficient *a* or *c* of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = c \cos dx$; on a graph of the cosine or sine function, the vertical distance from the *y*-coordinate of the maximum point on the graph to the midline of the cosine or sine curve	7.3	**amplitud** el coeficiente *a* o *c* del término de seno o coseno en una función de la forma $f(x) = a \sin bx$ o $g(x) = c \cos dx$; en un gráfico de la función seno o coseno, la distancia vertical desde la coordenada *y* del punto máximo en la gráfica hasta la línea media de la curva de seno o coseno
arc part of a circle's circumference	5.2	**arco** parte de la circunferencia de un círculo
arc length the distance between the endpoints of an arc; written as $m\overset{\frown}{AB}$	5.4 7.1	**longitud de arco** distancia entre los extremos de un arco; se expresa como $m\overset{\frown}{AB}$
argument the result of raising the base of a logarithm to the power of the logarithm, so that *b* is the argument of the logarithm $\log_a b = c$; the term $cx + d$ in a cosine or sine function of the form $f(x) = a + b \sin (cx + d)$ or $g(x) = a + b \cos (cx + d)$	2.3	**argumento** el resultado de elevar la base de un logaritmo a la potencia del logaritmo, de manera que *b* es el argumento del logaritmo $\log_a b = c$; el término $cx + d$ en una función coseno o seno de la forma $f(x) = a + b \sin (cx + d)$ o $g(x) = a + b \cos (cx + d)$
asymptote a line that a function gets closer and closer to as one of the variables increases or decreases without bound	6.1 6.2	**asíntota** una línea que una función se acerca cada vez más cerca de una de las variables aumenta o disminuye sin límite
average rate of change the ratio of the difference of output values to the difference of the corresponding input values: $\frac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval	3.8	**tasa de cambio promedio** proporción de la diferencia de valores de salida a la diferencia de valores correspondientes de entrada: $\frac{f(b)-f(a)}{b-a}$; medida de cuánto cambia una cantidad en cierto intervalo

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axis of rotation a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line	4.4	**eje de rotación** línea alrededor de la cual puede girar la figura de un plano en un espacio de tres dimensiones para crear una figura sólida, como un diámetro o una línea de simetría
axis of symmetry of a parabola the line through the vertex of a parabola about which the parabola is symmetric. The equation of the axis of symmetry is $x=\frac{-b}{2a}$.	3.2	**eje de simetría de una parábola** línea que atraviesa el vértice de una parábola sobre la que la parábola es simétrica. La≈ecuación del eje de simetría es $x=\frac{-b}{2a}$.
B		
base the quantity that is being raised to an exponent in an exponential expression; in a^x, a is the base; or, the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the equation $\log_2 g(x) = 3 - x$	2.3	**base** cantidad que es elevada a un exponente en una expresión exponencial; en a^x, a es la base; o, la cantidad que se eleva a un exponente que es el valor del logaritmo, tal que 2 en la ecuación $\log_2 g(x) = 3 - x$
bias leaning toward one result over another; having a lack of neutrality	8.3	**sesgo** inclinación por un resultado sobre otro; carecer de neutralidad
biased sample a sample in which some members of the population have a better chance of inclusion in the sample than others	8.3	**muestra sesgada** muestra en la cual algunos miembros de la población tienen una mayor posibilidad de ser incluidos en la muestra que otros
C		
Cavalieri's Principle The volumes of two objects of equal height are equal if the areas of their corresponding cross sections are in all cases equal.	4.5	**Principio de Cavalieri** Los volúmenes de dos objetos de igual altura son iguales si las superficies de sus correspondientes secciones transversales son en todos los casos iguales.
ceiling function also known as the least integer function; a function represented as $y=\lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x; for example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.	1.4	**función techo** también conocida como función del mínimo entero; función representada como $y=\lceil x \rceil$. Para cualquier entrada x, la salida es el entero más pequeño mayor que o igual a x; por ejemplo, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, y $\lceil -2.1 \rceil = -2$.

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center of a circle the point in the plane of the circle from which all points on the circle are equidistant. The center is not part of the circle; it is in the interior of the circle.	5.1	**centro de un círculo** punto en el plano del círculo desde el cual son equidistantes todos los puntos del círculo. El centro no es parte del círculo: se encuentra en el interior del círculo.
central angle an angle with its vertex at the center of a circle	5.2 5.4 7.1	**ángulo central** ángulo con su vértice en el centro de un círculo
centroid the intersection of the medians of a triangle	4.8	**centroide** intersección de las medianas de un triángulo
chance variation a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *sampling error*	8.3	**variación aleatoria** medida que muestra cómo una muestra refleja con precisión la población, con errores de muestreo más pequeños que resultan de muestras grandes y/o cuando los datos se agrupan estrechamente alrededor de la media; también llamada *error de muestreo*
chord a segment whose endpoints lie on the circumference of the circle	5.2	**cuerda** segmento cuyos extremos se ubican en la circunferencia del círculo
circle the set of all points in a plane that are equidistant from a reference point in that plane, called the center. The set of points forms a two-dimensional curve that measures 360°.	5.1 5.2	**círculo** conjunto de todos los puntos de un plano equidistantes desde un punto de referencia en ese plano, denominado centro. El conjunto de puntos forma una curva bidimensional que mide 360°.
circumcenter the intersection of the perpendicular bisectors of a triangle	4.8	**circuncentro** intersección de las bisectrices perpendiculares de un triángulo
circumference the distance around a circle; $C = 2\pi r$ or $C = \pi d$, for which C represents circumference, r represents the circle's radius, and d represents the circle's diameter	5.2 5.4	**circunferencia** distancia alrededor de un círculo; $C = 2\pi r$ o $C = \pi d$, en donde C representa la circunferencia, r representa el radio del círculo y d, su diámetro
circumscribed angle the angle formed by two tangent lines whose vertex is outside of the circle	5.6	**ángulo circunscrito** ángulo formado por dos líneas tangentes cuyo vértice está fuera del círculo
circumscribed circle a circle that contains all vertices of a polygon	4.8	**círculo circunscrito** círculo que contiene todos los vértices de un polígono

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closure a system is closed, or shows closure, under an operation if the result of the operation is within the system	3.1	**cierre** un sistema es cerrado, o tiene cierre, en una operación si el resultado de la misma está dentro del sistema
cluster sample a sample in which naturally occurring groups of population members are chosen for a sample	8.2	**muestreo en grupos** muestra en la cual se eligen para una muestra grupos naturalmente ya formados de miembros de la población
common denominator a quantity that is a shared multiple of the denominators of two or more fractions	6.6	**denominador común** cantidad que es un múltiplo compartido de los denominadores de dos o más fracciones
common logarithm a base-10 logarithm which is usually written without the number 10, such as $\log x = \log_{10} x$	2.3	**logaritmo común** logaritmo de base 10 que se escribe normalmente sin el número 10, como $\log x = \log_{10} x$
compound interest interest earned on both the initial amount and on previously earned interest	2.7	**interés compuesto** interés devengado tanto de la cantidad inicial como del interés previamente devengado
concave polygon a polygon with at least one interior angle greater than 180° and at least one diagonal that does not lie entirely inside the polygon	4.2	**polígono cóncavo** polígono con al menos un ángulo interior de más de 180° y con al menos una diagonal que no se ubica por completo dentro de él
concentric circles coplanar circles that have the same center	5.2	**círculos concéntricos** círculos coplanares que tienen el mismo centro
concurrent lines lines that intersect at one point	4.8	**rectas concurrentes** rectas con intersección en un punto
cone a solid or hollow object that tapers from a circular or oval base to a point	4.5	**cono** objeto sólido o hueco que se estrecha desde una base circular u ovalada hasta un punto
confounding variable an ignored or unknown variable that influences the result of an experiment, survey, or study	8.4	**variable de confusión** una variable ignorada o desconocida que influye sobre el resultado de un experimento, encuesta o estudio
congruent having the same shape, size, lines, and angles; the symbol for representing congruency between figures is $\cong$	4.4	**congruente** tener la misma forma, tamaño, rectas y ángulos; el símbolo para representar la congruencia entre números es $\cong$
congruent arcs two arcs that have the same measure and are either of the same circle or of congruent circles	5.2	**arcos congruentes** dos arcos que tienen la misma medida y son parte del mismo círculo o de círculos congruentes

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consecutive angles angles that lie on the same side of a figure	4.2	**ángulos consecutivos** ángulos ubicados en el mismo lado de una figura
continuous compounding interest that compounds infinitely	2.6	**composición continua** interés que se compone infinitamente
control group the group of participants in a study who are not subjected to the treatment, action, or process being studied in the experiment, in order to form a comparison with participants who are subjected to it	8.1	**grupo de control** grupo de participantes en un estudio que no están sujetos al tratamiento, acción o proceso que está en estudio en el experimento con el fin de establecer una comparación con participantes que sí lo están.
convenience sample a sample in which members are chosen to minimize the time, effort, or expense involved in sampling	8.2	**muestreo de conveniencia** muestreo en el cual se eligen los miembros para minimizar el tiempo, esfuerzo o gasto involucrado en este proceso
convex polygon a polygon with no interior angle greater than 180°; all diagonals lie within the polygon	4.2	**polígono cóncavo** polígono sin ángulo interior de más de 180°; todas las diagonales están dentro de polígono
cosine function a trigonometric function of the form $f(x) = a \cos [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$	7.3	**función del coseno** función trigonométrica de la forma $f(x) = a \cos [b(x - c)] + d$, donde a, b, c y d son constantes y x es una variable definida en radianes a lo largo del dominio $(-\infty,\infty)$
coterminal angles angles that, when drawn in standard position, share the same terminal side	7.1	**ángulos coterminales** ángulos que, cuando están trazados en una posición estándar, comparten el mismo lado terminal
cross section the plane figure formed by the intersection of a plane with a solid figure, where the plane is at a right angle to the surface of the solid figure	4.4	**sección transversal** figura del plano formada por la intersección de un plano con una figura sólida, donde el plano está en un ángulo recto a la superficie de la figura sólida
cube root function a function that contains the cube root of a variable. The general form is $f(x)=a\sqrt[3]{(x-h)}-k$, where a, h, and k are real numbers.	1.2	**función raíz cúbica** función que contiene la raíz cúbica de una variable. La forma general es $y=a\sqrt[3]{(x-h)}+k$, donde a, h, y k son números reales.
cycle the smallest representation of a cosine or sine function graph as defined over a restricted domain; equal to one repetition of the period of a function	7.3	**ciclo** la representación más pequeña de una gráfica de la función coseno o seno definida a través de un dominio restringido; igual a una repetición del período de una función

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	D	
data numbers in context	8.1	**datos** números en contexto
decay factor $1 - r$ in the exponential decay model $f(t) = a(1 - r)^t$, or b in the exponential function $f(t) = ab^t$ if $0 < b < 1$; the multiple by which a quantity decreases over time. The general form of an exponential function modeling decay is $f(t) = a(1 - r)^t$.	2.7	**factor de decaimiento 1** – r en el modelo de decaimiento exponencial $f(t) = a(1 - r)^t$, o b en la función exponencial $f(t) = ab^t$ si $0 < b < 1$; el múltiplo por el que una cantidad disminuye con el tiempo. La forma general de una función exponencial que determina decaimiento es $f(t) = a(1 - r)^t$.
decay rate r in the exponential decay model $f(t) = a(1 - r)^t$	2.6 2.7	**tasa de decaimiento** r en el modelo de decaimiento exponencial $f(t) = a(1 - r)^t$
delta **(Δ)** a Greek letter commonly used to represent the change in a value	3.8	***delta*** **(Δ)** letra griega utilizada comúnmente para representar el cambio en un valor
denominator the value located below the line of a rational expression or fraction; the divisor	6.3	**denominador** el valor ubicado debajo de la línea de una expresión racional o fracción; el divisor
density the amount, number, or other quantity per unit of area or volume of some substance or population being studied	4.6	**densidad** la cantidad, el número u otra cantidad por unidad de área o volumen de alguna sustancia o población que está siendo estudiada
dependent system a system of equations that intersect at every point	1.5	**sistema dependiente** sistema de ecuaciones que se cruzan en cada punto
dependent variable labeled on the y-axis; the quantity that is based on the input values of the independent variable; the output variable of a function	6.1	**variable dependiente** designada en el eje de y; cantidad que se basa en los valores de entrada de la variable independiente; variable de salida de una función
depressed polynomial the result of dividing a polynomial by one of its binomial factors	3.5	**polinomio deprimido** el resultado de dividir un polinomio por uno de sus factores binómicos
desirable outcome the data sought or hoped for, represented by p; also known as *favorable outcome* or *success*	8.6	**resultado deseado** datos buscados o esperados, representado por p; también conocido como *resultado favorable* o *éxito*
diagonal a line that connects nonconsecutive vertices	4.2	**diagonal** línea que conecta vértices no consecutivos

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diameter a line segment passing through the center of a circle connecting two points on the circle; twice the radius	5.2	**diámetro** línea recta que pasa por el centro de un círculo y conecta dos puntos en el círculo; dos veces el radio
directrix of a parabola a line that is perpendicular to the axis of symmetry of a parabola and that is in the same plane as both the parabola and the focus of the parabola; the fixed line referenced in the definition of a parabola	3.2	**directriz de una parábola** línea perpendicular al eje de simetría de una parábola que está en el mismo plano tanto de la parábola como de su foco; línea fija mencionada en la definición de parábola
distance formula formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	5.1	**fórmula de distancia** fórmula que establece la distancia entre los puntos (x_1, y_1) y (x_2, y_2) equivale a $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
domain the set of all input values (*x*-values) that satisfy the given function without restriction	1.10 6.1	**dominio** conjunto de todos los valores de entrada (valores de *x*) que satisfacen la función dada sin restricciones
double-blind study a study in which neither the researcher nor the participants know who has been subjected to the treatment, action, or process being studied, and who is in a control group	8.4	**estudio doble-ciego** estudio en el cual ni el investigador ni los participantes saben quién se sometió al tratamiento, acción o proceso que está siendo estudiado y quién está en un grupo de control
doubling time the amount of time it takes a value to double	2.1	**doblando tiempo** la cantidad de tiempo que toma un valor para duplicar
	E	
e an irrational number with an approximate value of 2.71828; *e* is the base of the natural logarithm (ln *x* or $\log_e x$)	2.3	***e*** número irracional con un valor aproximado de 2,71828; *e* es la base del logaritmo natural (In *x* o $\log_e x$)
empty set a set that has no elements, denoted by ∅; the solution to a system of equations with no intersection points, denoted by { }	1.5	**conjunto vacío** un conjunto que no tiene elementos, indicado por ∅; la solución a un sistema de ecuaciones sin puntos de intersección, indicado por { }
end behavior the behavior of the graph as *x* approaches positive or negative infinity	3.4 3.8 6.1	**comportamiento final** el comportamiento del gráfico a medida que *x* se acerca al infinito positivo o negativo

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even-degree polynomial function a polynomial function in which the highest exponent is an even number. Both ends of the graph of an even-degree polynomial function will extend in the same direction, either upward or downward.	3.4	**función polinómica de grado par** función polinómica en la cual el exponente mayor es un número par. Ambos extremos del gráfico de una función polinómica de grado par se extenderán en la misma dirección, hacia arriba o hacia abajo.
experiment a process or action that has observable results	8.1	**experimento** proceso o acción con resultados observables
exponential decay an exponential equation with a base, b, that is between 0 and 1 exclusive (that is, $0 < b < 1$); an example is the formula $y = a(1 - r)^t$, where a is the initial value, $(1 - r)$ is the base (with $0 < r < 1$), t is the variable exponent, and y is the final value	2.7	**decaimiento exponencial** una ecuación exponencial con una base, b, que está entre 0 y 1 exclusivo (es decir, $0 < b < 1$); un ejemplo es la fórmula $y = a(1 - r)^t$, donde a es el valor inicial, $(1 - r)$ es la base (con $0 < r < 1$), t es el exponente variable y y es el final valor
exponential decay model an exponential function, $f(t) = a(1 - r)^t$, where $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the percent decrease per time period (expressed as a decimal), and t is the number of time periods	2.7	**modelo de decaimiento exponencial** función exponencial, $f(t) = a(1 - r)^t$, en la que $f(t)$ es el valor de salida final despues de t períodos de tiempo, a es el valor inicial, r es el porcentaje de disminución por período (expresado como decimal), y t es la cantidad de períodos
exponential expression an expression that contains a base and a power/exponent	2.7	**expresión exponencial** expresión que incluye una base y una potencia o exponente
exponential function a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 5^x$	2.3 2.6 2.7 2.8	**función exponencial** función de la fórmula $f(x) = ab^{cx}$ en la cual a, b y c son constantes; una función que tiene una variable en el exponente, tal como $f(x) = 5^x$
exponential growth an exponential function with a base, b, greater than 1 ($b > 1$); can be represented by the formula $f(t) = a(1 + r)^t$, where a is the initial value, $(1 + r)$ is the growth rate, t is time, and $f(t)$ is the final value	2.7	**crecimiento exponencial** función exponencial con una base, b, mayor que 1 **(**$b > 1$); puede representarse la fórmula $f(t) = a(1 + r)^t$, en la que a es el valor inicial, $(1 + r)$ es la tasa de crecimiento, t es el tiempo, y $f(t)$ es el valor final

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exponential growth model an exponential function, $f(t) = a(1 + r)^t$, where $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the percent increase per time period (expressed as a whole number or decimal), and t is the number of time periods	2.7	**modelo de crecimiento exponencial** función exponencial, $f(t) = a(1 - r)^t$, en la que $f(t)$ es el valor de salida final despues de t períodos de tiempo, a es el valor inicial, r es el porcentaje de aumento por período (expresado como entero o decimal), y t es la cantidad de períodos
extraneous solution (extraneous root) of an equation a solution of an equation that arises during the solving process, but which is not a solution of the original equation	6.7	**solución extraña (raíz extraña) de una ecuación** solución de una ecuación que surge durante el proceso de resolución pero que no es una solución de la ecuación original
	F	
factored form of a quadratic function the intercept form of a quadratic equation, written as $f(x) = a(x - p)(x - q)$, where p and q are the x-intercepts of the function; also known as the *intercept form of a quadratic function*	3.2	**forma factorizada de una función cuadrática** forma de intercepto de una ecuación cuadrática, se expresa como $f(x) = a(x - p)(x - q)$, en la que p y q son los interceptos de x de la función; también se conoce como la *forma de intercepto de una función cuadrática*
family of functions a set of functions whose graphs have the same general shape as their parent function. The parent function is the function with a simple algebraic rule that represents the family of functions.	6.1	**familia de funciones** conjunto de funciones cuyas gráficas tienen la misma forma general que su función raíz. La función raíz es la función con una regla algebraica simple que representa la familia de funciones.
favorable outcome the data sought or hoped for, represented by p; also known as *desirable outcome* or *success*	8.6	**resultado favorable** datos buscados o esperados, representados por p; también conocido como *resultado deseado* o *éxito*
first difference in a set of data, the change in the y-value when the x-value is increased by 1	1.1	**primera diferencia** en un conjunto de datos, el cambio en el valor y cuando el valor x aumenta por 1
floor function also known as the greatest integer function; a function represented as $y = \lfloor x \rfloor$. For any input x, the output is the largest integer less than or equal to x; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.	1.4	**función piso** también conocida como la función del mayor entero; función representada como $y = \lfloor x \rfloor$. Para cualquier entrada x, la salida es el entero más grande que es menor que o igual a x; por ejemplo, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, y $\lfloor -2.1 \rfloor = -3$.

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focus of a parabola a fixed point on the interior of a parabola that is not on the directrix of the parabola but is on the same plane as both the parabola and the directrix; the fixed point referenced in the definition of a parabola	3.2	**foco de una parábola** punto fijo en el interior de una parábola que no está en la directriz de la parábola sino en el mismo plano que la parábola y la directriz; punto fijo mencionado en la definición de parábola
fraction a ratio of two expressions or quantities	6.3	**fracción** relación de dos expresiones o cantidades
frequency of a periodic function the reciprocal of the period for a periodic function; indicates how often the function repeats	7.3	**frecuencia de una función periódica** recíproca del período para una función periódica; indica con que frecuencia se repite la función
function a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of *x*, there is exactly one value of *y*	1.9 1.10	**función** relación en la que cada elemento del dominio se empareja con un único elemento del rango; es decir, para cada valor de *x*, existe exactamente un valor de *y*
function notation the use of $f(x)$, which means "function of *x*," instead of *y* or another dependent variable in an equation of a function; $f(x) = 2x + 1$ and $y = 2x + 1$ are equivalent functions	1.8	**notación de funciones** el uso de $f(x)$, que significa "función de *x*", en lugar de *y* u otra variable dependiente en la ecuación de una función; $f(x) = 2x + 1$ e $y = 2x + 1$ son funciones equivalentes
function transformation a movement or stretching of the graph of the function, caused by adding or multiplying a constant to the function	2.2	**transformación de funcion** un movimiento o estiramiento de la gráfica de la función, causada por la adición o multiplicación de una constante a la función
	G	
general form of an equation of a circle $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$	5.1	**forma general de ecuación de un círculo** $Ax^2 + By^2 + Cx + Dy + E = 0$, en la que $A = B$, $A \neq 0$, y $B \neq 0$
greatest integer function also known as the floor function; a function represented as $y = \lfloor x \rfloor$. For any input *x*, the output is the largest integer less than or equal to *x*; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.	1.4	**función del mayor entero** también conocida como función piso; función que se representa como $y = \lfloor x \rfloor$. Para cualquier entrada *x*, la salida es el entero más grande que es menor que o igual a *x*; por ejemplo, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, y $\lfloor -2.1 \rfloor = -3$.

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growth factor the multiple by which a quantity increases over time	2.7	**factor de crecimiento** múltiplo por el que una cantidad aumenta con el tiempo
growth rate the rate of increase in size per unit of time; r in the exponential growth model $f(t) = a(1 + r)^t$	2.6 2.7	**tasa de crecimiento** tasa de aumento de tamaño por unidad de tiempo; r en el modelo de crecimiento exponencial $f(t) = a(1 + r)^t$
	H	
half-life the amount of time it takes a value to decay to half its original value	2.1	**media vida** la cantidad de tiempo que toma un valor para decaer a la mitad de su valor original
horizontal asymptote a line defined as follows: The line $y = b$ is a horizontal asymptote of the graph of a function f if $f(x)$ gets closer to b as x either increases or decreases without bound.	6.1 6.2	**asíntota horizontal** línea recta que se define de la siguiente manera: La línea $y = b$ es una asíntota horizontal del gráfico de una función f si $f(x)$ se acerca a b a medida que x aumenta o disminuye sin límites.
	I	
incenter the intersection of the angle bisectors of a triangle	4.8	**incentro** intersección de las bisectrices del ángulo de un triángulo
independent variable labeled on the x-axis; the quantity that changes based on values chosen; the input variable of a function	6.1	**variable independiente** designada en el eje x; cantidad que cambia según los valores seleccionados; variable de entrada de una función
inference a conclusion reached upon the basis of evidence and reasoning	8.3	**inferencia** conclusión alcanzada sobre la base de evidencia y razonamiento
inflection point a point on a curve at which the sign of the curvature (i.e., the concavity) changes	1.2	**punto de inflexión** punto en una curva en el que cambia el signo de la curvatura (es decir, la concavidad)
initial side the stationary ray of an angle from which the measurement of the angle starts	7.1	**lado inicial** rayo fijo de un ángulo desde el cual comienza la medición del ángulo
inscribed angle an angle formed by two chords whose vertex is on the circle	5.2	**ángulo inscrito** ángulo formado por dos cuerdas cuyo vértice está en el círculo
inscribed circle a circle whose tangents form a triangle	4.8	**círculo inscrito** círculo cuyos tangentes forman un triángulo
integer a number that is not a fraction or a decimal	3.4	**entero** un número que no es una fracción ni un decimal

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English	Unit/Lesson	Español
intercepted arc an arc whose endpoints intersect the sides of an inscribed angle and whose other points are in the interior of the angle	5.2	**arco interceptado** arco cuyos extremos intersecan los lados de un ángulo inscrito y cuyos otros puntos se sitúan en el interior del ángulo
intercept form of a quadratic function the factored form of a quadratic equation, written as $f(x) = a(x - p)(x - q)$, where p and q are the x-intercepts of the function; also known as the *factored form of a quadratic function*	3.2	**forma de intercepto de una función cuadrática** forma factorizada de una ecuación cuadrática, expresada como $f(x) = a(x - p)(x - q)$, donde p y q son los interceptos de x de la función; también se conoce como la *forma factorizada de una ecuación cuadrática*
interval the set of all real numbers between two given numbers. The two numbers on the ends are the endpoints. The endpoints might or might not be included in the interval depending on whether the interval is open, closed, or half-open/half-closed.	1.1	**intervalo** conjunto de todos los números reales entre dos números dados. Los dos números en los finales son los extremos. Los extremos podrían o no estar incluidos en el intervalo, según si el intervalo está abierto, cerrado, o medio abierto o medio cerrado.
inverse function the function that results from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$	1.8 1.10	**función inversa** función que se produce como resultado de cambiar las variables x y y en una función determinada; la inversa de $f(x)$ se expresa como $f^{-1}(x)$
inverse operation an operation that reverses the effect of another operation. Addition and subtraction are inverse operations, and multiplication and division are inverse operations.	1.8	**operación inversa** operación que revierte el efecto de otra. La adición y la sustracción son operaciones inversas, y la multiplicación y división son operaciones inversas.
inverse relation a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function	1.10	**relación inversa** una relación $g(x)$ tal que $g(f(x)) = x$ y $f(g(y)) = y$ donde $f(x)$ es una función
invertible function a function $f(x)$ for which an inverse function $f^{-1}(x)$ exists; i.e., $f(x)$ and x have one-to-one correspondence	1.9	**función invertible** función $f(x)$ para la cual existe una función inversa $f^{-1}(x)$; es decir, $f(x)$ y x tienen una correspondencia uno a uno
irrational base *e* an irrational number with an approximate value of 2.71828; e is the base of the natural logarithm ($\ln x$ or $\log_e x$)	2.6	**base irracional *e*** número irracional con un valor aproximado de 2,71828; e es la base del logaritmo natural (In x o $\log_e x$)

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isosceles trapezoid a trapezoid with one pair of opposite parallel lines and congruent legs	4.3	**trapezoide isósceles** trapezoide con un par de líneas paralelas opuestas y catetos congruentes
	K	
kite a quadrilateral with two distinct pairs of congruent sides that are adjacent	4.3	**cometa** cuadrilátero con dos pares distintos de lados congruentes que son adyacentes
	L	
least common denominator (LCD) the least common multiple of the denominators of two or more fractions	6.6	**coeficiente líder** múltiplo común mínimo de los denominadores de dos o más.
least common multiple (LCM) of polynomials with two or more polynomials, the common multiple of the polynomials that has the least degree and the least positive constant factor	6.8	**mínimo común múltiplo (LCM) de polinomios** con dos o más polinomios, el múltiplo común de los polinomios que tiene el menor grado y el menor factor constante positivo
least integer function also known as the ceiling function; a function represented as $y=\lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x; for example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.	1.4	**función de mínimo entero** también conocida como función techo; función representada como $y=\lceil x \rceil$. Para cualquier entrada x, la salida es el entero más pequeño mayor que o igual a x; por ejemplo, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, y $\lceil -2.1 \rceil = -2$.
like terms terms that contain the same variables raised to the same power	3.1	**términos semejantes** términos que contienen las mismas variables elevadas a la misma potencia
linear function a function that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line; its solutions are the infinite set of points on the line.	1.1 2.8	**función lineal** función que puede expresarse en la forma $ax + by = c$, donde a, b y c son constantes; también puede escribirse como $f(x) = mx + b$, donde m es la pendiente y b es el intercepto de y. El gráfico de una función lineal es una línea recta; sus soluciones son el conjunto infinito de puntos en la línea.

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local maximum the greatest value of a function for a particular interval of the function; also known as a *relative maximum*	3.4	**máximo local** el mayor valor de una función para un intervalo específico de la función; también conocido como *máximo relativo*
local minimum the least value of a function for a particular interval of the function; also known as a *relative minimum*	3.4	**mínimo local** el menor valor de una función para un intervalo específico de la función; también conocido como *mínimo relativo*
logarithm a quantity that represents the power to which a base a must be raised in order to equal a quantity x; written $\log_a x$	2.5	**logaritmo** cantidad que representa la potencia a la cual se debe elevar una base a para que equivalga a una cantidad x; se escribe $\log_a x$
logarithmic function the inverse of an exponential function; for the exponential function $g(x) = 5^x$, the inverse logarithmic function is $x = \log_5 g(x)$	2.3	**función logarítmica** la inversa de una función exponencial; para la función exponencial $g(x) = 5^x$, la función logarítmica inversa es $x = \log_5 g(x)$
	M	
major arc part of a circle's circumference that is larger than its semicircle	5.2	**arco mayor** parte de la circunferencia de un círculo que es mayor que su semicírculo
margin of error the quantity that represents the level of confidence in a calculated parameter, abbreviated MOE. The margin of error can be calculated by multiplying the critical value by the standard deviation, if known, or by the SEM.	8.7	**margen de error** cantidad que representa el nivel de confianza en un parámetro calculado, abreviado MOE. El margen de error puede calcularse multiplicando el valor crítico por la desviación estándar, si se conoce, o por el SEM.
maximum the greatest value or highest point of a function	7.3	**máximo** el mayor valor o punto más alto de una función
measurement bias bias that occurs when the tool used to measure the data is not accurate, current, or consistent	8.8	**sesgo de medición** sesgo que se produce cuando la herramienta utilizada para medir los datos no es exacta, actual o constante
median of a triangle the segment joining the vertex to the midpoint of the opposite side	4.8	**mediana de un triángulo** segmento que une el vértice con el punto medio del lado opuesto

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midline in a cosine function or sine function of the form $f(x) = a + \sin x$ or $g(x) = a + \cos x$, a horizontal line of the form $y = a$ that bisects the vertical distance on a graph between the minimum and maximum function values	7.3	**línea media** en una función del coseno o en una función del seno de la forma $f(x) = a + \sin x$ o $g(x) = a + \cos x$, una línea horizontal de la forma $y = a$ que divide en dos la distancia vertical en un gráfico entre los valores de funciones mínimos y máximos
minimum the least value or lowest point of a function	7.3	**mínimo** el menor valor o el punto más bajo de una función
minor arc part of a circle's circumference that is smaller than its semicircle	5.2	**arco menor** parte de la circunferencia de un círculo que es menor que su semicírculo
multiple root a polynomial function with a root that occurs more than once; also known as a *root with multiplicity*	3.2	**raíz múltiple** función polinómica con una raíz que aparece más de una vez; también conocido como *raíz con multiplicidad*
	N	
natural logarithm a logarithm whose base is the irrational number e; usually written in the form "ln," which means "$\log_e$"	2.3	**logaritmo natural** logaritmo cuya base es el número irracional e; escrito normalmente en la forma "ln", que significa "$\log_e$"
neutral not biased or skewed toward one side or another; regarding surveys, neutral refers to phrasing questions in a way that does not lead the response toward one particular answer or side of an issue	8.1	**neutral** no sesgado hacia un lado u otro; respecto de las encuestas, neutral se refiere a la formulación de preguntas de una manera que no conduzca la respuesta hacia una respuesta o lado específico de un tema
nonresponse bias bias that occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results	8.8	**sesgo sin respuesta** sesgo que se produce cuando los encuestados de una encuesta tienen características diferentes de los no encuestados, dando pie a que la población que no responde sea subrepresentada en los resultados de la encuesta
numerator the value located above the line of a rational expression or fraction; the dividend	6.3	**numerador** el valor ubicado por encima de la línea de una expresión racional o fracción; el dividendo

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O		
observational study a study in which all data, including observations and measurements, are recorded in a way that does not change the subject that is being measured or studied	8.1	**estudio observacional** estudio en el cual todos los datos, incluyendo las observaciones y las mediciones, están registrados de tal manera que no cambian el objeto que está siendo medido o estudiado
odd-degree polynomial function a polynomial function in which the highest exponent is an odd number. One end of the graph of an odd-degree polynomial function will extend upward and the other end will extend downward.	3.4	**función polinómica de grado impar** función polinómica en la cual el exponente mayor es un número impar. Un extremo del gráfico de una función polinómica de grado impar se extenderá hacia arriba y el otro extremo se extenderá hacia abajo.
one-to-one a relationship wherein each point in a set of points is mapped to exactly one other point	1.8	**unívoca** relación en la que cada punto de un conjunto de puntos se corresponde con otro con exactitud
one-to-one correspondence the feature of a function whereby each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$, and if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence	1.10	**correspondencia uno a uno** la característica de una función por la cual cada valor del dominio corresponde a un único valor de función; es decir, si $x = a$ y $x = b$, los dos puntos serían $(a, f(a))$ y $(b, f(b))$, y si $a \neq b$, entonces $f(a) \neq f(b)$ para que una función exhiba una correspondencia uno a uno
orthocenter the intersection of the altitudes of a triangle	4.8	**ortocentro** intersección de las alturas de un triángulo
outcome a result of an experiment	8.1	**resultado** consecuencia de un experimento
P		
parabola the U-shaped graph of a quadratic equation; the set of all points that are equidistant from a fixed line, called the directrix, and a fixed point not on that line, called the focus. The parabola, directrix, and focus are all in the same plane. The vertex of the parabola is the point on the parabola that is closest to the directrix.	3.2	**parábola** gráfico de una ecuación cuadrática en forma de U; conjunto de todos los puntos equidistantes de una línea fija denominada directriz y un punto fijo que no está en esa línea, llamado foco. La parábola, la directriz y el foco están todos en el mismo plano. El vértice de la parábola es el punto más cercano a la directriz.

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parallelogram a special type of quadrilateral with two pairs of opposite sides that are parallel; denoted by the symbol ▱	4.2	**paralelogramo** un tipo especial de cuadrilátero con dos pares de lados opuestos paralelos; se expresa con el símbolo ▱
parameter numerical value(s) representing the data in a set, including proportion, mean, and variance	8.3	**parámetro** valores numéricos que representan los datos en un conjunto, incluyendo la proporción, la media y la varianza
parent function a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.	6.1	**función principal** función con una regla algebraica simple que representa una familia de funciones. Los gráficos de las funciones en la familia tienen la misma forma general que la función principal.
percent of change $\frac{\text{amount of change}}{\text{original amount}}$, written as a percent	2.7	**porcentaje de cambio** $\frac{\text{porcentaje de cambio}}{\text{cantidad original}}$, se expresa como porcentaje
perfect square trinomial a trinomial of the form $x^2+bx+\left(\frac{b}{2}\right)^2$ that can be written as the square of a binomial	5.1	**trinomio cuadrado perfecto** trinomio de la forma $x^2+bx+\left(\frac{b}{2}\right)^2$ que puede expresarse como el cuadrado de un binomio
period in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*	7.3	**período** en una curva de la función del seno o coseno, distancia horizontal desde un máximo a un máximo o desde un mínimo a un mínimo; una repetición del período de una función se llama *ciclo*
periodic function a function whose values repeat at regular intervals	7.3	**función periódica** función cuyos valores se repiten a intervalos regulares
periodic phenomena real-life situations that repeat at regular intervals and can be represented by a periodic function	7.4	**fenómenos periódicos** situaciones de la vida real que se repiten a intervalos regulares y se pueden representar mediante una función periódica
***pi* (*π*)** the ratio of circumference of a circle to the diameter; equal to approximately 3.14	5.2	***pi* (*π*)** proporción de la circunferencia de un círculo al diámetro; equivale aproximadamente a 3.14

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piecewise function a function that is defined by two or more expressions on separate portions of the domain	1.6	**función por partes** función definida por dos o más expresiones en porciones separadas del dominio
placebo a substance that is used as a control in testing new medications; the substance has no medicinal effect on the subject	8.1	**placebo** sustancia que se utiliza como control en las pruebas de medicamentos nuevos; la sustancia no tiene efecto medicinal sobre el sujeto
plane a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions	4.4	**plano** figura plana, bidimensional, sin profundidad, que tiene al menos tres puntos no colineales y se extiende infinitamente en todas direcciones
plane figure a two-dimensional shape on a plane	4.4	**figura plana** forma bidimensional sobre un plano
point(s) of intersection in a graphed system of equations, the ordered pair(s) where graphed functions intersect on a coordinate plane	1.5	**punto(s) de intersección** en un sistema de ecuaciones graficado, par(es) ordenado(s) donde las funciones de la gráfica se cruzan sobre un plano de coordenadas
point of concurrency a single point of intersection of three or more lines	4.8	**punto de concurrencia** punto único de intersección de tres o más líneas
polyhedron a three-dimensional object that has faces made of polygons	4.4 4.5	**poliedro** objeto tridimensional que tiene caras compuestas por polígonos
polynomial function 1. a function whose rule is a one-variable polynomial; $P(x)$ is a polynomial function if $P(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer and $a_n \neq 0$ 2. a function of the general form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial	3.2 3.7	**función polinómica** 1. función cuya regla es un polinomio de una variable; $P(x)$ es una función polinómica si $P(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, donde n es un entero no negativo y $a_n \neq 0$ 2. función de la forma general $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, donde a_1 es un número racional, $a_n \neq 0$ y n es un número entero no negativo y el grado más alto del polinomio

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population average the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population mean*	8.7	**promedio de la población** suma de todas las cantidades de una población, dividida por el número total de cantidades de la población; representada normalmente por μ; también se conoce como *media poblacional*
population mean the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population average*	8.7	**media poblacional** suma de todas las cantidades de una población, dividida por el número total de cantidades de la población; representada normalmente μ; también se conoce como *promedio de la población*
power the quantity that shows the number of times the base is being multiplied by itself in an exponential expression, such as x in the logarithmic function $x = \log_5 g(x)$ and its exponential function, $g(x) = 5^x$	2.3	**potencia** cantidad que muestra el número de veces que se multiplica la base por sí misma en una expresión exponencial, tal que x en la función logarítmica $x = \log_5 g(x)$ y su función exponencial $g(x) = 5^x$
proportion a statement of equality between two ratios	6.7	**proporción** afirmación de igualdad entre dos relaciones
pyramid a solid or hollow polyhedron object that has three or more triangular faces that converge at a single vertex at the top; the base may be any polygon	4.5	**pirámide** objeto poliedro sólido o hueco con tres o más caras triangulares que convergen en un único vértice en la parte superior; la base puede ser cualquier polígono
Pythagorean Theorem a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.	5.1	**Teorema de Pitágoras** teorema que relaciona la longitud de la hipotenusa de un triángulo rectángulo (c) con las longitudes de sus catetos (a y b). El teorema establece que $a^2 + b^2 = c^2$.
	Q	
quadratic equation an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$	6.3	**ecuación cuadrática** ecuación que se puede expresar en la forma $ax^2 + bx + c = 0$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$

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quadratic expression an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$	6.3	**expresión cuadrática** expresión algebraica que se puede expresar en la forma $ax^2 + bx + c$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$
quadratic formula a formula that states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. A quadratic equation in this form can have no real solutions, one real solution, or two real solutions.	6.3	**fórmula cuadrática** fórmula que establece que las soluciones de una ecuación cuadrática de la forma $ax^2 + bx + c = 0$ están dadas por $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Una ecuación cuadrática en esta forma tener ningún solución real, o tener una solución real, o dos soluciones reales.
quadratic function a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.	1.10 2.8	**función cuadrática** función definida por una expresión de segundo grado de la forma $f(x) = ax^2 + bx + c$, donde $a \neq 0$ y a, b y c son constantes. La representación gráfica de toda función cuadrática es una parábola.
quadrilateral a polygon with four sides	4.2	**cuadrilátero** polígono con cuatro lados
	R	
radian the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°	5.4 7.1	**radián** medida del ángulo central que intercepta un arco de longitud igual al radio del círculo; π radianes = 180°
radian measure the ratio of the arc intercepted by the central angle to the radius of the circle	5.4	**medida de radián** proporción del arco interceptado por el ángulo central al radio del círculo
radical function a function with the independent variable under a root. The general form is $y=a\sqrt[n]{(x-h)}+k$, where n is a positive integer root and a, h, and k are real numbers.	1.2	**función radical** función con la variable independiente bajo una raíz. La forma general es $y=a\sqrt[n]{(x-h)}+k$, donde n es una raíz de entero positivo y a, h, y k son números reales.

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radius the distance from the center to a point on the circle; equal to one-half the diameter	5.1 5.2	**radio** distancia desde el centro a un punto en el círculo; equivale a la mitad del diámetro
random the designation of a group or sample that has been formed without following any kind of pattern and without bias. Each group member has been selected without having more of a chance than any other group member of being chosen.	8.4	**aleatorio** designación de un grupo o muestra que se formó sin seguir ninguna clase de patrón y sin sesgo. Cada miembro del grupo se selecciono sin tener más probabilidades de ser elegido que cualquier otro miembro del grupo.
random number generator a tool to select a number without following a pattern, where the probability of any number in the set being generated is equal	8.3	**generador de números aleatorios** herramienta para seleccionar un número sin seguir un patrón, por lo que la probabilidad de generar cualquier número del conjunto es igual
randomization the selection of a group, subgroup, or sample without following a pattern, so that the probability of any item in the set being generated is equal; the process used to ensure that a sample best represents the population	8.4	**aleatorización** selección de un grupo, subgrupo o muestra sin seguir un patrón, de manera que la probabilidad de cualquier elemento en el conjunto que está siendo generado sea igual; proceso utilizado para asegurar que la muestra sea la que mejor represente a la población
range the set of all outputs of a relation or function; the set of y-values for which a function is defined	1.10 6.1	**rango** conjunto de todas las salidas de una función; conjunto de valores de y para el que se define una función
rate a ratio that compares measurements with different kinds of units	6.8	**tasa** proporción que compara medidas con distintos tipos de unidades
rate of change a ratio that describes how much one quantity changes with respect to the change in another quantity; also known as the slope of a line	3.8	**tasa de cambio** proporción que describe cuánto cambia una cantidad con respecto al cambio de otra cantidad; también se la conoce como pendiente de una recta
ratio the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage	6.7	**proporción** relación entre dos cantidades; puede expresarse en palabras, fracciones, decimales o como porcentaje
rational equation an algebraic equation that contains at least one rational expression	6.7	**ecuación racional** una ecuación algebraica que contiene al menos una expresión racional

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rational exponent an exponent of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$. If m and n are positive integers and a is a real number, then $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$.	2.7	**exponente racional** exponente de la forma $\frac{m}{n}$, donde m y n son enteros y $n \neq 0$. Si m y n son enteros positivos y a es un número real, entonces $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$.
rational expression an expression made of the ratio of two polynomials, in which a variable appears in the denominator	6.3	**expresión racional** expresión que resulta de la relación de dos polinomios, en la cual una variable aparece en el denominador
rational function a function that can be written in the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$	6.8	**función racional** función que puede expresarse en la forma $f(x)=\frac{p(x)}{q(x)}$, donde $p(x)$ y $q(x)$ son polinomios y $q(x) \neq 0$
rational number any number that can be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; any number that can be written as a decimal that ends or repeats	6.7	**números racionales** números que pueden expresarse como $\frac{m}{n}$, en los que m y n son enteros y $n \neq 0$; cualquier número que puede escribirse como decimal finito o periódico
reciprocal a number that, when multiplied by the original number, has a product of 1	6.5	**recíproco** número que multiplicado por el número original tiene producto 1
rectangle a special parallelogram with four right angles	4.3	**rectángulo** paralelogramo especial con cuatro ángulos rectos
reference angle the acute angle that the terminal side makes with the x-axis. The sine, cosine, and tangent of the reference angle are the same as that of the original angle (except for the sign, which is based on the quadrant in which the terminal side is located).	7.1	**ángulo de referencia** el ángulo agudo que forma el lado terminal con el eje x. El seno, el coseno y la tangente del ángulo de referencia son iguales a los del ángulo original (con excepción del signo, que se basa en el cuadrante en el que se ubica el lado terminal).

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regular polyhedron a polyhedron with faces that are all congruent regular polygons; the angles created by the intersecting faces are congruent, and the cross sections are similar figures	4.4	**poliedro regular** poliedro cuyas caras son todas polígonos regulares congruentes; los ángulos creados por las caras que se cruzan son congruentes y las secciones transversales son figuras similares
relation a relationship between two variables in which at least one value of the domain or independent variable, x, is matched with one or more values of the dependent or range variable, y	1.9 1.10	**relación** relación entre dos variables en la que al menos un valor del dominio o variable independiente, x, concuerda con uno o más valores de la variable de rango o dependiente, y
relative maximum the greatest value of a function for a particular interval of the function; also known as a *local maximum*	3.4	**máximo relativo** el mayor valor de una función para un intervalo en particular de la función; también conocido como *máximo local*
relative minimum the least value of a function for a particular interval of the function; also known as a *local minimum*	3.4	**mínimo relativo** el menor valor de una función para un intervalo en particular de la función; también conocido como *mínimo local*
reliability the degree to which a study or experiment performed many times would have similar results	8.3	**confiabilidad** grado en el cual un estudio o experimento realizado varias veces tendría resultados similares
Remainder Theorem For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.	3.5	**Teorema del Resto** para un polinomio $p(x)$ y un número a, dividiendo $p(x)$ por $x - a$ resulta un resto de $p(a)$, entonces $p(a) = 0$ si y solo si $(x - a)$ es un factor de $p(x)$.
representative sample a sample in which the characteristics of the people, objects, or items in the sample are similar to the characteristics of the population	8.3	**muestra representativa** muestra en la cual las características de las personas, los objetos o elementos en ella son similares a las características de la población
response bias bias that occurs when responses by those surveyed have been influenced in some manner	8.8	**sesgo de respuesta** sesgo que se produce cuando las respuestas de los encuestados fueron influenciadas de alguna manera
rhombus a special parallelogram with all four sides congruent	4.3	**rombo** paralelogramo especial con sus cuatro lados congruentes

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English	Unit/Lesson	Español
right prism a three-dimensional solid that has two congruent bases and rectangular faces that join the two bases at 90° angles	4.4	**prisma recto** un sólido tridimensional que tiene dos bases congruentes y caras rectangulares que unen las dos bases a ángulos de 90°
root the *x*-intercept of a function; also known as *zero*	3.4	**raíz** intercepto de *x* de una función; también conocida como *cero*
root(s) solution(s) of a quadratic equation	3.2	**raíces** soluciones de una ecuación cuadrática
root with multiplicity a polynomial function with a root that occurs more than once; also known as a *multiple root*	3.2	**raíz con multiplicidad** función polinómica con una raíz que aparece más de una vez;; también conocido como *raíz múltiple*
rotation in three dimensions, a transformation in which a plane figure is moved about one of its sides, a fixed point, or a line that is not located in the plane of the figure, such that a solid figure is produced	4.4	**rotación** en tres dimensiones, transformación en la cual una figura plana se mueve sobre uno de sus lados, un punto fijo o una línea que no está ubicada en el plano de la figura, de manera que se produce una figura sólida
	S	
sample average the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample mean*	8.7	**promedio de la muestra** suma de todas las cantidades en una muestra dividida por el número total de cantidades en la muestra, normalmente representada por $\overline{x}$; también se conoce como *media de la muestra*
sample mean the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample average*	8.7	**media de la muestra** suma de todas las cantidades en una muestra dividida por el número total de cantidades en la muestra, normalmente representada por $\overline{x}$; también se conoce como *promedio de la muestra*
sample population a portion of the population; the number of elements or observations in a sample population is represented by *n*	8.6	**población de la muestra** porción de la población; la cantidad de elementos u observaciones en una población de muestra se representa por *n*

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sample proportion the fraction of favorable results p from a sample population n; conventionally represented by $\hat{p}$, which is pronounced "p hat." The formula for the sample proportion is $\hat{p}=\frac{p}{n}$, where p is the number of favorable outcomes and n is the number of elements or observations in the sample population.	8.6	**proporción de la muestra** fracción de los resultados favorables p de una población de muestra n; convencionalmente representada por $\hat{p}$, que se pronuncia "p hat". La fórmula para la proporción de la muestra es $\hat{p}=\frac{p}{n}$, donde p es la cantidad de resultados favorables y n es la cantidad de elementos u observaciones en la población de la muestra.
sample survey a survey carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population	8.1	**encuesta de muestra** encuesta realizada utilizando un método de muestreo para encuestar solo una porción de la población en lugar de toda la población
sampling error a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *chance variation*	8.3	**error de muestreo** medición que demuestra qué tan precisamente refleja una muestra a una población, con pequeños errores de muestreo ocasionados por muestras grandes y/o cuando los datos se agrupan estrechamente alrededor de la media; también se llama *variación aleatoria*
secant line a line that intersects a circle at two points	5.6	**línea secante** recta que corta un círculo en dos puntos
second difference in a set of data, the change in successive first differences	1.1	**segunda diferencia** en un conjunto de datos, el cambio en sucesivas primeras diferencias
sector a portion of a circle bounded by two radii and their intercepted arc	5.5	**sector** porción de un círculo limitado por dos radios y el arco que cortan
semicircle an arc that is half of a circle	5.2	**semicírculo** arco que es la mitad de un círculo
similar figures two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$	4.4	**figuras similares** dos figuras que tienen la misma forma pero no necesariamente el mismo tamaño; el símbolo que representa la similitud entre figuras es $\sim$

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English	Unit/Lesson	Español
simulation a set of data that models an event that could happen in real life	8.5	**simulación** conjunto de datos que imita un evento que podría suceder en la vida real
sine curve a curve with a constant amplitude and period, which are given by a sine or cosine function; also called a *sine wave* or *sinusoid*	7.3	**curva del seno** curva con amplitud y período constantes que están dados por una función seno o coseno; también se denomina *onda de seno* o *sinusoide*
sine function a trigonometric function of the form $f(x) = a \sin [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$	7.3	**función seno** función trigonométrica de la forma $f(x) = a \sin [b(x - c)] + d$, en la cual a, b, c y d son constantes y x es una variable expresada en radianes sobre el dominio $(-\infty,\infty)$
sine wave a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sinusoid*	7.3	**onda senoidal** curva con amplitud y período constantes dados por una función seno o coseno; también se denomina *curva del seno* o *sinusoide*
sinusoid a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sine wave*	7.3	**sinusoide** curva con amplitud o período constantes dados por una función seno o coseno; también se denomina *curva del seno* u *onda senoidal*
skew to distort or bias, as in data	8.4	**sesgar** distorsionar o afectar, como en el caso de los datos
solid figure a three-dimensional object that has length, width, and height (depth)	4.4	**figura sólida** objeto tridimensional que tiene largo, ancho y altura (profundidad)
solution set the set of ordered pairs that represent all of the solutions to an equation or a system of equations	1.5	**conjunto de soluciones** conjunto de pares ordenados que representa todas las soluciones para una ecuación o sistema de ecuaciones
sphere a three-dimensional surface that has all its points the same distance from its center	4.5	**esfera** superficie tridimensional que tiene todos sus puntos a la misma distancia de su centro
spread refers to how data is spread out with respect to the mean; sometimes called *variability*	8.7	**dispersión** forma en que los datos se esparcen con respecto a la media; algunas veces se denomina *variabilidad*
square a special parallelogram with four congruent sides and four right angles	4.3	**cuadrado** paralelogramo especial con cuatro lados congruentes y cuatro ángulos rectos

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square root function a function that contains a square root of a variable	1.2	**función raíz cuadrada** función que contiene una raíz cuadrada de una variable
standard error of the mean the variability of the mean of a sample; given by $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the number of elements or observations in the sample population	8.7	**error estándar de la media** variabilidad de la media de una muestra; dado por $\text{SEM} = \frac{s}{\sqrt{n}}$, donde s representa la desviación estándar y n la cantidad de elementos u observaciones en la población de la muestra
standard error of the proportion the variability of the measure of the proportion of a sample, abbreviated SEP. The standard error (*SEP*) of a sample proportion $\hat{p}$ is given by the formula $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion determined by the sample and n is the number of elements or observations in the sample population.	8.6	**error estándar de la proporción** variabilidad de la medida de la proporción de una muestra, abreviada SEP. El error estándar (*SEP*) de una proporción de la muestra $\hat{p}$ está dado por la fórmula $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, donde $\hat{p}$ es la proporción de la muestra determinada por la muestra y n representa la cantidad de elementos u observaciones en la población de la muestra.
standard form of an equation of a circle $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius	5.1	**forma estándar de ecuación de un círculo** $(x - h)^2 + (y - k)^2 = r^2$, donde (h, k) es el centro y r es el radio
standard form of a quadratic function a quadratic function written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term	3.2	**forma estándar de función cuadrática** función cuadrática expresada como $f(x) = ax^2 + bx + c$, donde a es el coeficiente del término cuadrático, b es el coeficiente del término lineal, y c es el término constante

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statistics a branch of mathematics focusing on how to collect, organize, analyze, and interpret information from data gathered; numbers used to summarize, describe, or represent sets of data	8.1	**estadística** rama de la matemática enfocada en la manera de recabar, organizar, analizar e interpretar la información proveniente de los datos reunidos; números utilizados para resumir, describir o representar conjuntos de datos
step function a function that is a series of disconnected constant functions	1.4	**función escalonada** función que es una serie de funciones constantes desconectadas
stratified sample a sample chosen by first dividing a population into subgroups of people or objects that share relevant characteristics, then randomly selecting members of each subgroup for the sample	8.2	**muestra estratificada** muestra escogida dividiendo primero una población en subgrupos de personas u objetos que comparten características relevantes, luego seleccionando al azar miembros de cada subgrupo para la muestra
subtended arc the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc	7.1	**arco subtendido** sección de un arco formada por un ángulo central que pasa por el círculo, creando así los puntos extremos del arco
success the data sought or hoped for, represented by p; also known as *desirable outcome* or *favorable outcome*	8.6	**éxito** datos buscados o esperados, representados por p; también conocido como *resultado deseado* o *resultado favorable*
survey a study of particular qualities or attributes of items or people of interest to a researcher	8.1	**encuesta** estudio de las cualidades o atributos particulares de elementos o personas de interés para un investigador
synthetic division a shorthand way of dividing a polynomial by a linear binomial	3.5	**división sintética** forma abreviada de dividir un polinomio por un binomio lineal
synthetic substitution the process of using synthetic division to evaluate a function by using only the coefficients	3.5	**sustitución sintética** proceso que utiliza la división sintética para evaluar una función utilizando solo los coeficientes
systematic sample a sample drawn by selecting people or objects from a list, chart, or grouping at a uniform interval; for example, selecting every fourth person	8.2	**muestra sistemática** la muestra se obtiene mediante la selección de personas u objetos a partir de una lista, una tabla o mediante la agrupación a intervalos regulares; por ej., eligiendo una de cada cuatro personas
system of equations a set of equations with the same unknowns	1.5	**sistema de ecuaciones** conjunto de ecuaciones con las mismas incógnitas

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	T	
tangent line a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle	5.6	**recta tangente** línea que corta un círculo en exactamente un punto y es perpendicular al radio del círculo
terminal side for an angle in standard position, the movable ray of an angle that can be in any location and which determines the measure of the angle	7.1	**lado terminal** para un ángulo en posición estándar, el rayo móvil de un ángulo que puede estar en cualquier ubicación y que determina la medida del ángulo
***theta* (θ)** a Greek letter commonly used to refer to unknown angle measures	7.1	***theta* (θ)** letra griega generalmente utilizada para referirse a las medidas de un ángulo desconocido
translation in three dimensions, the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced	2.2 4.4	**traslación** en tres dimensiones, movimiento horizontal o vertical de una figura plana en una dirección que no está en el plano de la figura, de manera que se produzca una figura sólida
trapezoid a quadrilateral with exactly one pair of opposite parallel lines	4.3	**trapezoide** cuadrilátero con exactamente un par de líneas paralelas opuestas
turning point a point where the graph of the function changes direction, from sloping upward to sloping downward or vice versa	3.4	**punto de inflexión** punto en el cual la gráfica de función cambia de dirección, de una inclinación o pendiente ascendente a una descendente o viceversa
	V	
validity the degree to which the results obtained from a sample measure what they are intended to measure	8.3	**validez** el grado en el cual los resultados obtenidos de una muestra miden lo que se pretende que midan
variability refers to how data is spread out with respect to the mean; sometimes called *spread*	8.7	**variabilidad** hace referencia al modo en que se distribuyen los datos respecto de la media; algunas veces se denomina *dispersión*
vertex a point at which two or more lines meet	1.2 1.3	**vértice** punto en el que se encuentran dos o más líneas
vertex form a quadratic function written as $f(x) = a(x - h)^2 + k$, where the vertex of the parabola is the point (h, k); the form of a quadratic equation where the vertex can be read directly from the equation	3.2	**fórmula de vértice** función cuadrática que se expresa como $f(x) = a(x - h)^2 + k$, donde el vértice de la parábola es el punto (h, k); forma de una ecuación cuadrática en la que el vértice se puede leer directamente de la ecuación

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vertical asymptote a line defined as follows: The line $x = a$ is a vertical asymptote of the graph of a function f if $f(x)$ either increases or decreases without bound as x gets closer to a.	6.1 6.2	**asíntota vertical** recta definida de la siguiente manera: La línea $x = a$ es una asíntota vertical del gráfico de una función f si $f(x)$ aumenta o disminuye sin límites a medida que x se acerca a a.
voluntary response bias bias that occurs when the sample is not representative of the population due to the sample having the option of responding to the survey	8.8	**sesgo de respuesta voluntaria** sesgo que se produce cuando la muestra no es representativa de la población debido a que en la muestra existe la opción de responder a la encuesta (la respuesta es optativa)
	X	
***x*-intercept** the point at which the graph crosses the x-axis; written as $(x, 0)$	6.2	**intercepto de *x*** punto en el que el gráfico cruza el eje x; se expresa como $(x, 0)$
	Y	
***y*-intercept** the point at which the graph crosses the y-axis; written as $(0, y)$	6.2	**intercepto de *y*** punto en el que el gráfico cruza el eje y; se expresa como $(0, y)$
	Z	
zeros the x-values of a function for which the function value is 0	6.2	**ceros** valores de x de una función para la que el valor de la función es 0

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North Carolina Math 3

Custom
Teacher Resource
Unit 3: Polynomial Functions

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This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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A–APR.2 Understand and apply the Remainder Theorem.

A–APR.3 Understand the relationship among factors of a polynomial expression, the solutions of a polynomial equation and the zeros of a polynomial function.

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Identify and interpret parts of a piecewise, absolute value, polynomial, exponential and rational expressions including terms, factors, coefficients, and exponents.

F–BF.1 Write a function that describes a relationship between two quantities.★

a. Build polynomial and exponential functions with real solution(s) given a graph, a description of a relationship, or ordered pairs (include reading these from a table).

F–IF.7 Analyze piecewise, absolute value, polynomials, exponential, rational, and trigonometric functions (sine and cosine) using different representations to show key features of the graph, by hand in simple cases and using technology for more complicated cases, including: domain and range; intercepts; intervals where the function is increasing, decreasing, positive, or negative; rate of change; relative maximums and minimums; symmetries; end behavior; period; and discontinuities.★

F–IF.9 Compare key features of two functions using different representations by comparing properties of two different functions, each with a different representation (symbolically, graphically, numerically in tables, or by verbal descriptions).

F–BF.1 Write a function that describes a relationship between two quantities.★

a. Build polynomial and exponential functions with real solution(s) given a graph, a description of a relationship, or ordered pairs (include reading these from a table).

F–LE.3 Compare the end behavior of functions using their rates of change over intervals of the same length to show that a quantity increasing exponentially eventually exceeds a quantity increasing as a polynomial function.★

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G–MG.1 Apply geometric concepts in modeling situations.★

- Use geometric and algebraic concepts to solve problems in modeling situations.
- Use geometric shapes, their measures, and their properties, to model real-life objects.
- Use geometric formulas and algebraic functions to model relationships.
- Apply concepts of density based on area and volume.
- Apply geometric concepts to solve design and optimization problems.

N–CN.9 Use the Fundamental Theorem of Algebra to determine the number and potential types of solutions for polynomial functions.

Essential Questions

1. How are terms arranged in a polynomial expression?
2. How can you determine which terms can be combined when simplifying polynomial expressions?
3. What are key features of polynomial functions?
4. How can the degree and sign of the leading coefficient of a polynomial function be used to determine the end behavior of that function?
5. How can the degree and sign of the leading coefficient of a polynomial function be used to determine the number of turns of that function?
6. How can you use the number of sign changes in a function to determine the number and type of real zeros of that function?
7. How can synthetic substitution be used to find the value of a function?
8. How are factors, zeros, and *x*-intercepts of a polynomial function related?
9. How can you find the degree of a polynomial from its graph?
10. What does the remainder of polynomial division tell you about the divisor?

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WORDS TO KNOW

average rate of change (3.8)
axis of symmetry of a parabola (3.2)
closure (3.1)
delta (**Δ**) (3.8)
depressed polynomial (3.5)
directrix of a parabola (3.2)
end behavior (3.4, 3.8)
even-degree polynomial function (3.4)
factored form of a quadratic function (3.2)
focus of a parabola (3.2)
integer (3.4)
intercept form of a quadratic function (3.2)
like terms (3.1)
local maximum (3.4)
local minimum (3.4)
multiple root (3.2)
odd-degree polynomial function (3.4)
parabola (3.2)
polynomial function (3.2, 3.7)
rate of change (3.8)
relative maximum (3.4)
relative minimum (3.4)
Remainder Theorem (3.5)
root (3.4)
root(s) (3.2)
root with multiplicity (3.2)
standard form of a quadratic function (3.2)
synthetic division (3.5)
synthetic substitution (3.5)
turning point (3.4)
vertex form (3.2)

Recommended Resources

- Hotmath.com. "Rational Zeros and the Fundamental Theorem of Algebra."

  http://www.walch.com/rr/00165

  This site provides numerous practice problems centering on determining the zeros of a function using the Fundamental Theorem of Algebra. Users may click to show hints and step-by-step guidance toward solutions.

- Illuminations. "Function Matching."

  http://www.walch.com/rr/00166

  Given a graphed function, users manipulate this interactive applet to recreate the graph of the identical function. Users can choose the type of function to match, or match a random function generated by the site.

- Khan Academy. "Multiply Binomials by Polynomials."

  http://www.walch.com/rr/00158

  Users can practice multiplying polynomials at this site. Four possible answers are given as well as options to show hints. If a question is answered incorrectly, users can click "Show solution" to be guided through the steps to find the answer before moving on to the next problem.

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- MathIsFun.com. "Function Grapher and Calculator."

  http://www.walch.com/rr/00169

  This site provides an online graphing utility. Users can enter values to create their own system and then zoom in on the functions to determine approximate intersection points.

- MathIsFun.com. "Remainder Theorem and Factor Theorem."

  http://www.walch.com/rr/00167

  This site summarizes both the Remainder Theorem and the Factor Theorem, and features examples as well as practice problems with worked solutions.

- Math Warehouse. "Polynomial Equations."

  http://www.walch.com/rr/00159

  This site defines polynomials and their components, with examples and non-examples.

- Purplemath.com. "Solving Polynomials."

  http://www.walch.com/rr/00168

  This site offers a summary of solving large polynomials in addition to information on factoring and graphing polynomial functions.

- Quia. "Battleship for Polynomials—Adding and Subtracting."

  http://www.walch.com/rr/00160

  This site uses the classic game of Battleship to help players practice Introduction to Polynomial Functions. Competing against the computer, players must answer the problems correctly in order to "hit" a battleship and ultimately sink it. Users may set the computer's skill level to easy, medium, or hard.

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Conceptual Activities

- Desmos. "Constructing Polynomials."

  http://www.walch.com/ca/01054

  In this activity, students will consider properties of polynomial functions such as end behavior, leading terms, and properties of roots. They will explore connections between those properties and the factored forms of the equations of the polynomials.

- Desmos. "Polygraph: Polynomial Functions."

  http://www.walch.com/ca/01055

  This activity is designed to spark vocabulary-rich conversations about polynomial functions. Key vocabulary that may appear in student questions includes: degree, roots, end behavior, limit, quadrant, axis, increasing, decreasing, maximum, minimum, extrema, concave up, and concave down.

- Desmos. "Polynomial Equation Challenges."

  http://www.walch.com/ca/01056

  In this activity, students will create polynomial equations (of degree 2, 3, and 4) to match given zeros and points. Students will explore how the factored form of the equations relates to the zeros and the order of those zeros.

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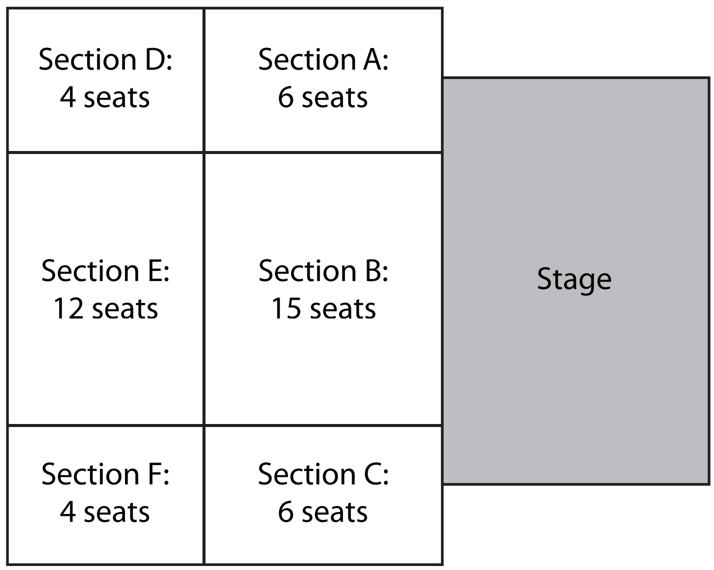
Lesson 3.1: Introduction to Polynomial Functions

Warm-Up 3.1

A band is holding a small concert. The seating is being sold in sections, and different sections have different numbers of seats. A diagram showing the sections and the number of seats in each section is shown as follows.



1. Mario bought all the seats in sections A and D. How many total seats did he buy?

2. Evan bought all the seats in sections B and C. How many total seats did he buy?

3. Tara is trying to buy enough seats for a total of 18 people. She would like her group to sit in sections E and F, which have all seats available. Will she be able to buy enough seats for her group?

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Instruction

Lesson 3.1: Introduction to Polynomial Functions

North Carolina Math 3 Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Identify and interpret parts of a piecewise, absolute value, polynomial, exponential and rational expressions including terms, factors, coefficients, and exponents.

Warm-Up 3.1 Debrief

1. Mario bought all the seats in sections A and D. How many total seats did he buy?

   Use the quantities shown in the diagram to find the total seats in sections A and D.

   Section A has 6 seats and Section D has 4 seats.

   The total number of seats bought is the sum of the seats in the two sections: $6 + 4 = 10$.

   Mario bought 10 total seats.

2. Evan bought all the seats in sections B and C. How many total seats did he buy?

   Use the quantities shown in the diagram to find the total seats in sections B and C.

   Section B has 15 seats and Section C has 6 seats.

   The total number of seats bought is the sum of the seats in the two sections: $15 + 6 = 21$.

   Evan bought 21 total seats.

3. Tara is trying to buy enough seats for a total of 18 people. She would like her group to sit in sections E and F, which have all seats available. Will she be able to buy enough seats for her group?

   Use the quantities shown in the diagram to find the total seats in sections E and F.

   Section E has 12 seats and Section F has 4 seats.

   The total number of seats in sections E and F is $12 + 4 = 16$.

   Therefore, there are 16 seats available in these two sections.

   Compare the number of seats available and the number of seats Tara needs.

   Tara's group needs 18 seats.

   $16 < 18$

   16 is less than 18; because there are only 16 seats available, and Tara needs 18 seats, Tara will not be able to buy enough seats in sections E and F for her group.

Connection to the Lesson

- Students will write expressions by writing a sum of terms, including constants.
- Students will write expressions using quantities represented in a diagram.

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Prerequisite Skills

This lesson requires the use of the following skills:

- using variables to represent unknown quantities (6.EE.2a)
- adding and subtracting real numbers (8.NS.2)
- calculating perimeter (3.MD.8)

Introduction

Polynomials, or expressions that contain variables, numbers, or combinations of variables and numbers, can be added and subtracted. Introduction to Polynomial Functions is a way to combine and simplify expressions in order to find a different, but equivalent, representation for a sum or difference.

Key Concepts

- **Like terms** are terms that contain the same variables raised to the same powers. For example, the terms $2a^3$ and $-9a^3$ are like terms because both contain the variable a raised to the third power.
- To add two polynomials, combine like terms by adding the coefficients of the terms. The variable and exponent do not change when adding like terms.
- When two polynomials are added, the result is another polynomial. Because the sum of two polynomials is also a polynomial, the system of polynomials is closed under the operation of addition. A system is closed, or shows **closure**, for a given operation if the result of the operation is within the system.
- For example, the set of integers is closed under the operation of multiplication because the product of two integers is always an integer. However, the set of integers is *not* closed under division, because in some cases the quotient of two integers is not an integer—for example, $3 \div 2 = 1.5$, and 1.5 is not an integer.
- To subtract two polynomials, first rewrite the subtraction using addition by distributing the subtraction to each term in the second polynomial. For example, $a + 1 - (a - 10) = a + 1 + (-a + 10)$. After rewriting using addition, combine like terms.
- Because subtraction of polynomials can be rewritten as addition, and polynomials are closed under the operation of addition, polynomials are also closed under the operation of subtraction.

Common Errors/Misconceptions

- combining two terms with the same variable raised to different powers
- incorrectly changing a subtraction problem to an addition problem by inaccurately distributing the negative sign

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Scaffolded Practice 3.1: Introduction to Polynomial Functions

Identify the terms in each expression, and note the coefficient, variable, and power of each term.

1. $4x^3 + 2x$

2. $-n^5 + 3n + 1$

3. $2r^2 - 10r + 15$

4. $-6v^9 - v^7 - v^5 - v^3$

5. $a^6 - 9a^4 + 7a^3 + 4a - 7$

continued

Write a polynomial function using the given terms. Determine the degree of each polynomial function.

6. $8, 16x^8, x^5$

7. $-20x, 9x^2$

8. x^3, x^2, x^4, x

9. $-3x^6, -x^5, 12, -x$

10. $14x, 11x^4, -30, 2x^2$

Guided Practice 3.1

Example 1

Simplify $(2x^2 + x + 10) + (7x^2 + 14)$.

1. Rewrite the sum so that like terms are together.

   The first polynomial, $2x^2 + x + 10$, has a term with an exponent of 2, a term with an exponent of 1, and a constant term. The second polynomial, $7x^2 + 14$, has a term with an exponent of 2 and a constant term. The terms with the same exponents and variables are like terms, and the constants are also like terms.

   Rearrange the terms of the given expression in descending order.

   $(2x^2 + x + 10) + (7x^2 + 14)$

   $= 2x^2 + 7x^2 + x + 10 + 14$

2. Find the sum of the constant terms.

   The previous expression contains two constant terms: 10 and 14.

   $2x^2 + 7x^2 + x + 10 + 14$

   $= 2x^2 + 7x^2 + x + 24$

3. Find the sum of any terms with the same variable raised to the same power by adding the coefficients of the terms.

$2x^2 + 7x^2 + x + 24$	Expression from the previous step
$= (2 + 7)x^2 + x + 24$	Combine like terms by adding coefficients.
$= 9x^2 + x + 24$	Simplify.

   $(2x^2 + x + 10) + (7x^2 + 14)$ is equivalent to $9x^2 + x + 24$.

Example 2

Simplify $(6x^4 - x^3 - 3x^2 + 20) + (10x^3 - 4x^2 + 9)$.

1. Rewrite subtraction using addition.

   Subtraction can be rewritten as adding a negative.

   $(6x^4 - x^3 - 3x^2 + 20) + (10x^3 - 4x^2 + 9)$

   $= [6x^4 + (-x^3) + (-3x^2) + 20] + [10x^3 + (-4x^2) + 9]$

2. Rewrite the sum so that like terms are together.

   Be sure to keep any negative signs with the coefficients.

   $[6x^4 + (-x^3) + (-3x^2) + 20] + [10x^3 + (-4x^2) + 9]$

   $= 6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 20 + 9$

3. Find the sum of the constant terms.

   The previous expression contains two constant terms: 20 and 9.

   $6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 20 + 9$

   $= 6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 29$

4. Find the sum of any terms with the same variable raised to the same power.

   The previous expression contains the following like terms: $(-x^3)$ and $10x^3$; $(-3x^2)$ and $(-4x^2)$.

   Add the coefficients of like terms, being sure to keep any negative signs with the coefficients.

$6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 29$	Expression from the previous step
$= 6x^4 + (-1 + 10)x^3 + [-3 + (-4)]x^2 + 29$	Combine like terms by adding coefficients.
$= 6x^4 + 9x^3 - 7x^2 + 29$	Simplify.

   $(6x^4 - x^3 - 3x^2 + 20) + (10x^3 - 4x^2 + 9)$ is equivalent to $6x^4 + 9x^3 - 7x^2 + 29$.

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Example 3

Simplify $(-x^6 + 7x^2 + 11) - (12x^6 + 4x^5 - 2x + 1)$.

1. Rewrite the difference as a sum.

   A difference can be written as adding a negative quantity.

   Distribute the negative sign in the second polynomial.

   $(-x^6 + 7x^2 + 11) - (12x^6 + 4x^5 - 2x + 1)$

   $= (-x^6 + 7x^2 + 11) + [(-12x^6) + (-4x^5) + 2x + (-1)]$

2. Rewrite the sum so that like terms are together.

   Be sure to keep any negative signs with the coefficients. Write the terms in descending order.

   $(-x^6 + 7x^2 + 11) + [(-12x^6) + (-4x^5) + 2x + (-1)]$

   $= -x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 11 + (-1)$

3. Find the sum of the constant terms.

   The previous expression contains two constant terms: 11 and –1.

   $-x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 11 + (-1)$

   $= -x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 10$

4. Find the sum of any terms with the same variable raised to the same power.

   The previous expression contains the following like terms: $-x^6$ and $(-12x^6)$.

$-x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 10$	Expression from the previous step
$= [(-1) + (-12)]x^6 + (-4x^5) + 7x^2 + 2x + 10$	Combine like terms by adding coefficients.
$= -13x^6 + (-4x^5) + 7x^2 + 2x + 10$	Simplify.
$= -13x^6 - 4x^5 + 7x^2 + 2x + 10$	Rewrite "$+ (-4x^5)$" as subtraction.

$(-x^6 + 7x^2 + 11) - (12x^6 + 4x^5 - 2x + 1)$ is equivalent to $-13x^6 - 4x^5 + 7x^2 + 2x + 10$.

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Problem-Based Task 3.1: Laying Tile

SMP
1 2
3 4 ✓
5 6 ✓
7 ✓ 8

A contractor is creating a design using different-sized rectangular tiles. The area of each tile is shown in the diagram. What is the total area of the shown strip of tile? (Diagram not shown to scale.)

36 in^2	$3x$ in^2	x^2 in^2

Problem-Based Task 3.1: Laying Tile

Coaching

a. How do you find the total area of the strip of tile?

b. What expression can be written to show the total area of the strip of tile?

c. Typically, in what order are the terms in a polynomial expression written?

d. What is the polynomial expression that shows the total area of the strip of tile?

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Problem-Based Task 3.1: Laying Tile

Coaching Sample Responses

a. How do you find the total area of the strip of tile?

To find the total area, add together the areas of each component.

b. What expression can be written to show the total area of the strip of tile?

The areas of each component of the strip of tile are 36 in^2, $3x$ in^2, and x^2 in^2. The total area is the sum of these three smaller areas, or $(36 + 3x + x^2)$ in^2.

c. Typically, in what order are the terms in a polynomial expression written?

In a polynomial expression, the terms are typically listed in descending order based on the powers of the variable; the term with the greatest exponent is listed first, followed by the term with the next greatest exponent, and so on. The constant term is listed last.

d. What is the polynomial expression that shows the total area of the strip of tile?

The term with the highest power is x^2, so it is listed first. This is followed by $3x$, and the constant term, 36, is last.

Therefore, the total area can be represented by the expression $(x^2 + 3x + 36)$ in^2.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.1: Introduction to Polynomial Functions A

Identify the terms in each expression, and note the coefficient, variable, and power of each term.

1. $y^4 + 13$

2. $8c^3 - c^2 + 8c$

3. $12z^5 + 9z^2 - z - 7$

4. $-5m^{10} + m^8 + 5m^6 - m^4$

Write a polynomial function using the given terms. Determine the degree of each polynomial function.

5. $30x, x^8, -3x^3$

6. $14x^2, -6x^3, 10, -x^6$

7. $-x^5, 2x, 22, x^7, 7x^3$

continued

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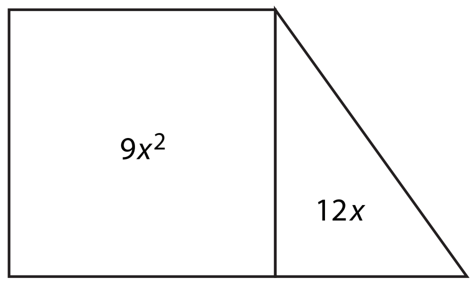
Each of the figures below is divided into separate parts with each area written within that part. Find the total area of each figure. All units are in square inches.

8.



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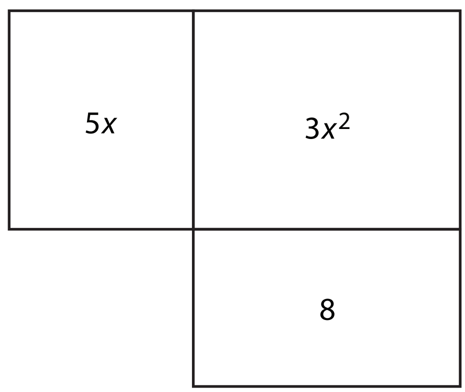

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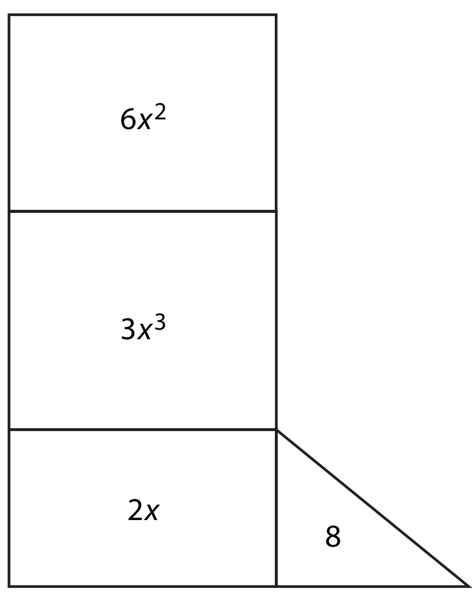

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Practice 3.1: Introduction to Polynomial Functions B

Identify the terms in each expression, and note the coefficient, variable, and power of each term.

1. $-k - 1$

2. $2p^3 + p^2 + 30$

3. $-4b^4 + 3b^3 + 2b^2 + 1$

4. $8x^{12} - 7x^2 + 6x + 2$

Write a polynomial function using the given terms. Determine the degree of each polynomial function.

5. $-5x^2, -3x^5, -6x^3$

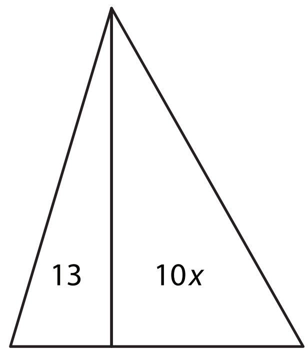
6. $15x, x^4, 9, -x^2$

7. $4x, -2x^6, 10x^3, 20, -x^5$

continued

Each of the figures below is divided into separate parts with each area written within that part. Find the total area of each figure. All units are in square inches.

8.



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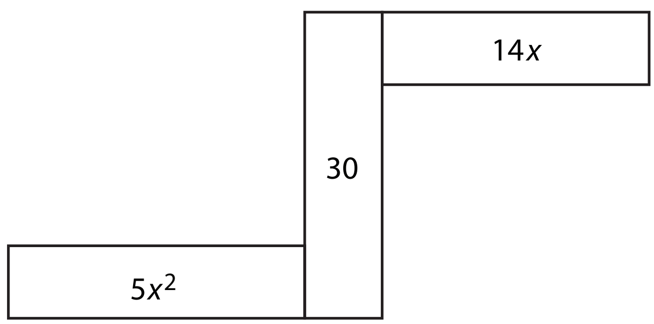

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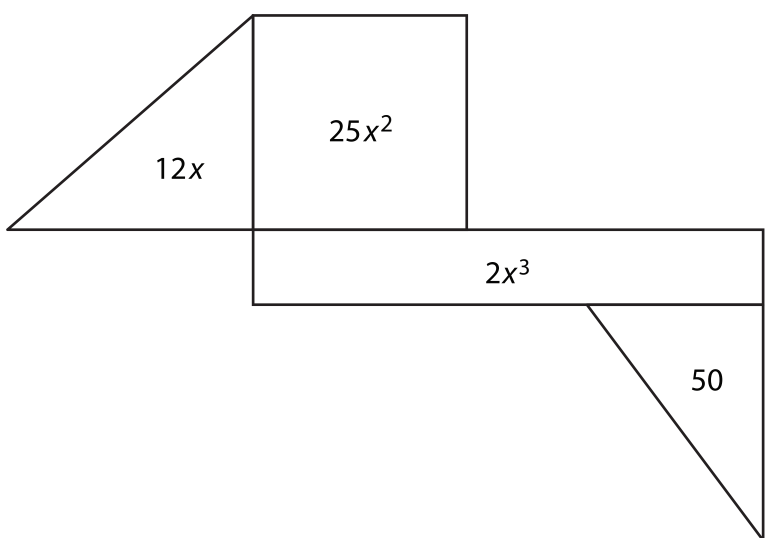

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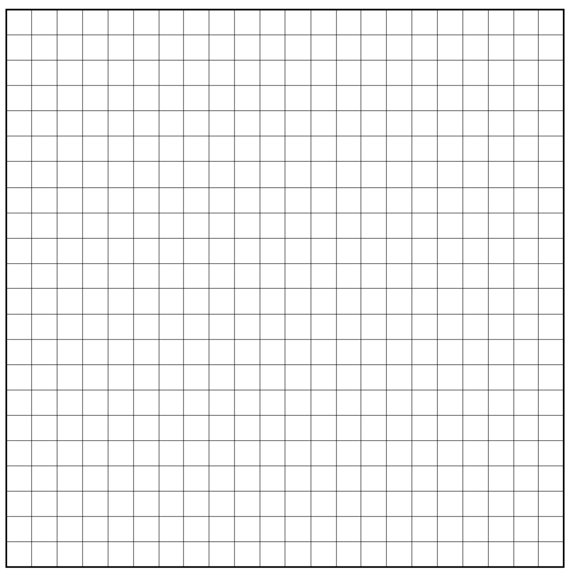
Lesson 3.2: Graphing Quadratic and Cubic Functions

Warm-Up 3.2

Lorenzo is sketching some designs for valley slopes in a new video game. Help Lorenzo complete the sketches by graphing each function. Create a table of values from $x = -5$ to $x = 5$ for each function to aid in your drawing.

1. $f(x) = (x+1)^2 - 2$



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2. $g(x) = (x+1)^3 + 2$

3. $h(x) = x^3 - 4x$

Lesson 3.2: Graphing Quadratic and Cubic Functions

North Carolina Math 3 Standard

F–IF.7 Analyze piecewise, absolute value, polynomials, exponential, rational, and trigonometric functions (sine and cosine) using different representations to show key features of the graph, by hand in simple cases and using technology for more complicated cases, including: domain and range; intercepts; intervals where the function is increasing, decreasing, positive, or negative; rate of change; relative maximums and minimums; symmetries; end behavior; period; and discontinuities.★

Warm-Up 3.2 Debrief

1. $f(x) = (x+1)^2 - 2$

First, build the table of values.

Construct the table and list the input values.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y											

Next, evaluate the function $f(x) = (x+1)^2 - 2$ at the input values.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	14	7	2	–1	–2	–1	2	7	14	23	34

To sketch the graph, plot each of the coordinate pairs from the table of values on a grid. Note that you might not be able to fit them all on the grid, depending on the dimensions you chose. In this case, (4, 23) and (5, 34) are off the visible grid.

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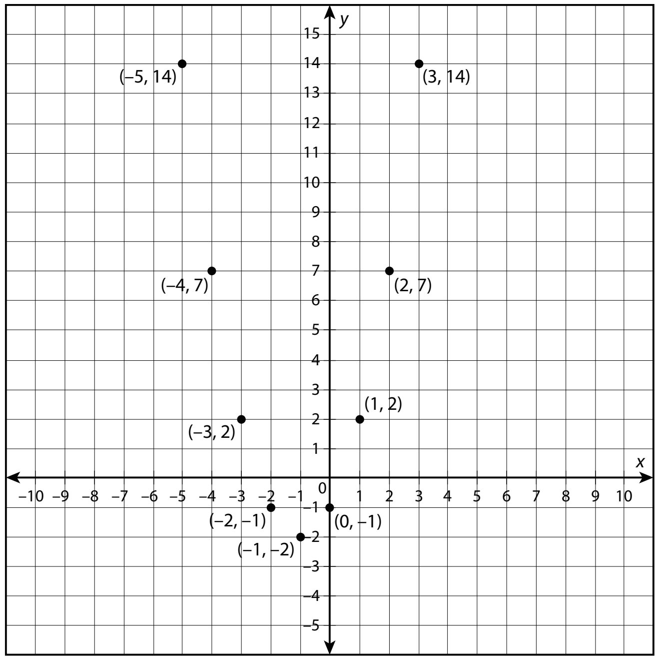
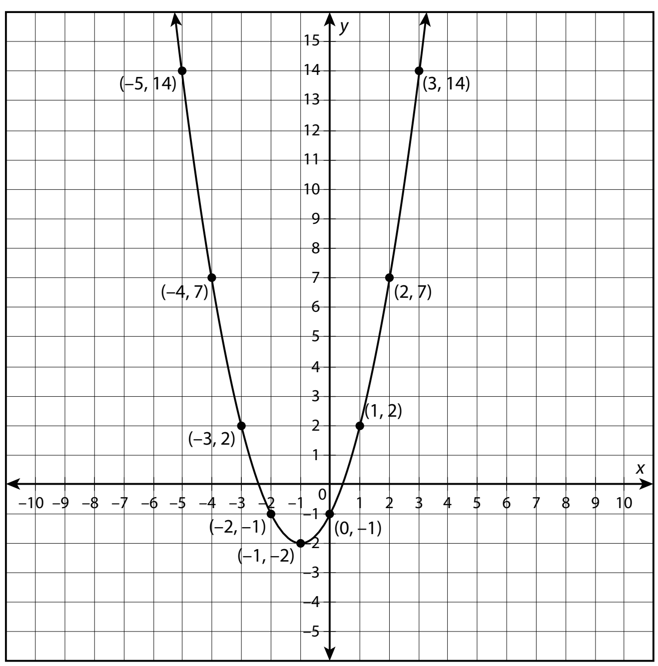


Finally, draw a curve passing through all the plotted points.



2. $g(x) = (x+1)^3 + 2$

Create the table of values by listing each input and the corresponding output below it.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–62	–25	–6	1	2	3	10	29	66	127	218

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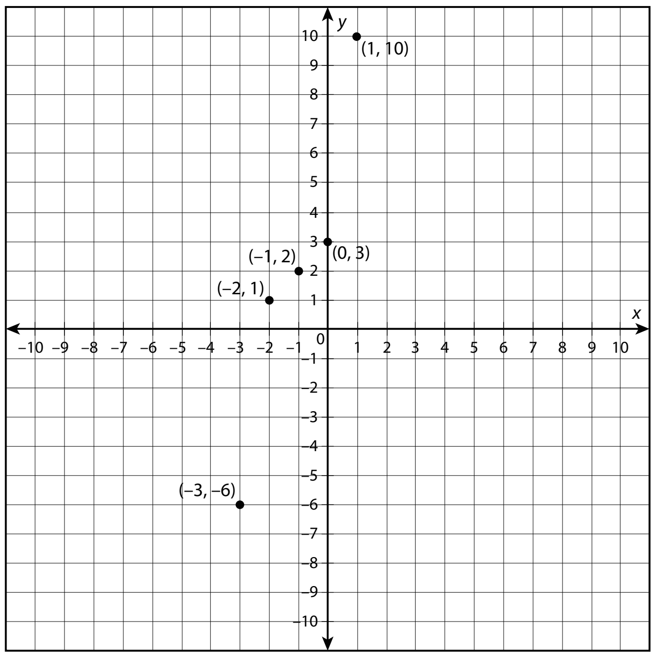
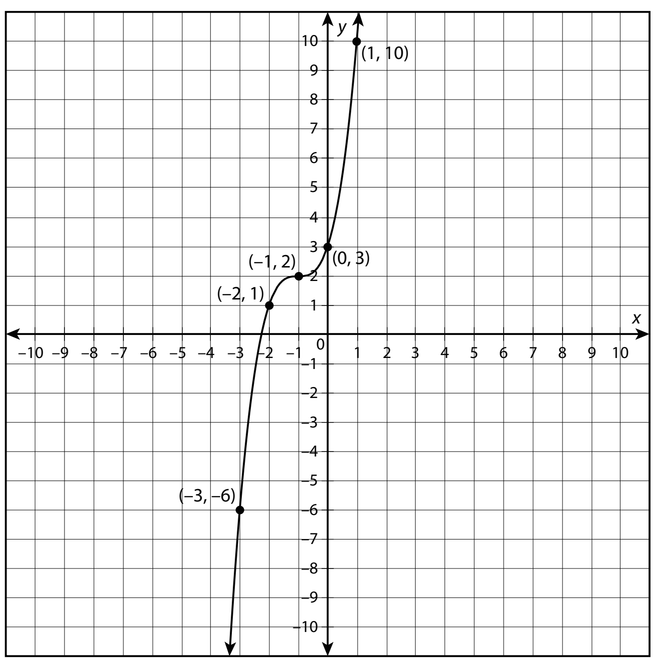
To sketch the graph, plot each of the coordinate pairs from the table of values on a grid. Note that you might not be able to fit them all on the grid, depending on the dimensions you chose. In this case, only values from x = –3 to x = 1 are visible.



Finally, draw a curve passing through all the plotted points.



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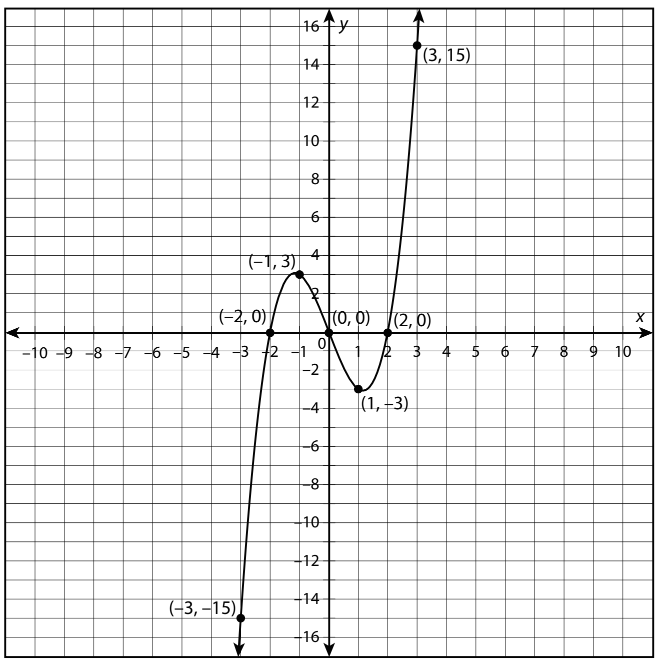
3. $h(x) = x^3 - 4x$

Create the table of values by listing each input and the corresponding output below it.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–105	–48	–15	0	3	0	–3	0	15	48	105

To sketch the graph, plot each of the coordinate pairs from the table of values on a grid. Then draw a curve passing through all the points.



Notice that this function has three zeros, or *x*-intercepts: *x* = –2, *x* = 0, and *x* = 2.

Connection to the Lesson

- Students will identify the roots of a function from a graph or a table.
- Students will graph quadratic and cubic functions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- rewriting expressions (A–SSE.3★)
- plotting points on a coordinate graph (6.NS.8)

Introduction

The U-shaped graph of a quadratic function is one of the most recognizable shapes of algebra. The shape of this graph has many properties that are useful for real-world applications, such as focusing light in a solar array or receiving radio signals.

Quadratic functions are actually members of a larger group of functions called **polynomial functions**. Polynomial functions are similar in structure to quadratic functions, but the largest power to which a variable is raised may be greater than 2. In this lesson, we will investigate two basic types of polynomial functions: quadratics and cubics.

Key Concepts

- A polynomial function has the general form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.
- The **degree** of a polynomial is the value of the exponent of the variable raised to the highest power.
- A quadratic function is a polynomial function of degree 2.
- A cubic function is a polynomial function of degree 3.
- It is important to understand behavior of quadratic and cubic functions because the graph of any polynomial function will generally resemble one of these two functions.

Graphing Quadratic Functions

- Quadratic functions can be written in different forms to reveal information about the graph.
- The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$. In this form, the value c is the y-intercept of the graph. The standard form is used as a base to convert between different forms.
- The **factored form**, or **intercept form**, of a quadratic function is $f(x) = a(x - p)(x - q)$. In this form, p and q are the x-intercepts of the function. If the function never touches the x-axis, p and q will be complex numbers.
- You can find the factored form from standard form by inspection, by completing the square, or by using the quadratic formula.

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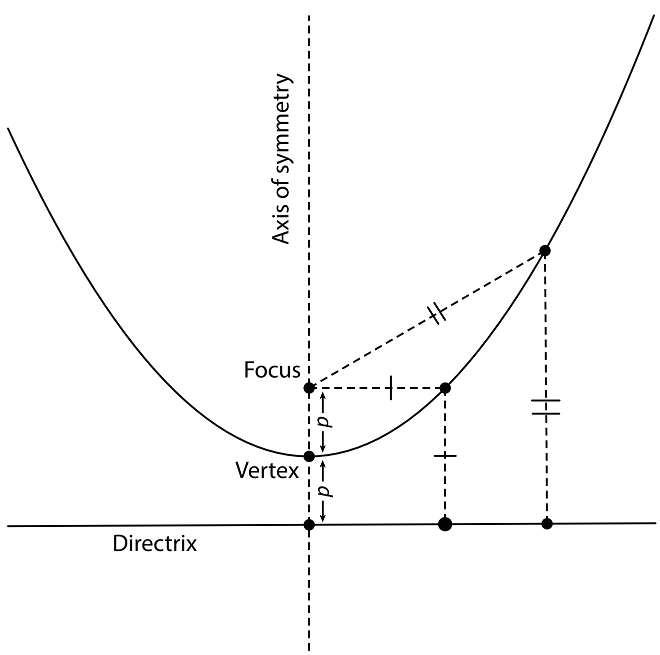
- The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$. In this form, the point (h, k) is the vertex of the graph.
- Every parabola has a vertex. The vertex is either a minimum (low point) or a maximum (high point).
- The vertex form also reveals the **axis of symmetry** of the parabola. The axis of symmetry is the line that divides the graph of the quadratic function into equal halves. The axis of symmetry is the line $x = h$.
- You can find the vertex of a quadratic function in standard form by completing the square and identifying (h, k). Or, you can use a formula.
- The formula for finding the vertex of a quadratic function from standard form is $(h,k) = \left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.
- In all three forms, when a is positive, the graph opens upward. When a is negative, the graph opens downward.
- The graph of a quadratic function is a **parabola**.
- Formally, it is defined as the set of all points that are equidistant from a given fixed point and a given fixed line in the plane.
- The given fixed line is called the **directrix** of the parabola.
- The given fixed point is called the **focus**.
- The vertex lies midway between the focus and the directrix on the axis of symmetry.



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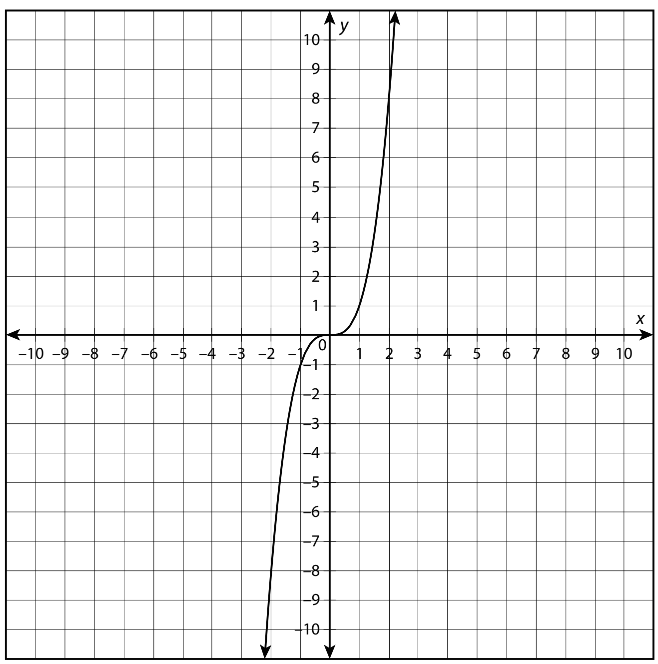
- There are two standard forms for the equation of a parabola, depending on whether the parabola opens upward/downward or left/right. In this lesson, we will only cover parabolas opening upward/downward.
- The standard form for a parabola opening up or down is $(x - h)^2 = 4p(y - k)$. In this equation, the point (h, k) is the vertex of the equation.
- The value $|p|$ represents the distance from the vertex to the focus. It is also the distance from the vertex to the directrix.
- The vertex form of a quadratic equation is related to the standard form for a parabola in that $a = \dfrac{1}{4p}$.

Graphing Cubic Functions

- The parent graph of a quadratic function is $f(x) = x^3$, graphed below.



- The general form of a cubic function is $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, and d are constants and $a \neq 0$.
- The constant d represents the y-intercept.
- The sign of a determines the overall behavior of the function.
- If $a > 0$, the function increases overall from left to right.
- If $a < 0$, the function decreases overall from left to right.

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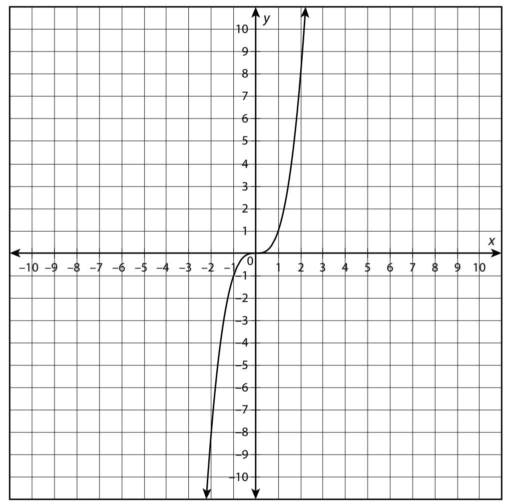
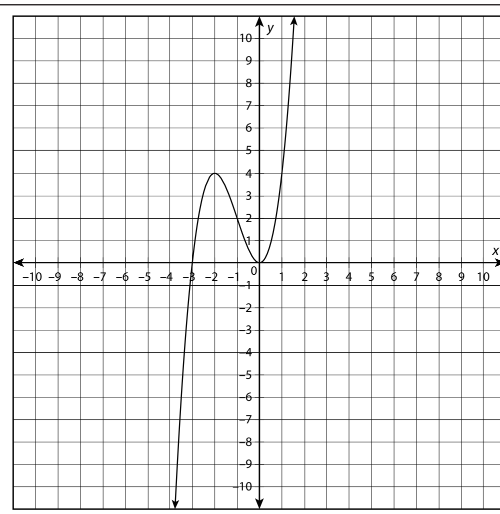
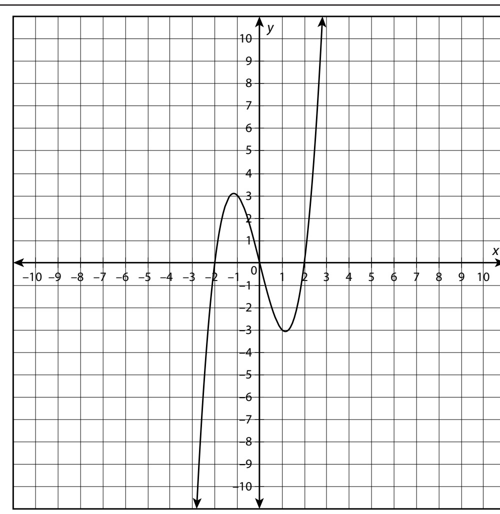
- The other two terms affect the number and shape of the "bumps" on the graph.
- Cubic functions can have one, two, or three x-intercepts.

One intercept	$f(x) = x^3$	
Two intercepts	$f(x) = x^3 + 4x^2 + 4x$	
Three intercepts	$f(x) = x^3 - 4x$	

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- The **roots** of a cubic function are the solutions to the equation $0 = ax^3 + bx^2 + cx + d$.
- The *x*-intercepts of a cubic function correspond to its real roots.
- A cubic function must have three roots. However, not all roots have to be real. We already know that there must be at least one real root because there must be at least one *x*-intercept. Since complex roots only come in pairs, there are two cases to consider:
    - The function has 3 real roots.
    - The function has 1 real and 2 complex roots.
- When a root is repeated, it is called a **multiple root** or a **root with multiplicity**. A cubic function can have a root with multiplicity 2 or 3.

Common Errors/Misconceptions

- incorrectly evaluating expressions when creating a table of values
- incorrectly plotting *x*-*y* coordinates
- reversing domain and range

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Scaffolded Practice 3.2: Graphing Quadratic and Cubic Functions

Complete each problem as directed.

1. Create a table of values for the function $f(x) = x^2 - 4x + 3$ using the domain $-5 \le x \le 5$.

2. Create a table of values for the function $f(x) = x^3 + x^2 - 4x + 1$ using the domain $-5 \le x \le 5$.

3. Given the function $f(x) = -x^2 + x + 12$, find the focus and the directrix.

4. Given the function $f(x) = -2x^2 + 8x - 2$, find the vertex and the axis of symmetry.

5. Does the function $f(x) = -3x^2 - 5x + 4$ open up or down?

6. Is the function $f(x) = -x^3 + 2x^2 + 5x + 1$ increasing or decreasing overall?

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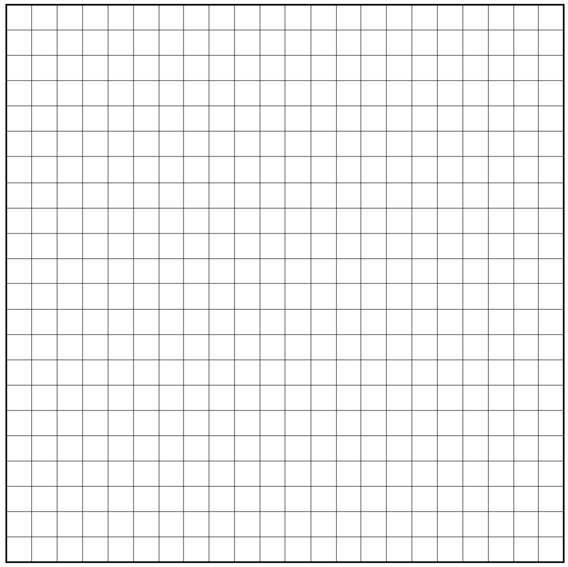
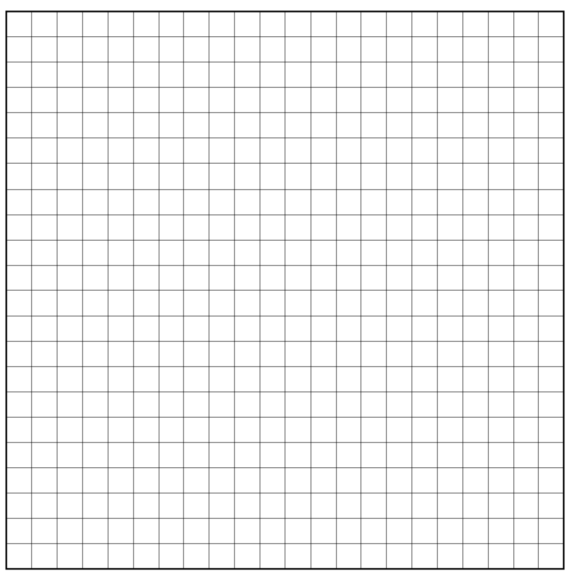
7. Sketch the graph of the function $f(x) = x^2 + 7x + 12$.



8. Sketch the graph of the function $f(x) = -x^2 - x + 12$.



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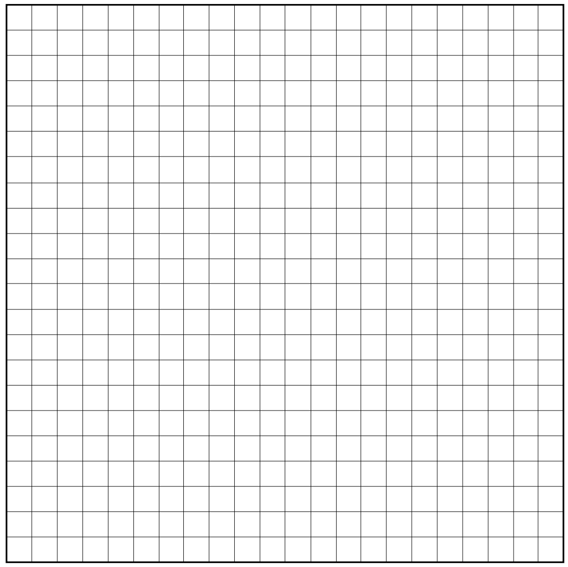
9. Sketch the graph of the function $f(x) = x^3 - 10x^2 + 20x$.



10. Using either a table of values or a graph, identify the zeros of the function $f(x) = x^3 - 12x^2 + 44x - 48$.

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Guided Practice 3.2

Example 1

Given the function $f(x) = -8x^2 - 10x + 3$, identify the roots, the vertex, and the axis of symmetry of the function.

1. Set the equation equal to zero and factor the equation to identify the roots.

$-8x^2 - 10x + 3 = 0$	Original equation
$(-2x - 3)(4x - 1) = 0$	Factored equation

The two equations to solve are $-2x - 3 = 0$ and $4x - 1 = 0$.

$-2x - 3 = 0$	$4x - 1 = 0$	Equations found in previous step
$-2x = 3$	$4x = 1$	Add.
$x = -\dfrac{3}{2}$	$x = \dfrac{1}{4}$	Divide.

The roots are $x = -\dfrac{3}{2}$ and $x = \dfrac{1}{4}$.

2. Find the vertex of the function.

The formula for the vertex is $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$. In our function $f(x) = -8x^2 - 10x + 3$, $a = -8$ and $b = -10$.

$$h = \frac{-(-10)}{2(-8)} = -\frac{5}{8}$$

$$k = f\left(-\frac{5}{8}\right) = -8\left(-\frac{5}{8}\right)^2 - 10\left(-\frac{5}{8}\right) + 3 = \frac{49}{8}$$

The vertex is $\left(-\dfrac{5}{8}, \dfrac{49}{8}\right)$.

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3. Identify the axis of symmetry.

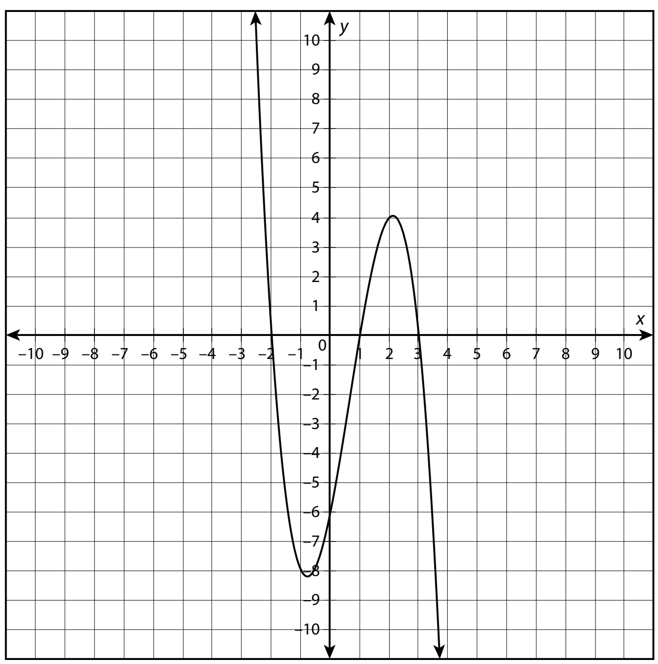
The axis of symmetry is always equal to the *x*-value of the vertex.

The axis of symmetry is $x = -\frac{5}{8}$.

Example 2

Graph the function $f(x) = -x^3 + 2x^2 + 5x - 6$ and describe whether it is overall increasing or decreasing.

1. Graph the function.



2. Identify the overall behavior of the function.

The function is overall decreasing from left to right.

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Example 3

A rectangular storage shed has a volume of 528 cubic feet. The shed is 3 feet deeper than it is wide. The shed is 2 feet shorter than it is wide. What are the dimensions of the shed?

1. Determine what the variable x should represent.

   The dimensions are given compared to the width of the shed. The variable x should represent the width of the shed.

2. Assign values for the other two dimensions in terms of x.

   Since the shed is 3 feet deeper than it is wide, this dimension can be represented by $x + 3$. The shed's height is 2 feet less than the depth, so this dimension can be represented by $x - 2$.

3. Set up an equation for the volume.

   The volume of the shed is the product of its dimensions. It is also equal to 528 cubic feet. Multiply the factors together and set them equal to 528.

   $$528 = x(x + 3)(x - 2)$$

   This equation states that the product of the factors must be equal to 528. Now we must find values of x that make the equation true. To do this, we would solve for x. However, we don't know how to solve cubic equations. Therefore, we will instead create a polynomial and identify the zeros.

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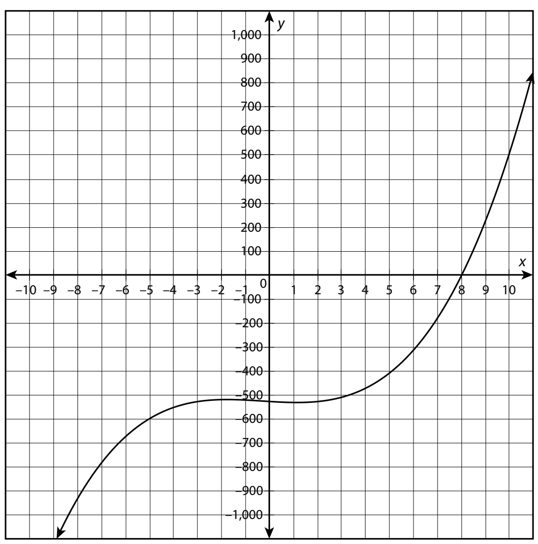
4. Create a polynomial.

   The expression on the right looks like a polynomial, but a number is on the left. Rearrange the equation so that the entire expression is equal to 0.

$528 = x(x + 3)(x - 2)$	Equation from the previous step
$528 = x(x^2 + x - 6)$	Multiply the binomials.
$528 = x^3 + x^2 - 6x$	Distribute.
$0 = x^3 + x^2 - 6x - 528$	Subtract 528 from both sides.

   This equation states that the product of the factors minus 528 must be zero. This is the same equation we would use if we were trying to find the x-intercepts of a polynomial. Therefore, we can solve the problem if we graph the polynomial $p(x) = x^3 + x^2 - 6x - 528$ and identify the zeros.

5. Graph the equation to determine the zero(s).



   The only x-intercept is (8, 0).

6. State the solution.

   The only x-intercept of the polynomial was (8, 0), so $x = 8$ must be the width of the shed. The other dimensions are 11 feet for the depth and 6 feet for the height. Thus, the dimensions are $8 \times 11 \times 6$ ft.

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Example 4

Meghan is designing a device that includes a parabolic trough mirror to focus light into a line. The entire device will fit inside a square box measuring $10 \times 10 \times 10$ cubic centimeters.

Consider a cross-section of the box where the edges of the box are located at (–5, 0) and (5, 0). Meghan wants the vertex of the parabola to lie in the middle of the box, so she has taken the directrix to be $y = 0$. If the light must be focused at the point (3, 10), what is a quadratic equation that represents the location of the mirror? Draw a picture of this situation.

1. Identify the given information.

   The focus is (3, 10). The directrix is $y = 0$.

2. Plan a strategy.

   We know that the vertex of a parabola is equidistant from the directrix and the focus. We also know that the vertex of a parabola lies on the axis of symmetry with the focus. In the general form of a parabola, $(x - h)^2 = 4p(y - k)$, p represents the distance from the focus to the vertex and (h, k) represents the vertex. We need to find the value of p from the focus and directrix and use it to find the vertex. Then, we can use the general form for the equation of a parabola to create a quadratic equation.

3. Find the vertex of the parabola.

   The distance from the focus to the directrix is 10 cm because the box is 10 cm high and the directrix and focus are located on opposite sides of the box. Since the vertex is equidistant from both, $|p| = \frac{10}{2} = 5$.

   As the vertex is located on the axis of symmetry with the focus (3, 10), its x-coordinate stays the same. The parabola opens upward, so the vertex will be located below the focus. Subtracting the value of p from the y-coordinate of the focus, we get $(h, k) = (3, 5)$ as the vertex of the parabola.

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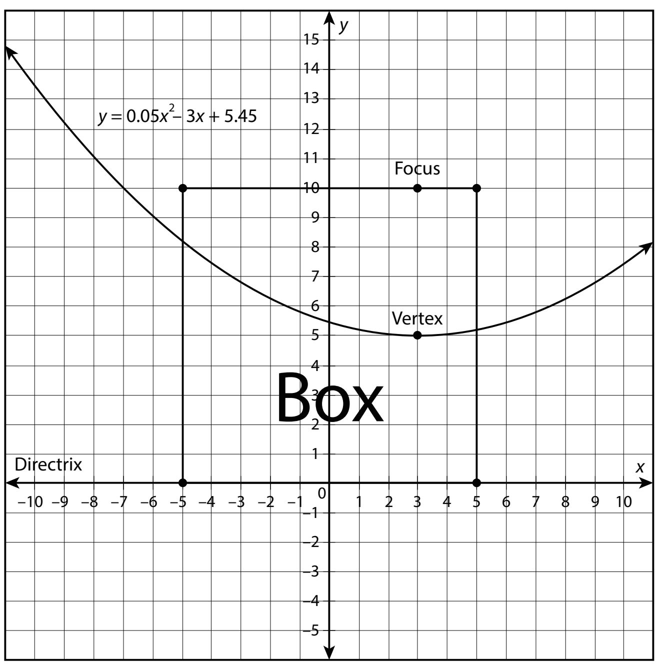
4. Find the quadratic equation.

We now know that $(h, k) = (3, 5)$ and $|p| = 5$. We also know the sign of p is positive because the parabola opens up. Therefore, we can substitute values into the general form for a parabola and rearrange to find a quadratic function.

$(x - h)^2 = 4p(y - k)$	General form
$(x - 3)^2 = 4(5)(y - 5)$	Substitute (3, 5) for (h, k) and 5 for p.
$(x - 3)^2 = 20(y - 5)$	Multiply constants.
$x^2 - 6x + 9 = 20y - 100$	Expand the parentheses.
$x^2 - 6x + 109 = 20\boldsymbol{y}$	Add 100 to both sides.
$0.05x^2 - 0.3x + 5.45 = \boldsymbol{y}$	Divide both sides by 20.

5. Draw the situation.



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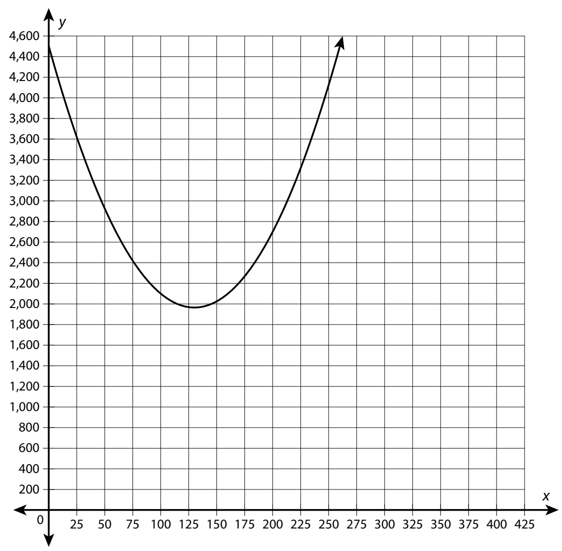
Example 5

A manufacturer of cell phones finds that the cost (in dollars) created by manufacturing x units per week is given by the function $f(x) = 0.15x^2 - 39x + 4500$. How many units should be manufactured to minimize the cost? Show your work graphically and algebraically.

1. Graph the function.



The vertex is the minimum of this graph. The point that will minimize the cost is the vertex.

2. Find the vertex algebraically.

Since the a-value of the quadratic is positive, the vertex represents the minimum of the graph.

The formula for the vertex (h, k) is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. In our function

$f(x) = 0.15x^2 - 39x + 4500$, $a = 0.15$ and $b = -39$.

$$h = \frac{-(-39)}{2(0.15)} = 130$$

$$k = f(130) = 0.15(130)^2 - 39(130) + 4500 = 1965$$

The vertex is (130, 1965).

3. Identify the solution.

Since the x-value represents the number of units sold, the minimum number sold should be 130. This will minimize the cost.

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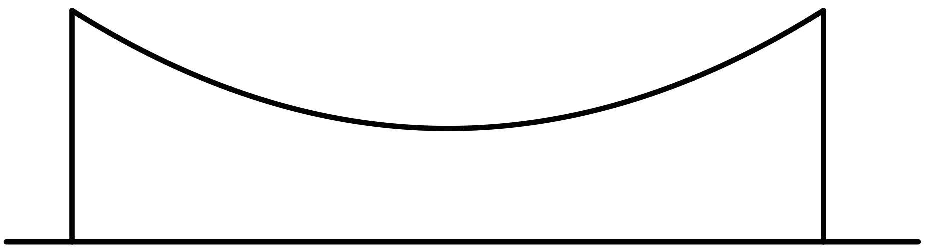
Problem-Based Task 3.2: Suspension Bridge

SMP
1 ✓ 2
3 4 ✓
5 6 ✓
7 ✓ 8

A certain suspension bridge has a center span that approximately resembles this sketch:

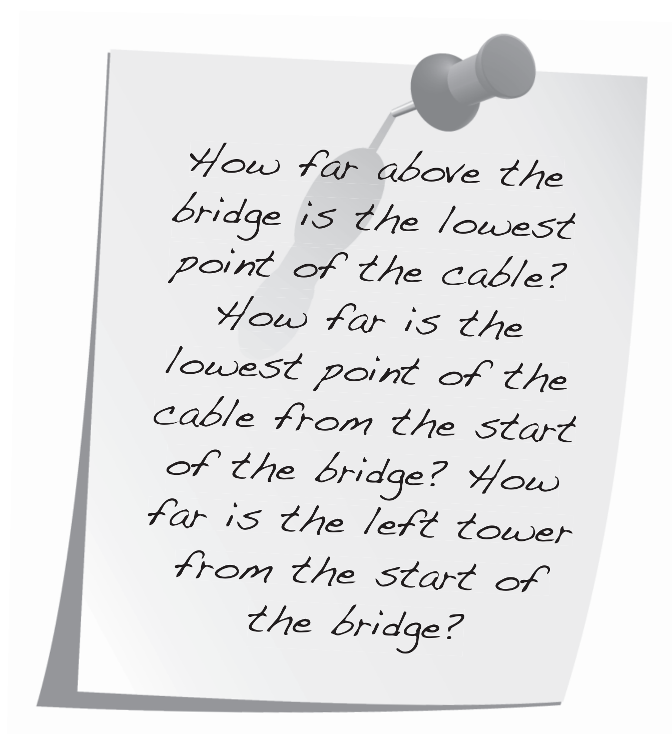


The supporting towers are 75 meters tall and are spaced 80 meters apart. The curve of the suspension cable is a parabola with a function equation of $f(x) = 0.01875x^2 - 1.875x + 91.875$. Use this information to answer the following questions.

1. How far above the bridge is the lowest point of the cable?

2. How far is the lowest point of the cable from the start of the bridge?

3. How far is the left tower from the start of the bridge?



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Problem-Based Task 3.2: Suspension Bridge

Coaching

a. How would you determine the lowest point of the support cable?

b. How would you determine the distance from the start of the bridge to the lowest point of the cable?

c. How would you determine the distance from the start of the bridge to the first tower?

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Problem-Based Task 3.2: Suspension Bridge

Coaching Sample Responses

a. How would you determine the lowest point of the support cable?

To determine the height above the bridge, you need to find the y-coordinate of the vertex of the parabola. Since the a-value of the quadratic is positive, the vertex is a minimum.

Use $\frac{-b}{2a}$ to find the x-coordinate.

$$h = \frac{-(-1.875)}{2(0.01875)} = 50$$

Now plug 50 into the function to find the y-coordinate.

$$k = f(50) = 0.01875(50)^2 - 1.875(50) + 91.875 = 45$$

Therefore, the lowest point of the cable is 45 meters above the bridge.

b. How would you determine the distance from the start of the bridge to the lowest point of the cable?

Since the center of the suspension cable is at the center of the bridge, the x-coordinate of the vertex is the distance from the start of the bridge.

Therefore, the distance from the start of the bridge to the center is 50 meters.

c. How would you determine the distance from the start of the bridge to the first tower?

The distance between the two towers is 80 meters. Since the vertex is halfway between the two towers, the distance from the tower to the center is $\frac{80}{2} = 40$ meters. The center of the cable is 50 meters from the start of the bridge. Therefore, the tower must be 50 – 40 = 10 meters from the start of the bridge.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Problem-Based Task 3.2 Implementation Guide: Suspension Bridge

North Carolina Math 3 Standard

F–IF.7 Analyze piecewise, absolute value, polynomials, exponential, rational, and trigonometric functions (sine and cosine) using different representations to show key features of the graph, by hand in simple cases and using technology for more complicated cases, including: domain and range; intercepts; intervals where the function is increasing, decreasing, positive, or negative; rate of change; relative maximums and minimums; symmetries; end behavior; period; and discontinuities.★

Task Overview

Focus

The focus of the suspension bridge task is to provide a relevant example of how the vertex and axis of symmetry of a quadratic graph can relate to a real life application.

This activity will provide practice with:

- relating the shape of a suspension bridge to a quadratic function
- analyzing the values of the vertex of the quadratic function
- interpreting coordinates in terms of the task situation
- examining the results to see if they make sense in the context of the problem

Introduction

This task should be used to explore the concept of the vertex and axis of symmetry in a real-life application. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of the lowest point of the suspension cable in relation to the quadratic function. Review the following terms: *vertex, axis of symmetry,* and *coordinates.*

vertex	the point where the parabola crosses its axis of symmetry; the aximum or minimum of the parabola
axis of symmetry	a vertical line that divides the parabola into two congruent halves
coordinates	a set of values that show a position relative to a starting point, usually denoted (0, 0); in a coordinate pair, (x, y), the x-coordinate is always the first coordinate listed and the y-coordinate is the second one listed.

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students should make the connection of the position of the vertex and how it relates to the corresponding position in the real-life example.

- **SMP 4:** Model with mathematics.

  Students should recognize and apply both the mathematics of finding the coordinates of the vertex and the equation of the axis of symmetry.

- **SMP 6:** Attend to precision.

  The accuracy of applying the formulas is critical to the accurate description of the resulting answer.

- **SMP 7:** Look for and make use of structure.

  The formulas for vertex and axis of symmetry are specific and need to be accurately applied.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly computing either the *x*- or *y*-coordinate of the vertex

  Have students plot the function on a graphing calculator and use the minimum feature to verify the computations.

- incorrectly listing the axis of symmetry

  Stress that the axis of symmetry runs through the *x*-value of the vertex. Graph that value in a graphing calculator to view the result.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- How do you find the vertex of a quadratic? (**Answer:** The vertex is always $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.)
- What is the axis of symmetry of a quadratic? (**Answer:** The axis of symmetry is $x = \frac{-b}{2a}$.)

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- How can you tell if the vertex is a minimum or a maximum of the quadratic? (**Answer:** If the *a*-value is positive, it is a minimum. If the *a*-value is negative, it is a maximum.)

  Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Make sure students understand the importance of accuracy when finding the numerical values in the task. Explain that if you were designing a bridge, inaccuracies could cause a catastrophic tragedy.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The standard form of a quadratic function is $f(x) = ax^2 + bx + c$.

  The standard form helps to determine if the *a*-value is positive or negative. It is the most versatile form, as it can be converted to either vertex form or intercept form.

- The vertex of a quadratic is found by using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

  It is necessary to accurately identify the vertex.

- The axis of symmetry for a quadratic function is $x = \frac{-b}{2a}$.

  The axis of symmetry is located in the "center" of the bridge. It is the line around which the bridge should be symmetric.

Extending the Task

To extend the task, have students go online and research the blueprints of different construction projects to view the necessity for accuracy. Some ideas for research projects are:

- blueprints for the Skyway Bridge in St. Petersburg, Florida
- blueprints for the Hoover Dam in Colorado
- blueprints for the Golden Gate Bridge in San Francisco, California

The National Archives offers a guide to researching their collection of architectural drawings:

- The National Archives. "Architectural drawings."

  http://www.walch.com/IG/10003

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I identified whether the given quadratic determined a minimum or maximum point on the graph. Then I determined how that point related to the task assigned.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used the formulas for the vertex and axis of symmetry of a quadratic. The vertex models a point on the bridge curve and the axis of symmetry marks the midpoint of the bridge.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I determined in advance whether the results should be exact values or rounded off answers to a specified accuracy.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I noticed that all quadratics have the same structural components of maximums or minimums and all are symmetric.)

Alternate Strategies or Solutions

- Students could design scaled pictures to measure the desired values.
- Students could use a program such as Geogebra or Desmos and trace the coordinates on the graph after plotting it in the program.

Technology

Students can use graphing calculators, or computer applications such as GeoGebra or Desmos.

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Practice 3.2: Graphing Quadratic and Cubic Functions A

Complete each problem as directed.

1. Create a table of values for the equation $y = 9 - 8x - x^2$ using the domain $-4 \le x \le 4$.

2. Create a table of values for the equation $y = x^3 + 4x^2 - 3x - 1$ using the domain $-4 \le x \le 4$.

3. Determine the focus and directrix of the equation $y = -8x + 7x^2 - 3$.

4. What is the equation of a parabola with focus (3, 1) and directrix $y = 0$?

5. Find the vertex and axis of symmetry for the equation $y = -3x^2 + 18x + 7$.

continued

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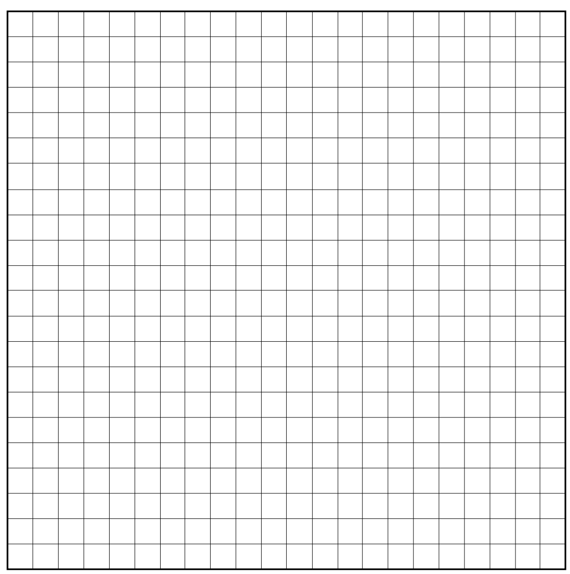
6. Identify the zeros of the function $f(x) = x^2 - 8x + 12$ by graphing.

7. Identify the zeros of the function $f(x) = x^3 + 3x^2 - 10x - 24$ by graphing.

8. Describe the overall behavior of the function $f(x) = -2x^3 - 6x + 4$.

9. Graph the function $f(x) = 2x^2 - 12x + 7$. Be sure to indicate the vertex of the graph.



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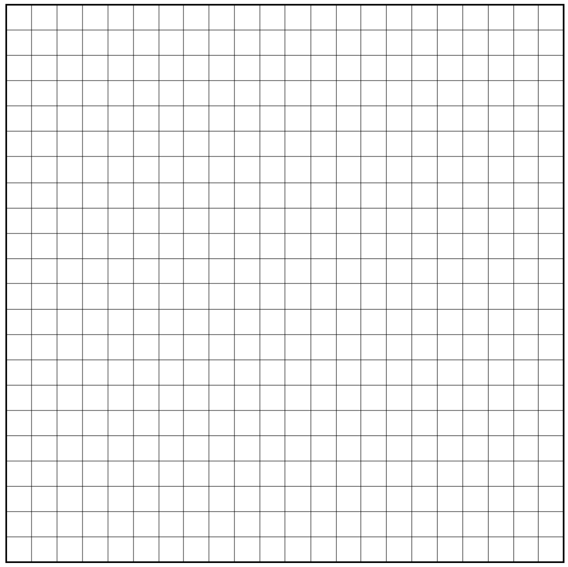
10. Graph the function $f(x) = -2x^3 + 8x$. Use a table of values to assist in plotting points.



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Practice 3.2: Graphing Quadratic and Cubic Functions B

Complete each problem as directed.

1. Create a table of values for the equation $y = 3x^2 + 8x + 6$ using the domain $-4 \le x \le 4$.

2. Create a table of values for the equation $y = x^3 - 16x$ using the domain $-4 \le x \le 4$

3. Determine the focus and directrix of the equation $y = 2x^2 + 4x + 5$.

4. What is the equation of a parabola with focus $\left(-1, \frac{131}{12}\right)$ and directrix $y = \frac{133}{12}$?

5. Find the vertex and axis of symmetry for the equation $y = 5x^2 - 10x + 3$.

continued

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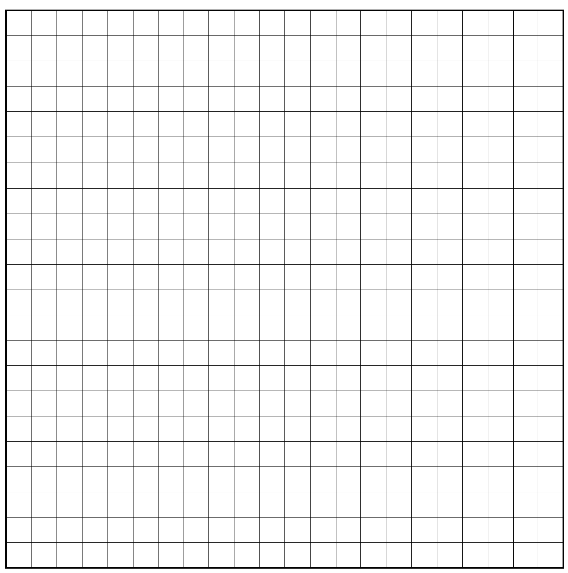
6. Identify the zeros of the function $f(x) = x^2 + 5x - 14$ by graphing.

7. Identify the zeros of the function $f(x) = x^3 - 10x^2 + 31x - 30$ by graphing.

8. Describe the overall behavior of the function $f(x) = -4 + 5x + 3x^2 + 5x^3$.

9. Graph the function $f(x) = -\frac{1}{2}x^2 + 2x + 5$. Be sure to indicate the vertex of the graph.



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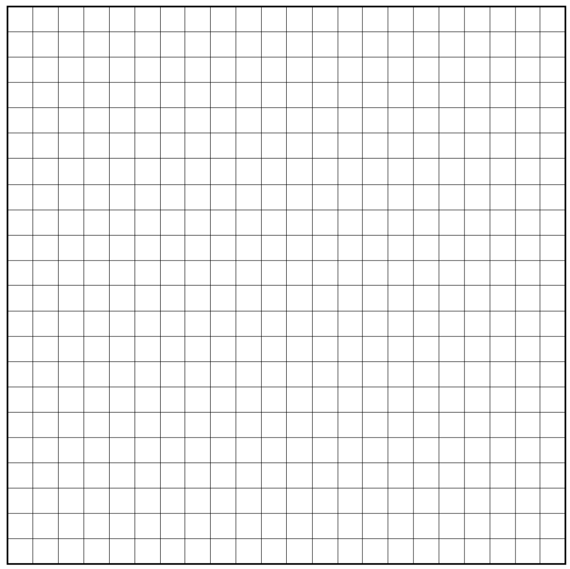
10. Graph the function $f(x) = x^3 - 2x^2 - 5x + 1$. Use a table of values to assist in plotting points.



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Lesson 3.3: Optimization of Volume

Warm-Up 3.3

A cocoa company sells its cocoa in a rectangular prism-shaped box that measures 5 in. × 5 in. × 6.25 in. The company would like to redesign its packaging so that the box is a cylinder with a diameter of 5 in. that holds the same volume of cocoa as the prism-shaped box. What is the height of the cylindrical box? Round your answer to the nearest hundredth of an inch.

1. What is the volume of the prism-shaped box?

2. What equation can you use to find the height of the cylindrical box?

3. Calculate the height of the cylindrical box.

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Instruction

Lesson 3.3: Optimization of Volume

North Carolina Math 3 Standard

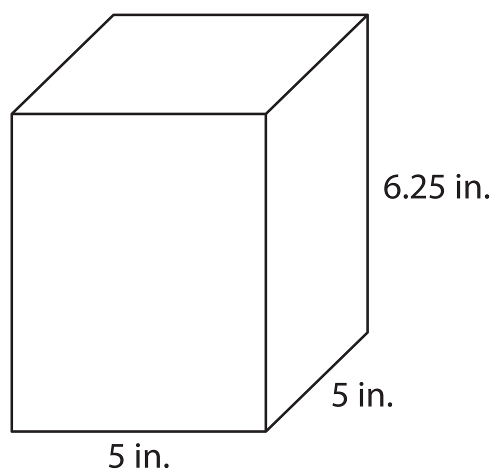
G–MG.1 Apply geometric concepts in modeling situations.★

- Use geometric and algebraic concepts to solve problems in modeling situations.
- Use geometric shapes, their measures, and their properties, to model real-life objects.
- Use geometric formulas and algebraic functions to model relationships.
- Apply concepts of density based on area and volume.
- Apply geometric concepts to solve design and optimization problems.

Warm-Up 3.3 Debrief

1. What is the volume of the prism-shaped box?

   The volume of a prism is found by multiplying the area of the base by the height.



Area of base: 5 • 5 = 25

Volume of prism: 25 • 6.25 = 156.25

The volume of this box is 156.25 in.^{3}.

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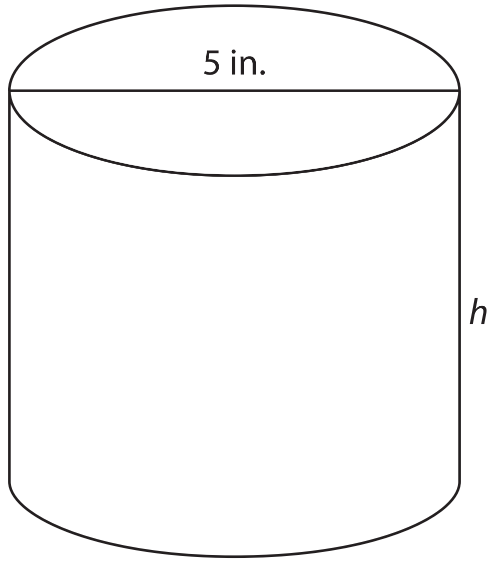
2. What equation can you use to find the height of the cylindrical box?



The cylindrical box has the same volume as the prism-shaped box, 156.25 in.^{3}.

The volume of a cylinder is found by multiplying the area of the base, πr^2, by the height.

$V = \pi r^2 h$

Substitute known values.

$156.25 = \pi(2.5^2)(h)$

The equation that can be used to find the height of the cylinder is $156.25 = \pi(2.5^2)(h)$.

3. Calculate the height of the cylindrical box.

Solve the equation from question 2.

$156.25 = \pi(2.5^2)(h)$

$$h = \frac{156.25}{\pi(2.5^2)}$$

$h \approx 7.96$

The height of the cylindrical box is about 7.96 in.

Connection to the Lesson

- Students will solve complex problems involving volume.
- Students will learn to calculate dimensions that will maximize or minimize volume in mathematical and real-world problems.

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Prerequisite Skills

This lesson requires the use of the following skills:

- understanding formulas for surface area and for volume of prisms and cylinders (8.G.9)
- knowing how to graph a function on a graphing calculator or other graphing technology (F–IF.4★)

Introduction

Polynomial functions can be used to model problems that involve geometry. In this lesson, you will investigate problems that seek to optimize volume and surface area. Optimization problems require you to maximize or minimize the volume or surface area of a container under given parameters. In future math courses, you can learn how to solve optimization problems using calculus, but in this lesson, you will learn how to solve them using polynomial functions and graphing technology.

Key Concepts

- Recall the formulas for the volume of a rectangular prism and a cylinder:
  - Volume of a rectangular prism with length l, width w, and height h: $V = lwh$
  - Volume of a cylinder with radius r and height h: $V = \pi r^2 h$ or V = (area of base)(height)
- Recall that to find the surface area of a 3-dimensional shape, find the sum of the areas of each surface.
- An optimization problem seeks to maximize or minimize a quantity. In this lesson, the quantity that is being optimized is related to volume. For example, you may want to maximize the volume of a box that can be constructed from a given amount of cardboard, or minimize the surface area of a tin can of a given volume.

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- To solve an optimization problem, follow these steps:

Solving Optimization Problems

1. Draw a diagram of the situation to help you visualize the problem, then label all known and unknown dimensions.
2. Determine the quantity that you are seeking to maximize or minimize.
3. Write a polynomial function for this quantity in terms of one variable.
4. Determine the domain for the function.
5. Use graphing technology to graph this function and find the maximum or minimum *y*-value within the domain.
6. Be sure to answer the question being asked.

On a TI-83/84:

Step 1: Press the [Y=] button.

Step 2: Type the function for the quantity to be maximized or minimized into Y_1.

Step 3: Press [WINDOW]. Determine a reasonable window for the graph. The Xmin and Xmax can be determined by the domain for the function. The Ymin can be 0 and the Ymax can be estimated by substituting a value in the middle of the domain into the function.

Step 4: Press [GRAPH]. If you cannot see the maximum (or minimum) point of the graph, adjust the window.

Step 5: To find a minimum value, press [2ND][CALC] and choose [3:minimum]. To find a maximum value, press [2ND][CALC] and choose [4:maximum].

Step 6: Use the arrow buttons to move the cursor to the left of the maximum (or minimum) *y*-value and press [ENTER]. Then use the arrow buttons to move the cursor to the right of the maximum (or minimum) value and press [ENTER]. Then press [ENTER] once more.

Step 7: The maximum (or minimum) value is the *y*-coordinate displayed at the bottom of the screen.

On a TI-Nspire:

Step 1: Press the [home] key. Choose the Graphs and Geometry Application, then enter the equation as *f1(x)*.

Step 2: Press [menu], then select 4: Window/Zoom and 1: Window Setting to adjust the window range so the vertex is visible.

Step 3: If the vertex is a maximum, press [menu]. Select 5: Trace, then 1: Graph Trace. Then, trace the graph using the navigation pad arrows until an uppercase "M" appears. The point where you see the "M" is the maximum, or the vertex.

Step 4: If the vertex is a minimum, press [menu]. Select 5: Trace, then 1: Graph Trace. Then, trace the graph using the navigation pad arrows until a lowercase "m" appears. The point where you see the "m" is the minimum, or the vertex.

Common Errors/Misconceptions

- confusing whether the function to be minimized is for volume or surface area
- attempting to maximize or minimize a function that is in terms of two variables, or being unsure how to rewrite the function in terms of just one variable
- entering the incorrect function into the graphing calculator
- being unsure how to interpret the results shown on the calculator

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Scaffolded Practice 3.3: Optimization of Volume

Use what you know about volume formulas to complete the problems.

1. What is the formula for the volume of a rectangular prism?

2. What is the formula for the volume of a cylinder?

3. What is the formula for the surface area of a rectangular prism?

4. What is the formula for the surface area of a cylinder?

5. You cut congruent squares with side length x from each corner of a 7 inch × 12 inch piece of cardboard and fold up the sides. Write a polynomial function that represents the volume of the box that is formed. What is the domain of the function?

continued

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6. You cut congruent squares with side length x from each corner of a 3.5 feet $\times$ 8 feet piece of aluminum and fold up the sides. Write a polynomial function that represents the volume of the box that is formed. What is the domain of the function?

7. Write a system of equations in terms of V, x, and y that represents the volume of a box that has a square base, no top, and a surface area of 260 cm^2. Assume that x is the side length of the base, y is the height, and V is volume.

8. Write a system of equations in terms of S, x, and y that represents the surface area of a cylinder that has a volume of 36 cubic inches. Assume that x is the radius of the base, y is the height, and S is surface area.

9. The function that represents the volume of a box with a square base, no top, and a surface area of 480 square inches is $V = 120x - 0.25x^3$. What is the domain of the function? Why?

10. The function that represents the surface area of a cylinder with a volume of 57 cubic centimeters is $S = 2\pi x^2 + \frac{114}{x}$. What is the domain of this function? Why?

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Guided Practice 3.3

Example 1

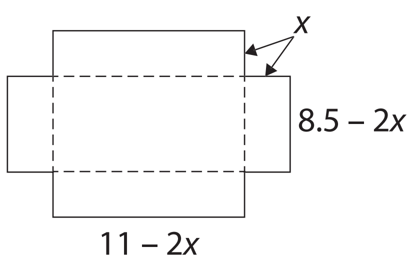
An 8.5 in. × 11 in. piece of cardboard is made into a box with no top by cutting congruent squares from each corner and folding the sides up. What are the dimensions of the box with the largest volume? What is the largest volume?

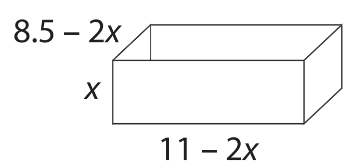
1. Draw and label a diagram.

   The side length of the squares being cut from the corners is x.

   The width of the cardboard before cutting out the squares was 8.5 inches, so the width with the corners cut out is $(8.5 - 2x)$ inches.

   The length of the cardboard before cutting out the squares was 11 inches, so the length with the corners cut out is $(11 - 2x)$ inches.

   The diagram on the left shows the sheet of cardboard with the corners cut out. The diagram on the right shows the box that is formed by folding the four flaps up on the dotted lines.

2. Write a function for the quantity that you are seeking to maximize.

   The quantity that is to be maximized is the volume. The volume of a box, or a rectangular prism, is length • width • height. Let $f(x)$ represent the volume of the box, and let x represent the height of the box. The width and length of the box are given in terms of x in the previous diagram.

   $f(x) = x(8.5 - 2x)(11 - 2x)$

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3. Determine the domain for this function in the context of the problem.

   The least possible value of x is 0, since a length measurement cannot be negative. Consider the largest square that can be cut from the corners of a piece of cardboard that measures 8.5 inches by 11 inches. The square must be less than half the shortest side of the cardboard, or $8.5 \div 2 = 4.25$. So, the greatest possible value of x is 4.25. The domain is $0 < x < 4.25$.

4. Graph the polynomial function on a graphing calculator, and calculate the maximum value of the function within the given domain.

   On a TI-83/84:

   Step 1: Press the [Y=] button.

   Step 2: Type the function for the volume of Y_1:
   $Y_1 = x(8.5 - 2x)(11 - 2x)$.

   Step 3: Press [WINDOW]. Set Xmin = 0 and Xmax = 5. Choose a value between 0 and 5, then substitute it in the function to get an idea of the possible y-values. If $x = 2$, $y = 63$; therefore, a reasonable window for y is 0 to 75. Set Ymin = 0 and Ymax = 75.

   Step 4: Press [GRAPH].

   Step 5: To find the maximum value of the function, press [2ND] [CALC] and choose [4:maximum].

   Step 6: Use the arrow buttons to move the cursor to the left of the maximum y-value and press [ENTER]. Then use the arrow buttons to move the cursor to the right of the maximum y-value and press [ENTER]. Then press [ENTER] once more. The maximum value is the y-coordinate displayed at the bottom of the screen.

(*continued*)

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On a TI-Nspire:

Step 1: Press the [home] key. Choose the Graphs and Geometry Application, then enter the function for volume, $Y_1 = x(8.5 - 2x)(11 - 2x)$, as $f1(x)$.

Step 2: Press [menu], then select 4: Window/Zoom and 1: Window Setting to adjust the window range so the vertex is visible.

Step 3: Press [menu]. Select 5: Trace, then 1: Graph Trace. Then, trace the graph using the navigation pad arrows until an uppercase "M" appears. The point where you see the "M" is the maximum, or the vertex.

The maximum value of the function in the given domain is approximately 66.148, when x is approximately 1.585.

5. Interpret the results in the context of the problem.

The value of y, or $f(x)$, represents the volume of the box, so the maximum volume of the box is approximately 66.148 in.^{2}.

The value x represents the side length of the square that is cut from the corners of the cardboard. The value of x that produced the box with the greatest volume is 1.585 in. Substitute this value of x into the two expressions for length and width to find the other two dimensions.

$$8.5 - 2(x) = 8.5 - 2(1.585) = 5.33$$

$$11 - 2(x) = 11 - 2(1.585) = 7.83$$

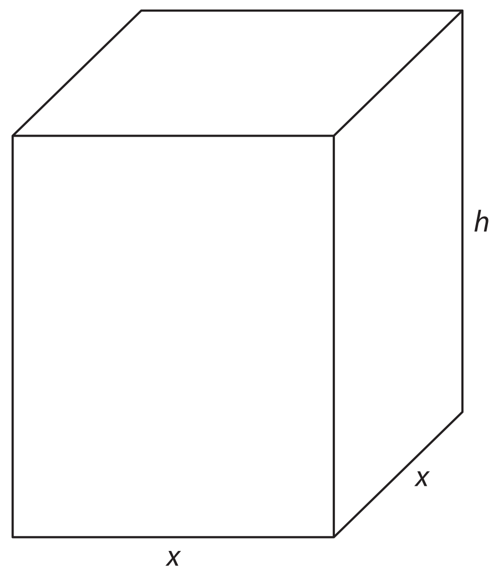
The approximate dimensions of the box are
1.585 in. × 5.33 in. × 7.83 in.

Example 2

Renée wants to make a box with a surface area of 144 cm^2. She would like the base of the box to be a square. What dimensions will maximize the volume of the box, and what is the maximum volume?

1. Draw and label a diagram.

   The base of the box must be a square, so label the length and width both x. Label the height h.



2. Write a function in terms of one variable for the quantity you are seeking to maximize.

   We would like to maximize the volume: $V = x^2h$. This function is in terms of two variables, x and h. Use information given in the problem to find an expression for x or h and substitute this expression into the volume function so that the volume function is in terms of just one variable.

   The problem states that the surface area must be 144 cm^2. The surface area of this box is made up of four sides with dimensions $x \times h$ and the top and bottom that have area x^2. Write an equation for the surface area.

   $4xh + 2x^2 = 144$

(continued)

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This is easier to solve for h than for x. Solve this equation for h and simplify:

$$4xh = 144 - 2x^2$$

$$h = \frac{144 - 2x^2}{4x}$$

$$h = \frac{36}{x} - \frac{x}{2}$$

Substitute this value of h in the volume function, then simplify:

$$V = x^2h$$

$$V = x^2\left(\frac{36}{x} - \frac{x}{2}\right)$$

$$V = 36x - \frac{x^3}{2}$$

The function in terms of one variable that represents the volume of the box is $V = 36x - \frac{x^3}{2}$.

3. Determine the domain of the function within the context of the problem.

   The smallest possible value of x is 0, since the length and width of the box cannot be negative. To determine the largest that x can be, imagine that the top and bottom of the box are so large that there isn't any height to the box. In this case, the top would be half of the surface area and the bottom would be half of the surface area.

(*continued*)

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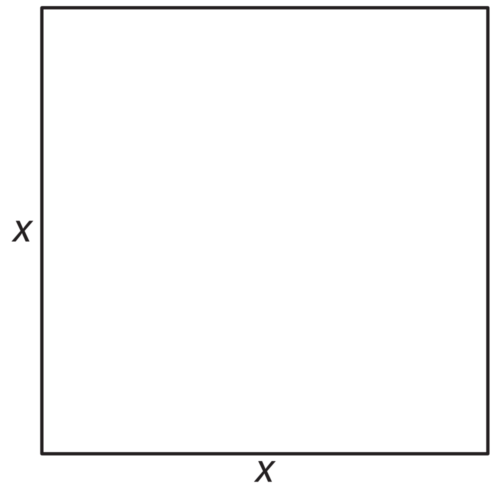
The surface area is 144, so the surface area of the top and bottom would be 72. Find the side length of a square with surface area 72:



$x^2 = 72$

$x = \sqrt{72} \approx 8.485$

The domain is $0 < x < 8.485$.

4. Graph the function on a graphing calculator, and calculate the maximum value of the function within the given domain.

On a TI-83/84:

Step 1: Press the [Y=] button.

Step 2: Type the function for the volume of Y_1: $Y_1 = 36x - x^3/2$.

Step 3: Press [WINDOW]. Set Xmin = 0 and Xmax = 10. Choose a value between 0 and 10, then substitute it in the function to get an idea of the possible y-values. If $x = 5$, $y = 117.5$; therefore, a reasonable window for y is 0 to 125. Set Ymin = 0 and Ymax = 125.

Step 4: Press [GRAPH].

(*continued*)

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Step 5: To find the maximum value of the function, press [2ND] [CALC] and choose [4:maximum].

Step 6: Use the arrow buttons to move the cursor to the left of the maximum y-value and press [ENTER]. Then use the arrow buttons to move the cursor to the right of the maximum y-value and press [ENTER]. Then press [ENTER] once more. The maximum value is the y-coordinate displayed at the bottom of the screen.

On a TI-Nspire:

Step 1: Press the [home] key. Choose the Graphs and Geometry Application, then enter the function for volume, $Y_1 = 36x - x^3/2$, as $f1(x)$.

Step 2: Press [menu], then select 4: Window/Zoom and 1: Window Setting to adjust the window range so the vertex is visible.

Step 3: Press [menu]. Select 5: Trace, then 1: Graph Trace. Then, trace the graph using the navigation pad arrows until an uppercase "M" appears. The point where you see the "M" is the maximum, or the vertex.

The maximum value of the function in the given domain is approximately 117.576, when x is approximately 4.899.

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5. Interpret the results in the context of the problem.

   The maximum volume of a box with a surface area of 144 cm^2 is about 117.576 cm^3. This maximum volume occurs when the value of x, the side length of the square base, is about 4.899 cm. To find the third dimension of the box, the height, solve the equation $V = lwh$, substituting in known values.

   $V = lwh$

   $117.576 = (4.899)(4.899)(h)$

   $\frac{117.576}{4.899^2} = h$

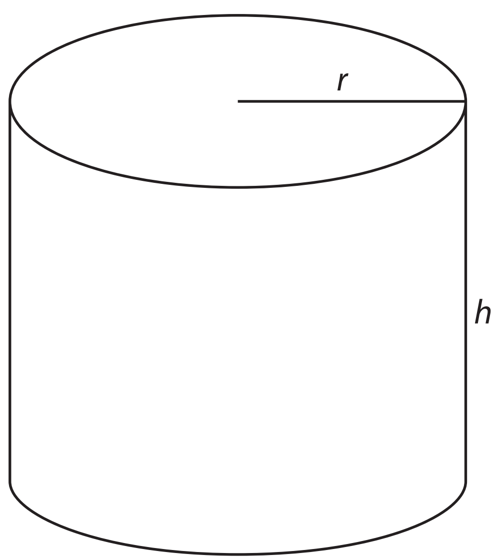
   $h \approx 4.899$

   The dimensions of the box are approximately $4.899 \times 4.899 \times 4.899$, so the box with the maximum volume is a cube.

Example 3

A manufacturer would like to make a tin can that has a volume of 25 cubic inches and that uses the least amount of tin possible. What is the height and radius of the can?

1. Draw and label a diagram.



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2. Write a function in terms of one variable for the quantity you are seeking to minimize.

   In this example, we want to minimize the surface area. Recall that the equation for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$. This function is in terms of two variables, r and h, so we have to find a way to express it using just one variable. Use the additional information given in the problem to write a relationship involving r and h. The problem states that the volume of the can must be 25 in.^{3}, and the volume of a cylinder can be found with the formula $V = 2\pi r^2 h$. Substitute 25 for V, then solve for h:

   $$25 = 2\pi r^2 h$$

   $$h = \frac{25}{2\pi r^2}$$

   Substitute this value for h into the function for surface area, then simplify:

   $$SA = 2\pi r^2 + 2\pi rh$$

   $$SA = 2\pi r^2 + 2\pi r\left(\frac{25}{2\pi r^2}\right)$$

   $$SA = 2\pi r^2 + \frac{25}{r}$$

   The function we are seeking to minimize, in terms of one variable, is $SA = 2\pi r^2 + \frac{25}{r}$. In terms of x and y, this can be written as $y = 2\pi x^2 + \frac{25}{x}$.

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3. Determine the domain of the function in the context of the problem.

   The variable in the function represents the radius of the can. The minimum value of the radius is 0. To determine a maximum possible radius, consider the fact that if the volume is fixed, then the larger the radius, the shorter the height of the can. Use the volume formula, $V = 2\pi r^2 h$, to look at the radii for very short can heights. If the height of the can is 0.5 inches: $25 = \pi r^2(0.5)$, so $r \approx 3.99$ inches.

   Logically, a can that is 8 inches in diameter and 0.5 inches tall will not use the minimum amount of tin. A more reasonable domain for this problem can be $0 < r < 5$, or in terms of x, $0 < x < 5$.

4. Graph the function on a graphing calculator, and calculate the minimum value of the function within the given domain.

   On a TI-83/84:

   Step 1: Press the [Y=] button.

   Step 2: Type the function for the volume of Y_1: $Y_1 = 2\pi x^2 + 25/x$.

   Step 3: Press [WINDOW]. Set Xmin = 0 and Xmax = 5. Choose a value between 0 and 5, then substitute it in the function to get an idea of the possible y-values. If $x = 2$, $y \approx 37.6$; therefore, a reasonable window for y is 0 to 50. Set Ymin = 0 and Ymax = 50.

   Step 4: Press [GRAPH].

   Step 5: To find the minimum value of the function, press [2ND] [CALC] and choose [3:minimum].

   Step 6: Use the arrow buttons to move the cursor to the left of the minimum y-value and press [ENTER]. Then use the arrow buttons to move the cursor to the right of the minimum y-value and press [ENTER]. Then press [ENTER] once more. The minimum value is the y-coordinate displayed at the bottom of the screen.

(*continued*)

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On a TI-Nspire:

Step 1: Press the [home] key. Choose the Graphs and Geometry Application, then enter the function for volume, $Y_1 = 2\pi x^2 + 25/x$, as $f1(x)$.

Step 2: Press [menu], then select 4: Window/Zoom and 1: Window Setting to adjust the window range so the vertex is visible.

Step 3: Press [menu]. Select 5: Trace, then 1: Graph Trace. Then, trace the graph using the navigation pad arrows until a lowercase "m" appears. The point where you see the "m" is the minimum, or the vertex.

The minimum value of the function in the given domain is approximately 29.816, when x is approximately 1.258.

5. Interpret the results in the context of the problem.

The minimal amount of tin needed to make a tin can that has a volume of 25 in.^{2} is about 29.816 square inches. This minimal surface area occurs when the radius of the can is about 1.258 inches. To find the height, substitute the known radius and volume into the volume formula and solve for h.

$$V = 2\pi r^2 h$$

$$25 = 2\pi(1.258)^2 h$$

$$h = \frac{25}{2\pi(1.258)^2}$$

$$h \approx 2.514$$

The height of the can is about 2.514 inches, and the radius of the can is about 1.258 inches. (Note that the height and the diameter of the can are approximately equal.)

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Problem-Based Task 3.3: Popcorn Can

SMP	
1 ✓	2 ✓
3	4 ✓
5 ✓	6
7	8 ✓

A popcorn company has asked two teams to design a new container that can hold 3,456 cm^3 of popcorn kernels. Team A is using a material that costs \$0.053 per 100 cm^2, and it wants to design a cylindrical can. Team B is using a material that costs \$0.048 per 100 cm^2, and it wants to design a rectangular prism-shaped can with a square base. The teams will use a plastic lid that will cost the same, so both cans will have no top. If both teams seek to minimize the cost of the can, which can will have a lower materials cost? Justify your answer.

Problem-Based Task 3.3: Popcorn Can

Coaching

a. What is the quantity that Team A and Team B are seeking to maximize or minimize?

b. What is the equation for the quantity that Team A wants to minimize?

c. What is the equation for the quantity that Team B wants to minimize?

d. How can you express the function for surface area for Team A in terms of just one variable?

e. How can you express the function for surface area for Team B in terms of just one variable?

f. What is the domain of the function for the surface area of the cylindrical can?

g. What is the domain of the function for the surface area of the rectangular prism-shaped can?

h. What is the minimum surface area for each can? Use a graphing calculator.

i. Which can will cost less for the materials needed?

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Problem-Based Task 3.3: Popcorn Can

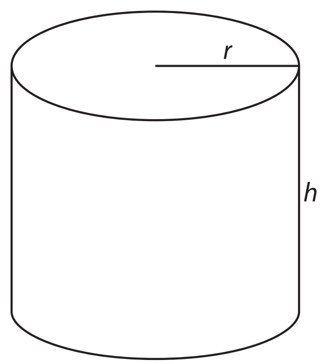
Coaching Sample Responses

a. What is the quantity that Team A and Team B are seeking to maximize or minimize?

Both teams want to minimize the cost of the can. The cost is based on the surface area. So, the teams need to minimize the surface area of their cans.

b. What is the equation for the quantity that Team A wants to minimize?

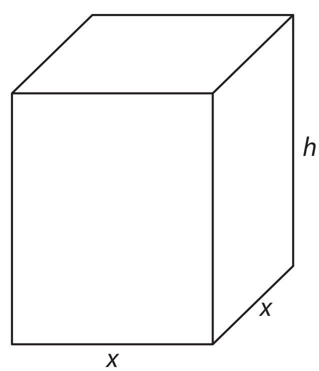
The can being designed by Team A is a cylinder without a top.



The equation for the surface area of this shape is $SA = \pi r^2 + 2\pi rh$.

c. What is the equation for the quantity that Team B wants to minimize?

The can being designed by Team B is a prism with a square base and without a top.



The equation for the surface area of this shape is $SA = 4hx + x^2$.

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d. How can you express the function for surface area for Team A in terms of just one variable?

The function for surface area for Team A is in terms of two variables, r and h. Use the information given in the problem to write an expression for h in terms of r, then substitute this value in the surface area function.

The given information states that the volume of the container must be 3,456 cm^3. The volume of a cylinder is $V = \pi r^2 h$. Substitute 3,456 for V, then solve for h:

$$3456 = \pi r^2 h$$

$$h = \frac{3456}{\pi r^2}$$

Substitute this expression for h into the surface area function:

$$SA = \pi r^2 + 2\pi r h$$

$$SA = \pi r^2 + 2\pi r\left(\frac{3456}{\pi r^2}\right)$$

This is a function for surface area of the open-topped cylinder in terms of just one variable, r.

This function can be written in terms of x and y as $y = \pi x^2 + 2\pi x\left(\frac{3456}{\pi x^2}\right)$.

e. How can you express the function for surface area for Team B in terms of just one variable?

The function for surface area for Team B is in terms of two variables, x and h. Use the information given in the problem to write an expression for h in terms of x, then substitute this value in the surface area function.

The given information states that the volume of the container must be 3,456 cm^3. The volume of a rectangular prism with a square base is $V = x^2 h$. Substitute 3,456 for V, then solve for h:

$$V = x^2 h$$

$$3456 = x^2 h$$

$$h = \frac{3456}{x^2}$$

Substitute this expression for h into the surface area function:

$$SA = 4hx + x^2$$

$$SA = 4\left(\frac{3456}{x^2}\right)x + x^2$$

This is a function for surface area of the open-topped prism in terms of just one variable, x. This function can be written in terms of x and y as $y = 4\left(\frac{3456}{x^2}\right)x + x^2$.

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f. What is the domain of the function for the surface area of the cylindrical can?

The variable for the formula for the surface area of the cylindrical can represents the radius. The radius must be greater than 0. To find the upper bound, imagine the radius is so large that the can is only 1 cm tall. The equation for the volume of this can would be $\pi r^2 h = 3456$. Substitute 1 for h and solve for r:

$$\pi r^2 h = 3456$$

$$\pi r^2(1) = 3456$$

$$r^2 = \frac{3456}{\pi}$$

$$r \approx 33.2$$

If we're trying to minimize the surface area, it makes sense that the radius must be less than 33. A reasonable domain for the function that represents the surface area of the cylindrical can is $0 < r < 33$.

g. What is the domain of the function for the surface area of the rectangular prism-shaped can?

The variable for the formula for the surface area of the prism-shaped can represents the width of the can. The width must be greater than 0. To find a reasonable upper bound, imagine that the can is only 1 cm high. The volume of the prism is expressed by the equation $x^2 h = 3456$. Substitute 1 for h and solve for x:

$$x^2 h = 3456$$

$$x^2(1) = 3456$$

$$x \approx 58.8$$

Because we're trying to minimize the surface area, it makes sense that the width of the can is less than 59 cm. A reasonable domain for the function that represents the surface area of the rectangular prism-shaped can is $0 < x < 59$.

h. What is the minimum surface area for each can? Use a graphing calculator.

First, determine the minimum surface area of the function that represents the cylindrical can, $y = \pi x^2 + 2\pi x\left(\frac{3456}{\pi x^2}\right)$.

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On a TI-83/84:

Step 1: Press the [Y=] button.

Step 2: Type the function for the volume of Y_1: $Y_1 = \pi x^2 + 2\pi x(3456/(\pi x^2))$.

Step 3: Press [WINDOW]. Set Xmin = 0 and Xmax = 33. Choose a value between 0 and 33, then substitute it in the function to get an idea of the possible *y*-values. If $x = 16$, $y \approx 1236$; therefore, a reasonable window for *y* is 0 to 1,500. Set Ymin = 0 and Ymax = 1500.

Step 4: Press [GRAPH].

Step 5: To find the minimum value of the function, press [2ND][CALC] and choose [3:minimum].

Step 6: Use the arrow buttons to move the cursor to the left of the minimum *y*-value and press [ENTER]. Then use the arrow buttons to move the cursor to the right of the minimum *y*-value and press [ENTER]. Then press [ENTER] once more. The minimum value is the *y*-coordinate displayed at the bottom of the screen.

On a TI-Nspire:

Step 1: Press the [home] key. Choose the Graphs and Geometry Application, then enter the function for volume, $Y_1 = \pi x^2 + 2\pi x(3456/(\pi x^2))$, as *f1(x)*.

Step 2: Press [menu], then select 4: Window/Zoom and 1: Window Setting to adjust the window range so the vertex is visible.

Step 3: Press [menu]. Select 5: Trace, then 1: Graph Trace. Then, trace the graph using the navigation pad arrows until a lowercase "m" appears. The point where you see the "m" is the minimum, or the vertex.

The minimum value of the function in the given domain is approximately 1,004.35, when *x* is approximately 10.32. Therefore, the minimum surface area of Team A's cylindrical can is about 1,004.35 cm^2, when the radius is about 10.32 cm.

Now, repeat these calculator steps for Team B's rectangular prism-shaped can using the function found in part e, $y = 4\left(\frac{3456}{x^2}\right)x + x^2$. This yields a minimal surface area of about 1,088.57 cm^2, when the width of the can is about 19.05 cm.

i. Which can will cost less for the materials needed?

The material that Team A plans to use costs \$0.053 per 100 cm^2. The surface area of this can is 1,004.35 cm^2. Divide by 100, then multiply by 0.053.

$1004.35 \div 100 = 10.0435$

$10.0435 \bullet 0.053 \approx 0.53$

The materials cost for this can is about \$0.53.

The material that Team B plans to use costs \$0.48 per 100 cm^2. The surface area of this can is 1088.57. Divide by 100, then multiply by 0.48.

$1088.57 \div 100 = 10.8857$

$10.8857 \bullet 0.048 \approx 0.52$

The materials cost for this can is about \$0.52.

The cost of materials for the prism-shaped can designed by Team B is slightly less than the cost of materials for the cylindrical can designed by Team A.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.3: Optimization of Volume A

Use the following information to complete problems 1–3.

1. Letitia has a rectangular sheet of cardboard that measures 12 in. × 18 in. She wants to create a box with no top by cutting congruent squares out of the corners and folding up the sides. What function does Letitia have to maximize to create the box with the largest volume? What is the domain of the function?

2. What size squares does Letitia have to cut from the corners of the cardboard to maximize the volume of the box?

3. What is the volume of the box?

Use what you learned about optimization of volume to solve problems 4–10.

4. What is the maximum volume of a box that has a square base, no top, and a surface area of 820 in.^{2}?

5. Levi is making a closed-top box that has a volume of 8,640 cm^3 and that is twice as long as it is wide. If he minimizes the surface area of this box, what are the dimensions of the box?

continued

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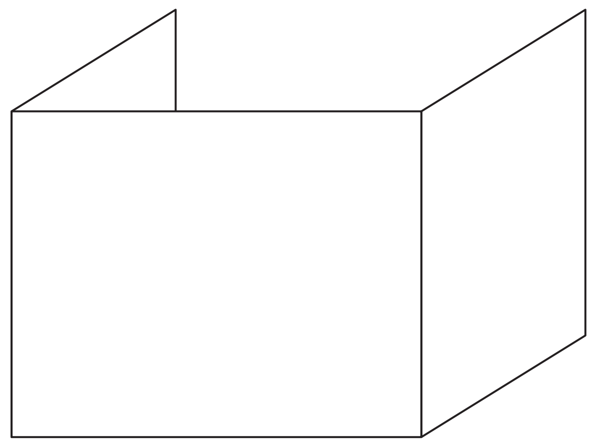
6. Paula would like to make a tin letter holder to mount on her kitchen wall. The letter holder will not have a top or a back. Paula would like it to be twice as tall as it is wide. If she has 81 in.^{2} of tin, what dimensions should she make the box if she wants to maximize its volume?



7. A company is designing a cylindrical tin can for peanuts. The tin can will have no top, since it will be fitted with a plastic lid. The volume of the can must be 1,357 cm^3. What is the minimum surface area of the tin can?

8. What is the maximum volume of a cylinder that has a surface area of 350 in.^{2}? What are the diameter and height of this cylinder?

9. Emil is building a fish tank that will be shaped like a rectangular prism. The top of the tank will be open, and the other 5 sides will be glass. He would like the length to be 1.5 times the width. If the surface area of the glass is 800 in.^{2}, what are the dimensions of the fish tank with the greatest possible volume? What is the maximum volume?

10. A company makes large cylindrical barrels to store cooking oil. The material used to make the top and bottom costs $0.05 per square inch, and the material used to make the curved side of the barrel costs $0.03 per square inch. The barrel must have a volume of 25 cubic feet. What radius and height will minimize the total cost of the barrel? (*Hint*: The equation to minimize represents the cost, not the volume.)

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Practice 3.3: Optimization of Volume B

Use the following information to complete problems 1–3.

1. Felix has a rectangular sheet of aluminum that measures 40 cm × 50 cm. He wants to create a box with no top by cutting congruent squares out of the corners and folding up the sides. What function does Felix have to maximize to create the box with the largest volume? What is the domain of the function?

2. What size squares does Felix have to cut from the corners of the aluminum to maximize the volume of the box?

3. What is the volume of the box?

Use what you learned about optimization of volume to solve problems 4–10.

4. A box has a surface area of 500 in.^{2}, a square base, and no top. If the volume of this box is maximized, what are the dimensions of the box?

5. Kate would like to make a box with a closed top that is three times as tall as it is wide and that has a volume of 5,224 cm^3. If the surface area of the box is minimized, what are the dimensions of the box?

continued

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6. Hamish is making a cardboard garage for his little brother's toy cars. The garage will be shaped like a rectangular prism with no bottom and no front. He would like the front of the garage to be three times as wide as it is tall. What are the dimensions of the garage that has the maximum volume if the surface area is 4,200 cm^2?

7. A soda can measures about 4.75 inches tall and 2.5 inches wide. A soda company would like to design a new soda can that has the same volume as a traditional soda can but that minimizes the amount of aluminum used to make the can. What would be the dimensions of this new can?

8. What is the maximum volume of a cylinder that has a surface area of 855 cm^2? What are the dimensions of this cylinder?

9. A box company wants to make an open box with the largest possible volume by cutting congruent squares from the corners of a sheet of cardboard that measures 24 in. × 48 in. and folding the sides up. What is the maximum possible volume?

10. A manufacturer makes cylindrical tins to store powdered milk. The material used to make the top and bottom costs $0.03 per square inch, and the material used to make the curved side of the tin costs $0.01 per square inch. The tin must have a volume of 231 cubic inches, which is approximately 1 gallon. What is the minimum cost of the tin? (*Hint*: The equation to minimize represents the cost, not the volume.)

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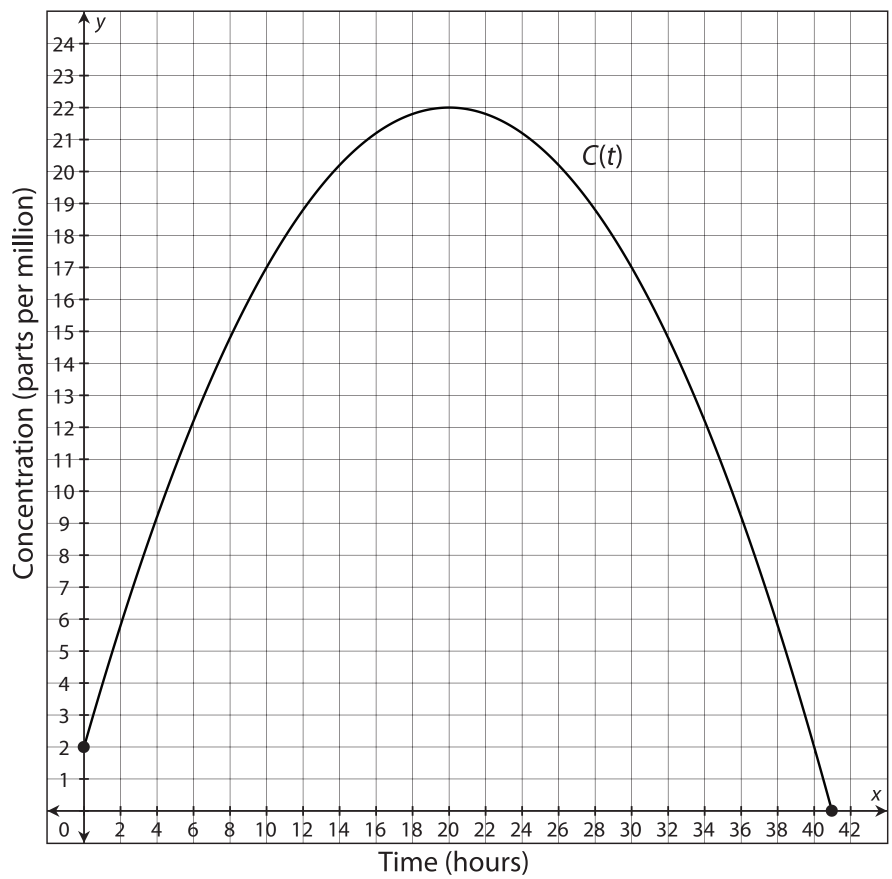
Lesson 3.4: Describing End Behavior and Turns

Warm-Up 3.4

The following graphed function represents the concentration, in parts per million, of a particular medication in the bloodstream after *t* hours. Use the graph to answer the questions that follow.



1. When is the concentration increasing? Decreasing?

2. After how many hours is the concentration of medicine at its highest? What is the concentration at this time?

3. What is the x-intercept and what does it represent?

4. What is the y-intercept and what does it represent?

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Instruction

Lesson 3.4: Describing End Behavior and Turns

North Carolina Math 3 Standards

F–IF.7 Analyze piecewise, absolute value, polynomials, exponential, rational, and trigonometric functions (sine and cosine) using different representations to show key features of the graph, by hand in simple cases and using technology for more complicated cases, including: domain and range; intercepts; intervals where the function is increasing, decreasing, positive, or negative; rate of change; relative maximums and minimums; symmetries; end behavior; period; and discontinuities.★

N–CN.9 Use the Fundamental Theorem of Algebra to determine the number and potential types of solutions for polynomial functions.

Warm-Up 3.4 Debrief

1. When is the concentration increasing? Decreasing?

   The function starts increasing when the amount of time passed is at 0, and continues increasing until the function reaches the vertex, (20, 22). From that point, the function is decreasing. This means that the concentration of medication is increasing when the amount of time passed is less than 20 hours, or $t < 20$. The concentration is decreasing when the amount of time passed is greater than 20 hours, or $t > 20$.

2. After how many hours is the concentration of medicine at its highest? What is the concentration at this time?

   Use the vertex of the function to determine the maximum concentration.

   The vertex of this function is (20, 22); therefore, 20 hours after a person takes the medication, the medication is at its highest concentration of 22 parts per million.

3. What is the *x*-intercept and what does it represent?

   The *x*-intercept is the point at which the function crosses the *x*-axis. In this graph, the *x*-intercept appears to be (41, 0) and represents when, 41 hours after the dose is first taken, the concentration of medicine in the bloodstream is 0 parts per million.

4. What is the *y*-intercept and what does it represent?

   The *y*-intercept is the point at which the function crosses the *y*-axis. In this graph, the *y*-intercept appears to be (0, 2). It represents the time when the dose is first taken, at $t = 0$, and the concentration in the bloodstream at that time, 2 parts per million.

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Connection to the Lesson

- Students will extend their understanding of when functions are increasing and decreasing.
- Students will also extend their understanding of maximums and minimums of quadratic functions to that of identifying local maximums and minimums of polynomial functions.
- Students will interpret maximums and minimums as well as *x*- and *y*-intercepts of polynomial functions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- determining the degree and leading coefficient of a polynomial (A–SSE.1)
- writing a polynomial in standard form (A–SSE.2)

Introduction

By this point in your mathematics experience, you have worked extensively with functions: determining the slopes of linear functions, identifying the vertices of quadratic functions as maxima or minima, determining the end behavior of exponential functions, and identifying intercepts of many types of functions. You can extend the skills you have used to analyze functions you have previously studied in order to understand the graphs of other polynomial functions.

Key Concepts

- Recall that a polynomial function is a function with a general form of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is the highest degree of the polynomial.
- Polynomial functions are defined for any function that contains positive integer exponents.
- Recall that **integers** are numbers that are not fractions or decimals.
- The degree of a polynomial function is the highest exponent to which the dependent variable is raised.
- For example, the equation $y = 3x^7 + 9x^3 - x + 4$ is a seventh-degree polynomial function because its highest exponent is 7 and all other exponents are positive whole numbers.
- $y=4x^{\frac{3}{5}}+6x-3$ is not a polynomial function because the exponent is not a whole number. $y=-3\sqrt{x}+4$ is also not a polynomial function since the square root of a number can be written as a power of $\frac{1}{2}$, which is not a whole number either. And $y=5x^3+2x^{-4}+6$ is not a polynomial function because not all exponents are non-negative integers.

End Behavior

- To determine the **end behavior** of a polynomial function, or the behavior of the graph as x approaches positive or negative infinity, consider the highest degree of the polynomial and its coefficient, ax^n.
- If n is even, the polynomial function is considered an **even-degree polynomial function**.

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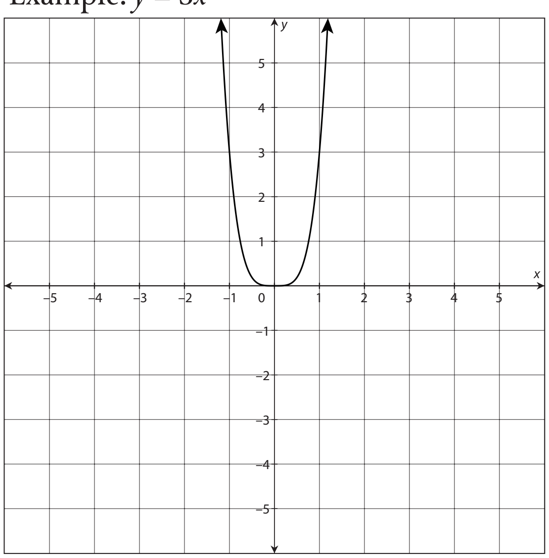
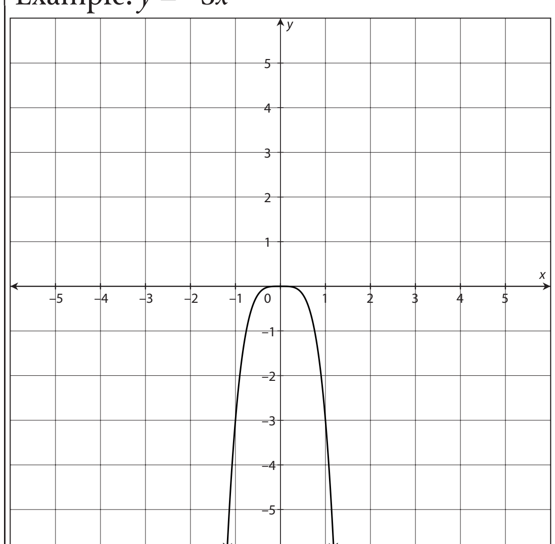
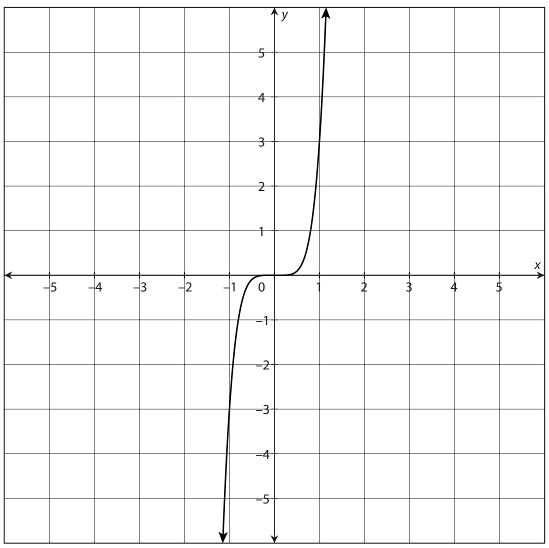
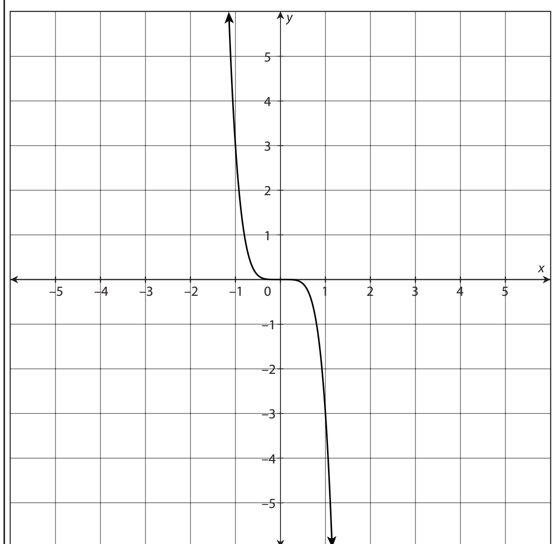
- When *n* is even and *a* is positive, then both ends of the function will extend upward. That is, the value of *f*(*x*) approaches positive infinity as *x* approaches negative infinity, and also when *x* approaches positive infinity. Symbolically, this can be written $f(x)\to+\infty$ as $x\to-\infty$ and $f(x)\to+\infty$ as $x\to+\infty$.
- When *n* is even and *a* is negative, then both ends of the function will extend downward. That is, the value of *f*(*x*) approaches negative infinity as *x* approaches negative infinity, and also when *x* approaches positive infinity. Symbolically, this can be written $f(x)\to-\infty$ as $x\to-\infty$ and $f(x)\to-\infty$ as $x\to+\infty$.
- If *n* is odd, the polynomial function is considered an **odd-degree polynomial function**.
- When *n* is odd and *a* is positive, then one end of the function will extend down to the left and the other end will extend up to the right. That is, the value of *f*(*x*) approaches positive infinity as *x* approaches positive infinity, and the value of *f*(*x*) approaches negative infinity as *x* approaches negative infinity. Symbolically, this can be written $f(x)\to-\infty$ as $x\to-\infty$ and $f(x)\to+\infty$ as $x\to+\infty$.
- When *n* is odd and *a* is negative, then one end of the function will extend up to the left and the other end will extend down to the right. That is, the value of *f*(*x*) approaches positive infinity as *x* approaches negative infinity, and the value of *f*(*x*) approaches negative infinity as *x* approaches positive infinity. Symbolically, this can be written $f(x)\to+\infty$ as $x\to-\infty$ and $f(x)\to-\infty$ as $x\to+\infty$.

Even-degree polynomials	
Positive leading coefficient Example: $y = 3x^4$	**Negative leading coefficient** Example: $y = -3x^4$
	
$f(x)\to+\infty$ as $x\to-\infty$ $f(x)\to+\infty$ as $x\to+\infty$	$f(x)\to-\infty$ as $x\to-\infty$ $f(x)\to-\infty$ as $x\to+\infty$

Odd-degree polynomials	
Positive leading coefficient Example: $y = 3x^5$	**Negative leading coefficient** Example: $y = -3x^5$
	
$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to +\infty$ as $x \to +\infty$	$f(x) \to +\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to +\infty$

Turning Points

- A **turning point** of a function is a point where the graph of the function changes from sloping upward to sloping downward or, alternatively, from sloping downward to sloping upward.
- To determine the maximum number of turning points of a function, subtract 1 from the highest degree of the polynomial. In other words, find $n - 1$.
- For instance, the polynomial function $y = 3x^7 + 9x^3 - x + 4$ can have no more than 7 – 1, or 6, turning points.
- The maximum number of turning points does not necessarily indicate the actual number of turning points of a function, just that it can have no more than that number. Some functions may have fewer turning points than the number calculated.
- A turning point corresponds to a **local maximum**, the greatest value of a function for a particular interval of the function, or a **local minimum**, the least value of a function for a particular interval of the function. A local maximum may also be referred to as a **relative maximum** and a local minimum may also be referred to as a **relative minimum**.

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Roots of a Polynomial Function

- The highest degree of the polynomial determines the maximum number of **roots**, or *x*-intercepts of a function.
- A polynomial function with a degree of 10 could have up to 10 roots, but could also have 0 to 9 roots, depending on the specific equation.
- Recall that real numbers include all rational and irrational numbers, but do not include imaginary and complex numbers.

Sketching a Polynomial Function

- Being able to identify the general end behavior, the possible number of turning points, and the maximum number of roots of a polynomial function can be helpful in creating a rough sketch of the function.
- Start by choosing at least six *x*-values that are both positive and negative. It is also useful to choose the value of 0.
- As you've done in previous courses, substitute each chosen *x*-value into the given function and evaluate to determine the corresponding *y*-value. Then, plot the points on a graph.
- Be sure to smoothly connect all chosen points to illustrate the graph of the function.
- Graphing calculators are especially helpful when sketching a complicated function.

Common Errors/Misconceptions

- assuming the maximum number of calculated turning points is the actual number of turning points of a function
- assuming the maximum number of roots is the actual number of roots of a function
- not using the highest degree term to identify the end behavior of the polynomial function
- not using the highest degree of the polynomial function to determine the maximum number of turns
- confusing even-degree and odd-degree functions with even and odd functions

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Scaffolded Practice 3.4: Describing End Behavior and Turns

For problems 1–5, determine the end behavior, the maximum number of turning points, and the maximum number of real roots of each function.

1. $f(x)=5x^3+2x^2-6$

2. $f(x)=-7x^4+5x^3+4$

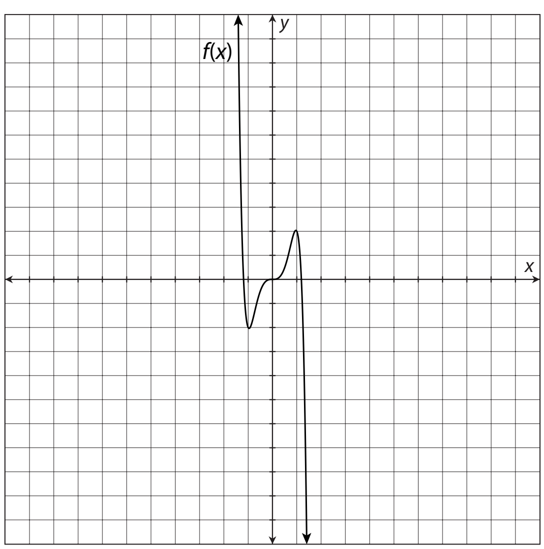
3. $g(x)=-2x^7-3x^6+4x^4+8$

4. $f(x)=-8x^6+x^4+3x^2-5x+1$

5. $h(x)=12x^{10}+x^9+3x^6+4x^3$

For problems 6–10, describe the end behavior of each graph. Determine whether the graph represents an even-degree or odd-degree function, and determine the number of real roots.

6.



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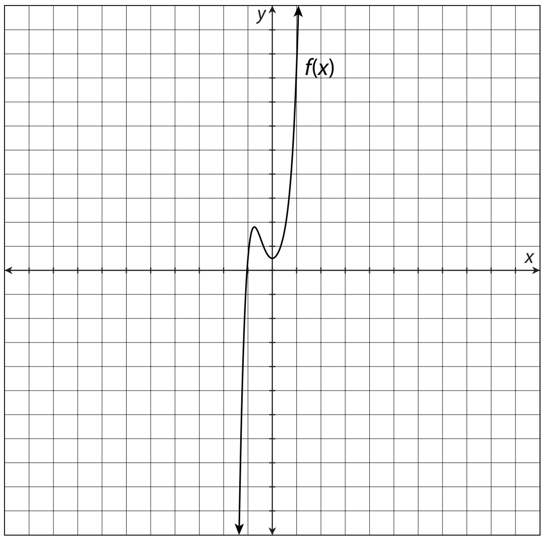
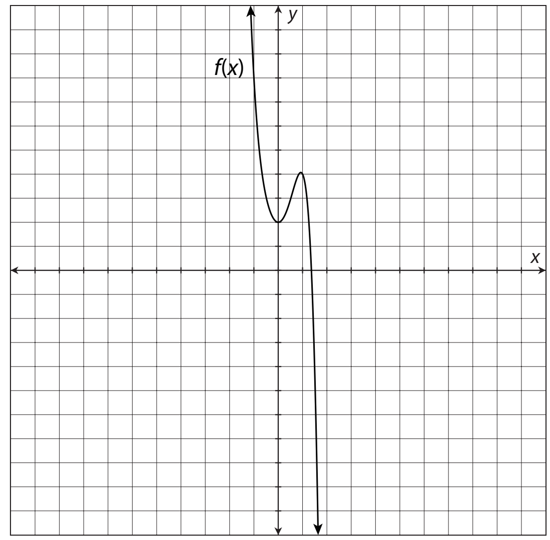
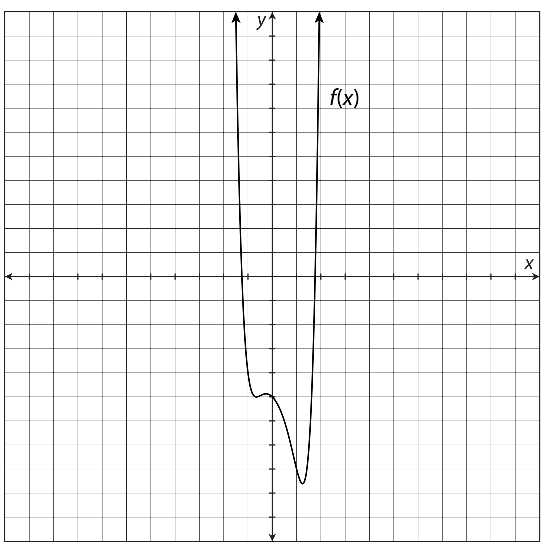
7.



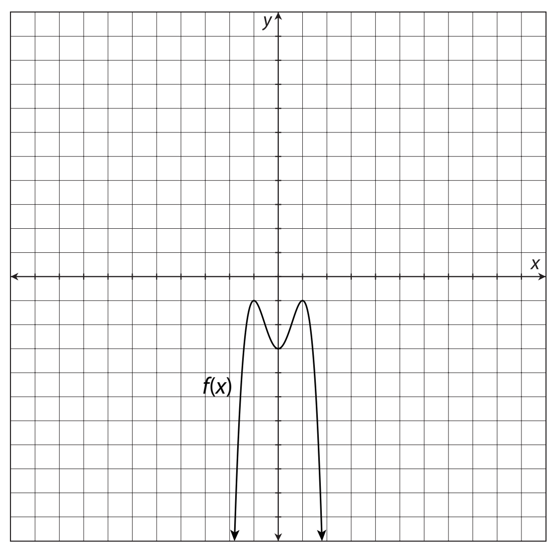
9.



8.



10.



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Guided Practice 3.4

Example 1

Determine the end behavior, maximum number of turning points, and maximum number of real roots of the function $f(x) = 6x^5 - 3x^4 + 2x + 7$.

1. Identify the leading coefficient and degree of the polynomial function.

   The function $f(x) = 6x^5 - 3x^4 + 2x + 7$ is a fifth-degree polynomial function because the highest exponent is 5.

   The coefficient of the term containing the highest exponent is 6; therefore, 6 is the leading coefficient.

2. Determine the end behavior of the function.

   The leading coefficient, 6, is positive and the degree of the function, 5, is odd; therefore, the graph of the function will extend down to the left and up to the right.

   Symbolically, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

3. Determine the maximum number of turning points of the polynomial function.

   To determine the maximum number of turning points, subtract 1 from the degree of the polynomial; that is, find $n - 1$.

   The degree of the polynomial is 5, and $5 - 1 = 4$.

   The maximum number of turning points in the graph is 4.

4. Determine the maximum number of real roots of the polynomial function.

   The maximum number of real roots is equal to the degree of the polynomial; therefore, the maximum number of roots of the function $f(x) = 6x^5 - 3x^4 + 2x + 7$ is 5.

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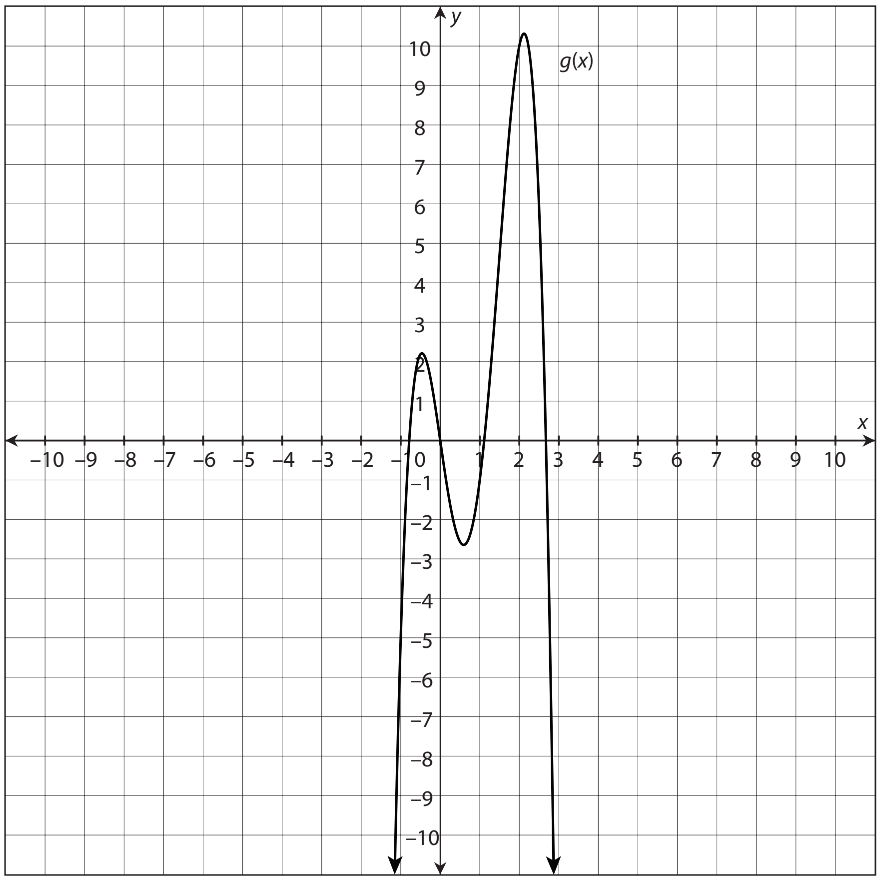
Example 2

Describe the end behavior of the given graph of $g(x)$. Determine whether the graph represents an even-degree or odd-degree function, and determine the number of real roots.



1. Describe the end behavior of the graphed function, $g(x)$.

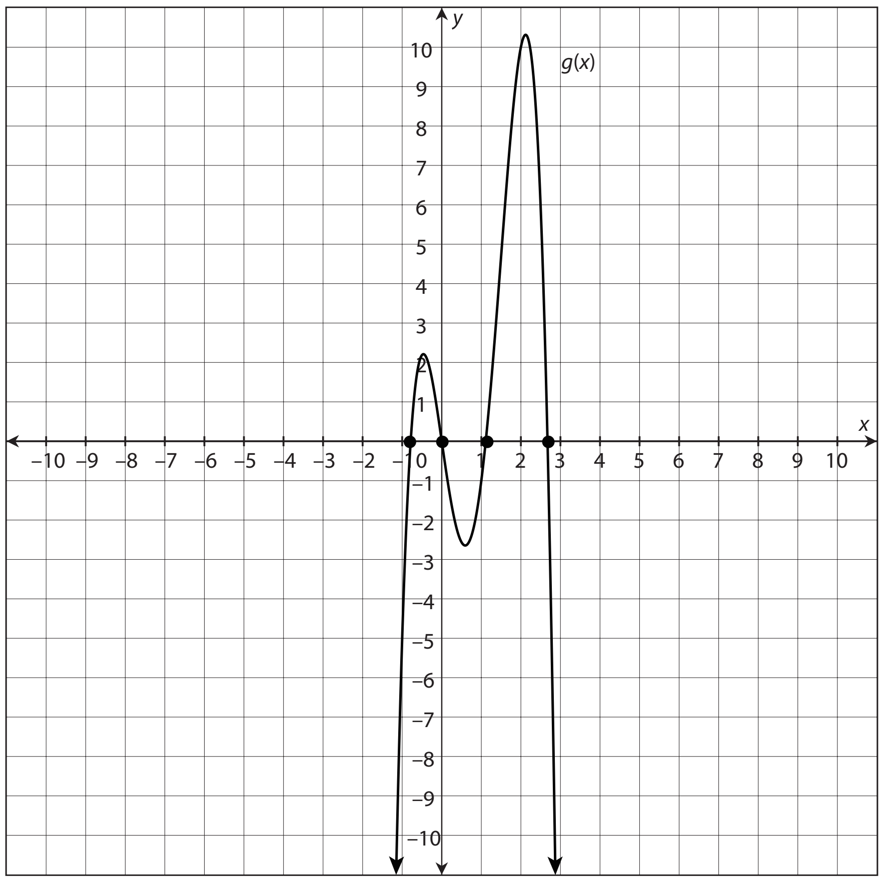
   Refer to the graph to determine the function's end behavior as x gets larger or smaller. The value of $g(x)$ approaches negative infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written as $g(x) \to -\infty$ as $x \to -\infty$ and $g(x) \to -\infty$ as $x \to +\infty$.

2. Determine whether the graph of $g(x)$ represents an even-degree or odd-degree function.

   Both ends of the graph are pointing in the same direction: downward. Therefore, the function must be an even-degree function.

3. Determine the number of real roots.

   Real roots are the points at which the graphed function intersects the *x*-axis. Because the *x*-axis contains only real numbers, only the real roots can be graphed. Determine the number of times the graph intersects the *x*-axis.



   The graph intersects the *x*-axis at four points, so there are four real roots.

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Example 3

Use a graphing calculator to graph the function $p(x) = -x^4 + 3x^2 + 4$. Summarize the end behavior and turning points of the function.

1. Graph the equation on your calculator.

   On a TI-83/84:

   Step 1: Press [Y=].

   Step 2: Type the function into Y_1. Use the [X, T, θ, n] button for the variable *x*. To enter exponents, choose [CATALOG] and arrow down to ^.

   Step 3: Press [WINDOW] to change the viewing window as needed.

   Step 4: Press [GRAPH].

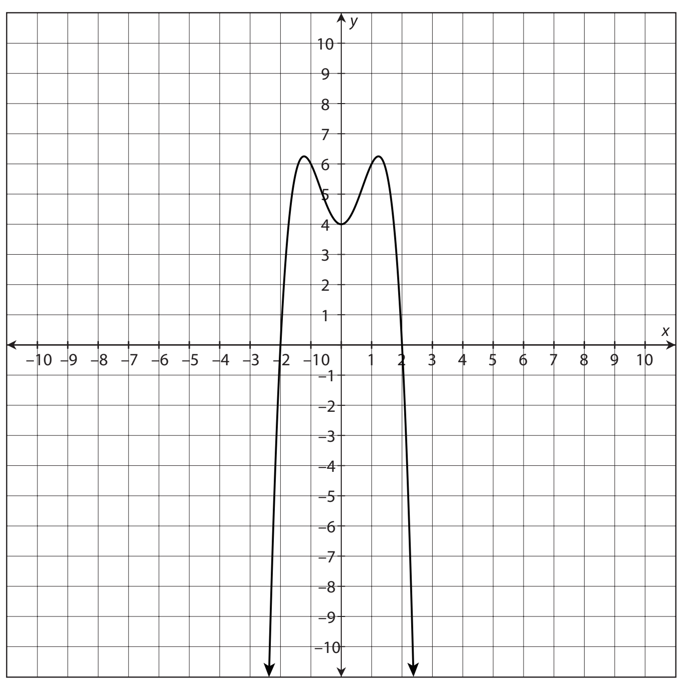
   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon and press [enter].

   Step 3: Type the function next to *f1*(*x*), or any available equation, and press [enter]. Use the [X] button for the letter *x*. Use the [^] button for exponents.

   Step 4: To change the viewing window, press [menu]. Select 1: Window Settings.



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2. Analyze the table of values.

Refer to the table of values for the function.

On a TI-83/84:

Step 1: Choose [TABLE]. Scroll up and down the table to view various points on the graph.

On a TI-Nspire:

Step 1: Choose [menu]. Navigate to 7: Table, 1: Split-Screen Table. Arrow up and down the table to view various points on the graph.

Points include:

x	*y*
–3	–50
–2	0
–1	6
0	4
1	6
2	0
3	–50

Both ends of the graph extend in the same direction, and the graph opens downward. Therefore, we know that this is an even-degree polynomial with a negative leading coefficient, so $p(x) \to -\infty$ as $x \to -\infty$, and $p(x) \to -\infty$ as $x \to +\infty$.

We can see from the graph that this function has three turning points. It can be seen from the graph and the table of values that one turning point is at (0, 4). $x = 0$ is less than the surrounding points, so it is a local (or relative) minimum. Two other approximate turning points are (–1, 6) and (1, 6). $x = -1$ and 1 are greater than the surrounding points, so they are both local (or relative) maximums.

The graph intersects the *x*-axis at two points, indicating that there are two real roots of this function.

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Example 4

Create a rough sketch of the graph of a sixth-degree polynomial function with a positive leading coefficient.

1. Determine the end behavior of the possible function.

   The function is a sixth-degree polynomial; therefore, it is an even-degree polynomial. Both ends of the graphed function will extend in the same direction. Because the leading coefficient is positive, both ends of the graphed function will extend upward. Symbolically, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

2. Determine the maximum number of turns.

   The maximum number of turns is defined by the degree of the polynomial minus 1.

   maximum number of turns = $n - 1$

   $= 6 - 1$

   $= 5$

   A sixth-degree polynomial will have no more than 5 turning points.

3. Determine the maximum number of real roots.

   The maximum number of real roots, or x-intercepts, is equal to the degree of the polynomial. This particular function will have no more than 6 real roots.

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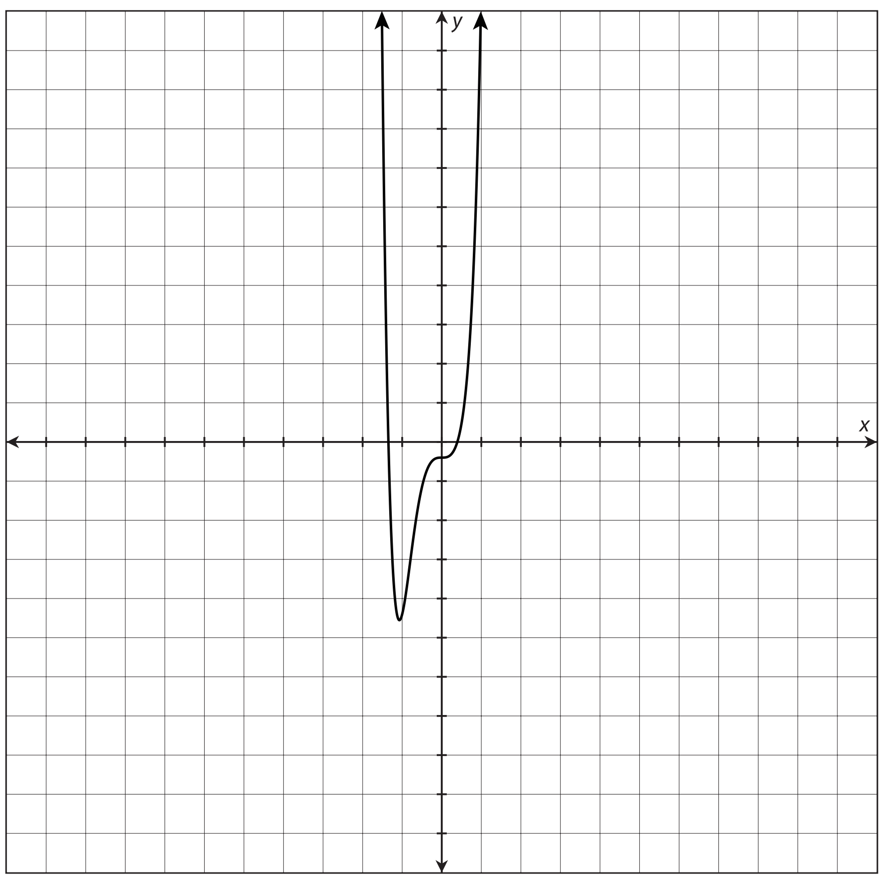
4. Use the information from the previous steps to sketch a possible graph of a sixth-degree polynomial with a positive leading coefficient.

   We have determined the following about a sixth-degree polynomial function with a positive leading coefficient:

   - Both ends of the function will extend upward.
   - It can have no more than 5 turning points.
   - It can have no more than 6 real roots.

   The following graph extends upward at both ends, has two turning points, and has two real roots. It satisfies all the requirements of a sixth-degree polynomial function with a positive leading coefficient, so it is a possible graph.



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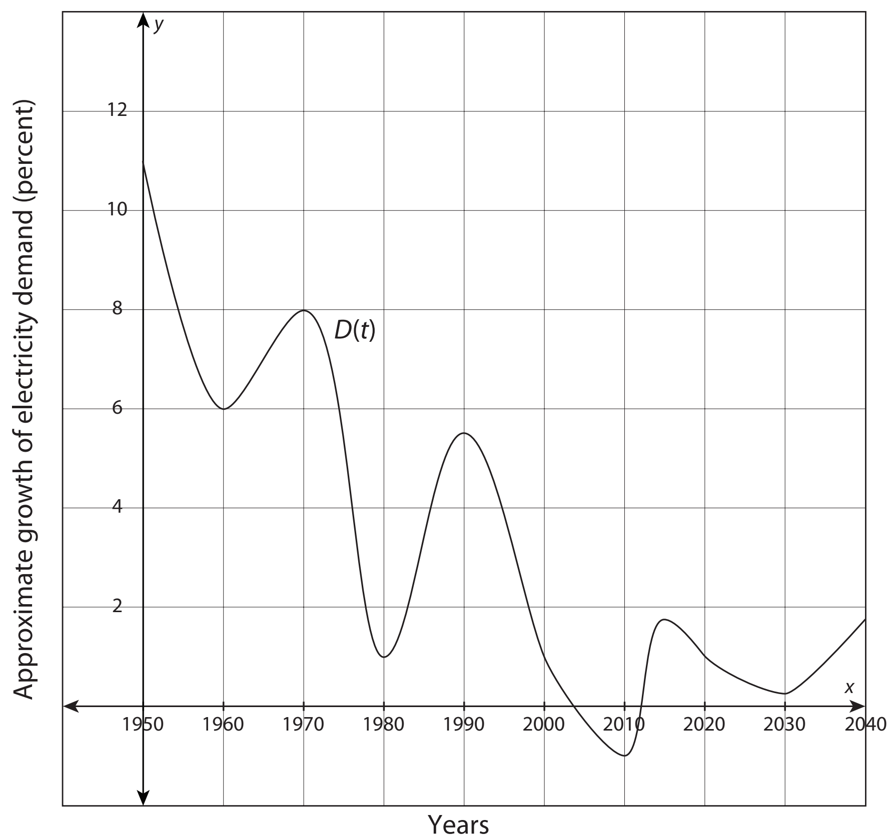
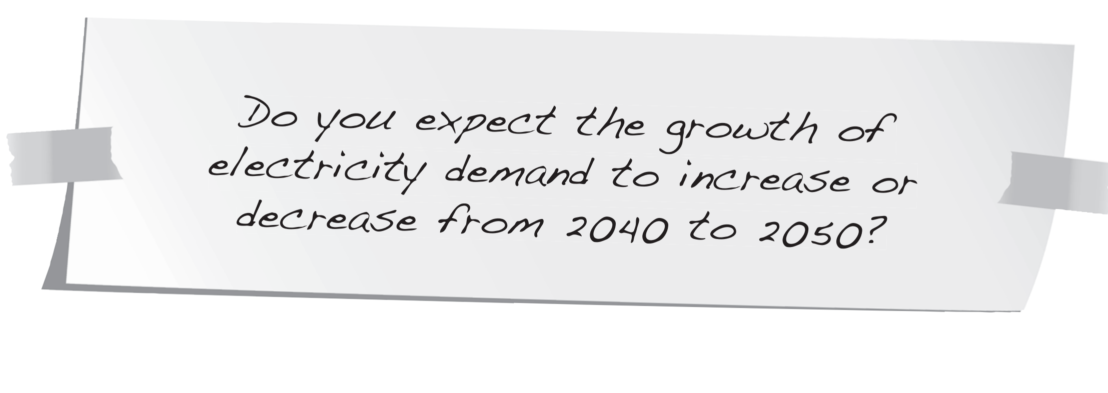
Problem-Based Task 3.4: It's Electric!

SMP
1 ✓ 2
3 ✓ 4 ✓
5 6
7 8

The growth of demand for electricity in the United States changes from year to year. Below is a sketch of a polynomial function that represents that approximate growth since 1950 and includes projections to 2040. Consider a possible polynomial function that could represent this information. Using end behavior, turning points, and roots, do you expect the growth of electricity demand to increase or decrease from 2040 to 2050? Explain your reasoning.



Source: U.S. Energy Information Administration: Annual Energy Outlook 2013



Problem-Based Task 3.4: It's Electric!

Coaching

a. What are the turning points of the polynomial function?

b. What does each turning point represent?

c. What are the real roots of this polynomial?

d. What do the real roots represent?

e. What is the behavior of this function as *t* approaches positive infinity?

f. Do you expect the growth of demand for electricity to increase or decrease from 2040 to 2050? Explain.

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Problem-Based Task 3.4: It's Electric!

Coaching Sample Responses

a. What are the turning points of the polynomial function?

The turning points of the graph are where the graph of the function changes from sloping upward to downward or vice versa. This function appears to have 7 turning points. The turning points are approximately (1960, 6), (1970, 8), (1980, 1), (1990, 5.5), (2010, –1), (2012, 1.75), and (2030, 0.25).

b. What does each turning point represent?

The point (1960, 6) is a local minimum; this indicates the growth of demand for electricity decreased from 1950 to 1960 and then began to increase until the year 1970, when the growth of demand reached a local maximum. After 1970, the growth of demand again decreased until it reached another local minimum in 1980. The point (1990, 5.5) represents another local maximum. The growth of demand decreased from this point until 2010, when the growth of demand was approximately –1%. The growth of demand increased until 2012 and is projected to decrease until the year 2030, when it is projected that the growth of demand will increase until at least 2040.

c. What are the real roots of this polynomial?

The real roots of this function are found at the x-intercepts. This function appears to have two real roots, at the points (2003, 0) and (2011, 0).

d. What do the real roots represent?

The real roots represent the years in which the growth of demand for electricity was 0%.

e. What is the behavior of this function as t approaches positive infinity?

It is difficult to tell from the graph what the end behavior of the function is, but it appears as though $D(t)$ approaches $+\infty$ as t approaches $+\infty$.

f. Do you expect the growth of demand for electricity to increase or decrease from 2040 to 2050? Explain.

Based on the number of turning points and the end behavior of the function represented by the graph, as well as possible advances in technology, it is possible that the growth in demand for electricity will continue to increase from 2040 to 2050. However, it is also possible that because of alternate power sources, the growth in demand for electricity will decline from 2040 to 2050.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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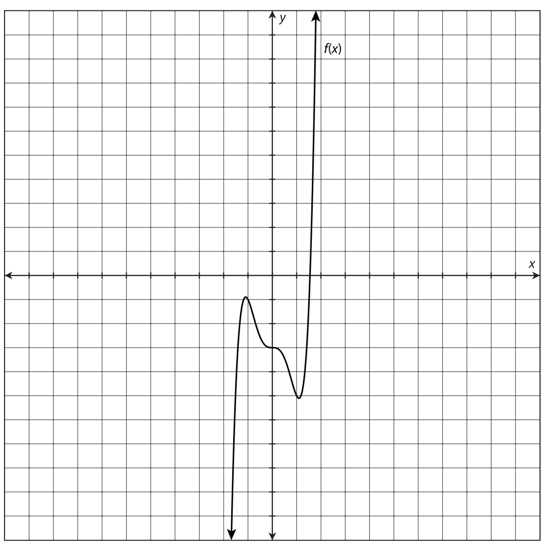
Practice 3.4: Describing End Behavior and Turns A

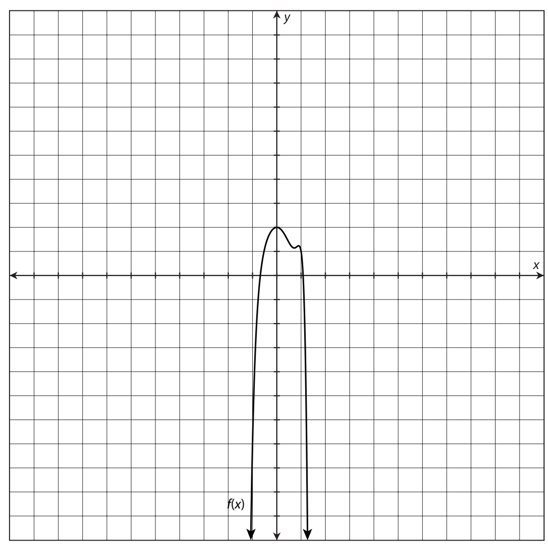
For problems 1 and 2, determine the end behavior, the maximum number of turning points, and the maximum number of real roots of each function.

1. $f(x) = -2x^4 + 6x^3 + 5x$

2. $g(x) = 4x^3 - 7x^2 + 8$

For problems 3 and 4, describe the end behavior of each graph. Determine whether the graph represents an even-degree or odd-degree function, and determine the number of real roots.

3. 

4. 

For problems 5 and 6, create a rough sketch of a possible graph of the function described.

5. a fifth-degree polynomial function with a negative leading coefficient

6. a fourth-degree polynomial function with a positive leading coefficient

continued

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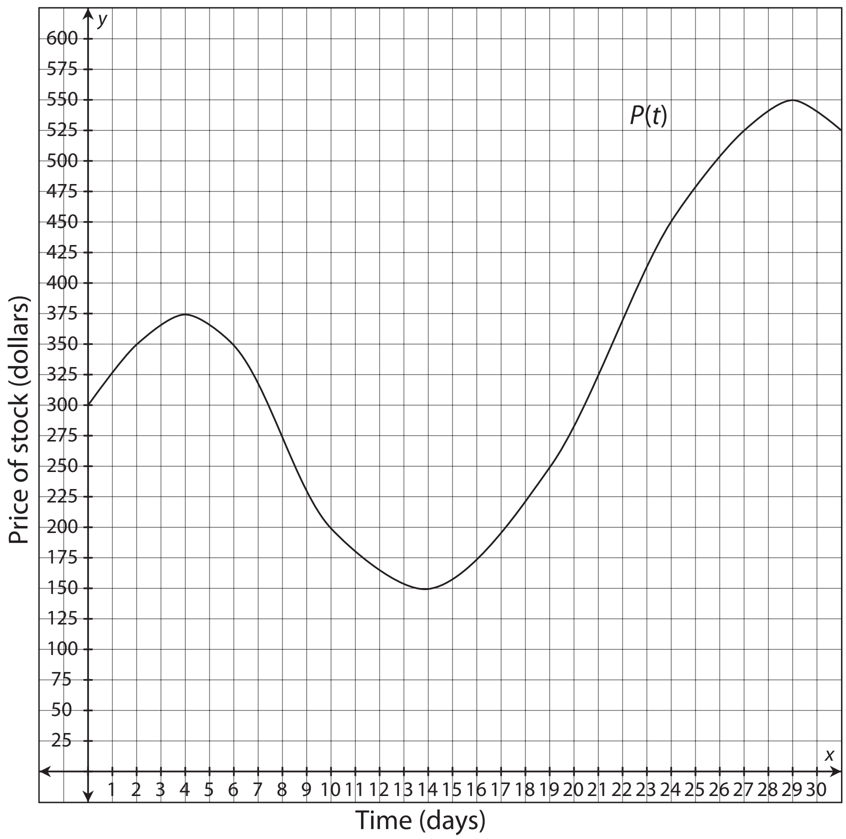
Lesson 3.4: Describing End Behavior and Turns

The following graph models the price of a particular stock over a period of time. Use this graph to complete problems 7–10.



7. Estimate the turning points of the graph of this function.

8. What do the turning points mean in terms of the price of the stock?

9. Describe the end behavior of this graph.

10. If this graph were modeled by a polynomial function, what is the least degree the equation could have? Explain your answer.

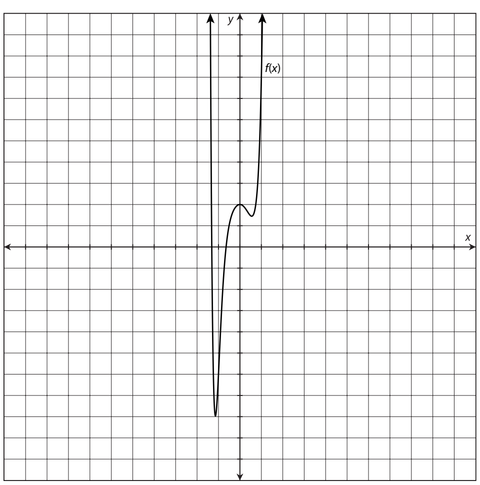
Practice 3.4: Describing End Behavior and Turns B

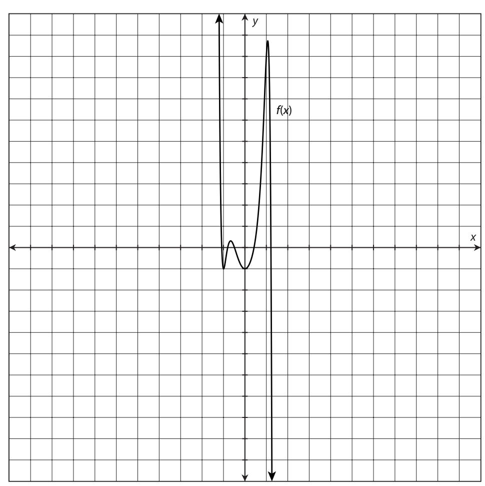
For problems 1 and 2, determine the end behavior, the maximum number of turning points, and the maximum number of real roots of each function.

1. $g(x) = -5x^7 + 3x^4 - 2x + 6$

2. $f(x) = 7x^8 + 2x^3 - 5x - 8$

For problems 3 and 4, describe the end behavior of each graph. Determine whether the graph represents an even-degree or odd-degree function, and determine the number of real roots.

3. 

4. 

For problems 5 and 6, create a rough sketch of a possible graph of the function described.

5. a ninth-degree polynomial with a positive leading coefficient

6. an eighth-degree polynomial with a negative leading coefficient

continued

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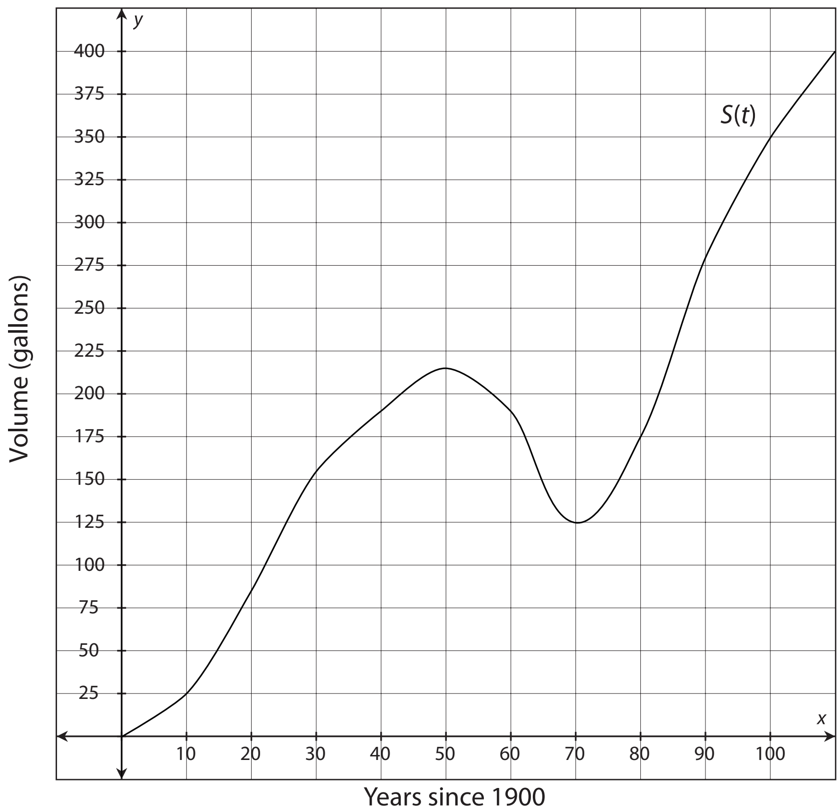
Name: Date:

UNIT 3 • POLYNOMIAL FUNCTIONS F–IF.7★, N–CN.9

Lesson 3.4: Describing End Behavior and Turns

The following graph models the volume of water in a water reservoir each year since 1900. Use this graph to complete problems 7–10.



7. Estimate the turning points of the graph of this function.

8. What do the turning points mean in terms of the volume of the reservoir?

9. Describe the end behavior of this graph.

10. If this graph were modeled by a polynomial function, what is the least degree the equation could have? Explain your answer.

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Lesson 3.5: The Remainder Theorem

Warm-Up 3.5

A factory produces sweatshirts for a sports team, and packages the sweatshirts in boxes of 20. Every hour, the factory produces 9,284 sweatshirts.

1. How many full boxes of sweatshirts are packaged in 1 hour?

2. How many sweatshirts are produced but not packaged in 1 hour?

3. If the remaining sweatshirts were set aside, what fraction of a box would they make up?

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Lesson 3.5: The Remainder Theorem

North Carolina Math 3 Standard

A–APR.2 Understand and apply the Remainder Theorem.

Warm-Up 3.5 Debrief

1. How many full boxes of sweatshirts are packaged in 1 hour?

   Divide the number of sweatshirts produced each hour, 9,284, by the number of sweatshirts that are in one box, 20.

$$\begin{array}{r} 464 \\ 20\overline{)9284} \\ \underline{80} \\ 128 \\ \underline{120} \\ 84 \\ \underline{80} \\ 4 \end{array}$$

   9,284 divided by 20 is approximately 464. It is unnecessary to continue the division past the whole number, because only the number of full boxes concerns us.

   464 full boxes of sweatshirts are packaged in 1 hour.

2. How many sweatshirts are produced but not packaged in 1 hour?

   First, determine the number of sweatshirts that are packaged in 1 hour: $464 \bullet 20 = 9280$.

   In 1 hour, 9,280 sweatshirts are packaged in 464 boxes.

   Now subtract the number of packaged sweatshirts from the number of produced sweatshirts: $9284 - 9280 = 4$.

   The remaining number of sweatshirts that are produced but not packaged is 4.

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3. If the remaining sweatshirts were set aside, what fraction of a box would they make up?

   At the end of 1 hour, 4 sweatshirts remain unpackaged. Each box holds 20 sweatshirts when full. Therefore, the remaining sweatshirts can be represented as the fraction $\frac{4}{20}$ or, in reduced form, $\frac{1}{5}$.

   The remaining unpackaged sweatshirts make up $\frac{1}{5}$ of a box.

Connection to the Lesson

- Students will extend their understanding of long division of whole numbers to division of polynomials.
- Students will write the remainders of division problems as fractions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying and dividing monomials (A–SSE.2)
- evaluating functions for a given value of *x* (F–IF.2)

Introduction

In mathematics, the word "remainder" is often used in relation to the process of long division. You are probably familiar with dividing whole numbers. For example, the result (or quotient) of 8 divided by 4 is 2; that is, $8 \div 4 = 2$. In other words, if the dividend of 8 units were divided by a divisor of 4, each group would have 2 units.

The process of division does not always result in a whole number. For instance, the result of 14 divided by 5, or $14 \div 5$, is 2 with a remainder of 4. Recall that this means that if a dividend of 14 units were divided by a divisor of 5, then each group would have 2 units and there would be 4 units left over. 4 is referred to as the remainder. In earlier grades, you often wrote the quotient and its remainder as 2R4 or $2\frac{4}{5}$. The idea of having remainders also extends to dividing polynomials. Sometimes when polynomials are divided, the result is a polynomial; other times there is a remainder.

Key Concepts

- Long division, used to divide whole numbers, can also be used to divide polynomials. Recall that the dividend is the quantity being divided, and the divisor is the quantity by which it is being divided: dividend ÷ divisor = quotient.
- Long division with polynomials can sometimes be long and tedious; fortunately, there is another way to divide polynomials.
- The process of **synthetic division**, a shorthand way of dividing a polynomial by a linear binomial by using only the coefficients, is often used instead.
- In order to use synthetic division, the dividend must be a polynomial written in standard form, ordered by the power of the variables, with the largest power listed first. If a term is missing, 0 must be used in its place.
- For example, compare $4x^2 + 16$ to the standard form of a polynomial, $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$. Notice that $4x^2 + 16$ does not have a value for $a_{n-1}x^{n-1}$, or in this case, ax. To prepare this polynomial for synthetic division, substitute 0 as the coefficient (a) in the ax term: $4x^2 + 0x + 16$. Now you can use the coefficients 4, 0, and 16 to perform the synthetic division.

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- For synthetic division, the divisor must be of the form $(x - a)$, where a is a real number.
- Use the following steps to divide polynomials using synthetic division. An example has been provided for clarity.

Synthetic Division of Polynomials	
Example: $(3x^2 - 20x + 12) \div (x - 3)$	
1. Write the coefficients of the dividend, $3x^2 - 20x + 12$: 3, –20, and 12.	$3 \quad -20 \quad 12$
2. Identify the value of a in the divisor, $(x - a)$. Write this value, 3, in the upper left corner.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \end{array}$
3. Create a horizontal line below the coefficients, allowing for space to write above and below the line.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \\ & & & \\ \hline \end{array}$
4. Write the first coefficient in the dividend below the horizontal line.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \\ & & & \\ \hline & 3 & & \end{array}$
5. Multiply the number below the horizontal line by the value of a and write the product under the next coefficient and above the horizontal line.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \\ & & 9 & \\ \hline & 3 & & \end{array}$
6. Add the numbers in the new column. Write the result below the horizontal line in that column.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \\ & & 9 & \\ \hline & 3 & -11 & \end{array}$
7. Repeat steps 5 and 6 until addition has been completed for all columns.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \\ & & 9 & -33 \\ \hline & 3 & -11 & -21 \end{array}$
8. Draw a box around the far right sum.	$\begin{array}{c\|ccc} 3 & 3 & -20 & 12 \\ & & 9 & -33 \\ \hline & 3 & -11 & \boxed{-21} \end{array}$
9. The numbers below the horizontal line represent the quotient. These numbers are the coefficients of the polynomial quotient in the order of decreasing degree. The boxed number is the remainder. Place the remainder over the divisor to express the final term of the polynomial.	$= 3x - 11 - \dfrac{21}{x-3}$

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Instruction

- If you are dividing by a linear function, $(x - a)$, the order of the quotient is 1 less than the dividend. The remainder, if any, is a constant.
- If the remainder is 0, then the divisor is a factor of the polynomial.
- Synthetic division can also be used to find the value of a function. This is known as **synthetic substitution**.
- To evaluate a polynomial using synthetic substitution, follow the same process described for synthetic division. For example, given the function $3x^2 - 20x + 12$, if you must determine the value of the function at $x = 3$, use 3 as the a value in the divisor of the synthetic division. The resulting remainder gives the value of the polynomial when evaluated at $x = 3$.
- If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder, r, is equal to $p(a)$.
- This process leads to the **Remainder Theorem**.

Remainder Theorem
For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

- To take a closer look at the Remainder Theorem, let's work with the same polynomial we used to demonstrate synthetic division of polynomials: $3x^2 - 20x + 12$.
- Let $p(x) = 3x^2 - 20x + 12$. We can use synthetic substitution (by following the process of synthetic division), to evaluate this function for $x = 3$. The result is the remainder, or –21. Because this result is a number other than 0, the Remainder Theorem allows us to conclude that $(x - 3)$ is not a factor of $p(x)$. Only when the remainder is 0 will $(x - a)$ be a factor of the polynomial. A remainder of any number other than 0 indicates that $(x - a)$ is not a factor of the given polynomial.
- If we evaluate the same function, $p(x) = 3x^2 - 20x + 12$, for $x = 6$, the remainder is 0. By the Remainder Theorem, $(x - 6)$ is a factor of the given polynomial.
- Dividing a polynomial by one of its binomial factors results in a **depressed polynomial**. When $3x^2 - 20x + 12$ is divided by 6, the depressed polynomial is $3x - 2$.
- It is sometimes easier to evaluate a function for a given value by using direct substitution. For instance, when evaluating the expression $3x^2 + 2$ when $x = 4$, directly substitute 4 into the polynomial and follow the order of operations: $3(4)^2 + 2 = 3(16) + 2 = 48 + 2 = 50$. Other times, when the polynomials are more complicated, synthetic substitution is more efficient. Both methods are helpful when working with polynomials.
- As you will see in this lesson, synthetic division and substitution can also be applied to real-world problems.

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Common Errors/Misconceptions

- not using the standard form of a polynomial before attempting to use synthetic division
- using the incorrect sign when substituting or dividing
- forgetting to use 0 as a placeholder for coefficients that are not present in the standard form of the dividend

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Scaffolded Practice 3.5: The Remainder Theorem

For problems 1–4, use synthetic division to find each quotient.

1. $(3x^2 + 7x + 10) \div (x + 2)$

2. $(2x^4 - 2x^3 - 6x^2 - 16x - 6) \div (x - 3)$

3. $(x^3 + 6x^2 + 13x + 20) \div (x + 4)$

4. $(x^2 - 18) \div (x - 6)$

For problems 5–8, use synthetic substitution to evaluate each function.

5. $p(x) = x^2 - 8x + 6$ for $x = 8$

continued

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6. $p(x) = 2x^2 + 4x + 9$ for $x = 2$

7. $p(x) = 3x^4 + 2x^3 - 6x - 10$ for $x = -2$

8. $p(x) = 6x^5 + 2x^3 - 5x + 1$ for $x = 1$

For problems 9 and 10, find the value of k.

9. $(x^2 + kx + 7) \div (x - 3)$ has a remainder of –20.

10. $(x^3 - 6x^2 + kx + 5) \div (x + 2)$ has a remainder of –21.

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Guided Practice 3.5

Example 1

Find the quotient of $(x^2 - 5x - 20) \div (x - 4)$ using polynomial long division.

1. Set up the division.

   The dividend is $x^2 - 5x - 20$ and the divisor is $x - 4$.

$$x-4\overline{)x^2-5x-20}$$

2. Divide the leading term of the dividend by the leading term of the divisor.

   The leading term of the dividend is x^2. The leading term of the divisor is x.

   The result of x^2 divided by x is x.

   Record the result in a manner similar to long division of whole numbers.

$$\begin{array}{r} x \\ x-4\overline{)x^2-5x-20} \end{array}$$

   Multiply the result, x, by the divisor, $x - 4$.

   $x(x - 4) = x^2 - 4x$

   Write the result under the dividend, lining up the terms of equal degree.

$$\begin{array}{r} x \\ x-4\overline{)x^2-5x-20} \\ \underline{x^2-4x} \end{array}$$

   Subtract the last line from the line above it.

   $(x^2 - 5x - 20) - (x^2 - 4x) = -x - 20$

   Record the result in the division.

$$\begin{array}{r} x \\ x-4\overline{)x^2-5x-20} \\ \underline{x^2-4x} \\ -x-20 \end{array}$$

(continued)

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Repeat the process. Divide the leading term of the new dividend, $-x - 20$, by the leading term of the divisor, $x - 4$.

The result of $-x$ divided by x is -1.

Record the result in the division.

$$\begin{array}{r|l} & x \quad -1 \\ \hline x-4 & x^2-5x-20 \\ & x^2-4x \\ \hline & \quad\quad -x-20 \end{array}$$

Multiply the result, -1, by the divisor, $x - 4$.

$-1(x - 4) = -x + 4$

Write the result under the division, again lining up the terms of equal degree.

$$\begin{array}{r|l} & x \quad -1 \\ \hline x-4 & x^2-5x-20 \\ & x^2-4x \\ \hline & \quad\quad -x-20 \\ & \quad\quad\quad x+4 \\ \hline \end{array}$$

Subtract the last line from the line above it.

$$\begin{array}{r|l} & x \quad -1 \\ \hline x-4 & x^2-5x-20 \\ & x^2-4x \\ \hline & \quad\quad -x-20 \\ & \quad\quad\quad x+4 \\ \hline & \quad\quad\quad\quad -24 \end{array}$$

Notice that x has been divided out from the problem. -24 represents the remainder of $(x^2 - 5x - 20) \div (x - 4)$.

Write the remainder over the divisor: $-\frac{24}{x-4}$. Remember to include the remainder in the result of the long division.

The result of $(x^2 - 5x - 20) \div (x - 4)$ using polynomial long division is $x-1-\frac{24}{x-4}$.

Example 2

Find the quotient of $(x^2 - 5x - 20) \div (x - 4)$ using synthetic division.

1. Identify the coefficients of the dividend.

   The dividend is $x^2 - 5x - 20$. The coefficients of this expression are 1, –5, and –20.

2. Identify the value of a.

   The divisor is $x - 4$, which is of the form $x - a$; therefore, the value of a is 4.

3. Record the coefficients of the dividend and the value of a in the divisor in the synthetic division format.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & & \\ \hline \end{array}$$

4. Carry out the process of synthetic division.

   Bring down the first coefficient of the dividend below the horizontal line.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & & \\ \hline & 1 & & \end{array}$$

   Multiply the first coefficient by the value of a and write the value under the second coefficient.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & 4 & \\ \hline & 1 & & \end{array}$$

   Add the second column.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & 4 & \\ \hline & 1 & -1 & \end{array}$$

(*continued*)

Continue multiplying and adding for the remaining column.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & 4 & -4 \\ \hline & 1 & -1 & -24 \end{array}$$

Draw a box around the remainder.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & 4 & -4 \\ \hline & 1 & -1 & \boxed{-24} \end{array}$$

5. Write the quotient.

   Each sum represents the coefficient of the quotient.

   Recall that the order of the quotient is 1 less than the dividend, x^2.

   Write the remainder over the divisor.

   The result of $(x^2 - 5x - 20) \div (x - 4)$ using synthetic division is $x-1-\dfrac{24}{x-4}$.

Example 3

Find the quotient of $(3x^3 + 16x^2 + 18x + 8) \div (x + 4)$ using synthetic division.

1. Identify the coefficients of the dividend.

   The dividend is $3x^3 + 16x^2 + 18x + 8$. The coefficients of this expression are 3, 16, 18, and 8.

2. Identify the value of a.

   Recall that, in general, the divisor is written in the form $(x - a)$. Here, the divisor is written as $(x + 4)$ or $[x - (-4)]$; therefore, the value of a is –4.

3. Record the coefficients of the dividend and the value of a in the divisor in the synthetic division format.

$$\begin{array}{r|rrrr} -4 & 3 & 16 & 18 & 8 \\ & & & & \\ \hline & & & & \end{array}$$

4. Carry out the process of synthetic division.

Bring down the first coefficient of the dividend below the horizontal line.

$$\begin{array}{r|rrrr} -4 & 3 & 16 & 18 & 8 \\ & & & & \\ \hline & 3 & & & \end{array}$$

Multiply the first coefficient by the value of a and write the value under the second coefficient.

$$\begin{array}{r|rrrr} -4 & 3 & 16 & 18 & 8 \\ & & -12 & & \\ \hline & 3 & & & \end{array}$$

Add the second column.

$$\begin{array}{r|rrrr} -4 & 3 & 16 & 18 & 8 \\ & & -12 & & \\ \hline & 3 & 4 & & \end{array}$$

Continue multiplying and adding for the remaining columns.

$$\begin{array}{r|rrrr} -4 & 3 & 16 & 18 & 8 \\ & & -12 & -16 & -8 \\ \hline & 3 & 4 & 2 & 0 \end{array}$$

Draw a box around the remainder.

$$\begin{array}{r|rrrr} -4 & 3 & 16 & 18 & 8 \\ & & -12 & -16 & -8 \\ \hline & 3 & 4 & 2 & \boxed{0} \end{array}$$

Notice that the remainder is 0. Therefore, according to the Remainder Theorem, $x + 4$ is a factor of the expression $3x^3 + 16x^2 + 18x + 8$.

5. Write the quotient.

Each sum represents a coefficient of the quotient.

$(3x^3 + 16x^2 + 18x + 8) \div (x + 4) = 3x^2 + 4x + 2$

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Example 4

Use synthetic substitution to evaluate $p(x) = x^2 - 32$ for $x = -7$.

1. Identify the coefficients of the polynomial function.

   The polynomial function is $p(x) = x^2 - 32$. The coefficients of this function are 1 and –32.

2. Use synthetic division to evaluate the function for $x = -7$.

   Write the divisor in the form $x - a$.

   The value of x is –7; therefore, it is written as $(x - a) = [x - (-7)]$ or $x + 7$.

   Determine the coefficients of the dividend.

   Compare $x^2 - 32$ to the standard form of a polynomial, $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$. Notice that $x^2 - 32$ does not have a value for $a_{n-1}x^{n-1}$, or in this case, ax. Therefore, be sure to include 0 in the setup of the synthetic division to represent the missing coefficient.

   The other coefficients are 1 and –32.

   Record the coefficients of the dividend and the value of a in the divisor in the synthetic division format.

   $$\begin{array}{r|rrr} -7 & 1 & 0 & -32 \\ & & & \\ \hline & & & \end{array}$$

   Work through the process of synthetic division.

   $$\begin{array}{r|rrr} -7 & 1 & 0 & -32 \\ & & -7 & 49 \\ \hline & 1 & -7 & 17 \end{array}$$

   Draw a box around the remainder.

   $$\begin{array}{r|rrr} -7 & 1 & 0 & -32 \\ & & -7 & 49 \\ \hline & 1 & -7 & \boxed{17} \end{array}$$

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3. Identify the value of $p(x)$ when $x = -7$.

   The remainder of the synthetic division is the value of $p(x)$ for the given value of x.

   The value of $p(-7)$ is equal to 17, the remainder found using synthetic division.

4. Verify your answer by substituting –7 for x in the polynomial function $p(x) = x^2 - 32$.

$p(x) = x^2 - 32$	Original function
$p(-7) = (-7)^2 - 32$	Substitute –7 for x.
$p(-7) = 49 - 32$	Simplify.
$p(-7) = 17$	

   The process of synthetic substitution reveals the same answer as direct substitution.

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Example 5

If the remainder of $(x^2 + kx + 34) \div (x - 5)$ is –21, what is the value of k?

1. Create a polynomial function given the dividend, $x^2 + kx + 34$.

   $p(x) = x^2 + kx + 34$

2. Determine the value of k.

   The divisor, $(x - 5)$, is of the form $(x - a)$.

   According to the Remainder Theorem, when the value of a, 5, is substituted into the polynomial function, the result is the remainder, –21. Write the dividend as a function, $p(x) = x^2 + kx + 34$.

$p(5) = -21$	Remainder Theorem
$p(x) = x^2 + kx + 34$	Original function
$p(5) = (5)^2 + k(5) + 34$	Substitute the value of a, 5, for x.
$-21 = (5)^2 + k(5) + 34$	Substitute the remainder, –21, for $p(5)$.
$-21 = 25 + 5k + 34$	Simplify.
$-21 = 59 + 5k$	Solve for k.
$-80 = 5k$	
$k = -16$	

   The value of k is –16 because $(x^2 - 16x + 34) \div (x - 5)$ gives a remainder of –21.

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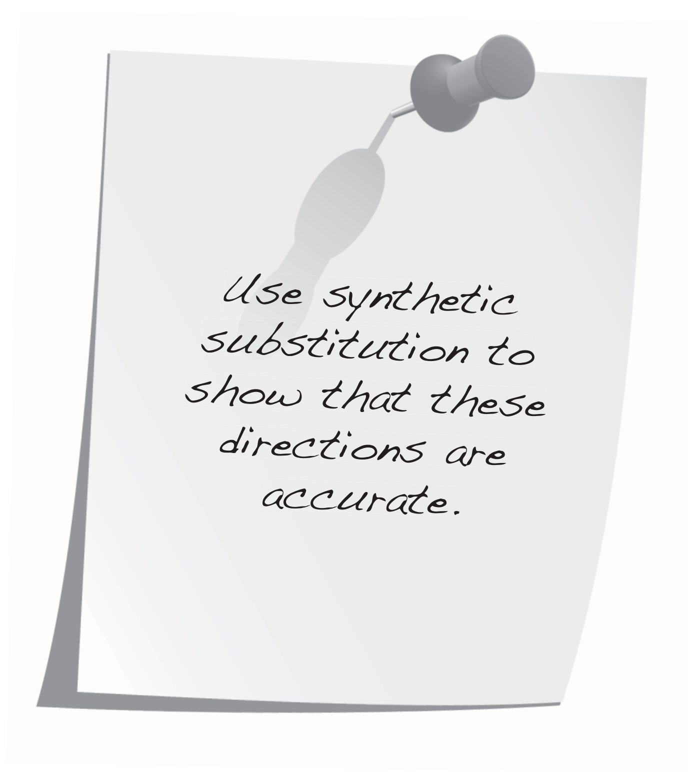
Problem-Based Task 3.5: When Is the Next Dose Due?

SMP	
1	2 ✓
3	4 ✓
5	6
7	8 ✓

The amount of a certain medication remaining in the bloodstream t hours after taking the medicine is modeled by the equation $M(t) = -x^3 + 5x^2 + 3x + 18$. Package directions recommend taking a second dose 4–6 hours after the initial dose. Use synthetic substitution to show that these directions are accurate.



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Problem-Based Task 3.5: When Is the Next Dose Due?

Coaching

a. Using synthetic substitution, what is $M(3)$?

b. Using synthetic substitution, what is $M(4)$?

c. Using synthetic substitution, what is $M(5)$?

d. Using synthetic substitution, what is $M(6)$?

e. For each value of t given in parts a–d, what does a remainder mean in terms of the context of the problem?

f. What does a remainder of 0 mean in the context of the problem?

g. After how many hours is the medication completely eliminated from the bloodstream?

h. Why would the package directions suggest taking a second dose after 4–6 hours?

i. Are these directions accurate?

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Problem-Based Task 3.5: When Is the Next Dose Due?

Coaching Sample Responses

a. Using synthetic substitution, what is $M(3)$?

By the Remainder Theorem, $M(3)$ should be the remainder when you divide the polynomial, $M(t) = -x^3 + 5x^2 + 3x + 18$, by $x - 3$.

$$\begin{array}{r|rrrr} 3 & -1 & 5 & 3 & 18 \\ & & -3 & 6 & 27 \\ \hline & -1 & 2 & 9 & \boxed{45} \end{array}$$

The remainder is 45; therefore, using synthetic substitution, $M(3) = 45$.

b. Using synthetic substitution, what is $M(4)$?

By the Remainder Theorem, $M(4)$ should be the remainder when you divide the polynomial, $M(t) = -x^3 + 5x^2 + 3x + 18$, by $x - 4$.

$$\begin{array}{r|rrrr} 4 & -1 & 5 & 3 & 18 \\ & & -4 & 4 & 28 \\ \hline & -1 & 1 & 7 & \boxed{46} \end{array}$$

The remainder is 46; therefore, using synthetic substitution, $M(4) = 46$.

c. Using synthetic substitution, what is $M(5)$?

By the Remainder Theorem, $M(5)$ should be the remainder when you divide the polynomial, $M(t) = -x^3 + 5x^2 + 3x + 18$, by $x - 5$.

$$\begin{array}{r|rrrr} 5 & -1 & 5 & 3 & 18 \\ & & -5 & 0 & 15 \\ \hline & -1 & 0 & 3 & \boxed{33} \end{array}$$

The remainder is 33; therefore, using synthetic substitution, $M(5) = 33$.

d. Using synthetic substitution, what is $M(6)$?

By the Remainder Theorem, $M(6)$ should be the remainder when you divide the polynomial, $M(t) = -x^3 + 5x^2 + 3x + 18$, by $x - 6$.

$$\begin{array}{r|rrrr} 6 & -1 & 5 & 3 & 18 \\ & & -6 & -6 & -18 \\ \hline & -1 & -1 & -3 & \boxed{0} \end{array}$$

The remainder is 0; therefore, using synthetic substitution, $M(6) = 0$.

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e. For each value of t given in parts a–d, what does a remainder mean in terms of the context of the problem?

A remainder of a value other than 0 indicates that the medication is still in the bloodstream.

f. What does a remainder of 0 mean in the context of the problem?

A remainder of 0 means that the medication is no longer in the bloodstream.

g. After how many hours is the medication completely eliminated from the bloodstream?

The medication is completely eliminated from the bloodstream after 6 hours.

h. Why would the package directions suggest taking a second dose after 4–6 hours?

Comparing the remainders for $M(3)$, $M(4)$, and $M(5)$ shows that the amount of medication in the bloodstream peaks at the 4-hour mark, then starts to decrease, meaning the medication is wearing off. Taking medication before 4 hours have passed could be harmful, since the first dose is still building up in the bloodstream. If the second dose is taken after the first dose starts wearing off but before it is completely eliminated from the bloodstream, then the patient is less likely to feel the effects of the ailment as the first dose wears off, since some medication will still be in the bloodstream.

i. Are these directions accurate?

Yes, the directions are accurate. Taking a second dose after 4 hours have passed but before 6 hours have passed will lessen the effects of the ailment, without resulting in an overdose.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.5: The Remainder Theorem A

For problems 1 and 2, use synthetic division to find each quotient.

1. $(x^3 - 7x^2 + 36) \div (x - 3)$

2. $(2x^4 - 6x^2 + 8x + 2) \div (x + 2)$

For problems 3 and 4, use synthetic substitution to evaluate each function.

3. $p(x) = 3x^2 + 6x + 10$ for $x = -4$

4. $p(x) = x^5 + 4x^3 + 2x - 16$ for $x = 2$

For problems 5 and 6, use synthetic division to find each quotient.

5. $(x^2 - 7x + 10) \div (x - 1)$

6. $(x^2 + 5x + 3) \div (x + 6)$

For problems 7–10, use the Remainder Theorem to solve each problem.

7. The area in square feet of a rectangular garden can be expressed as the product of the garden's length and width, or $A(x) = 3x^2 + 13x + 14$. If the width of the garden is $(x + 2)$ feet, what is the length of the garden?

8. The area in square meters of a rectangular patio can be expressed as the product of the patio's length and width, or $A(x) = 7x^2 - 34x + 24$. If the length of the patio is $(x - 4)$ meters, what is the width of the patio?

9. A generator produces voltage using levels of current modeled by $I(t) = t + 4$, where $t > 0$ represents the time in seconds. The power of the generator can be modeled by $P(t) = 0.5t^3 + 8t^2 + 24t$. If voltage is calculated by dividing $P(t)$ by $I(t)$, what expression represents the voltage of the generator?

10. A second generator produces voltage using levels of current modeled by $I(t) = t + 4$, where $t > 0$ represents the time in seconds. The power of the generator can be modeled by $P(t) = 0.2t^3 + 8.8t^2 + 32t$. What expression represents the voltage of this generator?

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Practice 3.5: The Remainder Theorem B

For problems 1 and 2, use synthetic division to find each quotient.

1. $(x^2 + 8x + 10) \div (x - 2)$

2. $(3x^5 - 4x^3 + 8x + 7) \div (x + 1)$

For problems 3 and 4, use synthetic substitution to evaluate each function.

3. $p(x) = x^2 + 5x + 10$ for $x = -2$

4. $p(x) = 6x^4 + 8x + 4x + 2$ for $x = 1$

For problems 5 and 6, use synthetic division to find each quotient.

5. $(x^2 - 6x + 4) \div (x + 1)$

6. $(2x^2 + 7x + 8) \div (x - 2)$

For problems 7–10, use the Remainder Theorem to solve each problem.

7. The area in square inches of a tabletop can be expressed as the product of the tabletop's length and width, or $A(x) = 4x^2 - 18x + 18$. If the length of the tabletop is $(x - 3)$ inches, what is the width of the tabletop?

8. The area in square centimeters of the front of an envelope can be expressed as the product of the envelope's length and width, or $A(x) = 5x^2 + 44x + 63$. If the width of the envelope is $(x + 7)$ centimeters, what is the length of the envelope?

9. A generator produces voltage using levels of current modeled by $I(t) = t + 2$, where $t > 0$ represents the time in seconds. The power of the generator can be modeled by $P(t) = 0.5t^3 + 9t^2 + 16t$. If voltage is calculated by dividing $P(t)$ by $I(t)$, what expression represents the voltage of the generator?

10. A second generator produces voltage using levels of current modeled by $I(t) = t + 5$, where $t > 0$ represents the time in seconds. The power of the generator can be modeled by $P(t) = 0.2t^3 + 9t^2 + 40t$. What expression represents the voltage of this generator?

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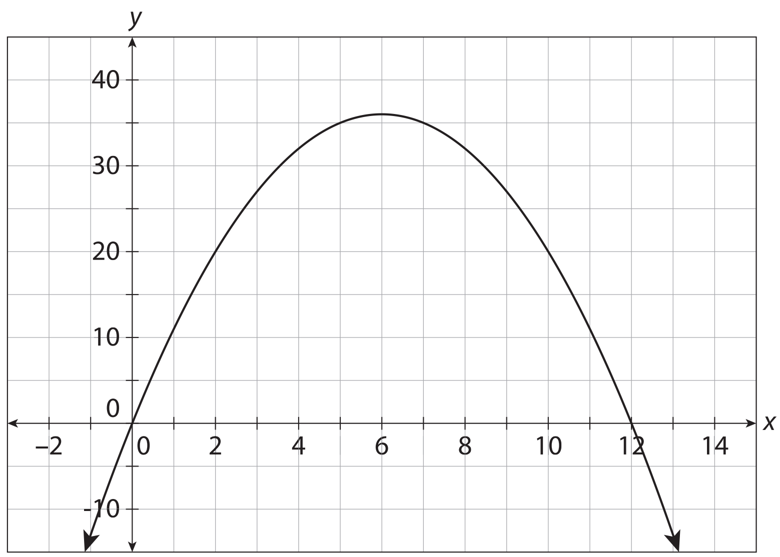
Lesson 3.6: Zeros of Polynomial Functions

Warm-Up 3.6

Rosa launched a model rocket at the park. The path of the rocket can be modeled by the function $f(x) = -x^2 + 12x$. The graph of this function is shown, and units are measured in feet.



1. Factor the quadratic expression in the function.

2. Set each factor equal to 0 and solve. What do these solutions tell you about the graph of the function?

3. What do these solutions tell you about the situation with the rocket?

4. What is a reasonable domain for this function in the context of the problem?

Lesson 3.6: Zeros of Polynomial Functions

North Carolina Math 3 Standard

A–APR.3 Understand the relationship among factors of a polynomial expression, the solutions of a polynomial equation, and the zeros of a polynomial function.

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8 ✓

Warm-Up 3.6 Debrief

1. Factor the quadratic expression in the function.

   The factored form of the function is $f(x) = -x(x - 12)$.

2. Set each factor equal to 0 and solve. What do these solutions tell you about the graph of the function?

   Set each factor equal to 0: $-x = 0$ and $x - 12 = 0$.

   The first equation gives a solution of $x = 0$, and the second equation gives a solution of $x = 12$. These are the two x-intercepts of the graph.

3. What do these solutions tell you about the situation with the rocket?

   The place where the rocket lands is 12 feet from where it was launched. If the rocket is launched at $x = 0$, it lands at $x = 12$.

4. What is a reasonable domain for this function in the context of the problem?

   Because the x-axis represents the ground, negative y-values do not make sense in the context of the problem. To avoid negative y values, x must be between 0 and 12. The domain is $0 \le x \le 12$.

Connection to the Lesson

- Students will factor polynomial expressions with degrees greater than 2.
- Students will find the zeros of a polynomial function and the solutions of a polynomial equation by factoring.
- Students will interpret the zeros of polynomial functions in the context of problem situations.

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Prerequisite Skills

This lesson requires the use of the following skills:

- understanding long division or synthetic division of polynomials (A–APR.3)
- understanding of the factor theorem (A–APR.3)
- ability to find the zeros of a quadratic function by factoring (F–IF.4)

Introduction

The factors of a polynomial expression and the solutions of a polynomial equation are concepts that are closely related to finding the zeros of a polynomial function. This lesson will explore how to find the zeros of a polynomial function, as well as the relationship among these closely related concepts.

Key Concepts

- Recall that a polynomial function is a function written in the following form: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a non-negative integer and the highest degree of the polynomial.
- As you have seen, a polynomial function can be graphed, and is defined for all real numbers. Any real number x can be substituted into the function to find the corresponding value of $f(x)$ or y, and this coordinate pair (x, y) is on the graph of the function.
- A polynomial function has an infinite number of solutions—all the points on the graph of the function.
- A polynomial equation is similar to a polynomial function, but instead of the polynomial being set equal to $f(x)$, the polynomial is equal to a number, generally 0. So a polynomial equation has only one variable, x, and a limited number of solutions.
- A polynomial equation in which the polynomial is set equal to 0 can be solved by factoring the polynomial and setting each factor equal to 0. Solve each of these small equations. These are the solutions to the polynomial equation, and are called the roots of the polynomial equation.
- If a factor is a quadratic expression that cannot be factored, it can be solved by setting it equal to 0 and using the quadratic formula.
- If you set a polynomial function equal to 0 and solve, the solutions tell you the zeros of the polynomial function. The zeros of the function are the same as the x-intercepts, the points at which the function intersects the x-axis.

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Instruction

- A polynomial equation of degree n (where n is the highest exponent on the variable) has, at most, n real solutions. Likewise, a polynomial function of degree n has, at most, n real zeros.
- These real zeros can be rational or irrational.
  - For example, consider the function $f(x) = x^2 + x - 6$. To find the zeros of this function, first factor $f(x) = (x - 2)(x + 3)$. Then set $f(x)$ equal to 0: $(x - 2)(x + 3) = 0$. Then set each factor equal to 0: $x - 2 = 0$ or $x + 3 = 0$. Solve these two equations: $x = 2$ or $x = 3$. This function has two rational solutions, 2 and 3.
  - Next, consider the function $f(x) = x^2 - 3$. To find the zeros, set the function equal to 0, $x^2 - 3 = 0$, and solve. $x^2 = 3$, so $x = \pm\sqrt{3}$. This function has two irrational solutions, $\sqrt{3}$ and $-\sqrt{3}$.
- The Rational Zero Theorem (also referred to as the Rational Root Theorem when referring to a polynomial equation) provides a way of determining a list of all the possible rational zeros of a polynomial function. The theorem states that for a polynomial in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a non-negative integer and the highest degree of the polynomial, every rational zero has the form $\dfrac{p}{q}$, where p is a factor of the constant, a_0, and q is a factor of the leading coefficient, a_n.
- Once you know a list of possible zeros of a polynomial, you can test each one using long division or synthetic division to see which are actual zeros of the function.
- If it is determined that m is a zero of the polynomial function, this means $(x - m)$ is a factor of the polynomial.
- To test whether $(x - m)$ is a factor of the polynomial, divide the polynomial by $x - m$ using long division or synthetic division. If the remainder is zero, then $x - m$ is a factor. This is known as the Factor Theorem.
- When trying to determine the zeros of a polynomial function, Descartes' Rule of Signs can be used to determine the number and sign of real zeros. (Note that polynomial functions may also have complex zeros, a topic that will be covered in a later lesson.)

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- Descartes' Rule of Signs tells us that:
  - the number of positive real zeros is the same as the number of sign changes of the terms in the polynomial expression $f(x)$, or less than this by an even number.
  - the number of negative real zeros is the same as the number of sign changes of the terms of $f(-x)$, or less than this by an even number.
- For example, consider the function $f(x) = 2x^5 + 3x^4 - x^2 + x - 1$. The signs of the terms change from positive to negative or negative to positive 3 times:

Positive to negative
Negative to positive
Positive to negative

$$f(x) = 2x^5 + 3x^4 - x^2 + x - 1$$

- So, there will be 3 real positive zeros or 1 real positive zero.
- Next, find $f(-x)$ and determine the number of sign changes. When you substitute $-x$ for an x that has an even exponent, the sign of the term doesn't change. When you substitute $-x$ for an x that has an odd exponent, the sign does change.

Negative to positive
Positive to negative

$$f(-x) = -2x^5 + 3x^4 + x^2 - x - 1$$

- So, there will either be 2 negative real zeros or no negative real zeros.

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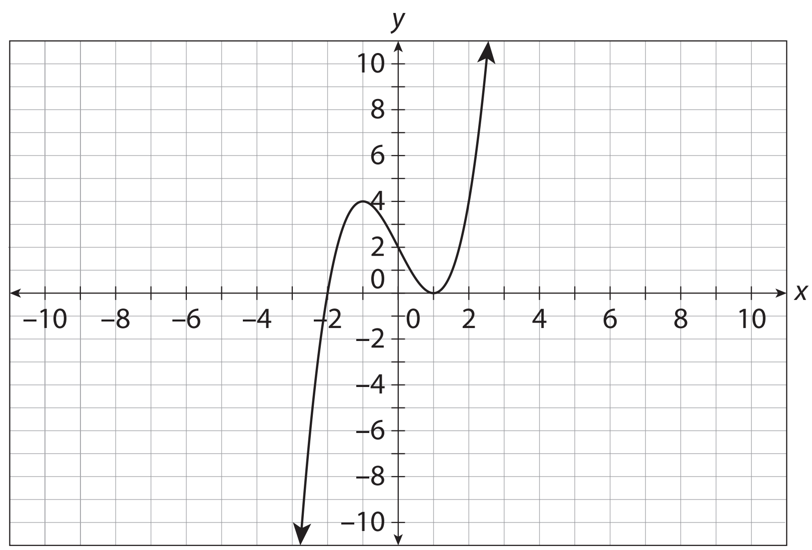
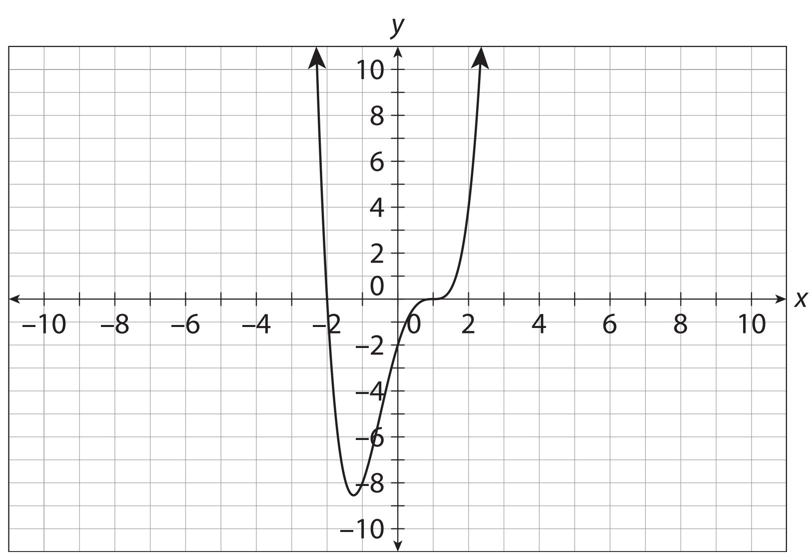
- If a function is completely factored and a factor appears more than once, this can give you information about the behavior of the graph. For example, if the factor $x - 1$ appears twice in the factored form of a polynomial function, the factor is said to have a multiplicity of 2. If the factor appears three times, the factor has a multiplicity of 3. If the multiplicity is an even number, rather than the graph crossing the x-axis at 1 (or whatever the zero happens to be), the graph will touch the x-axis at this point, but not cross it.
- Here is the graph of the function $f(x) = (x - 1)(x - 1)(x + 2)$. The factor $x - 1$ occurs twice, so the graph "bounces" on the x-axis where $x = 1$.



- If the multiplicity is an odd number, the graph will momentarily flatten out on the x-axis as it crosses it. Here is the graph of the function $f(x) = (x - 1)(x - 1)(x - 1)(x + 2)$:



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Instruction

- To find the real zeros of a polynomial function on a calculator, graph the function and determine the values of the x-intercepts.

On a TI-83/84:

Step 1: Press the Y= button.

Step 2: Enter the function.

Step 3: Set the window to a standard window, with both x and y going from –10 to 10, by pressing Zoom and choosing 6: ZStandard.

Step 4: If the x-intercepts are not visible, zoom out until you can see them by pressing Zoom and choosing 3: Zoom Out.

Step 5: Press GRAPH. Adjust the window again if necessary.

Step 6: Press 2ND–CALC and choose 2: Zero. Use the left or right arrow button to move the cursor to the left of the first x-intercept and press ENTER. Then use the arrow buttons to move the cursor to the right of the x-intercept and press ENTER again. Press ENTER a third time, and the x-intercept will be displayed at the bottom of the screen.

Step 7: Repeat this procedure for each x-intercept.

On a TI-Nspire:

Step 1: From HOME, choose Graphs and Geometry Application and enter the polynomial equation as $f1(x)$.

Step 2: Press MENU–#4 Window–#1 Window Setting to control the window to adjust the window range so the x-intercepts are visible.

Step 3: Use the NavPad to move the trace point to the x-axis. As you approach the x-axis, the spider will “jump” to the zero location. When the z appears, you have found a zero of the function.

Common Errors/Misconceptions

- confusing polynomial equations with polynomial functions
- confusing the meanings of zero, root, and solution
- not understanding how to use division to factor a polynomial expression
- when using the Rational Zero Theorem, giving up before finding a value that is a zero of the function
- not understanding how to use the factored form of a polynomial equation to find the solutions

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Scaffolded Practice 3.6: Zeros of Polynomial Functions

For problems 1–4, identify the zeros of each polynomial function.

1. $g(x) = (x - 3)(x + 2)(x - 4)$

2. $f(x) = (x + 3)(x - 1)(2x - 1)$

3. $h(x) = (x + 2)(x^2 - 5x - 6)$

4. $f(x) = (x + 1)(x - 1)(x^2 - 5)$

For problems 5–8, use the Rational Zero Theorem to determine the possible zeros of each polynomial function.

5. $f(x) = x^3 + 5x^2 + 2x - 8$

continued

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6. $h(x) = x^4 - 15x^2 + 38x - 60$

7. $f(x) = 4x^3 - 2x^2 - 2x - 3$

8. $g(x) = 3x^4 - 2x^2 + 6x - 10$

For problems 9 and 10, find two polynomial functions in factored form that have the zeros listed.

9. Zeros at 3, –2, and $\frac{3}{4}$

10. Zeros at $-\frac{1}{2}$, 1, and a double zero at 4

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Guided Practice 3.6

Example 1

What are the solutions to the polynomial equation $(x - 4)(2x + 6)(x^2 - 7x + 10) = 0$? What does this tell you about the graph of the function $f(x) = (x - 4)(2x + 6)(x^2 - 7x + 10)$?

1. Factor the polynomial expression completely.

   The polynomial is in factored form, but the quadratic factor, $x^2 - 7x + 10$, can be factored further.

   $x^2 - 7x + 10 = (x - 5)(x - 2)$

   The fully factored form of this polynomial is $(x - 4)(2x + 6)(x - 5)(x - 2) = 0$.

2. Set each factor equal to 0.

   If the product of factors is equal to 0, this means at least one of the factors is equal to 0. Set each factor equal to 0 and solve.

$x - 4 = 0$	$2x + 6 = 0$	$x - 5 = 0$	$x - 2 = 0$
$x = 4$	$2x = -6$	$x = 5$	$x = 2$
	$x = -3$		

   The solutions to this polynomial equation are $x = -3$, 2, 4, and 5. These are also called the roots of the equation.

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3. Determine how the solutions to the polynomial equation are related to the graph of the polynomial function $f(x) = (x - 4)(2x + 6)(x^2 - 7x + 10)$.

   If $f(x) = 0$, the result(s) tell the zeros of the function, or the x-intercepts. If we replace $f(x)$ with 0, the resulting equation is the same as the equation that was solved in Step 2. Therefore, the zeros of this function are –3, 2, 4, and 5. These are the x-intercepts of the graph of $f(x)$.

4. Verify the solution with a graphing calculator.

   On a TI-83/84:

   Step 1: Press Y= and enter the polynomial $(x - 4)(2x + 6)(x^2 - 7x + 10)$.

   Step 2: Set the window to a standard window, with both x and y going from –10 to 10, by pressing Zoom and choosing 6: ZStandard. The x-intercepts that we want to verify all fall between –10 and 10, so they will be visible in this view. To see the entire curve, you can set the Ymin to –375 and Ymax to 100.

   Step 3: Press GRAPH.

   Step 4: Visual inspection shows that the graph intersects the x-axis four times at roughly the values $x = -3$, 2, 4, and 5. To verify these values, Press 2ND–CALC and choose 2: zero. Use the left arrow button to move the cursor to the left of the first x-intercept and press ENTER. Then use the arrow buttons to move the cursor to the right of the first x-intercept and press ENTER again. Press ENTER a third time, and the x-intercept will be displayed at the bottom of the screen.

   Step 5: Repeat this procedure for each x-intercept. This will verify that the solution found algebraically is correct.

(*continued*)

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On a TI-Nspire:

Step 1: From HOME, choose Graphs and Geometry Application and enter the polynomial equation as $f1(x) = (x - 4)(2x + 6)(x^2 - 7x + 10)$.

Step 2: Press MENU–#4 Window–#1 Window Setting to control the window to adjust the window range so the x-intercepts are visible.

Step 3: Use the NavPad to move the trace point to the x-axis. As you approach the x-axis, the spider will "jump" to the zero location. When the z appears, you have found a zero of the function.

Example 2

What are the zeros of the function $f(x) = x^3 - 4x^2 - 7x + 10$?

1. Use the Rational Zero Theorem to determine the possible zeros of this function.

   The Rational Zero Theorem states that the possible zeros of a function are in the form $\frac{p}{q}$, where p is a factor of the constant and q is a factor of the lead coefficient. The constant is 10, which has for factors $\pm 1, \pm 2, \pm 5$, and ± 10. These are the possible values of p. The lead coefficient is 1, so the value of q is 1. Thus, the possible zeros of the function are $\pm 1, \pm 2, \pm 5$, and ± 10.

   Notice that the degree of the polynomial is 3, so this function has, at most, 3 zeros.

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2. Use synthetic division (or long division) to find out which of these possible zeros are actual zeros of the function.

   Try the first possibility in the list of possible zeros, 1. If 1 is a zero of the function, then $(x - 1)$ is a factor of the function. To determine if this is a factor of the function, divide the polynomial expression in the function by $(x - 1)$. If the remainder is 0, then $(x - 1)$ is a factor. If the remainder is not 0, then $(x - 1)$ is not a factor, and you can try the next possible rational zero, –1.

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -7 & 10 \\ & & 1 & -3 & -10 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

   The remainder of the division is 0, so $(x - 1)$ must be a factor of the polynomial, and 1 is a zero of the function.

   Note that instead of using synthetic or long division, you could simply substitute 1 into the original function to see if it gives a result of $f(x) = 0$. However, this is generally more work, and will not give you the depressed polynomial, which is the polynomial that is 1 degree less than the original polynomial and will help you further factor the function.

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3. Use the depressed polynomial to find the remaining zeros of the function.

   The three numbers before the 0 along the bottom of the synthetic division, 1, –3, and –10, are the coefficients of the depressed polynomial, $x^2 - 3x - 10$. This means the original function, $f(x) = x^3 - 4x^2 - 7x + 10$, can be factored as follows:

   $f(x) = (x - 1)(x^2 - 3x - 10)$

   Factor this further by factoring the quadratic, $x^2 - 3x - 10$:

   $f(x) = (x - 1)(x + 2)(x - 5)$

   Now that the function is completely factored, find the remaining zeros by setting each factor equal to 0 and solving.

$x - 1 = 0$	$x + 2 = 0$	$x - 5 = 0$
$x = 1$	$x = -2$	$x = 5$

   The zeros of this polynomial function are –2, 1, and 5. This can be verified by graphing the function on a graphing calculator and making sure the x-intercepts are –2, 1, and 5.

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Example 3

Find the zeros of the function $f(x) = 2x^4 - x^3 - 17x^2 + 16x + 12$.

1. Use the Rational Zero Theorem to determine the possible zeros of this function.

   Again, the Rational Zero Theorem states that the possible zeros of a function are in the form $\frac{p}{q}$, where p is a factor of the constant and q is a factor of the lead coefficient. In the given equation, the constant term is 12, and the lead coefficient is 2. The factors of 12 are ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12. These are the values of p. The factors of 2 are ± 1 and ± 2. These are the values of q. So, all possible zeros of the equation are all possible combinations of $\frac{p}{q}$: ± 1, ± 2, ± 3, ± 4, ± 6, ± 12, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

2. Use synthetic division or long division to find one of these possible zeros that is an actual zero of the function.

   There are quite a few possible zeros to try. If you use synthetic division to test whether or not 1 and –1 are zeros, you'll find that they are not. Next, try 2. If 2 is a zero of the function, then $(x - 2)$ is a factor of the original function.

$$\begin{array}{r|rrrrr} 2 & 2 & -1 & -17 & 16 & 12 \\ & & 4 & 6 & -22 & -12 \\ \hline & 2 & 3 & -11 & -6 & 0 \end{array}$$

(continued)

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The remainder is 0, so we know that 2 is a zero of the function, and that $(x - 2)$ is a factor. The numbers along the bottom of the synthetic division tell the coefficients of the depressed polynomial. The original polynomial is of degree 4, so the depressed polynomial is 1 less, or 3, with coefficients 2, 3, –11, and –6.

So far, we can factor the given function like this:
$f(x) = (x - 2)(2x^3 + 3x^2 + 11x - 6)$.

3. Continue factoring.

   Test more of the possible zeros from step 1. If you try –2 and 3, you'll find that they don't give a remainder of 0, so these are not zeros of the function. Try –3.

$$\begin{array}{r|rrrr} -3 & 2 & 3 & -11 & -6 \\ & & -6 & 9 & 6 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

   Because the remainder is 0, we know that –3 is a zero of the function, and that $(x + 3)$ is a factor. The depressed polynomial is a second-degree polynomial that can be determined by using the numbers along the bottom row of the synthetic division as the coefficients.

   At this point, we can factor the original function like this:

   $f(x) = (x - 2)(x + 3)(2x^2 - 3x - 2)$

   Now, the only factoring that remains is of the quadratic expression $2x^2 - 3x - 2$. This can be factored as $(2x + 1)(x - 2)$.

   The function in completely factored form is
   $f(x) = (x - 2)(x + 3)(2x + 1)(x - 2)$.

   Note that there are two identical factors, $(x - 2)$, so this factor has a multiplicity of 2. This is also called a "double zero." This tells us that the graph of the function will touch the x-axis at 2, but will not cross it.

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4. Set each factor equal to 0 and solve to find the zeros of the function.

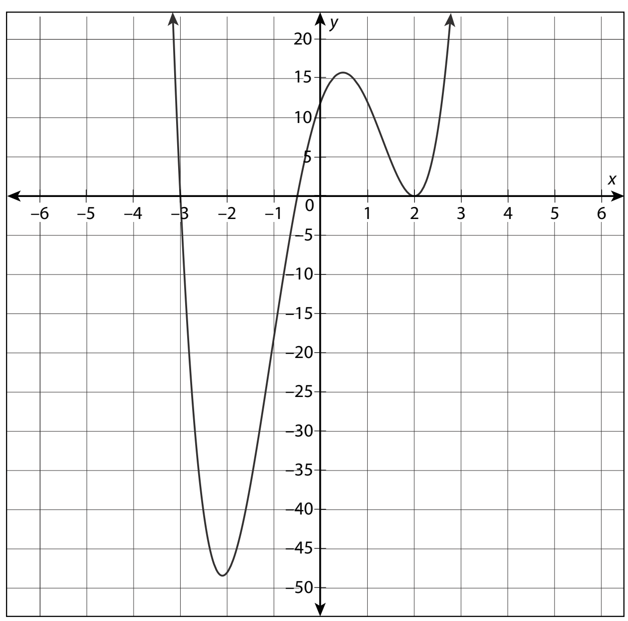
   The factored form of the function is $f(x) = (x - 2)(x + 3)(2x + 1)(x - 2)$. Set each of the factors equal to 0 and solve. The factor that appears twice will lead to the same zero, so it only has to be solved once.

$x - 2 = 0$	$x + 3 = 0$	$2x + 1 = 0$
$x = 2$	$x = -3$	$2x = -1$
		$x = -\dfrac{1}{2}$

   The zeros of the function are 2, –3, and $-\dfrac{1}{2}$.

5. Verify the solutions graphically.

   The graph of the original function should intersect the x-axis at –3 and $-\dfrac{1}{2}$, and it should touch but not cross the x-axis at 2. Graph the original function, $f(x) = 2x^4 - x^3 - 17x^2 + 16x + 12$, on a graphing calculator to visually verify that this is true. Here is the graph of this function:



   It appears that this curve intersects the x-axis as expected, at –3 and $-\dfrac{1}{2}$, and it touches, but does not cross, the x-axis at 2.

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Example 4

Find two polynomial functions in factored form that have zeros at 2 and $\frac{3}{4}$, and a double zero at –3.

1. Use the zeros to determine factors of the polynomial.

   If 2 is a zero of the function, we know that the function is equal to 0 when $x = 2$. Move all terms to the same side of the equation: $x - 2 = 0$. This means that $(x - 2)$ is a factor of the polynomial. This can be thought of as reversing the procedure used in earlier examples to find the zeros of a function.

   Repeat this process for $\frac{3}{4}$. If $\frac{3}{4}$ is a zero of the function, the function is equal to 0 when $x = \frac{3}{4}$. Move both terms to the same side of the equation: $x - \frac{3}{4} = 0$. You can eliminate the fraction by multiplying both sides of the equation by 4: $4x - 3 = 0$. Another factor of the polynomial is $(4x - 3)$. (This can be left as $x - \frac{3}{4}$, but it is generally easier to work with integers.)

   Finally, we have a double zero at –3. Repeat the process shown above: $x = -3$, so $x + 3 = 0$. There are two factors of $(x + 3)$.

   The known factors of this function are $(x - 2)(4x - 3)(x + 3)(x + 3)$.

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2. Determine two polynomial functions with the given zeros.

   One function is given by the factors found in the previous step:

   $f(x) = (x - 2)(4x - 3)(x + 3)(x + 3)$

   A second function can be found by adding a fifth factor. If the factor is a constant, the function will have zeros at the given values, 2 and $\frac{3}{4}$, and a double zero at –3. If the new factor contains the variable, there will be zeros at 2 and $\frac{3}{4}$, and double zeros at –3 and at the zero produced by this new factor. To avoid adding new zeros to the function, let's choose a new factor of 2:

   $f(x) = 2(x - 2)(4x - 3)(x + 3)(x + 3)$

3. Verify your solutions with a graphing calculator.

   Graph both of the functions found in step 2 on a graphing calculator to make sure the graph intersects the x-axis at 2 and $\frac{3}{4}$, and that it touches the x-axis but doesn't cross it at $x = -3$, since this zero has a multiplicity of 2.

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Example 5

A box manufacturer has been asked to make a rectangular box that has a volume of 297 in^3. In addition, the box must be 2 inches longer than it is wide, and the height must be one-third of the width. What will be the dimensions of the box?

1. Write a function for the volume of the box in terms of the width.

   The volume of a rectangular prism is V = (length)(width)(height), or $V = lwh$. In the problem, both the length and the height are expressed in terms of the width.

   $l = w + 2$

   $h = \frac{1}{3}w$

   Substitute these expressions into the formula for the volume:

   $V = (w + 2)(w)\left(\frac{1}{3}w\right)$

   Simplify:

   $V = (w^2 + 2w)\left(\frac{1}{3}w\right)$

   $V = \frac{1}{3}w^3 + \frac{2}{3}w^2$

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2. Substitute 297 for the volume, and put the equation in standard form.

   The problem states that the volume of the box must be 297 in^3. Substitute 297 for V and put the equation in standard form, with all the terms on one side and zero on the other.

$297=\frac{1}{3}w^3+\frac{2}{3}w^2$	Substitute 297 for V.
$3(297)=3\left(\frac{1}{3}w^3+\frac{2}{3}w^2\right)$	Multiply both sides of the equation by 3.
$891=w^3+2w^2$	Simplify.
$w^3+2w^2-891=0$	Subtract 891 from both sides.

3. Determine the nature of the solutions of the equation.

   The Rational Zero Theorem states that the possible roots of the equation (zeros of the function) are all possible combinations of $\frac{p}{q}$.

   The value of q is 1, so the possible rational roots are: ± 1, ± 3, ± 9, ± 11, ± 27, ± 33, ± 81, ± 99, ± 297, and ± 891.

   Furthermore, Descartes' Rule of Signs tells us there will be 1 real positive root and either 2 or 0 real negative roots, because the sign changes once in the expression w^3+2w^2-891, and the sign changes twice in $-w^3+2w^2-891$.

4. Determine a solution to the equation.

   Since we know there is one real positive root, try the positive values determined by the rational root theorem until we find one that is an actual root. Use synthetic or long division to see if 1 is a root. It isn't, since there is a remainder. Try 3. Again, this isn't a root, because there is a remainder. Try 9:

$$\begin{array}{c|cccc} 9 & 1 & 2 & 0 & -891 \\ & & 9 & 99 & 891 \\ \hline & 1 & 11 & 999 & 0 \end{array}$$

   This time, we get a remainder of 0, so we know that 9 is a root of the equation and therefore a solution of the equation. We know there is only 1 positive root to the equation due to Descartes' Rule of Signs, so this is the only positive solution. We do not need to be concerned with negative solutions, because the dimensions of a box must be positive.

5. Determine the dimensions of the box.

   Since the variable in the equation represents the width, the width of the box is 9 inches. The problem states that the length is 2 inches more than the width, so the length is 11 inches. The problem also states that the height is one-third the width, so the height is 3 inches. The dimensions of the box, in inches, are $11 \times 9 \times 3$.

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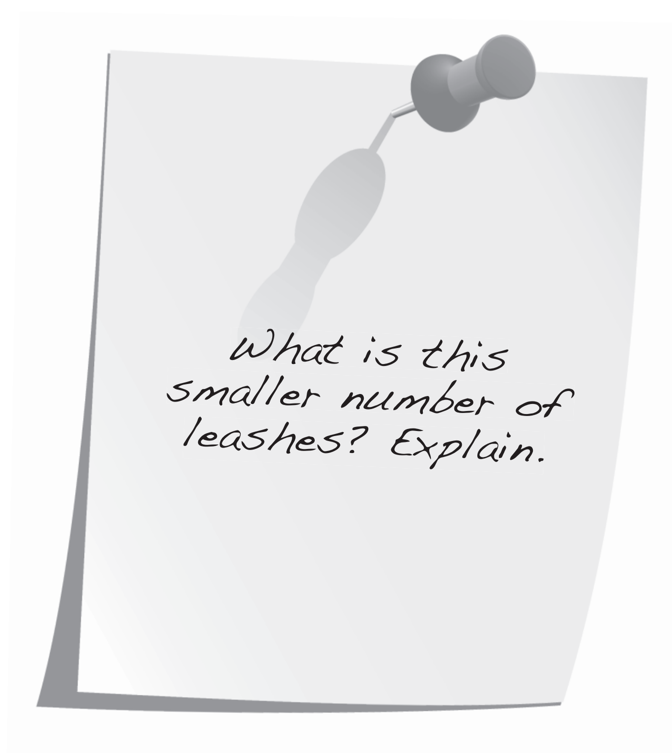
Problem-Based Task 3.6: Dog Leash Profit

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

A company makes dog leashes, and has determined that the amount it can charge for a particular style of leash is represented by the expression $32 - 2x^2$, where x is the number (in thousands) of leashes produced.

It costs the company $8 to make each leash. The profit that the company makes can be determined by subtracting the cost to make x thousand leashes from the cost to manufacture x thousand leashes.

Currently, the company makes 2.5 thousand leashes per year, and makes an annual profit of $28,750. A mathematician for the company determines there is a smaller number of leashes that the company could make per year while earning the same profit. What is this smaller number of leashes? Explain.



Problem-Based Task 3.6: Dog Leash Profit

Coaching

a. What is the function that represents the company's profit?

b. The profit of $28,750 is the profit for 1,000 leashes. What is the profit for one leash?

c. What equation results when you substitute the profit per leash into the profit function?

d. What are the solutions to the equation found in part c?

e. What is the other number of leashes the company could produce to receive a profit of $28,750?

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Problem-Based Task 3.6: Dog Leash Profit

Coaching Sample Responses

a. What is the function that represents the company's profit?

Profit is determined by subtracting the cost to make the leashes from the income earned by the sale of the leashes. The problem states that the company sells the leashes for $8 each. If the company makes x thousand leashes, the cost to make them is $8x$. The profit from selling the leashes is the amount charged per leash multiplied by the number of leashes: $x(32 - 2x^2)$. Therefore, the profit equation is:

$P(x) = x(32 - 2x^2) - 8x$

$P(x) = 32x - 2x^3 - 8x$

$P(x) = -2x^3 + 24x$

b. The profit of $28,750 is the profit for 1,000 leashes. What is the profit for one leash?

The profit for one leash is $28,750 ÷ 1000 = $28.75.

c. What equation results when you substitute the profit per leash into the profit function?

Substitute 28.75 for $f(x)$ in the profit function:

$28.75 = -2x^3 + 24x$

Move all terms to the same side of the equation so that it is equal to zero:

$-2x^3 + 24x - 28.75 = 0$

d. What are the solutions to the equation found in part c?

Find the solutions to the equation. We know that 2.5 is one solution, because $-2(2.5)^3 + 24(2.5) - 28.75 = 0$. Use synthetic division to find the depressed polynomial.

$$\begin{array}{r|rrrr} 2.5 & -2 & 0 & 24 & 28.75 \\ & & -5 & -12.5 & 28.75 \\ \hline & -2 & -5 & 11.5 & 0 \end{array}$$

The depressed polynomial is $-2x^2 - 5x + 11.5$, so we know that $(x - 2.5)(-2x^2 - 5x + 11.5) = 0$. The depressed polynomial cannot be factored, so use the quadratic formula to solve.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-2)(11.5)}}{2(-2)}$$

$x \approx -3.95$ or $x \approx 1.45$

The negative solution doesn't make sense in the context of the problem, since you can't make a negative number of leashes. The other solution is valid.

e. What is the other number of leashes the company could produce to receive a profit of $28,750?

If the company produces 1.45 thousand, or 1,450 leashes, the profit will be the same as if it produced 2.5 thousand, or 2,500 leashes. This would be a wise business move, as it can earn the same profit while making 1,050 fewer leashes.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.6: Zeros of Polynomial Functions A

Use what you have learned to complete problems 1 and 2.

1. What are the zeros of the function $f(x) = (x - 2)(2x + 1)(x + 3)$?

2. Find two polynomial functions, in factored form, that have zeros at $\frac{1}{2}$ and 3, and a double zero at –2.

Use the function $f(x) = x^4 + 5x^3 - 3x^2 - 13x + 10$ to complete problems 3–6.

3. Use the Rational Zero Theorem to find all possible zeros of this function.

4. Use Descartes' Rule of Signs to determine the number of real positive zeros and the number of real negative zeros of this function.

5. What is the factored form of $f(x)$?

6. What are the zeros of this function, and what do the zeros tell you about the graph of the function?

continued

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Use the following information to complete problems 7–10.

7. Solve the equation $2x^3 + 7x^2 + 2x - 3 = 0$.

8. Solve the equation $x^5 + 3x^4 - 11x^3 - 27x^2 + 10x + 24 = 0$.

9. Write the function $f(x) = x^3 + 6x^2 - 13x - 42$ as the product of linear factors. Then list all of its zeros.

10. The volume of a box is 340 in^3. The width of the box is five times the height. The length of the box is 15 inches more than the height. What are the dimensions of the box?

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Practice 3.6: Zeros of Polynomial Functions B

Use what you have learned to complete problems 1 and 2.

1. What are the zeros of the function $f(x) = (3x - 2)(x + 1)(x - 7)$?

2. Find two polynomial functions, in factored form, that have zeros at $\frac{3}{5}$ and 1, and a double zero at –4.

Use the function $f(x) = x^4 - 15x^3 + 79x^2 - 165x + 100$ to complete problems 3–6.

3. Use the Rational Zero Theorem to find all possible zeros of this function.

4. Use Descartes' Rule of Signs to determine the number of real positive zeros and the number of real negative zeros of this function.

5. What is the factored form of $f(x)$?

6. What are the zeros of this function, and what do the zeros tell you about the graph of the function?

continued

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Use the following information to complete problems 7–10.

7. Solve the equation $2x^3 - 5x^2 - 6x + 9 = 0$.

8. Solve the equation $x^5 + 7x^4 + 15x^3 + 5x^2 - 16x - 12 = 0$.

9. Write the function $f(x) = 3x^4 - 20x^3 + 38x^2 - 12x - 9$ as the product of linear factors. Then list all of its zeros.

10. The volume of a box is 96 in^3. The height of the box is half the width. The length of the box is 8 inches more than the width. What are the dimensions of the box?

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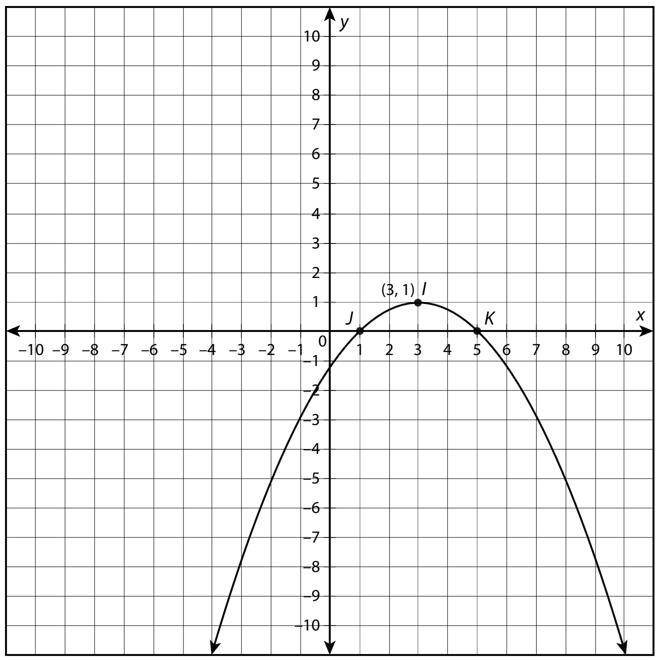
Lesson 3.7: Building Polynomial Functions

Warm-Up 3.7

A producer is using a computer program to design a realistic background for a movie set. The hill he designs has the graph shown:



1. What are the coordinates of the *x*-intercepts, or roots, of the graph?

2. What would be two factors of the equation of the graph?

3. What is the equation of the graph?

Instruction

Lesson 3.7: Building Polynomial Functions

North Carolina Math 3 Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

a. Build polynomial and exponential functions with real solution(s) given a graph, a description of a relationship, or ordered pairs (include reading these from a table)

Warm-Up 3.7 Debrief

1. What are the coordinates of the *x*-intercepts, or roots, of the graph?

   The roots and *x*-intercepts are located where the function crosses the *x*-axis. The function crosses the *x*-axis at both 1 and 5. The roots of the graph are (1, 0) and (5, 0).

2. What would be two factors of the equation of the graph?

   Two factors of the equation are $(x - 1)$ and $(x - 5)$. If $x = r$ is a root of a polynomial, then $(x - r)$ is a factor.

3. What is the equation of the graph?

   To write the equation of the graph with given factors, use the formula for the factored form of a quadratic equation, $y = a(x - p)(x - q)$, where p and q are the zeros of the function. Substitute another (x, y) point that you know on the graph to solve for a, the leading coefficient.

$y = a(x - p)(x - q)$	Factored form of a quadratic equation
$y = a(x - 1)(x - 5)$	Set up the equation.
$y = a(x^2 - 6x + 5)$	Distribute the factors.
$1 = a[(3)^2 - 6(3) + 5]$	Substitute the (x, y) values of another point for x and y in the equation. Another clear point on the graph is the vertex, (3, 1).
$1 = a(-4)$	Simplify.
$-\frac{1}{4} = a$	Divide both sides by –4 to find a.
$y = -\frac{1}{4}\left(x^2 - 6x + 5\right)$	Substitute $\frac{1}{4}$ for a back in the equation $a(x^2 - 6x + 5)$.
$y = -\frac{1}{4}x^2 + \frac{3}{2}x - \frac{5}{4}$	Distribute to write in standard form.

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The resulting equation, $y = -\frac{1}{4}x^2 + \frac{3}{2}x - \frac{5}{4}$, represents the graph of the hill on the movie set.

Connection to the Lesson

- The connection between roots of polynomials, factors, and their various representations will be essential to building polynomial functions.
- Students will also need to recognize that the coordinates of other points on a function represent the (x, y) variables in the function to solve for the a value.

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Prerequisite Skills

This lesson requires the use of the following skills:

- recognizing the relationship between factors and roots of polynomial functions
- multiplying binomials
- determining roots from several representations, such as algebraic factors, graphs, and tables
- understanding and applying the Fundamental Theorem of Algebra
- solving equations involving several steps

Introduction

When discussing the solutions to general polynomial equations, the solutions are assumed to be equal to 0. This could represent when a ball hits the ground, so the height equals 0; when a business breaks even, so profit equals 0; or algebraically, the x-intercepts on a graph. From these solutions and any other point that satisfies the equation, the degree and equation can be determined to predict other values.

Key Concepts

- The solutions to **polynomial functions** are represented by the independent variable, when the dependent variable is equal to 0.
- Solutions can be represented as the x-intercepts of a graph, independent values in a table where the dependent value is equal to 0, in word problems, or as the algebraic solutions where factors are equal to 0.
- Polynomial functions can be built by determining the roots, building factors from the roots, multiplying the factors, and substituting a point to solve for the leading coefficient.
- To determine real solutions from a graph, locate the x-intercepts.

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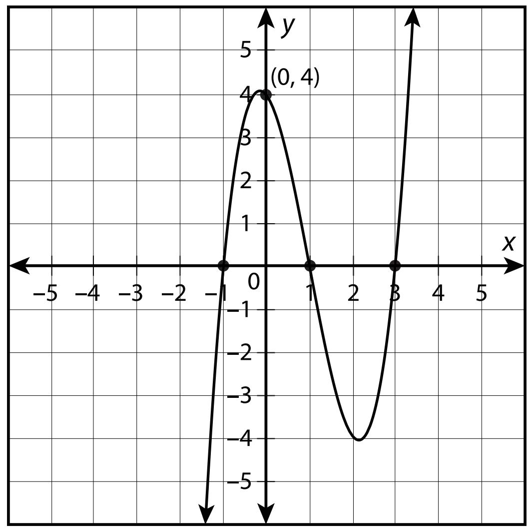
- If any roots are double roots, they are represented by an x-intercept that is a relative minimum or relative maximum on the graph. Those factors must be written and multiplied twice when distributing the factors.



- The solutions are –1, 1, and 3, and another point on the graph is (0, 4).
- To determine solutions from a table, find the independent values where the dependent value is equal to 0.

x	–3	–2	–1	0	1	2	3
y	–64	–20	0	4	0	–4	0

- The solutions are –1, 1, and 3, and another point that satisfies the equation is (0, 4). Note that it is not clear whether any of these are double solutions; in these types of situations, you'll need more information to determine what degree the polynomial has. Generally, you should build the polynomial with minimal degree.
- To determine the solutions from a word problem, determine the independent values when the dependent variable equals 0.
- To determine the equation from the roots and another point algebraically:

Step 1: Write the equation in terms of the dependent variable, usually y, using the factors of the equation built from the roots. Leave a variable a in front of the first factor.

Step 2: Substitute another point on the function for the x and y variables to solve for a.

Step 3: Substitute a into the original equation and distribute the binomial factors to write the equation in standard form

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Instruction

- To build a polynomial function using the regression feature of a graphing calculator, first determine if the equation is a quadratic, cubic, or quartic equation.
- Using the Fundamental Theorem of Algebra, an equation with 2 solutions is quadratic, 3 solutions is cubic, and 4 solutions is quartic. The solution can be found where the polynomial is equal to 0.
- If at least one point other than the solutions is given, the calculator can perform an accurate regression.
- To find the real zeros of a polynomial function on a calculator, graph the function and determine the values of the *x*-intercepts.

On a TI-83/84:

Step 1: Determine the real roots of the function and the other point you can determine.

Step 2: Press [STAT][EDIT] and select 1: Edit. Enter *x*-values in L_1 of the table and *y*-values in L_2.

Step 3: Press [STAT][CALC]. If the equation is quadratic, select 5: QuadReg. If the equation is cubic, select 6: CubicReg. If the equation is quartic, select 7: QuartReg. Press "Calculate", and the coefficients of the terms in the polynomial will be calculated.

Step 4: Substitute the coefficients before the variables in the standard form of the polynomial.

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On a TI-Nspire:

Step 1: Determine the real roots of the function and the other point you can determine. Remember, the roots have coordinates (x, 0).

Step 2: Enter data on the List and Spreadsheet App, with the x-coordinates in List A and the y-coordinates in List B. Name your lists as x for List A and y for List B.

Step 3: From the home menu, choose 5: Data and Statistics. Press [enter].

Step 4: Using the NavPad, move to the bottom and choose the x-variable list name. Then, move to the left and choose the y-variable list name to create the scatter plot.

Step 5: To determine the best-fit regression equation, press [menu], select 3: Actions, and select 5: Regression. If the equation is quadratic, select 4: Show Quadratic. If the equation is cubic, select 5: Show Cubic. If the equation is quartic, select 6: Show Quartic. The curve and equation will appear on the screen.

Common Errors/Misconceptions

- confusing x-intercepts and y-intercepts by not realizing that $x = 0$ in y-intercepts and $y = 0$ for x-intercepts
- failing to determine a point other than the x-intercepts to solve for the leading coefficient
- not including double roots twice in setting up factors to write a function

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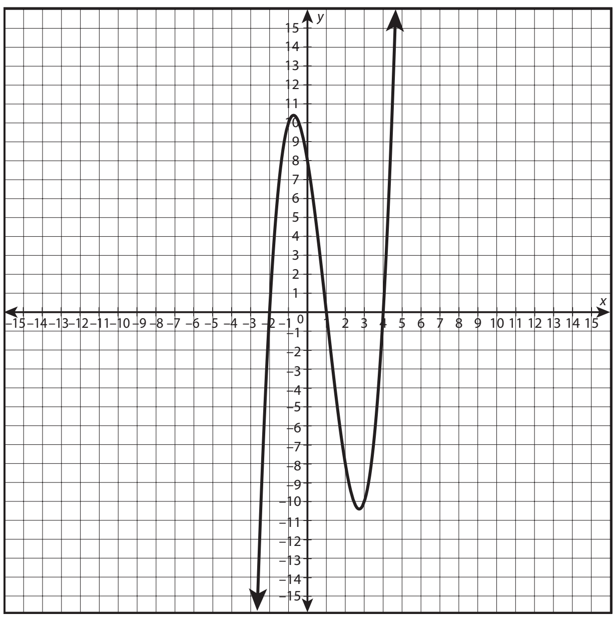
Scaffolded Practice 3.7: Building Polynomial Functions

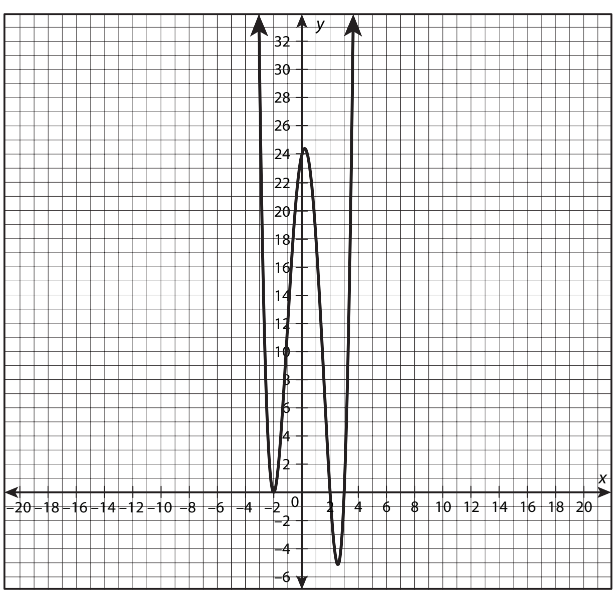
For problems 1–5, determine the degree of the polynomial equation based on the given information and the end behavior as *x* approaches infinity.

1. Equation through points (–4, –36), (–3, 0), (–1, 12), (0, 0), (3, –36), and (5, 0)

2.



3.



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4.

x	–2	–1	0	2	3
y	0	12	12	0	0

5. Equation through points (–2, 0), (0, 0) multiplicity of 2, (1, –9), and (4, 0)

For problems 6–10, write the polynomial equation for the given points or table of values.

6. Polynomial with x-intercepts at –2, 1, and 5, and a y-intercept at 10

7.

x	–3	–2	–1	0	1
y	–6	0	0	0	6

8. Polynomial with roots at –4, –1, and 1, and that passes through the point (–2, 18)

9. Polynomial with points (–3, –24), (–2, 0), (–1, 4), (0, 0), and (1, 0)

10.

x	–1	0	1	2	3
y	0	0	–12	–18	0

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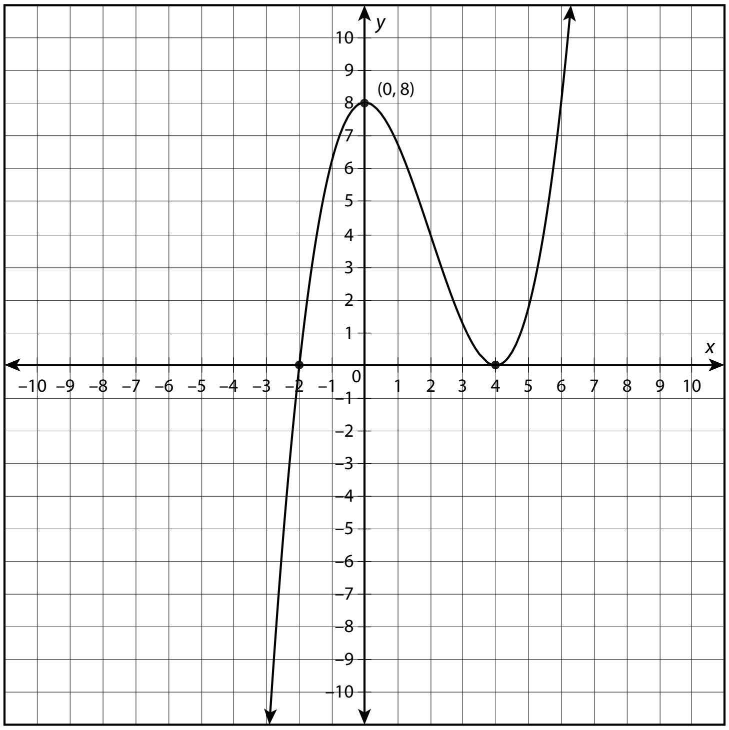
Guided Practice 3.7

Example 1

Determine the equation that represents the following graph using algebraic steps.



1. Determine the roots of the equation from the graph.

   The roots on the graph are represented by the x-intercepts. There is a root at –2 and a double root at 4.

2. Name another point on the graph.

   Often, either the y-intercept or a relative minimum or maximum are the clearest points to see on the graph. The y-intercept is at (0, 8).

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3. Set up and solve the equation using the factors determined from the roots, the variable a, and the (x, y) coordinates from the other point to find a.

The factor built from the root at –2 is $(x + 2)$, because $(-2 + 2) = 0$. The factor for the double root at 4 is $(x - 4)$, because $(4 - 4) = 0$. $(x - 4)$ is listed twice, because it is a double root. Then, (0, 8) is substituted for x and y to solve for a, the leading coefficient.

$y = a(x + 2)(x - 4)(x - 4)$	Set up the equation of the graph using the factors and variable a.
$8 = a(0 + 2)(0 - 4)(0 - 4)$	Substitute (0, 8) for (x, y).
$8 = a(32)$	Multiply the numerical factors on the right side of the equal sign.
$\frac{1}{4} = a$	Divide to isolate the variable.

4. Substitute the value of a back into the equation for the graph, and distribute to write the polynomial in standard form.

In the equation $y = a(x + 2)(x - 4)(x - 4)$, $a = \frac{1}{4}$, and x and y are variables, because they represent every point that satisfies the polynomial equation.

$y = a(x + 2)(x - 4)(x - 4)$	Equation
$y = \left(\frac{1}{4}\right)(x + 2)(x - 4)(x - 4)$	Substitute $\frac{1}{4}$ for a.
$y = \frac{1}{4}(x^2 - 2x - 8)(x - 4)$	Distribute the first two factors.
$y = \frac{1}{4}\left(x^3 - 6x^2 + 32\right)$	Distribute the remaining factor.
$y = \frac{1}{4}x^3 - \frac{3}{2}x^2 + 8$	Distribute the leading coefficient.

The equation $y = \frac{1}{4}x^3 - \frac{3}{2}x^2 + 8$ represents the graph.

Example 2

Determine the equation with roots at 0, 4, –5, and $-\frac{5}{2}$ that passes through point (–2, –54). Why does the equation have a constant of 0?

1. Determine the points determined by the roots, and determine the other point needed to write the equation.

   Remember, all roots are *x*-intercepts on the graph and have a *y*-coordinate of 0.

   The given roots are 0, 4, –5, and $-\frac{5}{2}$. Therefore, the *x*-intercept points determined by these roots are (0, 0), (4, 0), (–5, 0), and $\left(-\frac{5}{2}, 0\right)$.

   We are given that the graph also passes through the point (–2, –54).

2. Determine the degree of the polynomial represented by the equation through these points.

   The polynomial will be a quartic polynomial with degree 4, because there are 4 roots.

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3. Use your calculator's regression feature to write the quartic polynomial.

On a TI-83/84:

Step 1: Press [STAT][EDIT] and select 1: Edit. Enter *x*-values in L_1 of the table and *y*-values in L_2.

Step 2: Press [STAT][CALC] and select 7: QuartReg because the equation is quartic. Press "Calculate," and the coefficients of the terms in the polynomial will be calculated.

Step 3: Substitute the coefficients before the variables in the standard form of the polynomial. Press [ENTER].

On a TI-Nspire:

Step 1: Enter the coordinates on the List and Spreadsheet App, with the *x*-coordinates in List A and the *y*-coordinates in List B. Name your lists as *x* for List A and *y* for List B.

Step 2: From the home menu, choose 5: Data and Statistics. Press [enter].

Step 3: Using the NavPad, move to the bottom and choose the *x*-variable list name. Then, move to the left and choose the *y*-variable list name to create the scatter plot.

Step 4: To determine the best-fit regression equation, press [menu], select 3: Actions, and select 5: Regression. Then select 6: Show Quartic because the equation is quartic.

Either calculator will produce the curve and equation for the graph:

$$y = -3x^4 - \frac{21}{2}x^3 + \frac{105}{2}x^2 + 150x$$

4. Why does the equation have a constant of 0?

The constant is 0 because the root 0 would have a factor of *x*, meaning that *x* is multiplied times each other factor. When they are distributed, every term has a variable of *x*.

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Example 3

A 35-second roller coaster ride is tracked for its height (in cm) based on how long the ride is traveling (in seconds). The ride has a couple of inclines and declines, and some of the heights tracked are represented in the following table, where *t* represents time and *h* represents height. Use the table to answer each question that follows.

t	0	5	10	15	20	25	30	35
h	0	1875	0	0	3750	9375	11250	0

- What minimal degree polynomial equation represents the height (*h*) based on a given time (*t*)?
- What is the highest height the roller coaster reaches?
- What does it mean if the height is negative?
- If the function continued, would you expect the future values to be positive or negative?

1. Determine the degree of the equation representing the height.

   The table shows 4 solutions, as *x*-intercepts where the output, *h*, equals 0, so the minimal equation is quartic with a degree of 4.

2. Use the quartic regression feature on your calculator to determine the equation of the quartic polynomial.

   Enter the values from the table into your graphing calculator. The equation given by the quartic regression is

   $$y=-\frac{1}{4}x^4+15x^3-\frac{1025}{4}x^2+\frac{2625}{2}x.$$

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3. To find the highest height the roller coaster reaches, graph the equation in the calculator and use the maximum feature.

   Follow the steps appropriate to your calculator model to graph the equation.

   On a TI-83/84:

   Step 1: Press [Y=] and type the quartic equation $y=-\frac{1}{4}x^4+15x^3-\frac{1025}{4}x^2+\frac{2625}{2}x$ into Y_1.

   Step 2: Press [WINDOW]. Using the values in the table as a guide, set the minimum and maximum values as Xmin = 0, Xmax = 35, Ymin = 0, and Ymax = 15,000.

   Step 3: Press [GRAPH] to see the graph, then press [2ND][TRACE] to call up the CALC screen. Scroll to 4: maximum. Set the left bound anywhere between the two peaks in the graph, and set the right bound anywhere to the right of the second peak in the graph to calculate the highest point of the second peak. Press [ENTER].

   On a TI-Nspire:

   Step 1: From the home menu, choose the Graphs and Geometry Application. Enter the quartic equation $y=-\frac{1}{4}x^4+15x^3-\frac{1025}{4}x^2+\frac{2625}{2}x$ in $f1(x)$.

   Step 2: Press [menu], then select 4: Window and 1: Window Setting to control the window to any dimensions you enter. Set the minimum and maximum values as Xmin = 0, Xmax = 35, Ymin = 0, and Ymax = 15,000.

   Step 3: To find the maximum point, press [menu], select 5: Trace, and select 1: Graph Trace. Then trace the graph using the NavPad arrows until an M appears. The point when you see the M is the maximum.

(*continued*)

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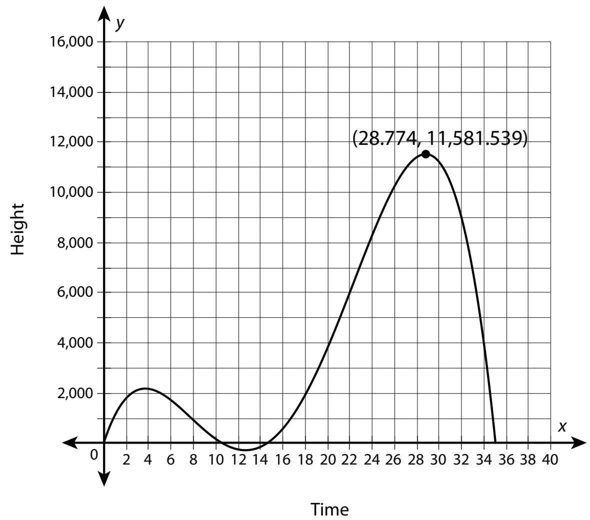
Either calculator should yield the point (28.774, 11,581.539) as the maximum. The maximum height of the roller coaster is the y-value, or about 11,582 cm.

The resulting graph should resemble the following:



4. What does it mean if the height is negative?

   Where the height is negative, the roller coaster dips lower than its starting point.

5. If the function continued, would you expect the future values to be positive or negative?

   Future values would be negative, because both end behaviors are decreasing as x approaches infinity for a quartic (even degree) equation with a negative leading coefficient.

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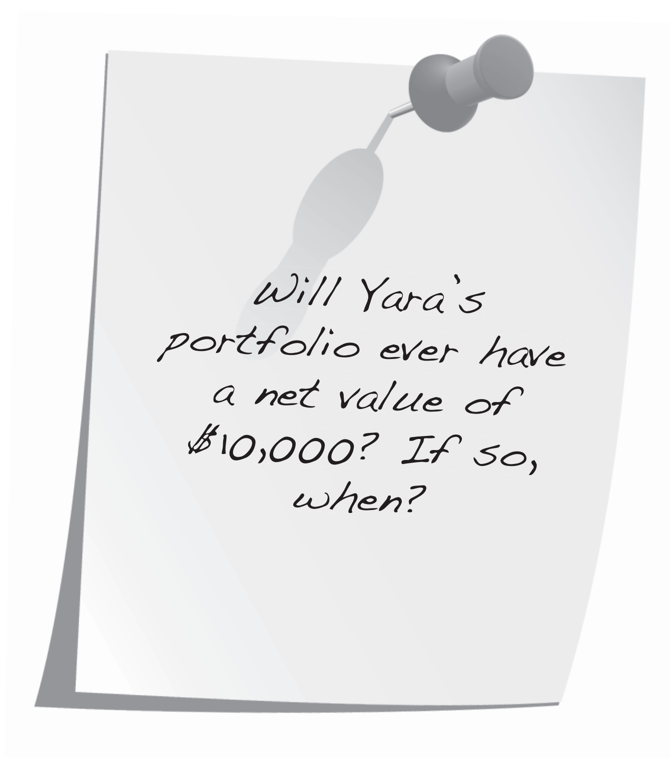
Problem-Based Task 3.7: Playing the Stock Market

SMP
1 ✓ 2 ✓
3 4
5 ✓ 6 ✓
7 ✓ 8

Yara is a recent college graduate. After reading about the success of investors such as Warren Buffett, Yara decides to invest money in the stock market. She saves $4,000 from her first year of work, and invests it on April 30. On this date, her portfolio's net value is –$4,000, because until her investment makes money, the $4,000 Yara "spent" is considered a loss. By May 15, she has made her money back, and her portfolio has a net value of $0. By May 31, her portfolio again has a net value of $0. On July 10, she once again has a net value of $0. Describe the performance of Yara's portfolio (when it was gaining money and when it was losing money) by building a minimal-degree polynomial for the situation. Assuming she leaves her money in the stock market, will Yara's portfolio ever have a net value of $10,000? If so, when?



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Problem-Based Task 3.7: Playing the Stock Market

Coaching

a. What points do you know will satisfy this equation based on the word problem?

b. What is the easiest way to write this equation?

c. Will the equation be quadratic, cubic, or quartic? How do you know?

d. What equation relates the number of days and the value of Yara's portfolio?

e. Graph the equation. Based on the graph, when is the portfolio's value rising, and when is it falling?

f. Based on the graph, when will the portfolio's value reach $10,000?

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Problem-Based Task 3.7: Playing the Stock Market

Coaching Sample Responses

a. What points do you know will satisfy this equation based on the word problem?

The points would be (0, –4,000), (15, 0), (31, 0) and (71, 0). The initial value, –$4,000, represents the *y*-intercept, and the days when the value equals $0 represent the *x*-intercepts.

As a table, this would be:

Day	0	15	31	71
Value	–4,000	0	0	0

b. What is the easiest way to write this equation?

The easiest way to write this is by using the regression feature in the calculator, because the numbers are large enough that the calculations become difficult by hand.

c. Will the equation be quadratic, cubic, or quartic? How do you know?

The minimum degree required for a polynomial with 3 *x*-intercepts is 3. The equation is cubic because it has 3 *x*-intercepts or solutions.

d. What equation relates the number of days and the value of Yara's portfolio?

Using the calculator cubic regression feature:

On a TI-83/84:

Step 1: Press [STAT][EDIT] and select 1: Edit. Enter *x*-values in L_1 of the table and *y*-values in L_2.

Step 2: Press [STAT][CALC] and select 6: CubicReg because the equation is cubic. Press "Calculate", and the coefficients of the terms in the polynomial will be calculated.

Step 3: Substitute the coefficients before the variables in the standard form of the polynomial.

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On a TI-Nspire:

Step 1: Enter the coordinates on the List and Spreadsheet App, with the x-coordinates in List A and the y-coordinates in List B. Name your lists as "Day" for List A and "Money" for List B.

Step 2: From the home menu, choose 5: Data and Statistics. Press [enter].

Step 3: Using the NavPad, move to the bottom and choose the x-variable list name. Then, move to the left and choose the y-variable list name to create the scatter plot.

Step 4: To determine the best-fit regression equation, press [menu], select 3: Actions, and select 5: Regression. Then select 5: Show Cubic because the equation is cubic.

Either calculator will produce the curve and equation for the graph: $y = 0.121x^3 - 14x^2 + 452.037x - 4000$.

e. Graph the equation. Based on the graph, when is the portfolio's value rising, and when is it falling?

By finding the relative maximum and minimum points on the graph, we can see on what days the values rise and fall.

On a TI-83/84:

Step 1: Press [Y=] and type the cubic equation $y = 0.121x^3 - 14x^2 + 452.037x - 4000$ into Y_1.

Step 2: Press [WINDOW]. Using the values in the table as a guide, set the range as Xmin = 0, Xmax = 100, Ymin = –15,000, and Ymax = 15,000.

Step 3: Press [GRAPH] to see the graph, then press [2ND][TRACE] to call up the CALC screen. Scroll to 4: maximum. Set the left bound anywhere before the first peak in the graph, and set the right bound just to the right of the first peak in the graph to calculate the highest point of the first peak. Press [ENTER] again to find the maximum.

Step 4: Press [2ND][TRACE] to call up the CALC screen. Scroll to 3: minimum. Set the left bound anywhere before the valley in the graph (about $x = 40$), and set the right bound anywhere to the right of the valley in the graph to calculate the lowest point of the valley. Press [ENTER] again to find the minimum.

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On a TI-Nspire:

Step 1: From the home menu, choose the Graphs and Geometry Application. Enter the equation $f1(x) = 0.121x^3 - 14x^2 + 452.037x - 4000$.

Step 2: Press [menu], select 4: Window and 1: Window Setting to control the window to any dimensions you enter. Set the minimum and maximum values as Xmin = 0, Xmax = 100, Ymin = –15,000, and Ymax = 15,000.

Step 3: To find the maximum point, press [menu], select 5: Trace, and select 1: Graph Trace. Then trace the graph using the NavPad arrows until an M appears. The point when you see the M is the maximum.

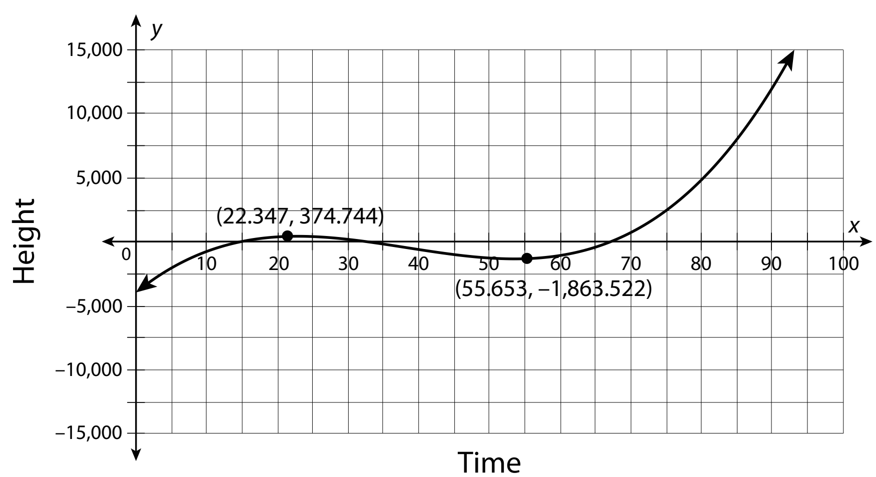
Step 4: To find the minimum point, press [menu], select 5: Trace, and select 1: Graph Trace. Then trace the graph using the NavPad arrows until a lowercase “m” appears.

Either calculator should give the maximum point as (22.347, 374.744). The day is the x-value, or about the 22nd day, May 22.

The minimum value is (55.653, –1863.522). The day is the x-value, or about the 56th day, June 25.

Therefore, the stock portfolio’s value increases from April 30 to May 22 (up to about \$375), decreases until June 25 (down to about –\$1,864), then increases for the duration of the investment.

The resulting graph should resemble the following:



f. Based on the graph, when will the portfolio’s value reach \$10,000?

To find the day when the value will be \$10,000, the equation must be set equal to 10,000. The calculator’s intersection feature can find this value.

On a TI-83/84:

Step 1: Press [Y=]. Leave the cubic equation $y = 0.121x^3 - 14x^2 + 452.037x - 4000$ in Y_1, and type $y = 10{,}000$ into Y_2.

Step 2: Do not change the window. Press [GRAPH] to observe a horizontal line that intersects the existing curve.

Step 3: Press [2ND][TRACE] to call up the CALC screen. Scroll to 5: intersect. Press [ENTER] to select the cubic equation, then press [ENTER] to select the horizontal line. Press [ENTER].

On a TI-Nspire:

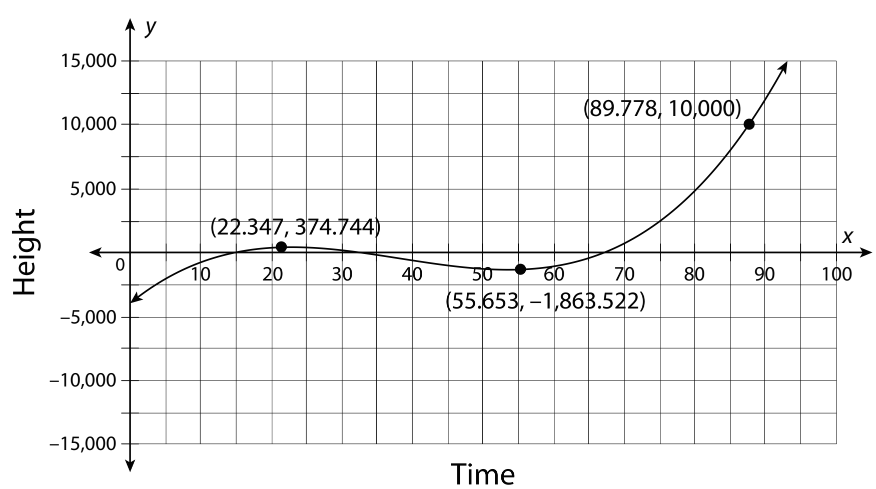
Step 1: From the Graphs and Geometry Application, type the cubic equation $f1(x) = 0.121x^3 - 14x^2 + 452.037x - 4000$ and type $f2(x) = 10{,}000$.

Step 2: Leave the graph window the same. Press [menu], select 6: Points & Lines, and select 3: Intersection Points.

Step 3: When the pointing hand appears, click on each graph and all intersection points will be displayed in the viewing window.

The point of intersection is at (89.778, 10,000). The *x*-value represents the day when the value is $10,000, or about the 90th day.

The graph will resemble the following:



The portfolio's value will reach $10,000 on about the 90th day after April 30, or about July 29.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

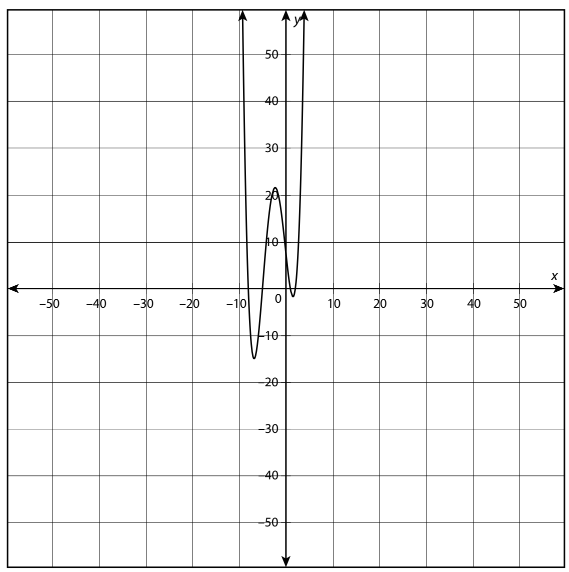
Practice 3.7: Building Polynomial Functions A

For problems 1 and 2, determine the degree of the polynomial equation based on the given information and the end behavior as *x* approaches infinity.

1. equation through the points:

x	0	6	20	34
y	19	0	0	0

2.



For problems 3 and 4, write the polynomial equation for the given points or table.

3. polynomial with *x*-intercepts at –3, 1, and 3, and a *y*-intercept at 10

4.

x	–5	–2.5	0	2.5	5	7.5	10
y	0	$-\frac{100}{3}$	–40	–30	$-\frac{40}{3}$	0	0

continued

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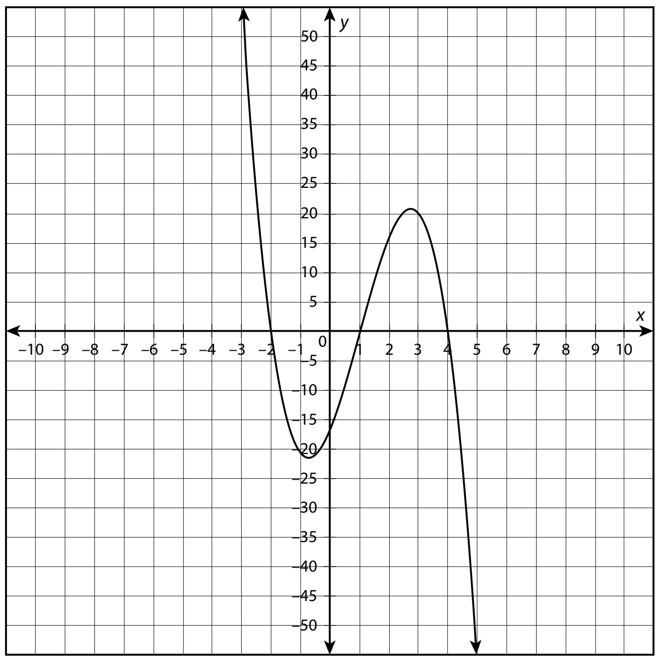
Use the following graph to complete problems 5–7.



5. What points can you determine from the intercepts of the graph?

6. What factors can you determine from the roots?

7. What is the equation of the polynomial represented by this graph?

continued

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Use the given information to complete problems 8–10.

In a very cold area near the North Pole, the average temperature barely reaches 0 degrees Fahrenheit (0°F). The following table represents the average temperatures for the given months in one year. (Assume January is at $x = 0$.)

Month	Jan.	March	May	July	Sept.	Nov.
(°F)	0	0	–6	–6	0	0

8. Numerically, what are the x-i ntercepts of the graph?

9. What minimal polynomial equation represents the average temperature in terms of the month?

10. Based on the equation, what will be the average temperature in December?

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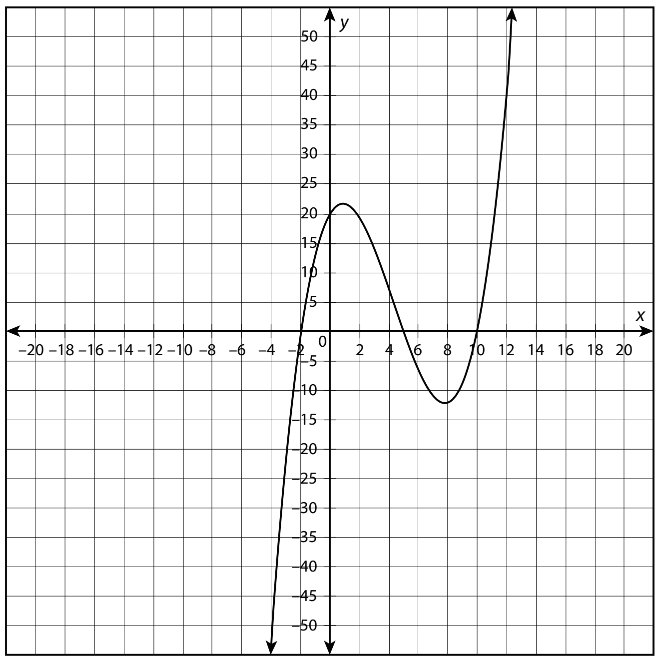
Practice 3.7: Building Polynomial Functions B

For problems 1 and 2, determine the degree of the minimal polynomial equation based on the given information and the end behavior as *x* approaches infinity.

1.



2. equation through points (–4, 0), (–1.25, 0), (5, 0), (10, 0), and (3, 9)

For problems 3 and 4, write the polynomial equation for the given points or table.

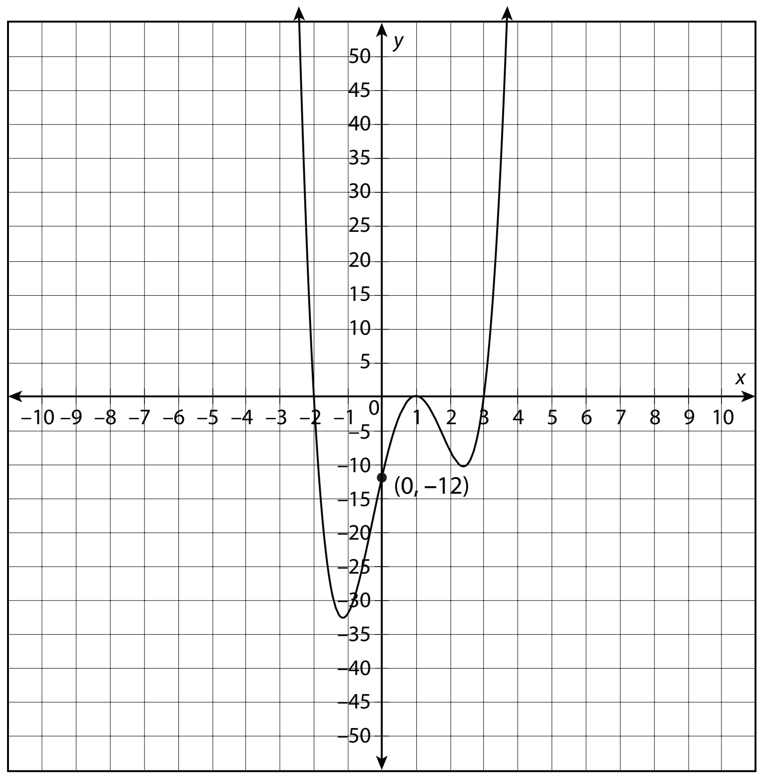
3. polynomial with *x*-intercepts at –5, –2, 2.5, and 4, and a *y*-intercept at –62

4.

x	–8	–4	0	4	8
y	–64	0	0	–16	0

continued

Use the following graph to complete problems 5–7.



5. What points can you determine from the intercepts of the graph?

6. What factors can you determine from the roots?

7. What is the equation of the polynomial represented by the graph?

continued

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Lesson 3.7: Building Polynomial Functions

Use the given information to complete problems 8–10.

In a shuttle run, an athlete runs a 40-yard sprint 4 times, sprinting from the starting line to a cone 40 yards away, back to the start, back to the cone, and back to the start. Generally, the final two sprints take longer than the initial sprint. The following table represents a sprinter's time and distance from the starting line.

x	0	5	10.4	16.7	23.1
y	0	40	0	40	0

8. What is a reasonable domain and range for this situation? What does this tell you about the degree of the minimal polynomial equation? *Hint:* (10.4, 0) is a double root).

9. What polynomial equation represents the sprinter's distance from the starting line depending on the time she's been running?

10. Graph the equation. Does the polynomial accurately model this situation?

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Lesson 3.8: End Behaviors of Functions

Warm-Up 3.8

Hari has two bank accounts, one at Uptown Bank and another at Downtown Bank. He deposits \$3,000 into each account. The amount in the Uptown Bank account grows by 3% every year according to the function $f(x) = 3000 \bullet 1.03^x$. The Downtown Bank account grows by \$100 every year according to the function $g(x) = 3000 + 100x$. In both equations, x represents how many years have passed since the initial deposit.

1. How much will be in each account after 1 year?

2. How much will be in each account after 3 years?

3. How much will be in each account after 5 years?

4. Which account will eventually have more money in it?

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Lesson 3.8: End Behaviors of Functions

North Carolina Math 3 Standards

F–LE.3 Compare the end behavior of functions using their rates of change over intervals of the same length to show that a quantity increasing exponentially eventually exceeds a quantity increasing as a polynomial function.★

F–IF.9 Compare key features of two functions using different representations by comparing properties of two different functions, each with a different representation (symbolically, graphically, numerically in tables, or by verbal descriptions).

Warm-Up 3.8 Debrief

1. How much will be in each bank account after 1 year?

   To find the amount in each bank account after 1 year, evaluate the functions at $x = 1$.

$f(x) = 3000 \bullet 1.03^x$	$g(x) = 3000 + 100x$	Original functions
$f(1) = 3000 \bullet 1.03^{(1)}$	$g(1) = 3000 + 100(1)$	Substitute 1 for x.
$f(1) = 3000 \bullet 1.03$	$g(1) = 3000 + 100$	Simplify.
$f(1) = 3090$	$g(1) = 3100$	Simplify.

   After 1 year, there will be \$3,090 in the Uptown Bank account and \$3,100 in the Downtown Bank account.

2. How much will be in each bank account after 3 years?

   To find the amount in each bank account after 3 years, evaluate the functions at $x = 3$.

$f(x) = 3000 \bullet 1.03^x$	$g(x) = 3000 + 100x$	Original functions
$f(3) = 3000 \bullet 1.03^{(3)}$	$g(3) = 3000 + 100(3)$	Substitute 3 for x.
$f(3) = 3000 \bullet 1.092727$	$g(3) = 3000 + 300$	Simplify.
$f(3) = 3278.181$	$g(1) = 3300$	Simplify.

   After 3 years, there will be \$3,278.18 in the Uptown Bank account and \$3,300 in the Downtown Bank account.

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3. How much will be in each bank account after 5 years?

   To find the amount in each bank account after 5 years, evaluate the functions at $x = 5$.

$f(x) = 3000 \bullet 1.03^x$	$g(x) = 3000 + 100x$	Original functions
$f(5) = 3000 \bullet 1.03^{(5)}$	$g(5) = 3000 + 100(5)$	Substitute 5 for x.
$f(5) = 3000 \bullet 1.159274$	$g(5) = 3000 + 500$	Simplify.
$f(5) = 3477.82$	$g(5) = 3500$	Simplify.

   After 5 years, there will be $3,477.82 in the Uptown Bank account and $3,500 in the Downtown Bank account.

4. Which bank account will eventually have more money in it?

   While the amount in the Downtown Bank account will be greater than the amount in the Uptown Bank account after 5 years, the amount in the Uptown Bank account will continue to increase more rapidly. Eventually, the function $f(x)$ will increase so rapidly that it will surpass $g(x)$; at this point, the Uptown Bank account will have more money in it.

Connection to the Lesson

- Students will compare the end behavior of two functions.
- Students will make an informal argument to show that exponential functions grow faster than linear function.

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Prerequisite Skills

This lesson requires the use of the following skills:

- evaluating functions (F–IF.2)
- calculating average rates of change (8.EE.5)
- graphing functions (A–CED.2★)
- understanding end behavior of polynomial and exponential functions (F–IF.2)

Introduction

Sometimes, in a graph comparing different functions, it isn't obvious that one function will eventually outgrow the other. You can analyze the functions to determine which will eventually be greater by comparing their rates of change over multiple intervals of the same length. For example, if two different production methods have different operating cost functions, you will be able to tell which production method is a better option in the long run by comparing the rates of change over time. Or, if two competing animal populations have different population functions, you can tell which population will eventually outcompete the other by finding which one eventually grows faster. In other words, you can use your understanding of the end behaviors of functions as a tool to make predictions about future behavior of models.

Key Concepts

End Behavior of Polynomial Functions

- Recall that to determine the **end behavior** of a polynomial function, or the behavior of the graph as x approaches positive or negative infinity, it is necessary to consider the highest degree of the polynomial and its coefficient, ax^n.
- If n is even, the polynomial function is considered an even-degree polynomial function.
- When n is even and a is positive, then both ends of the function will extend upward. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written as follows:

  $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$

- When n is even and a is negative, then both ends of the function will extend downward. That is, the value of $f(x)$ approaches negative infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written as follows:

  $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$

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- If n is odd, the polynomial function is considered an odd-degree polynomial function.
- When n is odd and a is positive, then the function will extend down to the left and up to the right. That is, the value of $f(x)$ approaches positive infinity as x approaches positive infinity, and the value of $f(x)$ approaches negative infinity as x approaches negative infinity. Written symbolically:

  $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$

- When n is odd and a is negative, then the function will extend up to the left and down to the right. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and the value of $f(x)$ approaches negative infinity as x approaches positive infinity. Written symbolically:

  $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$

End Behavior of Exponential Functions

- Recall that to determine the end behavior of an exponential function, it is necessary to consider the initial value, a, and the growth factor, b, of the exponential term of the function, $f(x) = a \bullet b^x + c$.
- If a is positive and b is greater than 1, the left end of the function will eventually become parallel to the line $y = c$, while the right end will extend upward. That is, the value of $f(x)$ approaches c as x approaches negative infinity, and the value of $f(x)$ approaches positive infinity as x approaches positive infinity. Written symbolically:

  $f(x) \rightarrow c$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$

- If a is positive and $0 < b < 1$, the left end of the function will extend upward, while the right end will eventually become parallel to the line $y = c$. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and the value of $f(x)$ approaches c as x approaches positive infinity:

  $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow c$ as $x \rightarrow +\infty$

- If a is negative and b is greater than 1, the left end of the function will eventually become parallel to the line $y = c$, while the right end will extend downward. That is, $f(x)$ approaches c as x approaches negative infinity, and $f(x)$ approaches negative infinity as x approaches positive infinity:

  $f(x) \rightarrow c$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$

- If a is negative and $0 < b < 1$, the left end of the function will extend downward, while the right end will eventually become parallel to the line $y = c$. That is, $f(x)$ approaches negative infinity as x approaches negative infinity, and $f(x)$ approaches c as x approaches positive infinity:

  $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to c$ as $x \to +\infty$

Comparing End Behavior Using Rates of Change

- The **rate of change** is a ratio that describes how much one quantity changes with respect to the change in another quantity.
- The uppercase Greek letter ***delta*** **(Δ)** is commonly used to represent the "change" in a value; for example, $\Delta f(x)$ can be read as "change in f of x." Hence, $\Delta f(x)$ and Δx are more concise ways of representing the numerator and denominator, respectively, in the rate of change formula.
- The **average rate of change** is the ratio of the difference of output values to the difference of the corresponding input values; this ratio is usually defined over an interval $[a, b]$. Mathematically, this can be represented by the formula $\dfrac{f(b)-f(a)}{b-a}=\dfrac{\Delta f(x)}{\Delta x}$.
- Note that the order in which the function values are compared must be the same as the order in which the domain values are compared. Switching the order of the values in the numerator with the order of the values in the denominator will result in an incorrect rate calculation. It will be off by a negative sign.
- You can compare the end behavior of different functions by comparing their average rates of change over the same interval, for multiple intervals.
- The intervals chosen need to be long enough that the end behaviors of the functions are obvious. For polynomials, choose intervals past the last turn of the end being examined. For exponentials, choose intervals well past any visible intersections. In general, longer intervals give a better picture of eventual end behavior, as do intervals farther from the origin. If the end behavior is not clear from the intervals chosen, you can redo the calculations with longer intervals.
- When comparing the right-end behavior of two functions approaching positive infinity, the function with a greater eventual rate of change will eventually approach positive infinity faster than the function with the lesser eventual rate of change. This means that the function with the greater eventual rate of change will eventually have greater values than the function with the lesser eventual rate of change as x approaches positive infinity.

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- For example, compare $f(x) = 5x^2$ and $g(x) = x^3$. Both functions will approach positive infinity as x approaches positive infinity. Compare the average rates of change over intervals of equal length to see that $g(x)$ will eventually be greater than $f(x)$. Looking at the coefficients of x for the two functions, 5 for $f(x)$ and 1 for $g(x)$, an interval of 5 units seems to be a reasonable choice for comparison.

Interval	[0, 5]	[5, 10]	[10, 15]	[15, 20]
$\frac{\Delta f(x)}{\Delta x}$	25	75	125	175
$\frac{\Delta g(x)}{\Delta x}$	25	175	475	925

- From the table, we can see that $g(x)$ has a greater rate of change than $f(x)$ as x increases. This means that $g(x)$ increases faster than $f(x)$ as x increases. In other words, even though the values of $f(x)$ and $g(x)$ are initially the same, $g(x)$ eventually has greater values than $f(x)$ as x approaches positive infinity.
- When comparing the right-end behavior of two functions approaching negative infinity, the function with the more negative eventual rate of change will eventually approach negative infinity faster than the function with the less negative eventual rate of change. This means that the function with the more negative eventual rate of change will eventually have more negative values than the function with the less negative eventual rate of change as x approaches positive infinity.
- For example, compare $f(x) = -1 \bullet 2^x$ and $g(x) = -x^5$. Both functions will approach negative infinity as x approaches positive infinity. We can compare the average rates of change over intervals of the same length to see that $g(x)$ will eventually be greater than $f(x)$. We will use lengths of 10 units to account for the relatively slow initial growth of an exponential function.

Interval	[0, 10]	[10, 20]	[20, 30]	[30, 40]
$\frac{\Delta f(x)}{\Delta x}$	–102.4	$-1.04755 \bullet 10^5$	$-1.07269 \bullet 10^8$	$-1.09844 \bullet 10^{12}$
$\frac{\Delta g(x)}{\Delta x}$	-10^4	$-3.1 \bullet 10^5$	$-2.11 \bullet 10^6$	$-7.81 \bullet 10^6$

- From the table, we can see that $g(x)$ has a less negative rate of change than $f(x)$ as x increases. This means that $f(x)$ decreases more quickly than $g(x)$ as x increases. In other words, $f(x)$ will eventually have more negative values than $g(x)$ as x approaches positive infinity.

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- When comparing the left-end behavior of two functions approaching positive infinity, the function with the more negative eventual rate of change will approach positive infinity faster than the function with the less negative eventual rate of change. This means that the function with the more negative rate of change will eventually have greater values than the function with the less negative rate of change as x approaches negative infinity.
- For example, compare $f(x) = -10x$ and $g(x) = x^2$. Both functions will approach positive infinity as x approaches negative infinity. Compare the average rates of change over intervals of 5 units to see that $g(x)$ will eventually be greater than $f(x)$. Although the parent function of the linear equation has been stretched by a factor of 10, use lengths of 5 units to give a more detailed picture of the function behavior.

Interval	[–20, –15]	[–15, –10]	[–10, –5]	[–5, 0]
$\frac{\Delta f(x)}{\Delta x}$	–10	–10	–10	–10
$\frac{\Delta g(x)}{\Delta x}$	–35	–25	–15	–5

- From the table, we can see that $f(x)$ has a less negative rate of change than $g(x)$ as x decreases. This means that $g(x)$ increases faster than $f(x)$ as x decreases. In other words, $g(x)$ will eventually have greater values than $f(x)$ as x approaches negative infinity.
- When comparing the left-end behavior of two functions approaching negative infinity, the function with the greater eventual rate of change will approach negative infinity faster than the function with the lesser eventual rate of change. This means that the function with the greater rate of change will eventually have lesser values than the function with the lesser rate of change as x approaches negative infinity.

- For example, compare $f(x) = -1 \cdot \left(\frac{1}{3}\right)^x$ and $g(x) = x^3$. Both functions will approach negative infinity as x approaches negative infinity. Compare the average rates of change over intervals of equal length to see that $g(x)$ will eventually be greater than $f(x)$. Use interval lengths of 5 (rather than the more convenient 3) to give a better picture of the function behavior farther from the origin.

Interval	[–20, –15]	[–15, –10]	[–10, –5]	[–5, 0]
$\frac{\Delta f(x)}{\Delta x}$	$6.94487 \cdot 10^8$	$2.85797 \cdot 10^6$	$1.17612 \cdot 10^4$	243
$\frac{\Delta g(x)}{\Delta x}$	925	475	175	25

- From the table, we can see that $f(x)$ has a greater rate of change than $g(x)$ as x decreases. This means that $f(x)$ decreases faster than $g(x)$ as x decreases. In other words, $f(x)$ will eventually have lesser values than $g(x)$.
- If two polynomial functions have the same right-end behavior, the polynomial with the greater degree eventually attains that right-end behavior faster. For example, $f(x) = x^3$ eventually approaches positive infinity faster than $g(x) = x^2$. The same holds for polynomial functions with the same left-end behavior.
- If a polynomial function and an exponential function have the same right-end behavior, the exponential function eventually attains that right-end behavior faster. For example, $f(x) = 2^x$ eventually approaches positive infinity faster than $g(x) = x^3$. The same is true for a polynomial function and an exponential function with the same left-end behavior.

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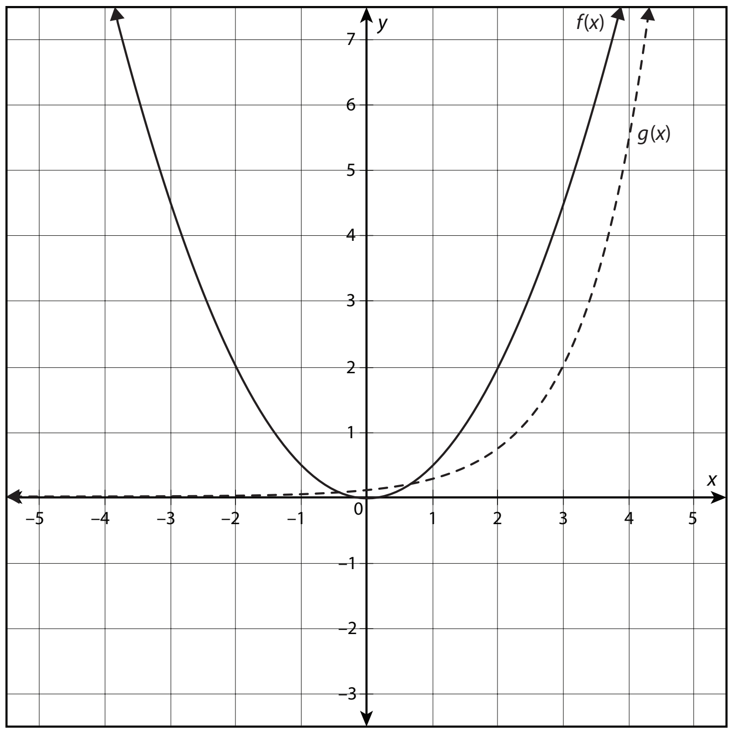
Comparing End Behavior by Graphing

- When comparing the graphs of functions on a graphing calculator, determine which function will eventually be greater by zooming out from the graph until it is clear that there will be no more intersections on the graph. Keep zooming until the conclusion is sure.
- For example, consider the following graph:



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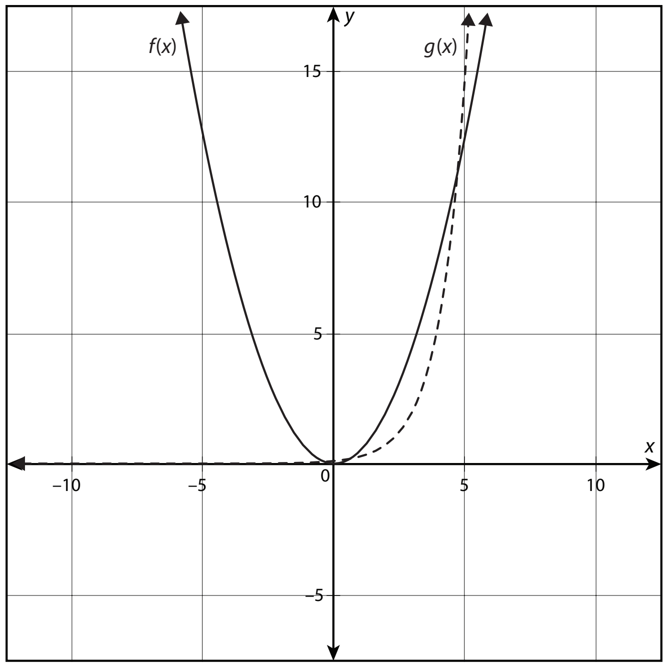
- The function $g(x)$, whose curve is farther to the right in the graph, is less than the function whose curve is farther to the left, $f(x)$. However, $g(x)$ is increasing faster than $f(x)$, so $g(x)$ will eventually be closer to the y-axis. Confirm this by zooming out:



- Now it is clear from the graph that $g(x)$ will eventually be greater than $f(x)$.

Common Errors/Misconceptions

- switching the numerator and denominator in a rate of change calculation
- switching the order of two function values in the numerator of a rate of change calculation from that of the two domain values in the denominator
- misidentifying the end behavior of a function
- misidentifying the function that eventually has greater values on its left end by forgetting to invert the direction of the average rate of change of a function as x approaches negative infinity

Scaffolded Practice 3.8: End Behaviors of Functions

For problems 1–8, let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ and $g(x) = a \bullet b^x + c$.

1. Describe the end behavior of $f(x)$ when a_n is positive and n is even.

2. Describe the end behavior of $f(x)$ when a_n is negative and n is even.

3. Describe the end behavior of $f(x)$ when a_n is positive and n is odd.

4. Describe the end behavior of $f(x)$ when a_n is negative and n is odd.

5. Describe the end behavior of $g(x)$ when a is positive and $b > 1$.

continued

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6. Describe the end behavior of $g(x)$ when a is negative and $b > 1$.

7. Describe the end behavior of $g(x)$ when a is positive and $0 < b < 1$.

8. Describe the end behavior of $g(x)$ when a is negative and $0 < b < 1$.

Use what you know about end behavior to complete problems 9 and 10.

9. For any two functions $p(x)$ and $q(x)$, how do you determine which function eventually grows faster as x approaches infinity?

10. What key feature should you look at to determine which function will eventually grow faster?

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Guided Practice 3.8

Example 1

Compare the end behaviors of the following functions by comparing the average rate of change over intervals of equal length and by graphing. Which function will eventually be the greatest as x approaches positive infinity and negative infinity?

$$\begin{cases} f(x)=4x \\ g(x)=x^2 \\ h(x)=\frac{1}{10}x^5-4 \end{cases}$$

1. Determine the end behavior of each function.

   First, examine each function to determine its type. When the function is a polynomial, examine the degree and sign of the leading coefficient of the function to determine the end behavior.

   The function $f(x) = 4x$ is linear with a positive slope. This means that the graph of the function will extend up to the right and down to the left.

   Symbolically, $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

   The function $g(x) = x^2$ is a second-degree polynomial with a positive leading coefficient, 1. This means that the graph of the function will extend up to the right and up to the left.

   Symbolically, $g(x) \to +\infty$ as $x \to +\infty$ and $g(x) \to +\infty$ as $x \to -\infty$.

   The function $h(x)=\frac{1}{10}x^5-4$ is a fifth-degree polynomial with a positive leading coefficient, $\frac{1}{10}$. This means that the graph of the function will extend up to the right and down to the left.

   Symbolically, $h(x) \to +\infty$ as $x \to +\infty$ and $h(x) \to -\infty$ as $x \to -\infty$.

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2. Determine appropriate intervals for calculating rates of change.

   Because we are interested in both the right- and left-end behavior of each function, we need intervals that will let us examine the rate of change for each function when x is increasing and when x is decreasing. It is not immediately obvious how quickly the functions are changing relative to one another, so longer intervals are better in this case. Choose length 10, because the leading coefficient of $h(x)$ makes this an easy number to work with. If this length does not give a good idea of the end behavior, redo the calculations with a different interval length. It's generally a good idea to choose an equal number of intervals for the case in which x is increasing and the case in which x is decreasing. Therefore, the following six intervals $[x_1, x_2]$ are reasonable for this problem:

   [–30, –20], [–20, –10], [–10, 0], [0, 10], [10, 20], and [20, 30]

3. Calculate the average rates of change over each interval for each function.

   Use the formula $\frac{\Delta f(x)}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$ to calculate the rates of change over each interval. Compile the results in a table to make comparison easy. Notice that because the intervals are all the same length, Δx will be the same for each interval $[x_1, x_2]$ using this formula. Calculate Δx, using [0, 10] as the sample interval, because this makes the calculation easy:

$\Delta x = x_2 - x_1$	General formula
$\Delta x = 10 - 0$	Substitute 10 for x_2 and 0 for x_1.
$\Delta x = 10$	

   (*continued*)

Now calculate the change in each function, $\Delta f(x)$ in the formula, starting with the interval farthest to the left on the x-axis, [–30, –20].

For $f(x) = 4x$:

$\Delta f(x) = f(x_2) - f(x_1)$	General formula
$\Delta f(x) = f(-20) - f(-30)$	Substitute –20 for x_2 and –30 for x_1.
$\Delta f(x) = -80 - (-120)$	Solve for function values.
$\Delta f(x) = -80 + 120$	Simplify.
$\Delta f(x) = 40$	

Therefore, $\frac{\Delta f(x)}{\Delta x} = \frac{40}{10} = 4$. Because $f(x)$ is a linear function, $\frac{\Delta f(x)}{\Delta x}$ will be the same for all intervals.

For $g(x) = x^2$:

$\Delta g(x) = g(x_2) - g(x_1)$	General formula
$\Delta g(x) = g(-20) - g(-30)$	Substitute –20 for x_2 and –30 for x_1.
$\Delta g(x) = 400 - 900$	Solve for function values.
$\Delta g(x) = -500$	Simplify.

Therefore, $\frac{\Delta g(x)}{\Delta x} = \frac{-500}{10} = -50$.

(*continued*)

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For $h(x)=\frac{1}{10}x^5-4$:

$\Delta h(x)=h(x_2)-h(x_1)$	General formula
$\Delta h(x)=h(-20)-h(-30)$	Substitute –20 for x_2 and –30 for x_1.
$\Delta h(x)=\left(\frac{-3.2\bullet 10^6}{10}-4\right)-\left(\frac{-2.43\bullet 10^7}{10}-4\right)$	Solve for function values.
$\Delta h(x)=-3.2\bullet 10^5-4-(-2.43\bullet 10^6-4)$	Simplify.
$\Delta h(x)=-3.2\bullet 10^5-4+2.43\bullet 10^6+4$	Simplify.
$\Delta h(x)=2.11\bullet 10^6$	Simplify.

Therefore, $\frac{\Delta h(x)}{\Delta x}=\frac{2.11\bullet 10^6}{10}=2.11\bullet 10^5$.

Repeating the process for the remaining intervals will yield the following values:

Interval	[–30, –20]	[–20, –10]	[–10, 0]	[0, 10]	[10, 20]	[20, 30]
$\frac{\Delta f(x)}{\Delta x}$	4	4	4	4	4	4
$\frac{\Delta g(x)}{\Delta x}$	–50	–30	–10	10	30	50
$\frac{\Delta h(x)}{\Delta x}$	$2.11\bullet 10^5$	$3.1\bullet 10^4$	1000	1000	$3.1\bullet 10^4$	$2.11\bullet 10^5$

4. Compare the functions' end behaviors by comparing the rates of change.

   We already know that all three functions approach positive infinity as x approaches positive infinity. However, they do not approach positive infinity at the same rate. The table shows that $g(x) = x^2$ eventually has a greater rate of change than $f(x) = 4x$, and that $h(x)=\frac{1}{10}x^5-4$ eventually has a greater rate of change than $g(x)$ as x approaches positive infinity. In other words, $g(x)$ eventually approaches positive infinity faster than $f(x)$, while $h(x)$ approaches positive infinity faster than $g(x)$ and $f(x)$. This means that $g(x)$ will eventually be greater than $f(x)$, while $h(x)$ will eventually be greater than $g(x)$ and $f(x)$ as x approaches positive infinity.

   On the other hand, as x approaches negative infinity, $g(x) = x^2$ approaches positive infinity, while $f(x) = 4x$ and $h(x)=\frac{1}{10}x^5-4$ approach negative infinity. This means that $g(x)$ will eventually be greater than $f(x)$ and $h(x)$ as x approaches negative infinity. However, $f(x)$ and $h(x)$ do not approach negative infinity at the same rate. The table shows that $h(x)$ eventually has a greater rate of change than $f(x)$ as x approaches negative infinity. In other words, $h(x)$ eventually approaches negative infinity faster than $f(x)$. This means that $h(x)$ will eventually be more negative than $f(x)$. Therefore, $g(x)$ will eventually be greater than $f(x)$ and $h(x)$, while $f(x)$ will eventually be greater than $h(x)$ as x approaches negative infinity.

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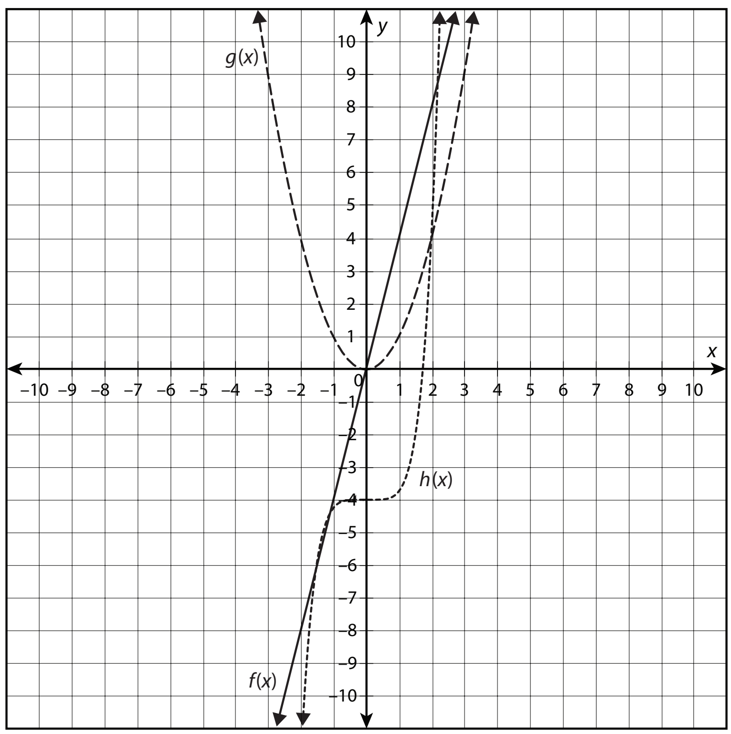
5. Graph the functions on the same graph and compare the end behaviors.

   Verify that comparing the graphs of the functions yields the same results as analyzing the functions by comparing rates of change. Graph the functions by hand or using a graphing calculator. If using a graphing calculator, use the standard viewing window for the initial view. Because the table of rates of change shows that the functions are changing fast enough in the first interval, it should be clear which function is growing or decreasing the fastest in this viewing pane.

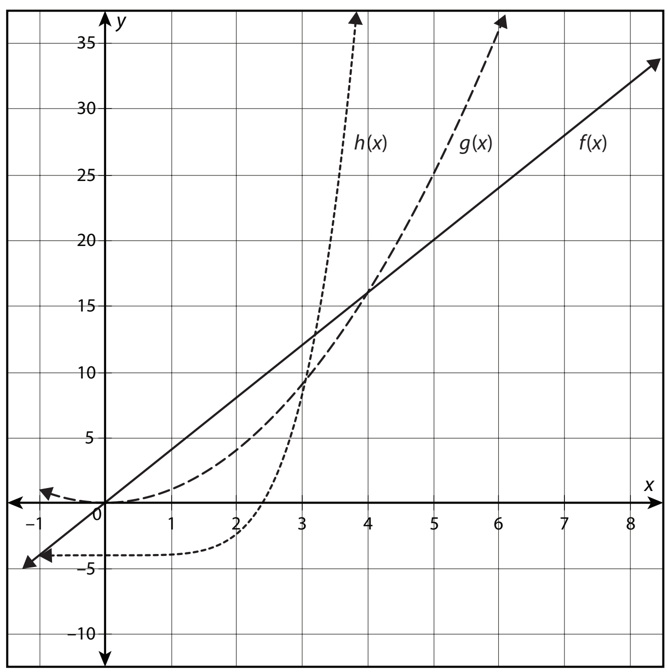
   The resulting graph should resemble the following:



(*continued*)

The graph shows that $h(x)=\frac{1}{10}x^5-4$ will eventually be greater than $f(x) = 4x$ and $g(x) = x^2$, and that $g(x)$ will eventually be greater than $f(x)$ and $h(x)$ on the left, while $f(x)$ will eventually be greater than $h(x)$ on the left. Although we can project that $g(x)$ will cross $f(x)$ again, and thus that $g(x)$ will eventually be greater than $f(x)$, we can also verify that $g(x)$ will eventually be greater than $f(x)$ by continuing the graph. We can do this by changing the viewing pane on a calculator, or by extending the y-axis if graphing by hand, until we can clearly see that $g(x)$ crosses $f(x)$ again.

Either method should yield a graph resembling the following:



It is now clear from the graph that $g(x) = x^2$ is indeed eventually greater than $f(x) = 4x$, so the results from step 4 are verified.

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6. Identify which function has the greatest values as x approaches positive infinity and which has the greatest values as x approaches negative infinity.

   We know from the previous steps that $h(x)=\frac{1}{10}x^5-4$ will eventually have the greatest values as x approaches positive infinity, while $g(x) = x^2$ will eventually have the greatest values as x approaches negative infinity.

Example 2

A team of scientists has come up with a model for the area of a small island, while another has come up with a model for the number of flowering plants on the island. Both models are predicted to be valid for the next 10 years, barring any extreme disturbance in the system. The square footage of a small island is modeled by the function $f(t) = (t - 10)^2 + 900$ ft^2, while the number of flowering plants on the island is modeled by the function $g(t) = 900 + 100 \bullet (0.75)^{t-2}$. In both equations, t represents time in years. Will there eventually be less than 1 flowering plant per square foot on the island? If so, is this a valid prediction?

1. Translate the problem into a mathematical question.

   To answer the question of whether there will eventually be less than 1 flowering plant per square foot, relate the square footage of the island to the number of flowering plants on the island as t increases. Thus, the statement should relate $f(t)$ to $g(t)$. More specifically, we are interested in whether the ratio of flowering plants to square footage will eventually be less than 1, which will occur if $\frac{g(t)}{f(t)}<1$, or $g(t) < f(t)$. So, we want to know whether $g(t)$ is eventually less than $f(t)$ as t approaches positive infinity, and if so, when.

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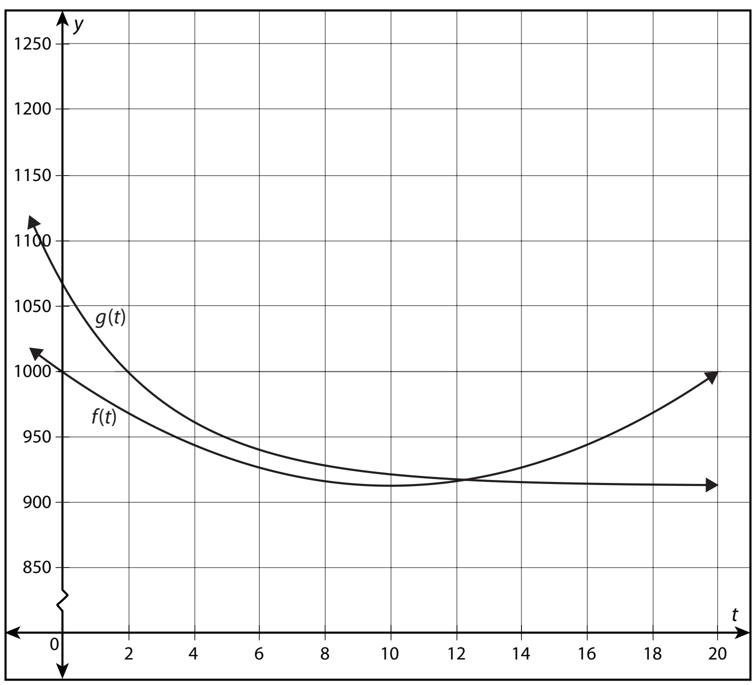
2. Analyze the functions to determine end behavior.

We are only interested in the right-end behavior of the functions, because the event we wish to examine will occur in the future. Because $f(t)$ is a quadratic function with a positive leading coefficient, the right end of $f(t)$ will approach positive infinity as t approaches positive infinity. Meanwhile, $g(t)$ is an exponential function with base $b < 1$. This means that the right end of $g(t)$ will approach a constant value as t approaches positive infinity. Now, we know that the exponential term of $g(t)$, $100 \bullet (0.75)^{t-2}$, will approach 0 as t approaches positive infinity because the term becomes smaller and smaller for increasingly large values of t. This means that $g(t) = 900 + 100 \bullet (0.75)^{t-2}$ will approach the constant term in the equation, 900. In other words, $g(t) \rightarrow 900$ as $t \rightarrow \infty$.

Verify this with a graph:



The graph shows that $f(t)$ approaches positive infinity as t approaches positive infinity, while $g(t)$ approaches 900 as t approaches positive infinity. Therefore, $g(t)$ will eventually be less than $f(t)$.

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3. Approximate when there will be less than 1 flowering plant per square foot on the island by finding the intersection point of $f(t)$ and $g(t)$.

   There will be less than 1 flowering plant per square foot after the point where the value of $f(t)$ becomes less than $g(t)$. This point will be the coordinates of the intersection point of the graphs. Find the approximate coordinates of the intersection either by graphing the function and estimating the intersection point, or by creating a table of values.

   Refer to the directions appropriate to your calculator model to estimate the intersection point by graphing.

   On a TI-83/84:

   Step 1: Plot both functions on the same graph.

   Step 2: Press [2ND][TRACE] to call up the CALC menu. Arrow down to 5: Intersect. Press [ENTER].

   Step 3: At the prompt, use the up/down arrow keys to select the Y1 equation. Press [ENTER].

   Step 4: At the prompt, use the up/down arrow keys to select the Y2 equation. Press [ENTER].

   Step 5: At the prompt, use the arrow keys to move the cursor close to an apparent intersection. Press [ENTER]. The coordinates of the intersection point are displayed.

   On a TI-Nspire:

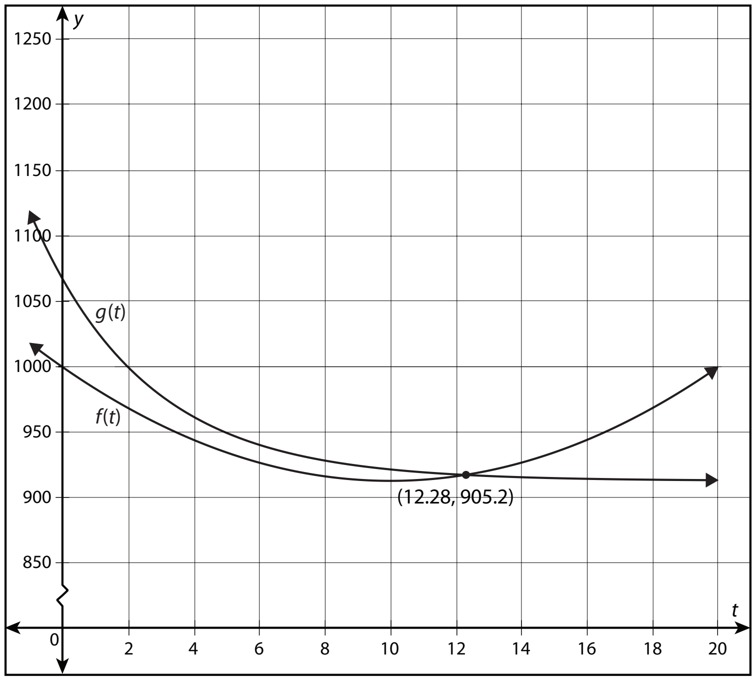
   Step 1: Plot both functions on the same graph.

   Step 2: Press [menu]. Arrow down to 6: Analyze Graph, then arrow right to 4: Intersection. Press [enter].

   Step 3: Use the pointing hand to click on each graphed line. The coordinates of the intersection point are displayed.

(*continued*)

The resulting graph should resemble the following:



The approximate coordinates are (12.28, 905.20).

Refer to the appropriate calculator directions for your calculator model to generate a table of values.

On a TI-83/84:

Step 1: Press [Y=] and enter both functions.

Step 2: Press [2ND][WINDOW] to call up the TABLE SETUP screen. Set TbleStart to 12.275 and ΔTbl to 0.001, then press [2ND][GRAPH] to call up the TABLE screen. A table of values for both equations will be displayed. Reset the TbleStart as desired to avoid excessive scrolling.

(*continued*)

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On a TI-Nspire:

Step 1: Define both functions. Type "define" followed by the function expression and press [enter] in the calculator mode, or plot both functions on a graph.

Step 2: Press the [home] key. Arrow over to the spreadsheets page, the fourth icon over, and press [enter].

Step 3: Press [ctrl][T] to switch to a table window. If for some reason the calculator has not stored the functions, press [home] again, switch to the calculator icon, and enter "define" and the function using the buttons on your keypad. Do this for both functions. Then go back to the spreadsheets page and switch to a table window. Select the function name representing $f(x)$ in the calculator from the list of defined functions that appears, and press [enter]. This will display a table of values for each value of x in the function $f(x)$.

Step 4: Press the right arrow key. Select the function name representing $g(x)$ from the list of defined functions that appears, and press [enter]. This will display a table of values for each value of x in the function $g(x)$.

Step 5: To choose what x-values the table uses, press [menu], arrow down to 2: Table, and select 5: Edit Table Settings. Set "Table Start" to 12.275, and set "Table Step" to 0.001. Arrow down to OK and press [enter] when finished. Repeat as desired to avoid excessive scrolling.

(*continued*)

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The resulting table should resemble the following:

t	$f(t)$	$g(t)$
12.275	905.176	905.203
12.276	905.180	905.202
12.277	905.185	905.200
12.278	905.189	905.199
12.279	905.194	905.197
12.280	905.198	905.195
12.281	905.203	905.194
12.282	905.208	905.193
12.283	905.212	905.191
12.284	905.217	905.190
12.285	905.221	905.188

The approximate coordinates found using a table of values are (12.28, 905.20), which matches the coordinates found by tracing the intersection of the functions on a graph.

4. Determine whether the result is valid within the context of the problem.

   The approximate intersection point of (12.28, 905.20) means that there will be less than 1 flowering plant per square foot on the island after about 12 years and three months. Because the problem states that the model is assumed to be accurate for 10 years, this prediction is questionable.

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Example 3

A wildlife manager wants to introduce pike to a lake. It is estimated that the lake can support no more than 2 large pike per cubic meter of water. Scientists have come up with prediction models for the volume of water in the lake and the population of pike, with reasonable validity for the next 12 years. Because of increased flow from the streams that drain into the lake, the volume of the lake, in cubic meters, is increasing according to the function $f(t) = 0.0001t^3 + 2t^2 + 2000t + 1000\text{ m}^3$, where t represents time in years. If the population of pike in the lake can be modeled by the function $g(t) = 1000 \bullet (1.5)^t$, would the lake ever be unable to support the pike population? If so, starting when? Should plans to introduce the pike be halted?

1. Translate the problem into a mathematical question.

   The population of pike and the volume of water in the lake are modeled by $f(t)$ and $g(t)$, respectively, so the mathematical question should relate $f(t)$ and $g(t)$. The pike population cannot be supported if there are more than 2 pike per cubic meter of water. In other words, we want to know if and when $\frac{g(t)}{f(t)} > \frac{2\text{ pike}}{\text{m}^3}$, or $g(t) > 2f(t)$.

   Therefore, determine whether $g(t)$ is eventually greater than $2f(t)$.

2. Analyze the functions to determine end behavior.

   We are only interested in the right-end behavior of the functions, because the event we wish to examine will occur in the future. Because $2f(t)$ is a cubic function with a positive leading coefficient, the right end of $2f(t)$ will approach positive infinity as t approaches positive infinity. Meanwhile, $g(t)$ is an exponential function with base $b > 1$. This means that the right end of $g(t)$ will also approach positive infinity as t approaches positive infinity.

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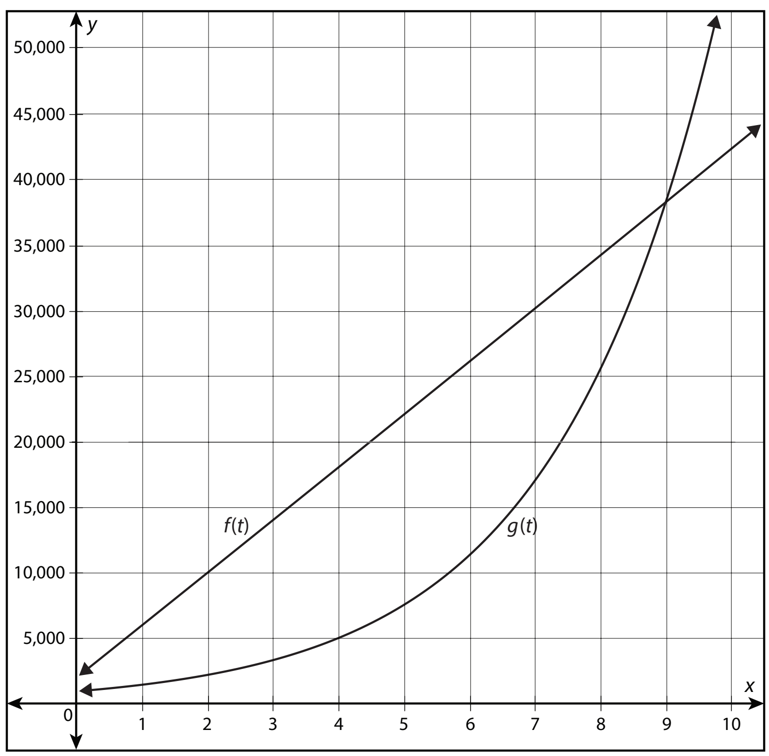
3. Calculate rates of change to compare the end behaviors of the functions.

Because both functions approach positive infinity as t approaches positive infinity, we need to determine which function will eventually increase faster. Do this by comparing the rates of change over intervals, and by looking at a graph. We will choose six intervals of length 2 each to give a detailed picture of the function behavior over the 12 years of function validity:

Interval	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
$\frac{\Delta 2f(t)}{\Delta t}$	4,008	4,024	4,040	4,056	4,072	4,088
$\frac{\Delta g(t)}{\Delta t}$	1,125	1,406	3,164	7,119	16,018	36,041



Both the table and the graph show that $g(t)$ will eventually increase faster than $2f(t)$. This means that, although $g(0) < 2f(0)$, eventually $g(t)$ will have greater values than $2f(t)$.

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4. Determine when the population of pike in the lake becomes unsustainable.

   To determine when the pike population becomes unsustainable, we need to know when $g(t)$ becomes greater than $2f(t)$, or when $g(t)$ crosses the graph of $2f(t)$. Do this by estimating the coordinates of the intersection of the two graphs, or by creating a table of values.

   Return to the graph of $g(t)$ and $2f(t)$ created in the previous step. Use your graphing calculator's trace function to determine that the estimated intersection point is at approximately (8.99, 38,300).

   The table of values for $g(t)$ and $2f(t)$ should resemble the following:

t	$2f(t)$	$g(t)$
8.985	38,263	38,210
8.986	38,267	38,226
8.987	38,271	38,241
8.988	38,275	38,257
8.989	38,279	38,272
8.990	38,283	38,288
8.991	38,287	38,303
8.992	38,292	38,319
8.993	38,296	38,334
8.994	38,300	38,350
8.995	38,304	38,366

   Using a table of values table yields the estimate (8.99, 38,300) for the point of intersection, which matches the result of tracing the graph.

5. Determine whether the result is valid within the context of the problem.

   The approximate intersection point (8.99, 38,300) means that the population of pike in the lake will become unsustainable after about 9 years. Because the problem states that the function models are valid for 12 years, this is probably a trustworthy prediction. Therefore, the plan to introduce pike to the lake should be reexamined.

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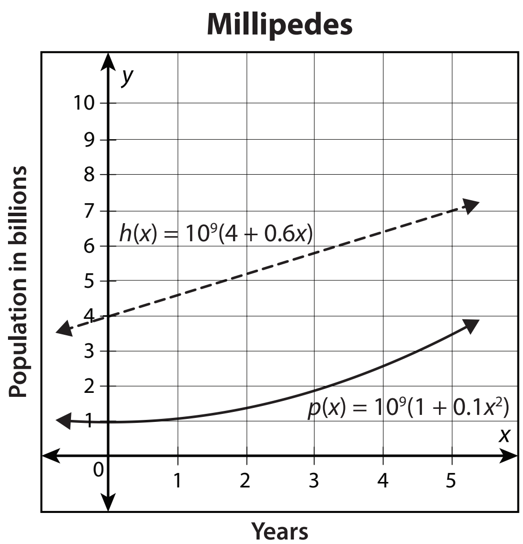
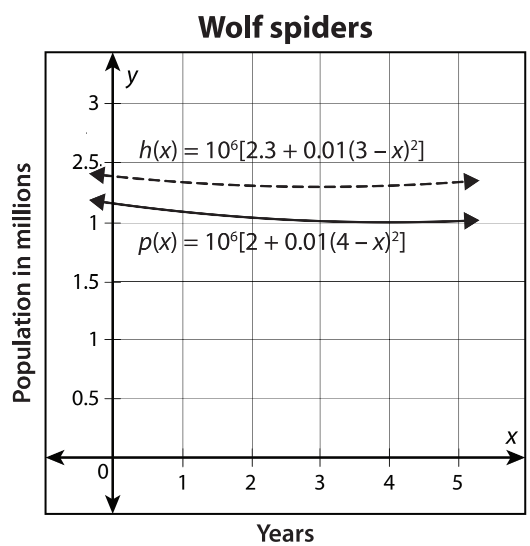
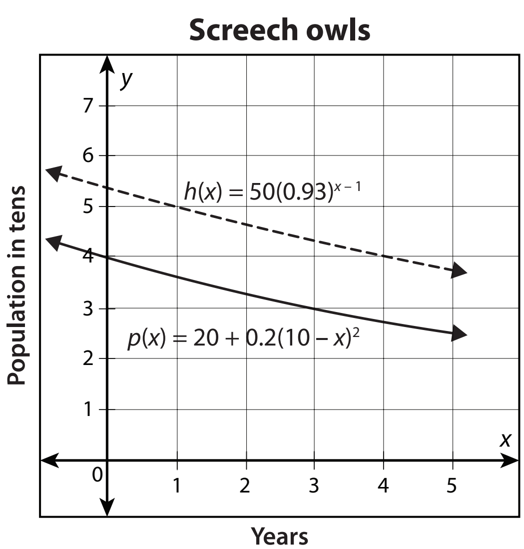
Problem-Based Task 3.8: Wildlife Management

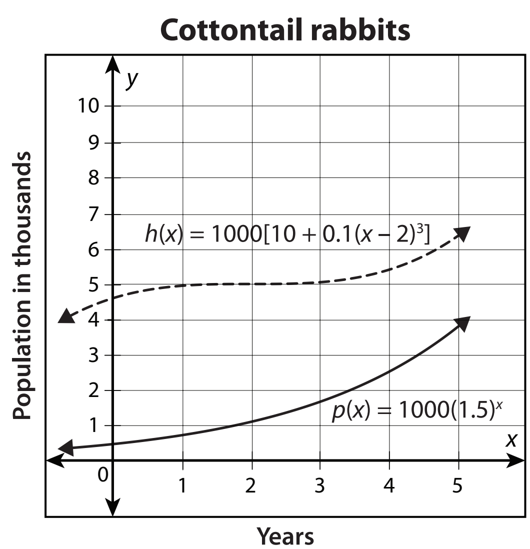
SMP
1 ✓ 2
3 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

You are the manager of a large wildlife preserve. If population of a species outgrows the limits of its habitat, catastrophic food chain collapse could result. The following charts represent the estimated populations and population limits of 4 different species in the preserve for the past 5 years, and the trends are predicted to continue according to the functions for 7 years if no action is taken. In each chart, the function $h(x)$ represents the estimated habitat limit for the species and $p(x)$ represents the population, with x representing time in years. Based on the population trends, will any of these species exceed their habitat limits? Which population trends are most concerning and why?









Which population trends are most concerning and why?

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Problem-Based Task 3.8: Wildlife Management

Coaching

a. How can we tell if a population is going to exceed its habitat limits?

b. Which end behavior should we study to determine whether a population is going to exceed its habitat limits?

c. Which populations, if any, will exceed their habitat limits?

d. If a population is going to exceed its habitat limit, how can we tell when this will occur?

e. When will any populations with unsustainable trends exceed their habitat limits?

f. Does this answer account for the fact that the chart shows information for the *past* 5 years?

g. Which population trends are most concerning? Why?

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Problem-Based Task 3.8: Wildlife Management

Coaching Sample Responses

a. How can we tell if a population is going to exceed its habitat limits?

A population will exceed its habitat limits if the population function eventually becomes greater than the limit function as time passes.

b. Which end behavior should we study to determine whether a population is going to exceed its habitat limits?

The events we are interested in will occur in the future. Because x represents time in years, we should compare the right-end behaviors of the functions to figure out which populations, if any, will eventually exceed their limits as x increases.

c. Which populations, if any, will exceed their habitat limits?

For millipedes, the habitat limit function $h(x) = 10^9 \bullet (4 + 0.6x)$ is an increasing linear function, while the population function $p(x) = 10^9 \bullet (1 + 0.1x^2)$ is a quadratic function with a positive leading coefficient. Hence, both functions approach positive infinity as x approaches positive infinity. We must compare the rates of change over intervals to decide if the population will become unsustainable. The following table shows the rates of change for $h(x)$ and $p(x)$ for intervals of 2:

Interval	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
$\frac{\Delta h(x)}{\Delta x}$	$6 \bullet 10^8$	$6 \bullet 10^8$	$6 \bullet 10^8$	$6 \bullet 10^8$	$6 \bullet 10^8$	$6 \bullet 10^8$
$\frac{\Delta p(x)}{\Delta x}$	$2 \bullet 10^8$	$6 \bullet 10^8$	10^9	$1.4 \bullet 10^9$	$1.8 \bullet 10^9$	$2.2 \bullet 10^9$

From the table, we see that $p(x)$ has a greater rate of change than $h(x)$ as x increases. Therefore, we may conclude that $p(x)$ will eventually exceed $h(x)$ as x increases. In other words, the millipede population will eventually become unsustainable with time.

For wolf spiders, both the habitat limit function $h(x) = 10^6 \bullet [2.3 + 0.01(3 - x)^2]$ and the population function $p(x) = 10^6 \bullet [2 + 0.01(4 - x)^2]$ are quadratic functions with a positive leading coefficient. Hence, both functions approach positive infinity as x approaches positive infinity. We must compare the rates of change over intervals to decide if the population will become unsustainable.

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The following table shows the rates of change for $h(x)$ and $p(x)$ for intervals of 2:

Interval	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
$\frac{\Delta h(x)}{\Delta x}$	$-4 \bullet 10^4$	0	$4 \bullet 10^4$	$8 \bullet 10^4$	$1.2 \bullet 10^5$	$1.6 \bullet 10^5$
$\frac{\Delta p(x)}{\Delta x}$	$-6 \bullet 10^4$	$-2 \bullet 10^4$	$2 \bullet 10^4$	$6 \bullet 10^4$	10^5	$1.4 \bullet 10^5$

From the table, we see that $h(x)$ has a greater rate of change than $p(x)$ as x increases. Therefore, we may conclude that $p(x)$ will not exceed $h(x)$ as x increases. In other words, the wolf spider population will remain sustainable with time.

For screech owls, the habitat limit function $h(x) = 50 \bullet (0.93)^{x-1}$ is an exponential function with $b < 1$, which approaches 0 as x approaches positive infinity. Meanwhile, the population function $p(x) = 20 + 0.2 \bullet (10 - x)^2$ is a quadratic with a positive leading coefficient, which means that $p(x)$ approaches positive infinity as x approaches positive infinity. Therefore, we may conclude that $p(x)$ will eventually exceed $h(x)$ as x increases. In other words, the screech owl population will eventually become unsustainable with time.

For cottontail rabbits, the habitat limit function $h(x) = 1000[10 + 0.1(x - 2)^3]$ is a cubic function with a positive leading coefficient and the population function $p(x) = 1000 \bullet (1.5)^x$ is an exponential function with $b > 1$. Hence, both functions approach positive infinity as x approaches positive infinity. We must compare the rates of change over intervals to decide if the population will become unsustainable.

The following table shows the rates of change for $h(x)$ and $p(x)$ for intervals of 2:

Interval	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
$\frac{\Delta h(x)}{\Delta x}$	400	400	2,800	7,600	14,800	24,400
$\frac{\Delta p(x)}{\Delta x}$	675	1,406	3,164	7,119	16,018	36,041

From the table, we see that $p(x)$ has a greater rate of change than $h(x)$ as x increases. Therefore, we may conclude that $p(x)$ will eventually exceed $h(x)$ as x increases. In other words, the cottontail rabbit population will eventually become unsustainable with time.

In conclusion, the populations of millipedes, screech owls, and cottontail rabbits will become unsustainable.

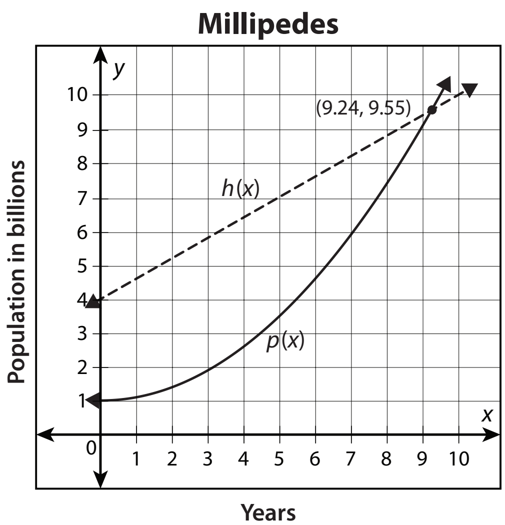
d. If a population is going to exceed its habitat limit, how can we tell when this will occur?

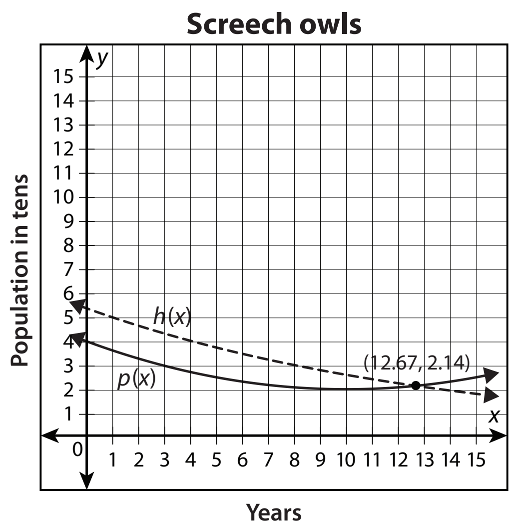
Because all four populations start out being less than their population limit, you can tell when a population outgrows its limit by finding the intersection of the graphs of each population function and its habitat limit function.

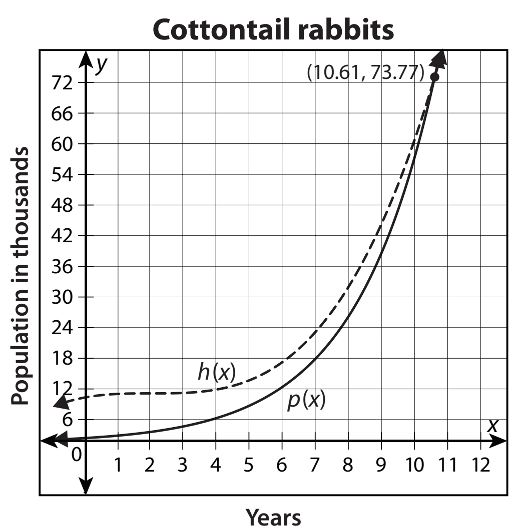
e. When will any populations with unsustainable trends exceed their habitat limits?

We know that millipedes, screech owls, and cottontail rabbits have unsustainable population trends. For each species, we must find where the graph of the population function crosses the graph of the habitat limit function. So, now we will plot the graph and estimate the intersection point.

The following graphs show the intersection point of the population function with the habitat limit function for each species:







From the graphs, we can see that $p(x)$ crosses $h(x)$ at $x = 9.24$ for millipedes, $p(x)$ crosses $h(x)$ at $x = 12.67$ for screech owls, and $p(x)$ crosses $h(x)$ at $x = 10.61$ for cottontail rabbits.

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f. Does this answer account for the fact that the chart shows information for the *past* 5 years?

Thus far, we have been treating year 1 on the chart as $x = 1$ of our function. However, because the chart extends 5 years into the past, we should start counting future years from the end of the chart, at $x = 5$. This means that "now" corresponds to $x = 5$, 1 year from now corresponds to $x = 6$, etc. Therefore, the millipede population becomes unsustainable after 9.24 – 5 = 4.24 years, the screech owl population becomes unsustainable after 12.67 – 5 = 7.67 years, and the cottontail rabbit population becomes unsustainable after 10.61 – 5 = 5.61 years.

Hence, the millipede, screech owl, and cottontail rabbit populations become unsustainable after 4, 7, and 5 years, respectively.

g. Which population trends are most concerning? Why?

The millipede population is most concerning because it will become unsustainable after 4 years. Some sort of action will need to be taken to control this population.

The cottontail rabbit population is also concerning because it is increasing exponentially. If uncontrolled, this could lead to serious habitat destruction from which it could take years for the preserve to recover. The preserve should act to control the rabbit population before this occurs.

The screech owl population is less concerning because it occurs right on the edge of the model's time frame for validity. However, it does have a decreasing habitat limit function, which could mean that the preserve would eventually be unable to support any population of screech owls. It might be useful to study food webs in the preserve to find out why the numbers are declining.

The spider population is not concerning because it is relatively sustainable compared to its habitat.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.8: End Behaviors of Functions A

For problems 1–4, determine the end behavior of $f(x)$ and $g(x)$, and state which function will eventually be greater as $x \to +\infty$ and as $x \to -\infty$.

1. $\begin{cases} f(x) = \dfrac{1}{10}x^2 - x + 12 \\ g(x) = -10x \end{cases}$

2. $\begin{cases} f(x) = e^x \\ g(x) = 12x^3 \end{cases}$

3. $\begin{cases} f(x) = -x^4 \\ g(x) = (9 - x)^3 \end{cases}$

4. $\begin{cases} f(x) = \left(\dfrac{1}{2}\right)^x - 3 \\ g(x) = 10x^6 \end{cases}$

Use the given information to complete problems 5–7.

The number of cells in a bacteria culture in a petri dish is modeled by the function $f(t) = 1000 \bullet 1.25^t$ cells, while the area of the petri dish that the culture covers is modeled by the function $g(t) = 5006[25 + \pi(0.5t)^2]$ μm^2. Because of the bacteria's own toxic secretions, the culture will die if the population density exceeds 1 cell per 10 μm^2, or if $\dfrac{f(t)}{g(t)} > \dfrac{\text{cells}}{10\ \mu\text{m}^2}$. Assume t represents time in hours, and note that 1000 μm = 1 mm.

continued

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5. What is the right-end behavior of $10f(t)$ and $g(t)$?

6. Determine the average rate of change for $10f(t)$ and $g(t)$ over the following intervals: [0, 6], [6, 12], [12, 18], [18, 24], and [24, 30].

7. Will the bacteria culture die in less than 1 day? Justify your answer.

Use the given information to complete problems 8–10.

As a blueberry bush grows, the number of berries and the number of twigs on the bush at harvest time increases. However, the productivity per twig tends to decrease because the bush spends more energy on maintaining its branches than producing berries. When the ratio of berries to twigs falls below 1, the plant will require vigorous pruning to regain productivity. The number of berries harvested from a blueberry bush increases according to the function $f(y) = 7.5y^2 - 13y + 20$, and the number of twigs on the bush at harvest time increases according to the function $g(y) = 7.75y^2 - 15y + 2$. Assume both models are valid for approximately 20 years, and that y represents time in years.

8. What is the right-end behavior of $f(y)$ and $g(y)$?

9. Determine the average rate of change for $f(y)$ and $g(y)$ over the following intervals: [0, 2], [2, 4], [4, 6], [6, 8], and [8, 10].

10. When will the blueberry bush need to be pruned? Justify your answer.

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Practice 3.8: End Behaviors of Functions B

For problems 1–4, determine the end behavior of $f(x)$ and $g(x)$, and state which function will eventually be greater as $x \to +\infty$ and as $x \to -\infty$.

1. $\begin{cases} f(x)=\dfrac{1}{3}x^2-10 \\ g(x)=30x \end{cases}$

2. $\begin{cases} f(x)=e^{-x} \\ g(x)=5x^4 \end{cases}$

3. $\begin{cases} f(x)=-x^2 \\ g(x)=(1-x)^5 \end{cases}$

4. $\begin{cases} f(x)=-\left(\dfrac{1}{10}\right)^x \\ g(x)=10x^3 \end{cases}$

Use the given information to complete problems 5–7.

The number of cells in a bacteria culture in a petri dish is modeled by the function $f(t) = 1000 \cdot 1.42^t$ cells, while the area of the petri dish that the culture covers is modeled by the function $g(t) = 17{,}001(30 + \pi(0.7t)^2)$ μm^2. Because of the bacteria's own toxic secretions, the culture will die if the population density exceeds 1 cell per 10 μm^2, or if $\dfrac{f(t)}{g(t)} > \dfrac{\text{cells}}{10\ \mu\text{m}^2}$. Assume t represents time measured in hours, and note that 1000 μm = 1 mm.

continued

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5. What is the right-end behavior of $10f(t)$ and $g(t)$?

6. Determine the average rate of change for $10f(t)$ and $g(t)$ over the following intervals: [0, 5], [5, 10], [10, 15], [15, 20], and [20, 25].

7. Will the bacteria culture die in less than 20 hours? Justify your answer.

Use the given information to complete problems 8–10.

The quality of wine grapes improves as the ratio of individual grapes to leaves increases. However, if the ratio becomes greater than 1, the strain on the vine will lead to decreased productivity for the next few years. According to current pruning practices, the average number of individual grapes harvested from a particular grape vine at a vineyard increases according to the function $f(y) = 10y^2 - 2y + 17$. Meanwhile, the average number of leaves on a vine at harvest time increases according to the function $g(y) = 9y^2 - 3y + 50$. Assume both models are valid for approximately 50 years, and that y represents time measured in years.

8. What is the right-end behavior of $f(y)$ and $g(y)$?

9. Determine the average rate of change for $f(y)$ and $g(y)$ over the following intervals: [0, 2], [2, 4], [4, 6], [6, 8], and [8, 10].

10. Should current pruning practices be changed? If so, when?

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Answer Key

Lesson 3.1: Introduction to Polynomial Functions (A–SSE.1a★)

Warm-Up 3.1, p. U3-1

1. 228 inches
2. $42 - x - x$, or $42 - 2x$
3. $72 - 2x - 2x$, or $72 - 4x$

Scaffolded Practice 3.1: Introduction to Polynomial Functions, p. U3-5

1. Terms: $4x^3$ and $2x$
   $4x^3$ has a coefficient of 4, a variable of x, and a power of 3.
   $2x$ has a coefficient of 2, a variable of x, and a power of 1.
2. Terms: $-n^5$, $3n$, and 1
   $-n^5$ has a coefficient of –1, a variable of n, and a power of 5.
   $3n$ has a coefficient of 3, a variable of n, and a power of 1.
   1 is a constant.
3. Terms: $2r^2$, $-10r$, and 15
   $2r^2$ has a coefficient of 2, a variable of r, and a power of 2.
   $-10r$ has a coefficient of –10, a variable of r, and a power of 1.
   15 is a constant.
4. Terms: $-6v^9$, $-v^7$, $-v^5$, and $-v^3$
   $-6v^9$ has a coefficient of –6, a variable of v, and a power of 9.
   $-v^7$ has a coefficient of –1, a variable of v, and a power of 7.
   $-v^5$ has a coefficient of –1, a variable of v, and a power of 5.
   $-v^3$ has a coefficient of –1, a variable of v, and a power of 3.
5. Terms: a^6, $-9a^4$, $7a^3$, $4a$, and –7
   a^6 has a coefficient of 1, a variable of a, and a power of 6.
   $-9a^4$ has a coefficient of –9, a variable of a, and a power of 4.
   $7a^3$ has a coefficient of 7, a variable of a, and a power of 3.
   $4a$ has a coefficient of 4, a variable of a, and a power of 1.
   –7 is a constant.
6. $f(x) = 16x^8 + x^5 + 8$; degree: 8
7. $f(x) = 9x^2 - 20x$; degree: 2
8. $f(x) = x^4 + x^3 + x^2 + x$; degree: 4
9. $f(x) = -3x^6 - x^5 - x + 12$; degree: 6
10. $f(x) = 11x^4 + 2x^2 + 14x - 30$; degree: 4

Practice 3.1 A: Introduction to Polynomial Functions, p. U3-13

1. Terms: y^4 and 13.
   y^4 has a coefficient of 1, a variable of y, and a power of 4.
   13 is the constant term.
2. Terms: $8c^3$, $-c^2$, and $8c$.
   $8c^3$ has a coefficient of 8, a variable of c, and a power of 3.
   $-c^2$ has a coefficient of –1, a variable of c, and a power of 2.
   $8c$ has a coefficient of 8, a variable of c, and a power of 1.
3. Terms: $12z^5$, $9z^2$, $-z$, and –7.
   $12z^5$ has a coefficient of 12, a variable of z, and a power of 5.
   $9z^2$ has a coefficient of 9, a variable of z, and a power of 2.
   $-z$ has a coefficient of –1, a variable of z, and a power of 1.
   –7 is the constant term.
4. Terms: $-5m^{10}$, m^8, $5m^6$, and $-m^4$.
   $-5m^{10}$ has a coefficient of –5, a variable of m, and a power of 10.
   m^8 has a coefficient of 1, a variable of m, and a power of 8.
   $5m^6$ has a coefficient of 5, a variable of m, and a power of 6.
   $-m^4$ has a coefficient of –1, a variable of m, and a power of 4.
5. $f(x) = x^8 - 3x^3 + 30x$; degree: 8
6. $f(x) = -x^6 - 6x^3 + 14x^2 + 10$; degree: 6
7. $f(x) = x^7 - x^5 + 7x^3 + 2x + 22$; degree: 7
8. $(9x^2 + 12x)$ in^2
9. $(3x^2 + 5x + 8)$ in^2
10. $(3x^3 + 6x^2 + 2x + 8)$ in^2

Practice 3.1 B: Introduction to Polynomial Functions, p. U3-15

1. Terms: $-k$ and –1.
   $-k$ has a coefficient of –1, a variable of k, and a power of 1.
   –1 is the constant term.
2. Terms: $2p^3$, p^2, and 30.
   $2p^3$ has a coefficient of 2, a variable of p, and a power of 3.
   p^2 has a coefficient of 1, a variable of p, and a power of 2.
   30 is the constant term.
3. Terms: $-4b^4$, $3b^3$, $2b^2$, and 1.
   $-4b^4$ has a coefficient of –4, a variable of b, and a power of 4.
   $3b^3$ has a coefficient of 3, a variable of b, and a power of 3.
   $2b^2$ has a coefficient of 2, a variable of b, and a power of 2.
   1 is the constant term.
4. Terms: $8x^{12}$, $-7x^2$, $6x$, and 2.
   $8x^{12}$ has a coefficient of 8, a variable of x, and a power of 12.
   $-7x^2$ has a coefficient of –7, a variable of x, and a power of 2.
   $6x$ has a coefficient of 6, a variable of x, and a power of 1.
   2 is the constant term.
5. $f(x) = -3x^5 - 6x^3 - 5x^2$; degree: 5
6. $f(x) = x^4 - x^2 + 15x + 9$; degree: 4
7. $f(x) = -2x^6 - x^5 + 10x^3 + 4x + 20$; degree: 6
8. $(10x + 13)$ in^2
9. $(5x^2 + 14x + 30)$ in^2
10. $(2x^3 + 25x^2 + 12x + 50)$ in^2

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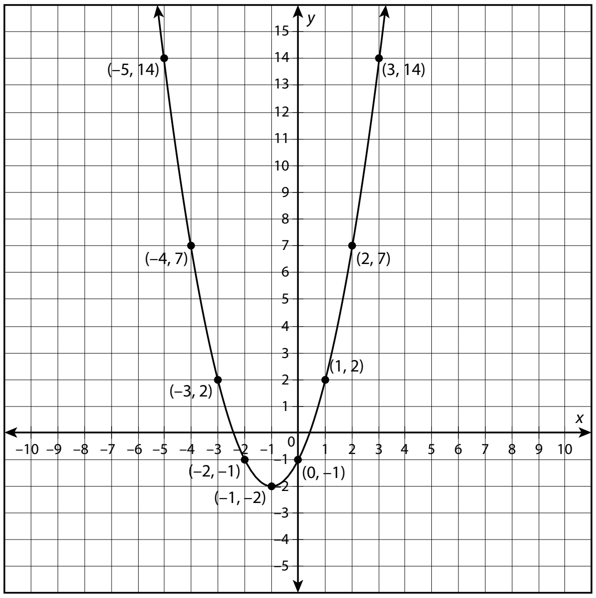
Lesson 3.2: Graphing Quadratic and Cubic Functions (F–IF.7★)

Warm-Up 3.2, p. U3-17

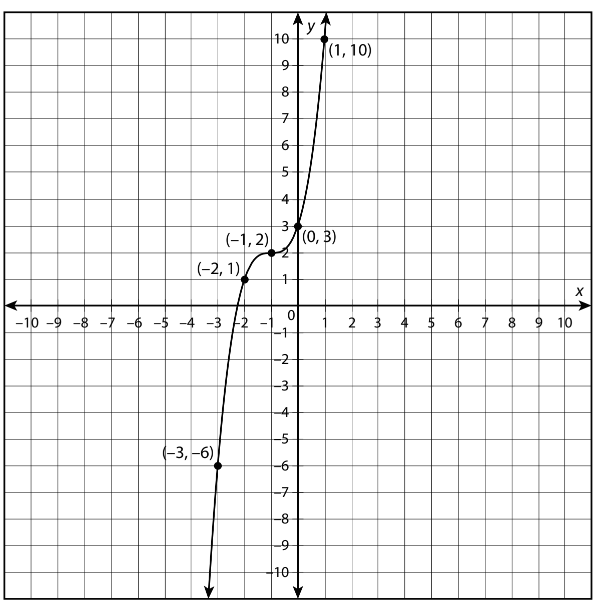
1.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	14	7	2	–1	–2	–1	2	7	14	23	34



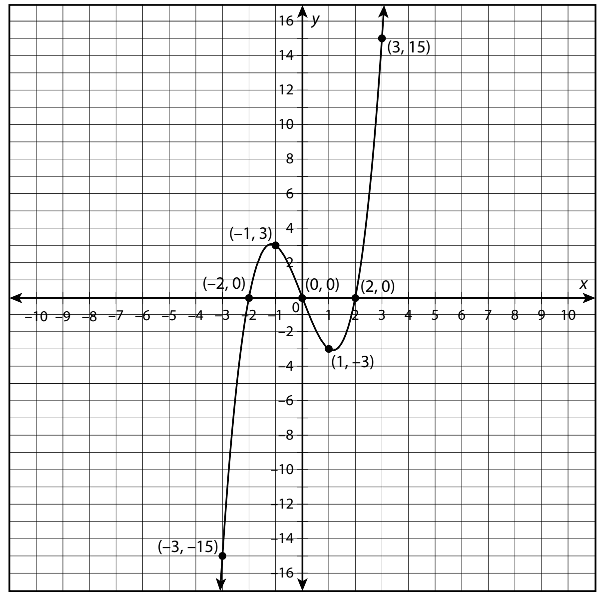
2.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–62	–25	–6	1	2	3	10	29	66	127	218



3.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–105	–48	–15	0	3	0	–3	0	15	48	105



Scaffolded Practice 3.2: Graphing Quadratic and Cubic Functions, p. U3-28

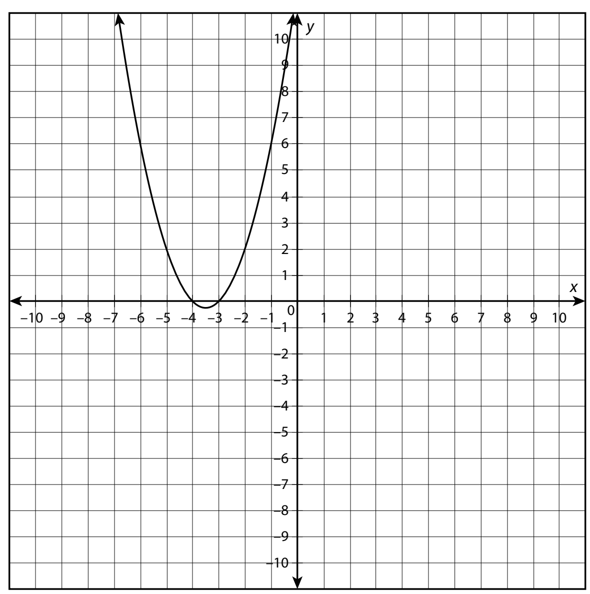
1.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	48	35	24	15	8	3	0	–1	0	3	8

2.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–79	–31	–5	5	5	1	–1	5	25	65	131

3. focus: (0.5, 12); directrix: $y = 12.5$
4. vertex: (2, 6); axis of symmetry: $x = 2$
5. down
6. decreasing
7.



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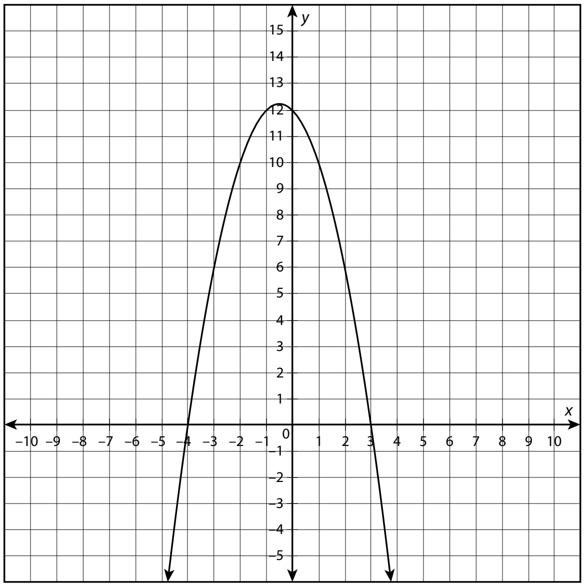
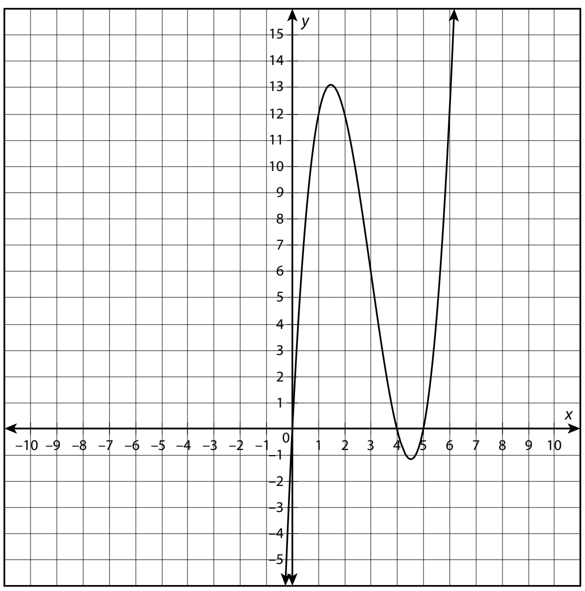
8.



9.



10. The zeros are (2, 0), (4, 0), and (6, 0).

Practice 3.2 A: Graphing Quadratic and Cubic Functions, p. U3-45

1.

x	–4	–3	–2	–1	0	1	2	3	4
y	25	24	21	16	9	0	–11	–24	–39

2.

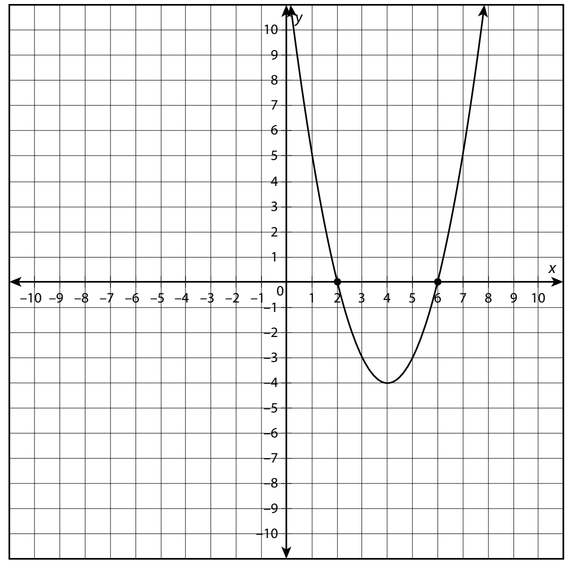
x	–4	–3	–2	–1	0	1	2	3	4
y	11	17	13	5	–1	1	17	53	115

3. focus: $\left(\frac{4}{7},-\frac{147}{28}\right)$; directrix: $y=-\frac{149}{28}$

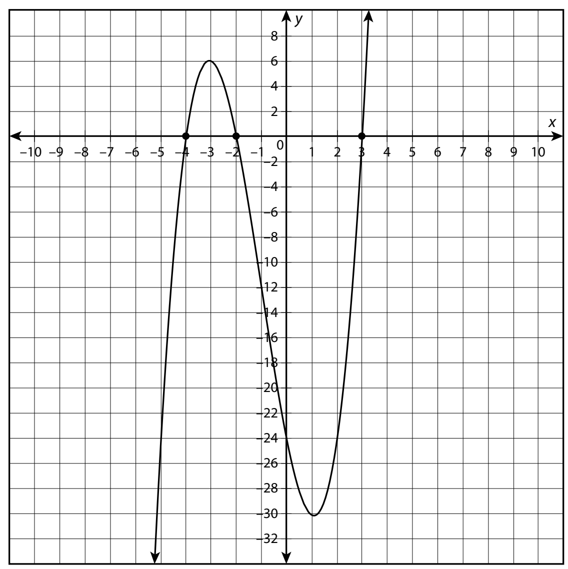
4. $f(x)=\frac{1}{2}x^2-3x+5$

5. (3, 34), $x = 3$

6. (2, 0) and (6, 0)



7. $(-4,0),(-2,0),(3,0)$



8. overall decreasing

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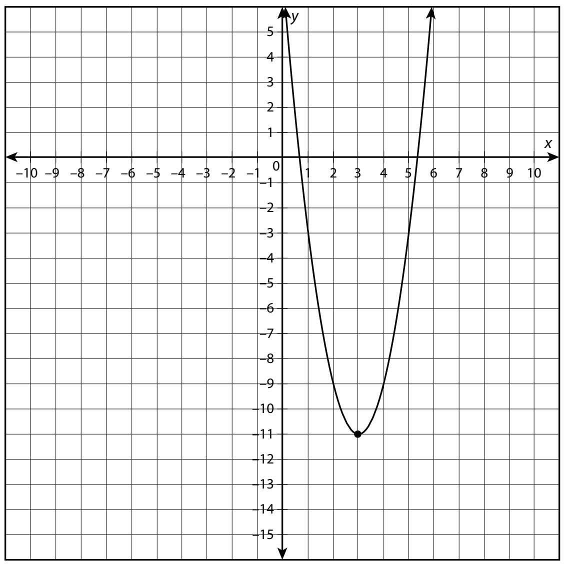
9.



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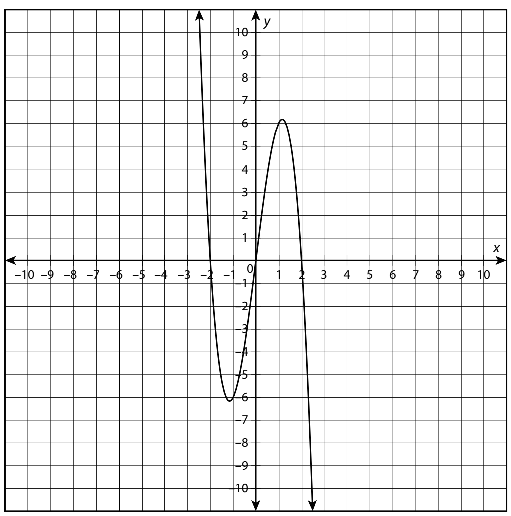


Practice 3.2 B: Graphing Quadratic and Cubic Functions, p. U3-48

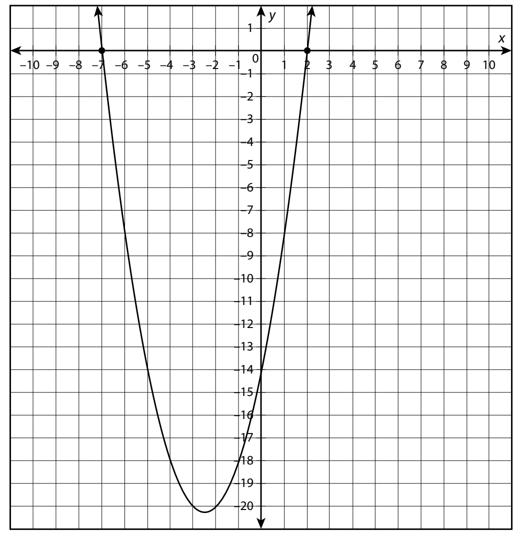
1.

x	–4	–3	–2	–1	0	1	2	3	4
y	22	9	2	1	6	17	34	57	86

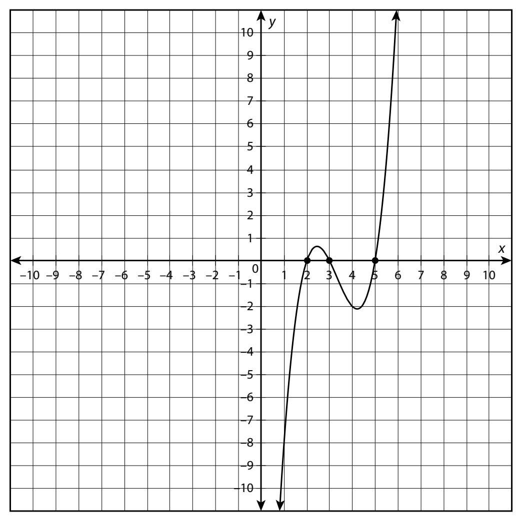
2.

x	–4	–3	–2	–1	0	1	2	3	4
y	0	21	24	15	0	–15	–24	–21	0

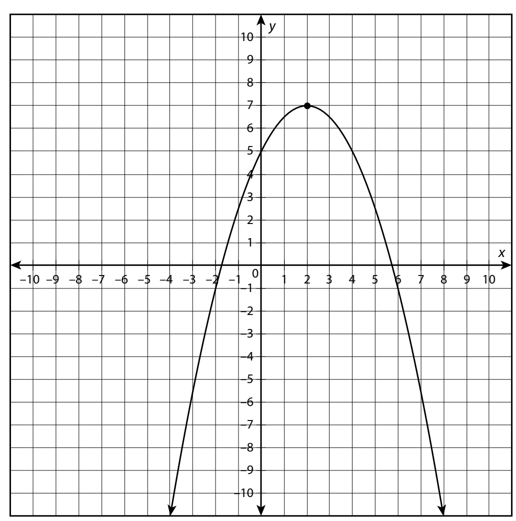
3. focus: (–1, 3.125); directrix: $y = 2.875$
4. $f(x)=8-6x-3x^2$
5. $(1,-2)$, $x=1$
6. (–7, 0) and (2, 0)



7. (2, 0), (3, 0) and (5, 0)



8. overall increasing
9.



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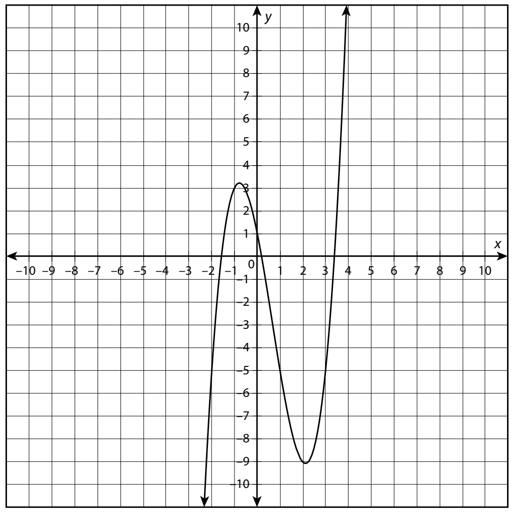
10.



Lesson 3.3: Optimization of Volume (G–MG.1★)

Warm-Up 3.3, p. U3-51

1. approximately 156.25 in.^{3}
2. $156.25 = \pi(2.5^2)(h)$
3. about 7.96 in.

Scaffolded Practice 3.3: Optimization of Volume, p. U3-57

1. $V = l \bullet w \bullet h$
2. $V = 2\pi r^2 h$
3. $SA = 2lw + 2lh + 2hw$
4. $SA = 2\pi r^2 + 2\pi rh$
5. $f(x) = x(7 - 2x)(12 - 2x)$; $0 < x < 3.5$
6. $f(x) = x(3.5 - 2x)(8 - 2x)$; $0 < x < 1.75$
7. $\begin{cases} 260 = x^2 + 4xy \\ V = x^2 y \end{cases}$
8. $\begin{cases} 36 = \pi x^2 y \\ S = 2\pi x^2 + 2\pi xy \end{cases}$
9. $0 < x < 21.91$; the volume must be greater than 0
10. $x > 0$; the surface area must be greater than 0

Practice 3.3 A: Optimization of Volume, p. U3-77

1. $V = x(18 - 2x)(12 - 2x)$; domain: $0 < x < 6$
2. squares with side length approximately 2.35 in.
3. approximately 228.16 in.^{3}
4. approximately 2,259.48 in.^{3}
5. approximately 14.80 cm × 29.59 cm × 19.73 cm
6. approximately 3.67 in. × 2.94 in. × 7.35 in.
7. 538.55 cm^2
8. maximum volume: 502.72 in.^{3}; diameter: 8.62 in.; height: 8.62 in.
9. dimensions: 20 in. × $13\frac{1}{3}$ in. × 8 in.; volume: $2{,}133\frac{1}{3}$ in.^{3}
10. radius: approximately 16.04 in.; height: approximately 53.45 in.

Practice 3.3 B: Optimization of Volume, p. U3-79

1. $V = x(40 - 2x)(50 - 2x)$; domain: $0 < x < 20$
2. squares with side length approximately 7.36 cm
3. approximately 6,564.23 cm^3
4. approximately 12.91 in. × 12.91 in. × 6.45 in.
5. approximately 10.51 cm × 31.53 cm × 15.76 cm
6. approximately 21.60 cm × 64.81 cm × 25.92 cm
7. radius: approximately 1.55 in.; height: approximately 3.09 in.
8. maximum volume: approximately 1,919.45 cm^3; diameter: 13.47 cm; height: 13.47 cm
9. approximately 2,660.43 in.^{3}
10. \$3.01

Lesson 3.4: Describing End Behavior and Turns (F–IF.7★, N–CN.9)

Warm-Up 3.4, p. U3-81

1. increasing: $t < 20$; decreasing: $t > 20$
2. (20, 22). The concentration is at its highest, 22 parts per million, 20 hours after a person takes the dose.
3. (41, 0). The concentration is 0 parts per million after 41 hours.
4. (0, 2). The concentration is 2 parts per million at the time of the dose.

Scaffolded Practice 3.4: Describing End Behavior and Turns, p. U3-88

1. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; turning points: 2; real roots: 3
2. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; turning points: 3; real roots: 4
3. $g(x) \to +\infty$ as $x \to -\infty$ and $g(x) \to -\infty$ as $x \to +\infty$; turning points: 6; real roots: 7
4. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; turning points: 5; real roots: 6
5. $h(x) \to +\infty$ as $x \to -\infty$ and $h(x) \to +\infty$ as $x \to +\infty$; turning points: 9; real roots: 10
6. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; odd-degree; 3 real roots
7. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; odd-degree; 1 real root
8. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; even-degree; 2 real roots
9. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; odd-degree; 1 real root
10. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; even-degree; 0 real roots

Practice 3.4 A: Describing End Behavior and Turns, p. U3-100

1. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; turning points: 3; real roots: 4
2. $g(x) \to -\infty$ as $x \to -\infty$ and $g(x) \to +\infty$ as $x \to +\infty$; turning points: 2; real roots: 3

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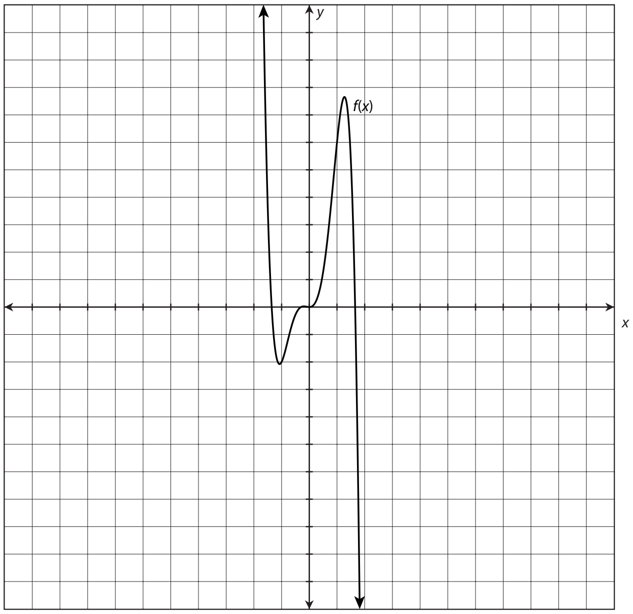
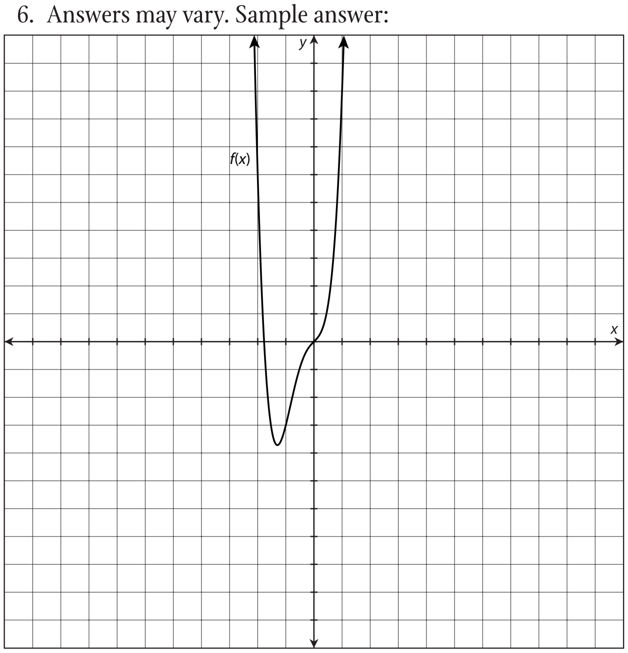
3. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; odd-degree; 1 real root
4. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; even-degree; 2 real roots
5. Answers may vary. Sample answer:



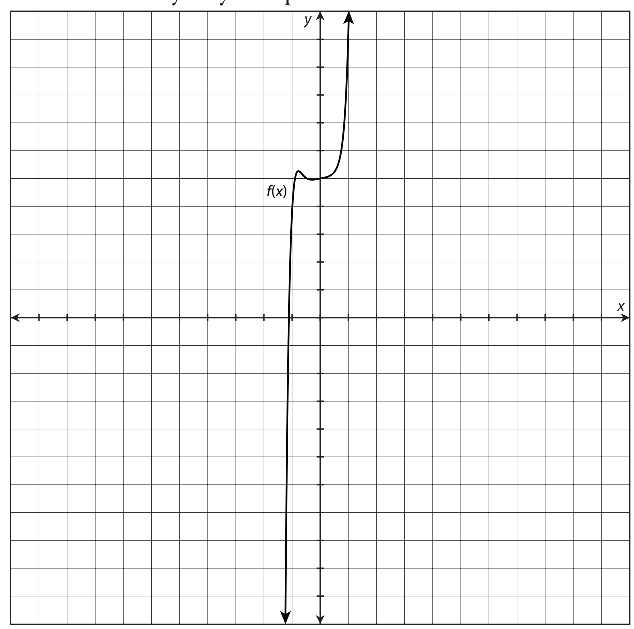
6. Answers may vary. Sample answer:



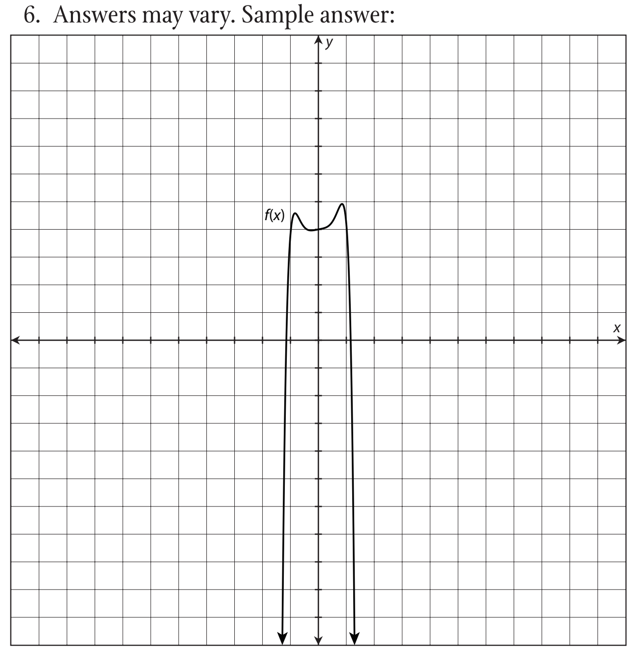
7. (4, 375), (14, 150), (29, 550)
8. (4, 375) is a local (relative) maximum; (14, 150) is a local (relative) minimum; (29, 550) is a local (relative) maximum. The price of the stock increases until day 4, as shown by turning point (4, 375); then the price decreases until day 14, as shown by turning point (14, 150); next, the price increases until day 29, as shown by turning point (29, 550); after this point, the price decreases.
9. It appears as though $P(t) \to -\infty$ as $t \to +\infty$ and $P(t) \to -\infty$ as $t \to -\infty$, but this is difficult to tell for sure since the domain is restricted to $0 \le t \le 31$.
10. The least degree is 4; there are 3 turning points, and the degree must be 1 more than the number of turning points.

Practice 3.4 B: Describing End Behavior and Turns, p. U3-102

1. $g(x) \to +\infty$ as $x \to -\infty$ and $g(x) \to -\infty$ as $x \to +\infty$; turning points: 6; real roots: 7
2. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; turning points: 7; real roots: 8
3. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; even-degree; 2 real roots
4. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; odd-degree; 5 real roots
5. Answers may vary. Sample answer:



6. Answers may vary. Sample answer:



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7. (50, 220), (70, 125)
8. (0, 0) is a local (relative) maximum; (50, 220) is a local (relative) maximum; (70, 125) is a local (relative) minimum. The volume increases until 1950, as shown by turning point (50, 220); then the volume decreases until 1970, as shown by turning point (70, 125); next, the volume increases until 2010, as shown by the point (110, 400).
9. It appears as though $V(t) \to +\infty$ as $t \to +\infty$ and $V(t) \to +\infty$ as $t \to -\infty$, but this is difficult to tell for sure since the domain is restricted to $0 \le t \le 110$.
10. The least degree is 3; there are 2 turning points, and the degree must be 1 more than the number of turning points.

Lesson 3.5: The Remainder Theorem (A–APR.2)

Warm-Up 3.5, p. U3-104

1. 464 boxes
2. 4 sweatshirts
3. $\frac{4}{20} = \frac{1}{5}$

Scaffolded Practice 3.5: The Remainder Theorem, p. U3-111

1. $3x + 1 + \frac{8}{x+2}$
2. $2x^3 + 4x^2 + 6x + 2$
3. $x^2 + 2x + 5$
4. $x + 6 - \frac{18}{x-6}$
5. $p(8) = 6$
6. $p(2) = 25$
7. $p(-2) = 34$
8. $p(1) = 4$
9. $k = -12$
10. $k = -3$

Practice 3.5 A: The Remainder Theorem, p. U3-125

1. $x^2 - 4x - 12$
2. $2x^3 - 4x^2 + 2x + 4 - \frac{6}{x+2}$
3. $p(-4) = 34$
4. $p(2) = 52$
5. $x - 6 + \frac{4}{x-1}$
6. $x - 1 + \frac{9}{x+6}$
7. $(3x + 7)$ ft
8. $(7x - 6)$ m
9. $0.5t^2 + 6t$
10. $0.2t^2 + 8t$

Practice 3.5 B: The Remainder Theorem, p. U3-126

1. $x + 10 + \frac{30}{x-2}$
2. $3x^4 - 3x^3 - x^2 + x + 7$
3. $p(-2) = 4$
4. $p(1) = 20$
5. $x - 7 + \frac{11}{x+1}$
6. $2x + 11 + \frac{8}{x-2}$
7. $(4x - 6)$ in
8. $(5x + 9)$ cm
9. $0.5t^2 + 8t$
10. $0.2t^2 + 8t$

Lesson 3.6: Zeros of Polynomial Functions (A–APR.3)

Warm-Up 3.6, p. U3-127

1. $f(x) = -x(x - 12)$
2. $x = 0, x = 12$
3. The rocket lands 12 feet from where it was launched.
4. $0 \le x \le 12$

Scaffolded Practice 3.6: Zeros of Polynomial Functions, p. U3-134

1. –2, 3, and 4
2. $-3, \frac{1}{2}$, and 1
3. –2, –1, and 6
4. –1, 1, and $\pm\sqrt{5}$
5. $\frac{p}{q}: \pm 1, \pm 2, \pm 4,$ and ± 8
6. $\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20, \pm 30,$ and ± 60
7. $\frac{p}{q}: \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{2},$ and $\pm\frac{3}{4}$
8. $\frac{p}{q}: \pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3},$ and $\pm\frac{10}{3}$
9. $f(x) = (x - 3)(x + 2)(4x - 3)$; $f(x) = 2(x - 3)(x + 2)(4x - 3)$
10. $f(x) = (2x + 1)(x - 1)(x - 4)(x - 4)$; $f(x) = 3(2x + 1)(x - 1)(x - 4)(x - 4)$

Practice 3.6 A: Zeros of Polynomial Functions, p. U3-153

1. $x = -3, -\frac{1}{2}, 2$
2. Possible solutions: $f(x) = (2x - 1)(x - 3)(x + 2)(x + 2)$; $f(x) = 3(2x - 1)(x - 3)(x + 2)(x + 2)$
3. ±1, ±2, ±5, ±10
4. This function has either 2 or 0 real positive zeros, and 2 or 0 real negative zeros.
5. $f(x) = (x - 1)(x - 1)(x + 2)(x + 5)$
6. The zeros of the function are –5, –2, and 1. The graph crosses the x-axis at –2 and –5. The graph touches the x-axis at 1 but does not cross, because this zero has a multiplicity of 2.
7. $x = -3, -1, \frac{1}{2}$
8. $x = -4, -2, -1, 1, 3$
9. $f(x) = (x + 2)(x + 7)(x - 3)$; zeros: –7, –2, and 3
10. $2 \times 10 \times 17$ inches

Practice 3.6 B: Zeros of Polynomial Functions, p. U3-155

1. $x=-1, \frac{2}{3}, 7$
2. Possible solutions: $f(x) = (5x - 3)(x - 1)(x + 4)(x + 4)$, $f(x) = -(5x - 3)(x - 1)(x + 4)(x + 4)$
3. ±1, ±2, ±4, ±5, ±10, ±20, ±25, ±50, ±100
4. This function has either 4, 2, or 0 real positive zeros, and it doesn't have any real negative zeros.
5. $f(x) = (x - 1)(x - 4)(x - 5)(x - 5)$
6. The zeros of the function are 1, 4, and 5. The graph crosses the x-axis at 1 and 4. The graph touches the x-axis at 5 but does not cross, because this zero has a multiplicity of 2.
7. $x=-\frac{3}{2}$, 1, 3
8. $x = -3, -2, -1, 1$
9. $f(x) = (3x + 1)(x - 3)(x - 3)(x - 1)$; zeros: $-\frac{1}{3}$, 3, and 1
10. $2 \times 4 \times 12$ inches

Lesson 3.7: Building Polynomial Functions (F–BF.1a★)

Warm-Up 3.7, p. U3-157

1. The roots of the graph are (1, 0) and (5, 0).
2. Two factors of the equation are $(x - 1)$ and $(x - 5)$.
3. $y=-\frac{1}{4}x^2+\frac{3}{2}x-\frac{5}{4}$

Scaffolded Practice 3.7: Building Polynomial Functions, p. U3-164

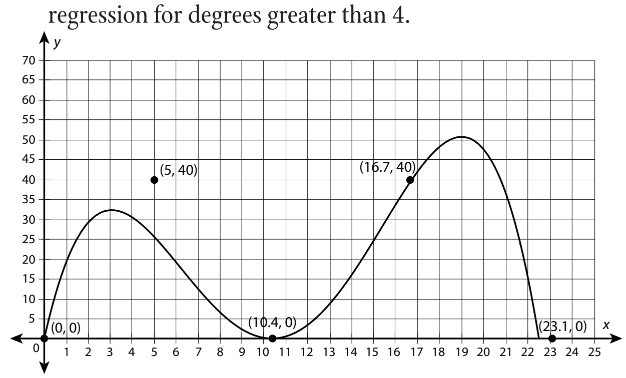
1. degree of 3
2. degree of 3
3. degree of 4
4. degree of 3
5. degree of 4
6. $y = x^3 - 4x^2 - 7x + 10$
7. $y = x^3 + 3x^2 + 2x$
8. $y = 3x^3 + 12x^2 - 3x - 12$
9. $y = 2x^3 + 2x^2 - 4x$
10. $y = 3x^3 - 6x^2 - 9x$

Practice 3.7 A: Building Polynomial Functions, p. U3-179

1. degree = 3; decreasing
2. degree = 4; increasing
3. $y=\frac{10}{9}x^3-\frac{10}{9}x^2-10x+10$
4. $y=-\frac{8}{75}x^3+\frac{4}{3}x^2+\frac{4}{3}x-40$
5. (–2, 0), (1, 0), (4, 0), and (0, –16)
6. $(x + 2)(x - 1)(x - 4)$
7. $y = -2x^3 + 6x^2 + 12x - 16$
8. (0, 0), (2, 0), (8, 0), and (10, 0)
9. $y=-\frac{1}{32}x^4+\frac{5}{8}x^3-\frac{29}{8}x^2+5x$
10. –9.281°F

Practice 3.7 B: Building Polynomial Functions, p. U3-182

1. degree = 3; increasing
2. degree = 4; increasing
3. $y = -0.62x^4 - 0.31x^3 + 15.81x^2 - 3.1x - 62$
4. $y=\frac{1}{8}x^3-\frac{1}{2}x^2-4x$
5. (–2, 0), (1, 0), (3, 0), and (0, –12)
6. $(x + 2)$, $(x - 1)$, $(x - 1)$, and $(x - 3)$
7. $y = 2x^4 - 6x^3 - 6x^2 + 22x - 12$
8. real-world domain: $0 \le x \le 23.1$; real-world range: $0 \le y \le 40$; degree = 4 because the function can't be negative on the real-world domain.
9. $y = -0.012x^4 + 0.547x^3 - 7.665x^2 + 34.166x$
10. The graph doesn't model the situation accurately. While the graph does pass close to 4 of the points, it implies that the sprinter ran just over 30 meters in the first run and 50 meters in the second run, whereas we know that the sprinter ran 40 meters each time. The equation maybe could be improved by adding more roots to the polynomial, but the calculator can't handle polynomial regression for degrees greater than 4.



Lesson 3.8: End Behaviors of Functions (F–LE.3★, F–IF.9)

Warm-Up 3.8, p. U3-185

1. After 1 year, there will be \$3,090 in the Uptown Bank account and \$3,100 in the Downtown Bank account.
2. After 3 years, there will be \$3,278.18 in the Uptown Bank account and \$3,300 in the Downtown Bank account.
3. After 5 years, there will be \$3,477.82 in the Uptown Bank account and \$3,500 in the Downtown Bank account.
4. Eventually, the function $f(x)$ will increase so rapidly that it will surpass $g(x)$; at this point, the Uptown Bank account will have more money in it.

Scaffolded Practice 3.8: End Behaviors of Functions, p. U3-196

1. As $x \to \infty, f(x) \to \infty$; as $x \to -\infty, f(x) \to \infty$
2. As $x \to \infty, f(x) \to -\infty$; as $x \to -\infty, f(x) \to -\infty$
3. As $x \to \infty, f(x) \to \infty$; as $x \to -\infty, f(x) \to -\infty$
4. As $x \to \infty, f(x) \to -\infty$; as $x \to -\infty, f(x) \to \infty$
5. As $x \to \infty, f(x) \to \infty$; as $x \to -\infty, f(x) \to c$
6. As $x \to \infty, f(x) \to -\infty$; as $x \to -\infty, f(x) \to c$

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7. As $x \to \infty$, $f(x) \to c$; as $x \to -\infty$, $f(x) \to \infty$
8. As $x \to \infty$, $f(x) \to c$; as $x \to -\infty$, $f(x) \to -\infty$
9. compare the end behaviors and growth rates
10. rate of change

Practice 3.8 A: End Behaviors of Functions, p. U3-220

1. As $x \to +\infty$, $f(x) \to +\infty$, $g(x) \to -\infty$, and $f(x) > g(x)$.
   As $x \to -\infty$, $f(x) \to +\infty$, $g(x) \to +\infty$, and $f(x) > g(x)$.
2. As $x \to +\infty$, $f(x) \to +\infty$, $g(x) \to +\infty$, and $f(x) > g(x)$.
   As $x \to -\infty$, $f(x) \to 0$, $g(x) \to -\infty$, and $f(x) > g(x)$.
3. As $x \to +\infty$, $f(x) \to -\infty$, $g(x) \to -\infty$, and $f(x) < g(x)$.
   As $x \to -\infty$, $f(x) \to -\infty$, $g(x) \to +\infty$, and $f(x) < g(x)$.
4. As $x \to +\infty$, $f(x) \to 0$, $g(x) \to +\infty$, and $f(x) < g(x)$.
   As $x \to -\infty$, $f(x) \to +\infty$, $g(x) \to +\infty$, and $f(x) > g(x)$.
5. As $t \to +\infty$, $f(t) \to +\infty$ and $g(t) \to +\infty$.
6.

Interval	[0, 6]	[6, 12]	[12, 18]	[18, 24]	[24, 30]
$\frac{\Delta 10\,f(t)}{\Delta t}$	4690	$1.79 \bullet 10^4$	$6.83 \bullet 10^4$	$2.60 \bullet 10^5$	$9.93 \bullet 10^5$
$\frac{\Delta g(t)}{\Delta t}$	$2.36 \bullet 10^4$	$7.08 \bullet 10^4$	$1.18 \bullet 10^5$	$1.65 \bullet 10^5$	$2.12 \bullet 10^5$

7. No, the bacteria culture will not die in less than a day. It dies after 25 hours.
8. As $y \to +\infty$, $f(y) \to +\infty$ and $g(y) \to +\infty$.
9.

Interval	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]
$\frac{\Delta f(y)}{\Delta y}$	2	32	62	92	122
$\frac{\Delta g(y)}{\Delta y}$	0.5	31.5	62.5	93.5	124.5

10. The bush will need to be pruned after 8 years, because the ratio will be less than 1 starting at $y = 8.5$.

Practice 3.8 B: End Behaviors of Functions, p. U3-222

1. As $x \to +\infty$, $f(x) \to +\infty$, $g(x) \to +\infty$, and $f(x) > g(x)$.
   As $x \to -\infty$, $f(x) \to +\infty$, $g(x) \to -\infty$, and $f(x) > g(x)$.
2. As $x \to +\infty$, $f(x) \to 0$, $f(x) \to +\infty$, and $f(x) < g(x)$.
   As $x \to -\infty$, $f(x) \to +\infty$, $g(x) \to +\infty$, and $f(x) > g(x)$.
3. As $x \to +\infty$, $f(x) \to -\infty$, $g(x) \to -\infty$, and $f(x) > g(x)$.
   As $x \to -\infty$, $f(x) \to -\infty$, $g(x) \to +\infty$, and $f(x) < g(x)$.
4. As $x \to +\infty$, $f(x) \to 0$, $g(x) \to +\infty$, and $f(x) < g(x)$.
   As $x \to -\infty$, $f(x) \to -\infty$, $g(x) \to -\infty$, and $f(x) < g(x)$.
5. As $t \to +\infty$, $f(t) \to +\infty$ and $g(t) \to +\infty$.
6.

Interval	[0, 5]	[5, 10]	[10, 15]	[15, 20]	[20, 25]
$\frac{\Delta 10\ f(t)}{\Delta t}$	9550	$5.51 \bullet 10^4$	$3.18 \bullet 10^5$	$1.84 \bullet 10^6$	$1.06 \bullet 10^7$
$\frac{\Delta g(t)}{\Delta t}$	$6.68 \bullet 10^4$	$2.00 \bullet 10^5$	$3.34 \bullet 10^5$	$4.67 \bullet 10^5$	$6.01 \bullet 10^4$

7. Yes, the bacteria culture will die in less than 20 hours. It dies after 17 hours.
8. As $y \to +\infty$, $f(y) \to +\infty$ and $g(y) \to +\infty$.
9.

Interval	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]
$\frac{\Delta f(y)}{\Delta y}$	18	58	98	138	178
$t \to +\infty$	15	51	87	123	159

10. Yes, pruning practices should be changed because the ratio of berries to vines will exceed 1 after 6 years. Practices should be changed before then.

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Instruction

Goal: To guide students to facility with evaluating, factoring, and graphing polynomials, both by hand and with technology

North Carolina Math 3 Standards

N–CN.9 Use the Fundamental Theorem of Algebra to determine the number and potential types of solutions for polynomial functions.

A–APR.2 Understand and apply the Remainder Theorem.

Student Activities Overview and Answer Key

Station 1

Working with groups, students use the Rational Root Theorem, the Fundamental Theorem of Algebra, and the Remainder Theorem to evaluate polynomials and find their factors.

Answers

1. a. 5
   b. no
   c. $\pm\dfrac{1,7}{1,3}$
   d. –5
2. a. 4
   b. $\pm\dfrac{1}{1,3}$
   c. no
   d. no
   e. no
3. a. 3
   b. $\pm\dfrac{1,2,5,10}{1,2}$
   c. no
   d. –290
4. a. 6
   b. ±1
   c. –2
   d. 6

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Station 2

Working with groups, students use technology to evaluate polynomials.

Answers

1. –12
2. 59,452
3. –1,354
4. 8
5. 40,905
6. 55,017
7. 36

Station 3

Working with groups, students work to factor polynomial expressions and functions, including polynomials with radical and complex roots.

Answers

1. a. 7
   b. $g(1) = 6$
   c. $g(-1) = 2$
2. a. 6
   b. $f(6) = 131{,}509$
   c. $f(1) = 3$
3. $(x-9)(x+6)(x-2)(x+1)$
4. $(x-2)(x+2)(x+3)(x-1)(x-4)$
5. $\left(x-\sqrt{2}\right)\left(x-4\right)(3x+2)$
6. $-3\sqrt{5}, -\dfrac{1}{4}, 2$
7. $1+2i, 3$
8. $2-2i, 2, -\dfrac{1}{2}$

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Instruction

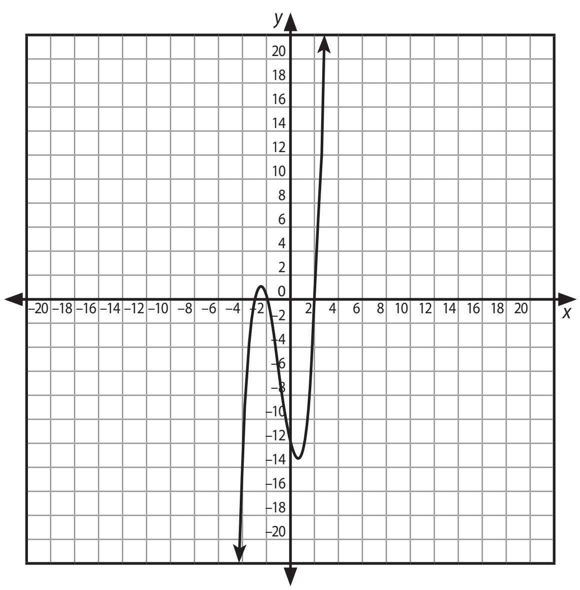
Station 4

Working with groups, students factor polynomial functions, find the zeros, and use technology to graph the functions. Students also use the Rational Root Theorem to find possible roots.

Answers

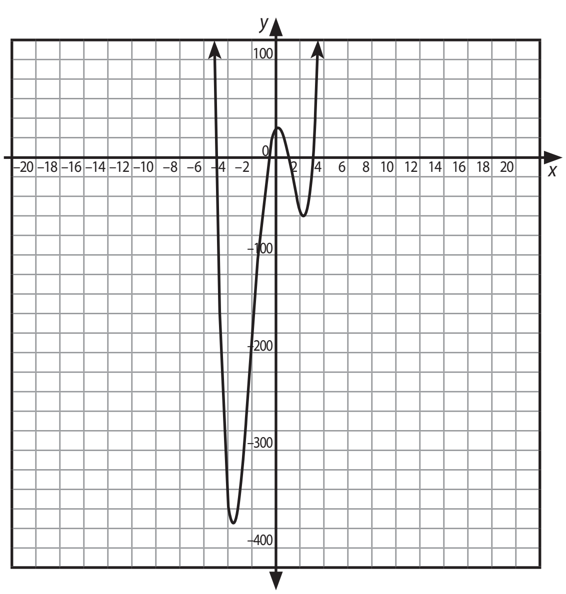
1. a. $f(x) = (x - 2)(x + 3)(x + 2)$; zeros are –3, –2, and 2

   b.



2. a. $f(x) = (x + 5)(x - 1)(3x + 2)(x - 3)$; zeros are –5, $-\frac{2}{3}$, 1, and 3

   b.



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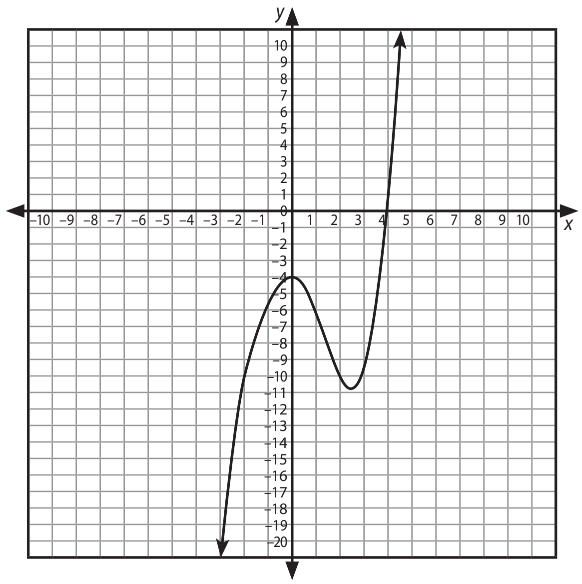
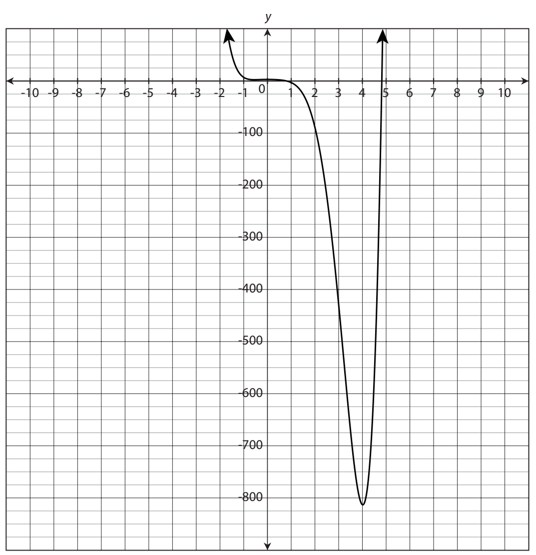
3. a. $(x + i)(x - i)(x - 4)$, zeros are $-i$, i, and 4

 b. 

4. a. ±1, ±3; there are no rational zeros for this function, only irrational and complex.

 b. 

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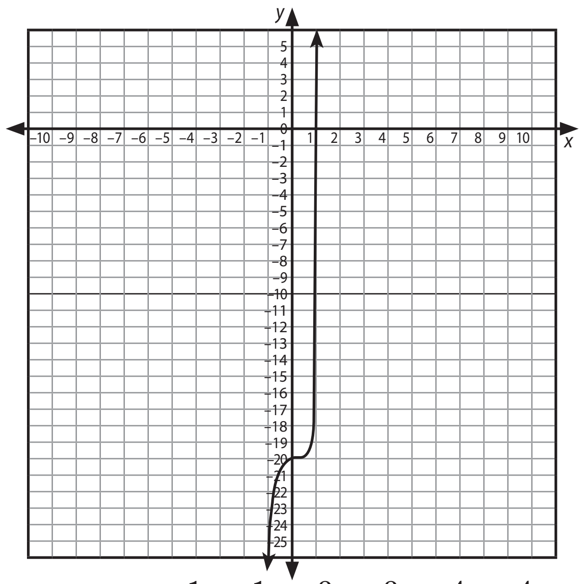
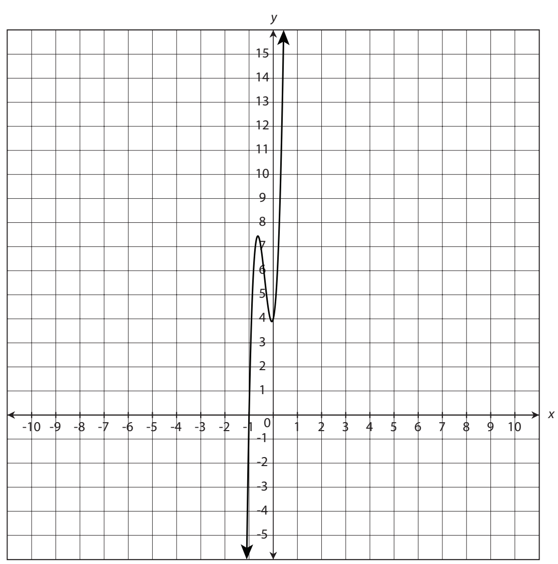
5. a. $\pm1, \pm2, \pm4, \pm5, \pm10, \pm20, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm\frac{1}{10}$; $x = 1$. The other zeros are complex.

b.



6. a. $\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{1}{9}, \pm\frac{2}{3}, \pm\frac{2}{9}, \pm\frac{4}{3}, \pm\frac{4}{9}$; $x = -1, \pm\frac{1}{3}i, \pm2i$

b.



Materials List/Setup

Station 1 calculator

Station 2 calculator

Station 3 calculator

Station 4 graphing calculator

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Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is the degree of a polynomial?
2. What is the Fundamental Theorem of Algebra?
3. What is a complex conjugate? What is a radical conjugate?
4. How are conjugates useful when you are factoring?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. The degree of a polynomial is the highest power to which x is raised.
2. Any polynomial of degree x^n will have n roots.
3. If you have a complex number in the form $a + bi$, the complex conjugate is $a - bi$. If you have a radical in the form $a + b\sqrt{c}$, the radical conjugate is $a - b\sqrt{c}$.
4. Complex conjugates allow you to treat $x^2 - n$ as the difference of two squares, even if n is not a perfect square. Additionally, if any polynomial equation has a complex or radical root, the root's conjugate will also be a root.

Possible Misunderstandings/Mistakes

- Incorrectly calculating the value of functions
- Incorrectly manipulating exponents
- Incorrectly factoring
- Not understanding the Rational Root Theorem
- Not extending the Rational Root Theorem far enough (i.e., not listing all possible factors of the constant and coefficient)

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- Not recognizing redundant factors in the list of possibilities from the Rational Root Theorem
- Incorrectly using graphing technology
- Attempting to find zeros from the polynomial's graph
- Not understanding or misusing polynomial long division and synthetic long division
- Not understanding the Remainder Theorem
- Making arithmetical errors

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Station 1

Work with your group to answer the questions about each polynomial function.

1. $f(x) = 3x^5 + x^3 - 2x^2 + 6x + 7$
   a. How many zeros does this function have?
   b. Is 3 a zero of this function?
   c. What are all possible rational zeros?
   d. What is $f(-1)$?

2. $f(x) = 3x^4 + 2x + 1$
   a. How many zeros does this function have?
   b. What are all possible rational zeros?
   c. Is $\frac{1}{3}$ a zero of this function?
   d. Is $-\frac{1}{3}$ a zero of this function?
   e. Is -1 a zero of this function?

3. $g(x) = -2x^3 - 3x^2 + 5x + 10$
   a. How many zeros does this function have?
   b. What are all possible rational zeros?
   c. Is 1 a zero of this function?
   d. What is $g(5)$?

4. $g(x) = x^6 + 4x^5 + 2x^4 - 1$
   a. How many zeros does this function have?
   b. What are all possible rational zeros?
   c. What is $g(-1)$?
   d. What is $g(1)$?

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Station 2

Work with your group to evaluate each polynomial function for the given value. Use a calculator to compute values.

1. $g(x) = -2x^3 + 2x^2 - 7x + 10$

   $g(2) =$

2. $f(x) = 4x^7 - 6x^5 + x^4 - 3x^3 - 4$

   $f(4) =$

3. $f(x) = -2x^8 - 7x^7 + x^6 - 2x^5 + x^4 + 5x^3 - x$

   $f(2) =$

4. $g(x) = x^4 - 5x + 2$

   $g(2) =$

5. $f(x) = 3x^6 - 2x^5 + x^4 - 2x^3 - 4x^2 + 5$

   $f(5) =$

6. $g(x) = 18x^5 - 10x^3 + 2x^2 - 6x - 3$

   $g(5) =$

7. $h(x) = 4x^4 - 3x^3 + 2x^2 - x - 10$

   $h(2) =$

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Station 3

Work with your group to answer the questions about each polynomial function. Show all your work.

1. $g(x) = 4x^7 - 2x + 4$
   a. How many zeros does the function have?
   b. What is $g(1)$?
   c. What is $g(-1)$?

2. $f(x) = 2x^6 + 5x^5 - 3x^3 - x^2 - x + 1$
   a. How many zeros does the function have?
   b. What is $f(6)$?
   c. What is $f(1)$?

Factor the following polynomials. Use any method.

3. $x^4 - 4x^3 - 53x^2 + 60x + 108$

4. $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48$

5. If $\left(x + \sqrt{2}\right)$ is a factor of $3x^4 - 10x^3 - 14x^2 + 20x + 16$, what are the other factors?

6. If $3\sqrt{5}$ is a zero of $f(x) = 4x^4 - 7x^3 - 182x^2 + 315x + 90$, what are the other zeros?

7. If $(1 - 2i)$ is a zero of $f(x) = x^3 - 5x^2 - 11x - 15$, what are the other zeros?

8. If $(2 + 2i)$ is a zero of $f(x) = 2x^4 - 11x^3 + 26x^2 - 16x - 16$, what are the other zeros?

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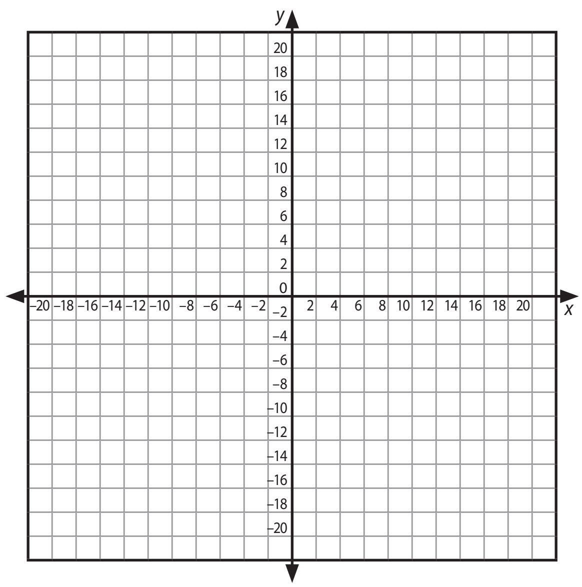
Station 4

Work with your group to answer the questions about each polynomial function. Then use a graphing calculator to find the graph of the function. Sketch the graphs.

1. $f(x) = x^3 + 3x^2 - 4x - 12$

   a. Factor the polynomial to find the zeros of the function.

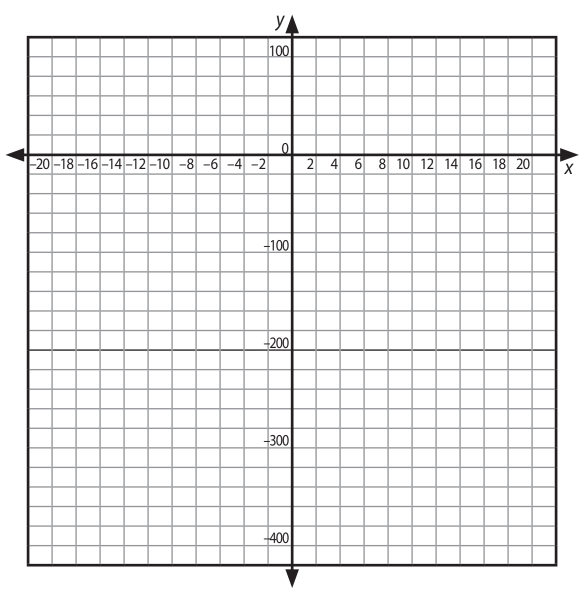
   b.



2. $f(x) = 3x^4 + 5x^3 - 49x^2 + 11x + 30$

   a. Factor the polynomial to find the zeros of the function.

   b.



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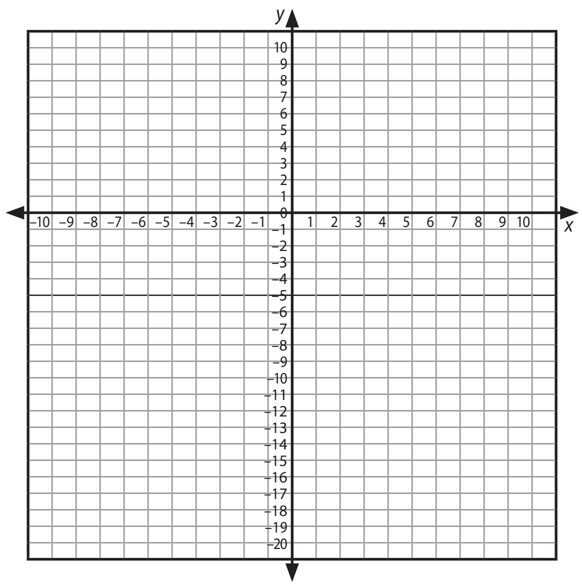
3. $f(x) = x^3 - 4x^2 + x - 4$

   a. Factor the polynomial to find the zeros of the function given that one of the zeros is $-i$.

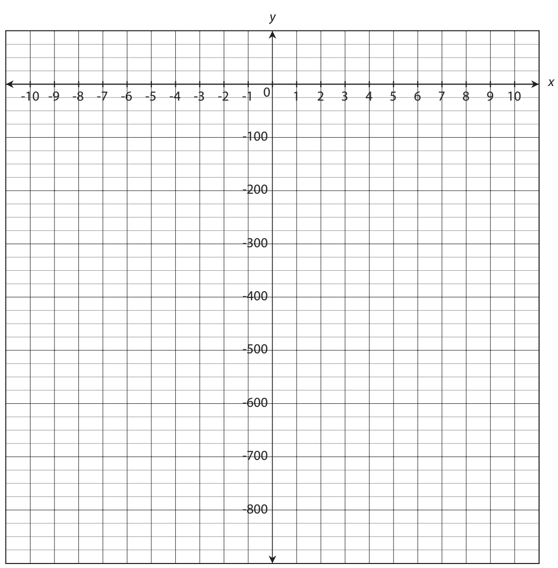
   b.



4. $f(x) = x^6 - 5x^5 + x^4 - 3x^2 + 3$

   a. State all possible rational zeros and find all rational zeros.

   b.



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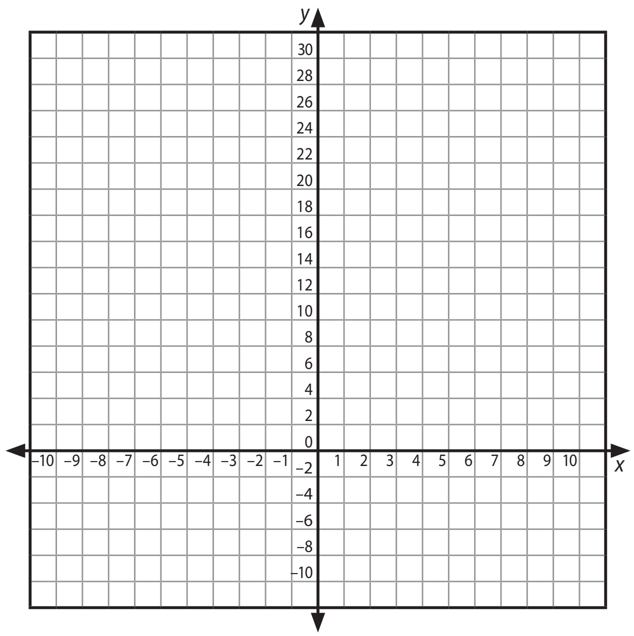
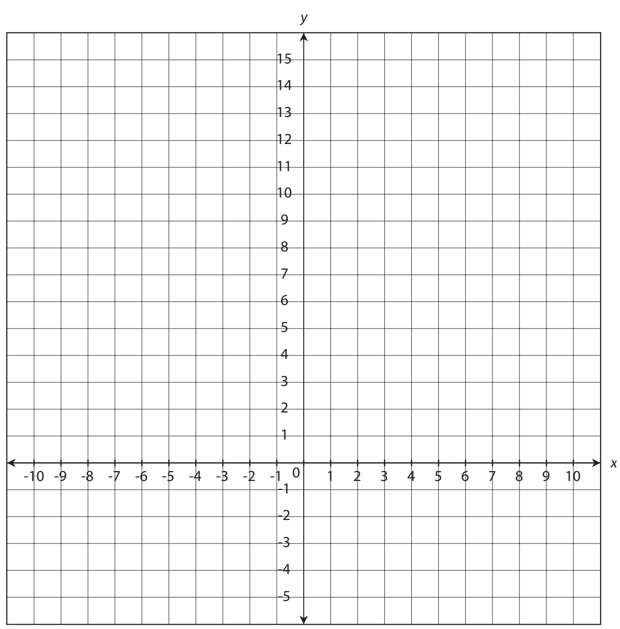
5. $f(x) = 10x^5 + 19x^3 - 10x^2 + x - 20$

a. State all possible rational zeros and find all rational zeros.

b. 

6. $f(x) = 9x^5 + 9x^4 + 37x^3 + 37x^2 + 4x + 4$

a. State all possible rational zeros and find ALL zeros. (*Hint*: Use the quadratic formula after finding a rational zero.)

b. 

Assessment

Unit 3 Mid-Unit Assessment

Circle the letter of the best answer.

1. What is the degree of the polynomial $5x^6 - 2x^4 + x^3 + 10x$?

   a. 1
   b. 5
   c. 6
   d. 10

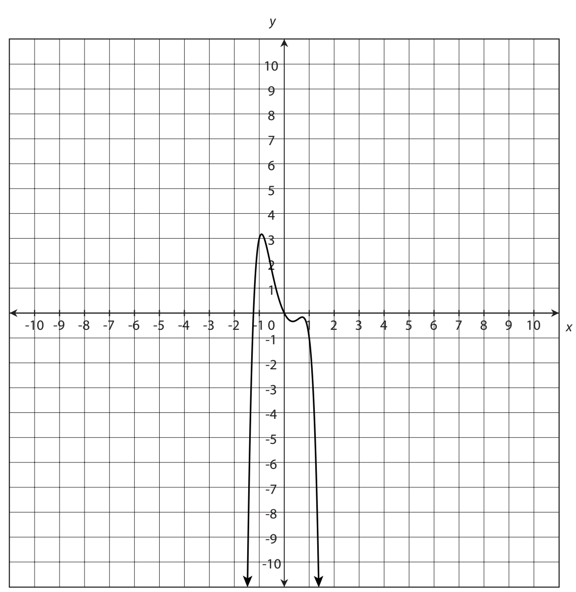
2. What is the result of $(4x^5 - 2x + 8) + (-7x^2 + 9x + 3)$?

   a. $11x^2 + 11x + 11$
   b. $3x^2 + 7x + 11$
   c. $-3x^5 + 7x + 11$
   d. $4x^5 - 7x^2 + 7x + 11$

3. What is the result of $(-x^6 + 10x^3 + 4x^2 + 22) - (5x^6 - 2x^2 + 16)$?

   a. $-6x^6 + 10x^3 + 6x^2 + 6$
   b. $4x^6 + 10x^3 + 2x^2 + 6$
   c. $5x^6 + 12x^3 + 4x^2 + 6$
   d. $4x^6 + 10x^3 + 2x^6 + 38$

4. What must be true about the degree and leading coefficient of the graphed polynomial?



   a. The polynomial is an odd-degree polynomial and has a positive coefficient.
   b. The polynomial is an odd-degree polynomial and has a negative coefficient.
   c. The polynomial is an even-degree polynomial and has a positive coefficient.
   d. The polynomial is an even-degree polynomial and has a negative coefficient.

continued

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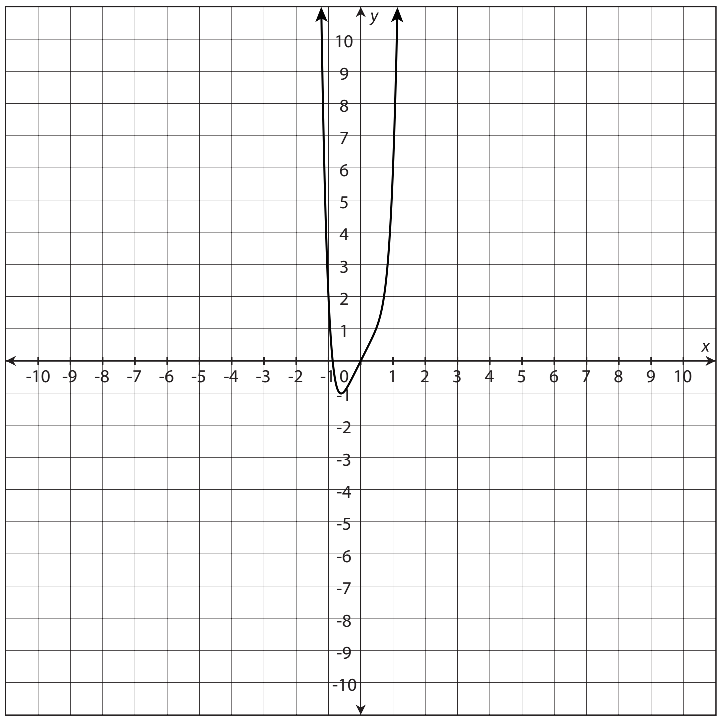
5. A polynomial has zeros of 2, 7, and 3. What is the minimum degree of the polynomial?

   a. 2
   c. 5
   b. 4
   d. 3

6. Which function could possibly represent the graphed function?



   a. $f(x) = -4x^6 - 2x$
   c. $f(x) = 4x^6 + 2x$
   b. $f(x) = 4x^5 + 8x$
   d. $f(x) = -4x^5 + 3x$

7. A canning company wants to transition from cylindrical tins to tins that are rectangular prisms. The cylindrical tins are 4 centimeters tall and 9 centimeters in diameter. The new tins are to be 3 centimeters tall and 12 centimeters long, and should hold the same volume. How wide should the tins be?

   a. about 3 cm
   c. about 7 cm
   b. about 6 cm
   d. about 28 cm

8. Delilah intends to make a cardboard box by cutting identical squares out of the corners of a piece of cardboard. The cardboard measures 12 inches by 20 inches. Which expression could represent the volume of the box if the length of a side of the square cut out is x inches?

   a. $240x^2$
   c. $x(12 - x)(20 - x)$
   b. $x^2(12 - x)(20 - x)$
   d. $x(12 - 2x)(20 - 2x)$

continued

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9. A company makes fancy boxes in the shape of rectangular prisms. The material to make the sides and bottom of each box costs \$0.12 per square centimeter, and the material used to make the lid costs \$0.04 per square centimeter. The volume of each box must be 1,200 cm^3, and the base must have a 2 : 3 side ratio. What dimensions will minimize the cost of the box?

   a. base length = 10 cm, base width = 5 cm, height = 24 cm

   b. base length = 10 cm, base width = 15 cm, height = 8 cm

   c. base length = 5 cm, base width = 5 cm, height = 48 cm

   d. base length = 5 cm, base width = 15 cm, height = 16 cm

Use what you have learned about polynomial functions to answer the following questions.

10. A third-degree polynomial with rational coefficients has the roots 3 and $(2 + i)$.

   a. What are the additional roots?

   b. What is this polynomial?

   c. Identify the end behavior and maximum number of turning points of the graph of the polynomial.

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Instruction

Unit 3 Mid-Unit Assessment Answer Key

Multiple Choice

Answer	Standard(s)
1. c	A–SSE.1a★
2. d	A–SSE.1a★
3. a	A–SSE.1a★
4. d	F–IF.7★
5. d	F–IF.7★, N–CN.9
6. c	F–IF.7★
7. c	G–MG.1★
8. d	G–MG.1★
9. b	G–MG.1★

Extended Response

Answer	Standard(s)
10. a. $2 - i$ b. $x^3 - 7x^2 + 17x - 15$ c. $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$; turning points: 2	F–IF.7★, N–CN.9

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Unit 3 End-of-Unit Assessment

Circle the letter of the best answer.

1. What is the result of $\left(-10x^4+3x^3-4\right)+\left(8x^4+2x^3+x\right)$?

   a. $-2x^4+5x^3-4x$

   b. $18x^4+5x^3+3x$

   c. $-2x^4+5x^3+x-4$

   d. $-10x^4+6x^3+x-4$

2. What is the result of $\left(12x^3+7x^2-1\right)-\left(-x^3+8x^2+x\right)$?

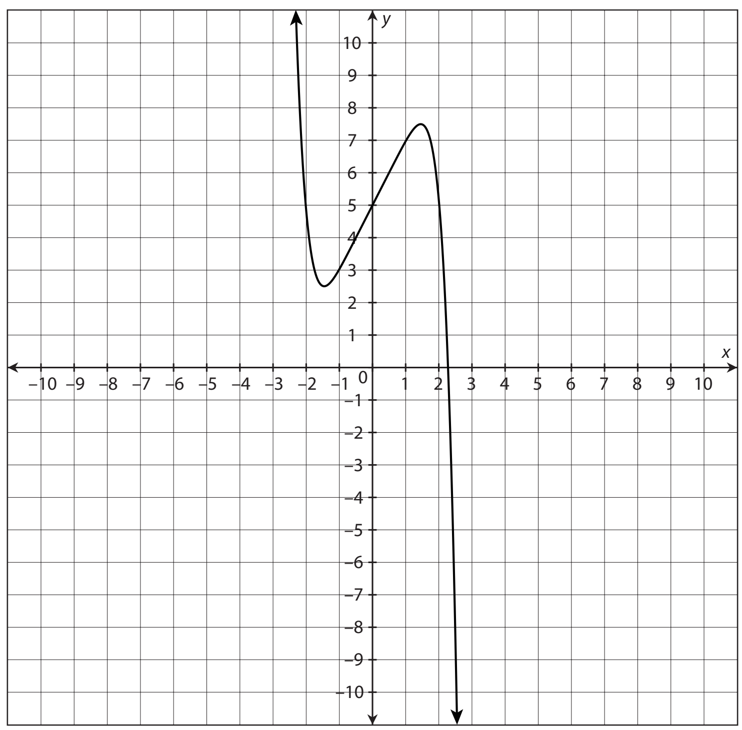
   a. $11x^3-x^2-x-1$

   b. $13x^3-x^2-x-1$

   c. $11x^3+15x^2+x-1$

   d. $13x^3-x^2-x$

3. What must be true about the degree and the leading coefficient of the graphed polynomial?



   a. The polynomial is an odd-degree polynomial and has a positive coefficient.

   b. The polynomial is an odd-degree polynomial and has a negative coefficient.

   c. The polynomial is an even-degree polynomial and has a positive coefficient.

   d. The polynomial is an even-degree polynomial and has a negative coefficient.

continued

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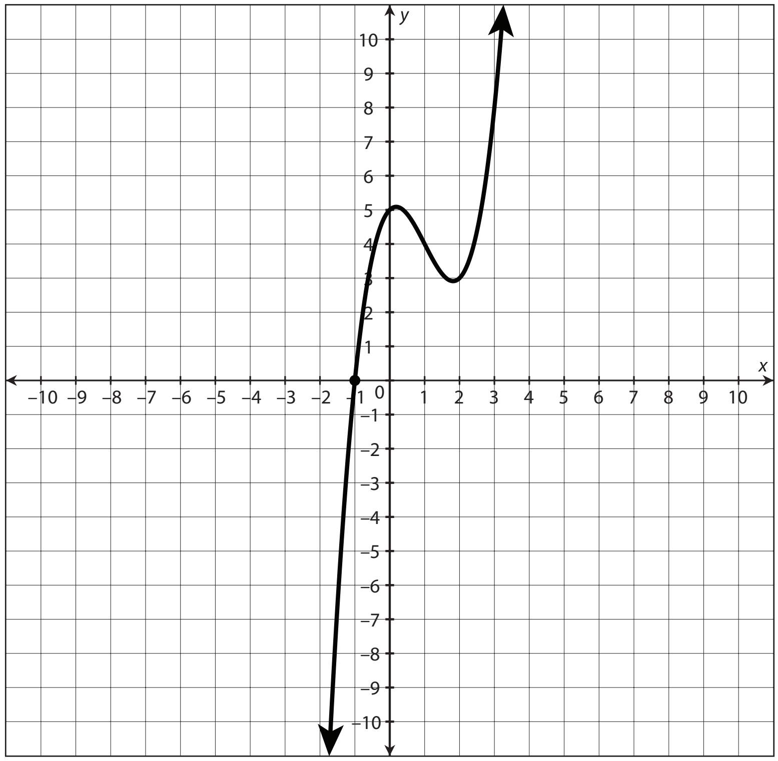
4. Which equation could possibly represent the graphed function?



a. $f(x) = x^3 - 3x^2 + x + 5$

b. $f(x) = -x^3 + 3x^2 - x - 5$

c. $f(x) = x^4 + x + 5$

d. $f(x) = -x^4 + 5$

5. What is the remainder when $(x^3 - 4x^2 + 10)$ is divided by $(x - 3)$?

a. 1

b. –53

c. 31

d. 7

6. A polynomial with rational coefficients has zeros of 3, 5, and $2i$. What is the minimum degree of the polynomial?

a. 3

b. 4

c. 5

d. 6

continued

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7. What are all the possible rational roots for the polynomial equation $3x^3 + 8x^2 - 4x - 12 = 0$?

   a. $\pm\frac{1}{12}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm1, \pm\frac{3}{2}, \pm3$

   b. $\pm\frac{1}{3}, \pm1, \pm4, \pm12$

   c. $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm1, \pm\frac{4}{3}, \pm2, \pm3, \pm4, \pm6, \pm12$

   d. $\pm\frac{1}{12}, \pm\frac{1}{4}, \pm1, \pm3$

8. Which polynomial has zeros at –3, 1, and 2, and passes through the point (3, 3)?

   a. $f(x) = (x + 3)(x - 1)(x - 2)$

   b. $f(x) = 0.25(x + 3)(x - 1)(x - 2)$

   c. $f(x) = -(x + 3)(x - 1)(x - 2)$

   d. $f(x) = -4(x + 3)(x - 1)(x - 2)$

9. Which polynomial is represented in the table?

x	–5	–1	0	2	4
$f(x)$	0	0	0	0	–9

   a. $f(x) = x(x + 5)(x + 1)(x - 2)(x - 4)$

   b. $f(x) = 0.025x(x + 5)(x + 1)(x - 2)$

   c. $f(x) = -0.025x(x + 5)(x + 1)(x - 2)$

   d. $f(x) = -x(x + 5)(x + 1)(x - 2)(x - 4)$

10. Let $f(x) = 4^x + 6$ and $g(x) = -3x^5 + 8$. Which statement best describes their end behaviors as $x \to -\infty$?

   a. As $x \to -\infty$, $f(x) \to 6$ and $g(x) \to -\infty$; $f(x)$ will eventually exceed $g(x)$.

   b. As $x \to -\infty$, $f(x) \to 4$ and $g(x) \to -\infty$; $f(x)$ is always greater than $g(x)$.

   c. As $x \to -\infty$, $f(x) \to 6$ and $g(x) \to \infty$; $g(x)$ will eventually exceed $f(x)$.

   d. As $x \to -\infty$, $f(x) \to 4$ and $g(x) \to \infty$; $g(x)$ is always greater than $f(x)$.

continued

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11. Let $f(x) = x^5 - 10$ and $g(x) = x^6 + 10$. Which statement best describes their end behaviors as $x \to \infty$?

   a. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$; $g(x)$ will eventually exceed $f(x)$.

   b. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$; $f(x)$ is always greater than $g(x)$.

   c. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$; $g(x)$ is always greater than $f(x)$.

   d. As $x \to \infty$, $f(x) \to -\infty$ and $g(x) \to \infty$; $g(x)$ is always greater than $f(x)$.

12. Roberto is making an open-topped slab-box out of clay. The surface area of the box must be 350 cm^2. The base of the box will be a right isosceles triangle. If x represents the length of one leg of the box's base, which equation could represent the volume of the box?

   a. $V = \dfrac{350x}{6} - \dfrac{x^3}{12}$

   b. $V = \dfrac{175x}{4} - \dfrac{x^3}{16}$

   c. $V = 350\left(2 - \sqrt{2}\right)x - \dfrac{\left(2 - \sqrt{2}\right)x^3}{4}$

   d. $V = \dfrac{175\left(2 - \sqrt{2}\right)x}{2} - \dfrac{\left(2 - \sqrt{2}\right)x^3}{8}$

13. A yogurt company is designing a container. The volume of the container must be 330 cm^3. The container will be a cylinder with an open top, which will be covered with foil. What are the dimensions of the container that minimize the surface area?

   a. radius ≈ 5.94 cm, height ≈ 6.30 cm

   b. radius ≈ 4.72 cm, height ≈ 9.43 cm

   c. radius = height ≈ 4.72 cm

   d. radius = height ≈ 5.94 cm

continued

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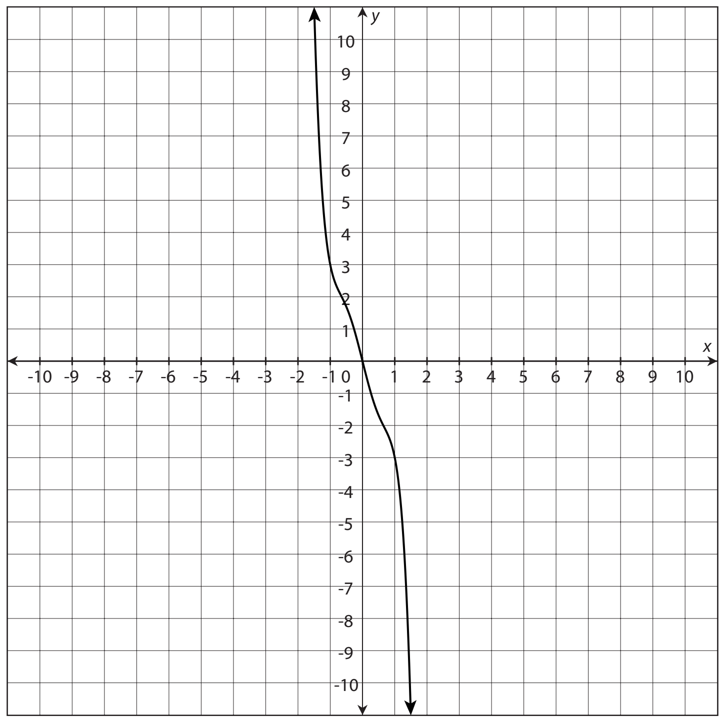
14. What must be true about the degree and leading coefficient of the graphed polynomial?



a. The polynomial is an odd-degree function and has a positive coefficient.

b. The polynomial is an odd-degree function and has a negative coefficient.

c. The polynomial is an even-degree function and has a positive coefficient.

d. The polynomial is an even-degree function and has a negative coefficient.

15. Which of the following is the result of using synthetic division to divide $(x^3 - 4x^2 + 9)$ by $(x - 1)$?

a. $x^2 - 3x - 3 + \dfrac{6}{x+1}$

b. $x - 3 + \dfrac{6}{x-1}$

c. $x^2 - 5x + 5 + \dfrac{4}{x-1}$

d. $x^2 - 3x - 3 + \dfrac{6}{x-1}$

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Instruction

Unit 3 End-of-Unit Assessment Answer Key

Multiple Choice

Answer	Standard(s)
1. c	A–SSE.1a★
2. b	A–SSE.1a★
3. b	F–IF.7★
4. a	F–IF.7★
5. a	A–APR.2
6. b	A–APR.3
7. c	A–APR.3
8. b	F–BF.1a★
9. c	F–BF.1a★
10. c	F–LE.3★
11. c	F–LE.3★
12. d	G–MG.1★
13. c	G–MG.1★
14. b	F–IF.7★
15. d	A–APR.2

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