Notes in

Structural Reliability Theory

And Risk Analysis

Content:

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Note 1: INTRODUCTION TO RISK ANALYSIS

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1 Introduction

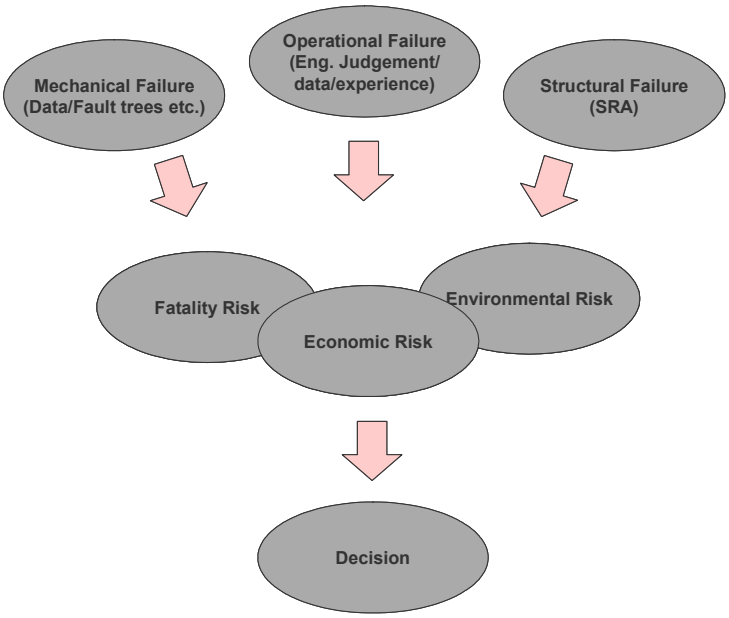


Figure 1. Elements of decision making for civil engineering structures.

Sustainable development related to conservation of the environment, the welfare and safety of the people have been subject to increasing concern of the society during the last decades. At the same time optimal allocations of available natural and financial resources are considered very important. Therefore methods of risk and reliability analysis in civil engineering developed during the last decades are

becoming more and more important as decision support tools in civil engineering applications. The decision process is illustrated in figure 1.

Civil engineering facilities such as bridges, buildings, power plants, dams and offshore platforms are all intended to contribute to the benefit and quality of life. Therefore when such facilities are planned it is important that the benefit of the facility can be identified considering all phases of the life of the facility, i.e. including design, manufacturing, construction, operation and eventually decommissioning.

Benefit has different meanings for different people in the society, simply because different people have different preferences. However, benefit for the society can be understood as

- economically efficient for a specific purpose
- fulfil given requirements with regard to safety of people directly or indirectly involved with and exposed to the facility
- fulfil given requirements to the effects of the facility on the community and environment

Taking into account these requirements it is seen that the task of the engineer is to make decisions or to provide the decision basis for others such that it may be ensured that engineering facilities are established, operated, maintained and decommissioned in such a way that they will optimise or enhance the possible benefits to society and individuals of society.

Activity	Approximate death rate ($\times 10^{-9}$ deaths/h exposure)	Typical exposure (h/year)	Typical risk of death ($\times 10^{-6}$/year)
Alpine climbing	30000 – 40000	50	1500-2000
Boating	1500	80	120
Swimming	3500	50	170
Cigarette smoking	2500	400	1000
Air travel	1200	20	24
Car travel	700	300	200
Train travel	80	200	15
Coal mining (UK)	210	1500	300
Construction work	70-200	2200	150-440
Manufacturing	20	2000	40
Building fires	1-3	8000	8-24
Structural failures	0.02	6000	0.1

Table 1. Some risks in society (from Melchers [1]).

For many years it has been assumed in design of structural systems that all loads and strengths are deterministic. The strength of an element was determined in such a way that it exceeded the load with a certain margin. The ratio between the strength and the load was denoted the safety factor. This number was considered as a measure of the reliability of the structure. In codes of practice for structural systems values for loads, strengths and safety factors are prescribed.

As described above structural analysis and design have traditionally been based on deterministic methods. However, uncertainties in the loads, strengths and in the modeling of the systems require that methods based on probabilistic techniques in a number of situations have to be used. A structure is usually required to have a satisfactory performance in the expected lifetime, i.e. it is required that it does

not collapse or becomes unsafe and that it fulfills certain functional requirements. Generally structural systems have a rather small probability that they do not function as intended, see table 1.

This note gives an introduction to the different aspect of risk analysis and risk acceptance criteria and is partly based on the JCSS working paper, Faber & Stewart [2].

2 Definition of Risk

Risk is here defined as the expected consequences associated with a given activity. Considering an activity with only one event with potential consequences risk R is thus the probability that this event will occur P multiplied with the consequences given the event occurs C i.e.

$$R = P \cdot C \tag{1}$$

For an activity with n events the risk is defined by

$$R = \sum_{i=1}^{n} P_i \cdot C_i \tag{2}$$

where P_i and C_i are the probability and consequence of event *i*.

This definition is consistent with the interpretation of risk used e.g. in the insurance industry (expected losses) and risk may e.g. be given in terms of DKKs, dollars, number of human fatalities, etc.

3 Framework for Risk Analysis

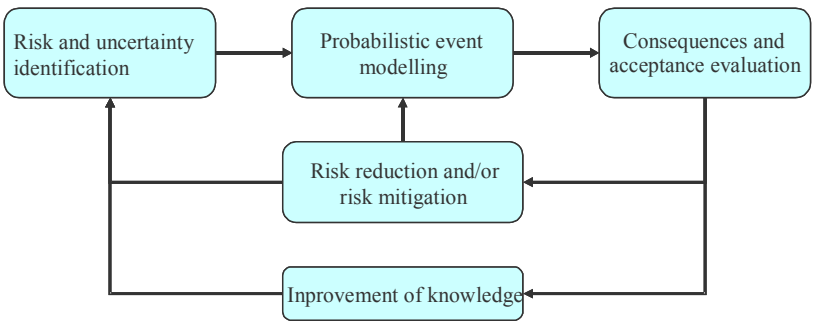


Figure 2. Principal flow diagram of risk assessment.

Risk assessment is used in a number of situations with the general intention to indicate that important aspects of uncertainties, probabilities and/or frequencies and consequences have been considered in some way or other. Decision theory provides a theoretical framework for such analyses, see Figure 2.

In typical decision problems encountered the information basis is often not very precise. In many situations it is necessary to use historical and historical data. The available historical information is often not directly related to the problem considered but to a somewhat similar situation. Furthermore, an important part of a risk assessment is to evaluate the effect of additional information, risk reducing measures and/or changes of the considered problem. It is therefore necessary that the framework for the decision analysis can take these types of information into account and allow decisions to be updated

based upon new information. This is possible if the framework of Bayesian decision theory is used, see e.g. Raiffa and Schlaifer [3] and Benjamin and Cornell [4].

A fundamental principle in decision theory is that optimal decisions must be identified as those resulting in the highest expected utility, see e.g. Ditlevsen and Madsen [5]. In typical engineering applications the utility may be related to consequences in terms of costs, fatalities, environmental impact etc. In these cases the optimal decisions are those resulting in the lowest expected costs, the lowest expected number of fatalities and so on.

4 Implementation of Risk Analysis

Risk analyses can be presented in a format, which is almost independent from the application. Figure 3 shows a general scheme for risk analysis, see Stewart and Melchers [6]. Maybe the most important step in the process of a risk analysis is to identify the context of the decision problem, i.e. the relation between the considered engineering system and/or activity and the analyst performing the analysis:

- Who are the decision maker(s) and the parties with interests in the activity (e.g. society, client(s), state and organizations).
- Which matters might have a negative influence on the impact of the risk analysis and its results.
- What might influence the manner in which the risk analysis is performed (e.g. political, legal, social, financial and cultural).

Furthermore the important step of setting the acceptance criteria must be performed. This includes the specification of the accepted risks in regard to economic, public or personnel safety and environmental criteria. In setting the acceptable risks – which might be considered a decision problem itself, due account should be taken to both international and national regulations in the considered application area. However, for risk analysis performed for decision making in the private or inter-company sphere with no potential consequences for third parties the criteria may be established without the consideration of such regulations. In these cases the issue of risk acceptance is reduced to a pure matter of cost or resource optimisation involving the preferences of the decision maker alone. Risk criteria are discussed in section 6.

System Definition

The system (or the activity) considered has to be described and all assumptions regarding the system representation and idealizations stated.

Identification of Hazard Scenarios

The next step is to analyse the system with respect to how the system might fail or result in other undesirable consequences. Three steps are usually distinguished in this analysis, namely the 1) Decomposition of the system into a number of components and/or sub-systems. 2) Identification of possible states of failure for the considered system and sub-systems – i.e. the hazards associated with the system. 3) Identification of how the hazards might be realized for the considered system and subsystems, i.e. the identification of the scenarios of failure events of components and subsystems which if they occur will lead to system failure.

A hazard is typically referred to as a failure event for the considered system or activity. Occurrence of a hazard is therefore also referred to as a system failure event. System failures may thus represent events such as collapse of a building structure, flooding of a construction site or explosion in a road or rail

tunnel. Identification of hazards is concerned about the identification of all events, which might have an adverse consequence to

- ¸ People
- Environment
- ¸ Economy

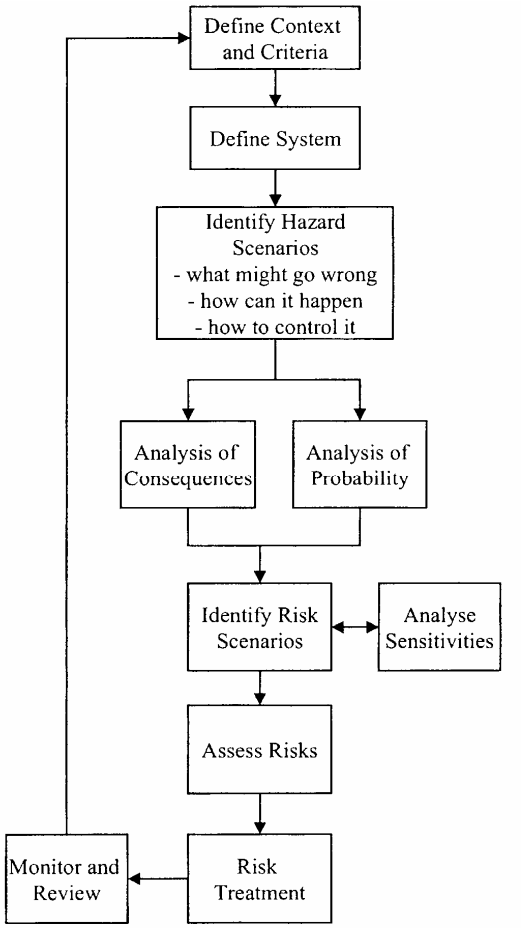


Figure 3. General scheme for risk-based decision analysis.

Different techniques for hazard identification have developed from various engineering application areas such as the chemical, nuclear power and aeronautical industries. Examples are:

- ¸ Preliminary Hazard Analysis (PHA)
- ¸ Failure Mode and Effect Analysis (FMEA)
- ¸ Failure Mode Effect and Criticality Analysis (FMECA)
- ¸ Hazard and Operability Studies (HAZOP)
- ¸ Risk Screening (Hazid sessions)

Analysis of Consequences

Typical consequences are economic consequences, loss of life and effects on the environment. The estimation of consequences given failure of the system of sub-systems requires a good understanding of the system and its interrelation with its surroundings and is thus best performed in collaboration with experts who have hands on experience with the considered type of activity.

Analysis of Probability
Evaluation of probabilities of failure for the individual components and sub-systems may be based on, in principle, two different approaches: failure rates for e.g. electrical and production systems or methods for structural reliability for structural systems as buildings and bridges.

Risk analyses are typically made on the basis of information, which is subject to uncertainty. These uncertainties may be divided in

- inherent or natural variability, e.g. the yield strength of steel.
- modelling uncertainty: (i) uncertainty related to the influence of parameters not included in the model, or (ii) uncertainty related to the mathematical model used.
- statistical uncertainty.

Identify Risk Scenarios
When consequences and probabilities are identified the risk can then be computed. Hazard scenarios, which dominate the risk may then be identified. The risk scenarios can bee ranked in accordance with the risk contribution.

Analyse Sensitivities
The sensitivity analysis is useful for analysis of the identified risk scenarios and normally includes an identification of the most important factors influencing the risks associated with the different risk scenarios. Also the sensitivity analysis may include studies of "what if" situations for the evaluation of the importance of various system simplifications performed under the definition of the system.

Risk Treatment
Calculated risks are compared with the accepted risks initially stated in the risk acceptance criteria. Should the risks not be acceptable in accordance with the specified risk acceptance criteria there are principally four different ways to proceed.

Risk mitigation: Risk mitigation is implemented by modification of the system such that the source of risk is removed. For example, the risk of fatalities from a ship collision with a bridge may be mitigated by traffic lights stopping traffic proceeding onto the bridge whenever a ship navigates under the bridge.

Risk reduction: Risk reduction may be implemented by reduction of the consequences and/or the probability of occurrence – in practice risk reduction is normally performed by a physical modification of the considered system.

Risk transfer: Risk transfer may be performed by e.g. insurance or other financial arrangements where a third party takes over the risk.

Risk acceptance: If the risks do not comply with the risk acceptance criteria and other approaches for risk treatment are not effective then risk acceptance may be an option.

Monitoring and Review
Risk analyses may be performed as already stated for a number of decision support purposes. For many engineering applications such as cost control during large construction projects and inspection and maintenance planning for bridge structures the risk analysis is a process where there is constant feed-

back of information from the system. Whenever new information is provided the risk analysis may be updated.

5 QRA methods

Quantitative Risk Analysis (QRA) is used in assessment of the risks. Three calculation methods are:

- Event Tree Analysis (ETA)
- Fault Tree Analysis (FTA)
- Risk matrix

These methods are described in the following partly based on [7].

5.1 Event Tree Analysis

An Event Tree is used to develop the consequences of an event. This method has been used for the first time in the Wash 14 "Rasmussen Report" in 1974 in order to represent the nuclear accident scenarios at the whole plant level and to gather the results obtained by using Fault Trees at emergency system level.

An Event Tree is constructed by defining an initial event and the possible consequences that result from this, when the emergency systems function or not. The initial event is usually placed on the left and branches are drawn to the right, each branch representing a different sequence of events and terminating in an outcome. The main elements of the tree are event definitions and branch points, or logic vertices.

The initial event is usually expressed as a frequency (events/year) and the subsequent splits as probabilities (events/demand), so that the final outcomes are also expressed as frequencies (event/year). Each branch of the Event Tree represents a particular scenario. An example of a simple Event Tree is shown in figure 4 and 5. The fire protection is provided by a sprinkler system. A detector will either detect the rise in temperature or it will not. If the detector succeeds the control box will either work correctly or it will not - and so on. There is only one branch in the tree that indicates that all the subsystems have succeeded:

Example: *Fire protection system*

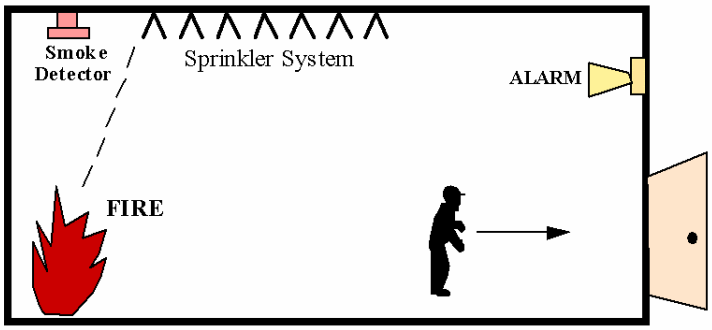


Fig 1

Figure 4. Example of a Fire Scenario.

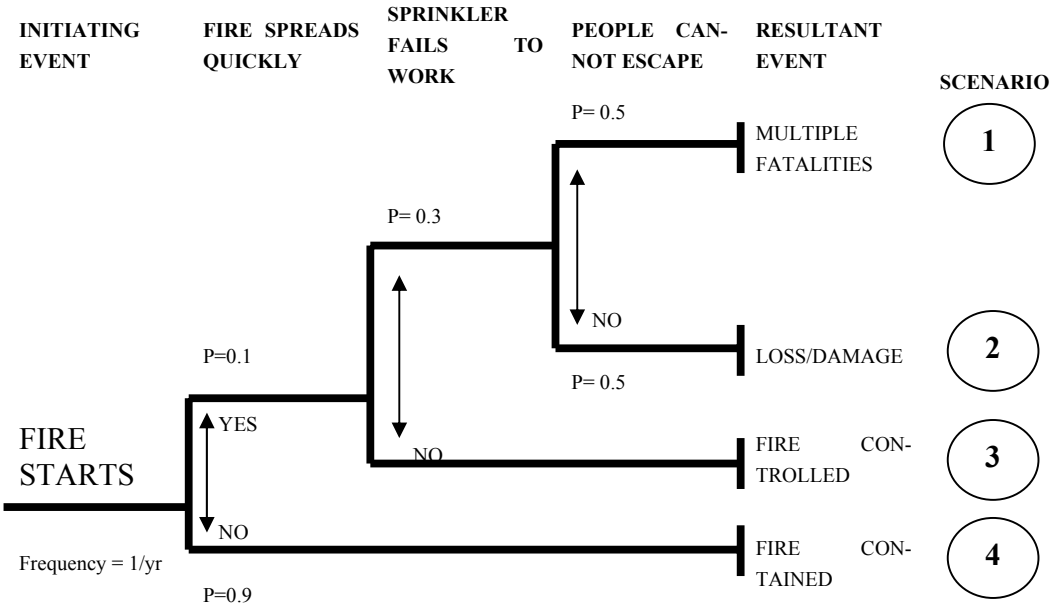


Figure 5. Event Tree Analysis for Building Protected by a Sprinkler System.

The results of the Event Tree are outcome event frequencies (probabilities) per year. The outcome frequencies may be processed further to obtain the following results:

Risk to Workforce:
- Annual risk
- Individual risk
- Fatal Accident Rate

Risk to Public:
Physical effects:
- Contours on site map
- Transects on site map

Individual risk:
- Contours on site map
- Transects on site map
- Annual risk at fixed location

Societal risk:
- FN table
- FN curve
- Equivalent annual fatalities

Risk to Workforce

Annual Risk

The annual risk may be expressed as Potential Loss of Life (*PLL*), where the *PLL* expresses the probability of fatalities per year for all the operation personnel. As such the *PLL* is a risk indicator which is valid for the whole installation, rather than for an individual. The calculation for a given event *i* is of the form

$$PLL_i = f_{0i} \cdot N_{bi} \cdot P_{fi} \qquad (3)$$

where

PLL_i	potential loss of life caused by event *i,* (1/year)
f_{0i}	outcome frequency of event *i* per year, (1/year)
N_{bi}	number of personnel exposed to the effects of event *i*
P_{fi}	probability of fatalities associated with event *i*

An example of *PLL* results is given in figure 6.

For all the outcome events

$$PLL = \sum PLL_i \tag{4}$$

Individual Risk

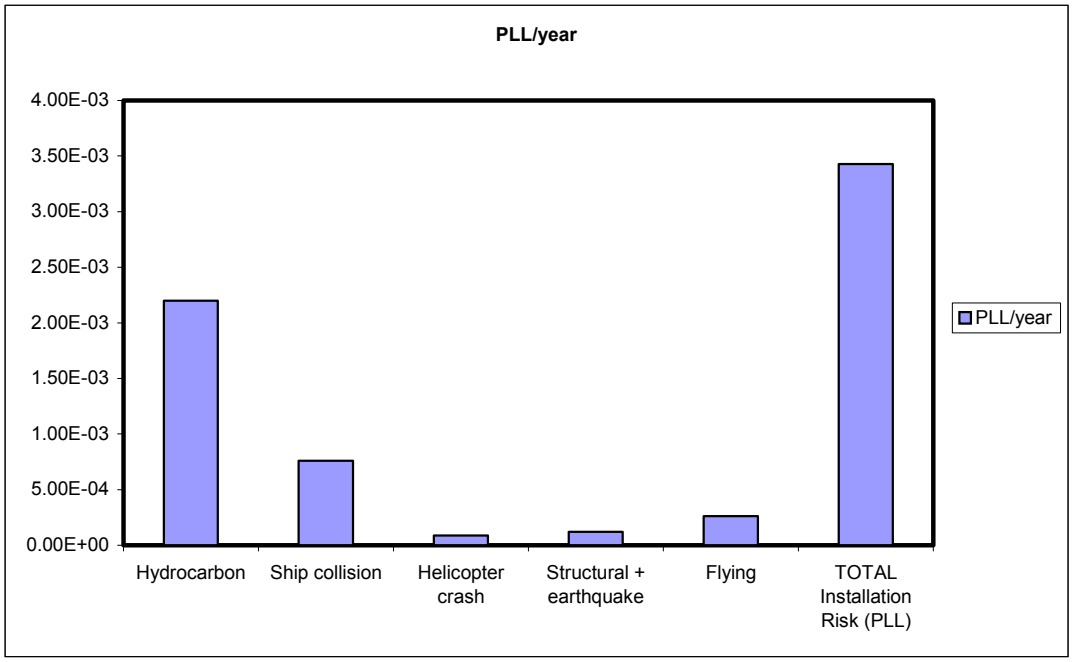
The Individual Risk (*IR*) expresses the probability per year of fatality for one individual. It is also termed as Individual Risk Per Annum (*IRPA*). The *IR* depends on the location of the individual at a given time and its contents of work. In practice, for the operating personnel of an installation an Average Individual Risk, *AIR* may be estimated for groups of persons taking into account the percentage of time of exposure to the hazard per year. For all the personnel involved in the annual operation of the installation, the AIR may be derived from the *PLL*

$$AIR = \frac{PLL}{N_b \cdot Q} \tag{5}$$

where

PLL	potential loss of life for the installation, (1/year)
N_b	number of personnel involved per year, (1/year)
Q	is average percentage of exposure

Scenario	PLL/year	%
Hydrocarbon	2.20E-03	64
Ship collision	7.60E-04	22
Helicopter crash	8.90E-05	3
Structural + earthquake	1.20E-04	3
Flying	2.60E-04	8
TOTAL Installation Risk (PLL)	3.43E-03	100



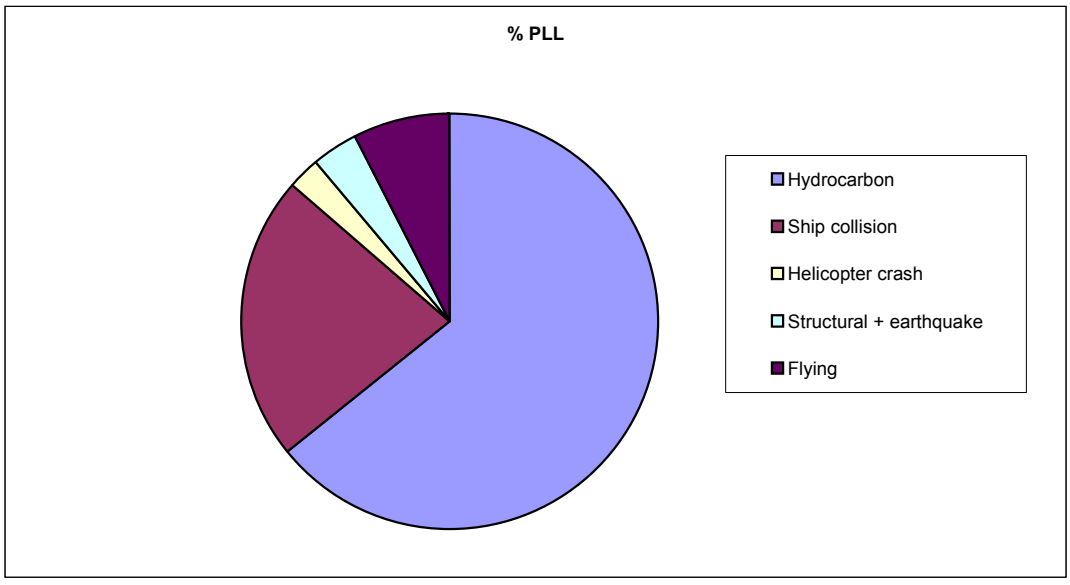


Figure 6. Example of Presentation of PLL Results.

Fatal Accident Rate

The Fatal Accident Rate (*FAR*) is defined as the potential number of fatalities in a group of people exposed for a specific exposure time to the activity in question. Generally, the *FAR* is expressed as a probability of fatality per 100 million exposure hours for a given activity. It is mainly used for comparing the fatality risk of activities. The 100 million exposure hours is to represent the number of hours at work in 1000 working lifetimes.

A location (area) specific *FAR* may be calculated as

$$FAR_{area} = \frac{PLL_{area} \cdot 10^8}{N_{area} \cdot 8760} \tag{6}$$

where

FAR_{area}	area specific Fatal Accident Rate, (1/year)
PLL_{area}	Potential Loss of Life in an area per year, (1/year)
N_{area}	manning in an area calculated as an average value over a typical year of operation
8760	number of hours per year

The area FAR is thus independent of manning level.

The installation *FAR* value is calculated as an average value over a year of operation as

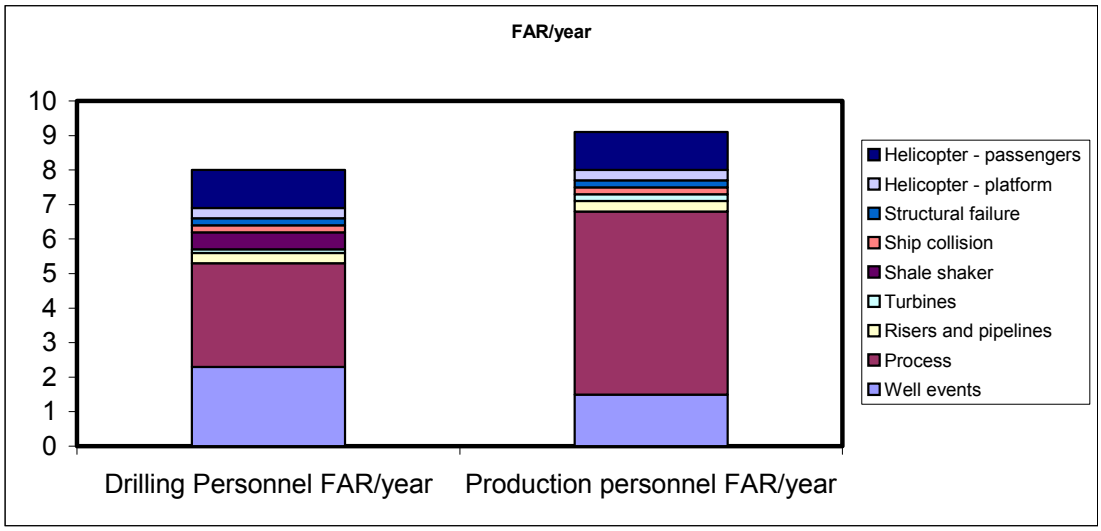
$$FAR_{installation} = \frac{PLL_{tot} \cdot 10^8}{N_{tot} \cdot 8760} \tag{7}$$

where

$FAR_{installation}$	whole installation *FAR*, (1/year)
PLL_{tot}	total *PLL* for the installation, (1/year)
N_{tot}	number of personnel on the installation

An example of the presentation of FAR results is shown in figure 7.

Scenario	Drilling Personnel FAR/year	Production personnel FAR/year
Well events	2.3	1.5
Process	3	5.3
Risers and pipelines	0.3	0.3
Turbines	0.1	0.2
Shale shaker	0.5	0
Ship collision	0.2	0.2
Structural failure	0.2	0.2
Helicopter - platform	0.3	0.3
Helicopter - passengers	1.1	1.1
TOTAL Installation Risk (FAR)	8	9.1



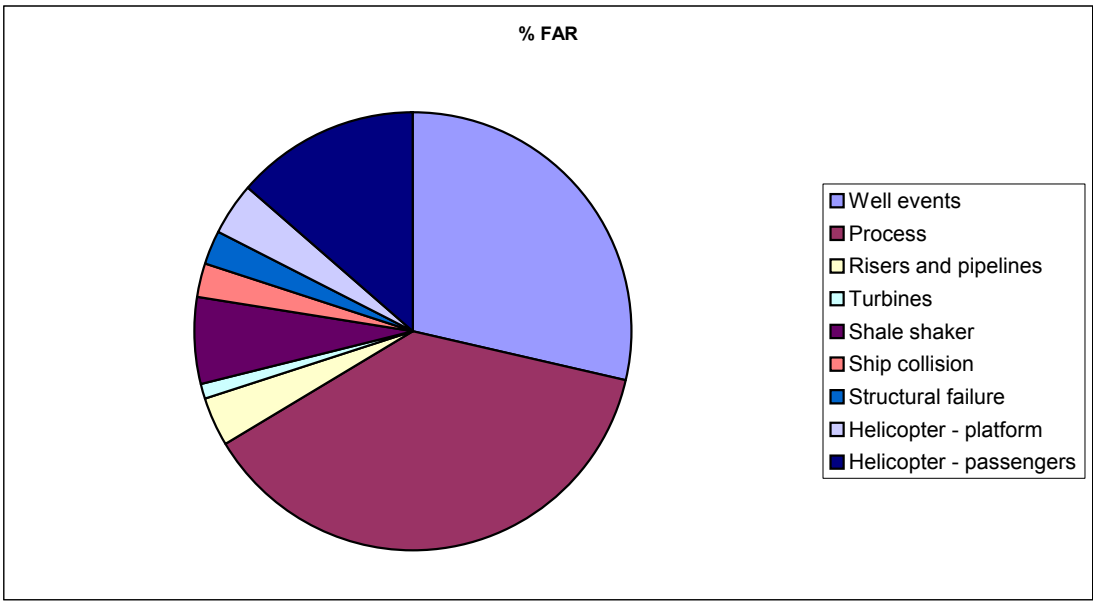


Figure 7. Example of Presentation of FAR Results.

Risk to Public
FN Curves

Scenario	Number (N) of Potential Fatalities	Frequency of Scenario per Year	Frequency of Incidents with Potential (N) or more Fatalities per Year
1	1	0.1	0.12021
2	20	0.014	0.01141
3	70	0.0075	0.00713
4	150	0.00023	0.00022
5	300	0.00009	0.00011
6	500	0.00001	0.00001

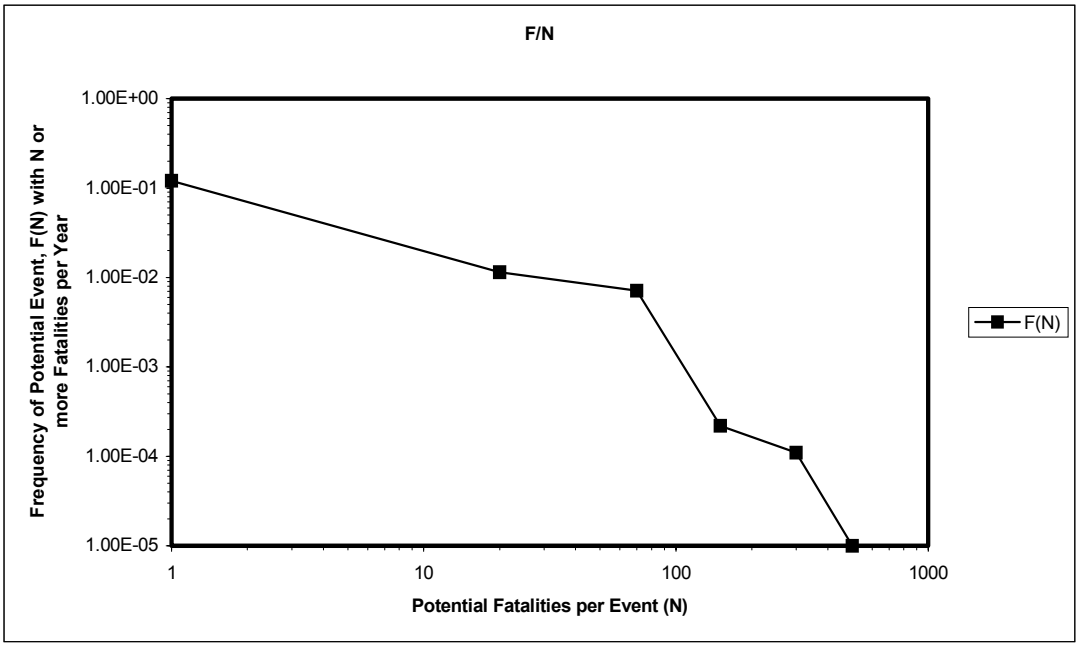


Figure 8. Example of F/N graph.

FN curves or F/N plots (generally also called the “Cumulative Frequency Graphs”) are probability versus consequence diagrams where “F” denotes frequency of a potential event and “N” the number of associated fatalities.

A Cumulative Frequency Graph shows the probability of N or more fatalities occurring. Such graphs tend to be of interest when the risk acceptance criterion selected, or, as is more often the case, imposed by the Regulator, includes an aversion to potential incidents that would result in, say, more than ten fatalities. In simple terms, risk aversion exists if society regards a single accident with 100 fatalities as in some sense worse than 100 accidents (e.g. road accidents) with a single fatality each. An example of a F/N graph is shown in figure 8.

5.2 Fault Tree Analysis

Compared to an Event Tree the "Fault Tree" analysis works in the opposite direction: It is a "deductive" approach, which starts from an effect and aims at identifying its causes. Therefore a Fault Tree is used to develop the causes of an undesirable event. It starts with the event of interest, the top event, such as a hazardous event or equipment failure, and is developed from the top down.

The Fault Tree is both a qualitative and a quantitative technique. Qualitatively it is used to identify the individual scenarios (so called paths or cut sets) that lead to the top (fault) event, while quantitatively it is used to estimate the probability (frequency) of that event.

A component of a Fault Tree has one of two binary states, either in the correct state or in a fault state. In other words, the spectrum of states from total integrity to total failure is reduced to just two states.

The application of a Fault Tree may be illustrated by considering the probability of a crash at a road junction and constructing a tree with AND and OR logic gates (figure 9). The Tree is constructed by deducing in turn the preconditions for the top event and then successively for the next levels of events, until the basic causes are identified.

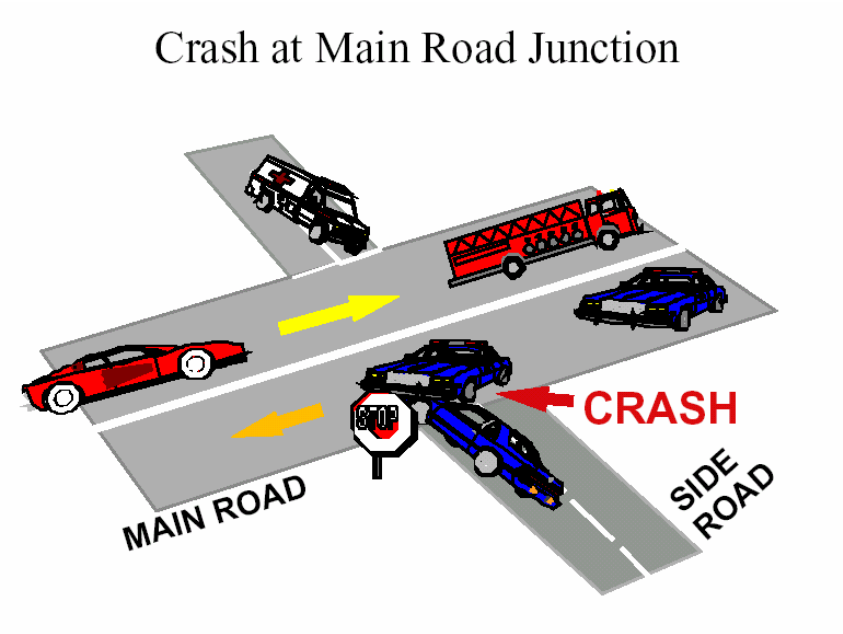


Figure 9. Example of Fault Tree.

Qualitative Analysis
By using the property of the Boolean algebra it is possible first to establish the combinations of basic (components) failures which can lead to the top (undesirable) event when occurring simultaneously. These combinations are so called "minimal cut sets" (or "prime impliquant") and can be derived from the logical equation represented by the Fault Tree.

Considering the Fault Tree representing figure 9, six scenarios can be extracted:

- Driving too fast AND Car at main road junction;
- Driver too ill AND Car at main road junction;
- Vision obscured AND Car at main road junction;
- Road too Slippery AND Car at main road junction;
- Brake failure AND Car at main road junction;
- Tyres worn AND Car at main road junction.

These 6 minimal cut sets are in first approach equivalent. However, a common cause failure analysis could show, for example that the "Road too slippery" increase the probability of "Car at main road junction" because it is too slippery from both side. Therefore the 4th cut set is perhaps more likely than the others.

Semi-Quantitative Analysis

The second step consists of calculating the probability of occurrence of each scenario. By ascribing probabilities to each basic event we obtain the next figures for our example:

- Driving too fast AND Car at main road junction : 0.1 x 0.01 = 10^{-3}
- Driver too ill AND Car at main road junction : 0.01 x 0.01 = 10^{-4}
- Vision obscured AND Car at main road junction : 0.01 x 0.01 = 10^{-4}
- Road too Slippery AND Car at main road junction : 0.01 x 0.01 = 10^{-4}
- Brake failure AND Car at main road junction : 0.001 x 0.01 = 10^{-5}
- Tyres worn AND Car at main road junction : 0.001 x 0.01 = 10^{-5}

Total = 1.32 10^{-3}

Now it is possible to sort the minimal cut sets in a more accurate way i.e. into three classes: One cut set at 10^{-3}, three at 10^{-4} and two at 10^{-5}. Of course, it is better to improve the scenarios with the higher probabilities first if we want to be efficient.

As by-product of this calculation, the global failure probability 1.32 10^{-3} is obtained by a simple sum of all the individual probabilities. But this simple calculation is a conservative approximation, which works well when the probabilities are sufficiently low (in case of safety, for example). It is less accurate when the probabilities increase and it can even exceed 1 when probabilities are very high. This is due to cross terms that are neglected. Therefore, this approach must be used with care.

Quantification in Fault Tree Analysis

As a Fault Tree represents a logical formula it is possible to calculate the probability of the top event by ascribing probabilities to each basic event, and by applying the probability calculation rules. When the events are independent, and when the probabilities are low it is possible to roughly estimate the probability of the output event if an OR gate is the sum of the probabilities of the events in input. An example is given in figure 10.

These simple calculations only work on the basis of the above hypothesis. For example, as soon as the Fault Tree contains repeated events (same events in several location in the Tree) the independence hypothesis is lost. Therefore the calculation becomes wrong and even worse it is impossible to know if the result is optimistic or pessimistic. On the other hand, the estimation of the top event probability is less and less accurate (more and more conservative) when the probabilities increase (even if the events are independent).

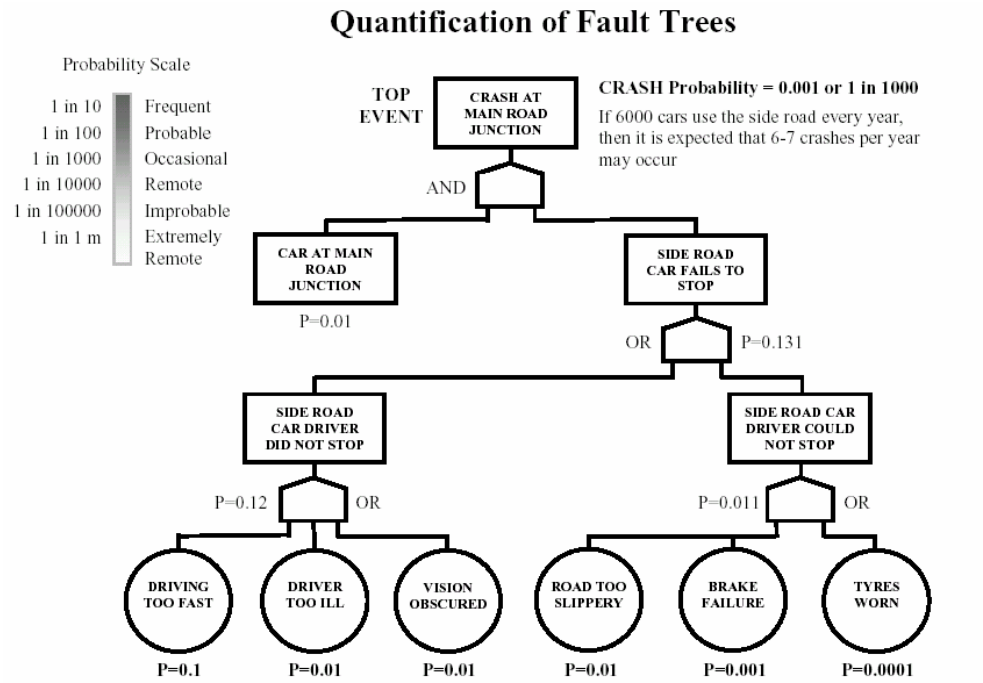


Figure 10.

5.3 Risk Matrix

The arrangement of accident probability and corresponding consequence in a Risk Matrix may be a suitable expression of risk in cases where many accidental events are involved or where single value calculation is difficult. As figure 11 shows the matrix is separated into three regions,-

- unacceptable risk,
- further evaluation or attention is required, and
- acceptable risk.

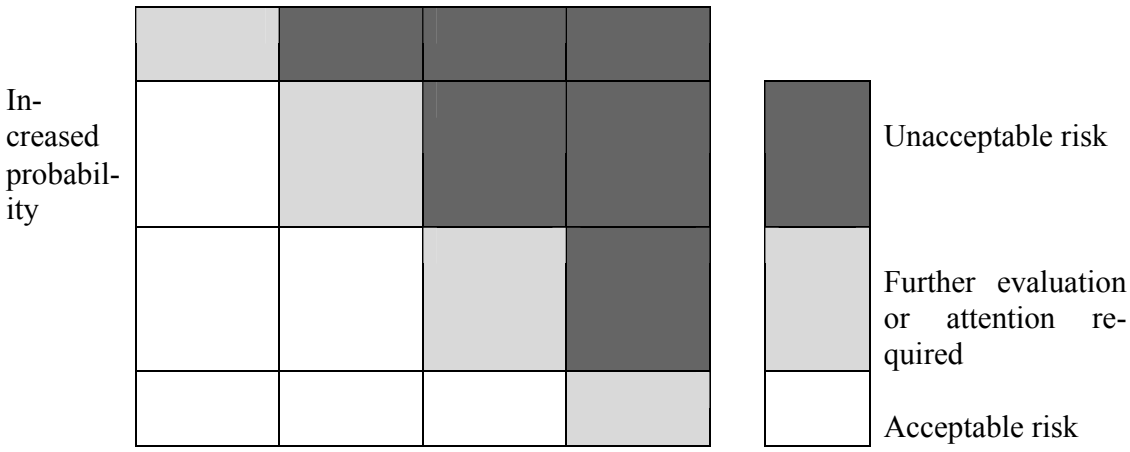


Figure 11. Risk Matrix.

Further evaluations have to be carried out for the region between acceptable and unacceptable risk, to determine whether further risk reduction is required or more studies should be performed.

The limit of acceptability is set by defining the regions in the matrix which represent unacceptable and acceptable risk. The Risk Matrix may be used for qualitative as well as quantitative studies. If probability is classified in broad categories such as "rare" and "frequent" and consequence in "small", "medium", "large" and "catastrophic", the results from a qualitative study may be shown in the Risk Matrix. The definition of the categories is particularly important in case of qualitative use. The categories and the boxes in the Risk Matrix may be replaced by continuous variables, implying a full quantification. An illustration of this is shown in Figure 12.

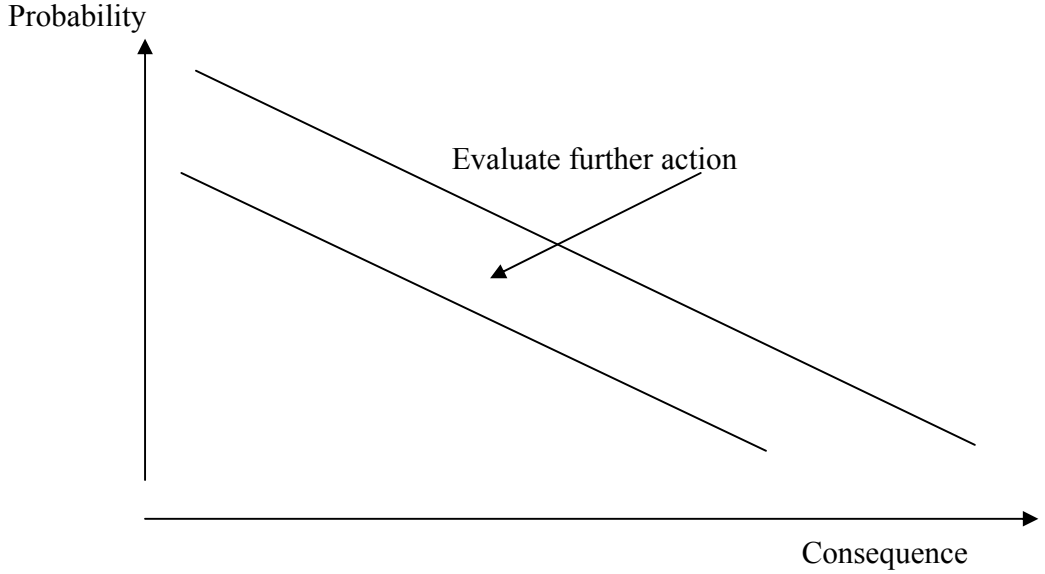


Figure 12. Risk Matrix Presentation with Continuous Variables.

The upper tolerability limit (figures 11 and 12) is almost always defined, whereas the lower limit is related to each individual risk reducing measure, depending on when the cost of implementing each measure becomes unreasonably disproportional to the reduction of risk.

Examples of the applications of Risk Matrix are evaluation of:

- Risk to safety of personnel for different solutions such as integrated versus separate quarters platform;
- Risk of operations such as exploration drilling;
- Risk of the use of a particular system such as mechanical pipe handling;
- Environmental risk.

5.4 Decision trees

Decision trees are used to illustrate decisions and consequences of decisions. Further, when probabilities are assigned to consequences expected costs / utilities of different alternatives can be determined. In figure 13 is shown an example of a decision tree where each possible decision and consequence are systematically identified – the example is taken from [4]. Two alternative designs for the structural deign of a building are considered. Design A is based on a conventional procedure with a probability of satisfactory performance equal to 99% and costs $1.5 million. Design B is a new concepts and will reduce the costs to $1 million. The reliability of B is not known, but the engineer estimates if the assumptions made are correct the probability of good performance to 0.99, whereas if the assumptions are wrong then the probability is only 0.9. He is only 50% sure of the assumptions. The cost of unsatisfactory performance is $10 million.

The expected costs of the two alternatives are:
A: *C=0.99 x 1.5 + 0.01 x 11.5 = 1.6*
B: *C=0.5 x (0.99 x 1.0 + 0.01 x 11.0) + 0.5 x (0.9 x 1.0 + 0.1 x 11.0) = 1.55*

According to decision theory the alternative with the lowest expected costs should be chosen, i.e. alternative B should be chosen here.

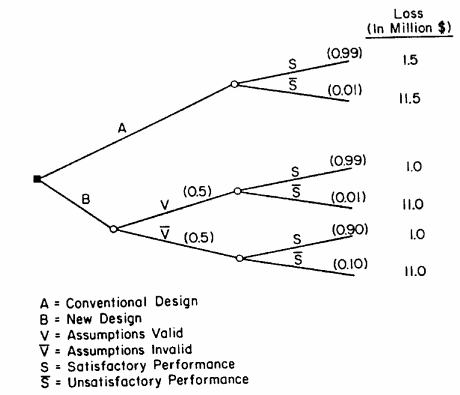


Figure 13. Decision tree for design problem.

The decision tree is constructed from left to right. Each consequence is associated with probabilities (summing up to 1) after each node. For each branch the expected cost /utility is determined by multiplying probabilities and costs/utilities for that branch.

6 Risk acceptance criteria

Acceptance of risk is basically problem of decision making, and is inevitably influenced by many factors such as type of activity, level of loss, economic, political, and social factors, confidence in risk estimation, etc. A risk estimate, in the most simplest form, is considered acceptable when below the level which divides the unacceptable from acceptable risks. For example, an estimate of individual risk per annum of 10^{-7} can be considered as "negligible risk"; similarly, an estimate of injuries occurring several times per year, can be considered as "unacceptable".

The "as low as reasonably practicable" (**ALARP**) principle is sometimes used in the industry as the only acceptance principle and sometimes in addition to other risk acceptance criteria.

The use of the ALARP principle may be interpreted as satisfying a requirement to keep the risk level as low as reasonably practicable, provided that the ALARP evaluations are extensively documented.
The ALARP principle is shown in Figure 14.

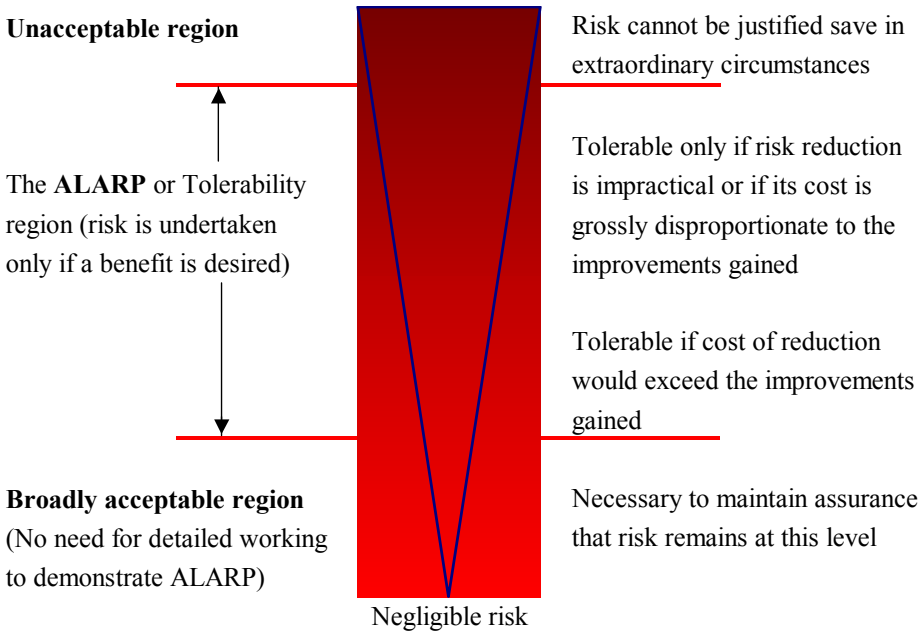


Figure 14. The ALARP Principle.

The risk level should be reduced as far as possible in the interval between acceptable and unacceptable risk. The common way to determine what is possible is to use cost-benefit evaluations as basis for decision on whether to implement certain risk reducing measures or not.

The upper tolerability limit is almost always defined, whereas the lower tolerability limit is sometimes defined, or not defined. The lower limit is individual to each individual risk reducing measure, depending on when the cost of implementing each measure becomes unreasonably disproportional to the risk reducing effect.

The ALARP principle is normally used for risk to safety of personnel, environment and assets.

The value for the upper tolerability limit derived from accident statistics, for example, indicate that "a risk of death around 1 in 1,000 per annum is the most that is ordinarily accepted by a substantial group of workers in any industry in the UK".

HSE (Health and Safety Excecutive), [8] suggested the upper maximum tolerable risk level as a line with a slope of -1 through point $n = 500$ (number of fatalities), $F = 2 \times 10^{-4}$ (frequency) per year. This line corresponds to $n = 1$ at $F = 10^{-1}$ per year, and $\underline{n} = 100$ at $F = 10^{-3}$ per year. However, in the document [9], HSE quotes that risk of a single accident causing the death of 50 people or more with the frequency of 5×10^{-3} per year is intolerable.

For the "negligible" level, the HSE recommends a line drawn three decades lower than the intolerable line. This line corresponds to one fatality, $n = 1$, in one per ten thousand per year, $F = 10^{-4}$ per year, and similarly, n = 100 corresponds to one in a million per year, $F = 10^{-6}$ per year.

Railway transport

For the railway area different railway operators in the UK has suggested the risk criteria in table 2.

Criterion			Railtrack	Union Railways	Eurotunnel
Passenger individual risk	Intolerable		10^{-4} per year		8×10^{-10} per passenger km and 9.5×10^{-9} per passenger journey
	Broadly acceptable		10^{-6} per year		8×10^{-13} per passenger km and 9.5×10^{-12} per passenger journey
	Target / Design values		10^{-5} per year	10^{-5} per year	N/A
Staff individual risk	Intolerable		10^{-3} per year		10^{-3} per year
	Broadly acceptable		10^{-6} per year		10^{-6} per year
	Target / Design values		2×10^{-4} per year		10^{-4} per year
Societal risk criteria	Intolerable	N>10 N>100			1.1×10^{-2} per year 9.1×10^{-4} per year
	Broadly acceptable	N>10 N>100			1.1×10^{-5} per year 9.1×10^{-7} per year
	Target / Design values	N>10 N>100		10^{-2} per year 10^{-3} per year	N/A

Table 2. Risk criteria adopted by different operators.

Road Transport

The current convention adopted by Department of Transport (DTLR) for the Value of Statistical Life (*VOSL*) is about £1,000,000 (2001 prices). This value is used in a cost-benefit evaluation of road improvement schemes in the UK. There are no risk criteria specific to road transport, but only to transport of hazardous material by road [8].

It is interesting to note that according to [8], the national scrutiny level of societal risk for road transport could be defined by a line passing through points defined by 10 fatalities corresponding to 0.7 times per year (7 in 10 years), 100 fatalities corresponding to 0.07 times per year (7 in 100 years), and 1,000 fatalities corresponding to 0.007 times per year (7 in 1,000 years).

Recently developed criteria for a tunnel under a river were based on the HSE's suggestion in [7], which defines the upper maximum tolerable risk level as a line with a slope of –1 passing through points N = 1 and F = 10^{-1} per year, and N = 100 and F = 10^{-3} per year. For the **negligible** level a line drawn three decades lower is suggested by the HSE. This line corresponds to one fatality, N = 1, in 1 per 10,000 per year (F = 10^{-4} per year), and similarly, N = 100 corresponds to 1 in a million per year (F = 10^{-6} per year). For the **broadly acceptable** level a line two decades lower that the maximum tolerable line is suggested. This line is defined as N = 1 and F = 10^{-3} per year, and N = 100 and F = 10-5 per year. The proposed criteria for the societal risk for the tunnel are presented in figure 15.

The Dutch criteria are compared with the proposed UK criteria in figure 16. It can be seen that up to N = 100, the both criteria can be assumed to have similar impact, in spite of the fact that the Dutch maximum tolerable level is more stringent that the proposed one for the UK. The equalising factor is the ALARP requirement in the UK which drives the risk towards the broadly acceptable level.

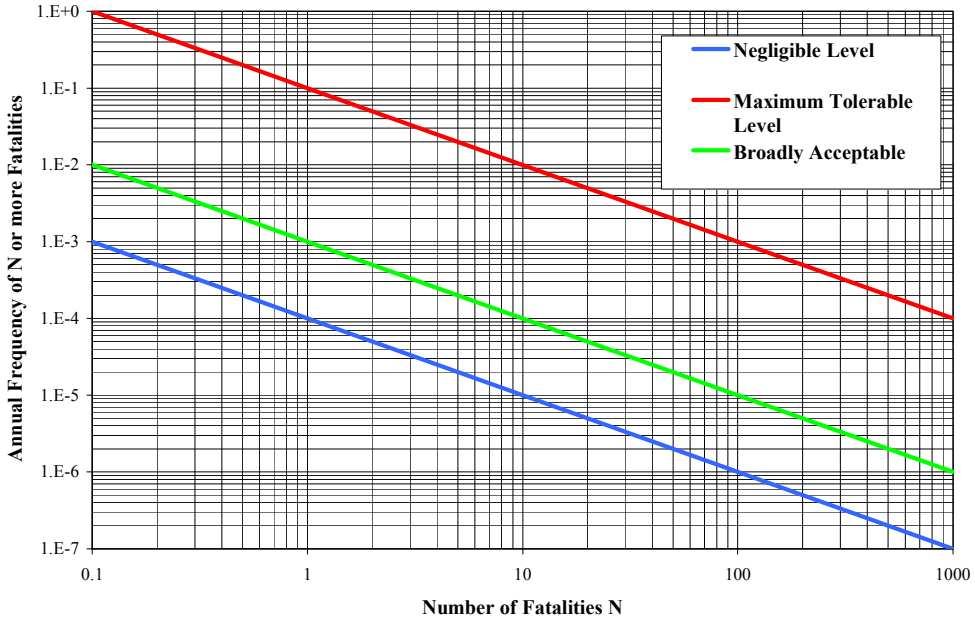


Figure 15. Proposed criteria for road tunnel users.

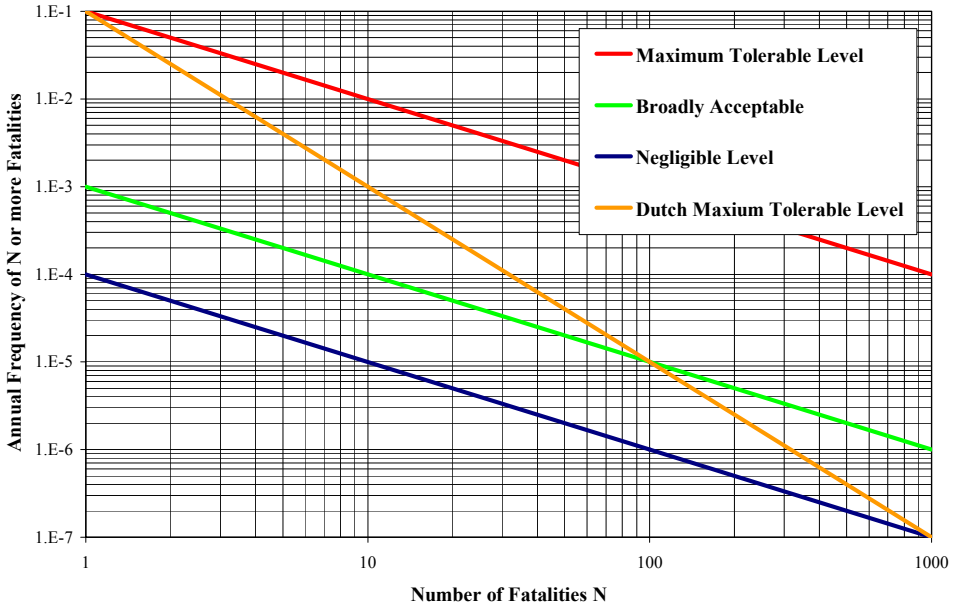


Figure 16. Comparison of proposed and Dutch criteria for road tunnel users.

Risk criteria used in the Netherlands are shown in table 3.

INDIVIDUAL RISK		Per Year
Maximum permissible risk	New residential buildings	1.00E-06
	Existing residential buildings	1.00E-05
SOCIETAL RISK		**Frequency Per Year**
Maximum permissible risk	> 10 deaths	1.00E-05
	> 100 deaths	1.00E-07
	> 1,000 deaths	1.00E-09

Table 3. Risk criteria in the Netherlands.

Risk criteria were developed for a 6,600 m long tunnel under the River Westerschelde completed in 2002. No risk criteria for road users were available. So the criteria for population in the vicinity of the road used for transportation of hazardous substances have been used as the basis. These criteria define the upper tolerability level as the individual risk per annum of 10^{-6}, and the societal risk of 10 fatalities once in 10,000 years, and 100 fatalities once in a million years. Since the risk to road users can be considered as voluntary, a factor of 10 has been used for the criteria for road users [10], as presented in table 4.

Number of Fatalities	MaximumAnnual Frequency per km
1	10^{-1}
10	10^{-3}
100	10^{-5}

Table 4. Societal risk criterion used for Westerschelde tunnel.

It is interesting to note that the risk criterion for the overall length of the tunnel is also mentioned, but it is not clear if this was supposed to be used to "average" risks along the tunnel, as it is also mentioned that risks vary significantly throughout the tunnel (i.e. exits and entrances and their slopes present a considerably greater risk than horizontal sections).

Building sector

Buildings are usually designed for a life time of 50 years and in practice most buildings have a much longer life, also because of maintenance and refurbishment activities which also for a part of the building sector. A number of 80 years is often mentioned as an average. For some buildings, of course, a smaller design life is chosen on purpose, especially for some kind of industrial buildings.

The Building sector generally involves many partners, e.g.

- the owner
- the user
- the principal
- the architect
- the structural designer
- a number of specialists (geotechnics, building physics, etc
- the contractor
- a number of material suppliers

- a number of suppliers of building elements
- a number of suppliers of installations
- a number of subcontractors
- the authorities.

Usually they do not belong to one firm and this may make communication quite difficult. It also contributes to the difficulty to make successful innovations in this sector. The following hazards are to be considered:

- Normal loads (self weight, floor loads, environmental loads, etc)
- Natural hazards (i.e. earthquakes, landslides, etc)
- Manmade hazards (fire, explosions, collisions)
- Degradation (by mechanical, chemical and physical or biological attack))
- Ecological aspects (energy consumption, durable and economic use of materials)
- Human errors in design, construction and use
- Terrorist action

Design usually takes care of all those issues, either implicitly or explicitly. The degree in which attention is paid depends very much on the nature and importance of the building.

Most buildings are designed without any special and explicit risk analysis and risk management. Implicitly, of course, the building codes can be considered as a relatively effective risk management tool.

Special attention is given to the aspect of fire safety, both the structural side and the human safety side. Most people in fires do not die from fire or structural collapse, but from smoke. The most important issue over here is a good system of detection, warning, early extinguishing and good possibilities of evacuation and escape. In this field an increasing interest in explicit risk analysis approach can be observed.

In most buildings codes safety classes are distinguished for various types of buildings and various types of building elements. A very common system is to divide the consequences of failure into small, medium and large. Some codes make a distinction between economic losses and danger to human lives. Based on these classes there is a safety differentiation in the code, which usually comes down to the prescription of different values for the partial factors or the introduction of a set of adjustment factors.

It is well known, but not always fully appreciated, that the reliability of a structure as estimated on the basis of a given set of probabilistic models for loads and resistances may have limited bearing to the actual reliability of the structure. This is the case when the probabilistic modeling forming the basis of the reliability analysis is highly influenced by subjectivity and then the estimated reliability should be interpreted as being a measure for comparison only. In these cases it is thus not immediately possible to judge whether the estimated reliability is sufficiently high without first establishing a more formalized reference for comparison.

Such a reference may be established by the definition of an optimal or best practice structure. The idea behind the "best practice" reference is that if the structure of consideration has been designed according to the "best practice" then the reliability of the structure is "optimal" according to agreed conventions for the target reliability. Typical values for the corresponding target annual failure probability are in the range of 10^{-6} to 10^{-7} depending on the type of structure and the characteristics of the considered failure

mode. Using this approach the target reliability is determined as the reliability of the "best practice" design as assessed with the given probabilistic model.

The determination of the "best practice" design can be performed in different ways. The simplest approach is to use the existing codes of practice for design as a basis for the identification of "best practice" design. Alternatively the "best practice design" may be determined by consultation of a panel of recognized experts.

In case where the probabilistic modeling does not rest on subjective assessments the most rational approach is to establish the optimal design on the basis of the economic decision theory.

In tables 1 target failure probabilities are given for ultimate limit states based on recommendations of JCSS [11]. Note that the values given correspond to a year reference period and the stochastic models recommended in JCSS [11].

Relative cost of safety measure	Minor consequences of failure	Moderate consequences of failure	Large consequences of failure
High	$P_F \approx 10^{-3}$	$P_F \approx 5\ 10^{-4}$	$P_F \approx 10^{-4}$
Normal	$P_F \approx 10^{-4}$	$P_F \approx 10^{-5}$	$P_F \approx 5\ 10^{-6}$
Low	$P_F \approx 10^{-5}$	$P_F \approx 5\ 10^{-5}$	$P_F \approx 10^{-6}$

Table 5. Tentative target failure probabilities) related to a one-year reference period and ultimate limit states

7 References

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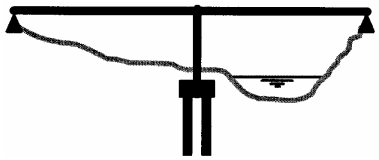
Opgave 1 – Risikoanalyse

I forbindelse med en eksisterende bro er der opstået tvivl om den har tilstrækkelig bæreevne. Derfor ønskes et svigttræ og et hændelsestræ opstillet. For at kunne vurdere forskellige alternative handlinger ønskes også et beslutningstræ opstillet.



Spørgsmål 1 – Svigttræ (failure tree)

Top hændelsen er svigt af broen. Broen består af 1 stålbjælke og et fundament med 2 pæle. Begge pæle har en svigtsandsynlighed på 5% og det kan antages at en pæl er tilstrækkelig til at bære belastningen. Stålbjælken kan ruste. Sandsynligheden for svigt af stålbjælken er 1% hvis den starter at ruste og 0.01% hvis den ikke ruster. Sandsynligheden for at bjælken starter at ruste i løbet af levetiden er 10%. Broen svigter stålbjælken eller fundamentet svigter. Alle hændelser kan antages uafhængige.

Opstil et svigttræ og vis at sandsynligheden for svigt af broen er 0.00359.

Spørgsmål 2 – Hændelsestræ (event tree)

Konsekvenserne af et svigt af broen ønskes undersøgt ved at et hændelsestræ opstilles. Hvis broen svigter antages det at sandsynligheden for at dette sker så langsomt at en hurtig opdagelse heraf og efterfølgende reparation er 70%. Dette vil indebære omkostninger af størrelsesordenen ½ mill. kr.

Hvis derimod svigtet sker øjeblikkeligt som et totalt kollaps indtræder, så er omkostningerne 10 mill. kr. På basis af trafikobservationer vides at sandsynligheden for at der ingen køretøjer er på broen er 95% i svigtøjeblikket. Med en sandsynlighed på 5% er der 10 køretøjer og 20 personer på broen på et tilfældigt tidspunkt. Konsekvenserne af tab af et menneskeliv antages at kunne ækvivaleres med 4 mill. kr.

Opstil et hændelsestræ og vis at de forventede omkostninger ved svigt af broen er 4.55 mill. kr.

Spørgsmål 3 – Beslutningstræ (decision tree)

En ingeniør spørges om at undersøge 3 forskellige alternativer:

1. Bygge en ny bro. Denne antages at koste 8 mill. kr. og at have en svigtsandsynlighed der kan sættes til 0.
2. Indføre trafikrestriktioner, som medfører at svigtsandsynligheden (fra Spørgsmål 1) halveres og at restriktionerne medfører en omkostning på 2 mill. kr.
3. Status quo – dvs. gøre ingenting.

Bestem de forventede omkostninger for hver af de 3 alternativer. Hvilket alternativ bør vælges.
De forventede omkostningerne for et svigt af den eksisterende bro tages fra spørgsmål 2.

Opstil et beslutningstræ og bestem de forventede omkostninger for hver af de 3 alternativer. Hvilket alternativ er bedst?

Note 1 + 2: STRUCTURAL RELIABILITY

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1 Introduction

For many years it has been assumed in design of structural systems that all loads and strengths are deterministic. The strength of an element was determined in such a way that it exceeded the load with a certain margin. The ratio between the strength and the load was denoted the safety factor. This number was considered as a measure of the reliability of the structure. In codes of practice for structural systems values for loads, strengths and safety factors are prescribed.

These values are traditionally determined on the basis of experience and engineering judgement. However, in new codes partial safety factors are used. Characteristic values of the uncertain loads and resistances are specified and partial safety factors are applied to the loads and strengths in order to ensure that the structure is safe enough. The partial safety factors are usually based on experience or calibrated to existing codes or to measures of the reliability obtained by probabilistic techniques.

Activity	Approximate death rate ($\times 10^{-9}$ deaths/h exposure)	Typical exposure (h/year)	Typical risk of death ($\times 10^{-6}$/year)
Alpine climbing	30000 – 40000	50	1500-2000
Boating	1500	80	120
Swimming	3500	50	170
Cigarette smoking	2500	400	1000
Air travel	1200	20	24
Car travel	700	300	200
Train travel	80	200	15
Coal mining (UK)	210	1500	300
Construction work	70-200	2200	150-440
Manufacturing	20	2000	40
Building fires	1-3	8000	8-24
Structural failures	0.02	6000	0.1

Table 1. Some risks in society (from Melchers [1]).

As described above structural analysis and design have traditionally been based on deterministic methods. However, uncertainties in the loads, strengths and in the modeling of the systems require that methods based on probabilistic techniques in a number of situations have to be used. A structure is usually required to have a satisfactory performance in the expected lifetime, i.e. it is required that it does not collapse or becomes unsafe and that it fulfills certain functional requirements. Generally structural systems have a rather small probability that they do not function as intended, see table 1.

Reliability of structural systems can be defined as the probability that the structure under consideration has a proper performance throughout its lifetime. Reliability methods are used to estimate the probability of failure. The information of the models which the reliability analyses are based on are

generally not complete. Therefore the estimated reliability should be considered as a nominal measure of the reliability and not as an absolute number. However, if the reliability is estimated for a number of structures using the same level of information and the same mathematical models, then useful comparisons can be made on the reliability level of these structures. Further design of new structures can be performed by probabilistic methods if similar models and information are used as for existing structures which are known to perform satisfactory. If probabilistic methods are used to design structures where no similar existing structures are known then the designer has to be very careful and verify the models used as much as possible.

The reliability estimated as a measure of the safety of a structure can be used in a decision (e.g. design) process. A lower level of the reliability can be used as a constraint in an optimal design problem. The lower level of the reliability can be obtained by analyzing similar structures designed after current design practice or it can be determined as the reliability level giving the largest utility (benefits – costs) when solving a decision problem where all possible costs and benefits in the expected lifetime of the structure are taken into account.

In order to be able to estimate the reliability using probabilistic concepts it is necessary to introduce stochastic variables and/or stochastic processes/fields and to introduce failure and non-failure behavior of the structure under consideration.

Generally the main steps in a reliability analysis are:

1. Select a target reliability level.
2. Identify the significant failure modes of the structure.
3. Decompose the failure modes in series systems of parallel systems of single components (only needed if the failure modes consist of more than one component).
4. Formulate failure functions (limit state functions) corresponding to each component in the failure modes.
5. Identify the stochastic variables and the deterministic parameters in the failure functions. Further specify the distribution types and statistical parameters for the stochastic variables and the dependencies between them.
6. Estimate the reliability of each failure mode.
7. In a design process change the design if the reliabilities do not meet the target reliabilities.
   In a reliability analysis the reliability is compared with the target reliability.
8. Evaluate the reliability result by performing sensitivity analyses.

The single steps are discussed below.

Typical failure modes to be considered in a reliability analysis of a structural system are yielding, buckling (local and global), fatigue and excessive deformations.

The failure modes (limit states) are generally divided in:

Ultimate limit states
Ultimate limit states correspond to the maximum load carrying capacity which can be related to e.g. formation of a mechanism in the structure, excessive plasticity, rupture due to fatigue and instability (buckling).

Conditional limit states
Conditional limit states correspond to the load-carrying capacity if a local part of the structure has failed. A local failure can be caused by an accidental action or by fire. The conditional limit states

can be related to e.g. formation of a mechanism in the structure, exceedance of the material strength or instability (buckling).

Serviceability limit states
Serviceability limit states are related to normal use of the structure, e.g. excessive deflections, local damage and excessive vibrations.

The fundamental quantities that characterize the behavior of a structure are called the *basic variables* and are denoted $\mathbf{X} = (X_1,...,X_n)$ where n is the number of basic stochastic variables. Typical examples of basic variables are loads, strengths, dimensions and materials. The basic variables can be dependent or independent, see below where different types of uncertainty are discussed. A stochastic process can be defined as a random function of time such that for any given point in time the value of the stochastic process is a random variable. Stochastic fields are defined in a similar way where the time is exchanged with the space.

The uncertainty modeled by stochastic variables can be divided in the following groups:

Physical uncertainty: or inherent uncertainty is related to the natural randomness of a quantity, for example the uncertainty in the yield stress due to production variability.

Measurement uncertainty: is the uncertainty caused by imperfect measurements of for example a geometrical quantity.

Statistical uncertainty: is due to limited sample sizes of observed quantities.

Model uncertainty: is the uncertainty related to imperfect knowledge or idealizations of the mathematical models used or uncertainty related to the choice of probability distribution types for the stochastic variables.

The above types of uncertainty are usually treated by the reliability methods which will be described in the following chapters. Another type of uncertainty which is not covered by these methods are gross errors or human errors. These types of errors can be defined as deviation of an event or process from acceptable engineering practice.

Generally, methods to measure the reliability of a structure can be divided in four groups, see Madsen et al. [2], p.30:

- *Level I methods*: The uncertain parameters are modeled by one characteristic value, as for example in codes based on the partial safety factor concept.
- *Level II methods*: The uncertain parameters are modeled by the mean values and the standard deviations, and by the correlation coefficients between the stochastic variables. The stochastic variables are implicitly assumed to be normally distributed. The reliability index method is an example of a level II method.
- *Level III methods*: The uncertain quantities are modeled by their joint distribution functions. The probability of failure is estimated as a measure of the reliability.
- *Level IV methods*: In these methods the consequences (cost) of failure are also taken into account and the risk (consequence multiplied by the probability of failure) is used as a measure of the reliability. In this way different designs can be compared on an economic basis taking into account uncertainty, costs and benefits.

Level I methods can e.g. be calibrated using level II methods, level II methods can be calibrated using level III methods, etc.

Level II and III reliability methods are considered in these notes. Several techniques can be used to estimate the reliability for level II and III methods, e.g.

- **simulation techniques**: Samples of the stochastic variables are generated and the relative number of samples corresponding to failure is used to estimate the probability of failure. The simulation techniques are different in the way the samples are generated.
- **FORM techniques**: In First Order Reliability Methods the limit state function (failure function) is linearized and the reliability is estimated using level II or III methods.
- **SORM techniques**: In Second Order Reliability Methods a quadratic approximation to the failure function is determined and the probability of failure for the quadratic failure surface is estimated.

In level IV methods the consequences of failure can be taken into account. In cost-benefit analyses (or RISK analyses) the total expected cost-benefits for a structure in its expected lifetime are maximized

$$\max_z W(z) = B(z) - C_I(z) - C_{IN}(z) - C_{REP}(z) - P_f(z)C_F \quad (1)$$

where z represents design/decision variables, B is the expected capitalized benefits, C_I is the initial (or construction) costs, C_{IN} is the expected capitalized inspection costs, C_{REP} is the expected capitalized repair costs and C_F is the capitalized failure costs. Cost-optimized inspection strategies are based on cost-benefit analyses where the costs due to inspection, repair and failure are minimized with e.g. inspection locations and inspection times and qualities as decision variables.

For a detailed introduction to structural reliability theory reference is made to the following textbooks: Melchers [1], Madsen, Krenk & Lind [2], Thoft-Christensen & Baker [3] and Ditlevsen & Madsen [4].

2 Basic Probability theory and Stochastic Variables

2.1 Events and basis probability rules

An event E is defined as a subset of the sample space (all possible outcomes of a random quantity) Ω. The failure event E of e.g. a structural element can be modeled by $E = \{R \le S\}$ where R is the strength and S is the load. The probability of failure is the probability $P_f = P(E) = P(R \le S)$. If a system is modeled by a number of failure events, failure of the system can be defined by a union or an intersection of the single failure events.

If failure of one element gives failure of the system, then a **union (series system)** is used to model the system failure, E:

$$E = E_1 \cup ... \cup E_m = \bigcup_{i=1}^{m} E_i \quad (2)$$

where E_i is the event modeling failure of element i and m is the number of events.

If failure of all elements are needed to obtain failure of the system, then an **intersection (parallel system)** is used to model the system failure, E:

$$E = E_1 \cap ... \cap E_m = \bigcap_{i=1}^{m} E_i \quad (3)$$

Disjoint / mutually exclusive events are defined by

$$E_1 \cap E_2 = \emptyset \quad (4)$$

where $\emptyset$ is the impossible event.

A complementary event $\overline{E}$ is denoted defined by

$$E \cap \overline{E} = \emptyset \quad \text{and} \quad E \cup \overline{E} = \Omega \tag{5}$$

The so-called De Morgan's laws related to complementary events are

$$E_1 \cap E_2 = \overline{\overline{E}_1 \cup \overline{E}_2} \tag{6}$$

$$E_1 \cup E_2 = \overline{\overline{E}_1 \cap \overline{E}_2} \tag{7}$$

Probabilities of events have to fulfill the following **fundamental axioms**:

Axiom 1: for any event E:

$$0 \le P(E) \le 1 \tag{8}$$

Axiom 2: for the sample space Ω

$$P(\Omega) = 1 \tag{9}$$

Axiom 3: for mutually exclusive events $E_1, E_2, ..., E_m$:

$$P\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} P(E_i) \tag{10}$$

The **conditional probability** of an event E_1 given another event E_2 is defined by:

$$P\left(E_1 \middle| E_2\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} \tag{11}$$

Event E_1 is statistically independent of event E_2 if

$$P\left(E_1 \middle| E_2\right) = P(E_1) \tag{12}$$

From (11) we have

$$P(E_1 \cap E_2) = P\left(E_1 \middle| E_2\right)P(E_2) = P\left(E_2 \middle| E_1\right)P(E_1) \tag{13}$$

Therefore if E_1 and E_2 are statistically independent:

$$P(E_1 \cap E_2) = P\left(E_1\right)P(E_2) \tag{14}$$

Using the multiplication rule in (13) and considering mutually exclusive events $E_1, E_2, ..., E_m$ the **total probability theorem** follows:

$$\begin{aligned} P(A) &= P\left(A \middle| E_1\right)P(E_1) + P\left(A \middle| E_2\right)P(E_2) + ... + P\left(A \middle| E_m\right)P(E_m) \\ &= P\left(A \cap E_1\right) + P\left(A \cap E_2\right) + ... + P\left(A \cap E_m\right) \end{aligned} \tag{15}$$

where A is an event.

From the multiplication rule in (13) it follows

$$P(A \cap E_i) = P\left(A \middle| E_i\right)P(E_i) = P\left(E_i \middle| A\right)P(A) \tag{16}$$

Using also the total probability theorem in (15) the so-called **Bayes theorem** follows from:

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{m} P(A|E_j)P(E_j)} \tag{17}$$

2.2.1 Example 1 – statically determinate structure

Consider a statically determinate structural system with 7 elements. The failure probabilities $P_i = P(F_i) = P(\text{failure of element } i)$ of each element are determined to:

i	1	2	3	4	5	6	7
P_i	0.02	0.01	0.02	0.03	0.02	0.01	0.02

It is assumed that the failure events $F_1, F_2, ..., F_7$ are independent. The probability of failure of the system becomes:

$$\begin{aligned} P_{failure} &= P(F_1 \cup F_2 \cup ... \cup F_7) \\ &= 1 - P(\text{structure safe}) \\ &= 1 - P(\overline{F}_1 \cap \overline{F}_2 \cap ... \cap \overline{F}_7) \\ &= 1 - (1 - P(F_1))(1 - P(F_2))...(1 - P(F_7)) \end{aligned}$$

Using the element failure probabilities

$P_{failure} = 1-(1-0.02)^4(1-0.01)^2(1-0.03)=0.12$

2.2.2 Example 2 – use of Bayes theorem

Consider concrete beams which are tested before use. Let E denote the event that the beam is perfect. Further, let A denote the event that the beam pass the test. Experience show that

$P(A) = 0.95$ 95% of the beams pass the test

$P(E|A) = 0.90$

$P(E|\overline{A}) = 0.10$ reliability of test

The probability that a perfect beam pass the test is obtained as

$$P(A|E) = \frac{P(E \cap A)}{P(E)} = \frac{= P(E|A)P(A)}{P(E|A)P(A) + P(E|\overline{A})P(\overline{A})} = \frac{0.90 \cdot 0.95}{0.90 \cdot 0.95 + 0.10 \cdot 0.05} = \frac{0.90 \cdot 0.95}{0.86} = 0.994$$

2.2 Continuous stochastic variables

Consider a continuous stochastic variable X. The **distribution function** of X is denoted $F_X(x)$ and gives the probability $F_X(x) = P(X \le x)$. A distribution function is illustrated in figure 1. The **density function** $f_X(x)$ is illustrated in figure 1 and is defined by

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{18}$$

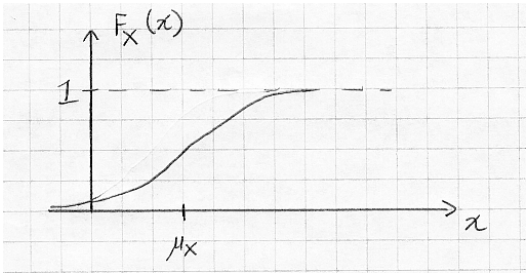


Figure 1. Distribution function $F_X(x)$.

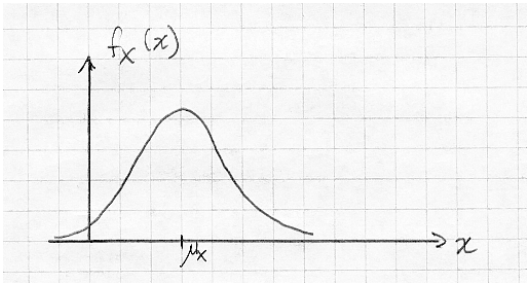


Figure 2. Density function $f_X(x)$.

The **expected value** is defined by

$$\mu = \int_{-\infty}^{\infty} x f_X(x) dx \tag{19}$$

The **variance** σ^2 is defined by

$$\sigma^2 = \int (x-\mu)^2 f_X(x) dx \tag{20}$$

where σ is the **standard deviation**.

The **coefficient of variation** $COV = V$ is

$$V = \frac{\sigma}{\mu} \tag{21}$$

The n th order central moment is

$$m_n = \int (x-\mu)^n f_X(x) dx \tag{22}$$

The **skewness** is defined by

$$\beta_1 = \frac{m_3^2}{m_2^3} \tag{23}$$

and the **kurtosis** is

$$\beta_2 = \frac{m_4}{m_2^2} \tag{24}$$

2.2.1 Example: Probability of failure – fundamental case

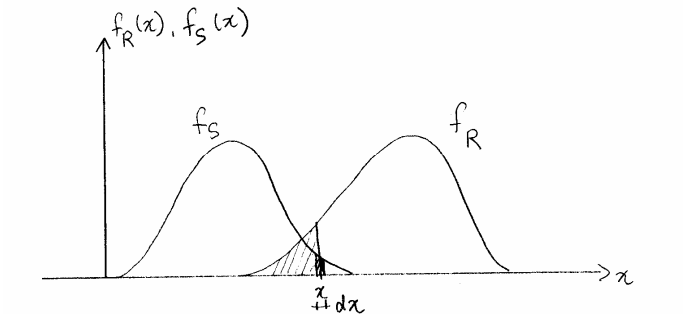


Figure 3. Density functions for fundamental case.

Consider a structural element with load bearing capacity R which is loaded by the load S. R and S are modeled by independent stochastic variables with density functions f_R and f_S and distribution functions F_R and F_S, see figure 3. The probability of failure becomes

$$P_F = P(failure) = P(R \le S) = \int_{-\infty}^{\infty} P(R \le x) P(x \le S \le x + dx) dx = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx$$

Alternatively the probability of failure can be evaluated by

$$P_F = P(failure) = P(R \le S) = \int_{-\infty}^{\infty} P(x \le R \le x + dx) P(S \ge x) dx = \int_{-\infty}^{\infty} f_R(x)(1 - F_S(x)) dx = 1 - \int_{-\infty}^{\infty} f_R(x) F_S(x) dx$$

It is noted that it is important that the lower part of the distribution for the strength and the upper part of the distribution for the load are modeled as accurate as possible.

2.2.2 Example: Normal distribution

The distribution function for a stochastic variable with expected value μ and standard deviation σ is denoted N(μ,σ), and is defined by

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\left(\frac{t-\mu}{\sigma}\right)^2\right) dt \qquad (25)$$

where $\Phi(u)$ is the standardized distribution function for a Normal distributed stochastic variable with expected value = 0 and standard deviation = 1 : N(0,1).

The Normal distribution has:

Skewness: $\beta_1 = 0$

Kurtosis: $\beta_2 = 0$

Figure 4 shows the density function for a Normal distributed stochastic variable with expected value 10 and standard deviation 3.

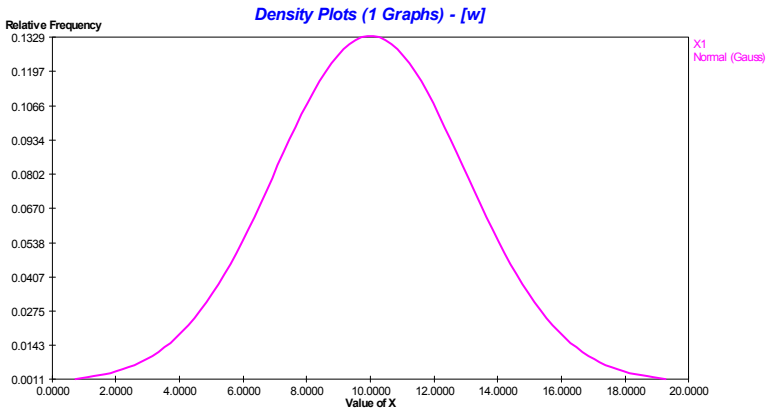


Figure 4. Normal distributed stochastic variable with expected value 10 and standard deviation 3.

2.2.3 Example: Lognormal distribution

The distribution function for a stochastic variable with expected value μ and standard deviation σ is denoted LN(μ,σ), and is defined by

$$F_X(x) = \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right) = \int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi}\sigma_Y t} \exp\left(-\left(\frac{t-\mu_Y}{\sigma_Y}\right)^2\right) dt \qquad (26)$$

where

$$\sigma_Y = \sqrt{\ln\left(\left(\frac{\sigma}{\mu}\right)^2 + 1\right)} \tag{27}$$

$$\mu_Y = \ln\mu - \frac{1}{2}\sigma_Y^2 \tag{28}$$

is the standard deviation and expected value for the Normal distributed stochastic variable $Y = \ln X$.

The Lognormal distribution has:
Skewness: $\beta_1 = V^2\left(V^2 + 3\right) \cong 3V^2$
Kurtosis: $\beta_2 = 3 + V^2\left(\left(V^2+1\right)^3 + 3\left(V^2+1\right)^2 + 6\left(V^2+1\right) + 6\right) \cong 3$

Figure 5 shows the density function for a Lognormal distributed stochastic variable with expected value 10 and standard deviation 3.

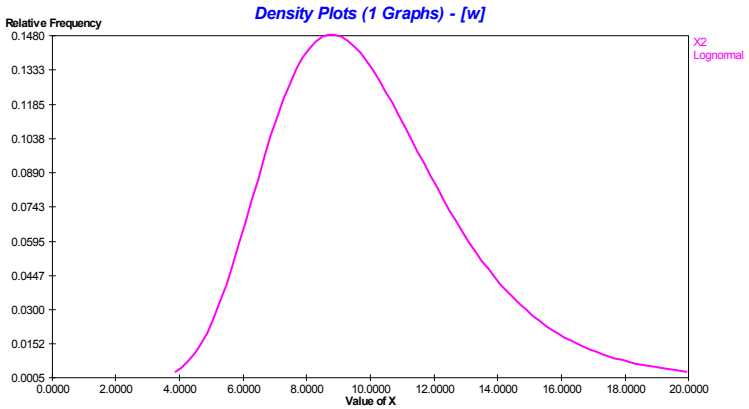


Figure 5. Lognormal distributed stochastic variable with expected value 10 and standard deviation 3.

If the coefficient of variation $V = \frac{\sigma}{\mu}$ is small (less than $\approx$ 0.25) then the standard deviation and expected value of $Y = \ln X$ can approximately be obtained from

$\sigma_Y \approx V$
$\mu_Y \approx \ln\mu$

The 5 % quantile $y_{0.05}$ defined such that $P(Y = \ln X \le y_{0.05}) = 0.05$ can be obtained from

$$y_{0.05} = \mu_Y - 1.645\sigma_Y$$

where -1.645 is obtained from the standard Normal distribution function such that $\Phi(-1.645) = 0.05$. Correspondly the 5% qualtile $x_{0.05}$ of the LogNormal variable can be obtained from

$$x_{0.05} = \exp(y_{0.05}) = \exp\left(\mu_Y - 1.645\sigma_Y\right)$$

which for small coefficients of variation becomes

$$x_{0.05} \approx \mu \exp(-1.645V)$$

2.2.4 Example: 2 parameter Weibull-fordeling

The distribution function for a stochastic variable with expected value μ and standard deviation σ is denoted W2(μ,σ), and is defined by:

$$F_X(x) = 1 - \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) \tag{29}$$

where α and β are the form- and shape-parameters. These are related to μ and σ by:

$$\mu = \beta\,\Gamma\left(1+\frac{1}{\alpha}\right) \tag{30}$$

$$\sigma = \beta\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)} \tag{31}$$

where $\Gamma(\cdot)$ is the Gamma distribution.

Figure 6 shows the density function for a 2-parameter Weibull distributed stochastic variable with expected value 10 and standard deviation 3.

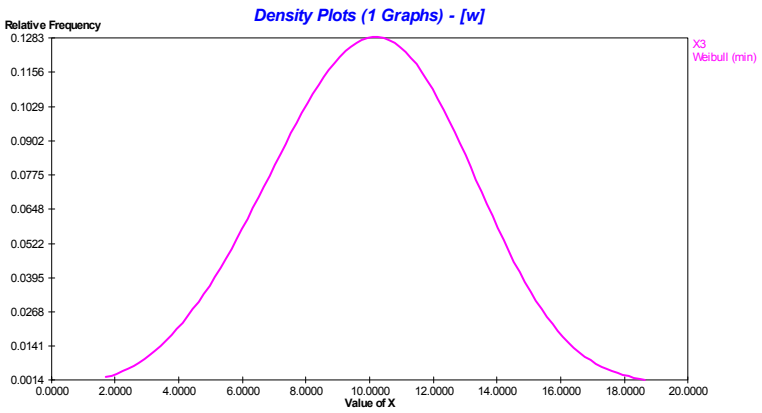


Figure 6. 2-parameter Weibull distributed stochastic variable with expected value 10 and standard deviation 3.

2.2.5 Example: 3 parameter Weibull-fordeling

The distribution function for a stochastic variable with expected value μ, standard deviation σ and lower threshold γ is denoted W3(μ,σ;γ), and is defined by:

$$F_X(x) = 1 - \exp\left(-\left(\frac{x-\gamma}{\beta-\gamma}\right)^{\alpha}\right) \qquad x \geq \gamma \tag{32}$$

where α and β are the form- and shape-parameters. These are related to μ and σ by:

$$\mu = (\beta-\gamma)\Gamma\left(1+\frac{1}{\alpha}\right) + \gamma \tag{33}$$

$$\sigma = (\beta-\gamma)\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)} \tag{34}$$

Figure 7 shows the density function for a 3-parameter Weibull distributed stochastic variable with expected value 10, standard deviation 3 and lower threshold γ=3.

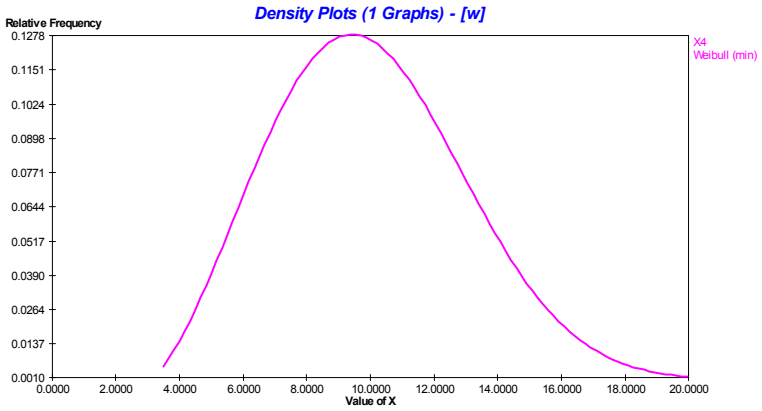


Figure 7. 3-parameter Weibull distributed stochastic variable with expected value 10, standard deviation 3 and lower threshold γ=3.

2.2.6 Example: Truncated Weibull distribution

The distribution function for a stochastic variable with expected value μ, standard deviation σ and lower threshold γ is denoted WT($\mu, \sigma; \gamma$), and is defined by:

$$F_X(x) = 1 - \frac{1}{P_0}\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) \quad , \quad x \geq \gamma \tag{35}$$

where

$$P_0 = \exp\left(-\left(\frac{\gamma}{\beta}\right)^{\alpha}\right) \tag{36}$$

α and β are the form- and shape-parameters. μ and σ has to be determined by numerical integration.

2.2.7 Example: Generalized Pareto distribution

The distribution function for a stochastic variable with expected value μ, standard deviation σ and lower threshold γ is denoted GP($\mu, \sigma; \gamma$), and is defined by:

$$F_X(x) = 1 - e^{-y} \quad , \quad y = \begin{cases} -\beta^{-1}\ln\left(1-\beta(x-\gamma)/\alpha\right) & \beta \neq 0 \\ \beta(x-\gamma)/\alpha & \beta = 0 \end{cases} \tag{37}$$

where α and β are the parameters. The allowable intervals for the parameters are:

$$\alpha > 0 \qquad -\infty < \gamma < \infty \qquad -\infty < \beta < \infty$$

$$\begin{aligned} &\text{if} \quad \beta > 0: \ \gamma \leq x \leq \gamma + \frac{\alpha}{\beta} \\ &\text{if} \quad \beta \leq 0: \ \gamma \leq x < \infty \end{aligned} \tag{38}$$

The parameters are related to μ and σ by:

$$\mu = \frac{\alpha}{1+\beta} + \gamma \tag{39}$$

$$\sigma = \frac{\alpha}{\sqrt{(1+\alpha)^2(1+2\alpha)}} \tag{40}$$

The generalized Pareto distribution has:

Skewness: $\beta_1 = \left(2(1-\beta)\sqrt{(1+2\beta)}/(1+3\beta)\right)^2$

Kurtosis: $\beta_2 = \frac{3(1+2\beta)(3-\beta+2\beta^2)}{(1+3\beta)(1+4\beta)}$

2.2.8 Example: Gumbel distribution

The distribution function for a stochastic variable with expected value μ and standard deviation σ is denoted G(μ , σ), and is defined by:

$$F_X(x) = \exp\left(-\exp\left(-\alpha(x-\beta)\right)\right) \tag{41}$$

where α and β are shape and scale parameters. These are related to μ and σ by:

$$\mu = \beta + \frac{0.5772}{\alpha} \tag{42}$$

$$\sigma = \frac{\pi}{\alpha\sqrt{6}} \tag{43}$$

The Gumbel distribution has:

Skewness: $\beta_1 = 1.3$

Kurtosis: $\beta_2 = 5.4$

Figure 8 shows the density function for a Gumbel distributed stochastic variable with expected value 10 and standard deviation 3.

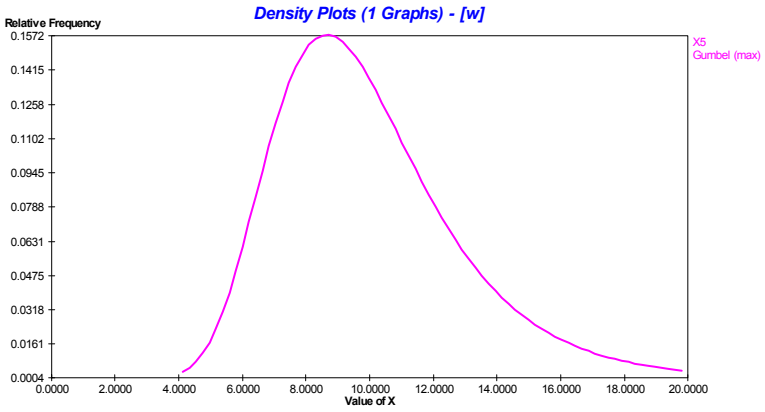


Figure 8. Gumbel distributed stochastic variable with expected value 10 and standard deviation 3.

The 98% quantile $x_{0.98}$ defined as $F_X(x_{0.98}) = 0.98$ becomes

$$x_{0.98} = u - \frac{1}{\alpha}\ln\left(-\ln(0.98)\right) = \mu\left(1 - V\frac{\sqrt{6}}{\pi}\left(0.5772 + \ln\left(-\ln(0.98)\right)\right)\right)$$

The distribution function for the maximum $Y = \max\{X_1, X_2, ..., X_n\}$ of n independent Gumbel distributed stochastic variables $X_1, X_2, ..., X_n$ becomes (see also section 3):

$$F_Y(y) = \left(F_X(y)\right)^n = \exp\left(-n\exp\left(-\alpha(y-\beta)\right)\right)$$

with expected value and standard deviation:

$$\mu_Y = \mu + \frac{\sqrt{6}}{\pi}\sigma \ln(n) = \mu\left(1 + V\frac{\sqrt{6}}{\pi}\ln(n)\right)$$

$$\sigma_Y = \sigma$$

2.3 Conditional distributions

The conditional distribution function for X_1 given X_2 is defined by

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} \tag{44}$$

X_1 and X_2 are statistically independent if $f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1)$ implying that

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \tag{45}$$

2.4 Covariance and correlation

The **covariance** between X_1 and X_2 is defined by

$$Cov[X_1,X_2] = E[(X_1-\mu_1)(X_2-\mu_2)] \tag{46}$$

It is seen that

$$Cov[X_1,X_1] = Var[X_1] = \sigma_1^2 \tag{47}$$

The **correlation coefficient** between X_1 and X_2 is defined by

$$\rho_{X_1,X_2} = \frac{Cov[X_1,X_2]}{\sigma_1\sigma_2} \tag{48}$$

and is a measure of linear dependence between X_1 and X_2. Further:

$$-1 \le \rho_{X_1,X_2} \le 1 \tag{49}$$

If $\rho_{X_1,X_2} = 0$ then X_1 and X_2 is uncorrelated, but not necessarily statistically independent.

For a stochastic vector $X = (X_1, X_2, \cdots, X_n)$ the covariance-matrix is defined by

$$\mathbf{C} = \begin{bmatrix} Var[X_1,X_1] & Cov[X_1,X_2] & \dots & Cov[X_1,X_n] \\ Cov[X_1,X_2] & Var[X_2,X_2] & \cdots & Cov[X_2,X_n] \\ \vdots & \vdots & \ddots & \vdots \\ Cov[X_1,X_n] & Cov[X_2,X_n] & \dots & Var[X_n,X_n] \end{bmatrix} \tag{50}$$

Correspondingly the correlation coefficient matrix is defined by

$$\rho = \begin{bmatrix} 1 & \rho_{X_1,X_2} & \cdots & \rho_{X_1,X_n} \\ \rho_{X_1,X_2} & 1 & \cdots & \rho_{X_2,X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{X_1,X_n} & \rho_{X_2,X_n} & \cdots & 1 \end{bmatrix} \tag{51}$$

The correlation coefficient matrix has to be positive definite.

Example 2.3.1 Linear combination of stochastic variables
Consider the following linear combination of the stochastic variables $X = (X_1, X_2, \cdots, X_n)$:
$Y = a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$

Y becomes a stochastic variable with expected value
$\mu_Y = a_0 + a_1 \mu_1 + a_2 \mu_2 + \cdots + a_n \mu_n$
where $\mu_1, \mu_2, \cdots, \mu_n$ are expected values of $X_1, X_2, \cdots, X_n$.

The variance of Y becomes
$\sigma_Y^2 = E\left[(Y - \mu_Y)^2\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_i \sigma_j$
where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are standard deviations of $X_1, X_2, \cdots, X_n$. ρ_{ij} is the correlation coefficient of X_i, X_j. σ_Y is the standard deviation of Y.

If the stochastic variables $X_1, X_2, \cdots, X_n$ are independent then
$\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

Finally, it can be shown that if $X_1, X_2, \cdots, X_n$ are Normal distributed then Y is also Normal distributed.

3 Estimation of distribution parameters

The following general comments can be made in relation to choice of distribution functions.

For **extreme loads** for example the annual maximum / extreme value of the load (wind velocity / wind pressure, significant wave height, ...) is the important value. If $X_1, X_2, \cdots, X_n$ are independent stochastic variables with identical distribution function F_X then the maximum value
$Y = \max\{X_1, X_2, \cdots, X_n\}$
has the distribution function
$F_Y(y) = P(Y \le y) = P(\max\{X_1, X_2, \cdots, X_n\} \le y)$
$= P(\{X_1 \le y\} \cap \{X_2 \le y\} \cap \ldots \cap \{X_n \le y\}) = (F_X(y))^n$

The density unction becomes
$f_Y(y) = n(F_X(y))^{n-1} f_X(y)$

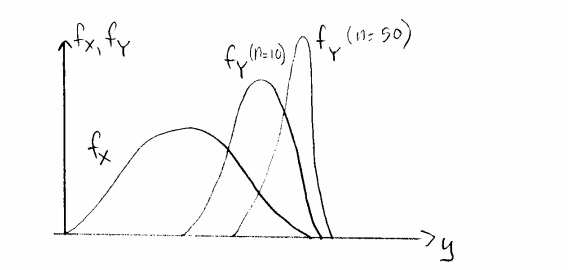


Figure 9. Density functions for extreme loads.

Figure 9 illustrates the density functions $f_X(y)$ and $f_Y(y)$. It is seen that as n increases the density function for the maximum value becomes more narrow (smaller coefficient of variation) and the expected value increases. It can be shown that in general $f_Y(y)$ approaches one of the so-called extreme distribution.

The following distributions can be relevant for extreme loads:

- Gumbel distribution. This distribution is recommended / used in JCSS [5], DS 410 [6], EN 1990 [7] (Basis of Design, annex C), ISO 2394 [8] for annual maximum wind pressure, snow load and temperature load.
- Weibull distribution (2-parameter / 3-parameter / truncated). This distribution is often used for significant wave heights in design and analysis of offshore structures.
- Generalised Pareto distribution. This distribution is recommended in e.g. van Gelder [9] for significant wave heights on shallow water. van Gelder [9] recommends on the basis of statistical analysis of measured data from a range of measurement stations placed at the coasts in the southern part of the North Sea, that a generalised Pareto distribution is used for the maximum significant wave height on shallow water.

For **fatigue** analysis where a good fit is important for the central part of the distribution of the load variations (stress ranges) the following distribution types will be relevant for wind velocities and significant wave heights:

- Normal distribution
- LogNormal distribution
- Weibull distribution. This distribution is used e.g. in Windatlas [10]

For **material strengths** the following distribution types can be considered:

- Normal distribution. If the strength can be considered as a sum of individual Normal distributed contributions, then example 2.3.1 shows that the sum becomes Normal distributed. For ductile materials this is a reasonable assumption. If the individual contributions are non-Normal distributed and no-one of them contributes much more than the others then according to the central limit theorem the sum becomes asymptotically Normal distributed. However, the normal distribution has the drawback that for strengths with large coefficient of variation there is a non-negligible probability for negative strengths. Therefore the Normal distribution cannot be recommended for materials with a high coefficient of variation.
- LogNormal distribution. If the strength can be considered as a product of individual LogNormal distributed contributions, then following example 2.3.1 the product becomes LogNormal distributed since $\ln X$ is Normal distributed if X is LogNormal distributed. Further if the individual contributions are non-LogNormal distributed and no-one of them contributes much more than the others then according to the central limit theorem the product becomes asymptotically LogNormal distributed. The LogNormal distribution is used / recommended in DS410 [6], Eurocodes (Basis of Design, annex C) [7] and ISO 2394 [8].
- Weibull distribution. This distribution is recommended for strengths where the largest defect is important for the value of the material strength (i.e. size effects are important), see e.g. Eurocodes (Basis of Design, annex C) [7] and ISO 2394 [8].

A number of methods can be used to estimate the statistical parameters in distribution functions, for example:

- The Maximum Likelihood method
- The Moment method
- The Least Square method
- Bayesian statistics

In general the Maximum Likelihood method or Bayesian statistics is recommended. The Maximum Likelihood method gives a consistent estimate of the statistical uncertainties. In Bayesian statistics it is possible to take consistently into account subjective / prior information.

3.1 Maximum Likelihood method

A Likelihood function is formulated which gives the probability that the actual data is an outcome of a given distribution with given statistical parameters. The statistical parameters are determined such that this probability is maximum. It is assumed that the given data are statistically independent.

As a example a truncated Weibull distribution is considered. The Log-Likelihood function becomes:

$$\ln L(\alpha,\beta,\gamma) = \ln\left(\prod_{i=1}^{n} f_X(x_i)\right) = \sum_{i=1}^{n} \ln\left(\frac{\alpha}{P_0}\left(\frac{x_i}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right) \tag{52}$$

where

$$P_0 = \exp\left(-\left(\frac{\gamma}{\beta}\right)^{\alpha}\right) \tag{53}$$

The statistical parameters are α, β and γ. x_i ,$i = 1,n$ are n data values. The optimization problem to obtain the maximum value of the log-Likelihood function, $\max_{\alpha,\beta,\gamma} \ln L(\alpha,\beta,\gamma)$ can be solved using non-linear optimization algorithms, e.g. NLPQL, [11]. The result is the best estimate of the statistical parameters α, β and γ.

Since the statistical parameters α, β and γ are determined from a limited number of data, the estimates will be subjected with statistical uncertainty. If the number of data is larger than 25-30 α, β and γ can be assumed to be asymptotically Normal distributed with expected values equal to the solution of the optimization problem and with the following covariance matrix, see e.g. Lindley, [4]

$$C_{\alpha,\beta,\gamma} = \left[-H_{\alpha\beta\gamma}\right]^{-1} = \begin{bmatrix} \sigma_\alpha^2 & \rho_{\alpha\beta}\sigma_\alpha\sigma_\beta & \rho_{\alpha\gamma}\sigma_\alpha\sigma_\gamma \\ \rho_{\alpha\beta}\sigma_\alpha\sigma_\beta & \sigma_\beta^2 & \rho_{\beta\gamma}\sigma_\beta\sigma_\gamma \\ \rho_{\alpha\gamma}\sigma_\alpha\sigma_\gamma & \rho_{\beta\gamma}\sigma_\beta\sigma_\gamma & \sigma_\gamma^2 \end{bmatrix} \tag{54}$$

where $H_{\alpha\beta\gamma}$ is the Hessian matrix with second derivatives of the Log-Likelihood function. σ_α, σ_β and σ_γ are standard deviations of α, β and γ. $\rho_{\alpha\beta}$ is the correlation coefficient between α and β.The Hessian matrix is determined by numerical differentiation.

3.2 Moment method

The unknown parameters in a given distribution function $F_X(x|\theta)$ for a stochastic variable X is denoted $\theta = (\theta_1,\theta_2,...,\theta_m)$. The theoretical statistical moments are with given $\theta = (\theta_1,\theta_2,...,\theta_m)$

$$m_j = \int x^j f_X(x|\theta)dx \tag{55}$$

On the basis of data / observations $\hat{\mathbf{x}} = (\hat{x}_1,\hat{x}_2,...,\hat{x}_n)$ the empirical moments are

$$\hat{m}_j = \frac{1}{n}\sum_{i=1}^{n} \hat{x}_i \tag{56}$$

Requiring that the theoretical moments are equal to the empirical moments the statistical parameters $\theta = (\theta_1, \theta_2, ..., \theta_m)$ can be determined.

It is noted that the method does not give an estimate of the statistical uncertainties and that it is not possible to include prior information. However, bootstrapping can in some situations be used to estimate the statistical uncertainties.

3.3 Least Squares method

The unknown parameters in a given distribution function $F_X(x|\theta)$ for a stochastic variable X is denoted $\theta = (\theta_1, \theta_2, ..., \theta_m)$.

On the basis f data / observations $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ an empirical distribution function is determined using e.g. the Weibull – plotting formula:

$$\hat{F}_i = \frac{i}{n+1} \quad , x = \hat{x}_i \tag{57}$$

The statistical parameters are determined by considering the optimization problem:

$$\min_{\theta} \sum_{i=1}^{n} \left(\hat{F}_i - F_X(x_i) \right)^2 \tag{58}$$

The solution of this optimization problem is a central estimate of the statistical parameters $\theta = (\theta_1, \theta_2, ..., \theta_m)$.

If the distribution has to be good in the tails of the distribution the summation in (58) can be reduced to e.g. the smallest 30% of the data.

3.4 Bayesian statistics

Bayesian statistics has the advantage that it is possible to determine the statistical parameters in a stochastic model (distribution function) such both the actual data (measurements) and prior knowledge can be used. Furthermore Bayesian statistics has the advantage that it is easy to make an updating if new data becomes available.

Consider a stochastic variable X with distribution function $F_X(x|\theta)$ which depends on the statistical parameters $\theta = (\theta_1, \theta_2, ...)$. For a Normal distribution the statistical parameters are equal to the expected value and the standard deviation.

It is assumed that one or more of the statistical parameters are uncertain, and that prior knowledge on this uncertainty can be expressed in a prior density function for parameters: $f'_\theta(\theta)$.

If data is available these can be used to update this prior knowledge. The updated – posterior density function for the statistical parameters can be determined by

$$f''_\theta(\theta|\hat{\mathbf{x}}) = \frac{f_X(\hat{\mathbf{x}}|\theta) f'_\theta(\theta)}{\int f_X(\hat{\mathbf{x}}|\theta) f'_\theta(\theta) d\theta} \tag{59}$$

where $f_X(\hat{\mathbf{x}}|\theta) = \prod_{i=1}^{n} f_X(\hat{x}_i|\theta)$ is the probability (Likelihood) for the given data / observations $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ if the statistical parameters is equal to θ.

The predictive (updated) density function for X given data $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ is determined by

$$f_X(x|\hat{\mathbf{x}}) = \int f_X(x|\theta) f_\theta''(\theta|\hat{\mathbf{x}}) d\theta \tag{60}$$

Prior, posterior and predictive distributions can be established in e.g. the following cases:

- Normal distribution with known standard deviation
- Normal distribution with known expected value
- Normal distribution with unknown expected value and standard deviation
- Lognormal distribution
- Gumbel distribution
- Weibull distribution
- Exponential distribution

It is noted that statistical uncertainty automatically is included in this modeling and that engineering judgments based on experience can be quantified rationally via prior distributions of the statistical parameters θ. Further it can be mentioned that in Eurocode 0, Basis of Design, annex D, [1] it is recommended that Bayesian statistics is used in statistical treatment of data and in design based on tests.

3.5 Example - timber

The following example is from a statistical analysis of timber strength data, see Sørensen & Hoffmeyer [12]. 1600 timber specimens of Norway spruce have been graded visually. 194 of the data has been graded as LT20. The bending strength has been measured, and on the basis of these test data the basic statistical characteristics have been determined, see table a. $x_{0.05}$ denotes the 5% fractile, i.e. $P(X \le x_{0.05}) = F_X(x_{0.05}) = 0.05$.

Number of data	194
Expected value	39.6
COV	0.26
Min. Value	15.9
Max. Value	65.3
$x_{0.05}$	21.6

Table a. Statistical data (in MPa).

Four different distribution types are fitted to the data

- Normal
- Lognormal
- 2 parameter Weibull
- 3-parameter Weibull with γ chosen as 0.9 times the smallest strength value.

The fitting is performed in two ways:

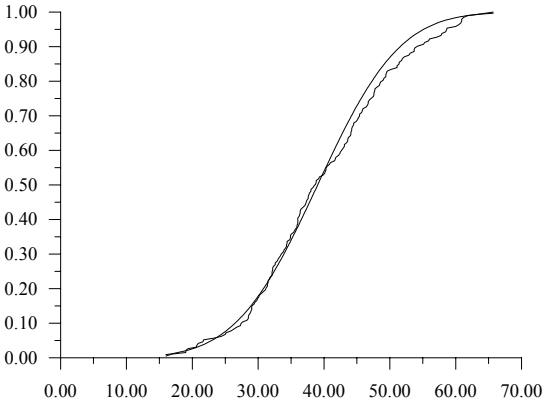
- a fit to all data. The Maximum Likelihood Method is used.
- a tail fit where only 30% of the data is used, namely those data with the lowest strengths, i.e. a fit to the lower tail of the distribution is made. The Least Square Technique

The results are shown in table b.

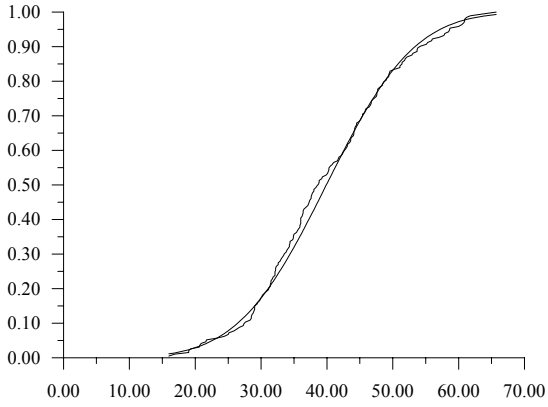
	COV	$x_{0.05}$
Non-parametric	0.26	21.6
Normal	0.26	22.4
Normal – tail	0.25	22.7
LogNormal	0.28	24.1
LogNormal - tail	0.38	22.8
Weibull-2p	0.27	21.3
Weibull-2p - tail	0.23	22.8
Weibull-3p	0.26	23.3
Weibull-3p - tail		

Table b. Statistical data (in MPa).

In figure c to f the distribution fits are shown.

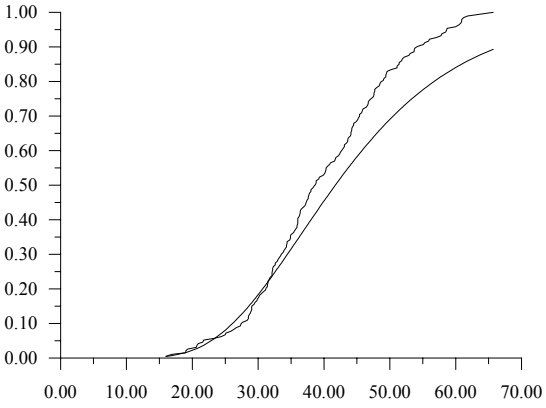


LT20: κ=30% truncation

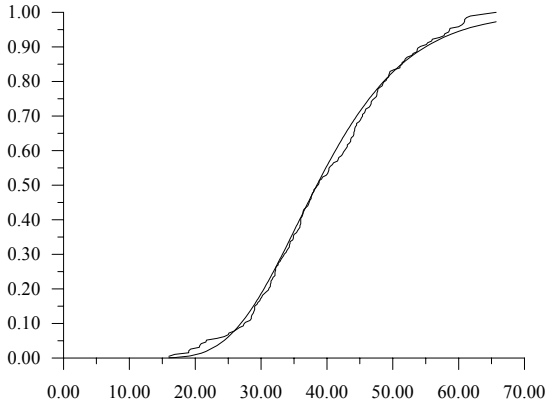


LT20: κ=100% truncation

Figure c. Fit to Normal distribution (in MPa).

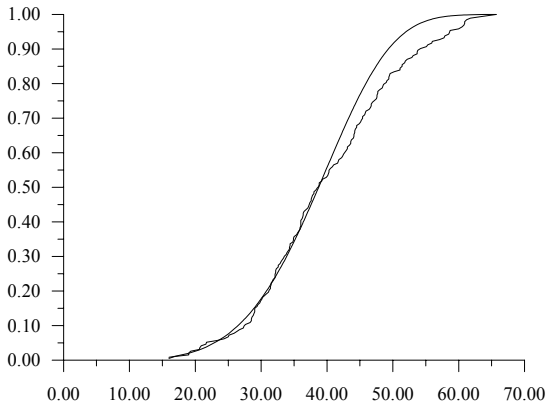


LT20: κ=30% truncation

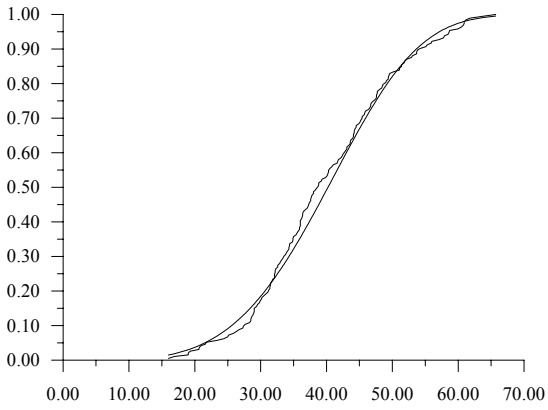


LT20: κ=100% truncation

Figure d. Distribution fits (in MPa). Lognormal distribution.

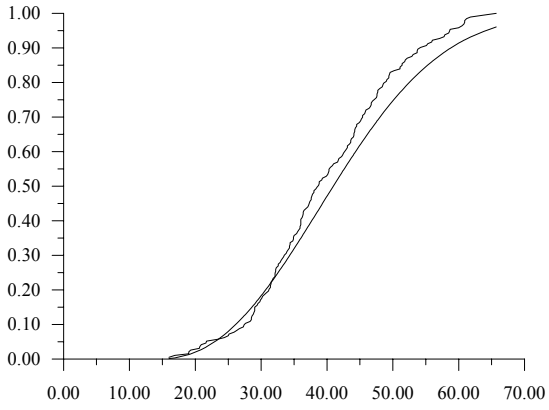


LT20: κ=30% truncation

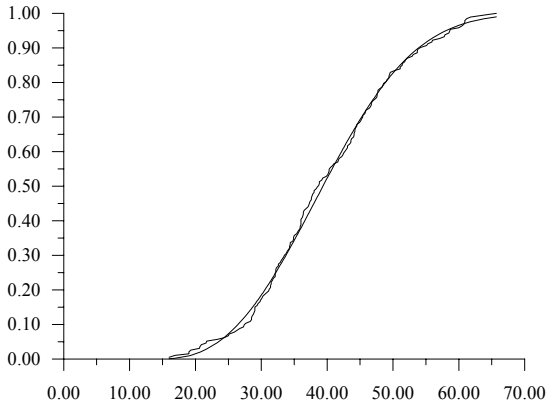


LT20: κ=100% truncation

Figure e. Distribution fits (in MPa). 2 parameter Weibull distribution.



LT20: κ=30% truncation



LT20: κ=100% truncation

Figure f. Distribution fits (in MPa). 3 parameter Weibull distribution.

From the results it is seen that

- the 2 parameter Weibull distribution gives the smallest COV
- the LogNormal distribution gives rather large COV's

3.6 Example – significant wave height

The following example is from Sørensen & Sterndorff [13]. Based on data from the Central part of the North Sea a distribution to the annual maximum significant wave height H_S is calibrated. It is assumed that data sets are available for the largest significant wave heights in each individual storm exceeding a certain threshold for a large number of years, i.e. POT (Peak Over Threshold) data sets. The threshold is determined partly on the basis of engineering judgement. The extreme significant wave heights H_S^* from each storm are assumed to follow a truncated Weibull distribution. The distribution function for the yearly maximum significant omnidirectional wave height H_S can then be written as follows assuming statistical independence between the storms:

$$F_{H_S}(h) = \left(1 - \frac{1}{P_0}\exp\left(-\left(\frac{h}{\beta}\right)^{\alpha}\right)\right)^{\lambda} \quad , \quad h \geq \gamma \qquad \text{and} \quad P_0 = \exp\left(-\left(\frac{\gamma}{\beta}\right)^{\alpha}\right) \tag{a}$$

where γ is the threshold, α is the shape parameter, β is the scale parameter and λ is the number of observed storms per year with H_S^* larger than the threshold.

The parameters α and β are determined using available data and are thus subject to statistical uncertainty. If the parameters are estimated by the Maximum Likelihood technique the uncertainty can be quantified and included in the stochastic model.

Data obtained by continuous simulations of significant wave heights and wave directions for the central part of the North Sea covering the period 1979 to 1993 are used. All storm events are identified and the maximum significant wave height within the eight directional sectors: N, NE, E, S, SW, W, and NW are determined. The simulated wave heights have been calibrated against available measurements from the same location. The calibrated statistical parameters and other optimal parameters are shown in table a together with estimates of the characteristic 100 year wave heights, $H_{S,100}$. In figure b and c empirical and fitted distribution functions are shown for the omnidirectional, SouthWest, West and NorthWest directions.

	α_j	β_j	λ_j	γ_j	$H_{S,100}$
N	3.06	4.25 m	1.20	4.0 m	7.8 m
NE	2.55	2.93 m	1.40	3.0 m	5.9 m
E	3.23	4.36 m	1.60	4.0 m	7.6 m
SE	3.00	3.90 m	1.07	4.0 m	7.0 m
S	3.53	4.75 m	1.53	5.0 m	8.1 m
SW	4.97	6.23 m	1.47	6.5 m	9.5 m
W	6.03	6.90 m	2.20	6.0 m	8.8 m
NW	4.98	6.25 m	1.80	5.25 m	9.1 m
Omni	5.52	6.64 m	3.73	6.0 m	9.2 m

Table a. Estimated statistical parameters and characteristic significant wave height: $H_{S,100}$.

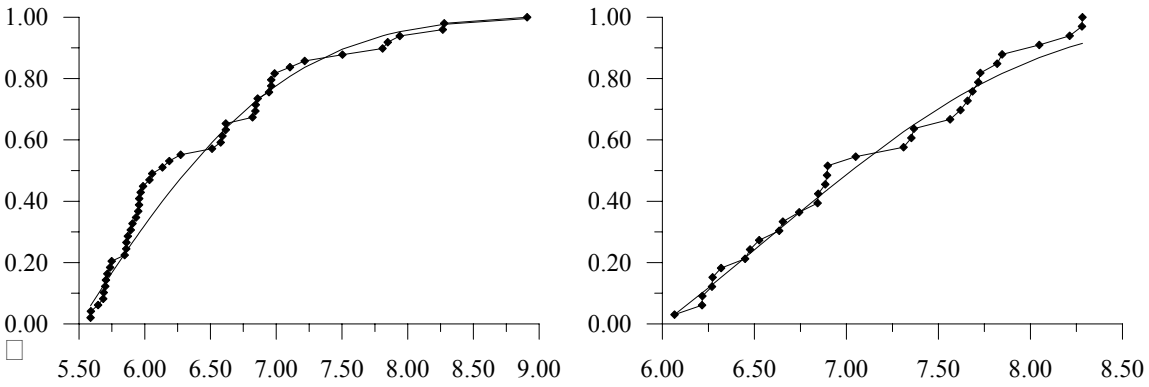


Figure b. Southwest (left) and West (right) empirical and fitted distribution functions.

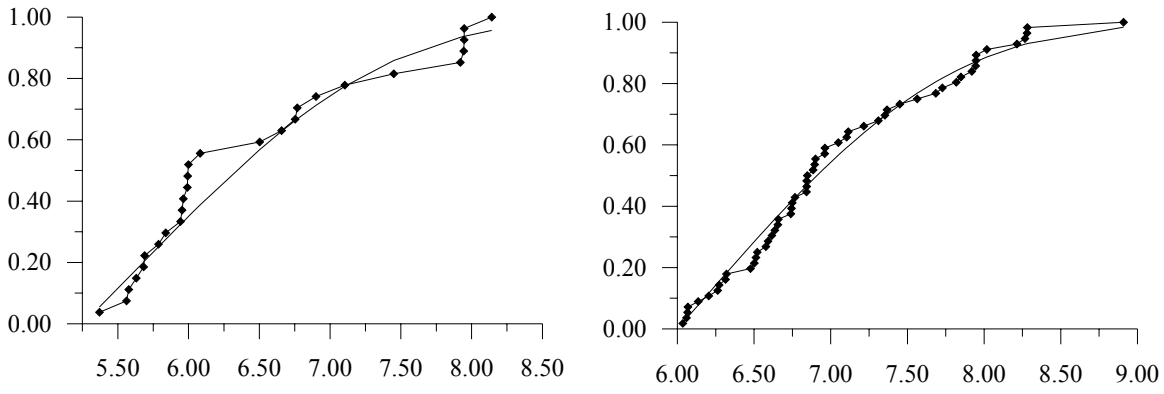


Figure c. Northwest (left) and Omnidirectional empirical and fitted distribution functions.

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Note 3: FIRST ORDER RELIABILITY METHODS

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3.1 Introduction

In this section the problem of estimating the reliability or equivalently the probability of failure is considered. Generally, methods to measure the reliability of a structure can be divided into four groups, see Madsen et al. [3.1], p.30:

- Level I methods: The uncertain parameters are modelled by one characteristic value, as for example in codes based on the partial coefficients concept.
- Level II methods: The uncertain parameters are modelled by the mean values and the standard deviations, and by the correlation coefficients between the stochastic variables. The stochastic variables are implicitly assumed to be normally distributed. The reliability index method is an example of a level II method.
- Level III methods: The uncertain quantities are modelled by their joint distribution functions. The probability of failure is estimated as a measure of the reliability.
- Level IV methods: In these methods the consequences (cost) of failure are also taken into account and the risk (consequence multiplied by the probability of failure) is used as a measure of the reliability. In this way different designs can be compared on an economic basis taking into account uncertainty, costs and benefits.

If the reliability methods are used in design they have to be calibrated so that consistent reliability levels are obtained. This is further discussed in a later note.

Level I methods can e.g. be calibrated using level II methods, level II methods can be calibrated using level III methods, etc.

In this note level II and III reliability methods are considered. Several techniques can be used to estimate the reliability for level II and III methods, e.g.

- Simulation techniques: Samples of the stochastic variables are generated and the relative number of samples corresponding to failure is used to estimate the probability of failure. The simulation techniques are different in the way the samples are generated. Simulation techniques are described in note 5.
- FORM techniques: In First Order Reliability Methods the *limit state function* (*failure function*, see below) is linearized and the reliability is estimated using level II or III methods. FORM techniques for level II methods are described in this note. FORM techniques for level III methods are described in note 4.
- SORM techniques: In Second Order Reliability Methods a quadratic approximation to the failure function is determined and the probability of failure for the quadratic failure surface is estimated. SORM techniques are discussed in note 5.

In section 3.2 basic variables and *failure functions* are defined. Next, a linear failure function is considered in section 3.3 and the reliability index β is defined. In section 3.4 non-linear failure functions are considered. The so-called invariance problem is discussed, and the Hasofer & Lind reliability index β is defined. A numerical algorithm for determination of the reliability index is

shown. Finally it is shown how a sensitivity analysis of the reliability index with respect to a deterministic parameter can be performed.

3.2 Basic Variables and Limit State Functions

It is assumed in this section and in section 4 and 5 (notes 4 and 5) that only one failure mode is considered and that a reliability measure related to this failure mode is to be estimated. Further, it is assumed that it is possible to give a mathematical formulation of this failure mode. An important step in a reliability analysis is to decide which quantities should be modelled by stochastic variables and which should be modelled by deterministic parameters. The stochastic variables are denoted $\mathbf{X} = (X_1,\ldots,X_n)$. The *n* stochastic variables could model physical uncertainty, model uncertainty or statistical uncertainty. The physical stochastic variables can be load variables (e.g. traffic load), resistance variables (e.g. yield strength) or geometrical variables (e.g. length or cross-sectional area of a beam). The variables in $\mathbf{X}$ are also denoted basic variables. Realizations of the basic variables are denoted $\mathbf{x} = (x_1 \ldots, x_n)$, i.e. $\mathbf{x}$ is a point in the *n*-dimensional basic variable space.

The joint density function for the stochastic variables $\mathbf{X}$ is denoted $f_{\mathbf{X}}(\mathbf{x})$. The elements in the vector of expected values and the covariance vector are:

$$\mu_i = E[X_i] \quad , \quad i = 1,\ldots,n \tag{3.1}$$

$$C_{ij} = \mathrm{Cov}[X_i, X_j] \quad , \quad i,j = 1,\ldots,n \tag{3.2}$$

The standard deviation of X_i is denoted σ_i. The variance of X_i is $\sigma_i^2 = C_{ii}$. The coefficient of correlation between X_i and X_j is defined by:

$$\rho_{ij} = \frac{C_{ij}}{\sigma_i \sigma_j} \quad , \quad i,j = 1,\ldots,n \tag{3.3}$$

It is easy to see that $-1 \le \rho_{ij} \le 1$.

Application of FORM, SORM and simulation methods requires as noted above that it is possible for given realizations $\mathbf{x}$ of the basic variables to state whether the structure (or component/failure mode) is in a safe state or in a failure state. The basic variable space is thus divided into two sets, the safe set ω_S and the failure set ω_F. The two sets are separated by the failure surface (limit state surface). It is assumed that the failure surface can be described by the equation:

$$g(\mathbf{x}) = g(x_1,\ldots,x_n) = 0$$

where $g(\mathbf{x})$ is denoted the failure function.

Usually the failure function is defined such that positive values of *g* correspond to safe states and negative values correspond to failure states, see figure 3.1.

$$g(\mathbf{x})\begin{cases} > 0 & , \quad \mathbf{x} \in \omega_s \\ \leq 0 & , \quad \mathbf{x} \in \omega_f \end{cases} \tag{3.4}$$

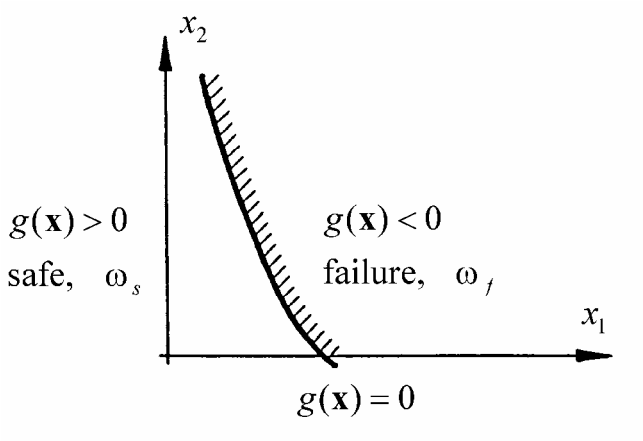

Figure 3.1. Failure function $g(\mathbf{x})$.

It is important to note that the failure surface does not define a unique failure function, i.e. the failure surface can be described by a number of equivalent failure functions. However, whenever possible, differentiable failure functions should be used. In structural reliability the failure function usually results from a mechanical analysis of the structure.

If, in the failure function $\mathbf{x}$ is replaced by the stochastic variables $\mathbf{X}$, the so-called safety margin *M* is obtained:

$$M = g(\mathbf{X}) \tag{3.5}$$

M is a stochastic variable. The probability of failure P_f of the component is:

$$P_f = P(M \leq 0) = P(g(\mathbf{X}) \leq 0) = \int_{\omega_f} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{3.6}$$

Example 3.1
In the fundamental case only two basic variables are used, namely the load variable *P* and the strength variable *S*. A failure function can then be formulated as:

$$g(s, p) = s - p \tag{3.7}$$

The failure surface $g(s, p) = 0$ is shown in figure 3.2. The safety margin corresponding to (3.7) is:

$$M = S - P \tag{3.8}$$

Instead of the failure function (3.7) the following equivalent failure function can be used:

$$g(s, p) = s^3 - p^3 \tag{3.9}$$

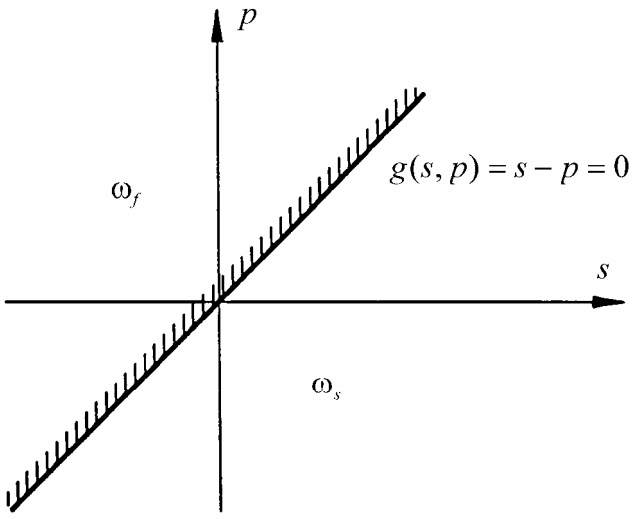

Figure 3.2. Failure function in fundamental case.

3.3 Reliability Analysis for Linear Safety Margins

A safety margin, which is linear in basic variables, can be written:

$$M = a_0 + a_1 X_1 + \cdots + a_n X_n \tag{3.10}$$

where $a_0, a_1, \ldots, a_n$ are constants. The expected value μ_M and the standard deviation σ_M are:

$$\mu_M = a_0 + a_1 \mu_{x_1} + \cdots + a_n \mu_{x_n} = a_0 + \mathbf{a}^T \mu_{\mathbf{X}} \tag{3.11}$$

$$\sigma_M = \sqrt{\mathbf{a}^T \mathbf{C} \mathbf{a}} \tag{3.12}$$

If the basic variables are independent (3.12) simplifies to:

$$\sigma_M = \sqrt{a_1^2 \sigma_{X_1}^2 + \cdots + a_n^2 \sigma_{X_n}^2} \tag{3.13}$$

As a measure of the reliability of a component with the linear safety margin (3.10) the reliability index β can be used:

$$\beta = \frac{\mu_M}{\sigma_M} \tag{3.14}$$

This definition of the reliability index was used by Cornell [3.2].

If the basic variables are normally distributed and the safety margin is linear then *M* becomes normally distributed. The probability of failure is, see figure 3.3:

$$P_f = P(M \le 0) = P\left(\mu_M + U\sigma_M \le 0\right) = P\left(U \le -\frac{\mu_M}{\sigma_M}\right) = \Phi(-\beta) \tag{3.15}$$

where Φ is the standard normal distribution function and U is a standard normally distributed variable with expected value zero and unit standard deviation $(\mu_U = 0, \sigma_U = 1)$.

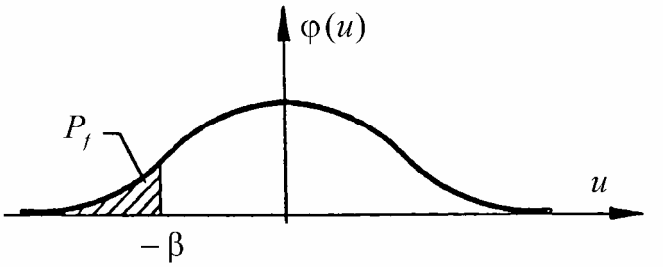

Figure 3.3. Illustration of reliability index and probability of failure. φ is the standard normal density function.

Example 3.2

Consider the fundamental case with the linear failure function (3.7). If the stochastic variables P and S are independent then the reliability index becomes:

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{\mu_S - \mu_P}{\sqrt{\sigma_S^2 + \sigma_P^2}}$$

Assume that P and S are normally distributed with expected values $\mu_P = 2,\ \mu_S = 3.5$ and standard deviations $\sigma_P = 0.3,\ \sigma_S = 0.25$.

The reliability index becomes:

$$\beta = \frac{3.5 - 2}{\sqrt{0.25^2 + 0.3^2}} = 3.84$$

Example 3.3 - Geometrical Interpretation of Reliability Index

Consider a simple problem with two basic independent variables X_1 and X_2 and a linear failure function:

$$g(\mathbf{x}) = a_0 + a_1 x_1 + a_2 x_2 \tag{3.16}$$

If normalized stochastic variables U_1 and U_2 with zero expected value and unit standard deviation are introduced by:

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad i = 1, 2 \tag{3.17}$$

then the failure function can be written:

$$g(\mathbf{u}) = a_0 + a_1(\mu_{X_1} + \sigma_{X_1} u_1) + a_2(\mu_{X_2} + \sigma_{X_2} u_2)$$
$$= a_0 + a_1\mu_{X_1} + a_2\mu_{X_2} + a_1\sigma_{X_1} u_1 + a_2\sigma_{X_2} u_2$$

or equivalently if the reliability index β is introduced:

$$g(\mathbf{u}) = \beta - \alpha_1 u_1 - \alpha_2 u_2$$

where:

$$\beta = \frac{a_0 + a_1\mu_{X_1} + a_2\mu_{X_2}}{\sqrt{a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2}}$$

$$\alpha_i = \frac{-a_i\sigma_{X_i}}{\sqrt{a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2}} \quad i = 1, 2$$

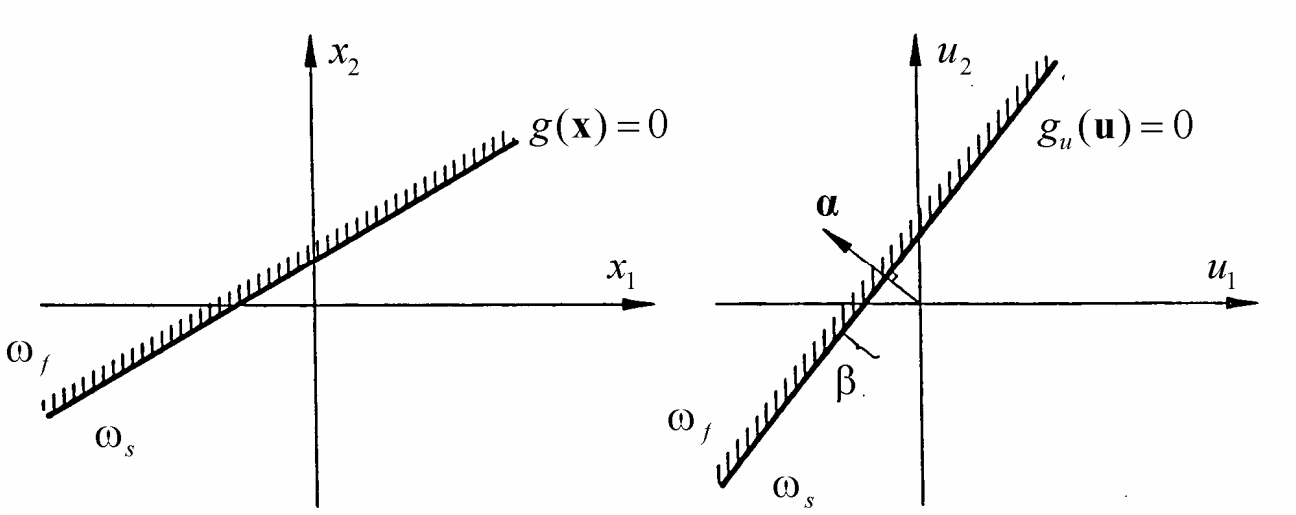

Figure 3.4. Linear failure function in the *x*-space and in the normalized *u*-space.

In figure 3.4 the failure function in the *x*-space and in the *u*-space is shown. It is seen that β is the shortest distance from origo to the failure surface in the normalized space and that the coefficients α_1 and α_2 are elements in a unit vector, $\boldsymbol{\alpha}$, normal to the failure surface.

3.4 Reliability Analysis with Non-Linear Failure Functions

In general the failure function is non-linear and the safety margin $M = g(\mathbf{X})$ is thus not normally distributed.

A first approximation to obtain an estimate of the reliability index in this case could be to linearize the safety margin with the point corresponding to the expected values as expansion point:

$$M \cong g(\boldsymbol{\mu}_{\mathbf{X}}) + \sum_{i=1}^{n} \left. \frac{\partial g}{\partial X_i} \right|_{\mathbf{X}=\boldsymbol{\mu}_{\mathbf{X}}} \left(X_i - \mu_{X_i} \right) \tag{3.18}$$

The reliability index can then be estimated from (3.11) - (3.14). However, as noted above, the failure surface $g(\mathbf{x}) = 0$ can be defined by many different but equivalent failure functions.

This implies that the reliability index based on the linearized safety margin becomes dependent on the mathematical formulation of the safety margin. This problem is also known as the *invariance problem*.

In 1974 Hasofer & Lind [3.3] proposed a definition of the reliability index which is invariant with respect to the mathematical formulation of the safety margin.

In this section it is assumed that the stochastic variables $X_i, i = 1,\ldots,n$ are independent. Further, it is implicitly assumed that the variables are normally distributed. The first step in calculation of the Hasofer & Lind reliability index β_{HL} is to define a transformation from $\mathbf{X}$ to stochastic variables $\mathbf{U}$ that are normalized. The normalized variables $U_i, i = 1,\ldots,n$ with expected values 0 and standard deviation 1 are defined by:

$$U_i = \frac{X - \mu_{X_i}}{\sigma_{X_i}} \quad i = 1,2,\ldots,n \tag{3.19}$$

By this transformation the failure surface in the new *u*-space is given by, see figure 3.5:

$$g(\mu_{X_1} + \sigma_{X_1} u_1, \ldots, \mu_{X_n} + \sigma_{X_n} u_n) = g_u(\mathbf{u}) = 0 \tag{3.20}$$

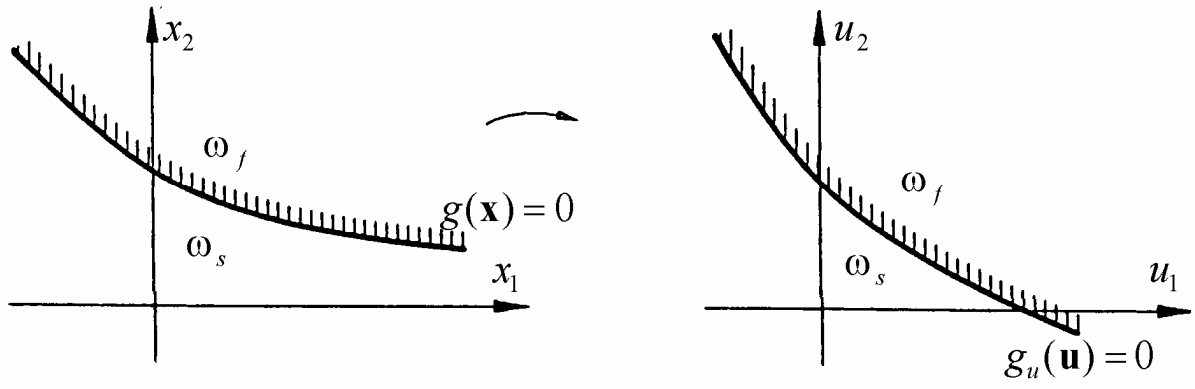

Figure 3.5. Failure functions in the *x*-space and the *u*-space.

It should be noted that the *u*-space is rotationally symmetric with respect to the standard deviations.

The Hasofer & Lind reliability index β is defined as the smallest distance from the origin O in the u-space to the failure surface $g_u(\mathbf{u}) = 0$. This is illustrated in figure 3.6. The point A on the failure surface closest to the origin is denoted the β*-point* or the *design point*. The Hasofer & Lind reliability index defined in the u-space is invariant to different equivalent formulations of the failure function because the definition of the reliability index is related to the failure surface and not directly to the failure function. The reliability index is thus defined by the optimization problem:

$$\beta = \min_{g_u(\mathbf{u})=0} \sqrt{\sum_{i=1}^{n} u_i^2} \tag{3.21}$$

The solution point for $\mathbf{u}$ is denoted $\mathbf{u}^*$, see figure 3.6.

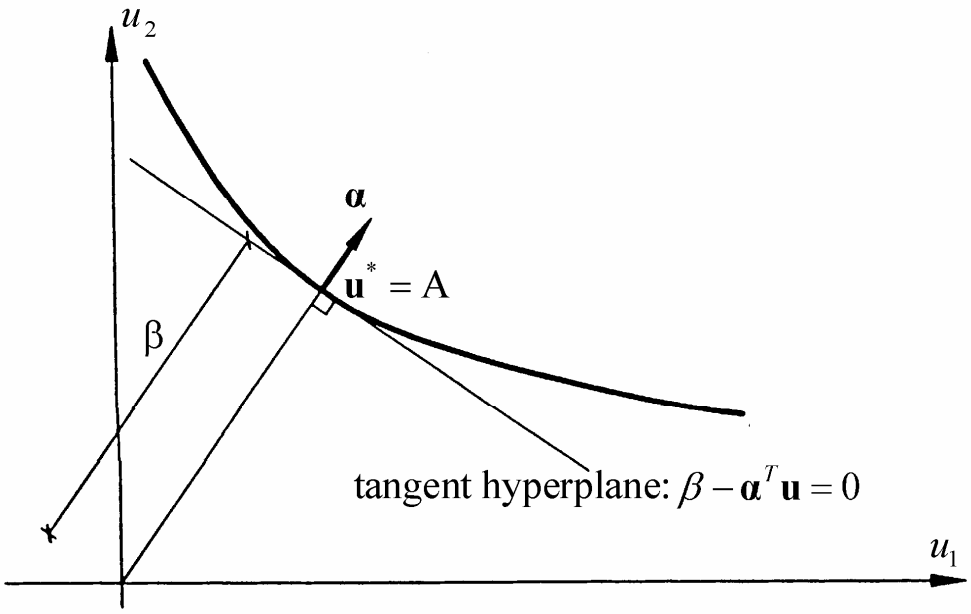

Figure 3.6. Geometrical illustration of the reliability index β.

If the failure surface is linear it is easy to see that the Hasofer & Lind reliability index is the same as the reliability index defined by (3.14). The Hasofer & Lind reliability index can thus be considered a generalization of the Cornell reliability index.

The numerical calculation of the reliability index β defined by (3.21) can be performed in a number of ways. (3.21) is an optimization problem with a quadratic objective function and one non-linear constraint. A number of algorithms exist for solution of this type of problem, e.g. the NLPQL algorithm by Schittkowski [3.4]. Here a simple iterative algorithm will be described. For simplicity the index u will be omitted on the failure function $g(\mathbf{u})$ in the following.

At the β point $\mathbf{u}^*$ it is seen that the following relation must be fulfilled:

$$\mathbf{u}^* = \lambda \nabla g(\mathbf{u}^*) \tag{3.22}$$

where λ is a proportionality factor. In order to formulate an iteration scheme it is assumed that a point $\mathbf{u}^0$ close to $\mathbf{u}^*$ is known, i.e.:

$$\mathbf{u}^* = \mathbf{u}^0 + \Delta\mathbf{u} \tag{3.23}$$

A first order approximation of $g(\mathbf{u})$ in $\mathbf{u}^0$ then gives:

$$g(\mathbf{u}^*) \cong g(\mathbf{u}^0) + \nabla g(\mathbf{u}^0)^T (\mathbf{u}^* - \mathbf{u}^0) = g(\mathbf{u}^0) + \nabla g(\mathbf{u}^0)^T \Delta\mathbf{u} \tag{3.24}$$

Application of (3.22) and (3.23) gives:

$$g(\mathbf{u}^*) \cong g(\mathbf{u}^0) + \nabla g(\mathbf{u}^0)^T (\mathbf{u}^* - \mathbf{u}^0) \cong g(\mathbf{u}^0) + \nabla g(\mathbf{u}^0)^T (\lambda \nabla g(\mathbf{u}^0) - \mathbf{u}^0) \tag{3.25}$$

from which λ can be determined using that $g(\mathbf{u}^*) = 0$:

$$\lambda = \frac{\nabla g(\mathbf{u}^0)^T \mathbf{u}^0 - g(\mathbf{u}^0)}{\nabla g(\mathbf{u}^0)^T \nabla g(\mathbf{u}^0)} \tag{3.26}$$

The following iteration scheme can then be formulated

1. Guess $(\mathbf{u}^0)$
   Set $i = 0$
2. Calculate $g(\mathbf{u}^i)$
3. Calculate $\nabla g(\mathbf{u}^i)$
4. Calculate an improved guess of the β point using (3.22) and (3.23)
   $$\mathbf{u}^{i+1} = \nabla g(\mathbf{u}^i) \frac{\nabla g(\mathbf{u}^i)^T \mathbf{u}^i - g(\mathbf{u}^i)}{\nabla g(\mathbf{u}^i)^T \nabla g(\mathbf{u}^i)} \tag{3.27}$$
5. Calculate the corresponding reliability index
   $$\beta^{i+1} = \sqrt{(\mathbf{u}^{i+1})^T \mathbf{u}^{i+1}} \tag{3.28}$$
6. If convergence in β (e.g. if $\left|\beta^{i+1} - \beta^i\right| \le 10^{-3}$), then stop, else $i = i + 1$ and go to 2.

If a unit normal vector $\boldsymbol{\alpha}$ to the failure surface at the β point $\mathbf{u}^*$ is defined by:

$$\boldsymbol{\alpha} = -\frac{\nabla g(\mathbf{u}^*)}{\left|\nabla g(\mathbf{u}^*)\right|} \tag{3.29}$$

then the β-point $\mathbf{u}^*$ can be written, see (3.22):

$$\mathbf{u}^* = \beta\boldsymbol{\alpha} \tag{3.30}$$

It is noted that $\boldsymbol{\alpha}$ is directed towards the failure set. The safety margin corresponding to the tangent hyperplane obtained by linearizing the failure function at the β point can then be written:

$$M = \beta - \boldsymbol{\alpha}^T \mathbf{U} \tag{3.31}$$

Further, using that $\boldsymbol{\alpha}^T \boldsymbol{\alpha} = 1$ it is seen from (3.30) that the reliability index β can be written:

$$\beta = \boldsymbol{\alpha}^T \mathbf{u}^* \tag{3.32}$$

For fixed $\boldsymbol{\alpha}$ it is seen that:

$$\left. \frac{d\beta}{du_i} \right|_{\mathbf{u}=\mathbf{u}^*} = \alpha_i \tag{3.33}$$

i.e. the components in the $\boldsymbol{\alpha}$ vector can be considered measures of the relative importance of the uncertainty in the corresponding stochastic variable on the reliability index. However, it should be noted that for dependent (correlated) basic variables the components in the $\boldsymbol{\alpha}$-vector cannot be linked to a specific basic variable, see the next section.

An important sensitivity measure related to α_i is the socalled *omission sensitivity factor* ς_i suggested by Madsen [3.5]. This factor gives the relative importance on the reliability index by assuming that stochastic variable no. i, i.e. it is considered a deterministic quantity. If variable no. i is applied to the value u_i^0, then the safety margin in the normalized space is written:

$$M_i' = \beta - \alpha_i u_i^0 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j U_j \tag{3.34}$$

with the reliability index:

$$\beta_i' = \frac{\beta - \alpha_i u_i^0}{\sqrt{1-\alpha_i^2}} \tag{3.35}$$

The omission sensitivity factor ς_i is defined by:

$$\varsigma_i = \frac{\beta_i'}{\beta} = \frac{1 - \alpha_i u_i^0 / \beta}{\sqrt{1-\alpha_i^2}} \tag{3.36}$$

If especially $u_i^0 = 0$ is chosen, then:

$$\varsigma_i = \frac{1}{\sqrt{1-\alpha_i^2}} \tag{3.37}$$

It is seen that if $|\alpha_i| < 0.14$, then $\varsigma_i - 1 < 0.01$, i.e. the error in the reliability index is less than 1% if a variable with $|\alpha| < 0.14$ is fixed. The omission sensitivity factor can be generalized to non-normal and dependent stochastic variables, see Madsen [3.5].

In this section it is assumed that the stochastic variables are normally distributed. The normalized variables $\mathbf{U}$ defined by the linear transformation (3.19) are thus also normally distributed. If the failure function in the *u*-space is not too non-linear, then the probability of failure P_f can be estimated from:

$$P_f = P(M \le 0) \cong P(\beta - \boldsymbol{\alpha}^T \mathbf{U} \le 0) = \Phi(-\beta) \tag{3.38}$$

where Φ is the standard normal distribution function. The accuracy of (3.38) is further discussed in section 5.

Example 3.4

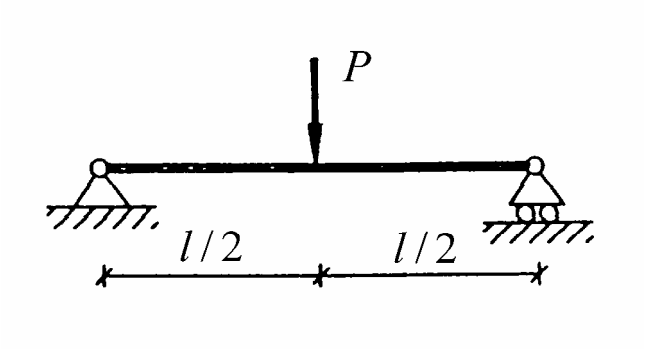

Figure 3.7. Linear elastic beam.

Consider the structure in figure 3.7. The maximum deflection is:

$$u_{\max} = \frac{1}{48}\frac{pl^3}{ei}$$

where *e* is the modulus of elasticity and *i* the moment of inertia. *p, l, e* and *i* are assumed to be outcomes of stochastic variables *P, L , E* and *I* with expected values μ and standard deviations σ.

	$\mu[\cdot]$	$\sigma[\cdot]$
P	2 kN	0.6 kN
L	6 m	~ 0 m
E	$2 \cdot 10^7$ kN/m^2	$3 \cdot 10^6$ kN/m^2
I	$2 \cdot 10^{-5}$ m^4	$2 \cdot 10^{-6}$ m^4

The failure criterion is assumed to be:

$$\frac{u_{\max}}{l} \geq \frac{1}{100}$$

The failure function can then be formulated as follows with $l = 6$ m:

$$g(p,l,e,i) = 48ei - 100pl^2 = 48ei - 3600p$$

The three stochastic variables $X_1 = P, \quad X_2 = E$ and $X = I$ are normalized by:

$$U_1 = \frac{P-2}{0.6} \rightarrow P = 0.6U_1 + 2$$

$$U_2 = \frac{E - 2\cdot 10^7}{0.3\cdot 10^7} \rightarrow E = (0.3U_2 + 2)10^7$$

$$U_3 = \frac{I - 2\cdot 10^{-5}}{0.2\cdot 10^5} \rightarrow I = (0.2U_3 + 2)10^{-5}$$

The failure function in *u*-space becomes:

$$g_u(\mathbf{u}) = 48(0.3u_2 + 2)\,(0.2u_3 + 2)100 - 3600(0.6u_1 + 2)$$

The derivatives with respect to u_1, u_2 and u_3 are:

$$a_1 = \frac{\partial g_u}{\partial g_1} = -2160$$

$$a_2 = \frac{\partial g_u}{\partial g_2} = 1440(0.2\,u_3 + 2)$$

$$a_3 = \frac{\partial g_u}{\partial g_3} = 960(0.3\,u_2 + 2)$$

Using (3.26) – (3.28) the following iteration scheme can be used:

Iteration	1	2	3	4	5
u_1	1.00	1.29	1.90	1.91	1.90
u_2	1.00	-1.89	-2.20	-2.23	-2.25
u_3	1.00	-1.32	-1.21	-1.13	-1.12
β	1.73	2.64	3.15	3.15	3.15
a_1	-2160	-2160	-2160	-2160	
a_2	3168	2500	2532	2555	
a_3	2208	1376	1286	1278	
$\sum a_i^2$	$19.58 \cdot 10^6$	$12.81 \cdot 10^6$	$12.73 \cdot 10^6$	$12.83 \cdot 10^6$	
$\sum a_i u_i$	3216	-9328	-11230	-11267	
$g_u(\mathbf{u})$	14928	1955	3.5	8.1	
λ	$-0.598 \cdot 10^{-3}$	$-0.881 \cdot 10^{-3}$	$-0.882 \cdot 10^{-3}$	$-0.879 \cdot 10^{-3}$	

The reliability index is thus $\beta = 3.15$ and the corresponding $\boldsymbol{\alpha}$-vector is:

$$\boldsymbol{\alpha} = \tfrac{1}{\beta}\mathbf{u}^* = (0.60, -0.71, -0.36).$$

The β point in basic variable space is:

$$\begin{aligned}(p^*, \quad e^*, \quad i^*) &= \left(0.6 \cdot 1.90 + 2, (-0.3 \cdot 2.25 + 2) \cdot 10^7, (-0.2 \cdot 1.12 + 2) \cdot 10^{-5}\right) \\ &= \left(3.14, 1.33 \cdot 10^7, 1.78 \cdot 10^{-5}\right)\end{aligned}$$

The omission sensitivity factor ς_3 corresponding to a fixed variable $u_3 = 0$ is, see (3.37):

$$\varsigma_3 = \frac{1}{\sqrt{1-(-0.36)^2}} = 1.07$$

i.e. the error in β is approximately 7% by assuming U_3 deterministic.

* * *

Another very important sensitivity measure is the *reliability elasticity coefficient* defined by:

$$e_p = \frac{d\beta}{dp}\frac{p}{\beta} \tag{3.39}$$

where p is a parameter in a distribution function (e.g. the expected value or the standard deviation) or p is a constant in the failure function. From (3.39) it is seen that if the parameter p is changed by 1%, then the reliability index is changed by e_p %. $d\beta/dp$ is determined as follows:

The failure function is written:

$$g(\mathbf{u}, p) = 0 \tag{3.40}$$

If the parameter p is given a small increment then β and the β-point change, but (3.40) still has to be satisfied, i.e.:

$$\sum_{i=1}^{n} \frac{\partial g}{\partial u_i} \frac{\partial u_i}{\partial p} + \frac{\partial g}{\partial p} = 0 \tag{3.41}$$

$d\beta/dp$ is determined from:

$$\begin{aligned} \frac{d\beta}{dp} &= \frac{d}{dp} \sqrt{\sum_{i=1}^{n} u_i^2} \\ &= \frac{1}{\beta} \sum_{i=1}^{n} u_i \frac{\partial u_i}{\partial p} \end{aligned} \tag{3.42}$$

Using (3.29) - (3.30) and (3.41), $d\beta/dp$ becomes:

$$\begin{aligned} \frac{d\beta}{dp} &= \frac{1}{\beta} \sum_{i=1}^{n} \frac{-\beta}{|\nabla g|} \frac{\partial g}{\partial u_i} \frac{\partial u_i}{\partial p} \\ &= \frac{1}{|\nabla g|} \frac{\partial g}{\partial p} \end{aligned} \tag{3.43}$$

i.e. $d\beta/dp$ can be estimated on the basis of a partial differentiation of the failure function with respect to the parameter p. $|\nabla g|$ is already determined in connection with calculation of β.

What is the reliability elasticity coefficient e_l for the length l in example 3.4? Using (3.43), $d\beta/dl$ is:

$$\begin{aligned} \frac{d\beta}{dl} &= \frac{1}{|\nabla g|} \frac{\partial g}{\partial l} \\ &= \frac{1}{\sqrt{\sum a_i^2}} (-200\, p \cdot l) \\ &= -1.05 \end{aligned}$$

and thus:

$$e_l = -1.05 \frac{l}{\beta} = -2.00$$

i.e. if the length is increased by 1%, then the reliability index decreases approximately by 2%.

3.5 References

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Note 4: FIRST ORDER RELIABILITY ANALYSIS WITH CORRELATED AND NON-NORMAL STOCHASTIC VARIABLES

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4.1 Introduction

In note 3 it was described how a first order reliability analysis can be performed for uncorrelated and normally distributed stochastic variables. The reliability method which is also named the "First Order Reliability Method" (FORM) results in a reliability index β. In this note it is described how a reliability index β can be determined when the stochastic variables are correlated and non-normally distributed.

4.2 Reliability Index for Correlated, Normally Distributed Variables

Let the stochastic variables X_i, $i = 1,\ldots,n$ be normally distributed with expected values $\mu_{X_1},\ldots,\mu_{X_n}$, standard deviations $\sigma_{X_1},\ldots,\sigma_{X_n}$ and with correlation coefficients $\rho_{ij}, i,j = 1,\ldots,n.$ Further, let a failure function $g(\mathbf{x})$ be given. In order to determine a reliability index for this failure mode a transformation from correlated to uncorrelated stochastic variables is added to the procedure described in section 3.4. This transformation can be performed in several ways, e.g. by determining eigenvalues and eigenvectors, see Thoft-Christensen & Baker [4.1]. Here Choleski triangulation is used. The procedure described in the following requires that the correlation coefficient matrix $\boldsymbol{\rho}$ is positive definite.

The first step is to determine normalized variables $Y_i, i = 1,\ldots,n$ with expected value 0 and standard deviation 1:

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad , \quad i = 1,\ldots,n \tag{4.1}$$

It is easy to see that $\mathbf{Y}$ will have a covariance matrix (and correlation coefficient matrix) equal to $\boldsymbol{\rho}$.

The next step is to define a transformation from $\mathbf{Y}$ to uncorrelated and normalized variables $\mathbf{U}$ with expected values 0 and standard deviations 1. The transformation is written:

$$\mathbf{Y} = \mathbf{T}\mathbf{U} \tag{4.2}$$

where $\mathbf{T}$ is a lower triangular matrix (i.e. $T_{ij} = 0$ for $j > i$). It is seen that the covariance matrix $\mathbf{C}_\mathbf{Y}$ for $\mathbf{Y}$ can be written:

$$\mathbf{C_Y} = E[\mathbf{Y}\mathbf{Y}^T] = E[\mathbf{T}\mathbf{U}\mathbf{U}^T\mathbf{T}^T] = \mathbf{T}E[\mathbf{U}\mathbf{U}^T]\mathbf{T}^T = \mathbf{T}\mathbf{T}^T = \boldsymbol{\rho} \tag{4.3}$$

The elements in $\mathbf{T}$ are then determined from $\mathbf{T}\mathbf{T}^T = \boldsymbol{\rho}$ as:

$$T_{11} = 1$$

$$T_{21} = \rho_{12} \quad T_{22} = \sqrt{1 - T_{21}^2} \tag{4.4}$$

$$T_{31} = \rho_{13} \quad T_{32} = \frac{\rho_{23} - T_{21}T_{31}}{T_{22}} \quad T_{33} = \sqrt{1 - T_{31}^2 - T_{32}^2}$$

etc.

Example 4.1
Let the three normalized stochastic variables $\mathbf{Y} = (Y_1, Y_2, Y_3)$ have the correlation coefficient matrix:

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & 0.5 & 0.2 \\ 0.5 & 1 & 0.4 \\ 0.2 & 0.4 & 1 \end{bmatrix}$$

The transformation matrix $\mathbf{T}$ is then calculated using (4.4):

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.87 & 0 \\ 0.2 & 0.34 & 0.92 \end{bmatrix}$$

The stochastic variables $\mathbf{Y}$ can thus be written:

$$Y_1 = U_1$$
$$Y_2 = 0.5U_1 + 0.87U_2$$
$$Y_3 = 0.2U_1 + 0.34U_2 + 0.92U_3$$

where (U_1, U_2, U_3) are uncorrelated and normalized variables.

* * *

The transformation form $\mathbf{X}$ to $\mathbf{U}$ can now be written:

$$\mathbf{X} = \boldsymbol{\mu}_\mathbf{X} + \mathbf{D}\mathbf{T}\mathbf{U} \tag{4.5}$$

where $\mathbf{D}$ is a diagonal matrix with standard deviations in the diagonal. Using (4.5) the failure function can be written $g(\mathbf{x}) = g(\boldsymbol{\mu}_\mathbf{X} + \mathbf{D}\mathbf{T}\mathbf{u})$ and a reliability index β can be determined as shown in section 3.4.

Example 4.2
A failure mode is modelled by a failure function with three normally distributed variables X_1, X_2, X_3:

$$g(\mathbf{x}) = x_1 - x_2 x_3^2$$

where $\mu_{X_1} = 25.0$, $\sigma_{X_1} = 0.25$, $\mu_{X_2} = 4.0$, $\sigma_{X_2} = 0.2$, $\mu_{X_3} = 2.0$ and $\sigma_{X_3} = 0.1$. The variables are correlated as the variables in example 4.1. The standardized normalized and uncorrelated *u*-variables are obtained from example 4.1 and (4.5) as:

$$X_1 = \mu_{X_1} + \sigma_{X_1} U_1$$
$$X_2 = \mu_{X_2} + \sigma_{X_2}(0.5U_1 + 0.87U_2)$$
$$X_3 = \mu_{X_3} + \sigma_{X_3}(0.2U_1 + 0.34U_2 + 0.92U_3)$$

The failure function in the *u*-space can then be written:

$$g(\mathbf{u}) = 25.0 + 0.25u_1 - \big(4.0 + 0.2(0.5u_1 + 0.87u_2)\big)\big(2.0 + 0.1(0.2u_1 + 0.34u_2 + 0.92u_3)\big)^2$$

The failure function can be used to find β as explained in section 3.4 by the iteration scheme used in example 3.4.

The solution is $\beta = 3.86$ ($P_f = 5.67 \cdot 10^{-5}$), $\mathbf{u}^* = \{1.051, 2.426, 2.812\}$ and $\boldsymbol{\alpha} = \{0.27, 0.63, 0.73\}$.

* * *

4.3 Reliability Index for Independent, Non-Normally Distributed Variables

Generally the stochastic variables are not normally distributed. In order to determine a measure of the reliability of a component (failure mode) with non-normally distributed variables it is natural, as for normally distributed variables, to establish a transformation to standardized (uncorrelated and normalized) normally distributed variables and to determine a Hasofer & Lind reliability index β.
A simple transformation from X_i to U_i can be defined by the identity:

$$\Phi(U_i) = F_{X_i}(X_i) \tag{4.6}$$

where F_{X_i} is the distribution function for X_i. Given a realisation $\mathbf{u}$ of $\mathbf{U}$ a realization $\mathbf{x}$ of $\mathbf{X}$ can be determined by:

$$\begin{aligned} x_1 &= F_{X_1}^{-1}\big(\Phi(u_1)\big) \\ &\vdots \\ x_n &= F_{X_n}^{-1}\big(\Phi(u_n)\big) \end{aligned} \tag{4.7}$$

and the failure surface can be written:

$$g(x_1,\ldots,x_n) = g\left(F_{X_1}^{-1}(\Phi(u_1)),\ldots,F_{X_n}^{-1}(\Phi(u_n))\right) = 0 \tag{4.8}$$

In the algorithm for determination of β (see section 3.4) the gradient of the failure function with respect to u_i is needed. From (4.8):

$$\frac{\partial g}{\partial u_i} = \frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial u_i} = \frac{\partial g}{\partial x_i}\frac{\varphi\left(\Phi^{-1}\left(F_{X_i}(x_i)\right)\right)}{f_{X_i}(x_i)} \tag{4.9}$$

where $f_{X_i}(x_i) = dF_{X_i}(x_i)/dx_i$ = is the density function for X_i.

Example 4.3 Lognormal Variable
For a lognormally distributed variable *X* with expected value μ and standard deviation σ the distribution function is:

$$F_X(x) = \Phi\left(\frac{\ln x - \mu_L}{\sigma_L}\right) \tag{4.10}$$

where:

$$\sigma_L = \sqrt{\ln\left(\frac{\sigma^2}{\mu^2}+1\right)} \qquad \text{and} \qquad \mu_L = \ln\mu - \tfrac{1}{2}\sigma_L^2$$

The transformation (4.7) becomes:

$$x = \exp(\sigma_L u + \mu_L) \tag{4.11}$$

* * *

Example 4.4 Gumbel Variable
For a Gumbel distributed variable *X* with expected value μ and standard deviation σ the distribution function is:

$$F_X(x) = \exp\left[-\exp\left[-a(x-b)\right]\right] \tag{4.12}$$

where:

$$a = \frac{\pi}{\sqrt{6}\sigma} \qquad \text{and} \qquad b = \mu - \frac{0.5772}{a}$$

The transformation (4.7) becomes:

$$x = b - \frac{1}{a}\ln\left[-\ln\Phi(u)\right] \tag{4.13}$$

* * *

The inverse transformation to (4.7) is:

$$\begin{aligned} u_1 &= \Phi^{-1}\left(F_{X_1}(x_1)\right) \\ &\vdots \\ u_n &= \Phi^{-1}\left(F_{X_n}(x_n)\right) \end{aligned} \tag{4.14}$$

When the transformation defined above is applied in connection with the β-algorithm in section 3.4 it is also known under the name of *principle of normal tail approximation*. In the normal tail approximation a normal distribution with parameters μ_i' and σ_i' is determined for each non-normal stochastic variable such that the distribution function values and the density function values are the same at a point x_i' :

$$\Phi\left(\frac{x_i' - \mu_i'}{\sigma_i'}\right) = F_{X_i}(x_i') \tag{4.15}$$

$$\frac{1}{\sigma_i'}\varphi\left(\frac{x_i' - \mu_i'}{\sigma_i'}\right) = f_{X_i}(x_i') \tag{4.16}$$

where f_{X_i} is the density function for X_i.

The solution to (4.15) - (4.16) is:

$$\sigma_i' = \frac{\varphi\left(\Phi^{-1}\left(F_{X_i}(x_i')\right)\right)}{f_{X_i}(x_i')} \tag{4.17}$$

$$\mu_i' = x_i' - \sigma_i'\Phi^{-1}\left(F_{X_i}(x_i')\right) \tag{4.18}$$

Normalized variables are defined by:

$$u_i = \frac{x_i - \mu_i'}{\sigma_i'} \tag{4.19}$$

and the failure function is written:

$$g(x_1,\ldots,x_n) = g(\mu_1' + \sigma_1' u_1,\ldots,u_n' + \sigma_n' u_n) \tag{4.20}$$

The gradient of the failure function with respect to u_i is:

$$\begin{aligned}\frac{\partial g}{\partial u_i} &= \frac{\partial g(\mathbf{x})}{\partial x_i}\frac{\partial x_i}{\partial u_i} \\ &= \frac{\partial g(\mathbf{x})}{\partial x_i}\sigma_i' \\ &= \frac{\partial g(\mathbf{x})}{\partial x_i}\frac{\varphi\left(\Phi^{-1}\left(F_{X_i}(x_i')\right)\right)}{f_{X_i}(x_i')}\end{aligned} \tag{4.21}$$

At the β-point, $\mathbf{u}^*$, and the corresponding point $\mathbf{x}^*$ in the *x*-space the gradient estimated by (4.9) is equal to the gradient estimated by (4.21) if $x_i' = x_i^*, i = 1,2,\ldots,n$. This indicates that if the current guess of the β-point in the algorithm $\mathbf{u}^i$ is used as $\mathbf{u}'$ in (4.17) - (4.21) and if the points $\mathbf{u}^1, \mathbf{u}^2, \ldots$ converge to $\mathbf{u}^*$ then the transformation defined by (4.7) is equivalent to the transformation defined by the normal tail approximation, see Ditlevsen [4.2] for further details.

Example 4.5

Consider the safety margin:

$$M = g(\mathbf{X}) = X_1 - 2X_2^2$$

where:

X_1: is log-normally distributed with expected value $\mu_1 = 10$ and standard deviation $\sigma_1 = 3$ (or LN(10.0, 3.0)). From (4.10) (μ_L, σ_L) = (2.26, 0.294) is obtained.

X_2: is Gumbel distributed with expected value $\mu_1 = 1$ and standard deviation $\sigma_1 = 0.1$ (or EX1(1.0, 0.1)). From (4.12) (*a, b*) = (12.8, 0.955) is obtained.

The transformation from the physical *x*-space to the standard normal *u*-space is found from (4.11) and (4.13):

$$g(\mathbf{u}) = \exp(\sigma_L u_1 + \mu_L) - 2\left(b - \frac{1}{a}\ln\left[-\ln\Phi(u_2)\right]\right)^2$$

By application of the β-iteration scheme explained in section 3.4 β can be found as β = 4.040 and $\mathbf{u}^*$ = {– 2.587, 3.103}, $\boldsymbol{\alpha}$ = {– 0.640, 0.768}.

* * *

4.4 Reliability Index for Dependent, Non-Normally Distributed Variables

In this section two techniques are described, which can be used to determine a reliability index when the stochastic variables are dependent and non-normally distributed, namely methods based on the Rosenblatt transformation, see [4.3] and the Nataf transformation, see [4.4]. It should be noted that if all the stochastic variables are normally and log-normally distributed then the technique described in section 4.2 can be used because the log-normal variables can easily be transformed to normal variables, see example 4.6.

Example 4.6

Consider 3 stochastic variables X_i, $i = 1, 2, 3$ with expected values $\mu[\cdot]$, standard deviations $\sigma[\cdot]$ and coefficients of variation $V[\cdot]$ as shown in this table:

	$\mu[\cdot]$	$\sigma[\cdot]$	$V[\cdot]$
X_1	μ_{X_1}	σ_{X_1}	σ_{X_1}/μ_{X_1}
X_2	μ_{X_2}	σ_{X_2}	σ_{X_2}/μ_{X_2}
X_3	μ_{X_3}	σ_{X_3}	σ_{X_3}/μ_{X_3}

and correlation matrix $\boldsymbol{\rho}$

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & & \text{sym.} \\ \rho_{X_2X_1} & 1 & \\ \rho_{X_3X_1} & \rho_{X_3X_2} & 1 \end{bmatrix}$$

X_1 is assumed to be normally distributed, but X_2 and X_3 are log-normally distributed. Two new variables are defined by $Y_i = \ln X_i$, $i = 2, 3$. They become normally distributed. The expected values and standard deviations of the normally distributed variables X_1, Y_2 and Y_3 become, see example 4.3,

	$\mu[\cdot]$	$\sigma[\cdot]$
X_1	μ_{X_1}	σ_{X_1}
Y_2	$\mu_{Y_2} = \ln \mu_{X_2} - \frac{1}{2}\sigma_{Y_2}^2$	$\sigma_{Y_2} = \sqrt{\ln(V_{X_2}^2 + 1)}$
Y_3	$\mu_{Y_3} = \ln \mu_{X_3} - \frac{1}{2}\sigma_{Y_3}^2$	$\sigma_{Y_3} = \sqrt{\ln(V_{X_3}^2 + 1)}$

The new correlation matrix $\boldsymbol{\rho}'$ of correlation coefficients between X_1, Y_2 and Y_3 can be obtained from the definition of the covariance between two stochastic variables:

$$\boldsymbol{\rho}' = \begin{bmatrix} 1 & & \text{sym.} \\ \dfrac{\rho_{X_2X_1}V_{X_2}}{\sigma_{Y_2}} & 1 & \\ \dfrac{\rho_{X_3X_1}V_{X_3}}{\sigma_{Y_3}} & \dfrac{\ln(1+\rho_{X_2X_3}V_{X_2}V_{X_3})}{\sigma_{Y_2}\sigma_{Y_3}} & 1 \end{bmatrix}$$

Example 4.7

Consider a normally distributed variable X_1 and two log-normally distributed variables X_2 and X_3 with the statistic parameters:

	$\mu[\cdot]$	$\sigma[\cdot]$	$V[\cdot]$
X_1	10.0	2.0	0.20
X_2	5.0	2.5	0.50
X_3	7.0	0.35	0.05

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & & (\text{sym.}) \\ 0.2 & 1 & \\ 0.5 & 0.3 & 1 \end{bmatrix}$$

From example 4.6 the following parameters are obtained for X_1, $Y_2 = \ln X_2$ and $Y_3 = \ln X_3$

	$\mu[\cdot]$	$\sigma[\cdot]$
X_1	10.0	2.0
Y_2	1.50	0.472
Y_3	1.94	0.05

and:

$$\boldsymbol{\rho}' = \begin{bmatrix} 1 & & (\text{sym.}) \\ 0.21 & 1 & \\ 0.50 & 0.37 & 1 \end{bmatrix}$$

It is seen that the absolute value of the correlation coefficients become higher (which will always be the case). Furthermore, it is seen from the example and the expressions in the $\boldsymbol{\rho}'$-matrix that the difference between ρ'_{ij} and ρ_{ij} vanishes for small coefficients of variation V, which is also the reason why the difference between ρ'_{ij} and ρ_{ij} is sometimes neglected.

From this example it is concluded that a failure function of normally and log-normally distributed stochastic variables can be transformed to a failure function of normally distributed variables. The failure function in the u-space can then be obtained from $\boldsymbol{\rho}'$ and the transformation explained in section 5.2. Next the reliability index β can be obtained as usual.

* * *

For dependent stochastic variables $X_i, i = 1,\ldots,n$ the *Rosenblatt transformation*, see [4.3], can be used to define a transformation to the u-space of uncorrelated and normalized normally distributed variables $U_i, i = 1,\ldots,n$. The transformation is defined as, see also (4.7):

$$\begin{aligned} x_1 &= F_{X_1}^{-1}\left(\Phi(u_1)\right) \\ x_2 &= F_{X_2|X_1}^{-1}\left(\Phi(u_2) \mid X_1 = x_1\right) \\ &\vdots \\ x_n &= F_{X_n|X_1\cdots X_{n-1}}^{-1}\left(\Phi(u_n) \mid X_1 = x_1,\ldots,X_{n-1} = x_{n-1}\right) \end{aligned} \tag{4.22}$$

where $F_{X_i|X_1\cdots X_{i-1}}\left(x_i \mid X_1 = x_1,\ldots,X_{i-1} = x_{i-1}\right)$ is the distribution function of X_i given $X_1 = x_1,\ldots,X_{i-1} = x_{i-1}$:

$$F_{X_i|X_1\cdots X_{i-1}}\left(x_i \mid X_1 = x_1,\ldots,X_{i-1} = x_{i-1}\right) = \frac{\int_{-\infty}^{x_i} f_{X_1\cdots X_{i-1}X_i}(x_1,\cdots,x_{i-1},t)dt}{f_{X_1\cdots X_{i-1}}(x_1,\ldots,x_{i-1})} \tag{4.23}$$

$f_{X_1\cdots X_i}(x_1,\ldots,x_i)$ is the joint density function of $X_1,\ldots,X_i$. The transformation starts for given $u_1,\ldots,u_n$ by determination of x_1. Next x_2 is calculated using the value of x_1 determined in the first step. $x_3,\ldots,x_n$ are then calculated in the same stepwise manner.

The inverse transformation from $x_1,\ldots,x_n$ to $u_1,\ldots,u_n$ is defined by:

$$\begin{aligned} u_1 &= \Phi^{-1}\left(F_{X_1}(x_1)\right) \\ u_2 &= \Phi^{-1}\left(F_{X_2|X_1}\left(x_2 \mid X_1 = x_1\right)\right) \\ &\vdots \\ u_n &= \Phi^{-1}\left(F_{X_n|X_1\cdots X_{n-1}}\left(x_n \mid X_1 = x_1,\ldots,X_{n-1} = x_{n-1}\right)\right) \end{aligned} \tag{4.24}$$

The Rosenblatt transformation is very useful when the stochastic model for a failure mode is given in terms of conditional distributions. For example, this is often the case when statistic uncertainty is included. Examples 4.8 and 4.9 show how the Rosenblatt transformation can be used.

Example 4.8. Evaluation of Maximum Wave Height

The wave surface elevation $\eta(t)$ can for short periods (8 hours) be assumed to be modelled by a stationary Gaussian stochastic process with zero mean. The wave surface is then fully described, if the spectral density $S_{\eta\eta}(\omega)$ of the elevation process is known. ω is the frequency. A commonly used spectral density is the JONSWAP spectrum, see [4.5]:

$$S_{\eta\eta}(\omega) = \frac{4k_b^4 k_\gamma H_S^2 \pi^3}{\omega^5 (k_p T_Z)^4} \exp\left(-\frac{1}{\pi}\left(\frac{2\pi k_b}{\omega k_p T_Z}\right)^4\right)\gamma^a \tag{a}$$

where $\gamma = 3$, $k_b = 1.4085$, $k_p = 0.327\exp(-0.315\gamma) + 1.17$ and $k_\gamma = 1 - 0.285\ln(\gamma)$. H_S is the significant wave height and T_Z is the zero crossing period. The superscript *a* is:

$$a = \exp\left(-\frac{(k_p T_Z \frac{\omega}{2\pi} - 1)^2}{2\sigma_a^2}\right)$$

where:

$$\sigma_a = \begin{cases} 0.07 \quad \text{for} \quad \omega < \dfrac{2\pi}{k_p T_Z} \\ 0.09 \quad \text{for} \quad \omega \geq \dfrac{2\pi}{k_p T_Z} \end{cases}$$

The distribution function of the maximum wave elevation H_m within a given time period [0, T] can be estimated from, see Davenport [4.6]:

$$F_{H_m}(h_m) = \exp\left(-\nu_0 T \exp\left[-\tfrac{1}{2}\left(\frac{h_m}{\sigma} \right)^2 \right] \right) \tag{b}$$

where :

$$\nu_0 = \sqrt{\frac{m_2}{m_0}} \tag{c}$$

$$\sigma = \sqrt{m_0} \tag{d}$$

and $m_i, i = 0, 2$ is the ith spectral moment:

$$m_i = \frac{1}{(2\pi)^i} \int_0^\infty \omega^i S_{\eta\eta}(\omega) d\omega \tag{e}$$

H_S and T_Z are usually modelled as stochastic variables. Here H_S is modelled by a Rayleigh distribution with the parameter s:

$$F_{H_S}(h) = 1 - \exp\left[-\frac{1}{2}\left(\frac{h}{s} \right)^2 \right] \quad , \quad h \geq 0 \tag{f}$$

and T_Z by a conditional distribution given H_S:

$$F_{T_Z|H_S}(t \mid H_S = h) = 1 - \exp\left[-\left(\frac{t}{k(h)} \right)^{\gamma(h)} \right] \tag{g}$$

where:

$$k(h) = 6.05 \exp(0.07h) \tag{h}$$

$$\gamma(h) = 2.35 \exp(0.21h) \tag{i}$$

The probability that H_m is larger than h_m is:

$$P(H_m > h_m) = P\left(h_m - H_m(H_S, T_Z) \leq 0 \right) \tag{j}$$

The distribution function for H_m given H_S and T_Z is given by (b). The distribution function for T_Z given H_S is given by (g), and the distribution function for H_S is given by (f). (j) can then be estimated by FORM using the failure function:

$$g = h_m - H_m(H_S, T_Z) \tag{k}$$

g is given by the three stochastic variables H_m, H_S and T_Z. The transformation to standardized variables U_1, U_2 and U_3 can be established by the Rosenblatt transformation:

$$\begin{aligned} \Phi(U_1) &= F_{H_S}(H_S) \\ \Phi(U_2) &= F_{T_Z|H_S}(T_Z \mid H_S) \\ \Phi(U_3) &= F_{H_m|H_S,T_Z}(H_m \mid H_S, T_Z) \end{aligned} \tag{l}$$

The reliability index β for (k) is determined by the algorithm in section 3.4 and:

$$P(H_m > h_m) \cong \Phi(-\beta) \tag{m}$$

For the parameters $s = 4.0$ m, $T = 8$ hours, β as a function of h_m is shown in figure 4.1.

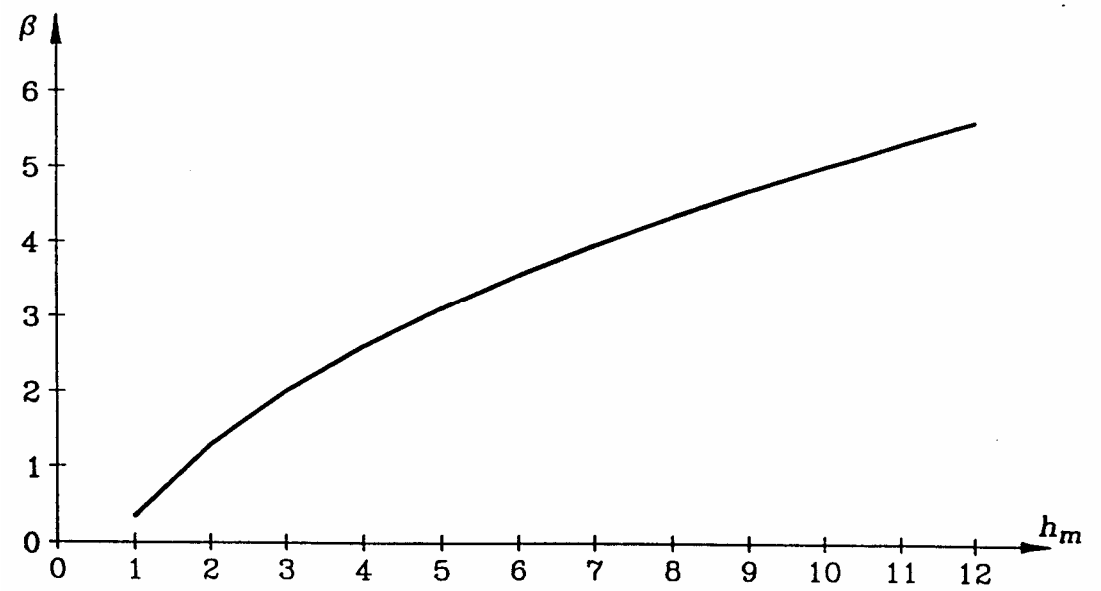

Figure 4.1. β as a function of h_m

Example 4.9
Consider a failure function with two stochastic variables X_1 and X_2: (Madsen et al. [4.7], p. 77)

$$g(\mathbf{x}) = 18 - 3x_1 - 2x_2 \tag{a}$$

X_1 and X_2 are dependent with a joint two-dimensional exponential distribution function:

$$F_{X_1X_2}(x_1,x_2) = 1-\exp(-x_1)-\exp(-x_2)+\exp[-(x_1+x_2+x_1x_2)] \quad , x_1>0, x_2>0 \tag{b}$$

and the corresponding probability density function:

$$f_{X_1X_2}(x_1,x_2) = \frac{\partial^2 F_{X_1X_2}(x_1,x_2)}{\partial x_1 \partial x_2} = (x_1+x_2+x_1x_2)\exp[-(x_1+x_2+x_1x_2)] \quad , x_1>0, x_2>0 \tag{c}$$

Realisations u_1 and u_2 of standard normal variables U_1 and U_2 are obtained from the Rosenblatt transformation as:

$$\begin{aligned} u_1 &= \Phi^{-1}\left(F_{X_1}(x_1)\right) \\ u_2 &= \Phi^{-1}\left(F_{X_2|X_1}(x_2 \mid x_1)\right) \end{aligned} \tag{d}$$

where:

$$f_{X_1}(x_1) = \int_0^\infty f_{X_1X_2}(x_1,x_2)dx_2 = \exp(-x_1) \quad , \quad x_1>0 \tag{e}$$

$$F_{X_1}(x_1) = \int_0^{x_1} f_{X_1}(x_1)dx_1 = 1-\exp(-x_1) \quad , \quad x_1>0 \tag{f}$$

Then it is possible to obtain $F_{X_2|X_1}(x_2 \mid X_1 = x_1)$ as:

$$F_{X_2|X_1}(x_2 \mid x_1) = \frac{\int_0^{x_2} f_{X_1X_2}(x_1,x_2)dx_2}{f_{X_1}(x_1)} = 1-(1+x_2)\exp\left[-(x_2+x_1x_2)\right] \quad , x_1>0, x_2>0 \tag{g}$$

For the transformation from the *x*-space to the *u*-space the formulas become:

$$\begin{aligned} x_1 &= F_{X_1}^{-1}\left(\Phi(u_1)\right) = -\ln\left[1-\Phi(u_1)\right] \\ x_2 &= F_{X_2|X_1}^{-1}\left(\Phi(u_2) \mid X_1 = x_1\right) \end{aligned} \tag{h}$$

from which x_2 can be found as the solution to:

$$1-(1+x_2)\exp[-(x_2+x_1x_2)] = \Phi(u_2) \tag{i}$$

The obtained failure function in the *u*-space is seen in figure 4.2.

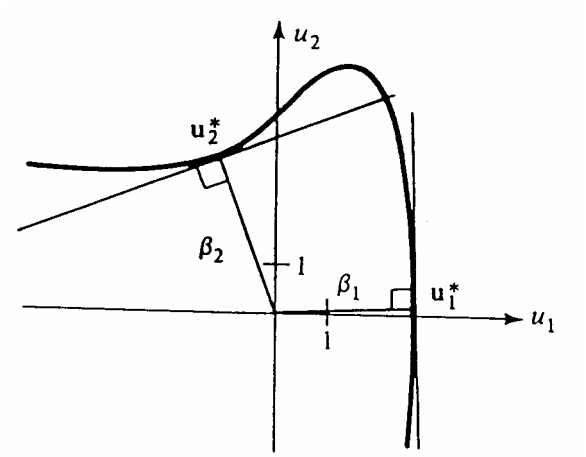

Figure 4.2. Failure surface in standard normal space.

The β-optimization problem includes a local and a global minimum. The β-point (which is also the global minimum) is $\mathbf{u}_1^* = \{2.78, 0.1\}$ with $\beta_1 = 2.78$ and $P_f \approx 2.68 \cdot 10^{-3}$. Further, the local minimum point $\mathbf{u}_2^* = \{-1.30, 3.25\}$ is identified with $\beta_2 = 3.50$.

* * *

An alternative way to define the transformation from the *u*-space to the *x*-space is to use the *Nataf transformation*, see [4.4] and [4.8]. This transformation is in general only an approximate transformation. The basic idea is to establish the marginal transformations defined in section 4.3 (as if the stochastic variables were independent) and to use a correlation coefficient matrix $\boldsymbol{\rho}^e$ in an *y*-space, which is obtained from the correlation coefficient matrix $\boldsymbol{\rho}$ in the *x*-space by multiplying each correlation coefficient by a factor *F,* which depends on distribution types and the statistical parameters. To describe the Nataf transformation it is thus sufficient to consider two stochastic variables X_i and X_j.

Marginal transformations of X_i and X_j to normally distributed variables Y_i and Y_j with expected value 0 and standard deviation 1 is, see (4.7):

$$X_i = F_{X_i}^{-1}\left(\Phi(Y_i)\right)$$
$$X_j = F_{X_j}^{-1}\left(\Phi(Y_j)\right) \tag{4.25}$$

The stochastic variables Y_i and Y_j have an (equivalent) correlation coefficient ρ_{ij}^e, which in the Nataf transformation is determined such that dependence between X_i and X_j is approximated as well as possible.

ρ_{ij}^e is determined as follows. Normalized variables Z_i and Z_j are introduced by:

$$Z_k = \frac{X_k - \mu_{X_k}}{\sigma_{X_k}} \quad k = i, j \tag{4.26}$$

The correlation coefficient ρ_{ij} between X_i and X_j is $\rho_{ij} = E[Z_i Z_j]$. From (4.25) and (4.26) it is seen that:

$$z_k = \frac{F_{X_k}^{-1}\left(\Phi(y_k)\right) - \mu_{X_k}}{\sigma_{X_k}} \quad k = i, j \tag{4.27}$$

Further, from (4.2) it is seen that uncorrelated variables U_i and U_j can be introduced by:

$$\begin{aligned} y_i &= u_i \\ y_j &= \rho_{ij}^e u_i + \sqrt{1-(\rho_{ij}^e)^2}\, u_j \end{aligned} \tag{4.28}$$

ρ_{ij} can then be related to the (unknown) equivalent correlation coefficient ρ_{ij}^e by:

$$\begin{aligned} \rho_{ij} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \varphi_2(y_i, y_j, \rho_{ij}^e)\, dy_i dy_j \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F_{X_i}^{-1}\left(\Phi(y_i)\right) - \mu_{X_i}}{\sigma_{X_i}} \frac{F_{X_j}^{-1}\left(\Phi(y_j)\right) - \mu_{X_j}}{\sigma_{X_j}} \varphi_2(y_i, y_j, \rho_{ij}^e)\, dy_i dy_j \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F_{X_i}^{-1}\left(\Phi(u_i)\right) - \mu_{X_i}}{\sigma_{X_i}} \frac{F_{X_j}^{-1}\left(\Phi\left(\rho_{ij}^e u_i + \sqrt{1-(\rho_{ij}^e)^2}\, u_j\right)\right) - \mu_{X_j}}{\sigma_{X_j}} \varphi(u_i)\varphi(u_j)\, du_i du_j \end{aligned} \tag{4.29}$$

where $\varphi_2(\,)$ is the two-dimensional normal density function. From (4.29) ρ_{ij}^e can be determined by iteration.

Based on ρ_{ij}^e the following approximate joint density function $f_{X_i X_j}^e(x_i, x_j)$ is obtained:

$$f_{X_i X_j}^e(x_i, x_j) = \frac{f_{X_i}(x_i) f_{X_j}(x_j)}{\varphi(y_i)\varphi(y_j)} \varphi_2(y_i, y_j, \rho_{ij}^e) \tag{4.30}$$

where $y_i = \Phi^{-1}\left(F_{X_i}(x_i)\right)$.

(4.29) has been solved for ρ_{ij}^e by der Kiureghian & Liu [4.8] for a number of distribution functions and approximations for the factor:

$$F = \frac{\rho_{ij}^e}{\rho_{ij}} \tag{4.31}$$

has been obtained. With $\rho = \rho_{ij}$ and $V_i = \sigma_{X_i} / \mu_{X_i}$ examples of approximations for *F* are shown in table 4.1.

For $n = 2$ it should be checked that $\left|\rho_{12}^{e}\right| \le 1$. For $n > 2$ the corresponding requirement is that $\boldsymbol{\rho}^{e}$ is positive definite. In der Kiureghian & Liu [4.8] or Ditlevsen & Madsen [4.9], approximations for F are also shown for Gamma, Frechet, Uniform, Rayleigh and Gumbel distributions.

X_i	X_j	F
normal	log-normal	$V_j \Big/ \sqrt{\ln(1+V_j^2)}$
log-normal	log-normal	$\ln(1+\rho V_i V_j) \Big/ (\rho\sqrt{\ln(1+V_i^2)\ln(1+V_j^2)})$
exponential	log-normal	$1.098 + 0.003\rho + 0.025\rho^2 + 0.019V_j + 0.303V_j^2 - 0.437\rho V_j$
Weibull	log-normal	$1.031 + 0.052\rho + 0.002\rho^2 + 0.011V_j + 0.220V_j^2 - 0.210V_i$ $+ 0.350V_i^2 + 0.005\rho V_j - 0.174\rho V_i + 0.009V_i V_j$
exponential	normal	1.107
Weibull	normal	$1.031 - 0.195V_i + 0.328V_i^2$
exponential	exponential	$1.229 - 0.367\rho + 0.153\rho^2$
Weibull	exponential	$1.147 + 0.145\rho + 0.010\rho^2 - 0.271V_i + 0.459V_i^2 - 0.467V_i\rho$
Weibull	Weibull	$1.063 - 0.004\rho - 0.001\rho^2 - 0.2V_i + 0.337V_i^2 - 0.2V_j + 0.337V_j^2$ $+ 0.007(\rho V_i + \rho V_j - V_i V_j)$

Table 4.1. The factor F for some commonly used distributions

Example 4.10
Consider the same problem as in example 4.9 but use the Nataf transformation instead of the Rosenblatt transformation. The correlation coefficient between X_1 and X_2 is:

$$\rho = \frac{1}{\sigma_{X_1}\sigma_{X_2}}\left[E[X_1X_2] - \mu_{X_1}\mu_{X_2}\right]$$

$$= \frac{1}{\sigma_{X_1}\sigma_{X_2}}\left[\int_0^\infty\int_0^\infty x_1x_2 f_{X_1X_2}(x_1x_2)dx_1dx_2 - \mu_{X_1}\mu_{X_2}\right]$$

$$= \int_0^\infty\int_0^\infty x_1x_2(x_1 + x_2 + x_1x_2)\exp\left[-(x_1 + x_2 + x_1x_2)\right]dx_1dx_2 - 1$$

$$= -0.40366$$

where:

$$\mu_{X_2} = \mu_{X_1} = E[x_1] = \int_0^\infty x_1 f_{X_1}(x_1)dx_1 = \int_0^\infty x_1\exp(-x_1)dx_1 = 1$$

$$\sigma_{X_2}^2 = \sigma_{X_1}^2 = E[x_1^2] - \mu_{X_1}^2 = \int_0^\infty x_1^2 f_{X_1}(x_1)dx_1 - \mu_{X_1}^2 = \int_0^\infty x_1^2\exp(-x_1)dx_1 - 1 = 1$$

The factor F for two exponentially distributed variables is:

$$F = 1.229 - 0.367\rho + 0.153^2\rho = 1.402$$

The equivalent coefficient thus is:

$$\rho^e = F\rho = -0.566$$

The transformation form (u_1, u_2) to (x_1, x_2) is given by (4.25) and (4.2) (or (4.28) for two stochastic variables)

$$x_1 = -\ln[1 - \Phi(u_1)]$$

$$x_2 = -\ln\left[1 - \Phi\left(\rho^e u_1 + \sqrt{1-(\rho^e)^2}\, u_2\right)\right]$$

Using the failure function in example 4.9 the two β-points are determined as:

$\beta_1 = 2.797 \quad \mathbf{u}_1^* = (2.80, \quad 0.07) \quad \boldsymbol{\alpha}_1 = (0.99, \quad 0.02)$
$\beta_2 = 3.658 \quad \mathbf{u}_2^* = (-2.02, \quad 3.05) \quad \boldsymbol{\alpha}_2 = (-0.55, \quad 0.83)$

* * *

4.5 Sensitivity Measures

As described in note 3 three important sensitivity measures can be used to characterize the sensitivity of the reliability index with respect to parameters and the stochastic variables, namely:

$\boldsymbol{\alpha}$-vector
The elements in the $\boldsymbol{\alpha}$-vector characterize the importance of the stochastic variables. From the linearized safety margin $M = \beta - \boldsymbol{\alpha}^T \mathbf{U}$ it is seen that the variance of M is:

$$\sigma_M^2 = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1 \tag{4.32}$$

For independent stochastic variables α_i^2 thus gives the percentage of the total uncertainty associated with U_i (and X_i). If for example X_2, X_3 and X_4 are dependent, then $\alpha_2^2 + \alpha_3^2 + \alpha_4^2$ gives the percentage of the total uncertainty which can be associated with X_2, X_3 and X_4 altogether.

Reliability elasticity coefficient e_p
e_p is defined by (3.39). For a parameter p in the failure function $g(\mathbf{u}, p) = 0, \quad e_p$ is obtained from (3.43):

$$e_p = \frac{1}{|\nabla g|} \frac{\partial g}{\partial p} \frac{p}{\beta} \tag{4.33}$$

For parameters p in the distribution function for $\mathbf{X}$, which is related to standardized variables $\mathbf{U}$ by $\mathbf{U} = \mathbf{T}(\mathbf{X}, p)$, e_p is obtained as:

$$e_p = \frac{1}{\beta}(\mathbf{u}^*)^T \frac{\partial \mathbf{T}(\mathbf{x}^*, p)}{\partial p} \frac{p}{\beta} \tag{4.34}$$

where $\mathbf{u}^*$ and $\mathbf{x}^*$ are the β-points in the *u*-space and the *x*-space.

Omission sensitivity factors ξ

As described in section 3.4 the factor:

$$\xi_i = \frac{1}{\sqrt{1-\alpha_i^2}} \tag{4.35}$$

gives a measure of the change in the reliability index if stochastic variable no. *i* is fixed. This stochastic variable is assumed to be independent of the other stochastic variables. As described in Madsen [4.10], the omission sensitivity factor can be generalized to dependent stochastic variables.

4.6 References

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Note 5: SORM AND SIMULATION TECHNIQUES

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First Order Reliability Methods can be expected to give reasonable results when the failure functions are not too non-linear. FORM techniques are described in notes 3 and 4. If the failure functions in the standardized *u*-space are rather non-linear then Second Order Reliability Methods (SORM) techniques, where a second order approximation of the failure function is established, can be used. These techniques are described in section 5.1.

Other techniques, which can be used for such types of problems, are simulation techniques. Simulation methods, which are described in sections 5.2 - 5.7, can also be powerful when the failure functions in the *u*-space have more than one β-point, i.e. there are several local, probable failure regions.

In simulation methods realisations (outcomes) $\hat{\mathbf{x}}$ of the stochastic variables $\mathbf{X}$ are generated for each sample. When simulation methods are used to estimate P_f the failure function is calculated for each realisation $\hat{\mathbf{x}}$ and if the realisation is in the failure region, then a contribution to the probability of failure is obtained. In section 5.2 different techniques to generate realisations of stochastic variables are described. In the literature a large number of simulation methods are described. Sections 5.3 to 5.7 contain a description of some of the most important methods, namely:

- Crude Monte Carlo simulation
- Importance sampling
- Importance sampling based on the β-point
- Monte Carlo sampling by excluding part of safe area
- Directional simulation
- Latin hypercube simulation
- Adaptive simulation

Finally in section 5.8 it is described how sensitivity measures can be obtained by simulation.

5.1 Second Order Reliability Method (SORM)

Compared with a FORM estimate of the reliability of a component (or failure mode) an improved estimate can be obtained by using a second order approximation of the failure surface at the β-point $\mathbf{u}^*$ in the *u*-space:

$$g(\mathbf{u}) \cong \nabla g(\mathbf{u}^*)^T(\mathbf{u}-\mathbf{u}^*) + \tfrac{1}{2}(\mathbf{u}-\mathbf{u}^*)^T\mathbf{D}(\mathbf{u}-\mathbf{u}^*) = 0 \qquad (5.1)$$

where $\mathbf{D}$ the Hessian matrix of second order partial derivatives of the failure surface at the β-point:

$$D_{ij} = \left.\frac{\partial^2 g}{\partial u_i \partial u_j}\right|_{\mathbf{u}=\mathbf{u}^*} \quad , \quad i,j = 1,2,\ldots,n \tag{5.2}$$

In the following it is described how a second order reliability index can be determined.

The β-point and the gradient vector can be written, see (3.29) and (3.30):

$$\mathbf{u}^* = \beta\boldsymbol{\alpha} \qquad \nabla g(\mathbf{u}^*) = -\left|\nabla g(\mathbf{u}^*)\right|\boldsymbol{\alpha} \tag{5.3}$$

An orthogonal transformation from $\mathbf{u}$ to $\mathbf{y}$ is defined by:

$$\mathbf{y} = \mathbf{R}\mathbf{u} \tag{5.4}$$

where the nth row in $\mathbf{R}$ is equal to $\boldsymbol{\alpha}$:

$$R_{ni} = \alpha_i \quad , \quad i = 1,\ldots,n \tag{5.5}$$

The remaining rows in $\mathbf{R}$ can be found by standard Gram-Schmidt orthogonalization.

(5.1) can then be written:

$$\beta - y_n + \frac{1}{2\left|\nabla g(\mathbf{u}^*)\right|}\tilde{\mathbf{y}}^T\mathbf{R}\mathbf{D}\mathbf{R}^T\tilde{\mathbf{y}} = 0 \tag{5.6}$$

where $\tilde{\mathbf{y}} = (y_1, y_2, \ldots, y_{n-1}, y_n - \beta)^T$.

The solution of (5.6) with respect to y_n using up to second order terms in $y_1, y_2, \ldots, y_{n-1}$ gives the hyperparabolic surface:

$$y_n = \beta + \mathbf{y'}^{\mathbf{T}}\,\mathbf{A}\mathbf{y'} \tag{5.7}$$

where $\mathbf{y'} = (y_1, \ldots, y_{n-1})^T$ and the elements in $\mathbf{A}$ are:

$$A_{ij} = \frac{1}{2\left|\nabla g(\mathbf{u}^*)\right|}\left\{\mathbf{R}\mathbf{D}\mathbf{R}^T\right\}_{ij} \quad i,j = 1,2,\ldots,n-1 \tag{5.8}$$

A second orthogonal transformation from $\mathbf{y'}$ to $\mathbf{v}$ is defined by:

$$\mathbf{y}' = \mathbf{H}\,\mathbf{v} \tag{5.9}$$

where the columns in $\mathbf{H}$ are the eigenvectors of $\mathbf{A}$. (5.7) can then be written:

$$y_n = \beta + \sum_{i=1}^{n-1}\lambda_i v_i^2 \tag{5.10}$$

where $\lambda_i, i = 1,2,\ldots,n-1$ are the eigenvectors in $\mathbf{A}$. The eigenvectors and eigenvalues can e.g. be found by Jacobi-iteration or subspace-iteration for large problems, where only the largest eigenvalues are important, see e.g. [5.11].

The probability of failure P_f estimated using the second-order approximation of the failure surface is:

$$P_f^{SO} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(v_1) \cdots \varphi(v_{n-1}) \int_{\beta + \Sigma \lambda_i v_i^2}^{\infty} \varphi(y_n) dy_n dv_1 \cdots dv_{n-1} \tag{5.11}$$

The approximation is illustrated in figure 5.1, which also shows the first-order approximation (see (3.38)) to the exact probability of failure $P_f = P(g(\mathbf{U}) \le 0)$.

$$P_f^{FO} = \Phi(-\beta) \tag{5.12}$$

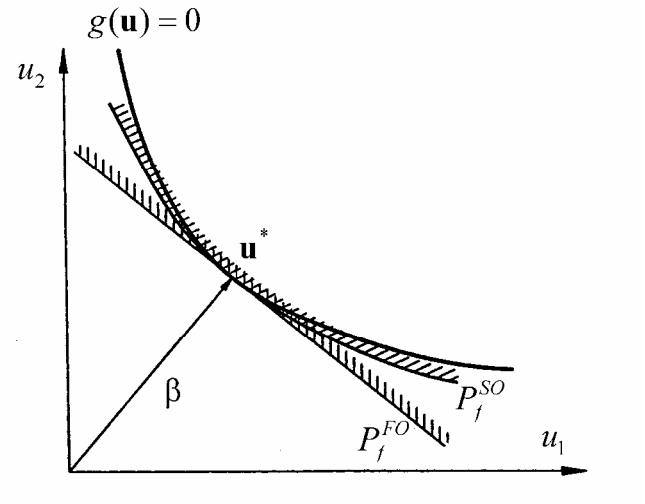

Figure 5.1. Illustration of first and second order approximations of the failure surface.

It should be noted that due to the rotational symmetry of the normal density function the points in the area close to the β-point (which is the point closest to origo) has the largest probability density. Therefore, the largest contributions to the probability of failure come from this area. Further, it is noted that the *u*-dimensional normal density function for uncorrelated variables $\varphi_n() \propto \exp(-r^2/2)$ decreases fast with the distance *r* from origo. If the failure surface is rather non-linear then a second order approximation of the failure surface can be expected to give a much better estimate of the probability of failure than the first-order approximation. Finally it should be noted that for $\beta \to \infty$ the first (and second) order estimates of the probability converge to the exact result: $P_f^{FO} \to P_f$.

Based on (5.11) Breitung [5.1] has derived an approxim ation to P_f^{SO}:

$$P_f^{SO} \approx \Phi(-\beta) \prod_{j=1}^{n-1} (1 + 2\beta\lambda_j)^{-\frac{1}{2}} \tag{5.13}$$

Improvements to (5.13) have been suggested by for example Tvedt [5.2] and [5.3].

A second order reliability index β^{SO} can be defined by:

$$\beta^{SO} = -\Phi^{-1}(P_f^{SO}) \tag{5.14}$$

The approximation in (5.13) - (5.14) assumes that the matrix $\mathbf{I} + 2\beta\mathbf{A}$ is positive definite.

5.2 Simulation of Stochastic Variables

A necessary tool in simulation techniques for estimation of the probability of failure is to simulate outcomes of stochastic variables with an arbitrary distribution. For this a method to generate uniformly distributed numbers is first described. Next it is shown how the inverse method can be used to generate outcomes of stochastic variables with a general distribution. Finally methods to generate outcomes of normally distributed variables are described.

Simulation of uniformly distributed numbers

The numbers generated by algorithms implemented on computers are usually not real random but only pseudo-random numbers. The reason is that they are generated by a rule (equation) such that the sequence of numbers is repeated after a number of outcomes. Further the same sequence of numbers is obtained if the generator is started again with the same starting conditions.

In this subsection a stochastic variable V which is uniformly distributed between 0 and 1 is considered. The distribution function is:

$$F_V(v) = \begin{cases} v & \text{if} \quad 0 \le \mathrm{v} \le 1 \\ 0 & \text{else} \end{cases} \tag{5.15}$$

The most widely used techniques to simulate (generate) pseudo-random numbers of V is the multiplicative congruential generators, see Hammersley & Handscomb [5.4] and the XOR generator, see Ditlevsen & Madsen [5.5]. In multiplicative congruential generators the pseudo-random numbers are determined sequentially by:

$$v_i = a v_{i-1} + c \left(\text{modulo } m\right) \tag{5.16}$$

where m is a large integer (usually a large power of 2) and a, c and v_{i-1} are integers between 0 and $m-1$. The starting seed number is v_0. The numbers v_i/m are then used as pseudo-random numbers uniformly distributed between 0 and 1. The sequence of numbers repeat after at most m steps. The full period m is obtained if:

1. c and m have no common divisor
2. $a \equiv$ (modulo p) for every prime factor p of m
3. $a \equiv$ (modulo 4) if m is a multiple of 4.

On many computers the following generator is used:

$$v_i = 89069 v_{i-1} + 1 \left(\text{modulo } 2^{32}\right) \tag{5.17}$$

The numbers generated by (5.16) are not completely independent. It can be shown that the correlation between successive numbers lies in the interval, see Hammersley & Handscomb [5.4]:

$$\left[\frac{1}{a}-\frac{6c}{am}(1-\frac{c}{m})-\frac{a}{m},\frac{1}{a}-\frac{6c}{am}(1-\frac{c}{m})+\frac{a}{m}\right] \tag{5.18}$$

Numerical investigations have shown that if the multiplicative congruential generator is used to generate outcomes of stochastic vectors then the generated vectors are not uniformly distributed in the n-dimensional space. An algorithm which generates numbers much more random in the n-dimensional space is the so-called XOR random number generator, see Ditlevsen & Madsen [5.5].

Simulation of random numbers by the inverse method

For a general stochastic variable X the distribution function is $F_X(x)$. In the inverse method two steps are needed to generate an outcome $\hat{x}$ of X:

1. generate an outcome $\hat{v}$ of V (e.g. using a multiplicative congruence generator)
2. determine the outcome of $\hat{x}$ by :

$$\hat{x} = F_X^{-1}\left(F_V(\hat{v})\right) = F_X^{-1}(\hat{v}) \tag{5.19}$$

The method is illustrated in figure 5.2. It is seen that the distribution function for $\hat{X}$ with outcomes generated by this procedure is:

$$F_{\hat{X}}(x) = P(\hat{X} \le x) = P\left(F_X^{-1}(V) \le x\right) = P\left(V \le F_X(x)\right) = F_X(x) \tag{5.20}$$

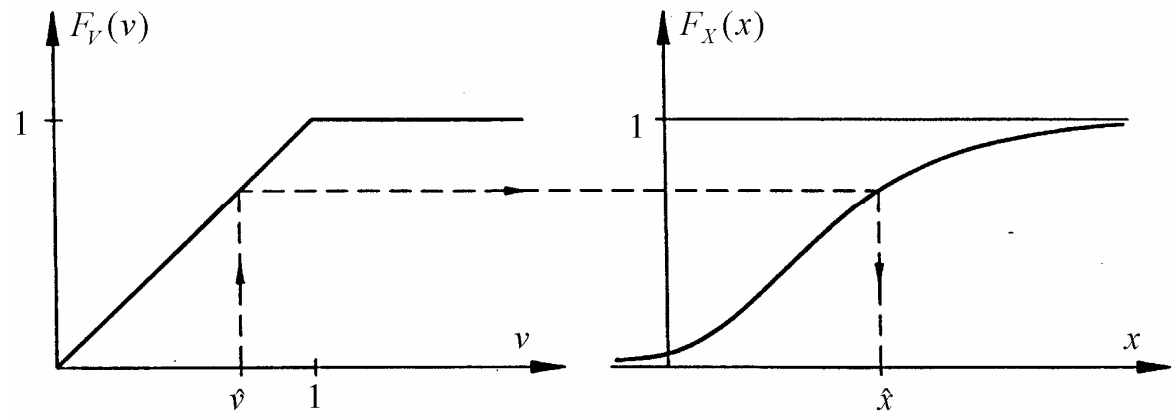

Figure 5.2. Illustration of the inverse method.

Example 5.1

Let X be exponential distributed with distribution function:

$$F_X(x) = 1 - \exp(-\lambda x)$$

Outcomes of X can be generated by:

$$\hat{x} = -\frac{1}{\lambda}\ln(1-\hat{v})$$

where the number $\hat{v}$ are generated by (5.16).

* * *

The Box-Muller method to simulation of normal distributed numbers

Outcomes $\hat{u}_1$ and $\hat{u}_2$ two independent normally distributed stochastic variables U_1 and U_2 both with expected value $\mu = 0$ and standard deviation $\sigma = 1$ can be generated using:

$$\begin{cases} U_1 = \sqrt{-2\ln V_1}\cos(2\pi V_2) \\ \\ U_2 = \sqrt{-2\ln V_1}\sin(2\pi V_2) \end{cases} \tag{5.21}$$

where V_1 and V_2 are independent stochastic variables uniformly distributed between 0 and 1.

Outcomes are determined by the following two steps :

1) generate outcomes $\hat{v}_1$ and $\hat{v}_2$ of V_1 and V_2
2) calculate the outcomes $\hat{u}_1$ and $\hat{u}_2$ using (5.21)

It is easy to show that U_1 and U_2 defined by (5.21) are independent and normally distributed.

Simulation of normally distributed numbers using the central limit theorem

From the central limit theorem it follows that:

$$V_1 + V_2 + \cdots + V_n - a \quad \rightarrow \quad U \quad \text{for} \quad n \rightarrow \infty \tag{5.22}$$

where $V_1, V_2, \ldots$ are independent equidistributed random variables uniformly distributed between 0 and 1 (expected value $\mu_V = \frac{1}{2}$ and variance $\sigma_V^2 = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$).

U is asymptotically normally distributed with expected value $\mu_U = n\frac{1}{2} - a$ and variance $\sigma_U^2 = n\frac{1}{12}$.

A reasonable choice is $a = \frac{n}{2}$ and n = 12. Then U becomes approximately normal with expected value 0 and standard deviation 1.

Simulation of correlated normally distributed numbers

A vector $\mathbf{X} = (X_1, \ldots, X_n)$, which is normally distributed with expected value $\boldsymbol{\mu}_\mathbf{X}$ and covariance matrix $\mathbf{C}_\mathbf{X}$ can be written, see (4.5):

$$\mathbf{X} = \boldsymbol{\mu}_{\mathbf{X}} + \mathbf{DTU} \tag{5.23}$$

where the elements in $\mathbf{U}$ are uncorrelated with zero mean and unit standard deviation. Using the techniques described above to generate outcomes of normally distributed variables and (5.23) realisations of $\mathbf{X}$ can be generated.

* * *

In the following sections different simulation methods to estimate the probability of failure are described:

$$P_f = P\big(g(\mathbf{U})\big) \le 0 \tag{5.24}$$

where the failure function g is assumed to be modelled in the u-space.

5.3 Crude Monte Carlo simulation

In crude Monte Carlo simulation P_f is estimated by:

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\hat{\mathbf{u}}_j)] \tag{5.25}$$

where N is the number of simulations and $\hat{\mathbf{u}}_j$ is sample no. j of a standard normally distributed stochastic vector $\mathbf{U}$ The indicator function $I[g(\mathbf{u})]$ is defined by:

$$I[g(\mathbf{u})] = \begin{cases} 0 & \text{if} \quad g(\mathbf{u}) > 0 \quad \text{(safe)} \\ 1 & \text{if} \quad g(\mathbf{u}) \le 0 \quad \text{(failure)} \end{cases}$$

The standard error of $\hat{P}_f$ is estimated by:

$$s = \sqrt{\frac{\hat{P}_f(1-\hat{P}_f)}{N}} \tag{5.26}$$

Confidence intervals for the estimate of the probability of failure can be determined using that $\hat{P}_f$ becomes normally distributed for $N \to \infty$.

5.4 Importance sampling

The idea in importance sampling is to concentrate the sampling in the area of the total sample space which has the largest contribution to the probability of failure. In this way the standard error of the estimate of P_f can be reduced significantly. P_f is written:

$$P_f = \int\cdots\int I[g(\mathbf{u})] f_{\mathbf{U}}(\mathbf{u}) du_1 \cdots du_n = \int\cdots\int I[g(\mathbf{y})] \frac{f_{\mathbf{U}}(\mathbf{y})}{f_{\mathbf{S}}(\mathbf{y})} f_{\mathbf{S}}(\mathbf{y}) dy_1 \cdots dy_n \tag{5.27}$$

where $f_{\mathbf{S}}(\mathbf{y})$ is the sampling density function and $f_{\mathbf{U}}(\mathbf{y}) = \varphi(y_1)\cdots\varphi(y_n)$ is the standard normal density function for $\mathbf{U}$.

In theory, if the sampling density $f_{\mathbf{S}}$ is chosen to be proportional to $f_{\mathbf{U}}$ in the failure region then the standard error on P_f would be zero. Unfortunately, this choice is not possible because P_f is not known beforehand. In the following it is shown how $f_{\mathbf{S}}$ can be chosen reasonably.

Using importance sampling P_f is estimated by:

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\hat{\mathbf{y}}_j)]\frac{f_{\mathbf{U}}(\hat{\mathbf{y}}_j)}{f_{\mathbf{S}}(\hat{\mathbf{y}}_j)} \tag{5.28}$$

where $f_{\mathbf{S}}(\mathbf{y})$ is the sampling density function from which the sample vectors $\hat{\mathbf{y}}_j$ are generated.

The standard error of the estimate $\hat{P}_f$ is:

$$s = \sqrt{\frac{1}{N(N-1)}\left\{\sum_{j=1}^{N}\left(I[g(\hat{\mathbf{y}}_j)]\frac{f_{\mathbf{U}}(\hat{\mathbf{y}}_j)}{f_{\mathbf{S}}(\hat{\mathbf{y}}_j)}\right)^2 - \frac{1}{N}\left[\sum_{j=1}^{N} I[g(\hat{\mathbf{y}}_j)]\frac{f_{\mathbf{U}}(\hat{\mathbf{y}}_j)}{f_{\mathbf{S}}(\hat{\mathbf{y}}_j)}\right]^2\right\}} \tag{5.29}$$

Example 5.2 Estimation of the probability of failure
Let X_1 be the load on an element and X_2 the strength of an element. Failure occurs if $X_1 \geq X_2$. If a failure function g is defined by:

$$g(x_1, x_2) = x_2 - x_1$$

then the probability of failure is:

$$P_f = P(X_2 - X_1 \leq 0) = P\big(g(\mathbf{X}) \leq 0\big) = \int_0^\infty \int_0^\infty I[g(\mathbf{x})] f_{\mathbf{X}}(\mathbf{x}) dx_1 dx_2$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint density function of the stochastic variables modelling the load and the strength.

In importance sampling the simulations are concentrated in the area which contributes most to the probability of failuire. P_f is estimated by (5.28):

$$P_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\hat{\mathbf{y}}_j)]\frac{f_{\mathbf{X}}(\hat{\mathbf{y}}_j)}{f_{\mathbf{Y}}(\hat{\mathbf{y}}_j)}$$

where $f_{\mathbf{Y}}(\mathbf{y})$ is the sampling density function from which the sample vector $\hat{\mathbf{y}}_j$ is generated. Figure 5.3 shows the general difference between crude Monte Carlo simulation and importance sampling.

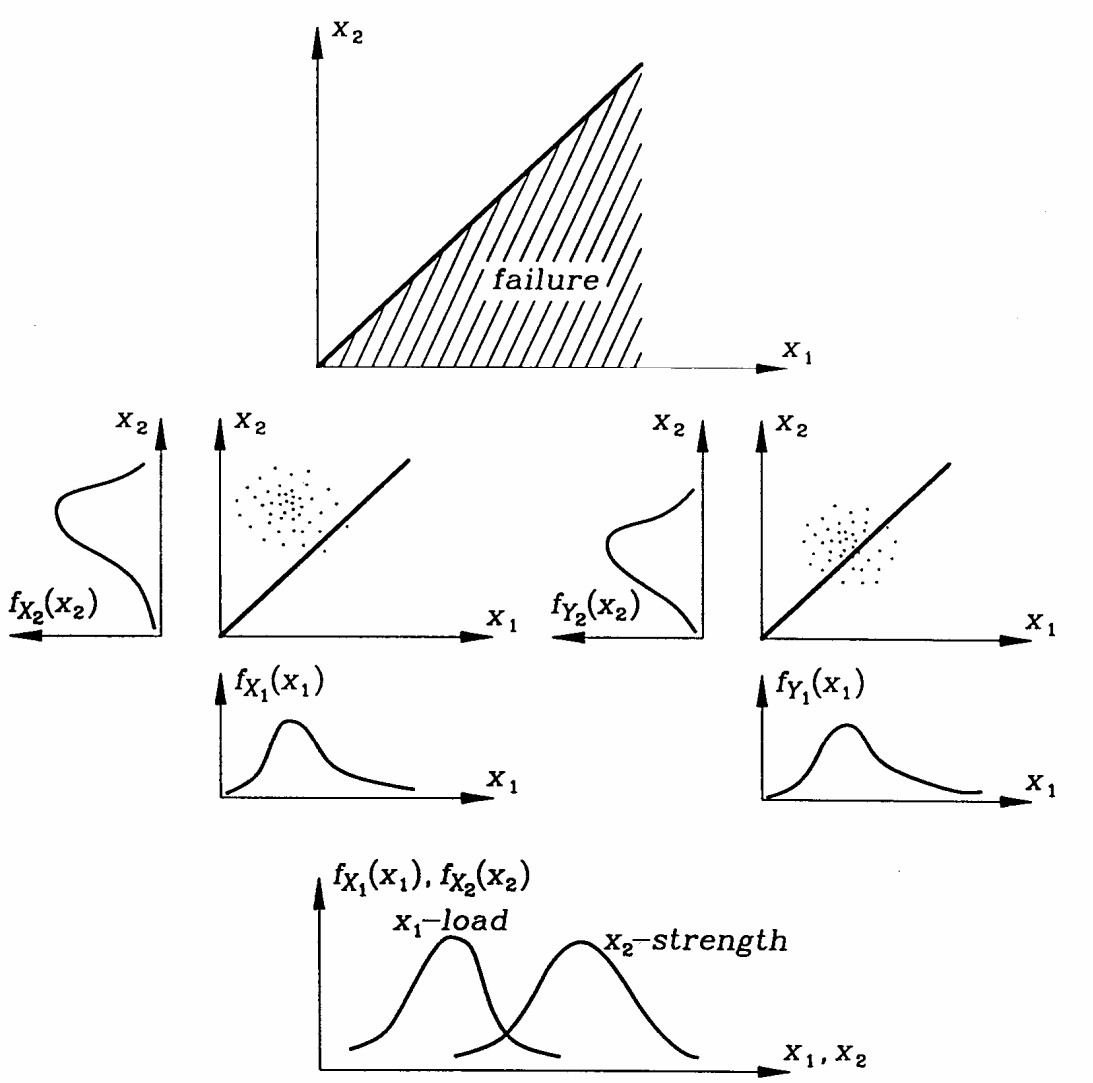

Figure 5.3. Crude Monte Carlo simulation and importance sampling.

Example 5.3
Consider the example from Madsen et al. [5.6], where the cross-section of a reinforced concrete beam is analysed. $n = 7$ stochastic variables are used. The failure function is written

$$g(\mathbf{x}) = x_2 x_3 x_4 - \frac{x_5 x_3^2 x_4^2}{x_6 x_7} - x_1$$

	VARIABLE	DIST.	μ	V
x_1	bending moment	N	0.01 MNm	0.3
x_2	eff. depth of reinforcement	N	0.3 m	0.05
x_3	yield stress of reinforcement	N	360 MPa	0.1
x_4	area of reinforcement	N	$226 \cdot 10^{-6}\,\mathrm{m}^2$	0.05
x_5	factor	N	0.5	0.1
x_6	width of beam	N	0.12 m	0.05
x_7	compressive strength of concrete	N	40 MPa	0.15

Table 5.1. Statistical data. μ is the expected value and $v = \sigma / \mu$ is the coefficient of variation. *N* indicates normal (Gauss) distribution.

The statistical data are shown in table 5.1. The stochastic variables are assumed to be independent.

A transformation to normalized stochastic variables (with expected value 0 and standard deviation 1) $U_i, i = 1,2,\ldots,7$ is established by:

$$X_i = \sigma_i U_i + \mu_i \quad , \quad i = 1,\ldots,7$$

The failure function is now written:

$$g(\mathbf{u}) = (\sigma_2 u_2 + \mu_2)(\sigma_3 u_3 + \mu_3)(\sigma_4 u_4 + \mu_4) - \frac{(\sigma_5 u_5 + \mu_5)(\sigma_3 u_3 + \mu_3)^2 (\sigma_4 u_4 + \mu_4)^2}{(\sigma_6 u_6 + \mu_6)(\sigma_7 u_7 + \mu_7)} - (\sigma_1 u_1 + \mu_1)$$

Crude Monte Carlo simulation and importance sampling are used.

In importance sampling P_f is estimated by (5.28) with $\hat{\mathbf{y}} = \hat{\mathbf{u}} + \mathbf{u}^*$ and $f_{\mathbf{S}}(\hat{\mathbf{y}}) = f_{\mathbf{U}}(\hat{\mathbf{y}} - \mathbf{u}^*)$, i.e. the samples are concentrated around the point $\mathbf{u}^*$. $\hat{\mathbf{u}}$ is a sample generated from the standard normal vector $\mathbf{U}$. In this example $\mathbf{u}^*$ is chosen as (see the next section)

$$\mathbf{u}^* = (2.5, -1, -2, -1, 0, 0, 0)$$

The standard error is estimated by (5.29).

N	CRUDE M C	IMP. SAMP.
1000	0 (0)	0.000344 (0.000016)
10000	0.000300 (0.000173)	0.000333 (0.000005)
100000	0.000350 (0.000059)	0.000337 (0.000002)

Table 5.2. Result of Monte Carlo simulation.

The numerical results are shown in table 5.2 with standard errors in ( ). It is seen that the standard error in importance sampling decreases much faster than in crude Monte Carlo simulation.

5.5 Importance sampling based on the β-point

If the β-point has been determined before simulation techniques are used importance sampling can be very effective with the β-point as the point around which the samplings are concentrated, see figure 5.4. Such a technique is described in this section. The sampling density function f_S in (5.28) is the normal density of uncorrelated variables with expected values $\mathbf{u}_i^*, i = 1,2,\ldots,n$ and common standard deviations, σ.

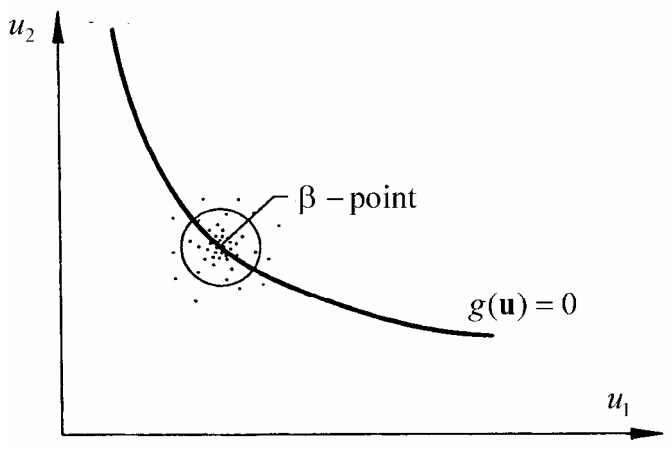

Figure 5.4. Importance sampling around the β-point.

P_f is estimated by:

$$P_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\sigma\hat{\mathbf{u}}_j + \mathbf{u}^*)]\frac{f_{\mathbf{U}}(\sigma\hat{\mathbf{u}}_j + \mathbf{u}^*)}{f_{\mathbf{U}}(\hat{\mathbf{u}}_j)}\sigma^n \tag{5.30}$$

where $f_{\mathbf{U}}(\mathbf{u})$ is the standardized normal density function and $\hat{\mathbf{u}}_j$ is a sample generated from standardized normal variables.

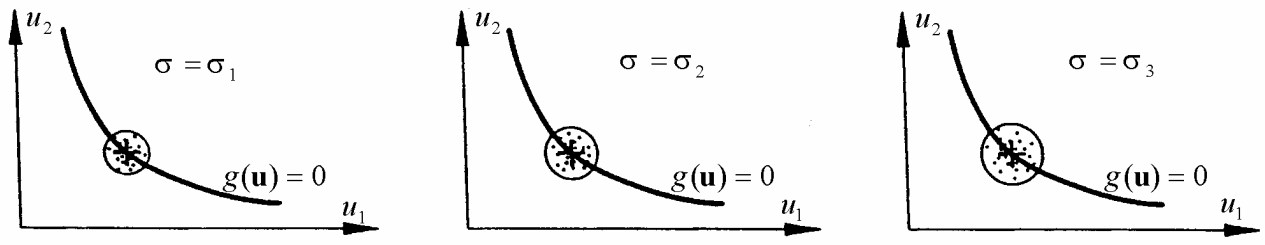

Figure 5.5. Different standard deviations of the sampling density, $\sigma_1 < \sigma_2 < \sigma_3$.

The standard error is estimated by (5.29). The efficiency of the importance sampling can be expected to be dependent on the choice of standard deviation of the sampling density, see figure 5.5.

It should be noted that if a failure mode has multiple β-points importance sampling based on only one β-point is not efficient. In this case more general methods have to be used, see section 5.7.

5.6 Monte Carlo sampling by excluding part of safe area

In this technique the space is separated into two disjoint regions D_1 and D_2, see figure 5.6. It is assumed that D_1 is selected such that no failure occurs in this region. Here D_1 is chosen as the region inside a sphere with radius β. The probability of being in D_1 is:

$$p_1 = P\left(\sum_{i=1}^{n} U_1^2 \le \beta^2\right) = \chi^2(n, \beta^2) \tag{5.31}$$

where $\chi^2(n, \beta^2)$ is the χ^2 distribution function with *n* degrees of freedom.

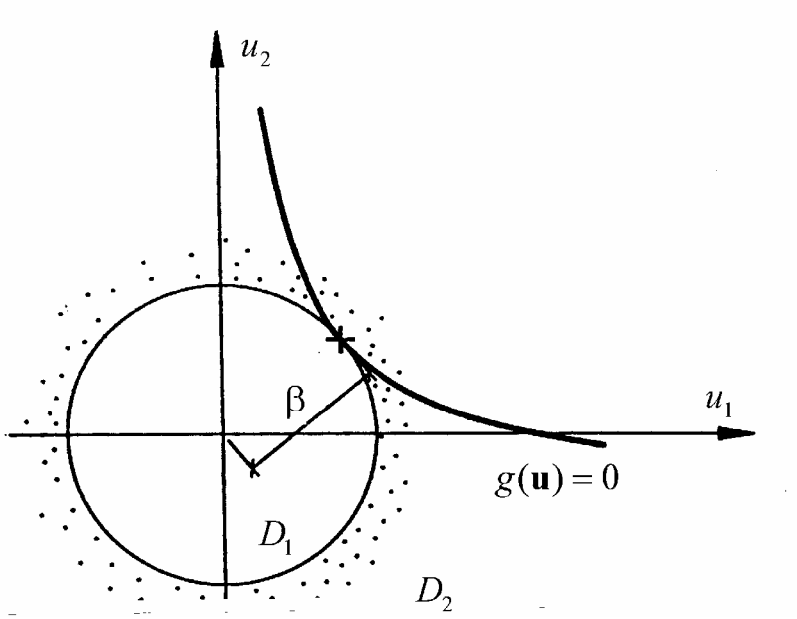

Figure 5.6. Monte Carlo sampling by excluding part of safe area.

The probability of failure is estimated from:

$$\hat{P}_f = \frac{1 - p_1}{N} \sum_{j=1}^{N} I[g(\hat{\mathbf{u}}_j)] \tag{5.32}$$

where $\hat{\mathbf{u}}_j$ is sample no. j from D_2 (simulated from a standard normally distributed stochastic vector $\mathbf{U} = (U_1, \ldots, U_n)$, but only those samples outside D_1 are used).
The standard error is:

$$s = (1 - p_1)\sqrt{\frac{\hat{P}_f(1 - \hat{P}_f)}{N}} \tag{5.33}$$

The standard error is thus reduced by a factor $(1 - p_1)$ when compared with crude Monte Carlo simulation. Usually this is a significant reduction. However, it should be taken into account that it is more difficult to generate the samples to be used. If the samples are generated by taking the samples from simulation of normal distributed variables with $|\hat{\mathbf{u}}| > \beta$ then on average $\frac{1}{1-p_1}$ samples should

be generated before one sample is outside the β-sphere. So only in cases where the failure function require much more computational work than the generation of the samples $\hat{\mathbf{u}}$ it can be expected that this technique is efficient.

Example 5.4
Consider an example where the failure surface in standardized coordinates can be written:

$$g(\mathbf{u}) = 2u_2u_3 + 20u_2 + 8u_3 - 3u_1 + 71 = 0$$

The reliability index is determined as $\beta = 3.305$ and the design point is $\mathbf{u}^* = (0.540, -3.548, -0.188)$. The estimate of the failure probability using (3.38) is:

$$P_f = \Phi(-3.305) = 0.000228$$

The failure probability is estimated by simulation using the following techniques:

- Crude Monte Carlo (C.M.C.) simulation.
- Importance sampling (Imp.samp.) using the design point. The standard deviation σ of the sampling density is chosen to ½, 1 and 2.
- Crude Monte Carlo simulation by excluding the β-sphere (C.M.C. – β).

The simulation results are shown in table 5.3 with standard errors in ( ). It is seen that importance sampling and Crude Monte Carlo simulation by excluding the β-sphere are much better than crude Monte Carlo simulation. Further it is seen that in this example $\sigma = 1$ is the best choice for importance sampling.

N	100	1000	10 000
C.M.C	0 (0)	0 (0)	0.000200 (0.000141)
Imp.samp. $\sigma = \frac{1}{2}$	0.000306 (0.000193)	0.000196 (0.000021)	0.000195 (0.000010)
Imp.samp. $\sigma = 1$	0.000146 (0.000034)	0.000215 (0.000014)	0.000232 (0.000005)
Imp.samp. $\sigma = 2$	0.000153 (0.000070)	0.000163 (0.000024)	0.000234 (0.000011)
C.M.C.- β	0.000129 (0.000073)	0.000219 (0.000003)	

Table 5.3. Result of Monte Carlo simulation.

5.7 Other Simulation Techniques

In this section some other simulation methods are described, namely directional sampling, Latin hypercube simulation and adaptive simulation techniques.

Directional simulation

Instead of formulating the reliability problem in rectangular coordinates it is possible to formulate it in polar coordinates. Directional simulation methods are based on such a formulation and were first suggested by Deak [5.7] in connection with evaluation of the multinormal distribution function.

The *n*-dimensional standardized normal vector $\mathbf{U}$ is written:

$$\mathbf{U} = R\mathbf{A} \tag{5.34}$$

where the radial distance $R > 0$ is a stochastic variable and $\mathbf{A}$ is a unit vector of independent stochastic variables, indicating the direction in the *u*-space.

In uniform directional simulation $\mathbf{A}$ is uniformly distributed on the *n*-dimensional unit (hyper-) sphere. It then follows that the radial distance *R* has a distribution such that R^2 is chi-square distributed with *n* degrees of freedom. If *R* is independent of $\mathbf{A}$ then the probability of failure can be written:

$$P_f = P\left(g(\mathbf{U}) \le 0\right) = \int_{\text{unit sphere}} P\left(g(R\mathbf{A}) \le 0 \middle| \mathbf{A} = \mathbf{a}\right) f_{\mathbf{A}}(\mathbf{a}) d\mathbf{a} \tag{5.35}$$

where $f_{\mathbf{A}}(\mathbf{a})$ is the constant density of $\mathbf{A}$ on the unit sphere.

It is now assumed that the origin $\mathbf{u} = \mathbf{0}$ is in the safe area $\left(g(\mathbf{0}) > 0\right)$ and that the failure region defined by $\left\{\mathbf{u} : g(\mathbf{u}) \le 0\right\}$ is star shaped with respect to the $\mathbf{u} = \mathbf{0}$, i.e. every half-line starting form $\mathbf{u} = \mathbf{0}$ only crosses the failure surface once.

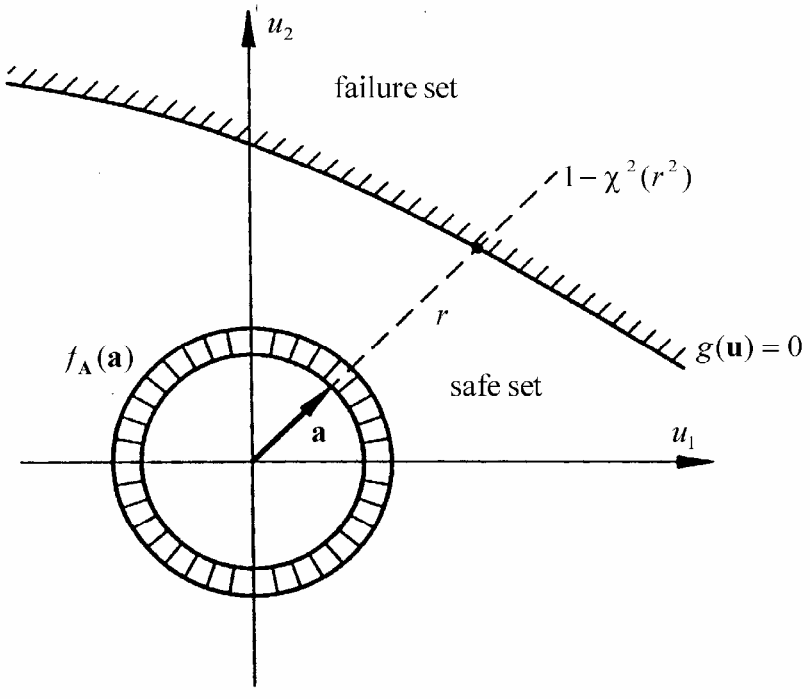

Figure 5.7. Uniform directional simulation.

The probability $P\left(g(R\mathbf{A}) \le 0 \middle| \mathbf{A} = \mathbf{a}\right)$ in (5.35) can then be calculated by:

$$P\left(g(R\mathbf{A}) \le 0 \middle| \mathbf{A} = \mathbf{a}\right) = \int_{r(\mathbf{a})}^{\infty} f_R(s \middle| \mathbf{A} = \mathbf{a}) ds = 1 - \chi_n^2\left(r(\mathbf{a})^2\right) \tag{5.36}$$

where $\chi_n^2(\,)$ is the χ_n^2 distribution with n degrees of freedom. $r(\mathbf{a})$ is the distance from the origin $\mathbf{u} = \mathbf{0}$ to the failure surface, i.e. $g\left(r(\mathbf{a})\mathbf{a}\right) = 0$ in the $\mathbf{a}$ direction.

An unbiased estimator of P_f is:

$$\hat{P}_f \approx E[\hat{P}_f] = \frac{1}{N}\sum_{j=1}^{N} \hat{p}_j = \frac{1}{N}\sum_{j=1}^{N}\left(1 - \chi_n^2(r(\hat{\mathbf{a}}_j)^2)\right) \tag{5.37}$$

where N is the number of simulations and $\hat{\mathbf{a}}_j$ is a simulated sample of $\mathbf{A}$. Several generalisations are possible, e.g. to include importance sampling, see Melchers [5.8] and Ditlevsen & Madsen [5.5].

Latin hypercube simulation method

The description of the Latin hypercube simulation method is based on McKay et al. [5.9].

The basic idea in this method is to assure that the entire range of each variable is sampled, in order to obtain an efficient estimate of the probability of failure. The range of each variable is divided into m intervals. The probability of an outcome in each interval should be equal.

In the simulation procedure the samples are generated in such a way that an interval of each variable will be matched just one time with an interval from each of the rest of the variables. In figure 5.8 the Latin hypercube method is illustrated by an example with $n = 2$ stochastic variables and $m = 7$ intervals.

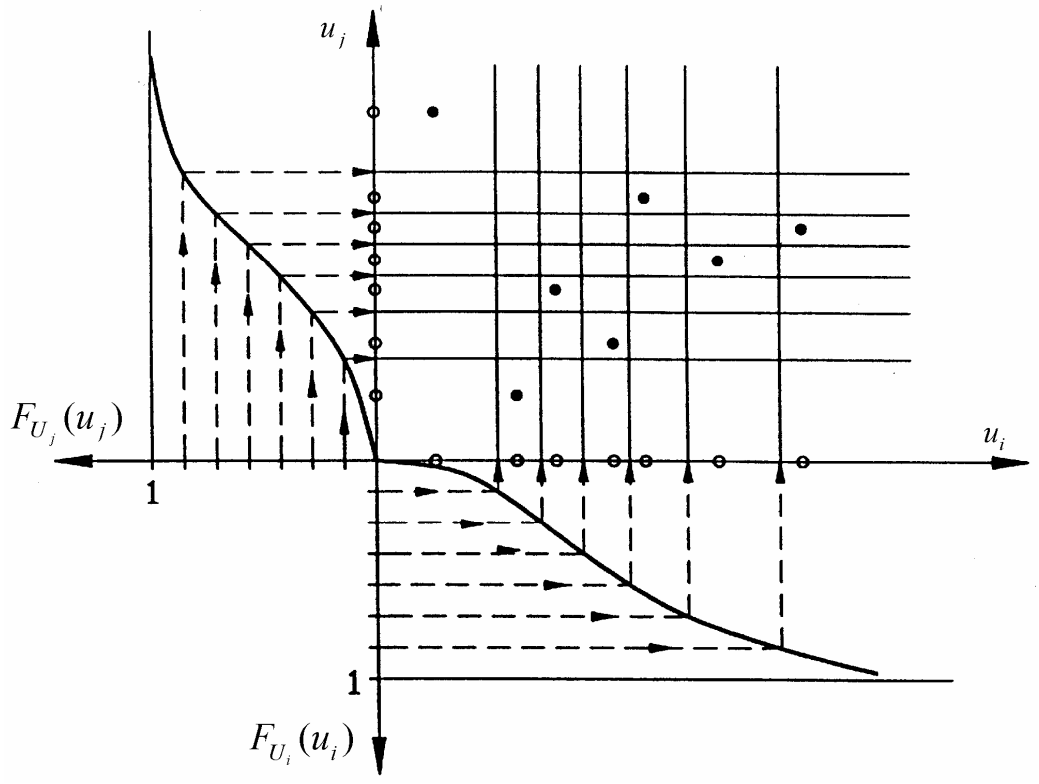

Figure 5.8. Latin hypercube simulation method.

The simulation procedure for the Latin hypercube method is :

1. For each variable generate one point from each of the intervals. $\hat{u}_{ij}, \quad j = 1,2,\ldots,m$ thus represents the m points for variable i.

2. The first point $\hat{\mathbf{u}}_j^1$ in the Latin hypercube sample is generated by sampling one value $\hat{u}_{ij}^1$ from each axis u_i. The second point is generated in the same way, except that the values $\hat{u}_{ij}^1$ are deleted from the sample. In this way m points are generated.

3. The probability of failure from this sample is estimated from:

$$\hat{P}_f = \frac{1}{m}\sum_{j=1}^{m} I[g(\hat{\mathbf{u}}^j)]$$

4. This procedure is repeated N times and the final estimate of P_f is:

$$\hat{P}_f = \frac{1}{Nm}\sum_{k=1}^{N}\sum_{j=1}^{m} I[g(\hat{\mathbf{u}}^{kj})]$$

where $\hat{\mathbf{u}}^{kj}$ is realisation no. j in the kth Latin hypercube sample.

There is no simple form for the standard error of this simulation method but in general the standard error is of the magnitude $\frac{1}{mN}$ times the standard error of crude Monte Carlo simulation.

Adaptive simulation methods

The description of the adaptive simulation methods is based on Melchers [5.8] and Karamchandani [5.10]. In order to develop a good importance sampling density it is necessary to know the region of the failure domain in which the probability density is relatively large. Usually our knowledge of this nature is poor. However, if the sample points are spread out (i.e. not clustered together), the value of the probability density of the points will vary. The regions that have higher probability densities can then be identified and the sampling density can be modified to generate sample points in these regions. However, it is still desirable to generate sample points that are spread out in order to explore the extent of the failure region in which the probability density is relatively large.

The initial sampling density is suggested to be standard normal with standard deviation 1 but with the expected value point moved to a point $\hat{\mathbf{u}}^{(0)}$ in or close to the failure region. This can be difficult, but based on the initial knowledge of which variables represents load variables and which variables represents strength variables such a point can be selected (for strength variables $\hat{u}^{(0)}$ should be negative and for load variables $\hat{u}^{(0)}$ should be positive). The initial density is used until a sample point is generated in the failure domain.

When multiple points in the failure region are generated the sampling density is modified such that the regions around the points with the largest probability density are emphasized. The simplest approach is to locate the expected value point at the point in the failure region with the largest probability density.

Another approach is to use a so-called multi-modal sampling density which generates samples around a number of points in the failure region, but emphasizes the region around a point in proportion to the probability density at the point. This allows us to emphasize more than one point and is

closer to the ideal sampling density (which is proportional to the probability density at each point in the failure domain). Let $\{\hat{\mathbf{u}}^{(1)}, \hat{\mathbf{u}}^{(2)}, \ldots, \hat{\mathbf{u}}^{(k)}\}$ be the set of points in the failure region, which are used to construct the multi-modal sampling density. The corresponding multi-modal density is:

$$h_{\mathbf{U}}^{k}(\mathbf{u}) = \sum_{j=1}^{k} w_j f_{\mathbf{U}}^{(j)}(\mathbf{u}) \tag{5.38}$$

where $f_{\mathbf{U}}^{(j)}(\mathbf{u})$ is the density function of a normally distributed stochastic vector with uncorrelated variables, standard deviations 1 and expected value point equal to $\hat{\mathbf{u}}^{(j)}$. The weights are determined by:

$$w_j = \frac{f_{\mathbf{U}}(\hat{\mathbf{u}}^{(j)})}{\sum_{i=1}^{k} f_{\mathbf{U}}(\hat{\mathbf{u}}^{(i)})} \tag{5.39}$$

The multimodal sampling density is illustrated in figure 5.9.

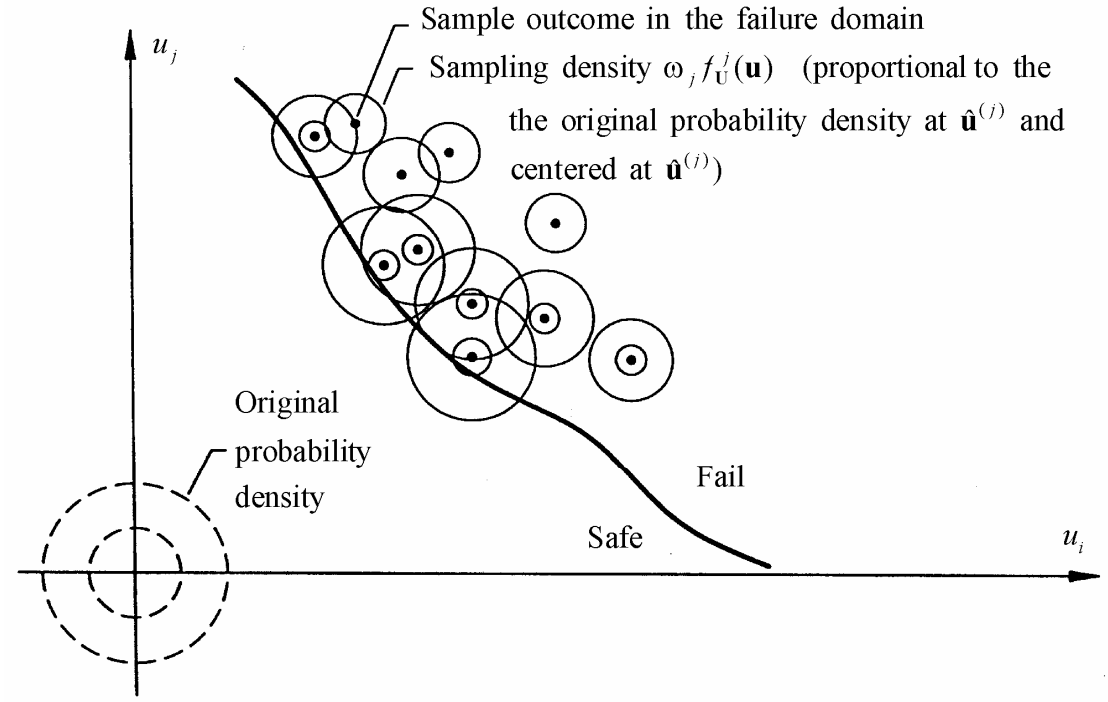

Figure 5.9. Multimodal sampling density (from [5.10]).

An estimate of the probability of failure can now be obtained on the basis of *N* simulations where the importance sampling technique is used :

$$\hat{P}_f = \frac{1}{N} \sum_{j=1}^{N} \frac{f_{\mathbf{U}}(\hat{\mathbf{u}}^{(j)})}{h_{\mathbf{U}}^{j}(\hat{\mathbf{u}}^{(j)})} I\left[g(\hat{\mathbf{u}}^{(j)})\right] \tag{5.40}$$

5.8 Sensitivity Measures

In many cases it is very interesting to know how sensitive an estimated probability of failure is with respect to a change of a parameter *p*. *p* is here assumed to be the expected value or the standard de-

viation of a stochastic variable. The transformation from the basic stochastic variables $\mathbf{X}$ to standardized normal variables is written:

$$\mathbf{X} = \mathbf{T}(\mathbf{U}, p) \tag{5.41}$$

and the probability of failure is defined by:

$$P_f = P\big(g(\mathbf{X}) \le 0\big) = \int I\big[g(\mathbf{x})\big] f_{\mathbf{X}}(\mathbf{x})\mathbf{dx} = \int I\big[g(\mathbf{T}(\mathbf{u}, p))\big] f_{\mathbf{U}}(\mathbf{u})\mathbf{du} \tag{5.42}$$

In crude Monte Carlo simulation P_f is estimated by:

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} I\big[g(\mathbf{T}(\hat{\mathbf{u}}^j, p))\big] \tag{5.43}$$

By direct differentiation the gradient $\frac{\partial P_f}{\partial p}$ of P_f with respect to p can be estimated by introducing a small change Δp in p and calculating:

$$\frac{\partial P_f}{\partial p} \approx \frac{\Delta \hat{P}_f}{\Delta p} = \frac{1}{\Delta p}\left(\frac{1}{N}\sum_{j=1}^{N} I\big[g\big(\mathbf{T}(\hat{\mathbf{u}}^j, p + \Delta p)\big)\big] - \frac{1}{N}\sum_{j=1}^{N} I\big[g\big(\mathbf{T}(\hat{\mathbf{u}}^j, p)\big)\big]\right) \tag{5.44}$$

The two terms in (5.44) are estimated separately. This estimate of ΔP_f can be expected to be both inaccurate because it is the difference between two "uncertain" estimates and time consuming because two sets of samples has to be generated.

Alternatively, $\frac{\partial P_f}{\partial p}$ can be written:

$$\begin{aligned}\frac{\partial P_f}{\partial p} &= \frac{\partial}{\partial p}\int I\big[g\big(\mathbf{T}(\mathbf{u}, p)\big)\big] f_{\mathbf{U}}(\mathbf{u}) d\mathbf{u} = \frac{\partial}{\partial p}\int I\big[g(\mathbf{x})\big] f_{\mathbf{X}(p)}(\mathbf{x}) d\mathbf{x} \\ &= \int I\big[g(\mathbf{x})\big]\frac{\partial f_{\mathbf{X}(p)}(\mathbf{x})}{\partial p} d\mathbf{x} \\ &= \int I\big[g(\mathbf{x})\big]\frac{\partial f_{\mathbf{X}(p)}(\mathbf{x})}{\partial p}\frac{1}{f_{\mathbf{X}(p)}(\mathbf{x})} f_{\mathbf{X}(p)}(\mathbf{x}) d\mathbf{x}\end{aligned} \tag{5.45}$$

where $f_{\mathbf{X}(p)}(\mathbf{x})$ is the density function of $\mathbf{X}$ with the parameter p. Corresponding to (5.43) and (5.45) the following estimates can be obtained by simulation:

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} I\big[g(\hat{\mathbf{x}}^j)\big] \tag{5.46}$$

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} I\big[g(\hat{\mathbf{x}}^j)\big]\frac{\partial f_{\mathbf{X}(p)}(\hat{\mathbf{x}}^j)}{\partial p}\frac{1}{f_{\mathbf{X}(p)}(\hat{\mathbf{x}}^j)} \tag{5.47}$$

The samples $\hat{\mathbf{x}}^j$ are generated from the density function $f_{\mathbf{X}(p)}(\mathbf{x})$ using for example the inverse

simulation method. The advantage of this formulation is that the same samples can be used to estimate both $\hat{P}_f$ and $\frac{\partial \hat{P}_f}{\partial p}$. This increases the accuracy and reduces the computational effort compared with direct differentiation. Similar formulations can be derived for other simulation types.

5.9 References

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Note 6: RELIABILITY EVALUATION OF SERIES SYSTEMS

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6.1 Introduction

So far, in the previous notes, only reliabilities of individual failure modes or limit states have been considered. In this note it is described how the individual limit states interact on each other and how the overall systems reliability can be estimated when the individual failure modes are combined in a series system of failure elements.

In section 6.2 a series system is defined, followed by section 6.3 where it is explained how the FORM-approximation of the reliability of a series system is obtained and how the correlation between failure elements are interpreted. In section 6.4 it is described how the multi-dimensional normal distribution function needed for the series system reliability estimation can be evaluated using bounds and approximations. Finally, section 6.5 introduces sensitivity analysis of series systems.

6.2 Modelling of Series Systems

A failure element or component, see figure 6.1, can be interpreted as a model of a specific failure mode at a specific location in the structure.

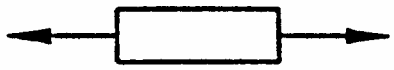

Figure 6.1. Failure element.

The combination of failure elements in a series system can be understood from the statically determinate (non-redundant) truss-structure in figure 6.2 with n structural elements (trusses). Each of the n structural elements is assigned 2 failure elements. One with a failure function modelling material yielding failure and one with a failure function modelling buckling failure.

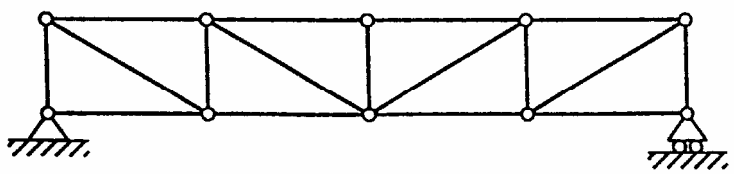

Figure 6.2. Statically determinate truss structure.

For such a statically determinate structure it is clear that the whole structural system fails as soon as any structural element fails, i.e. the structure has no load-carrying capacity after failure of one of the structural elements. This is called a weakest link system and is modelled as a series system. The series system which then becomes the systems reliability model consists of $2n$ failure elements shown in figure 6.3.

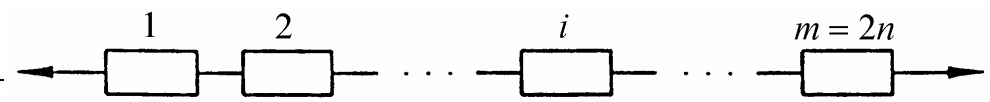

Figure 6.3. Weakest link system modelled as a series system of failure elements.

It is in this connection important to notice the difference between structural components and failure elements and the difference between a structural system and a systems reliability model.

If failure of one failure element is defined as systems failure the reliability of the series system can be interpreted as the reliability of failure. That also includes the case of statically indeterminate structures where failure of more than one failure element cannot be accepted.

6.3 FORM Approximation of the Reliability of a Series System

Consider a structural system where the system reliability model is a series system of *m* failure elements. Each of the failure elements is modelled with a safety margin:

$$M_i = g_i(\mathbf{X}) \quad , \quad 1,2,\ldots,m \tag{6.1}$$

The transformation between the standard normal stochastic $\mathbf{U}$-variables and the stochastic variables $\mathbf{X}$ can be obtained as explained in note 4 and is symbolically written as $\mathbf{X} = \mathbf{T(U)}$. Furthermore, it is known from notes 3 and 4 that the FORM probability of failure for failure element *i* can be written:

$$\begin{aligned} P_{f_i} &= P(M_i \le 0) = P\big(g_i(\mathbf{X}) \le 0\big) = P\big(g_i\big(\mathbf{T(U)}\big) \le 0\big) \\ &\approx P(\beta_i - \boldsymbol{\alpha}_i^T \mathbf{U} \le 0) = \Phi(-\beta_i) \end{aligned} \tag{6.2}$$

The series system fails if just one of the elements fails, i.e. the probability of failure of the series system is:

$$P_f^S = P\left(\bigcup_{i=1}^{m}\{M_i \le 0\}\right) = P\left(\bigcup_{i=1}^{m}\{g_i(\mathbf{X}) \le 0\}\right) = P\left(\bigcup_{i=1}^{m}\{g_i\big(\mathbf{T(U)}\big) \le 0\}\right) \tag{6.3}$$

Thus, if all the failure functions as in (6.2) are linearized at their respective β-points the FORM approximation of P_f^S of a series system can be written:

$$P_f^S \approx P\left(\bigcup_{i=1}^{m}\left\{-\boldsymbol{\alpha}_i^T \mathbf{U} \le -\beta_i\right\}\right) \tag{6.4}$$

which by use of De Morgan's laws can be written:

$$P_f^S \approx 1 - P\left(\bigcap_{i=1}^{m}\left\{-\boldsymbol{\alpha}_i^T \mathbf{U} > -\beta_i\right\}\right) = 1 - P\left(\bigcap_{i=1}^{m}\left\{\boldsymbol{\alpha}_i^T \mathbf{U} < -\beta_i\right\}\right) = 1 - \Phi_m(\boldsymbol{\beta}; \boldsymbol{\rho}) \tag{6.5}$$

where Φ_m is the *m*-dimensional normal distribution function (see the following section 6.4). It has been used that the correlation coefficient ρ_{ij} between two linearized safety margins $M_i = \beta_i - \boldsymbol{\alpha}_i^T \mathbf{U}$ and $M_j = \beta_j - \boldsymbol{\alpha}_j^T \mathbf{U}$ is:

$$\rho_{ij} = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j \tag{6.6}$$

From (6.5) a formal or so-called generalized series systems reliability index β^S can be introduced from:

$$P_f^S = 1 - \Phi_m(\boldsymbol{\beta}, \boldsymbol{\rho}) = \Phi(-\beta^S) \tag{6.7}$$

or:

$$\beta^S = -\Phi^{-1}(P_f^S) = -\Phi^{-1}\left(1 - \Phi_m(\boldsymbol{\beta}; \boldsymbol{\rho})\right) \tag{6.8}$$

Example 6.1 Illustration of the FORM approximation

Consider the two-dimensional case with 3 failure functions $g_i(\mathbf{T}(\mathbf{u})) = 0, i = 1,2,3$ shown in figure 6.4.

In figure 6.4 the exact failure domain, which is the union of the individual element failure domains is hatched. Furthermore, the reliability indices $\beta_i, i = 1,2,3$ and the safety margins linearized at their corresponding β-points $\mathbf{u}_i^*, i = 1,2,3$ are shown. It is seen that (6.7) or (6.8) is an approximation when the failure functions are non-linear in the *u*-space.

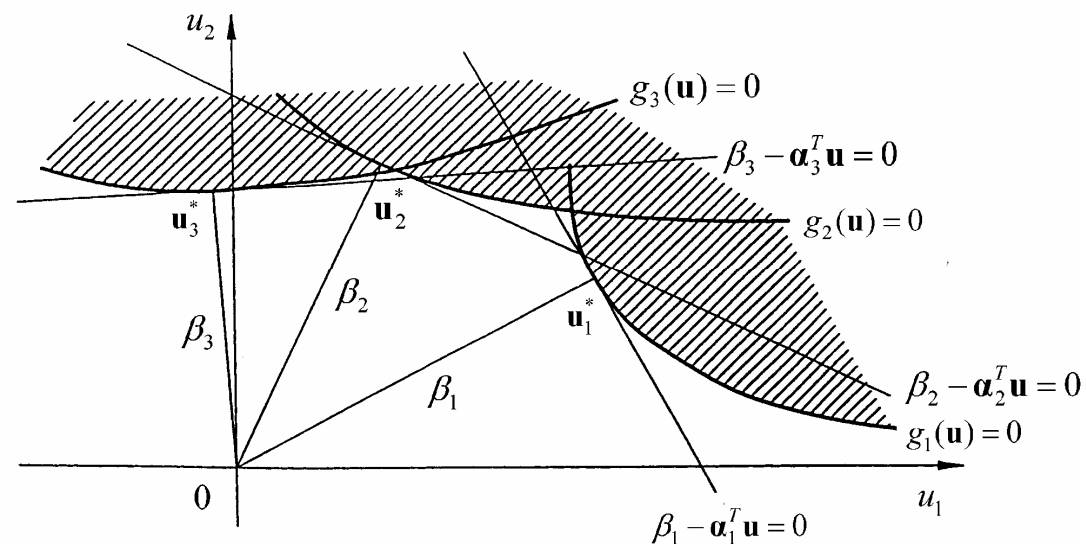

Figure 6.4. Illustration of the FORM-approximation.

* * *

Example 6.2 The Meaning of ρ_{ij}

Consider the two linearized safety margins $M_i = \beta_i - \boldsymbol{\alpha}_i^T \mathbf{U}$ and $M_j = \beta_j - \boldsymbol{\alpha}_j^T \mathbf{U}$ shown in figure 6.5

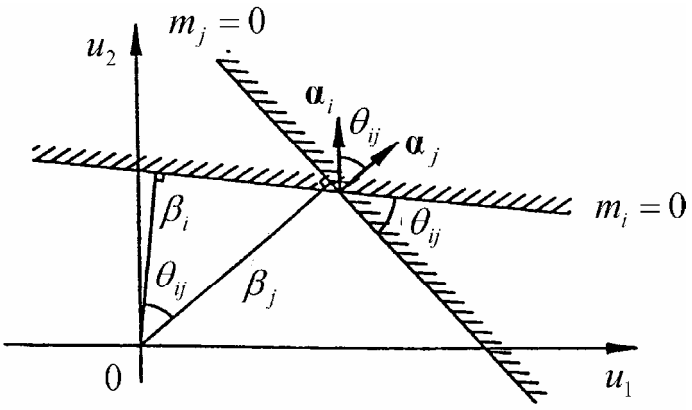

Figure 6.5. Illustration of ρ_{ij}.

From figure 6.5 it is seen that:

$$\cos\theta_{ij} = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j = \rho_{ij}$$

where θ_{ij} is the angle between the $\boldsymbol{\alpha}$-vectors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ or simply between the linearized safety margins. I.e., the correlation coefficients ρ_{ij} can be comprehended as a measure of the angle between the linearized safety margins and hereby as a measure of the extent of the failure domain.

* * *

Example 6.3 The Importance of ρ_{ij} in a Series System

Again the safety margins M_i and M_j from the previous example are considered. In figure 6.6 four cases are shown with $\beta_i = 3.0,\quad \beta_j = 3.0$ and ρ_{ij} equal – 1.0, 0.0, $\sqrt{0.5}$ and 1.0, respectively.

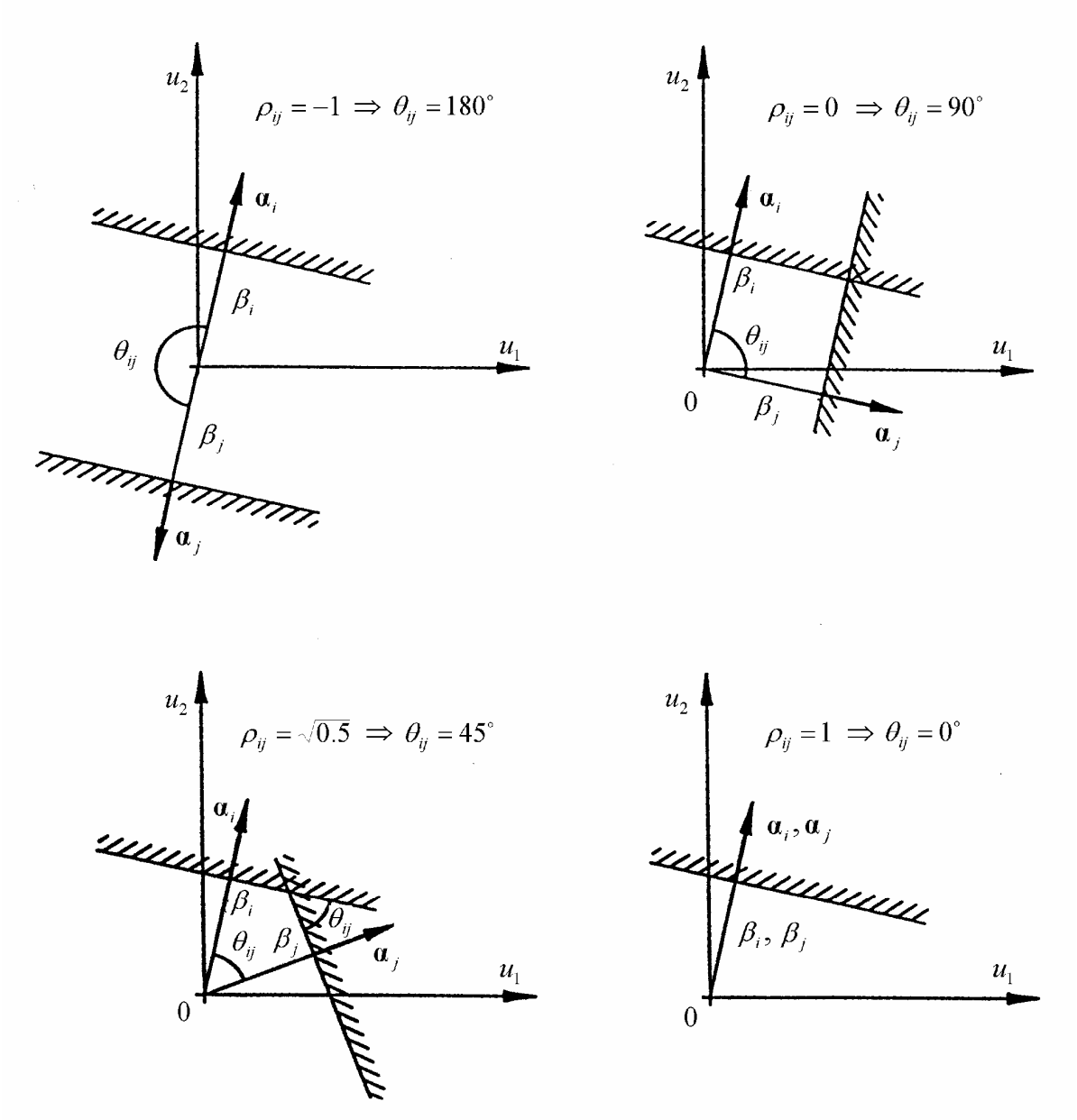

Figure 6.6. Illustration of ρ_{ij}.

The generalized systems reliability index β^S of the four cases in figure 6.6 can be found from (6.8) as 2.782, 2.783, 2.812 and 3.000, respectively.

In figure 6.7 $\beta^S = -\Phi^{-1}\left(1 - \Phi_2(3.0, 3.0; \rho)\right)$ is shown as a function of ρ.

From figures 6.6 and 6.7 it is seen that 2.782 = $\Phi^{-1}\left(2[1-\Phi(-3)]\right) \le \beta^S \le \Phi^{-1}\left([1-\Phi(-3)]\right)$ = 3.000 corresponding to the correlation $\rho = -1.0$ and the fully correlated case $\rho = 1.0$, respectively, i.e. it is always unsafe to assume that the failure elements are fully correlated if this is not the case.

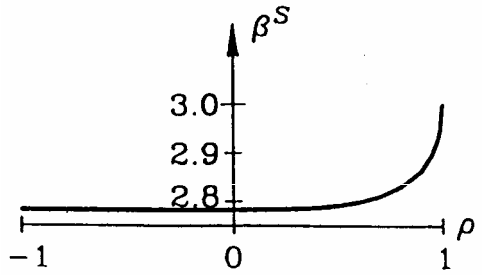

Figure 6.7. $\beta^S = -\Phi^{-1}\left(1 - \Phi_2(3.0, 3.0; \rho)\right)$ as a function of ρ.

* * *

6.4 Evaluation of Series Systems Reliabilities

From the previous section it is obtained that if β_i and ρ_{ij}, $i, j = 1,2,\cdots,m$ are known the problem is to evaluate the *m*-dimensional normal distribution function $\Phi_m(\mathbf{\beta;\rho})$ in (6.8) for the FORM approximation of β^S.

$\Phi_m(\mathbf{\beta;\rho})$ is defined as:

$$\Phi_m(\mathbf{\beta;\rho}) = \int_{-\infty}^{\beta_1}\int_{-\infty}^{\beta_2}\cdots\int_{-\infty}^{\beta_m}\varphi_m(\mathbf{x;\rho})dx_1\,dx_2\ldots dx_m \tag{6.9}$$

where φ_m is the *m*-dimensional normal density function:

$$\varphi_m(\mathbf{x;\rho}) = \frac{1}{(2\pi)^{m/2}\left|\mathbf{\rho}\right|^{1/2}}\exp(-\frac{1}{2}\mathbf{x}^T\mathbf{\rho}^{-1}\mathbf{x}) \tag{6.10}$$

The multi-dimensional integral in (6.9) can only in special cases be solved analytically and will for even small dimensions, say five, be too costly to evaluate by numerical integration. Instead, so-called bounds methods are used for hand calculations and so-called asymptotic approximate methods are used for computational calculations.

6.4.1 Reliability Bounds for Series Systems

In the following, so-called simple bounds and Ditlevsen bounds will be introduced as bounds for the reliability of series systems.

Simple Bounds
Simple bounds can be introduced as:

$$\max_{i=1}^{m} P(M_i \le 0) \le P_f^S \le \sum_{i=1}^{m}\left(P(M_i \le 0)\right) \tag{6.11}$$

where the lower bound corresponds to the exact value of P_f^S if all the elements in the series system are fully correlated.

In the terms of reliability indices (6.11) can be written:

$$-\Phi^{-1}\left(\sum_{i=1}^{m}\Phi(-\beta_i)\right) \le \beta^S \le \min_{i=1}^{m}\beta_i \tag{6.12}$$

When the failure of one failure element is not dominating in relation to the other failure elements the simple bounds are generally too wide and therefore often of minor interest for practical use.

Ditlevsen Bounds

Much better bounds are obtained from the second-order bounds called Ditlevsen bounds [6.4]. The derivation of the Ditlevsen bounds can be seen in [6.1], [6.4], [6.6], [6.7] or [6.8]. The bounds are:

$$P_f^S \ge P(M_1 \le 0) + \sum_{i=2}^{m}\max\left\{P(M_i \le 0) - \sum_{j=1}^{i-1}P(M_i \le 0 \bigcap M_j \le 0), 0\right\} \tag{6.13a}$$

$$P_f^S \le \sum_{i=1}^{m}P(M_i \le 0) - \sum_{i=2}^{m}\max_{j<i}\left\{P(M_i \le 0 \bigcap M_j \le 0)\right\} \tag{6.13b}$$

and in terms of the FORM approximation in reliability indices:

$$\Phi(-\beta^S) \ge \Phi(-\beta_1) + \sum_{i=2}^{m}\max\left\{\Phi(-\beta_i) - \sum_{j=1}^{i-1}\Phi_2(-\beta_i, -\beta_j; \rho_{ij}), 0\right\} \tag{6.14a}$$

$$\Phi(-\beta^S) \le \sum_{i=1}^{m}\Phi(-\beta_i) - \sum_{i=2}^{m}\max_{j<i}\left\{\Phi_2(-\beta_i, -\beta_j; \rho_{ij})\right\} \tag{6.14b}$$

The numbering of the failure elements influences the bounds. However, experience suggests that it is a good choice to arrange the failure elements according to decreasing probability of failure, i.e. $P(M_1 \le 0) \ge P(M_2 \le 0) \ge \cdots \ge P(M_m \le 0)$. The Ditlevsen bounds are usually much more precise than the simple bounds in (6.11) - (6.12), but require the estimation of $\Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ in (6.14).

From (6.9) it follows that:

$$\frac{\partial^2\Phi_2(\beta_i, \beta_j; \rho_{ij})}{\partial\beta_i\partial\beta_j} = \frac{\partial\Phi_2(\beta_i, \beta_j; \rho_{ij})}{\partial\rho_{ij}} \tag{6.15}$$

Therefore:

$$\Phi_2(\beta_i,\beta_j;\rho_{ij}) = \Phi_2(\beta_i,\beta_j;0) + \int_0^{\rho_{ij}} \frac{\partial \Phi_2(\beta_i,\beta_j;t)}{\partial t}\Big|_{t=z} dz$$
$$= \Phi(\beta_i)\Phi(\beta_j) + \int_0^{\rho_{ij}} \varphi_2(\beta_i,\beta_j;z)dz \qquad (6.16)$$

Hereby only a one-dimensional integral has to be solved for the evaluation of $\Phi_2(\beta_i,\beta_j;\rho_{ij})$. It is also possible to estimate $\Phi_2(-\beta_i,-\beta_j;\rho_{ij}) = P(M_i \le 0 \bigcap M_j \le 0)$ from simple bounds, which are derived from figure 6.8.

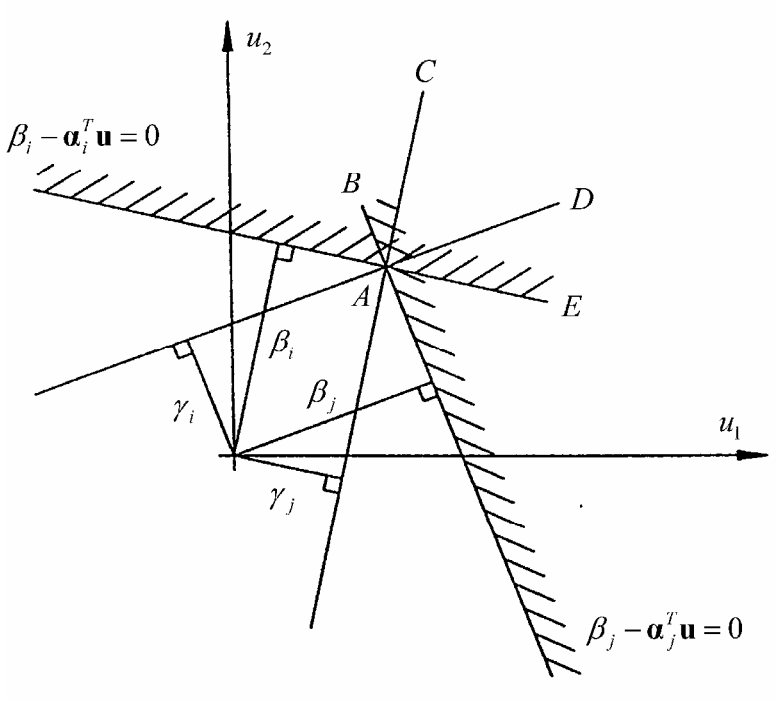

Figure 6.8. Figure for simple bounds of $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$.

From figure 6.8 it is seen that $P(M_i \le 0 \bigcap M_j \le 0)$ equals the probability contents in the hatched angle *BAE*. *P* is greater than the probability content in the angle *BAD* and in the angle *CAE*. However, *P* is less than the sum of the probability contents in the angles *BAD* and *CAE*. This observation makes it possible to derive simple bounds for $P_{ij} = \Phi_2(-\beta_i,-\beta_j;\rho_{ij})$.

The probability contents p_i and p_j in the angles *CAE* and *BAD*, respectively, are:

$$p_i = \Phi(-\beta_i)\Phi(-\gamma_j) \qquad \text{and} \qquad p_j = \Phi(-\beta_j)\Phi(-\gamma_i) \qquad (6.17)$$

where γ_i and γ_j can be found from figure 6.8 as:

$$\gamma_i = \frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}} \qquad \gamma_j = \frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}} \qquad (6.18)$$

Therefore, for $\rho_{ij} > 0$, the following bounds exist:

$$\max(p_i, p_j) \le \Phi_2(-\beta_i, -\beta_j; \rho_{ij}) \le p_i + p_j \tag{6.19}$$

and similarly for $\rho_{ij} < 0$:

$$0 \le \Phi_2(-\beta_i, -\beta_j; \rho_{ij}) \le \min(p_i, p_j) \tag{6.20}$$

These bounds are easy to use and P_{ij} can be approximated as the average of the lower and the upper bounds. If the gap between the lower and the upper bounds is too wide, a more accurate method, such as numerical integration of (6.16) should be used.

Example 6.4 Simple Illustration of Ditlevsen Bounds
Consider a simple example with 3 failure elements in a series system. Each of the elements $i = 1,2,3$ has a finite failure domain D_i with uniform and equal probability density as shown in figure 6.9

The lower Ditlevsen bound on $P_f^S = P(D_1 \bigcup D_2 \bigcup D_3)$ is:

$$P_f^S \ge P(D_1) + P(D_2) - P(D_2 \bigcap D_1) + P(D_3) - P(D_3 \bigcap D_1) - P(D_3 \bigcap D_2)$$

from which it is seen that the hatched domain in figure 6.9 is the difference between the lower Ditlevsen bound and the exact P_f^S.

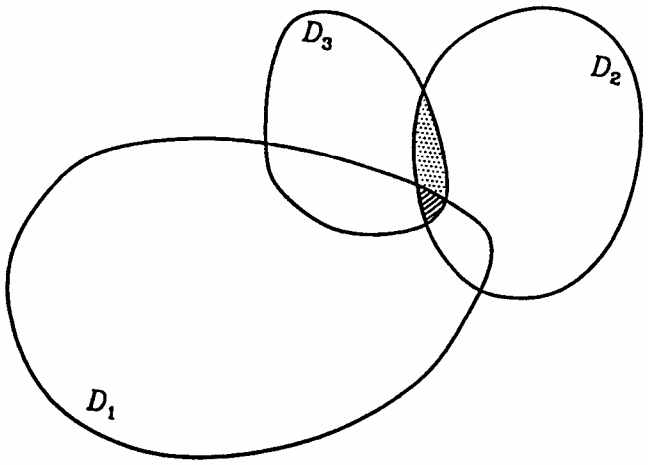

Figure 6.9. Illustration of Ditlevsen bounds.

The upper Ditlevsen bound on $P_f^S = P(D_1 \bigcup D_2 \bigcup D_3)$ is:

$$P_f^S \le P(D_1) + P(D_2) + P(D_3) - P(D_2 \bigcap D_1) - P(D_3 \bigcap D_1)$$

From which it is seen that the dotted domain in figure 6.9 is the difference between the upper Ditlevsen bound and the exact P_f^S.

* * *

Example 6.5 FORM Evaluation of β^S of a Series System

Consider a series system of 4 failure elements. After the transformation of the stochastic (physical) variables X_1 and X_2 into the standard normal space of variables U_1 and U_2 the four failure elements are described by the following failure functions:

$$g_1(\mathbf{u}) = \exp(u_1) - u_2 + 3$$

$$g_2(\mathbf{u}) = u_1 - u_2 + 5$$

$$g_3(\mathbf{u}) = \exp(u_1 + 4) - u_2$$

$$g_4(\mathbf{u}) = 0.1u_1^2 - u_2 + 4$$

The failure functions $g_i(\mathbf{u}) = 0 \quad i = 1,\ldots,4$ are shown in figure 6.10.

The reliability indices β_i with the corresponding P_{fi}, $\boldsymbol{\alpha}$-vectors and β-points are shown in table 6.1.

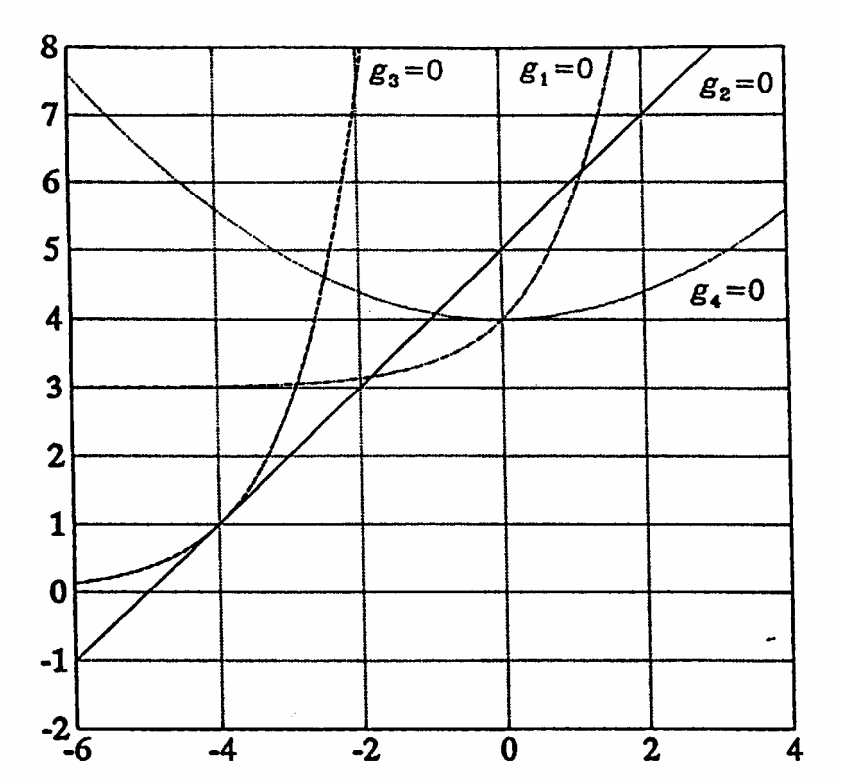

Figure 6.10. Four failure functions for a series system.

i	β_i	$\Phi(-\beta_i)$	α_{i1}	α_{i2}	u_{i1}^*	u_{i2}^*
1	3.51	$2.276\cdot10^{-4}$	–0.283	0.959	–0.99	3.36
2	3.54	$2.035\cdot10^{-4}$	–0.707	0.707	–2.50	2.50
3	3.86	$5.738\cdot10^{-5}$	–0.875	0.483	–3.38	1.86
4	4.00	$3.174\cdot10^{-5}$	0.000	1.000	0.00	4.00

Table 6.1 Information concerning failure elements.

From table 6.1 the correlation matrix $\boldsymbol{\rho}$ can be obtained from (6.6):

$$\boldsymbol{\rho} = \begin{bmatrix} 1.000 & & & \text{sym.} \\ 0.878 & 1.000 & & \\ 0.712 & 0.961 & 1.000 & \\ 0.962 & 0.714 & 0.492 & 1.000 \end{bmatrix}$$

Simple Bounds

From (6.12) the simple bounds of β^S can be obtained as:

$$\beta^S \geq -\Phi^{-1}(2.276\cdot10^{-4} + 2.035\cdot10^{-4} + 5.738\cdot10^{-5} + 3.174\cdot10^{-5}) = 3.28$$

$$\beta^S \leq \min\{3.51;3.54;3.86;4.00\} = 3.51$$

Ditlevsen Bounds

For Ditlevsen bounds it is necessary to evaluate $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$, i, j = 1,…,4, for $j < i$, which can be done approximately by (6.17) - (6.20). In the following matrix γ_i and γ_j from (6.18) are shown. (γ_i from (6.18) is shown in the lower triangle and γ_j is shown in the upper triangle)

$$\begin{bmatrix} - & 0.839 & 1.082 & -1.253 \\ 0.956 & - & -0.617 & 0.971 \\ 1.938 & 1.659 & - & 2.170 \\ 2.297 & 2.107 & 2.415 & - \end{bmatrix}$$

From (6.17) -(6.20) it is then possible to obtain the following table with bounds of $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$.

i, j	2,1	3,1	4,1	3,2	4,2	4,3
p_i	4.09	0.801	2.84	4.18	0.526	0.0476
p_j	3.86	0.599	0.246	0.535	0.220	0.0451
$a = \max\{p_i, p_j\}$	4.09	0.801	2.84	4.18	0.526	0.0476
$b = p_i + p_j$	7.95	1.40	3.09	4.71	0.776	0.0927
$0.5(a+b)$	6.02	1.10	2.96	4.45	0.636	0.0702

Table 6.2 List of probabilities ($p\cdot10^{-5}$).

Ditlevsen Lower Bound
In the lower Ditlevsen bound the upper bounds of $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$ are used, i.e.:

$$\begin{aligned}\Phi(-\beta^S) \geq\ & 2.276\cdot10^{-4} + \max\left\{2.035\cdot10^{-4} - 7.95\cdot10^{-5}, 0\right\} \\ & + \max\left\{5.738\cdot10^{-5} - (1.40+4.71)\cdot10^{-5}, 0\right\} \\ & + \max\left\{3.174\cdot10^{-5} - (3.09+0.776+0.0927)\cdot10^{-5}, 0\right\} \\ =\ & 3.52\cdot10^{-4}\end{aligned}$$

Ditlevsen Upper Bound
In the upper Ditlevsen bound the lower bounds of $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$ are used, i.e.:

$$\begin{aligned}\Phi(-\beta^S) \leq\ & 2.276\cdot10^{-4} + 2.035\cdot10^{-4} + 5.738\cdot10^{-5} + 3.174\cdot10^{-5} \\ & - 4.09\cdot10^{-5} - \max\left\{0.801\cdot10^{-5}, 4.18\cdot10^{-5}\right\} \\ & + \max\left\{2.84\cdot10^{-5}, 0.526\cdot10^{-5}, 0.0476\cdot10^{-5}\right\} \\ =\ & 4.09\cdot10^{-4}\end{aligned}$$

corresponding to:

$$3.36 \leq \beta^S \leq 3.39$$

If instead the average approximations of $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$ in the bottom row of table 6.2 are used only approximations of the bounds are obtained (i.e, there is no guarantee that β^S is within the bounds):

$$3.36 \leq \beta^S \leq 3.37$$

If $\Phi_2(-\beta_i,-\beta_j;\rho_{ij})$ is calculated exactly from (6.16) the following exact bounds are obtained:

$$3.381 \leq \beta^S \leq 3.383$$

It is seen that the Ditlevsen bounds in this case are narrow. This will often be the case.

* * *

Example 6.6 Failure Element with Two β-Points

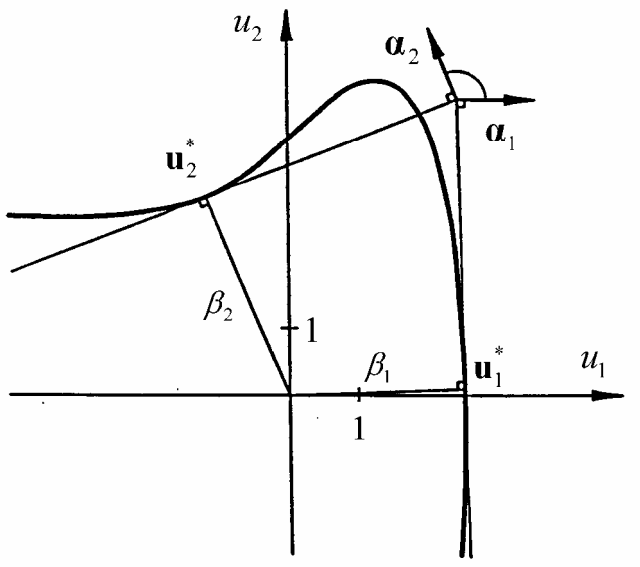

Figure 6.11. Failure functions from example 4.9.

Consider again example 4.9 where the failure function in the *u*-space was found as shown in figure 6.11.

Instead of estimating the probability of failure as $P_{f_1} = \Phi(-\beta_1) = \Phi(-2.78) = 2.68 \cdot 10^{-3}$, the probability of failure is estimated as $P_f = P(M_1 \le 0 \bigcup M_2 \le 0)$ where M_1 and M_2 are safety margins from linearization at the β-points $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$, respectively, (see figure 6.11). The safety margins are written $M_1 = \beta_1 - \boldsymbol{\alpha}_1^T \mathbf{U}$ and $M_2 = \beta_2 - \boldsymbol{\alpha}_2^T \mathbf{U}$.

With $\beta_1 = 2.784$, $\beta_2 = 3.501$ ($P_{f_2} = 2.31 \cdot 10^{-4}$) and the $\boldsymbol{\alpha}$-vectors $\boldsymbol{\alpha}_1 = (0.999, 0.036)$ and $\boldsymbol{\alpha}_2 = (-0.370, 0.929)$, the correlation coefficient is $\rho_{12} = \boldsymbol{\alpha}_1^T \boldsymbol{\alpha}_2 = -0.337$. The probability of failure is then obtained as $P_f = 1 - \Phi_2(\beta_1, \beta_2; \rho_{12})$, which is:

$$P_f = \Phi(-\beta_1) + \Phi(-\beta_2) - \Phi_2(-\beta_1, -\beta_2; \rho_{12})$$

$\Phi_2(-\beta_1, -\beta_2; \rho_{12})$ is estimated from (6.17) -(6.20). From (6.18) it can be obtained that $\gamma_1 = 4.2101$ and $\gamma_2 = 4.715$, which by use of (6.17) results in $p_1 = 3.25 \cdot 10^{-9}$ and $p_2 = 2.960 \cdot 10^{-9}$. An average estimate from (6.20) is then obtained as $\Phi_2(-\beta_1, -\beta_2; \rho_{12}) = 1.48 \cdot 10^{-9}$. P_f then is $P_f = 2.68 \cdot 10^{-3} + 2.32 \cdot 10^{-4} - 1.48 \cdot 10^{-9} = 2.91 \cdot 10^{-3}$, which corresponds to $\beta^S = 2.758$. Compared to the exact result $\beta^S = 2.755$ obtained by numerical integration with formula (c) in example 4.9 inserted into (3.6) this is a satisfactory estimate.

* * *

6.4.2 Numerical Methods for Evaluation of Φ_m

Approximation based on the average correlation coefficient
If as a special case all the correlations between the elements are the same, i.e. $\rho_{ij} = \rho$, $i, j = 1,2,\ldots,m, i \neq j$ then it can be shown that, see [6.7] or [6.10]:

$$\Phi_m(\boldsymbol{\beta};\boldsymbol{\rho}) = \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^{m} \Phi\left(\frac{\beta_i - \sqrt{\rho} t}{\sqrt{1-\rho}} \right) dt \tag{6.21}$$

For series systems the probability of failure then is:

$$P_f^S = 1 - \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^{m} \Phi\left(\frac{\beta_i - \sqrt{\rho} t}{\sqrt{1-\rho}} \right) dt \tag{6.22}$$

when the correlation coefficients are not all equal an approximation of the probability of failure can be obtained by using an average correlation coefficient $\bar{\rho}$ as ρ in (6.22). $\bar{\rho}$ is determined from:

$$\bar{\rho} = \frac{2}{m(m-1)} \sum_{i=1}^{m} \sum_{j=1}^{i-1} \rho_{ij} \tag{6.23}$$

The approximation based on the average correlation coefficient can be considered as the first term in a Taylor expansion of P_f^S at the average correlation coefficient point with respect to the correlation coefficients.

Using (6.22) with $\rho = \bar{\rho}$, an approximation of P_f^S is obtained. The approximation will in many cases be conservative.

Example 6.7
Consider the series system of example 6.5 again. The average correlation coefficient becomes:

$$\bar{\rho} = \frac{1}{6}(0.878 + 0.712 + 0.961 + 0.962 + 0.714 + 0.492) = 0.786$$

with $\boldsymbol{\beta} =$ (3.51, 3.54, 3.85, 4.00) in (6.22) the average correlation coefficient approximation becomes $P_f^S = 4.28 \cdot 10^{-4}$ corresponding to $\beta^S = 3.33$, which from example 6.5 is seen to give a conservative estimate of the series system reliability.

* * *

Advanced Asymptotic Methods
It has already been mentioned that the bounds methods in section 6.4.1 can be used in hand calculations. However, in professional reliability programs other more precise and more refined methods

are used. Two of these methods are the Hohenbichler approximation, see [6.5], and the approximation by Gollwitzer and Rackwitz [6.3]. These methods are in general very precise and make it possible to calculate Φ_m within reasonable computer time.

6.5 Sensitivity Analysis of Series Systems Reliabilities

From (6.8) it can be shown that the sensitivity of β^S with respect to a model parameter p can be found as:

$$\frac{d\beta^S}{dp} = \frac{1}{\varphi(\beta^S)} \sum_{i=1}^{m} \left\{ \frac{\partial \Phi_m(\mathbf{\beta};\mathbf{\rho})}{\partial \beta_i} \frac{d\beta_i}{dp} + 2 \sum_{j=1}^{i-1} \frac{\partial \Phi_m(\mathbf{\beta};\mathbf{\rho}) d\rho_{ij}}{\partial \rho_{ij}} \frac{d\rho_{ij}}{dp} \right\} \tag{6.24}$$

However, to get an estimate of the sensitivity of a systems reliability index β^S it is often sufficient to use:

$$\frac{d\beta^S}{dp} \approx \frac{1}{\varphi(\beta^S)} \sum_{i=1}^{m} \frac{\partial \Phi_m(\mathbf{\beta};\mathbf{\rho})}{\partial \beta_i} \frac{d\beta_i}{dp} \tag{6.25}$$

where $d\beta_i/dp$ can be obtained as already described in note 4 and $\partial \Phi_m(\mathbf{\beta},\mathbf{\rho})/\partial \beta_i$ can be determined either numerically by finite differences or by the semi-analytical methods described in [6.9] where also details of sensitivity analysis can be found.

6.6 References

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Note 7: RELIABILITY EVALUATION OF PARALLEL SYSTEMS

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7.1 Introduction

In this note it is described how the reliability of a system can be evaluated when more than one failure element have to fail before the whole system is defined to be in a state of failure. This is performed by introduction of parallel systems in section 7.2, followed by sections 7.3 and 7.4 where the FORM approximation of the reliability of a parallel system and reliability evaluation techniques are introduced, respectively. In section 7.5 it is described how the parallel systems are combined into a systems reliability model of a series system of parallel systems and finally, in section 7.6, it is shown, how the corresponding reliability evaluations can then be performed.

7.2 Modelling of Parallel Systems

The introduction and the necessity of parallel systems for the reliability modelling of some structural systems can be illustrated by considering the statically indeterminate (redundant) truss-structure in figure 7.1 with N structural elements (trusses). Two failure elements are assigned to each of the N structural elements, one with a failure function modelling material yielding failure and one with a failure function modelling buckling failure.

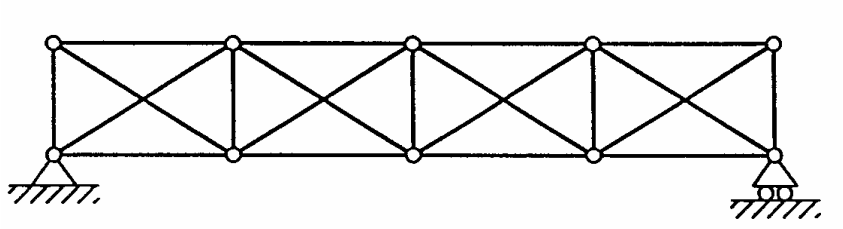

Figure 7.1. Statically indeterminate truss structure.

For such a statically indeterminate (redundant) structure it is clear that the whole structural system will not always fail as soon as one of structural element fails, because the structure has a load-carrying capacity after failure of some of the structural elements. This load-carrying capacity is obtained after a redistribution of the load effects in the structure after the element failure. Failure of the entire redundant structure will then often require failure of more than one structural element. (It is in this connection very important to define exactly what is understood by failure of the structural system). Clearly the number of systems failure modes in a redundant structure is generally high. Each of these system failure modes can be modelled by a parallel system consisting of generally n elements, where n is the number of failure elements which have to fail in the specific systems failure mode before the entire structure is defined to be in a state of failure. The parallel system with n elements is shown in figure 7.2.

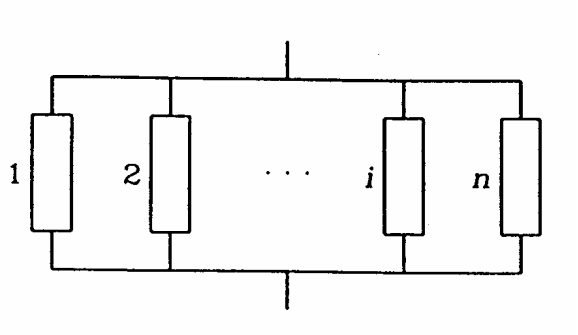

Figure 7.2. Failure mode of a redundant structure modelled as a parallel system.

Since a redistribution of the load effects has to take place in a redundant structural system after failure of one or more of the structural elements it becomes very important in parallel systems to describe the behaviour of the failed structural elements after failure has taken place. If the structural element has no strength after failure the element is said to be *perfectly brittle*. If the element after failure has a load-bearing capacity equal to the load at failure, the element is said to be *perfectly ductile.*

In figure 7.3 a perfectly brittle and a perfectly ductile element are shown with an example of the behaviours and the symbols used for perfectly brittle and perfectly ductile elements, respectively.

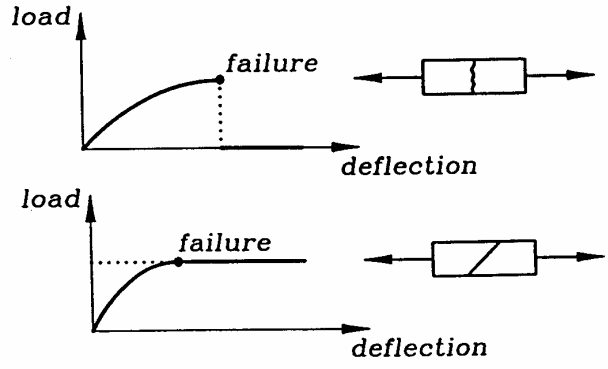

Figure 7.3. Perfectly brittle and perfectly ductile elements with symbols.

Clearly all kinds of structural components and material behaviours cannot be described as perfectly brittle or perfectly ductile. All kinds of combinations in between exist, i.e. some, but not all, of the failure strength capacity is retained. One of these modellings are the elastic-residual model shown in figure 7.4.

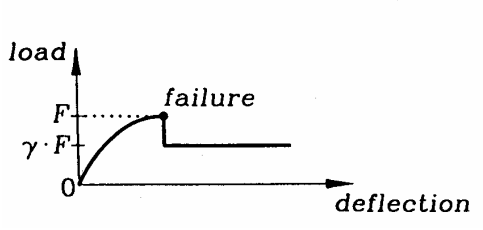

Figure 7.4. Elastic-residual element behaviour.

Before the reliability modelling in a parallel system of failure elements can be performed the structural behaviour of the considered failure mode must be clarified. More specifically the failure of the

structural elements and consequences with determination of residual load-carrying capacity and load redistribution in each step in the structural element failure sequence must be described. Then the failure functions of the failure elements in the parallel system can be formulated. Failure function no. 1 models failure in parallel system element no. 1 without failure in any other elements. Failure function no. 2 models failure in parallel system element no. 2 with failure in the structural element corresponding to failure element no. 1 (i.e. after redistribution of loads). Failure function no. 3 then models failure of parallel system element no. 3 with failure in the structural elements corresponding to failure element nos. 2 and 1, etc. etc.

The obtained failure functions can then be used in the reliability evaluations of the parallel system without further consideration of the structural system and structural behaviour.

Example 7.1 Structural Parallel Systems
As a special case of parallel systems so-called structural parallel systems as fibre bundles are considered in this example.

Consider a fibre bundle with *n* perfectly ductile fibres modelled by a parallel system. The strength $R_i, i = 1,2,\ldots,n$ of the individual fibres is identically normal distributed $\mathrm{N}(\mu,\sigma)$ with a common correlation coefficient ρ. The fibre bundle is loaded by a deterministic load $S = nS_e$, where S_e is the constant load on each fibre. The reliability indices of the fibre are the same for all fibres and equal to:

$$\beta = \frac{\mu - S_e}{\sigma}$$

The strength *R* of the ductile fibre bundle is obtained as the sum of the individual fibre strengths, i.e. *R* is normally distributed with:

$$\mu_R = n\mu \qquad \text{and} \qquad \sigma_R^2 = n\sigma^2 + n(n-1)\rho\sigma^2$$

The reliability index of the parallel system (fibre bundle) then is:

$$\beta^P = \frac{\mu_R - S}{\sigma_R} = \frac{n\mu - n(\mu - \beta\sigma)}{\sqrt{n\sigma^2 + n(n-1)\sigma^2\rho}} = \beta\sqrt{\frac{n}{1+\rho(n-1)}}$$

where it is used that $S = nS_e = n(\mu - \beta\sigma)$.

It is also possible to obtain β^P of a ductile fibre bundle when the fibres are not correlated by a common correlation coefficient ρ. This can e.g. be done by use of the average correlation coefficient defined in (6.23) and used in the above expression, see [7.4].

Another case of a fibre bundle is the Daniels system [7.7] of *n* perfectly brittle fibres. The strengths of the *n* fibres are $r_1, r_2, \ldots, r_n$, where $r_1 \le r_2 \le \cdots \le r_n$. The strength of the fibre bundle then is:

$$r_s = \max\{nr_1, (n-1)r_2, \ldots, 2r_{n-1}, r_n\}$$

Now, let r_i, $i = 1, 2, \ldots, n$ be realizations of independent random variables R_i with identical distribution functions. Similarly, r_s is the realization of R_s. Daniels showed that R_s is normally distributed $\mathrm{N}(\mu_{R_s}, \sigma_{R_s})$ for $n \to \infty$, where:

$\mu_{R_s} = nr_0[1 - F_R(r_0)]$ and $\sigma_{R_s}^2 = nr_0^2 F_R(r_0)[1 - F_R(r_0)]$

where r_0 is the maximum point of the function $r[1 - F_R(r)]$. The result is valid under the condition that r_0 is unique and $r[1 - F_R(r)] = 0$ for $r \to \infty$.

For a closer description also for small values of *n*, see [7.8 p. 249].

* * *

7.3 FORM Approximation of the Reliability of a Parallel System

After the failure functions of the failure elements in a parallel system have been formulated it is possible to estimate the reliability by FORM from the following description.

Consider a parallel system of *n* failure elements each modelled with a failure function and a safety margin:

$$M_i = g_i(\mathbf{X}), \quad i = 1, 2, \ldots, n \tag{7.1}$$

The transformation between the standard normal $\mathbf{U}$-variables and the stochastic variables $\mathbf{X}$ can be obtained as explained in note 4 and is symbolically written as $\mathbf{X} = \mathbf{T}(\mathbf{U})$.

The parallel system fails if all of the elements fail, i.e. the probability of failure of the parallel system is defined as the intersection of the individual failure events:

$$P_f^P = P\left(\bigcap_{i=1}^{n}\{M_i \le 0\}\right) = P\left(\bigcap_{i=1}^{n}\{g_i(\mathbf{X}) \le 0\}\right) = P\left(\bigcap_{i=1}^{n}\{g_i(\mathbf{T}(\mathbf{U})) \le 0\}\right) \tag{7.2}$$

Then a so-called joint β-point is introduced as the point in the failure domain (defined from (7.2)) closest to the origin, see figure 7.5. The n_A out of the *n* failure functions which equal zero at $\mathbf{u}^*$ are then linearized at $\mathbf{u}^*$:

$$M_i = \beta_i^J - \boldsymbol{\alpha}_i^T \mathbf{U}, \quad i = 1, 2, \ldots, n_A \tag{7.3}$$

where:

$$\boldsymbol{\alpha}_i = \frac{-\nabla_u g_i(\mathbf{T}(\mathbf{u}^*))}{\left|\nabla_u g_i(\mathbf{T}(\mathbf{u}^*))\right|} \quad \text{and} \quad \beta_i^J = \boldsymbol{\alpha}_i^T \mathbf{u}^* \tag{7.4}$$

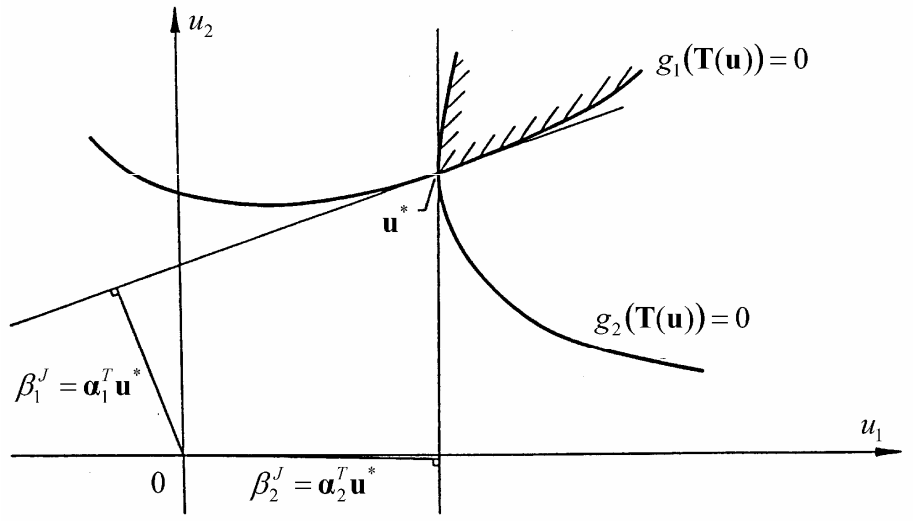

Figure 7.5. Illustration of the FORM-approximation of a parallel system.

Thus, $\boldsymbol{\beta}^J$ is an n_A-vector of indices at element level $\boldsymbol{\beta}^J=(\beta_1^J,\beta_2^J,\ldots,\beta_{n_A}^J)$ calculated from (7.4) by use of the joint β-point and not the individual β-points as in calculation of an element reliability index β.

The FORM-approximation of P_f^P of a parallel system can then be written:

$$P_f^P \approx P\left(\bigcap_{i=1}^{n_A}\left\{\beta_i^J-\boldsymbol{\alpha}_i^T\mathbf{U}\le 0\right\}\right)=P\left(\bigcap_{i=1}^{n_A}\left\{-\boldsymbol{\alpha}_i^T\mathbf{U}\le-\beta_i^J\right\}\right)=\Phi_{n_A}(-\boldsymbol{\beta}^J;\boldsymbol{\rho}) \tag{7.5}$$

where Φ_{n_A} is the n_A-dimensional normal distribution function and the correlation coefficient ρ_{ij} between two linearized safety margins $M_i=\beta_i^J-\boldsymbol{\alpha}_i^T\mathbf{U}$ and $M_j=\beta_j^J-\boldsymbol{\alpha}_j^T\mathbf{U}$ is:

$$\rho_{ij}=\boldsymbol{\alpha}_i^T\boldsymbol{\alpha}_j \tag{7.6}$$

From (7.5) a formal generalized parallel systems reliability β^P can be introduced by:

$$P_f^P=\Phi_{n_A}(-\boldsymbol{\beta}^J;\boldsymbol{\rho})=\Phi(-\beta^P) \tag{7.7}$$

as:

$$\beta^P=-\Phi^{-1}(P_f^P)=-\Phi^{-1}\left(\Phi_{n_A}(-\boldsymbol{\beta}^J;\boldsymbol{\rho})\right) \tag{7.8}$$

The joint β-point is from its definition determined as the solution of the following optimization problem:

$$\begin{aligned}&\min_{\mathbf{u}} \quad \gamma=\tfrac{1}{2}\mathbf{u}^T\mathbf{u}\\ &\text{s.t.} \quad g_i(\mathbf{u})\le 0 \quad , \quad i=1,2,\ldots,n\end{aligned} \tag{7.9}$$

The solution of the joint β-point problem can be obtained by a general non-linear optimization algorithm as NLPQL [7.1] or the problem specific algorithm JOINT3 described in [7.2].

Example 7.2 Illustration of the FORM-approximation

Consider the two-dimensional case with 3 failure functions $g_i(\mathbf{T}(\mathbf{u})) = 0,\ i = 1, 2, 3$ shown in figure 7.6.

In figure 7.6 the exact failure domain as the intersection of the individual element failure domains is hatched. Furthermore, the $n_A = 2$ active safety margins linearized at the joint β-point $\mathbf{u}^*$ are shown.

It is seen that (7.7) or (7.8) is an approximation when the failure functions are non-linear in the *u*-space or if so-called secondary joint β-points exist (a secondary β-point is shown in figure 7.6 as $\mathbf{u}_2$). For high reliability levels the approximation in (7.8) including the n_A active constraints of (7.9) is often sufficiently accurate.

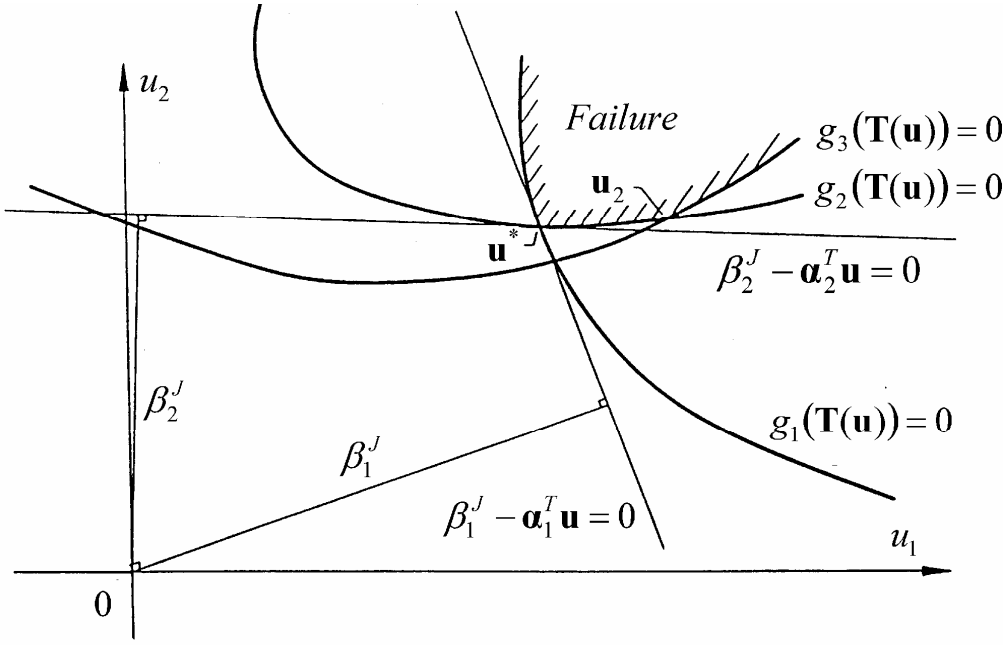

Figure 7.6. Illustration of the FORM-approximation.

* * *

The formulation in (7.9) requires that at least one of the failure functions is greater than zero in the origin. If this is not the case the problem can be converted to a series system problem by writing the safe domain as a union. For further explanation and inclusion of the secondary joint β-points for a more precise estimation, see [7.3].

In some references a cruder and older formulation of the FORM parallel system reliability is utilized. The failure domain is estimated as the intersection of the linearized failure functions at the individual β-points, i.e. only the individual β-point optimization problems are solved and not the joint β-point problem in (7.9).

Example 7.3 The Importance of ρ_{ij} in a Parallel System

For illustration of the importance of ρ_{ij} consider the margins $M_i = \beta_i^J - \boldsymbol{\alpha}_i^T\mathbf{U}$ and $M_j = \beta_j^J - \boldsymbol{\alpha}_j^T\mathbf{U}$. In figure 7.7 four cases are shown with $\beta_i = 3.0$, $\beta_j = 3.0$ and ρ_{ij} equal

–1.0, 0.0, $\sqrt{0.5}$ and 1.0, respectively.

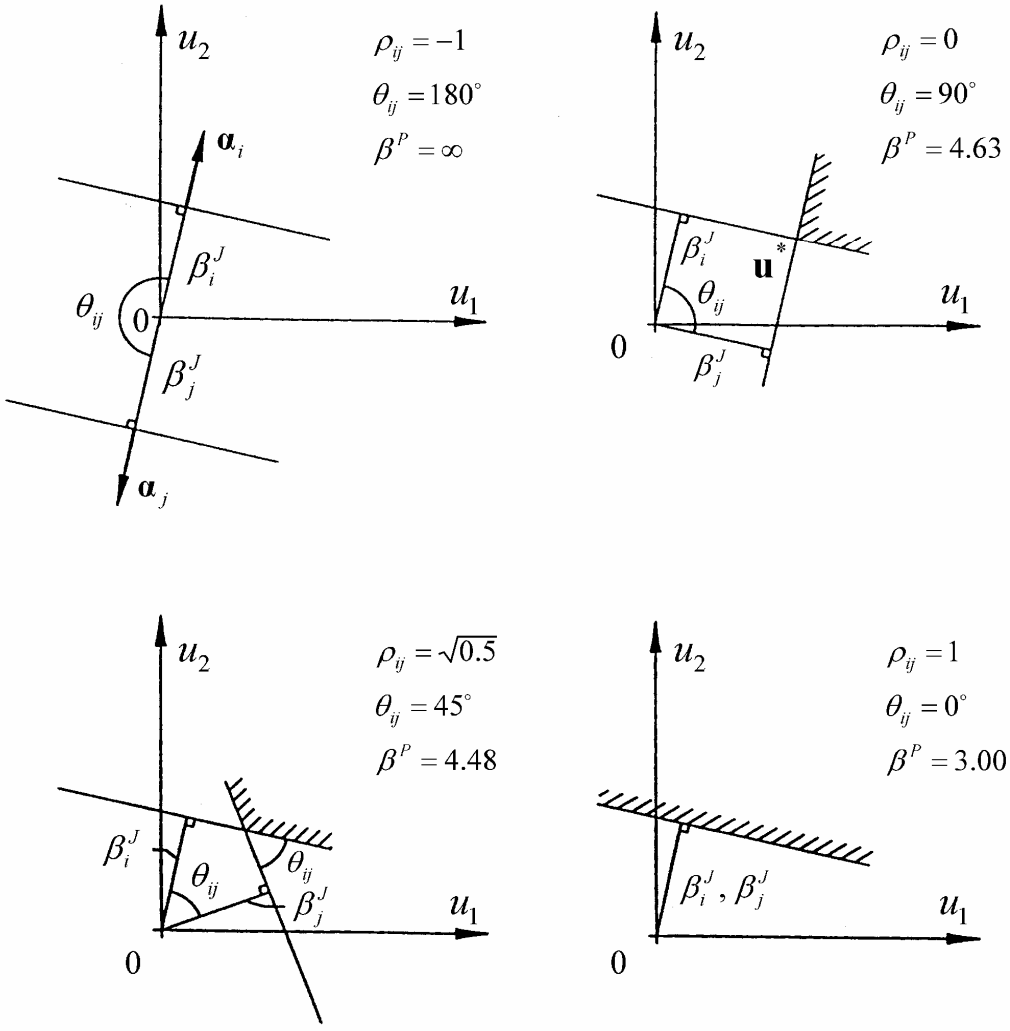

Figure 7.7. Illustration of ρ_{ij}.

The generalized parallel systems reliability index β^P of the four cases in figure 7.7 can be found from (7.8) as ∞, 4.63, 4.48 and 3.0, respectively.

In figure 7.8 $\beta^P = -\Phi^{-1}\left(\Phi_2(-3.0, -3.0; \rho)\right)$ is shown as a function of ρ.

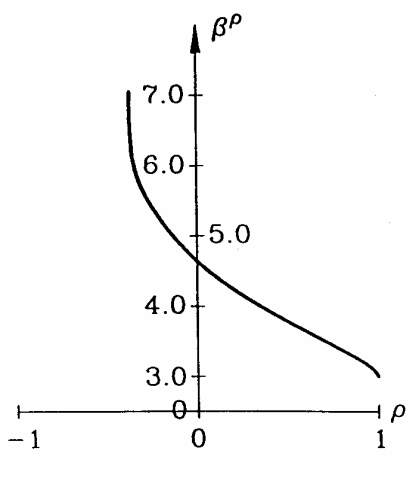

Figure 7.8. $\beta^P = -\Phi^{-1}\left(\Phi_2(-3.0, -3.0; \rho)\right)$ as a function of ρ.

From figure 7.8 it is seen that $3.0 \le \beta^P \le \infty$ corresponding to the fully positive correlated and the fully negative correlated cases, respectively.

* * *

7.4 Evaluation of Parallel Systems Reliabilities

The result from the previous section is that if β_i^J and $\rho_{ij}, 1, 2, \ldots, n_A$ are known the problem is to evaluate the n_A-dimensional normal distribution function $\Phi_{n_A}(-\boldsymbol{\beta}^J; \boldsymbol{\rho})$ in (7.8) for the FORM approximation of β^P. As described in note 6, this can generally not be performed by numerical integration within a reasonable computing time for higher dimensions. Instead bounds or approximate methods are used.

In the following, simple bounds and a second order bound will be introduced as bounds for the reliability of parallel systems.

Simple Bounds
If only the active constraints of (7.9) are assumed to influence the reliability of the parallel system the simple bounds can be introduced as:

$$0 \le P_f^P \le \min_{i=1}^{n_A}\left(P(M_i^J \le 0)\right) \tag{7.10}$$

where M_i^J, $i = 1, \ldots, n_A$ are the linearized safety margins at the joint β-point. The upper bound corresponds to the exact value of P_f^P if all the n_A elements are fully correlated with $\rho_{ij} = 1$.

In the terms of reliability indices β^J (7.10) can be written:

$$\max_{i=1}^{n_A} \beta_i^J \le \beta^P \le \infty \tag{7.11}$$

If all correlation coefficients ρ_{ij} between the n_A elements are higher than zero, the following simple bounds are obtained:

$$\prod_{i=1}^{n_A} P(M_i^J \le 0) \le P_f^P \le \min_{i=1}^{n_A} P(M_i^J \le 0) \tag{7.12}$$

where the lower bound corresponds to uncorrelated elements. i.e. $\rho_{ij} = 0$, $i \neq j$. In terms of β^J, (7.12) becomes:

$$\max_{i=1}^{n_A} \beta_i^J \le \beta^P \le -\Phi^{-1}\left(\prod_{I=1}^{n_A} \Phi(-\beta_i^J) \right) \tag{7.13}$$

The simple bounds will in most cases be so wide that they are of little practical use.

Second-Order Upper Bound

A second-order upper bound of P_f^P can be derived as:

$$P_f^P \le \min_{i,j=1}^{n_A} P(M_i^J \le 0 \bigcap M_j^J \le 0) \tag{7.14}$$

The corresponding lower bound of β^P is:

$$\beta^P \ge -\Phi^{-1}\left(\max_{i,j=1}^{n_A} \Phi_2(-\beta_i^J, -\beta_j^J, \rho_{ij}) \right) \tag{7.15}$$

In (7.15) it is seen that the probability of failure of a parallel system of two elements $\Phi_2(-\beta_i^J, -\beta_j^J, \rho_{ij})$ is necessary. These probabilities are the same as the probabilities used in the Ditlevsen bounds for series systems, see note 6. In note 6 both a method by numerical integration (6.16) and a bounds method (6.17) - (6.20) are described. Hereby the tools for evaluation of the bounds are described.

More refined and complicated bounds can also be developed, see [7.4], but will not be shown here.

Example 7.4 FORM Evaluation of β^P of a Parallel System

Consider a parallel system of 4 failure elements. After the transformation of the stochastic (physical) variables X_1 and X_2 into the standard normal space of variables U_1 and U_2 the four failure elements are described by the following failure functions:

$$g_1(\mathbf{u}) = \exp u_1 - u_2 + 1$$

$$g_2(\mathbf{u}) = u_1 - u_2 + 1$$

$$g_3(\mathbf{u}) = \exp(u_1 + 2) - u_2$$

$$g_4(\mathbf{u}) = 0.1u_1^2 - u_2 + 2$$

The failure functions $g_i(\mathbf{u}) = 0$, $i = 1, \ldots, 4$ are shown in figure 7.9.

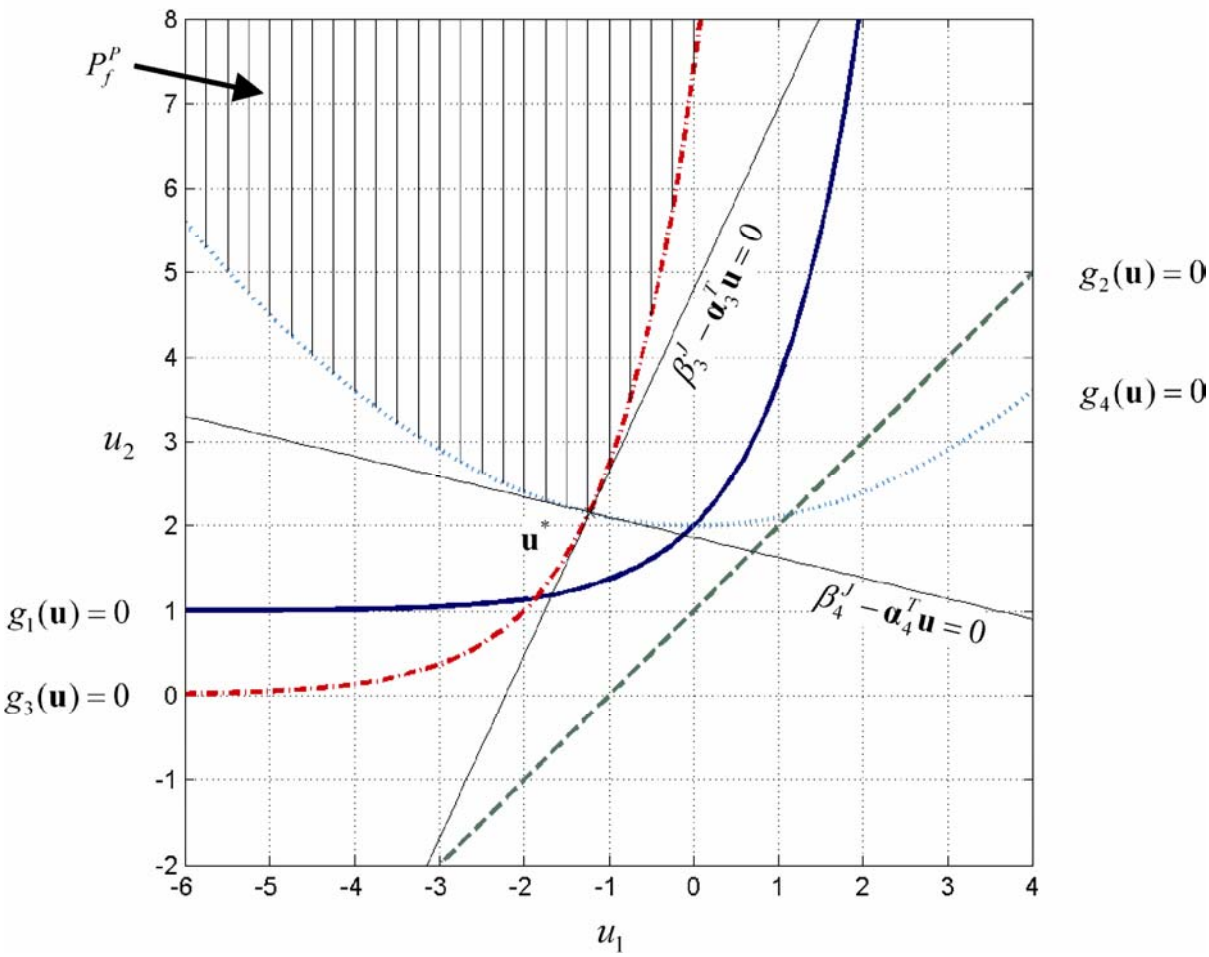

Figure 7.9. Four failure functions for a parallel system and joint β-point, $\mathbf{u}^*$.

It is seen directly from figure 7.9 that $n_A = 2$ and the joint β-point is the intersection between g_3 and g_4. The joint β-point can be found to be $\mathbf{u}^* = (-1.23; 2.16)$. The $\boldsymbol{\alpha}$-vectors are found from (7.4) as $\boldsymbol{\alpha}_3 = (-0.908; 0.420)$ and $\boldsymbol{\alpha}_4 = (0.233; 0.971)$, i.e the correlation coefficient from (7.6) is $\rho_{34} = 0.18$. From (7.3) $\boldsymbol{\beta}^J = (2.02; 1.81)$.

The simple bounds are obtained from (7.13):

$$\max\{1.81, 2.02\} \le \beta^P \le -\Phi^{-1}\left(\Phi(-1.8)\Phi(-2.02)\right)$$

or:

$$2.02 \le \beta^P \le 3.17$$

The second order lower bound will in this two-dimensional case be exact if $\Phi_2(-\beta_3^J, -\beta_4^J; \rho_{34})$ s evaluated to be exact. The result is:

$$\beta^P = 2.92$$

If instead the bounds technique from note 6, (6.17)-(6.20) is used, the bounds are obtained as 2.84 $\le \beta^P \le$ 3.04 or by taking the average of the bounds in (6.19) $\beta^P = 2.92$.

* * *

Advanced Asymptotic Methods
The bounds methods can be used in hand calculations. However, as described in note 6 (section 6.4.2) for series systems, other more precise and more refined methods are used in professional reliability programs.

7.5 General Systems Reliability

It is clear that a real redundant structural system generally has many failure modes, i.e. different sequences of element failure. Each sequence can then be modelled by a parallel system. If one of these parallel systems fails then the whole system fails, i.e. the overall systems reliability model is a series system of the failure modes or parallel systems. This is schematically shown in figure 7.10.

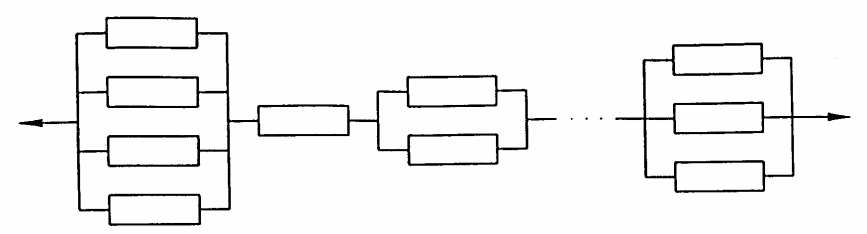

Figure 7.10. Systems reliability model as a series system of parallel systems.

It is also possible to formulate the systems reliability model as a parallel system of series systems, see [7.5].

Example 7.5 Systems Reliability Model of a Truss Structure
Consider the truss structure with two applied concentrated loads shown in figure 7.11.

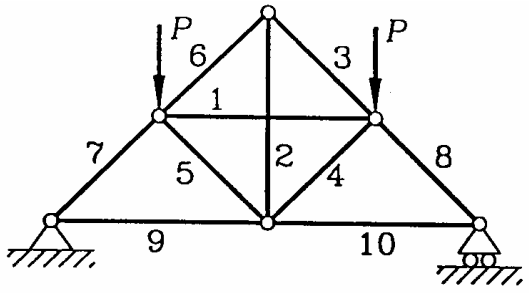

Figure 7.11. Statically indeterminate truss structure.

It is seen in figure 7.11 that the truss structure becomes statically determinate if any of the elements 1,2,3,4,5 or 6 is removed (fails). It is furthermore seen that the structure fails if any pair of the elements 1,2,3,4,5 and 6 fails. The structure also fails if one of the elements 7,8,9 or 10 fails. The systems reliability model is then a series system with 19 elements where 15 of the elements are parallel systems each with two failure elements. The elements in the series system are: {1,2}, {1,3}, {1,4}, {1,5}, {1,6}, {2,3}, {2,4}, {2,5}, {2,6}, {3,4}, {3,5}, {3,6}, {4,5}, {4,6}, {5,8}, {7}, {8}, {9} and {10}.

* * *

7.6 Reliability of Series Systems of Parallel Systems

The probability of failure of series systems of n_P parallel systems each with m_i, $i = 1, 2, \ldots, n_P$ failure elements can be written as a union of intersections:

$$P_f^S = P\left(\bigcup_{i=1}^{n_P} \bigcap_{j=1}^{m_i} \left\{ g_{ij}(\mathbf{X}) \le 0 \right\} \right) \tag{7.16}$$

where g_{ij} is the failure function of element *j* in parallel system *i*.

The FORM estimate of the generalized systems reliability index β^S is written as in note 6, see (6.1) - (6.8):

$$\beta^S = -\Phi^{-1}\left(1 - \Phi_{n_P}(\boldsymbol{\beta}^P; \boldsymbol{\rho}^P)\right) \tag{7.17}$$

where $\boldsymbol{\beta}^P$ is an n_P-vector of generalized reliability indices for the individual parallel systems calculated as in (7.8) and $\boldsymbol{\rho}^P$ is a matrix of the corresponding approximate correlation coefficients between the parallel systems.

For approximation of the coefficients in the correlation matrix $\boldsymbol{\rho}^P$ each of the parallel systems is approximated by a failure element with a linear safety margin, see [7.6]:

$$M_{P_i} = \beta_i^P - \left(\boldsymbol{\alpha}_i^P\right)^T \mathbf{U} \quad , \quad i = 1, 2, \ldots, n \tag{7.18}$$

where the vectors $\boldsymbol{\alpha}_i^P$, $i = 1, 2, \ldots, n$ are determined such that the sensitivity of β^P with respect to changes in the joint β-point: $\nabla_{u^*}\beta^P$ are equivalent when obtained from (7.18) (formulated as $\left(\beta_i^P\right)^T \mathbf{u}^*$) and when obtained from (7.8). Furthermore, a normalization is performed for calculation of correlations:

$$\boldsymbol{\alpha}_i^P = \frac{\mathbf{a}_i^P}{\left|\mathbf{a}_i^P\right|} \quad , \quad i = 1, 2, \ldots, n \tag{7.19}$$

where, the elements of $\mathbf{a}_i^P$ are obtained as:

$$a_{ij}^P = \frac{1}{\varphi(-\beta_i^P)} \sum_{k=1}^{n_{A_i}} \left(\alpha_{kj}^i + \left(\frac{d\boldsymbol{\alpha}_k^i}{du_j^*} \right)^T \mathbf{u}^{i^*} \right) \frac{\partial \Phi_{n_{A_i}}(-\boldsymbol{\beta}^{J^i}, \boldsymbol{\rho}^i)}{\partial \beta_k^{J^i}} \tag{7.20}$$

In (7.20) the influence on β^P in (7.18) of the correlations $\boldsymbol{\rho}^i$ are neglected. n_{A_i} is the number of active constraints in the *i*th parallel system. $d\boldsymbol{\alpha}_k / du_j^*$ is obtained from differentiation of (7.4):

$$\frac{d\boldsymbol{\alpha}_k}{du_j^*} = \left(\frac{-\mathbf{I}}{\left|\nabla_u g_{ik}\right|} + \frac{\nabla_u g_{ik} \nabla_u g_{ik}^T}{\left|\nabla_u g_{ik}\right|^3} \right) \frac{\partial \nabla_u g_{ik}}{\partial u_j^*} \tag{7.21}$$

The elements in the matrix of correlation coefficients between the parallel systems are then calculated from:

$$\rho_{mn}^P = \left(\boldsymbol{\alpha}_m^P\right)^T \boldsymbol{\alpha}_n^P \tag{7.22}$$

Now β^S can be estimated from (7.17). For further explanations and details of reliability estimation of series systems of parallel systems, see [7.6].

Comments on General Systems Reliability Models
The reliability modelling of a general system as a series system of parallel systems is healthy seen from a reliability theoretical point of view but from a structural engineering point of view in many cases unrealistic. This is due to the fact that the parallel systems reliabilities are dependent on the history of the load effects in the individual elements or in other words on 1) the residual load carrying capacity of a failed element or elements and 2) how the overall load effects in the entire structure are redistributed at each step in a sequence of element failures. This leads to the conclusion that failure of more than one structural element of major importance often cannot be treated in a realistic manner. More generally it can be said that the systems reliability model is totally dependent of the structural response model and thus it should not be refined more than the structural response model justifies.

7.7 Sensitivity Analysis of General Systems

The sensitivities for evaluation of the obtained systems reliability indices in (7.17) or (7.9) can in principle be obtained as explained in section 7.5. The sensitivity evaluation of a generalized reliability index of series system of parallel systems or of a parallel system, however, requires much more numerical effort and several perturbation analyses of optimality conditions of the included optimization problems, see [7.6].

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Note 8: Structural reliability: Level 1 approaches

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1 Introduction

During the last two decades calibration of partial safety factors in level 1 codes for structural systems and civil engineering structures has been performed on a probabilistic basis in a number of codes of practice, see e.g. OHBDC (Ontario Highway Bridge Design Code) [1], NBCC (National Building Code of Canada) [2], Ravindra & Galambos [3], Ellingwood et al. [4] and Rosenblueth & Esteva [5].

The calibration is generally performed for a given class of structures, materials and/or loads in such a way that the reliability measured by the first order reliability index β estimated on the basis of structures designed using the new calibrated partial safety factors are as close as possible to the reliability indices estimated using existing design methods. Procedures to perform this type of calibration of partial safety factors are described in for example Ravindra & Lind [6], Thoft-Christensen & Baker [7].

A code calibration procedure usually includes the following basic steps, see e.g. Nowak [8]:

- definition of scope of the code,
- definition of the code objective,
- selection of code format,
- selection of target reliability index levels, calculation of calibrated partial safety factors and
- verification of the system of partial safety factors.

A first guess of the partial safety factors is obtained by solving an optimization problem where the objective is to minimize the difference between the reliability for the different structures in the class considered and a target reliability level. In order to ensure that all the structures in the class considered have a satisfactory reliability, constraints are imposed on the reliability for the whole range of structures. In this note it is shown how this optimization problem can be formulated and solved. Next, the partial safety factors determined in this way are adjusted taking into account current engineering judgement and tradition.

In section 2 the partial safety factor method is briefly described. In section 3 it is shown how partial safety factors can be determined for a single failure mode using the results from a first order reliability method (FORM). In section 4 a general procedure for estimating partial safety factors is described. This procedure can be used to calibrate partial safety factors for a class of structures. In section 5 the 'design value method' is presented and illustrated by an example. Finally section 6, describes the calibration of partial safety factors in the Danish structural codes from 1999, [24].

2 Design values for loads and strengths

In the partial safety factor method single structural elements are usually considered and it has to be verified that the design resistance R_d is larger than the design load effect S_d for the structural element considered:

$$R_d > S_d \tag{1}$$

This requirement has to be verified for a number of different load combinations, see below in equation (5) and in section 6. The design value of the load effect is determined on the basis of permanent actions, variable actions and accidental loads.

Design value for permanent actions is determined by:

$$G_d = \gamma_G G_c \tag{2}$$

where

γ_G partial safety factor

G_c characteristic value for permanent actions, typically the 50 % quantile

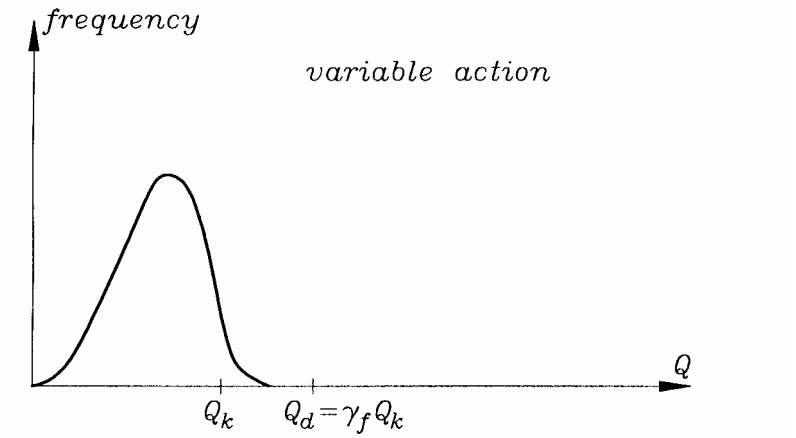

Figure 1. Characteristic and design values for variable action.

Design values for variable actions are determined by, see figure 1:

$$Q_d = \gamma_Q Q_c \tag{3}$$

where

γ_Q partial safety factor

Q_c characteristic value for variable actions, typically the 98 % quantile

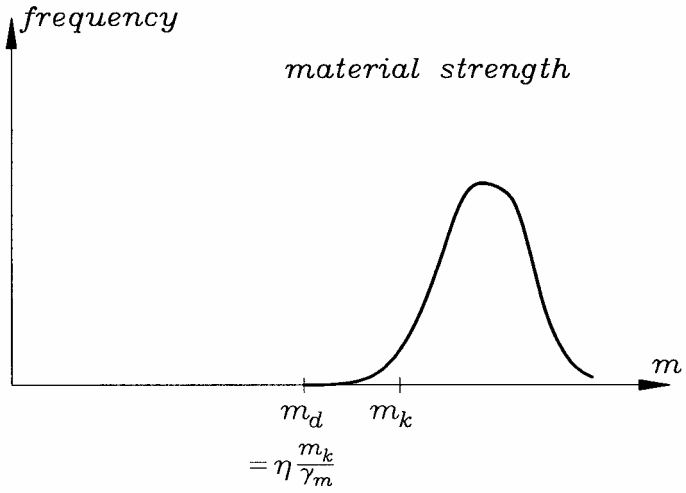

Figure 2. Characteristic and design values for strength parameter.

Design values for accidental loads are determined by

$$A_d = \gamma_A A_c \tag{4}$$

where

γ_A partial safety factor

A_c prescribed value for accidental load

The design value of the load effect S_d is obtained considering the following load combination

$$S_d = S\left(\gamma_G G_c, \gamma_Q Q_{c1}, \psi_2 Q_{c2}, ..., \psi_m Q_{cn}, \gamma_A A_c\right) \tag{5}$$

where n is the number of variable actions, $\gamma_Q Q_{c1}$ is the design value for the dominating variable action, ψ_j is the load combination factor for non-dominating variable action no j and Q_{cj} is the characteristic value for variable action no j. Table 24 in section 6.5 shows the load combinations to be verified in the Danish structural codes, see DS409 [24].

Design values for strength parameters are determined from:

$$m_d = \eta \frac{m_c}{\gamma_m} \tag{6}$$

where

γ_m partial safety factor

m_c characteristic value, typically the 5 % quantile

η conversion factor taking into account differences between the conditions in the structure and the conditions at the determination of the characteristic value. This includes for example load duration, scale, moisture and temperature effects. Normally η=1 but for example design values of timber strengths are determined using values of η between 0.6 and 1.1.

Design value for resistances R_d are usually determined using design values for material and geometrical parameters.

3 Estimation of partial safety factors for one failure mode

In code calibration based on first order reliability methods (FORM) it is assumed that the limit state function can be written

$$g(\mathbf{x},\mathbf{p},\mathbf{z}) = 0 \tag{7}$$

where $\mathbf{x} = (x_1,...,x_n)$ is a realization of $\mathbf{X} = (X_1,...,X_n)$ modeling n stochastic variables describing the uncertain quantities. External loads (e.g. wave), strength parameters and model uncertainty variables are examples of uncertain quantities. $\mathbf{p} = (p_1,...,p_M)$ are M deterministic parameters, for example well defined geometrical quantities. $\mathbf{z} = (z_1,...,z_N)$ are N design variables which are used to design the actual structure. Realizations $\mathbf{x}$ of $\mathbf{X}$ where $g(\mathbf{x},\mathbf{p},\mathbf{z}) \le 0$ corresponds to failure states, while $g(\mathbf{x},\mathbf{p},\mathbf{z}) > 0$ corresponds to safe states.

Using FORM (First Order Reliability Methods) the reliability index β is determined. The corresponding estimate of the probability of failure is

$$P_f = \Phi(-\beta) \tag{8}$$

where Φ is the standard normal distribution function.

If the partial safety factors and if the number of design variables is $N = 1$ then the design (modeled by z) can be determined from the **design equation**

$$G(\mathbf{x}_c,\mathbf{p},\mathbf{z},\gamma) \ge 0 \tag{9}$$

$\mathbf{x}_c = (x_{c1},...,x_{cn})$ are characteristic values corresponding to the stochastic variables $\mathbf{X}$. $\gamma = (\gamma_1,...,\gamma_m)$ are m partial safety factors. The partial safety factors γ are usually defined such that $\gamma_i \ge 1,\ i = 1,...,m$. In the most simple case, $m = n$.

The design equation is closely connected to the limit state function (7). In most cases the only difference is that the state variables $\mathbf{x}$ are exchanged by design values $\mathbf{x}^d$ obtained from the characteristic values $\mathbf{x}_c$ and the partial safety factors γ.

The characteristic values are for **load variables** usually the 90 %, 95 % or 98 % quantiles of the distribution function of the stochastic variables, e.g.

$$x_{ci} = F_{X_i}^{-1}(0.98)$$

where $F_{X_i}(x_i)$ is the distribution function for X_i. The design values for load variables are then obtained from

$$x_{di} = \gamma_i\, x_{ci} \tag{10}$$

The characteristic values are for **strength variables** usually the 10 %, 5 % or 2 % quantiles of the distribution function of the stochastic variables. The design values for strength variables are then obtained from

$$x_{di} = \frac{x_{ci}}{\gamma_i} \tag{11}$$

For **geometrical variables** usually the median (50 % quantile) is used and the design values are

$$x_{di} = \gamma_i \, x_{ci} \tag{12}$$

If $m = n = 2$, x_1 is a load variable and x_2 is a strength variable, the design equation can be written:

$$G\left((x_{c1}, x_{c2}), \mathbf{p}, \mathbf{z}, (\gamma_1, \gamma_2)\right) = G\left((x_{d1}, x_{d2}), \mathbf{p}, \mathbf{z}\right) = G\left(\left(\gamma_1 x_{c1}, \frac{x_{c2}}{\gamma_2}\right), \mathbf{p}, \mathbf{z}\right) \tag{13}$$

A reliability analysis by FORM with the limit state function (7) gives the reliability index β and the β-point $\mathbf{x}^*$. Partial safety factors can then be obtained from

$$\gamma_i = \frac{x_{ci}}{x_i^*} \qquad \text{for strength variables}$$

$$\gamma_i = \frac{x_i^*}{x_{ci}} \qquad \text{for load variables}$$

If more than one variable load type are important then e.g. the Turkstra rule can be used to model the combined effect, see e.g. Thoft-Christensen & Baker [7]. Let $X_1, ..., X_v$ model v different variable load variables. The variables modeling permanent loads are denoted $X_{v+1}, ..., X_{v+p}$ and the remaining stochastic variables are denoted $X_{v+p+1}, ..., X_n$. The design equation is written

$$G\left(\mathbf{x}_c, \mathbf{p}, \mathbf{z}, \gamma\right) = G\left(\left(\gamma_1 \psi_1 x_{c1}, ..., \gamma_v \psi_v x_{cv}, \gamma_{v+1} x_{c,v+1}, ..., \gamma_{v+p} x_{c,v+p}, \frac{x_{c,v+p+1}}{\gamma_{v+p+1}}, ..., \frac{x_{cn}}{\gamma_n}\right), \mathbf{p}, \mathbf{z}\right) = 0 \tag{14}$$

where $\psi_i \le 1$ is a load combination factor for the variable load X_i. Usually v different load combinations are investigated where in combination j, $\psi_j = 1$ and $\psi_i < 1$ for $j \ne i$.

Example Fundamental case

The limit state function corresponding to the fundamental case is written:

$$g = x_2 - x_1$$

where x_1 is a load variable and x_2 is a strength variable. The design equation becomes:

$$G = x_{d2} - x_{d1} = \frac{x_{c2}}{\gamma_2} - \gamma_1 x_{c1}$$

* * * * * * * *

4 General procedure for estimating partial safety factors

Code calibration can be performed by judgement, fitting, optimization or a combination of these, see Madsen et al. [11]. Calibration by **judgement** has been the main method until 10-20 years ago. **Fitting** of partial safety factors in codes is used when a new code format is introduced and the parameters in this code are determined such that the same level of safety is obtained as in the old code. The level of safety can be measured by the reliability index β. In **code optimization** the following steps are generally performed, see [11] and [8]:

1. *Definition of the scope of the code*, i.e. the class of structures to be considered is defined.

2. *Definition of the code objective.* The code objective may be defined at any higher level than the level of the reliability method used in the code. In a level 1 reliability method (which uses a single characteristic value of each uncertain quantity and partial safety factors) the objective may be to obtain on average the same reliability (measured by the target reliability index β as obtained by a reliability method on a higher level.

3. *Definition of code format*. The code format includes:
   - how many partial safety factors to be used
   - where to use the partial safety factors in the design equations
   - rules for load combinations

4. *Determination of the frequency at which each type of safety check is performed.*

5. *Definition of a measure of closeness* between code realizations and the code objective.

6. *Determination of the "best" code format*, i.e. calculation of the 'optimal' partial safety factors which gives the closest fit to the objective measured by the closeness criteria.

7. *Verification* of the system of partial safety factors.

Structural ***failure modes*** (limit states) are generally divided in:

Serviceability limit states
Serviceability limit states are related to normal use of the structure, e.g. excessive deflections, local damage and excessive vibrations.

Ultimate limit states
Ultimate limit states correspond to the maximum load carrying capacity which can be related to e.g. formation of a mechanism in the structure, excessive plasticity, rupture due to fatigue and instability.

Conditional limit states
Conditional limit states correspond to the load-carrying capacity if a local part of the structure has failed. A local failure can be caused by an accidental action or by fire. The conditional limit states can be related to e.g. formation of a mechanism in the structure, exceedance of the material strength or instability.

In general, the target reliability index can be determined by calibration to the reliability level of existing similar structures. Alternatively or supplementary the target reliability indices can be selected on the basis of e.g. the recommended minimum reliability indices specified in ISO [19] or

NKB [10]. The maximum probability of failure (or equivalently the minimum reliability) are assumed to be related to the consequences of failure specified by safety classes and failure types.

The following ***safety classes*** are considered, see NKB [10] and DS409 [24]:

Less serious: 1- and 2-storey buildings, which only occasionally hold persons, for instance stock buildings, sheds, and some agricultural buildings, small pylons, roofs and internal walls.

Serious: Buildings of more than two stories and hall structures which only occasionally hold people, small 1- and 2-storey buildings often used for people, for example houses, offices or productions buildings, tall pylons, scaffolds and moulds, external walls, staircases and rails.

Very serious: Buildings of more than two stories, hall structures, and stages which will often hold many persons and e.g. be used for offices, sports or production. 1- and 2-storey buildings with large spans often used by many persons, stands, pedestrian bridges, road bridges, railroad bridges.

The following ***failure types*** are considered (see NKB [10] and DS409 [24]):

Failure type I: Ductile failures where it is required that there is an extra carrying capacity beyond the defined resistance, i.e. in the form of strain hardening.

Failure type II: Ductile failures without an extra carrying capacity.

Failure type III: Failures such as brittle failure and instability failure.

For ultimate limit states NKB [10] recommends the maximum probabilities of failure shown in table 1 based on a reference period of 1 year. The corresponding minimum reliability indices are shown in table 2.

Safety class	Failure type I	Failure type II	Failure type III
Less serious	10^{-3}	10^{-4}	10^{-5}
Serious	10^{-4}	10^{-5}	10^{-6}
Very serious	10^{-5}	10^{-6}	10^{-7}

Table 1. Maximum annual probabilities of failure.

Safety class	Failure type I	Failure type II	Failure type III
Less serious	3.1	3.7	4.3
Serious	3.7	4.3	4.7
Very serious	4.3	4.7	5.2

Table 2. Target (minimum) annual reliability indices β.

As explained above calibration of partial safety factors is generally performed for a given class of structures, materials or loads in such a way that the reliability measured by the first order reliability index β estimated on the basis of structures designed using the new calibrated partial safety factors is as close as possible to the target reliability index or to the reliability indices estimated using existing design methods, see Thoft-Christensen & Baker [7], Ditlevsen & Madsen [12], Östlund [13], Shinozuka et al. [14], Vrouwenvelder [15] and Hauge et al. [16]. Procedures to perform this type of calibration of partial safety factors are described in e.g. Thoft-Christensen & Baker [7].

In the following this procedure is described and extended in some directions. For each failure mode the **limit state function** is written, see (7)

$$g(\mathbf{x},\mathbf{p},\mathbf{z}) = 0 \tag{15}$$

Using FORM (First Order Reliability Methods) the reliability index β can be determined.

If the number of design variables is $N = 1$ then the design can be determined from the **design equation**, see (9)

$$G(\mathbf{x}_c,\mathbf{p},\mathbf{z},\gamma) \geq 0 \tag{16}$$

If the number of design variables is $N > 1$ then a design optimization problem can be formulated:

$$\begin{aligned} \min \quad & C(z) \\ s.t. \quad & c_i(z) = 0 \quad ,i = 1,...,m_e \\ & c_i(z) \geq 0 \quad ,i = m_e + 1,...,m \\ & \mathrm{z_i^l} \leq \mathrm{z_i} \leq \mathrm{z_i^u} \quad ,i = 1,...,N \end{aligned} \tag{17}$$

C is the objective function and $c_i \quad ,i = 1,...,m$ are the constraints. The objective function C is often chosen as the weight of the structure. The m_e equality constraints in (17) can be used to model design requirements (e.g. constraints on the geometrical quantities) and to relate the load on the structure to the response (e.g. finite element equations). Often equality constraints can be avoided because the structural analysis is incorporated directly in the formulation of the inequality constraints. The inequality constraints in (17) ensure that response characteristics such as displacements and stresses do not exceed codified critical values as expressed by the design equation (16). The inequality constraints may also include general design requirements for the design variables. The lower and upper bounds, z_i^l and z_i^u, to z_i in (17) are so-called simple bounds. Generally, the optimization problem (17) is non-linear and non-convex.

The application area for the code is described by the set I of L different vectors $\mathbf{p}_i \quad ,i = 1,...,L$. The set I may e.g. contain different geometrical forms of the structure, different parameters for the stochastic variables and different statistical models for the stochastic variables.

The partial safety factors γ are calibrated such that the reliability indices corresponding to the L vectors $\mathbf{p}$ are as close as possible to a target probability of failure P_f^t or equivalently a target reliability index $\beta_t = -\Phi^{-1}\left(P_f^t\right)$. This is formulated by the following optimization problem

$$\min_{\gamma} \; W(\gamma) = \sum_{j=1}^{L} w_j \left(\beta_j(\gamma) - \beta_t\right)^2 \tag{18}$$

where $w_j \quad ,j = 1,...,L$ are weighting factors ($\sum_{j=1}^{L} w_j = 1$) indicating the relative frequency of appearance / importance of the different design situations. Instead of using the reliability indices in (18) to measure the deviation from the target, for example the probabilities of failure can be used. Also, a nonlinear objective function giving relatively more weight to reliability indices smaller than the target compared to those larger than the target can be used. $\beta_j(\gamma)$ is the reliability index for

combination j obtained as described below. In (18) the deviation from the target reliability index is measured by the squared distance.

The reliability index $\beta_j(\gamma)$ for combination j is obtained as follows. First, for given partial safety factors γ the optimal design is determined by solving the design equation (16) if $N=1$ or by solving the design optimization problem (17) if $N>1$. Next, the reliability index $\beta_j(\gamma)$ is estimated by FORM on the basis of (15) using the design z from (16) or (17).

It should be noted that, following the procedure described above for estimating the partial safety factors two (or more) partial safety factors are not always uniquely determined. They can be functionally dependent, in the simplest case as a product, which has to be equal to a constant.

In the above procedure there is no lower limit on the reliability. An improved procedure which has a constraint on the reliability and which takes the non-uniqueness problem into account can be formulated by the optimization problem

$$\begin{aligned} \min_{\gamma} \quad & W(\gamma) = \sum_{j=1}^{L} w_j \left[\left(\beta_j(\gamma) - \beta_t\right)^2 + \delta \sum_{i=1}^{m} \left(\gamma_i - \gamma_{ji}^*\right)^2 \right] \\ s.t. \quad & \beta_j(\gamma) \geq \beta_t^{\min} \quad , j = 1,...,L \\ & \gamma_i^l \leq \gamma_i \leq \gamma_i^u \quad , i = 1,...,m \end{aligned} \tag{19}$$

where $w_j \quad , j = 1,...,L$ are weighting factors ($\sum_{j=1}^{L} w_j = 1$). δ is a factor specifying the relative importance of the two terms in the objective function. $\beta_j(\gamma)$ is the reliability index for combination j obtained as described above. γ_{ji}^* is an estimate of the partial safety factor obtained by considering combination j in isolation. The second term in the objective function (19) is added due to the non-uniqueness-problem and has the effect that the partial safety factors are forced in the direction of the "simple" definition of partial safety factors. For load variables: $\gamma = \frac{x^*}{x_c}$. If only one combination is considered then $\gamma_{ji}^* = \frac{x_{ji}^*}{x_{c,ji}}$ where x_{ji}^* is the design point. Experience with this formulation has shown that the factor δ should be chosen to be of magnitude one and that the calibrated partial safety factors are not very sensitive to the exact value of δ. The constraints have the effect that no combination has a reliability index smaller than $\beta_t^{\min}$.

This type of code calibration has been used in Burcharth [17] for code calibration of rubble mound breakwater designs. These structures are known to have reliabilities which vary considerably. The reason is that the structures are used under widely different conditions.

As discussed above a first guess of the partial safety factors is obtained by solving these optimization problems. Next, the final partial safety factors are determined taking into account current engineering judgement and tradition.

Example 1

In this example partial safety factors are determined for one failure mode in one application $(L=1)$. Consider the limit state function:

$$g = zR - G - Q$$

where z is a design variable, R a resistance, G a permanent load and Q a variable load. The stochastic model in table 3 is used.

	Distribution	Expected value	Coefficient of variation
R	Lognormal	1 kN/m^2	0.15
G	Normal	2 kN	0.1
Q	Gumbel	3 kN	V

Table 3. Statistical parameters.

If the target reliability index is chosen to β_t=3.8 and V=0.4 then z=15.6 m^2. The corresponding β-point in basic variable space is $(r^*, g^*, q^*) = (0.76,\ 2.04,\ 9.83)$.

Characteristic values are chosen to:

R 5 % quantile: r_c=0.77

G 50 % quantile: g_c=1.0

Q 98 % quantile: $q_c = \mu_Q\left(1 - V\frac{\sqrt{6}}{\pi}\left[0.5772 + \ln\{-\ln(0.98)\}\right]\right) = 2.04\mu_Q$=6.12

Partial safety factors are then

$$\gamma_R = \frac{r_c}{r^*} = \frac{0.77}{0.76} = 1.01$$

$$\gamma_G = \frac{g^*}{g_c} = \frac{1.02}{1.0} = 1.02$$

$$\gamma_Q = \frac{q^*}{q_c} = \frac{9.83}{6.12} = 1.61$$

In table 4 results are shown for other coefficients of variation, V and target reliability indices. It is seen that the partial safety factors for the variable load become rather large compared with the two other partial safety factors, especially for large values of V.

β_t	V	z	q_c / μ_Q	r^*	g^*	q^*	γ_R	γ_G	γ_Q
3.8	0.2	12.8	1.52	0.70	2.08	5.82	1.10	1.04	1.28
4.3	0.2	14.5	1.52	0.68	2.08	6.53	1.13	1.04	1.43
4.8	0.2	11.3	1.52	0.65	2.08	7.30	1.18	1.04	1.60
3.8	0.3	13.4	1.78	0.74	2.05	7.86	1.04	1.02	1.47
4.3	0.3	15.5	1.78	0.71	2.05	8.96	1.08	1.02	1.68
4.8	0.3	17.9	1.78	0.68	2.05	10.2	1.12	1.02	1.91
3.8	0.4	15.6	2.04	0.76	2.04	9.83	1.01	1.02	1.61
4.3	0.4	18.3	2.04	0.73	2.04	11.3	1.05	1.02	1.84
4.8	0.4	21.4	2.04	0.70	2.04	13.0	1.10	1.02	2.12

Table 4. Partial safety factors obtained by direct reliability-based calibration.

* * * * * * * *

5 Design value format in Eurocodes

In the Eurocodes [18] and ISO [19] the so-called design value format is proposed to estimate partial safety factors. According to that format the design value x_d of an uncertain variable X is estimated from

$$F_X(x_d) = \Phi(-\alpha\beta^t) \tag{20}$$

where F_X is the distribution function for X and β^t is the target reliability index, e.g. β^t=3.8. α is the α-coefficient associated with the type and importance of the stochastic variable considered. The following values are recommended:

For strength variables: α = 0.8
For dominating loads: α = -0.7
For non-dominating loads: α = -0.4 x 0.7 = -0.28

When the design value have been estimated the partial safety factor is determined from:

$\gamma = \theta \dfrac{x_c}{x_d}$ for strength variables

$\gamma = \theta \dfrac{x_d}{x_c}$ for load variables

where θ is an uncertainty factor, typically = 1.05. x_c is the characteristic value.

The following distribution types are recommended:

For permanent loads: Normal distribution: $x_d = \mu_X(1 + 0.7\beta^t V)$

For variable loads: Gumbel distribution: $x_d = \mu_X\left(1 - V\dfrac{\sqrt{6}}{\pi}\left[0.5772 + \ln\left\{-\ln\left(\Phi\left(0.7\beta^t\right)\right)\right\}\right]\right)$

For strength: Lognormal distribution: $x_d = \mu_X \exp\left(-0.8\beta^t V\right)$

Example 2

Example 1 is considered again but now the partial safety factors are determined using the design value method. The result is shown in table 5. When compared to the results in table 4 it is seen that the partial safety factors for resistance and permanent loads are larger than those obtained in example 1 whereas the partial safety factors for the variable load are smaller. If the design value and corresponding reliability index are determined using the partial safety factors in table 5, it is seen that for V =0.4 the reliability indices are almost the same as the target reliability indices. However, for smaller values of V the reliability indices become larger than the target reliability indices when using the design value method.

β_t	V	q_c / μ_Q	r_d	g_d	q_d	γ_R	γ_G	γ_Q	z	$\beta(\gamma_R, \gamma_G, \gamma_Q)$
3.8	0.2	1.52	0.63	2.53	5.31	1.22	1.27	1.16	12.4	4.19
4.3	0.2	1.52	0.60	2.60	5.85	1.28	1.30	1.28	14.0	4.67
4.8	0.2	1.52	0.56	2.67	6.39	1.38	1.34	1.40	16.2	5.25
3.8	0.3	1.78	0.63	2.53	6.48	1.22	1.27	1.21	14.3	4.03
4.3	0.3	1.78	0.60	2.60	7.26	1.28	1.30	1.36	16.4	4.50
4.8	0.3	1.78	0.56	2.67	8.10	1.38	1.34	1.52	19.4	5.08
3.8	0.4	2.04	0.63	2.53	7.65	1.22	1.27	1.25	16.1	3.90
4.3	0.4	2.04	0.60	2.60	8.67	1.28	1.30	1.42	18.8	4.39
4.8	0.4	2.04	0.56	2.67	9.81	1.38	1.34	1.60	22.4	4.94

Table 5. Partial safety factors obtained using the design value method. z is the design value obtained using the values of γ_R, γ_G and γ_Q in columns 7-9. $\beta(\gamma_R, \gamma_G, \gamma_Q)$ is the corresponding reliability index.

* * * * * * * *

6 Calibration of Partial Safety Factors for Danish Structural Codes

This section describes the main steps in the probabilistic code calibration performed for the Danish Structural Codes (1999). It is based on [20]. First, the reliability level is evaluated in a number of typical, simple structures designed according to the current (old) Danish structural codes (1982) and with a reasonable stochastic model for the uncertain quantities. The reliability analyses show a non-uniform reliability level for different materials and actions. Next, new partial safety factors in a slightly modified code format are calibrated such that the safety level is the same in the new code as in the current codes, i.e. it is assumed that the reliability level in the old structural codes is satisfactory. Using the optimized partial safety factors a more uniform reliability level is obtained for different types of materials / structures and for different types of loads. The calibrations are performed with the assumption that characteristic values for actions and strengths are the same in the old and the new codes, except for changes in some of the quantile percentages used.

6.1 Characteristic values for loads and strengths

Characteristic values are determined as follows:

- permanent actions: 50 % quantiles
- variable actions: 98 % quantiles
- strength parameters: 5 % quantiles

6.2 Partial safety factors for loads and strengths

Three main load combinations have to be checked by design. Load combination 2 consists of four combinations and load combination 3 consists of three combinations:

Load combination 1: Serviceability limit states
Load combination 2: Ultimate limit states
- 2.1: Permanent and variable actions unfavorable – variable actions dominating
- 2.2: Permanent actions favorable
- 2.3: Permanent and variable actions unfavorable – permanent actions dominating
- 2.4: Fatigue

Load combination 3: Accidental actions
- 3.1: Impact, explosion and vertical actions on air raid shelters
- 3.2: Removal of a structural element
- 3.3: Fire

The partial safety factors for strength parameters γ_m are determined by

$$\gamma_m = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 \tag{21}$$

where

γ_0 takes into account the consequences of failure see table 6

γ_1 takes into account the type of failure, see table 7

γ_2 takes into account the possibility of unfavorable differences from the characteristic value of the material parameter, see table 8. The values in the table are indicated by ? since they are determined on the basis of a calibration, see the following subsections

γ_3 takes into account the uncertainty in the computational model, see table 10

γ_4 takes into account the uncertainty in connection with determination of the material parameter in the structure on the basis of the controlled material parameter, see table 11

γ_5 takes into account the amount of control at the working place (in excess of the statistical quality control), see table 12

Safety class	Low	Normal	High
γ_0	0.90	1.00	1.10

Table 6. γ_0 - factor for different safety classes.

Failure type	Ductile		Brittle
	With reserve	Without reserve	
γ_1	0.90	1.00	1.10

Table 7. γ_1 - factor for different failure types.

δ	<0.05	0.10	0.15	0.20	0.25	0.30
γ_2	?	?	?	?	?	?

Table 8. γ_2 as function of the coefficient of variation δ (for 5 % quantiles).

For other quantile values than the 5 % quantile, the γ_2 values should be multiplied with $\exp((1{,}65 - k_\gamma)\delta)$

Quantile, %	20	10	5	2.5	1	0.1
k_γ	0.84	1.28	1.65	1.96	2.33	3.00

Table 9. Factor k_γ

Accuracy of computational model	Good	Normal	Bad
γ_3	0.95	1.00	1.10

Table 10. γ_3 - factor taking into account the accuracy of the computational model. Normal accuracy corresponds to usual calculations of normal structures and structural elements.

Certainty in determination of material parameter	Large	Average	Small
γ_4	0.95	1.00	1.10

Table 11. γ_4 - factor taking into account the uncertainty in connection with determination of the material parameter in the structure on the basis of the controlled material parameter.

Control class	Extended	Normal	Reduced
γ_5	0.95	1.00	1.10

Table 12. γ_5 - factor for different levels of control for material identity and construction.

6.3 Stochastic models

The stochastic model used for calibration of partial safety factors is shown in table 13 and is partly based on the following references: SAKO [21], DGI [22] and Foschi et al. [23].

Variable	Coefficient of variation	Distribution type	Quantiles in old code	Quantile in New code
Permanent loads:				
Permanent action	10 %	N	50 %	50 %
self-weight: concrete	6 %	N	50 %	50 %
self-weight: steel	4 %	N	50 %	50 %
self-weight: timber	6 %	N	50 %	50 %
Variable loads:				
Imposed load	20 %	G	98 %	98 %
Environmental load	40 %	G	98 %	98 %
Strengths:				
Concrete compression strength	15 %	LN	10 %	5 %
Reinforcement	5 %	LN	0.1 %	5 %
Steel	5 %	LN	5 %	5 %
glued laminated timber	15 %	LN	5 %	5 %
eff. Friction angle – sand	3.3 %	LN	5 %	5 %
Undrained shear strength – clay	16 %	LN	5 %	5 %
Model uncertainty: concrete	5 %	LN	50 %	50 %
Model uncertainty: steel	3 %	LN	50 %	50 %
Model uncertainty: timber	5 %	LN	50 %	50 %
Model uncertainty: foundation	15 %	LN	50 %	50 %

Table 13. Stochastic model. Distributions types: N: normal, LN: lognormal, G: Gumbel.

For timber the Lognormal distribution is used. Also the Weibull distribution has been considered, but since the partial safety factors in the old codes implicitely were based on Lognormal distributions for the strengths this distribution is also used in the following calibrations. The Lognormal distribution is considered to be reasonable for laminated timber. For single lumber members theoretical considerations and statistical analysis of available data indicate that a Weibull distribution should be considered. A Weibull distribution usually results in significantly smaller reliability indices than the Lognormal distribution. Note, that in the new Danish codes all quantile values for material strengths are defined as 5 % values.

For the variable action, two action types are used, namely imposed action and environmental action (e.g. wind and snow).

6.4 Probabilistic calibration of partial safety factors for load combination 2.1 and 2.3

6.4.1 Description of example structures

Three types of structures are considered for calibration of the partial safety factors, namely a simply supported beam, a short column and typical geotechnical structures. The structures are:

- Simply supported reinforced concrete beam
- Simply supported steel beam
- Simply supported glued laminated beam
- Short concrete column
- Short steel column
- Short glued laminated column
- Central loaded footing (foundation) on sand
- Central loaded footing (foundation) on clay
- Concrete gravity wall

The following six load cases with different ratios between permanent and variable actions are considered for beams and columns of concrete, steel and timber and for the footings on sand and clay:

1) $(G_c, Q_{c,i}, Q_{c,e})$ = (30 kN/m, 0 kN/m, 0 kN/m)
2) $(G_c, Q_{c,i}, Q_{c,e})$ = (24 kN/m, 6 kN/m, 4.5 kN/m)
3) $(G_c, Q_{c,i}, Q_{c,e})$ = (24 kN/m, 18 kN/m, 13.5 kN/m)
4) $(G_c, Q_{c,i}, Q_{c,e})$ = (3.6 kN/m, 6 kN/m, 4.5 kN/m)
5) $(G_c, Q_{c,i}, Q_{c,e})$ = (3.6 kN/m, 18 kN/m, 13.5 kN/m)
6) $(G_c, Q_{c,i}, Q_{c,e})$ = (0 kN/m, 30 kN/m, 22.5 kN/m)

where G_c, $Q_{c,i}$ and $Q_{c,e}$ are characteristic values for permanent actions, variable imposed action and environmental action, respectively. For the gravity wall only load case 2.1 is considered. Therefore, in total 98 different structures are used in the calibration. The limit state functions for the considered failure modes are described in [20].

6.4.2 Reliability analysis with old partial safety factors

Each structure is first designed according to the old structural codes (1982). As described in section 6.2, the material partial safety factors are determined as a product of a number of factors, where γ_2 models the physical uncertainty related to a given type of material strength parameter. In table 14 is shown the partial safety factors γ_2 for materials and γ_f for actions corresponding to the old code.

γ_f/γ_2		Partial safety factor γ_f/γ_2	Quantile used to define characteristic value
γ_{G21}	Permanent action (load combination 2.1)	1.0	50 %
$\gamma_{Q21,i}$	Imposed action (load combination 2.1)	1.3	98 %
$\gamma_{Q21,e}$	Environmental action (load combination 2.1)	1.3	98 %
γ_{G23}	Permanent action (load combination 2.3)	1.15	50 %
$\gamma_{Q23,i}$	Imposed action (load combination 2.3)	-	
$\gamma_{Q23,e}$	Environmental action (load combination 2.3)	-	
γ_a	Reinforcement	1.32	0,1 %
γ_b	Concrete	1.73/1.1 = 1.58	10 %
γ_s	Steel	1.28/0.9 = 1.42	5 %
γ_t	Timber (glued laminated)	$1.35/0.95^2$ =1.49	5 %
γ_φ	Friction angle	1.2	5 %
γ_{c_u}	Undrained shear strength	1.8	5 %

Table 14. Partial safety factors in old code where material partial safety factors are modified for failure type, degree of control etc.

Next, reliability indices are determined for each structure using the partial safety factors in table 14. In table 15 the average and the standard deviation of the reliability indices are shown for each of the 9 groups of structures considered and in figure 3 the distribution of reliability indices is shown. Table 16 shows the annual probability of failure corresponding to some typical reliability indices. It is seen that:

- The reliability for concrete and steel structures is larger than for glued laminated timber structures.
- For the geotechnical problems the reliability levels are slightly lower than for concrete, steel and timber structures.
- The reliability indices for the concrete beam (where the reinforcement strength is important) are significantly larger than for the concrete column (where the concrete strength is important).

From table 15 it is also seen that the average reliability index for all example structures are 4.79. In the next section new partial safety factors are calibrated such that the average reliability index will remain equal to 4.79, i.e the target reliability index is β_t=4.79. In table 17 the average reliability index is shown for the six different load cases described in section 6.4.1. For each load case is shown the parameter α (=0 if all load is permanent and = 1 if all load is variable) defined by:

$$\alpha = \frac{Q_c}{Q_c + G_c} \tag{22}$$

It is seen that

- the largest reliability indices are obtained for α=0, i.e. when all the load is permanent load
- the smallest reliability indices are obtained for α=1, i.e. when all the load is variable load
- relatively high reliability indices are obtained for α=0.2 – 0.5

In summary the reliability analysis using the old partial safety factors shows a rather non-uniform reliability level and therefore some changes in the partial safety factors can be expected if a homogenous reliability level is to be obtained. Especially, it can be expected that the partial safety factors for timber and variable actions are increased and the partial safety factors for concrete and steel are decreased.

	Average value	Standard deviation
Beam – concrete	5.39	0.62
Beam – steel	5.06	0.64
Beam – timber	4.58	0.27
Column – concrete	4.64	0.26
Column – steel	5.10	0.66
Column – timber	4.58	0.27
Foundation on sand	4.61	0.45
Foundation on clay	4.37	0.76
Gravity wall	4.89	
Total	4.79	0.56

Table 15. Reliability indices for example structures using old partial safety factors.

Reliability index β	Probability of failure P_f
3.1	10^{-3}
3.7	10^{-4}
4.3	10^{-5}
4.7	10^{-6}
5.2	10^{-7}

Table 16. Reliability indices β and corresponding probability of failure P_f.

Load case	α	Average reliability index		
		concrete	steel	timber
1	0	5.32	5.82	4.36
2	0.2	5.08	5.02	4.36
3	0.43	5.20	5.27	4.87
4	0.63	5.11	4.79	4.80
5	0.83	4.79	4.52	4.63
6	1	4.60	4.33	4.50

Table 17. Reliability indices for example structures using old partial safety factors for the six load cases described in section 6.4.1.

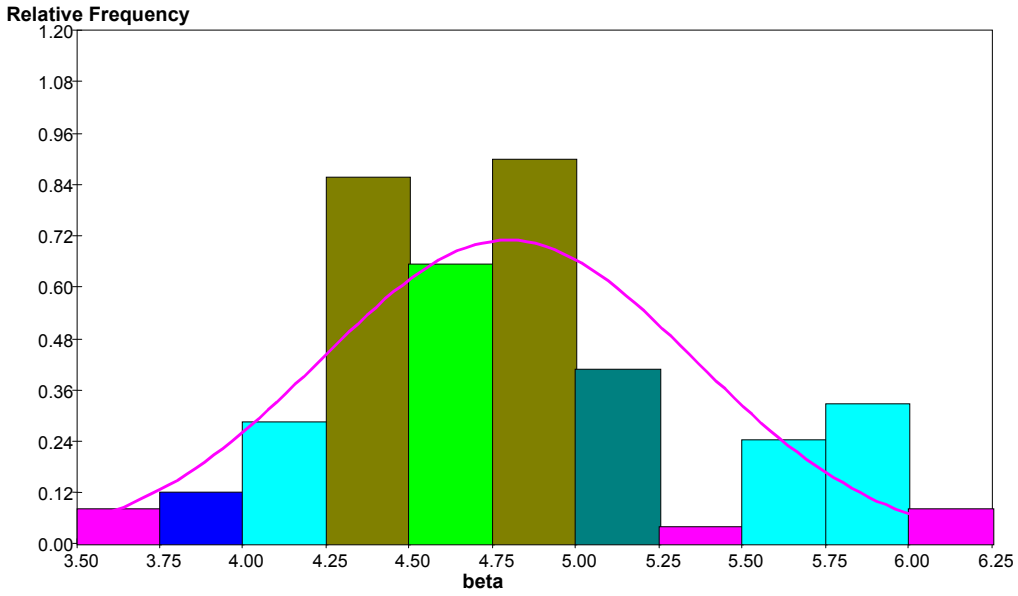

Figure 3. Reliability indices for the 98 example structures.

6.4.3 Calibration of new partial safety factors for load combination 2.1 and 2.3

The code format used is basically the same as in the current codes, except for two changes:

- Separate partial safety factors for imposed variable actions and for environmental variable actions (wind and snow) are introduced.
- In load combination 2.3 (permanent action dominating) also partial safety factors for the variable actions are introduced.

The partial safety factor γ_{G21} for permanent action in load combination 2.1 is chosen to 1.0 (as in the current code). The partial safety factors corresponding to load combination 2.1 (variable action dominating) and 2.3 (permanent action dominating) are:

Load combination 2.1:

$\gamma_{G21}=1, \quad \gamma_Q=\gamma_{Q21,i}$ (imposed actions) or $\gamma_Q=\gamma_{Q21,e}$ (environmental actions)

Load combination 2.3:

$\gamma_G=\gamma_{G23}, \quad \gamma_Q=\gamma_{Q23,i}$ (imposed actions) or $\gamma_Q=\gamma_{Q23,e}$ (environmental actions)

Structures of concrete, steel and timber are checked by both load combination 2.1 and 2.3. Geotechnical problems are checked by load combination 2.1 only.

The total set of partial safety factors to be calibrated is: $\gamma_{Q21,i}$, $\gamma_{Q21,e}$, γ_{G23}, $\gamma_{Q23,i}$, $\gamma_{Q23,e}$, γ_a, γ_c, γ_s, γ_t, γ_φ and γ_{c_u}. The optimal partial safety factors are obtained by minimizing the deviation between the target reliability index β_t and the reliability indices obtained from the 98 example structures. Each reliability index is determined in the following way. First, the considered structure is designed using the deterministic design equation with characteristic values for the variables and the actual guess on the partial safety factors. Next with the obtained design a reliability analysis is performed, now treating the uncertain quantities as stochastic variables.

The mathematical formulation of the optimization problem is:

$$\begin{aligned} \min_{\gamma} \quad & W(\gamma)=\sum_{i=1}^{L}\omega_i\left(\beta_i(\gamma)-\beta_t\right)^2 \\ \text{subject to} \quad & \gamma_{Q21,i}\geq 1,\gamma_{Q21,e}\geq 1,\gamma_{Q23,i}\geq 1,\gamma_{Q23,e}\geq 1 \\ & \gamma_c\geq 1,\gamma_a\geq 1,\gamma_s\geq 1,\gamma_t\geq 1,\gamma_\varphi\geq 1,\gamma_{c_u}\geq 1 \end{aligned} \tag{23}$$

where $\beta_i(\gamma)$ is the reliability index for example structure no i designed with partial safety factors γ. $L=98$ is the number of example structures. $\omega_i=1$ is assumed for all structures.

Results

Using a target reliability index β_t equal to the average of the reliability indices of the example structures and a standard nonlinear optimization program, the results shown in table 18 are obtained. The table shows the optimized partial safety factors directly and modified with the factors from table 14. The modification consists of multiplying the optimized partial safety factors γ_2 with the relevant values of the γ_1, γ_3, γ_4 and γ_5 factors in the old codes, see table 14. Further, also optimized and modified partial safety factors obtained by rounding and fixing the action partial safety factors are shown.

	old code γ_f/γ_m	optimized γ_f/γ_2	optimized γ_f/γ_m	optimized γ_f/γ_2	optimized γ_f/γ_m
γ_{G21} (2.1 permanent)	1	1	1	1	1
$\gamma_{Q21,i}$ (2.1 imposed)	1.3	1.28	1.28	1.3 (fixed)	1.3 (fixed)
$\gamma_{Q21,e}$ (2.1 env.)	1.3	1.52	1.52	1.5 (fixed)	1.5 (fixed)
γ_{G23} (2.3 permanent)	1.15	1.13	1.13	1.15 (fixed)	1.15 (fixed)
$\gamma_{Q23,i}$ (2.3 imposed)	-	1.00	1.00	1.0 (fixed)	1.0 (fixed)
$\gamma_{Q23,e}$ (2.3 env.)	-	1.00	1.00	1.0 (fixed)	1.0 (fixed)
γ_a (reinforcement)	1.4	1.25	1.25	1.23	1.23
γ_c (concrete)	1.8	1.50	1.65	1.49	1.64
γ_s (steel)	1.28	1.30	1.17	1.29	1.16
γ_t (glued lam. timber)	1.35	1.53	1.53	1.51	1.51
γ_φ	1.2	1.21	1.21	1.21	1.21
γ_{c_u}	1.8	1.90	1.90	1.90	1.90

Table 18. Partial safety factors for old and optimized code (β_t=4.79).

Corresponding to the optimal partial safety factors table 19 shows the average reliability indices and the standard deviation of the reliability indices for each of the 9 groups of structures considered. Figure 4 shows the distribution of the reliability indices obtained using the optimized partial safety factors in table 19. Table 20 shows the reliability indices for different values of α (relative ratio of variable load).

It is seen that

- the partial safety factor for imposed actions in load combination 2.1 is almost unchanged while the partial safety factor for environmental actions should be increased from 1.3 to 1.5.
- In load combination 2.3 the partial safety factor for permanent action is unchanged 1.15.
- the partial safety factors for reinforcement, concrete and steel can be decreased.
- the partial safety factor for glued laminated timber should be increased
- the partial safety factors for geotechnical parameters are almost unchanged.

	Average value	Standard deviation
Beam – concrete	4.69	0.34
Beam – steel	4.64	0.39
Beam – timber	4.81	0.22
Column – concrete	4.81	0.20
Column – steel	4.67	0.41
Column – timber	4.81	0.22
Foundation on sand	4.81	0.48
Foundation on clay	4.68	0.50
Gravity wall	5.10	
Total	4.79	0.35

Table 19. Reliability indices using the optimized partial safety factors.

Load case	α	Average reliability index		
		concrete	steel	timber
1	0	4.58	4.70	4.42
2	0.2	5.02	5.36	4.80
3	0.43	4.89	4.83	5.06
4	0.63	4.86	4.56	4.99
5	0.83	4.65	4.33	4.85
6	1	4.51	4.19	4.73

Table 20. Reliability indices for example structures using optimized partial safety factors for the six load cases described in section 4.1.

Generally, the partial safety factors for actions should be increased while the partial safety factors for the material strengths can be decreased. Further it is, as expected, seen that the difference in reliability levels in the example structures are much smaller using the optimized partial safety factors than using the partial safety factors in the old code (1982).

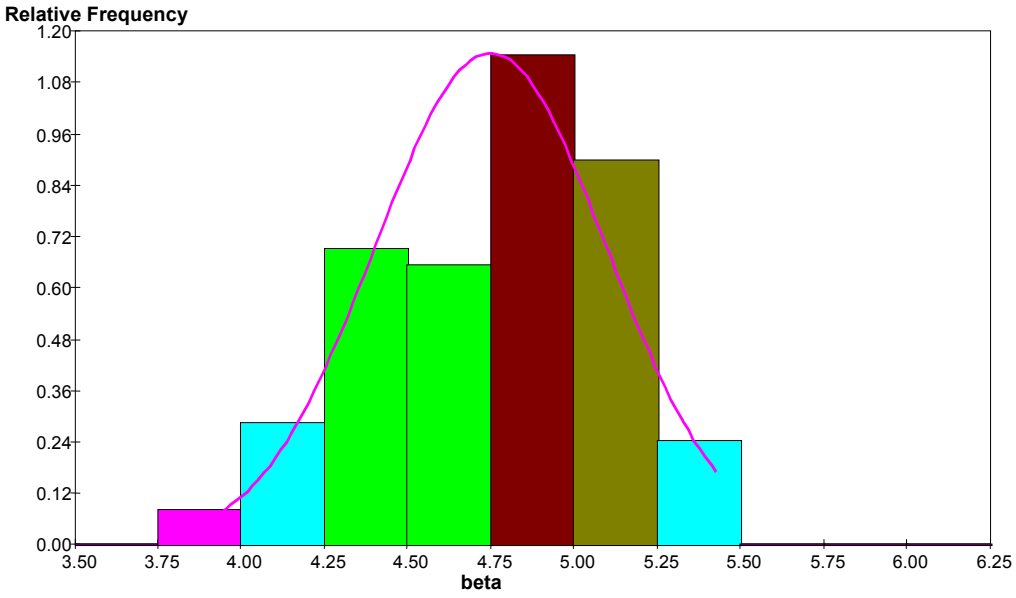

Figure 4. Reliability indices for the 98 example structures using calibrated partial safety factors.

The material partial safety factors in the new structural codes are based on the above calibration results. The values in the following table have been chosen.

Material	Coefficient of variation	Calibrated partial safety factor γ_2	Partial safety factor γ_2 in structural codes
Steel	5 %	1.29	1.30
Reinforcement	5 %	1.23	1.30
Concrete	15 %	1.49	1.50
Timber (glued laminated)	15 %	1.51	1.50

Table 21. Partial safety factors γ_2 in structural codes.

6.4.4 Evaluation of safety level for a simple limit state

In order to evaluate the reliability level in load combination 2.1 and 2.3 the following simple, but representative limit state function is considered:

$$g = zRX_R - \left((1-\alpha)G + \alpha Q\right) \tag{24}$$

where

R	strength (modeled by a stochastic variable)
X_R	model uncertainty (modeled by a stochastic variable)
z	design variable, e.g. a cross-sectional area
G	permanent action (modeled by a stochastic variable)
Q	variable action (modeled by a stochastic variable)
α	factor between 0 and 1, giving the relative importance of the variable action.

As examples only steel and concrete structures are considered. As variable actions both environmental and imposed actions are used. The following stochastic model, based on table 13, is used:

Variable		Distribution type	Coefficient of variation	Quantile
Permanent action		N	0.10	50 %
Variable action	-environmental	G	0.40	98 %
	-imposed	G	0.20	98 %
Strength	-concrete	LN	0.15	5 %
	-steel	LN	0.05	5 %
Model uncertainty	-concrete	N	0.05	50 %
	-steel	N	0.03	50 %

Table 22. Stochastic model. N: Normal, G: Gumbel, LN: Lognormal.

The design variable z is determined by considering load combination 2.1 and 2.3. The design equations can then be written:

$$z_1 R_c / \gamma_m - \left((1-\alpha)\gamma_{G_1} G_c + \alpha\gamma_{Q_1} Q_c\right) = 0 \tag{25}$$

$$z_3 R_c / \gamma_m - \left((1-\alpha)\gamma_{G_3} G_c + \alpha\gamma_{Q_3} Q_c\right) = 0 \tag{26}$$

z is determined as $z = \max(z_1, z_3)$. Index c indicates a characteristic value. The partial safety factors in table 23 are determined with $\gamma_m = \gamma_2$.

	Partial safety factor	
	Load combination 2.1	Load combination 2.3
Permanent action	$\gamma_{G_1} = 1.0$	$\gamma_{G_3} = 1.15$
Variable action –environmental	$\gamma_{Q_1} = 1.5$	$\gamma_{Q_3} = 1.0$
-imposed	$\gamma_{Q_1} = 1.3$	$\gamma_{Q_3} = 1.0$
Strength -concrete	$\gamma_m = \gamma_2 = 1.5$	
-steel	$\gamma_m = \gamma_2 = 1.3$	

Table 23. Partial safety factors.

The reliability index β is determined as function of α. The result is shown in figure 5. It is seen that an almost uniform distribution of the reliability index is obtained as a function of α. Only steel structures have a larger reliability when permanent actions are dominating (small α). However, in practice variable actions will usually be dominating for steel structures.

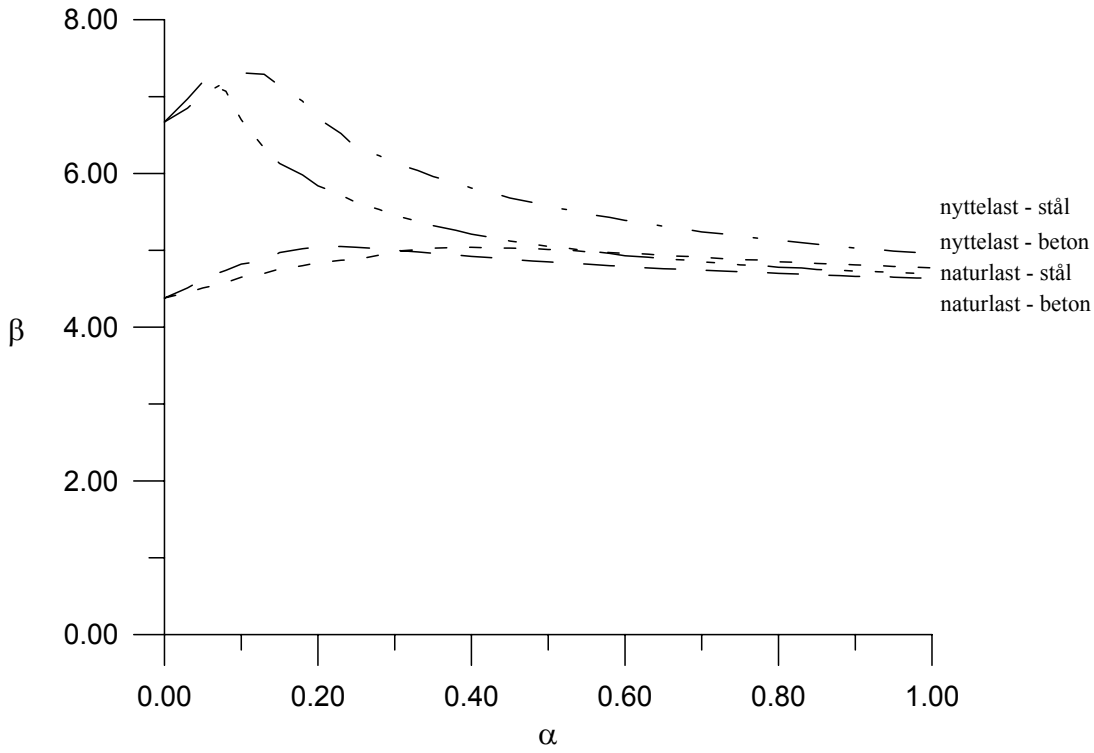

Figure 5. Reliability index β for different combinations of variable actions and material type.

In deterministic evaluations of the reliability the design equation is often rewritten as

$$r = \gamma_m \left(\gamma_G (1-q) + \gamma_Q q \right) \tag{27}$$

where

$$r = \frac{zR_c}{(1-\alpha)G_c + \alpha Q_c} \tag{28}$$

$$q = \frac{\alpha Q_c}{(1-\alpha)G_c + \alpha Q_c} \tag{29}$$

It is seen that r can be considered as a 'total safety factor', namely as a product of the material partial safety factor and the weighted action partial safety factor. q is seen to be a measure of the characteristic variable action compared to the total characteristic action.

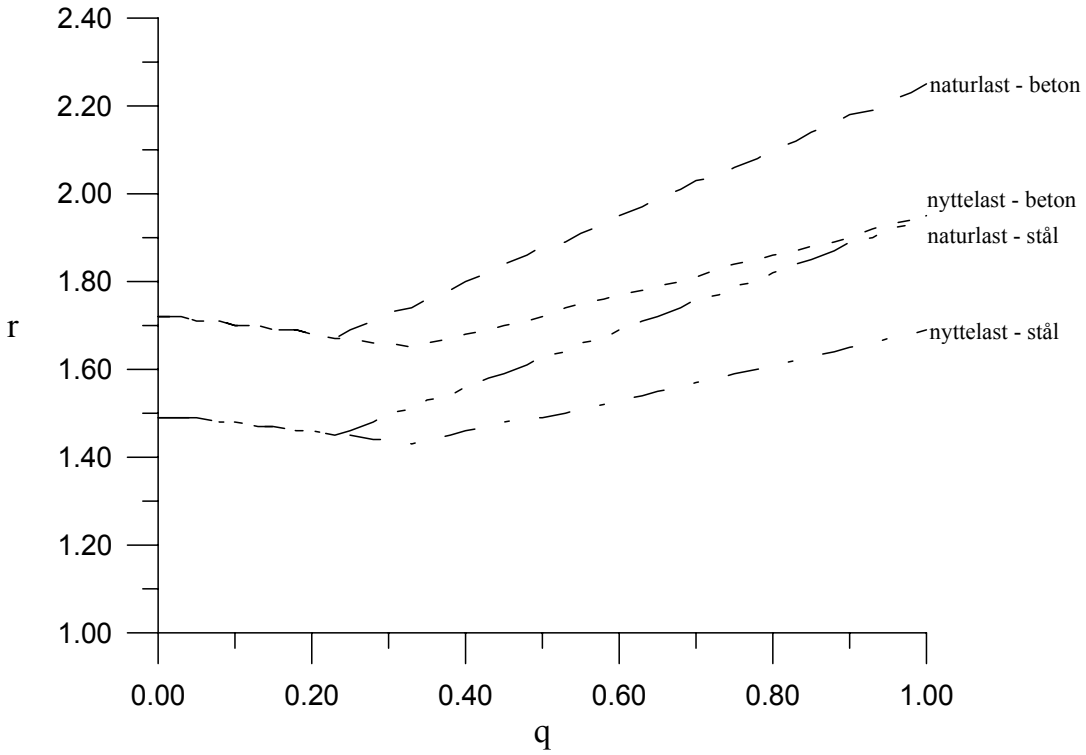

Figure 6. 'total safety factor' r as function of q for different combinations of variable actions and material type.

In figure 6 the 'total safety factor' r is shown as a function of q for different combinations of variable actions and material type. It is noted that the smallest value of the 'total safety factor' r is obtained for values of q between 0.2 and 0.3. A comparison with figure 5 shows that it is not in this interval that the smallest reliability index is obtained; in fact the largest reliability indices are obtained in this interval. The reasons for this are among others that

- the variable actions have a larger coefficient of variation than the permanent actions
- the 'total safety factor' r is based on characteristic values which have some 'safety' included since they are obtained as quantile values in the distribution functions for actions and strengths.

It is thus concluded that

- the reliability level is almost constant for different combinations of variable and permanent actions and different material types
- the 'total safety factor' r is not a good indicator of the reliability. Evaluations of the reliability level require that probabilistic calculations are performed.

6.5 Partial safety factors for actions in new DS 409

The final partial safety factors for actions in DS409 [24] are given in table 24.

	Load combination							
	Serv.	Ultimate				Accidental		
Load type	1	2.1	2.2[1)]	2.3	2.4	3.1	3.2	3.3
Permanent action								
Self weight								
G_c	1.0	1.0	0.8	0.9	1.0	1.0	1.0	1.0
$0.25\,G_c$ free action	-	-	-	1.0	-	-	-	-
Weight of soil and ground water	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Variable action								
Variable action								
imposed action	-	1.3	1.3	1.0	1.0-1.3	ψ	ψ	ψ
environmental action	-	1.5	1.5	1.0	1.0-1.3	-	-	ψ
Other variable actions	-	ψ	ψ	ψ	1.0-1.3	ψ	ψ	ψ
Horizontal mass load	-	1.0	1.0	1.0	-	-	-	0.25
Accidental action – impact, etc.	-	-	-	-	-	1.0	-	-
Accidental load – fire	-	-	-	-	-	-	-	1.0

Table 24. Load combinations, partial safety factors and load combination factors ψ. [1)] The partial safety factors for load combination 2.2 are for normal safety class. For low and high safety classes the partial safety factors for variable action have to be multiplied by γ_0, see table 6.

The final partial safety factors for materials can be determined using (21) in section 6.2 with γ_2 selected according to the coefficient of variation δ for the material considered. In table 25 γ_2 values are shown for different values of δ. The γ_2 values are determined such that the average reliability level is obtained for the relevant value of δ. The values for δ=0.05 and 0.15 shown in the table are derived in section 6.4.3.

δ	<0.05	0.10	0.15	0.20	0.25	0.30
γ_2	1.30	1.38	1.50	1.64	1.83	2.06

Table 25. γ_2 as function of the coefficient of variation δ (for 5 % quantiles).

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Note 9: TIME-VARIANT RELIABILITY

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1 Introduction

In the previous lectures it has been assumed that all variables could be considered either to be time-invariant stochastic variables or deterministic parameters. However, loads such as wave-loads, snow-loads and wind-loads are usually modeled as time-varying stochastic processes. In this case we are usually interested in determining the probability that the load within a given period of time exceeds a given threshold, the so-called barrier crossing problem. Further, it is of interest to determine the distribution of the maximum and minimum values of the process. This note is partly based on an earlier lecture note by S. Engelund.

2 Stochastic processes

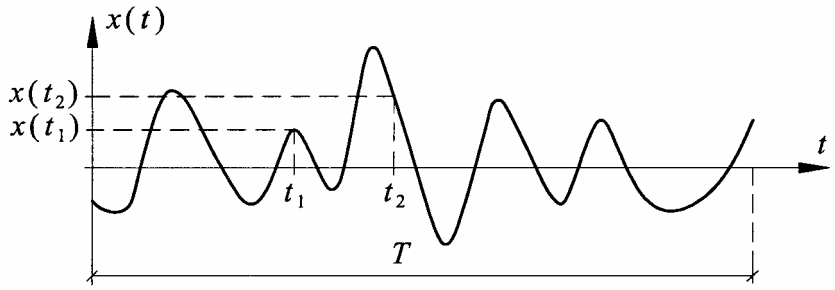

Figure 1. Realization of stochastic process.

A stochastic process is an indexed set of random variables $\{X(t), t \in [0,T]\}$ defined in the sample space Ω. The index variable t is here assumed to be time, defined on the time interval $[0,T]$. Figure 1 shows a realization of a stochastic process.

At time t the stochastic variable $X(t)$ with realization $x(t)$ is described by the distribution function

$$F_X(x;t) = P(\{X = X(t) \le x\}) \tag{1}$$

At times t_1 and t_2 the joint distribution function for the two random variables $X(t_1)$ and $X(t_2)$ is, see figure 1

$$F_X(x_1, x_2; t_1, t_2) = P(\{X_1 = X(t_1) \le x_1\} \cap \{X_2 = X(t_2) \le x_2\}) \tag{2}$$

Correspondingly, if n times are considered the joint distribution function is

$$F_X(x_1,x_2,...,x_n;t_1,t_2,...,t_n) = P\left(\{X_1 = X(t_1) \le x_1\} \cap \{X_2 = X(t_2) \le x_2\} \cap ... \cap \{X_n = X(t_n) \le x_n\}\right) \tag{3}$$

The corresponding joint density function of order n is defined by

$$f_X(x_1,x_2,...,x_n;t_1,t_2,...,t_n) = \frac{\partial^n F_X(x_1,x_2,...,x_n;t_1,t_2,...,t_n)}{\partial x_1 \partial x_2 ... \partial x_n} \tag{4}$$

The stochastic process is fully described by the distribution functions $F_X(x)$, $F_X(x_1,x_2)$,….
The ***expected value function*** $\mu_X(t)$ is defined by

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x,t) dx \tag{5}$$

The ***autocorrelation function*** $R_{XX}(t_1,t_2)$ is defined by

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1,x_2;t_1,t_2) dx_1 dx_2 \tag{6}$$

The ***autocovariance function*** $C_{XX}(t_1,t_2)$ is defined by

$$C_{XX}(t_1,t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \tag{7}$$

The ***variance function*** $\sigma_X^2(t)$ is defined by ($t_1 = t_2$):

$$\sigma_X^2(t) = C_{XX}(t,t) = R_{XX}(t,t) - \mu_X^2(t) \tag{8}$$

The ***autocorrelation coefficient function*** $\rho_{XX}(t_1,t_2)$ is defined by

$$\rho_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)} \tag{9}$$

It is seen that $-1 \le \rho_{XX}(t_1,t_2) \le 1$.

If all finite dimensional distribution functions $F_X(x)$, $F_X(x_1,x_2)$,…. are invariant to a linear translation of the time origin then the process is called ***strictly stationary***. If this invariance assumption only holds for $F_X(x)$ and $F_X(x_1,x_2)$ then the process is called ***weakly stationary***. For a stationary process $F_X(x;t)$ becomes independent on time and $F_X(x_1,x_2;t_1,t_2)$ only becomes dependent on the time difference $\tau = t_1 - t_2$. Similarly, $R_{XX}(t_1,t_2)$, $C_{XX}(t_1,t_2)$ and $\rho_{XX}(t_1,t_2)$ will only be dependent on $\tau = t_1 - t_2$.

For a stationary stochastic process, the ***spectral density*** is related to the covariance function by the Wiener-Khintchine equations:

$$S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{XX}(\tau)\exp(-i\omega\tau) d\tau \tag{10}$$

$$C_{XX}(\tau) = \int_{-\infty}^{\infty} S_X(\omega) \exp(i\omega\tau) d\omega \tag{11}$$

where ω is the circular frequency in radians per second. From (8) then follows that the variance of the stationary process is:

$$C_X^2 = C_{XX}(0) = \int_{-\infty}^{\infty} S_X(\omega) d\omega \tag{12}$$

If measurements of a stationary stochastic process are made, then usually only one realization becomes available. In that case the expected value is estimated by:

$$\mu = \frac{1}{T} \int_0^T x(\tau) d\tau \tag{13}$$

If this time average approaches μ_X for $T \to \infty$ the process is ***ergodic in mean value***. Similarly if

$$R(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} x(t+\tau) x(t) d\tau \tag{14}$$

approaches $R_{XX}(\tau)$ for $T \to \infty$ the process is ***ergodic in correlation.*** If this property holds for all moments then the process is called ***ergodic***.

A stochastic process $\{X(t), t \in [0,T]\}$ is ***Gaussian*** if the random variables $X(t_1)$, $X(t_2), \ldots, X(t_n)$ are jointly Normal distributed for any n. The joint density function can then be written:

$$f_X(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n) = \frac{1}{(2\pi)^{n/2} \sqrt{|\mathbf{C}|}} \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} \left(x_i - \mu_X(t_i)\right) \left[\mathbf{C}^{-1}\right]_{ij} \left(x_j - \mu_X(t_j)\right) \right) \tag{15}$$

where $\mathbf{C}$ is the covariance matrix:

$$\mathbf{C} = \begin{bmatrix} C_{XX}(t_1,t_1) & C_{XX}(t_1,t_2) & \ldots & C_{XX}(t_1,t_n) \\ C_{XX}(t_2,t_1) & C_{XX}(t_2,t_2) & \ldots & C_{XX}(t_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ C_{XX}(t_n,t_1) & C_{XX}(t_n,t_2) & \cdots & C_{XX}(t_n,t_n) \end{bmatrix} \tag{16}$$

and $[\mathbf{C}^{-1}]_{ij}$ denotes element i, j in the inverse covariance matrix $\mathbf{C}^{-1}$. A Gaussian process is thus completely determined by $\mu_X(t)$ and $C_{XX}(t_1, t_2)$. Therefore a stationary Gaussian process is strictly stationary. The derivative process is also Gaussian:

$$\dot{X}(t) = \frac{d}{dt} X(t) \tag{17}$$

For a stationary process it can be shown that

$$E[\dot{X}] = 0 \tag{18}$$

$$E[\dot{X}^2] = -\left.\frac{d^2 R_{XX}(t)}{dt^2}\right|_{t=0} = -R''_{XX}(0) \tag{19}$$

$$E[X\,\dot{X}] = 0 \tag{20}$$

Consider a stationary Gaussian process with mean value μ_X and standard deviation σ_X. Since $X(t)$ is a stationary process the mean value of $\dot{X}$ is $\mu_{\dot{X}} = 0$, see (18). The standard deviation of $\dot{X}$ is denoted $\sigma_{\dot{X}}$. The joint density function of X and $\dot{X}$ is then

$$f_{X\dot{X}}(x,\dot{x}) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left(-\frac{1}{2}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{X}}}\right)^2\right)\right) \tag{21}$$

3 Barrier Crossing

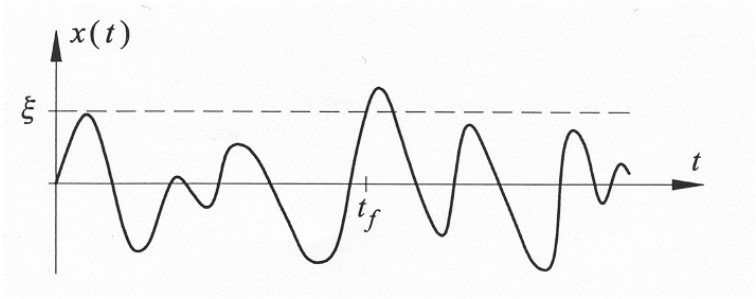

Figure 2. Realization of stochastic process.

In many engineering applications it is necessary to determine the reliability of structural components subject to stochastic process loading. Then the probability that the structural component enters, during some given time interval, a critical state (failure) must be determined. Let failure occur when the process $X(t)$ exceeds some threshold ξ, see figure 2 where failure occurs at time t_f. The probability of failure in the interval $[0;T]$ is

$$P_f(t) = 1 - P\big(X(t) < \xi, \quad \forall t \in [0;T]\big) \tag{22}$$

In the following a number of different methods by which estimates of (22) can be obtained are presented.

3.1 Simulation

Monte Carlo simulation of stochastic processes has attracted much attention in the recent years. Partly because the development of more efficient computers the method has become more attractive and partly because it often is the only available method to determine the reliability of complicated nonlinear structural systems. The most commonly used method for simulating Gaussian processes is the so-called spectral representation method proposed by Borgman [1].

$$X_M(t_j) = \sum_{k=0}^{M-1}\sqrt{2S_X(\omega_k)\Delta\omega}\,\cos(\omega_k t_j + \Theta_k) \tag{23}$$

where $S_X(\omega)$ is the one-sided spectrum of the stochastic process and $\omega_k = k\Delta\omega$. The phases, Θ_k, are stochastic variables, independent and uniformly distributed in the interval $[0;2\pi]$. The process $X_M(t)$ is asymptotically Gaussian as M becomes large due to the central limit theorem. Further, it is important to notice that the process $X_M(t)$ is periodic with the period $\frac{2\pi}{\Delta\omega}$. It is evident that for longer time histories and finer spectral resolution the computation time becomes excessive. Fortunately, this problem can be overcome by performing the summation in (23) by Fast Fourier Transformation (FFT). The failure probability now can be determined by simulating a large number of realizations of $X(t)$ and determining the relative number of times $X(t)$ exceeds the threshold value, ξ.

$$P_f = \frac{N_C}{N} \tag{24}$$

where N_C denotes the number of realizations which exceeds the threshold value and N denotes the number of realizations of $X(t)$.

The simulation method is not restricted to Gaussian processes. It is, however, more complicated to simulate Non-Gaussian processes. The major disadvantage of the method is the fact that it requires a very large number of simulations in order to determine an out-crossing probability if the out-crossings events are rare. In that case the method is very inefficient even if the Fast Fourier Transformation is applied to perform the summation.

3.2 Rice's In- and Exclusion Series

Let p_k denote the probability of exactly k out-crossings in the interval $[0;T]$. It is then evident that the probability of no out-crossings or the complementary first passage probability is

$$\begin{aligned}
p_0 &= 1 - P_f \\
&= 1 - \sum_{k=1}^{\infty} p_k \\
&= 1 + \sum_{k=1}^{\infty} p_k \sum_{i=1}^{\infty} (-1)^i \binom{k}{i} \\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k=1}^{\infty} i! \binom{k}{i} p_k \\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k=1}^{\infty} k(k-1)...(k-i+1) p_k \\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} m_i \\
&= m_0 - m_1 + \frac{1}{2} m_2 - \frac{1}{6} m_3 + ...
\end{aligned} \tag{25}$$

where m_i denotes the i th factorial moment of the number of out-crossings, i.e.

$$m_0 = 1$$
$$m_i = \sum_{k=1}^{\infty} k(k-1)...(k-i+1)p_k \quad \text{for } i \geq 1 \tag{26}$$

and where it has been used that

$$\binom{k}{i} = 0 \quad \text{for } i > k \tag{27}$$

(25) is the so-called Rice's " in- and exclusion " series (see Rice [4] which provides an exact solution to the barrier crossing problem. Of course, the moments m_i $(i = 1,2,...)$ must exist and the series in (25) must converge in order to make (25) a valid representation. The series provides lower and upper bounds for the survival probability upon truncation after an odd or even term, respectively. The computational effort involved in evaluating $P_f(t)$ according to this method, however, is extensive. Further an increasing number of terms has to be taken into account as m_1 increases. Normally the series is truncated after the first term. This provides an upper limit for the failure probability

$$P_f \leq m_1 \tag{28}$$

where m_1 is nothing but the mean number of out-crossings. It is evident that $P_f(t)$ can only be approximated by m_1 if the out-crossing probability is very small, i.e. $P_f << 1$.

3.3 The Poisson Assumption

Let the process $N^+(t,\xi)$ be a process that increases by one each time the process $X(t)$ exceeds the threshold ξ and let $N^+(0,\xi) = 0$. Obviously $N^+(t,\xi)$ is a counting process which counts the number of exits of $X(t)$ across ξ.

If it is now assumed that the probability of having two or more out-crossings in $]t, t+\Delta t]$ is negligible compared to the probability of having exactly one out-crossing, if Δt is sufficiently small, and further that the out-crossings in $]t, t+\Delta t]$ are independent of the previous out-crossings in $]0, t]$, then $N^+(t)$ is a Poisson process. The probability that the number of out-crossings $N^+(t,\xi)$ is equal to n can be determined as

$$P\left(N^+(t,\xi) = n\right) = \frac{1}{n!}\left(\lambda(t,\xi)\right)^n \exp(-\lambda(t)) \tag{29}$$

where $\lambda(t,\xi)$ is the mean value of $N^+(t,\xi)$ in the interval $]0,t]$,

$$\lambda(t,\xi) = E[N^+(t,\xi)] = m_1 \tag{30}$$

The probability of failure now is

$$P_f(t) = 1 - P\left(N^+(t,\xi) = 0\right) = 1 - \exp(-m_1) \tag{31}$$

For broad-banded processes the correlation length is of the magnitude equal to the zero up-crossing period. In this case the maxima between succeeding zero-upcrossings are virtually uncorrelated. Hence, the out-crossings from the safe domain related to these maxima will also be independent and (31) is valid.

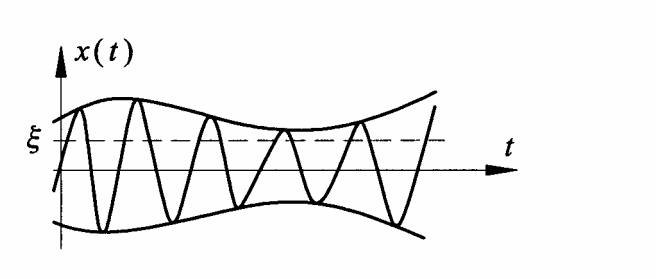

Figure 3. Out-crossings of a narrow-band process.

For narrow-banded processes, the out-crossings in case of low to medium barrier levels tend to occur in clumps, see figure 3. In this case the crossing events are highly correlated, and (31) is no longer appropriate. However, at higher barrier levels only the highest peak in a clump is likely to imply an out-crossing. This suggests that the out-crossings tend to become independent as $\xi \to \infty$. Actually, this hypothesis can be formally proved for Gaussian processes, see Cramer and Leadbetter [3].

3.4 Initial Conditions

By (25) and (31) one determines the probability that $X(t)$ at some time crosses the threshold, ξ. It has not been taken into account that the process might start in the failure region, i.e. $X(0) > \xi$. By taking the initial condition into account the failure probability can be defined as

$$P_f(T) = 1 - (1 - P_f(0))\, P\left(X(t) < \xi \;\; \forall t \in [0,T] \,\middle|\, X(0) < \xi\right) \tag{32}$$

where $P_f(0) = P\left(X(0) < \xi\right)$ is a simple time-invariant reliability problem.

By differentiation of (32) one obtains

$$\frac{dP_f(T)}{dT} = f_1(t)\, P\left(X(0) < \xi\right) \tag{33}$$

where $f_1(t)$ is the probability density function of the time to the first barrier crossing conditional on $X(0) < \xi$. No exact solutions for $f_1(t)$ are available even for very simple problems. Hence, it is necessary to determine some approximation by which the failure probability can be determined.

4 Mean Number of Out-crossings

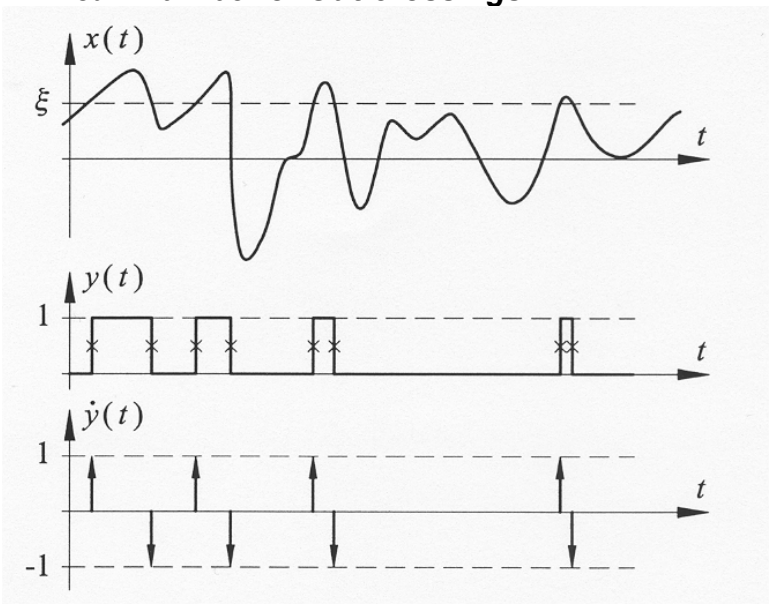

Figure 4. Realizations of $X(t)$, $Y(t)$ and $\dot{Y}(t)$.

In order to determine the mean number of exits of $X(t)$ across the level ξ it is convenient to consider the stochastic process $Y(t)$ given by

$$Y(t) = H(X(t) - \xi) \tag{34}$$

where $H(.)$ is Heavisides step function. By differentiation of $Y(t)$ the derivative process $\dot{Y}(t)$ can be determined by

$$\dot{Y}(t) = \dot{X}(t)\delta(X(t) - \xi) \tag{35}$$

where $\delta(.)$ denotes the dirac delta function. In (35) it has been assumed that $X(t)$ is a differentiable process. For a realization of $X(t)$ the corresponding realizations of $Y(t)$ and $\dot{Y}(t)$ are shown in figure 4. It is seen that $\dot{Y}(t)$ consists of a series of unit pulses which occurs each time an out-crossing of $X(t)$ occurs. The number of out-crossings, $N(T,\xi)$, within the time interval $]0,T]$ can be determined by integrating the absolute value of $\dot{Y}(t)$

$$N(T,\xi) = \int_0^T \left|\dot{Y}(\tau)\right| d\tau = \int_0^T \left|\dot{X}(\tau)\right| \delta(X(\tau) - \xi) d\tau \tag{36}$$

The mean number of out-crossings is

$$\begin{aligned} E\left[N(T,\xi)\right] &= \int_0^T E[\left|\dot{X}(\tau)\right| \delta(X(\tau) - \xi)] d\tau \\ &= \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left|\dot{x}(t)\right| \delta(x(t) - \xi) f_{X\dot{X}}(x, \dot{x}, \tau) dx\, d\dot{x}\, d\tau \\ &= \int_0^T \int_{-\infty}^{\infty} \left|\dot{x}(t)\right| f_{X\dot{X}}(\xi, \dot{x}, \tau)\, d\dot{x} d\tau \end{aligned} \tag{37}$$

where $f_{X\dot{X}}$ is the joint density function of X and $\dot{X}$. It should be noted that by deriving (37) both the up-crossings and down-crossings have been taken into account. However, for a stationary process it is reasonable to assume that any positive crossing is followed by a negative crossing:

$$E\left[N^{+}(T,\xi)\right]=E\left[N^{-}(T,\xi)\right]=\frac{1}{2}E\left[N(T,\xi)\right] \tag{38}$$

where $N^{-}(T,\xi)$ counts the number of down-crossings of X and $\dot{X}$ of the level ξ. This implies that

$$\begin{aligned} E\left[N^{+}(T,\xi)\right] &= \int_0^T\int_0^\infty \dot{x} f_{X\dot{X}}(\xi,\dot{x},\tau)\,d\dot{x}d\tau \\ &= m_1 \end{aligned} \tag{39}$$

It is often convenient to consider the rate of out-crossings pr unit time, $\nu^{+}(t,\xi)$ which is defined by

$$\nu^{+}(t,\xi)=\int_0^\infty \dot{x}\, f_{X\dot{X}}(\xi,\dot{x},t)\,d\dot{x} \tag{40}$$

which is the so-called ***Rice's formula***, see [4]. For stationary processes the out-crossing intensity does not depend on t i.e. $\nu^{+}(t,\xi)=\nu^{+}(\xi)$.

From (28) follows that an upper bound of the probability of failure in the time interval $]0,T]$ is

$$P_f(T)\le m_1=\int_0^T \nu^{+}(t,\xi)dt \tag{41}$$

If $\nu^{+}(t,\xi)=\nu^{+}(\xi)$ then

$$P_f(T)\le \nu^{+}(\xi)T \tag{42}$$

Higher order factorial moments and factorial moments of the number of out-crossing of a given safe domain by a vector process can be determined on the basis on the so-called ***Belyaev's formula***, see [1]. This formula, however, can only be solved analytically in a few special cases and a numerical solution is generally a non-trivial task.

4.1 Initial Conditions

We have now determined the mean number of out-crossings of $X(t)$ without taking into account the initial conditions. The mean number of $N^+(t)$ given $X(0) < \xi$ is often approximated by the unconditional mean value, m_1. By using (11) one then obtains

$$P_f(T) = 1 - (1 - P_f(0)) \exp\left(- \mathrm{E}[\,\mathrm{N}^+(\mathrm{T}, \xi)]\right) \tag{43}$$

It has, however, been shown that a better approximation for the mean number of out-crossings given $X(0) < \xi$ is given by

$$\mathrm{E}\left[\mathrm{N}^+(\mathrm{T}, \xi) \middle| \mathrm{X}(0) < \xi\right] \approx \frac{\mathrm{E}[\,\mathrm{N}^+(\mathrm{T}, \xi)]}{1 - P_f(0)} \tag{44}$$

whereby

$$P_f(T) = 1 - (1 - P_f(0)) \exp\left(- \frac{\mathrm{E}[\,\mathrm{N}^+(\mathrm{T}, \xi)]}{1 - P_f(0)} \right) \tag{45}$$

This expression has been shown to yield very accurate results even for relatively low threshold levels, where the out-crossings are not independent.

4.2 Gaussian Processes

Let $X(t)$ be a stationary Gaussian process with density function given by (21). For a given threshold ξ the out-crossing intensity now can be determined on the basis of Rice's formula (40):

$$\begin{aligned}
\nu^+(\xi) &= \int_0^\infty \dot{x} f_{X\dot{X}}(\xi, \dot{x})\, d\dot{x} \\
&= \int_0^\infty \dot{x} \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left(-\frac{1}{2}\left(\left(\frac{\xi - \mu_X}{\sigma_X} \right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{X}}} \right)^2 \right) \right) d\dot{x} \\
&= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left(-\frac{1}{2}\left(\frac{\xi - \mu_X}{\sigma_X} \right)^2 \right) \int_0^\infty \dot{x} \exp\left(-\frac{1}{2}\left(\frac{\dot{x}}{\sigma_{\dot{X}}} \right)^2 \right) d\dot{x} \\
&= \frac{\sigma_{\dot{X}}}{2\pi\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{\xi - \mu_X}{\sigma_X} \right)^2 \right)
\end{aligned} \tag{46}$$

For $\xi = \mu_X$ one finds the zero-crossing intensity

$$\nu^+(\mu_X) = \frac{1}{2\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \tag{47}$$

Example 1

Consider a stationary Gaussian process with

$$\mu_X = 1 \qquad \sigma_X = 0.3 \qquad \sigma_{\dot{X}} = 0.2$$

If the critical barrier is

$$\xi = 2$$

then the out-crossing intensity is, see (46)

$$\nu^+(2) = \frac{0.2}{2\pi 0.3}\exp\left(-\frac{1}{2}\left(\frac{2-1}{0.3}\right)^2\right) = 0.00041$$

and the zero-crossing intensity becomes

$$\nu^+(1) = \frac{0.2}{2\pi 0.3} = 0.11$$

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5 Distribution of Local Extremes

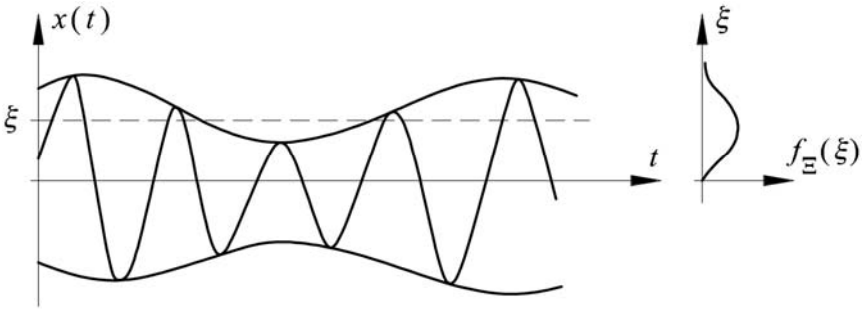

Figure 5. Realization of narrow-banded stochastic process and density function of local extremes.

First, consider the simple case of a ***stationary narrowband Gaussian process***, $X(t)$. A realization of a narrow-band process is shown in figure 5. For an ideally narrow-band process the rate of zero-crossings is equal to the rate of occurrence of maxima. Further the rate of crossings of the level ξ is equal to the rate of occurrence of maxima above ξ. Therefore, the ratio $\nu^+(\xi)/\nu^+(0)$ may be interpreted as the complementary distribution function of the local maxima, Ξ

$$F_\Xi(\xi) = 1 - P(\Xi > \xi) = 1 - \frac{\nu^+(\xi)}{\nu^+(0)} = 1 - \exp\left(-\frac{1}{2}\left(\frac{\xi - \mu_X}{\sigma_X}\right)^2\right) \quad , \xi \geq \mu_X \tag{48}$$

Differentiation of (48) yields the density function of the local maxima

$$f_{\Xi}(\xi) = \frac{\xi - \mu_X}{\sigma_X^2} \exp\left(-\frac{1}{2}\left(\frac{\xi - \mu_X}{\sigma_X} \right)^2 \right) \quad , \ \xi \geq \mu_X \tag{49}$$

which is the density function of the Rayleigh distribution.

Example 2

Using the same data as in example 1 and assuming that the process is narrow-banded, the density function of local maxima (peaks) becomes

$$f_{\Xi}(\xi) = \frac{\xi - 1}{0.3^2} \exp\left(-\frac{1}{2}\left(\frac{\xi - 1}{0.3} \right)^2 \right) \qquad \xi \geq 1$$

and the expected number of peaks between 1.5 and 1.6 becomes

$$P(1.5 \leq \Xi \leq 1.6) = \exp\left(-\frac{1}{2}\left(\frac{1.5-1}{0.3} \right)^2 \right) - \exp\left(-\frac{1}{2}\left(\frac{1.6-1}{0.3} \right)^2 \right) = 0.249\text{-}0.135 = 0.114$$

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Example 3

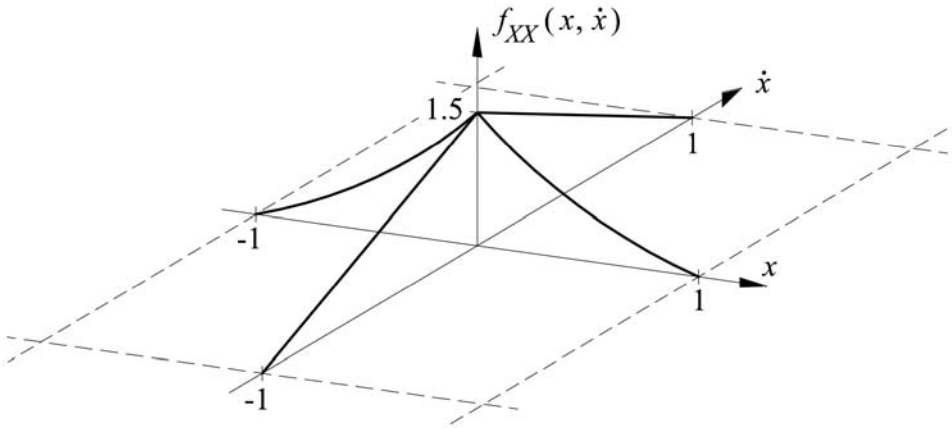

Figure 6. Joint density function $f_{X\dot{X}}(x, \dot{x})$.

Consider a stationary stochastic process where the joint density function of X and $\dot{X}$ is:

$$f_{X\dot{X}}(x, \dot{x}) = \begin{cases} c\left(|x| - 1\right)^2 \left(1 - |\dot{x}|\right) & \text{for } (x, \dot{x}) \in [-1;1] \times [-1;1] \\ 0 & \text{else} \end{cases}$$

From the condition that $\int_{-1}^{1}\int_{-1}^{1} f_{X\dot{X}}(x, \dot{x}) dx d\dot{x} = 1$ the constant c is determined:

$$\int_{-1}^{1}\int_{-1}^{1} c\left(|x|-1\right)^2\left(1-|\dot{x}|\right)dxd\dot{x}=1 \quad \Rightarrow$$

$$4c\int_{0}^{1}\int_{0}^{1}\left(x-1\right)^2\left(1-\dot{x}\right)dxd\dot{x}=1 \quad \Rightarrow$$

$$c=1.5$$

Thus

$$f_{X\dot{X}}(x,\dot{x})=\begin{cases}1.5\left(|x|-1\right)^2\left(1-|\dot{x}|\right) & \text{for } (x,\dot{x})\in[-1;1]\times[-1;1]\\ 0 & \text{else}\end{cases}$$

The density function is illustrated in figure 6. The marginal density functions are

$$f_X(x)=\int_{-1}^{1} f_{X\dot{X}}(x,\dot{x})d\dot{x}=1.5\left(|x|-1\right)^2 \quad \text{for } x\in[-1;1]$$

$$f_{\dot{X}}(\dot{x})=\int_{-1}^{1} f_{X\dot{X}}(x,\dot{x})dx=\left(1-|\dot{x}|\right) \quad \text{for } \dot{x}\in[-1;1]$$

Expected values and standard deviations are:

$$\mu_X=\int_{-1}^{1} x1.5\left(|x|-1\right)^2dx=0 \qquad \sigma_X=\sqrt{\int_{-1}^{1}(x-0)^2 1.5\left(|x|-1\right)^2dx}=\sqrt{\frac{1}{10}}$$

$$\mu_{\dot{X}}=\int_{-1}^{1}\dot{x}\left(1-|\dot{x}|\right)d\dot{x}=0 \qquad \sigma_{\dot{X}}=\sqrt{\int_{-1}^{1}(\dot{x}-0)^2\left(1-|\dot{x}|\right)d\dot{x}}=\sqrt{\frac{1}{6}}$$

The out-crossing intensity of the level ξ=0.8 is determined using Rice's formula, see (40):

$$\nu^+(\xi=0.8)=\int_{0}^{\infty}\dot{x}f_{X\dot{X}}(\xi=0.8,\dot{x})\,d\dot{x}=\int_{0}^{1}\dot{x}\,1.5\left(0.8-1\right)^2\left(1-\dot{x}\right)d\dot{x}=0.0100$$

If the process is approximated by a Gaussian process with the same expected values and standard deviations then out-crossing intensity becomes, see (46):

$$\nu^+(0.8)=\frac{\sqrt{1/6}}{2\pi\sqrt{1/10}}\exp\left(-\frac{1}{2}\left(\frac{0.8-0}{\sqrt{1/10}}\right)^2\right)=0.0089$$

and the expected number of peaks above 0.8 becomes, see (48):

$$P(0.8\le\Xi)=\exp\left(-\frac{1}{2}\left(\frac{0.8-0}{\sqrt{1/10}}\right)^2\right)=0.041$$

* * * * * * * *

In the following is considered a normalized process X_m with expected value equal to zero and unit standard deviation. A realization x_m is obtained from:

$$x_m = \frac{x - \mu_X}{\sigma_X} \tag{50}$$

For ***non-narrowband Gaussian processes*** an expression for the distribution of local maxima can be derived on the basis of Rice's formula, (40). Using the fact that the occurrence of a maxima of $X(t)$ implies a down-crossing of $\dot{X}(t)$ of the level $\xi = \mu_X$, and by introducing the so-called irregularity factor

$$\alpha = \frac{\text{expected number of zero crossings of } X(t)}{\text{expected number of peaks of } X(t)} = \frac{N}{N_m} \tag{51}$$

Rice [4] has derived the following expression for the density function of the local maxima:

$$f_{X_m}(x_m) = \sqrt{1-\alpha^2}\,\varphi\left(\frac{x_m}{\sqrt{1-\alpha^2}}\right) + \alpha x_m \exp\left(-\frac{x_m^2}{2}\right)\Phi\left(\frac{\alpha x_m}{\sqrt{1-\alpha^2}}\right) \tag{52}$$

where $\Phi(.)$ denotes the standard Normal distribution and $\varphi(.)$ denotes the standard Normal density function. The irregularity factor α takes on values in the interval between zero and one. It can be shown that when α=1 (an ideally narrow-band process) (52) gives the Rayleigh distribution, eq. (49). When α is approximately equal to zero, the density function of the local extremes, eq. (52), tends to the Gaussian density function with zero mean and standard deviation σ_X. This shows that maxima occur randomly and with equal probability of being above and below zero.

6 Global Extremes

6.1 Gaussian process

It is often on interest to have information about the largest of the maxima in an interval $[0,T]$. In this interval the expected number of local maxima is $N = \alpha N_m$, where N denotes the expected number of zero-crossings. Again consider a Gaussian process with zero mean and unity standard deviation. The distribution function, $F_T(x_m)$ of the extreme value in the interval $[0,T]$ can be obtained from

$$F_T(x_m) = F_{X_m}(x_m)^{N_m} = (1-(1-F_{X_m}(x_m)))^{N_m} \tag{53}$$

Integration of (52) gives

$$1 - F_{X_m}(x_m) = 1 - \Phi\left(\frac{x_m}{\sqrt{1-\alpha^2}}\right) + \alpha \exp\left(-\frac{x_m^2}{2}\right)\Phi\left(\frac{\alpha x_m}{\sqrt{1-\alpha^2}}\right) \tag{54}$$

Assuming that x_m is large leads to the asymptotic result

$$1 - F_{X_m}(x_m) \approx \alpha \exp\left(-\frac{x_m^2}{2} \right) \tag{55}$$

where it has been used that for large z

$$\Phi(z) \approx 1 - \varphi(z)\left(z^{-1} - z^{-3} + ... \right) \tag{56}$$

Now introduce the variable y given by

$$y = N_m(1 - F_{X_m}(x_m)) = \mathrm{N} \exp\left(-\frac{x_m^2}{2} \right) \tag{57}$$

and using the fact that the largest of N_m observed maxima is located around the $1/N_m$ fractile, which implies that the variable y is of order unity for increasing N_m, we obtain

$$\begin{aligned} F_T(x_m) &= \left(1 - \frac{y}{N_m} \right)^{N_m} \\ &= \exp\left(N_m \log\left(1 - \frac{y}{N_m} \right) \right) \\ &\approx \exp(-y) \\ &= \exp\left(-N \exp\left(-\frac{x_m^2}{2} \right) \right) \end{aligned} \tag{58}$$

The mean value and the standard deviation of the maximum value in the interval $[0, T]$ now can be determined on the basis of (58). It is found that

$$\mu_{\max} = \sqrt{2 \log N} + \frac{0.577}{\sqrt{2 \log N}} \tag{59}$$

$$\sigma_{\max} = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \log N}} \tag{60}$$

In the Danish codes of practice for wind loads the characteristic wind load is determined using (59) and (60), see [5].

6.2 Loads modeled by Poisson 'spike' process

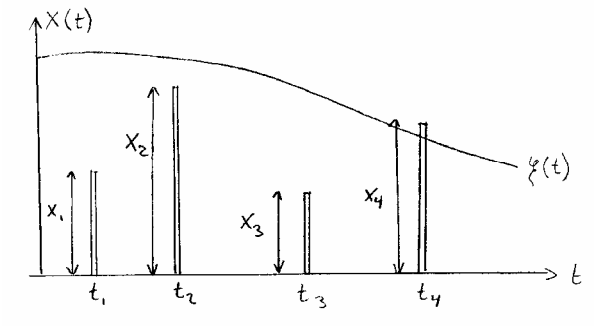

Figure 7. Poisson 'spike' process.

A stochastic process $\{X(t)\}$ is considered where the length of the single events is very small and can be considered as spikes. The size of the spikes are modeled as independent stochastic variables X with distribution function $F_X(x)$. The times $t_1, t_2, ...$ of the events are assumed to be modeled by a Poisson process with intensity $\lambda(t)$, see section 3.3. In figure 7 is shown the case with a time varying threshold $\xi(t)$. The events where the 'spikes' exceed the threshold can be considered as a new Poisson process with intensity

$$\lambda_\xi(t) = \lambda(t)\left[1 - F_X(\xi(t))\right] \tag{61}$$

$\{X(t)\}$ could for example model a load process and the threshold $\xi(t)$ could model the strength of an element.

Using (31) and (39)-(40) the probability of failure becomes:

$$P_f(t) = 1 - \exp\left(-\int_0^t \lambda_\xi(\tau)d\tau\right) \tag{62}$$

If the threshold $\xi(t)$ is constant $\xi(t) = \xi$ the probability of failure becomes:

$$P_f(t) = 1 - \exp\left(-\int_0^t \lambda(\tau)\left[1 - F_X(\xi)\right]d\tau\right) \tag{63}$$

When the threshold is constant the probability of no failure in the time interval $[0, T]$ is equal to the distribution function $F_T(\xi)$ of the maximum value of the stochastic process. Therefore

$$F_T(\xi) = P\left(\max_{t\in[0,T]}\{X(t)\} \le \xi\right) = 1 - P_f(T) = \exp\left(-\int_0^T \lambda(\tau)\left[1 - F_X(\xi)\right]d\tau\right) \tag{64}$$

If the intensity $\lambda(t)$ is time independent, then

$$F_T(\xi) = \exp\left(-\left[1 - F_X(\xi)\right]\lambda T\right) \tag{65}$$

6.3 Loads modeled by Poisson square-wave process

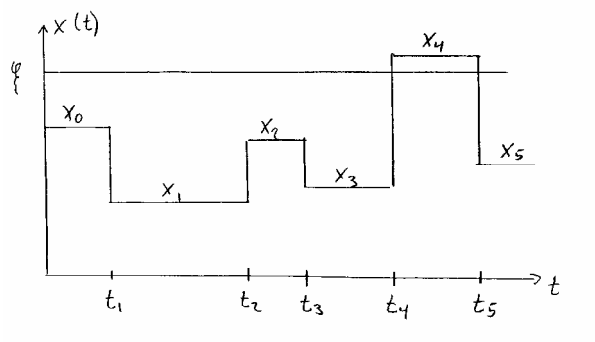

Figure 8. Poisson square wave process.

In figure 8 is shown a so-called square-wave Poisson process. The process $\{X(t)\}$ is constant between the changes in size at time $t_1, t_2, \ldots$. The times where the process changes its value are assumed to be modeled by a Poisson process with intensity $\lambda(t)$, see section 3.3. The threshold is assumed to be constant equal to ξ.

It can be shown that the distribution function $F_T(\xi)$ of the maximum value of the stochastic process in the time interval $[0,T]$ is, see e.g. Madsen et al. [6]

$$F_T(\xi) = F_X(\xi)\exp\left(-\left[1-F_X(\xi)\right]\int_0^T \lambda(\tau)d\tau\right) \tag{66}$$

If the intensity $\lambda(t)$ is time independent, then the distribution function becomes

$$F_T(\xi) = F_X(\xi)\exp\left(-\left[1-F_X(\xi)\right]\lambda T\right) \tag{67}$$

7 References

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[3] Cramer, H. & M.R. Leadbetter: Stationary and Related Stochastic Processes, Wiley, New York, 1967.

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[5] DS410:1998 – Code of practice for loads for the design of structures. Danish Standard, 1999.

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Note 10: LOAD COMBINATIONS

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1 Introduction

In this note the load combination problem is considered. The situation is considered where two or more variable loads act on a structure. How is for example the annual maximum combined load described and how are characteristic and design values determined? These are questions considered in this note, which is partly based on [1] and [2].

2 Exact model

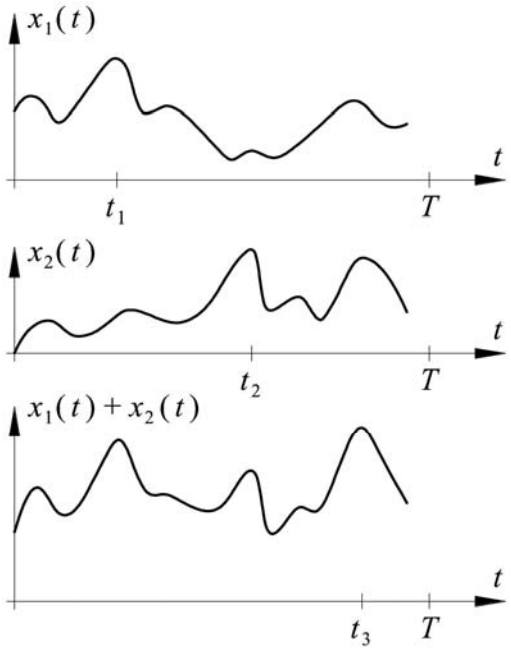

Figure 1. Realizations of two variable loads and their sum.

Consider two independent and stationary stochastic processes $\{X_1(t), t \in [0,T]\}$ and $\{X_2(t), t \in [0,T]\}$ with joint densities functions $f_{X_1\dot{X}_1}(x_1,\dot{x}_1)$ and $f_{X_2\dot{X}_2}(x_2,\dot{x}_2)$ for $(X_1,\dot{X}_1)$ and $(X_2,\dot{X}_2)$. The sum of the two processes is

$$X(t) = X_1(t) + X_2(t) \tag{1}$$

with derivative $\dot{X}(t) = \dot{X}_1(t) + \dot{X}_2(t)$. Figure 1 shows a realization of the stochastic processes. Note that in the time interval considered the maximum value for $X(t)$ does not occur at the same time as the maximum values for $X_1(t)$ or $X_2(t)$. The maximum value of $X(t)$ in the time interval $[0,T]$ is

denoted $X_{\max,T}$. T could for example be 1 year. The distribution function for $X_{\max,T}$ is equal to 1 minus the probability that the maximum value exceeds a threshold ξ in $[0,T]$:

$$\begin{aligned}
F_{X_{\max,T}}(\xi) &= 1 - P(\max X(t) > \xi \ , t \in [0,T]) \\
&= 1 - P(X(0) > \xi) - P(\text{one or more out - crossings of the barrier } \xi \,|\, X(0) \le \xi) \\
&\cong 1 - P(X(0) > \xi) - \sum_{n=1}^{\infty} P(n \text{ out - crossings of the barrier } \xi) \\
&\ge 1 - P(X(0) > \xi) - \sum_{n=1}^{\infty} nP(n \text{ out - crossings of the barrier } \xi) \\
&= 1 - P(X(0) > \xi) - \nu_X(\xi)T \\
&\approx 1 - \nu_X(\xi)T \\
&= 1 - T\int_0^{\infty} \dot{x} f_{X\dot{X}}(\xi, \dot{x}) d\dot{x}
\end{aligned} \tag{2}$$

where $\nu_X(\xi)$ is the out-crossing intensity determined by Rice's formula ($\nu_X(\xi) = \int_0^{\infty} \dot{x} f_{X\dot{X}}(\xi, \dot{x}) d\dot{x}$). To calculate $\nu_X(\xi)$ we need the joint density junction $f_{X\dot{X}}(x, \dot{x})$. First, it is seen that the distribution and density functions for X can be obtained from the convolution integrals:

$$\begin{aligned}
F_X(x) &= P(X_1 + X_2 \le x) \\
&= \int_{x_1 + x_2 \le x} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{x - x_1} f_{X_2}(x_2) dx_2 \right] f_{X_1}(x_1) dx_1 \\
&= \int_{-\infty}^{\infty} F_{X_2}(x - x_1) f_{X_1}(x_1) dx_1
\end{aligned} \tag{3}$$

$$\begin{aligned}
f_X(x) &= \frac{\partial F_X(x)}{\partial x} \\
&= \int_{-\infty}^{\infty} \frac{\partial F_{X_2}(x - x_1)}{\partial x} f_{X_1}(x_1) dx_1 \\
&= \int_{-\infty}^{\infty} f_{X_2}(x - x_1) f_{X_1}(x_1) dx_1
\end{aligned} \tag{4}$$

Similarly:

$$f_{X\dot{X}}(x, \dot{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1\dot{X}_1}(x_1, \dot{x}_1) f_{X_2\dot{X}_2}(x - x_1, \dot{x} - \dot{x}_1) dx_1 d\dot{x}_1 \tag{5}$$

where $x = x_1 + x_2$ and $\dot{x} = \dot{x}_1 + \dot{x}_2$.

Further, the out-crossing intensity $\nu_X(\xi)$ is determined by the following generalization of Rice's formula:

$$\begin{aligned}
\nu_X(\xi) &= \int_0^\infty \dot{x} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1\dot{X}_1}(x_1,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x} - \dot{x}_1) dx_1 d\dot{x}_1 d\dot{x} \\
&= \int_0^\infty \int_{-\infty}^{\infty} \int_{-\dot{x}_1}^{\infty} (\dot{x}_1 + \dot{x}_2) f_{X_1\dot{X}_1}(x_1,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x}_2) d\dot{x}_2 dx_1 d\dot{x}_1 \\
&= \int_{-\infty}^{\infty} \int_{\omega}^{\infty} \dot{x}_1 f_{X_1\dot{X}_1}(x_1,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x}_2) d\omega dx_1 + \\
&\quad \int_{-\infty}^{\infty} \int_{\omega}^{\infty} \dot{x}_2 f_{X_1\dot{X}_1}(x_1,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x}_2) d\omega dx_1
\end{aligned} \tag{6}$$

where ω is shown in figure 2.

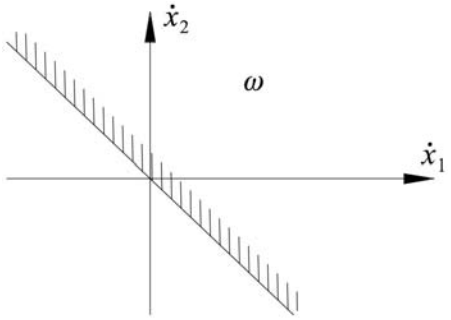

Figure 2. Domain ω.

An upper bound can then be determined from

$$\begin{aligned}
\nu_X(\xi) &\le \int_{-\infty}^{\infty} \int_{\omega_1} \dot{x}_1 f_{X_1\dot{X}_1}(x_1,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x}_2) d\omega_1 dx_1 + \\
&\quad \int_{-\infty}^{\infty} \int_{\omega_2} \dot{x}_2 f_{X_1\dot{X}_1}(x_1,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x}_2) d\omega_2 dx_1 \\
&= \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \dot{x}_1 f_{X_1\dot{X}_1}(x,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\dot{x}_2 d\dot{x}_1 dx + \\
&\quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \dot{x}_2 f_{X_1\dot{X}_1}(x,\dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\dot{x}_2 d\dot{x}_1 dx \\
&= \int_{-\infty}^{\infty} \nu_{X_1}(x) f_{X_2}(\xi - x) dx + \int_{-\infty}^{\infty} \nu_{X_2}(\xi - x) f_{X_1}(x) dx
\end{aligned} \tag{7}$$

where ω_1 and ω_2 are shown in figure 3.

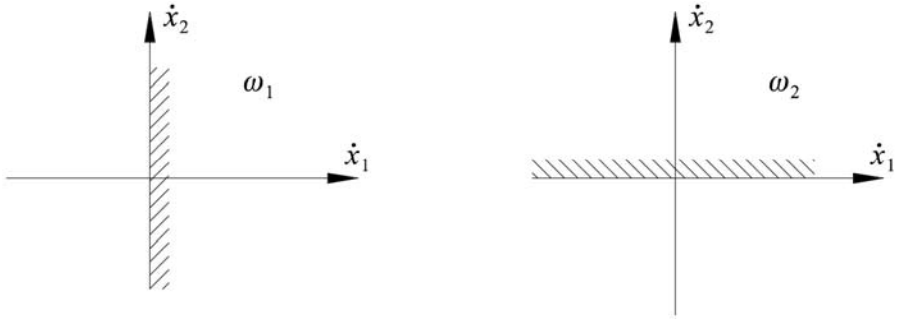

Figure 3. Domains ω_1 and ω_2.

For a sum of three processes the following generalization applies:

$$\begin{aligned}\nu_X(\xi) \le & \int_{-\infty}^{\infty} \nu_{X_1}(x_1) f_{X_2+X_3}(\xi - x_1) dx_1 + \\ & \int_{-\infty}^{\infty} \nu_{X_2}(x_2) f_{X_1+X_3}(\xi - x_2) dx_2 + \\ & \int_{-\infty}^{\infty} \nu_{X_3}(x_3) f_{X_1+X_2}(\xi - x_3) dx_3 +\end{aligned} \tag{8}$$

where $f_{X_i+X_j}(x)$ is determined by the convolution integral:

$$f_{X_i+X_j}(x) = \int_{-\infty}^{\infty} f_{X_i}(t) f_{X_j}(x-t) dt \tag{9}$$

Example 1

Consider two independent stochastic processes with joint density functions:

$$f_{X_1\dot{X}_1}(x_1, \dot{x}_1) = \begin{cases} 0.5\cos(x_1)(1 - |\dot{x}_1|) & \text{for } (x_1, \dot{x}_1) \in [-0.5\pi; 0.5\pi) \times [-1;1] \\ 0 & \text{else} \end{cases}$$

$$f_{X_2\dot{X}_2}(x_2, \dot{x}_2) = \begin{cases} \frac{1}{24}(9 - x_2^2)(1 - |\dot{x}_2|)^2 & \text{for } (x_2, \dot{x}_2) \in [-3;3] \times [-1;1] \\ 0 & \text{else} \end{cases}$$

The marginal density functions for X_1 and X_2 are:

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty} f_{X_1\dot{X}_1}(x_1, \dot{x}_1) d\dot{x}_1 \\ &= \begin{cases} 2\int_0^1 0.5\cos(x_1)(1 - |\dot{x}_1|) d\dot{x}_1 & \text{for } (x_1, \dot{x}_1) \in [-0{,}5\pi; 0.5\pi] \times [-1;1] \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 0.5\cos(x_1) & \text{for } x_1 \in [-0{,}5\pi; 0.5\pi] \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$\begin{aligned} f_{X_2}(x_2) &= \int_{-\infty}^{\infty} f_{X_2\dot{X}_2}(x_2, \dot{x}_2) d\dot{x}_2 \\ &= \begin{cases} \frac{1}{36}(9 - x_2^2) & \text{for } x_2 \in [-3;3] \\ 0 & \text{else} \end{cases} \end{aligned}$$

The out-crossing intensities X_1 and X_2 are:

$$\begin{aligned} \nu_{X_1}(\xi) &= \int_0^{\infty} \dot{x}_1 f_{X_1\dot{X}_1}(\xi, \dot{x}_1) d\dot{x}_1 \\ &= \begin{cases} \frac{1}{12}\cos(\xi) & \text{for } \xi \in [-0.5\pi; 0.5\pi] \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$\nu_{X_2}(\xi) = \int_0^\infty \dot{x}_2 f_{X_2\dot{X}_2}(\xi, \dot{x}_2) d\dot{x}_2$$
$$= \begin{cases} \frac{1}{288}\left(9 - \xi^2\right) & \text{for } \xi \in [-3;3] \\ 0 & \text{else} \end{cases}$$

An upper bound for the out-crossing intensity for $X(t) = X_1(t) + X_2(t)$ for $3 - \frac{\pi}{2} \le \xi \le 3 + \frac{\pi}{2}$ can then be determined from, see integration limits in figure 4:

$$\nu_X(\xi) \le \int_{-\infty}^{\infty} \nu_{X_1}(x) f_{X_2}(\xi - x) dx + \int_{-\infty}^{\infty} \nu_{X_2}(\xi - x) f_{X_1}(x) dx$$
$$= \int_{\xi-3}^{\pi/2} \frac{1}{12}\cos(x)\frac{1}{36}\left(9 - (\xi - x)^2\right) dx + \int_{\xi-3}^{\pi/2} \frac{1}{288}\left(9 - (\xi - x)^2\right)\frac{1}{2}\cos(x) dx$$
$$= -\frac{7\cos(\xi - 3)}{288} - \frac{7\sin(\xi - 3)}{864} - \frac{7\left(\pi^2 - 4\xi\pi + 4\xi^2 - 44\right)}{6912} \quad , \; 3 - \frac{\pi}{2} \le \xi \le 3 + \frac{\pi}{2}$$

The out-crossing intensity of $\xi = 2$ is then approximated by

$$\nu_X(2) \le -\frac{7\cos(2-3)}{288} - \frac{7\sin(2-3)}{864} - \frac{7\left(\pi^2 - 8\pi - 28\right)}{6912} = 0.0375$$

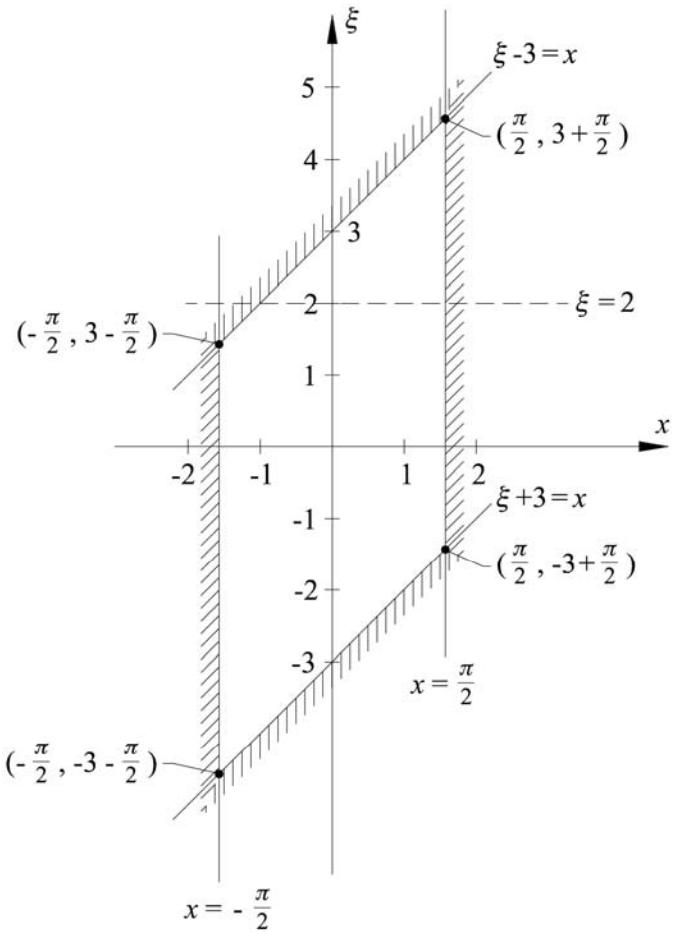

Figure 4. Integration domain for example 1.

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Example 2

Consider two independent, stationary Gaussian stochastic processes $X_1(t)$ and $X_2(t)$ with statistical parameters:

$$\mu_{X_1} = 1 \qquad \sigma_{X_1} = 3 \qquad \sigma_{\dot{X}_1} = 2$$
$$\mu_{X_2} = 0 \qquad \sigma_{X_2} = 2 \qquad \sigma_{\dot{X}_2} = 1$$

Further, the combined process is:

$$X(t) = 2X_1(t) + X_2(t)$$

Since $X_1(t)$ and $X_2(t)$ are Gaussian also $X(t)$ is Gaussian with statistical parameters:

$$\mu_X = 2 \cdot 1 + 0 = 2$$
$$\sigma_X = \sqrt{2^2 \cdot 3^2 + 2^2} = 6.325$$
$$\sigma_{\dot{X}} = \sqrt{2^2 \cdot 2^2 + 1^2} = 4.123$$

The out-crossing intensities of $X(t)$ with the critical barrier $\xi = 12$ is

$$\nu^+(12) = \frac{4.123}{2\pi\, 6.325} \exp\left(-\frac{1}{2}\left(\frac{12-2}{6.325} \right)^2 \right) = 0.0297$$

Note, that in this case with Gaussian processes and a linear combination of the processes, the out-crossing intensity can be determined without evaluation of the integrals in (7).

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3 The Ferry Borges-Castanheta load model

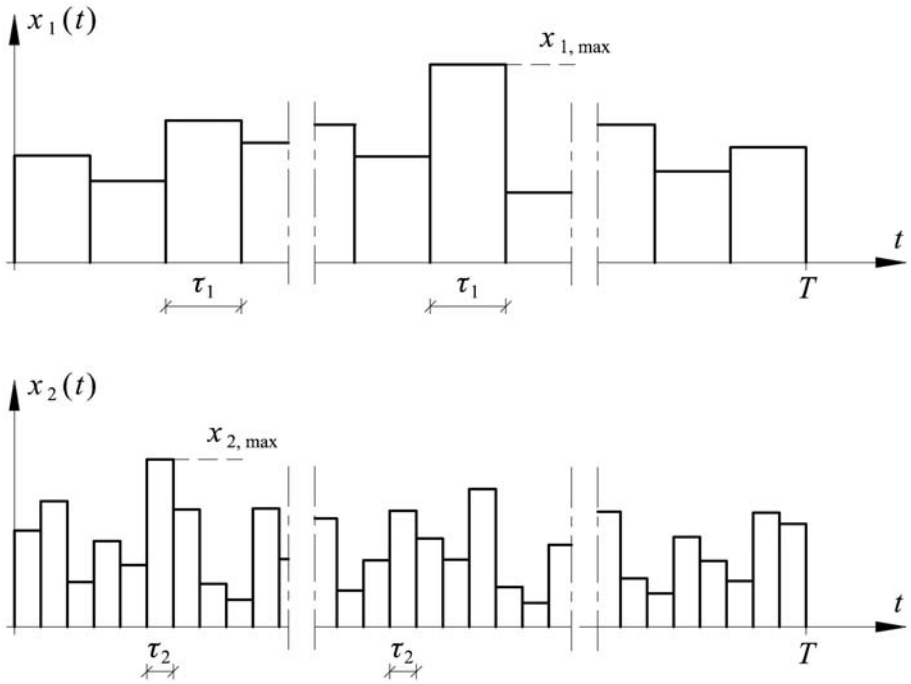

Figure 5. The Ferry Borges-Castanheta load model.

In the Ferry Borges-Castanheta load model, [3] it is assumed that the variable load processes in the load combination problem can be approximated by 'square-wave' processes, see figure 5 for the case of two load processes $X_1(t)$ and $X_2(t)$. The following description is based on ISO [2].

It is further assumed that:

- $\{X_1(t)\}$ and $\{X_2(t)\}$ are stationary, ergodic stochastic processes
- All intervals τ_1 with constant load for load process $\{X_1(t)\}$ are equal and all intervals τ_2 for load process $\{X_2(t)\}$ are equal.
- $\tau_1 \geq \tau_2$
- $r_1 = T/\tau_1$ and $r_2 = T/\tau_2$ are integers
- r_2/r_1 is an integer
- X_1 and X_2 are constant during each interval τ_1 and τ_2, respectively
- The values of X_1 for different intervals are mutually independent. The same holds for X_2.
- X_1 and X_2 are independent.

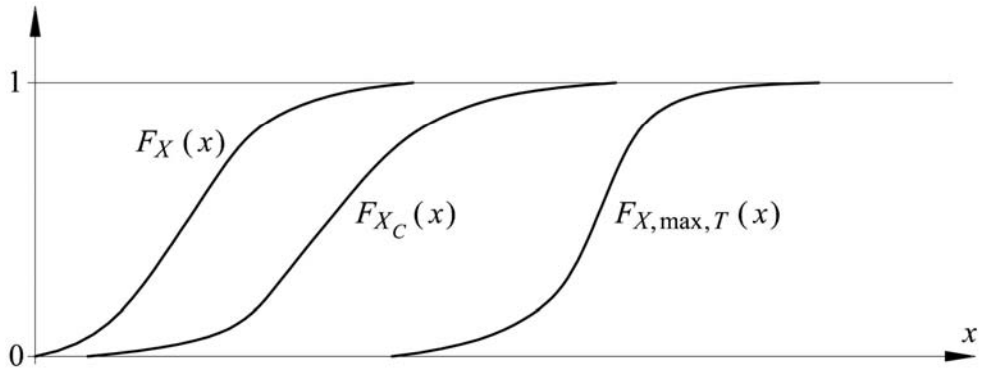

Figure 6. Distribution functions for load combination problems.

For each process (load) three different stochastic variables are defined:

1. An arbitrary point in time stochastic variable for load no j: X_j^* with distribution function $F_{X_j}(x_j)$.
2. The maximum value for load no j: $X_{j,\max,T}$ during the reference time T with the distribution function: $F_{X_j,\max,T}(x_j) = [F_{X_j}(x_j)]^{r_j}$.
3. For each load a combination load is defined:
   - for X_2 the ***combination load*** is denoted X_{2C} and is equal to the maximum value occurring during the interval τ_1. The distribution function becomes: $F_{X_{2C}}(x_2) = [F_{X_2}(x_2)]^{r_2/r_1}$
   - for X_1 the ***combination load*** is denoted X_{1C} and is equal to the arbitrary point in time variable X_1^*. The distribution function thus becomes: $F_{X_{1C}}(x_1) = F_{X_1}(x_1)$.

These stochastic variables and quantiles of them can be used in reliability analyses and in defining design values if the partial safety factor method is used, see next two sections. The distribution functions are illustrated in figure 6.

Note that the combination load X_C is not the same as the characteristic value of a stochastic variable, x_c.

4 The Turkstra rule

As can be seen from section 2 it is generally very difficult to obtain an exact / analytic expression for the out-crossing intensity (and the probability of failure). Therefore a number of approximations have been suggested. In practice (and in codes) the Turkstra rule, [4] is often used in load combination problems to estimate the probability of failure and to establish design values to be checked in a level 1 safety approach.

Instead of $X_{\max,T} = \max_T \{X_1(t) + X_2(t) + ... + X_r(t)\}$ the following r stochastic variables obtained from r different combinations of the loads (or load effects) are considered:

$$\begin{aligned} Z_1 &= \max_T \{X_1(t)\} + X_2(t^*) + ... + X_r(t^*) \\ Z_2 &= X_1(t^*) + \max_T \{X_2(t)\} + ... + X_r(t^*) \\ &\vdots \\ Z_r &= X_1(t^*) + X_2(t^*) + ... + \max_T \{X_r(t)\} \end{aligned} \tag{10}$$

where t^* is an arbitrary point in time and $\max_T \{X_j(t)\}$ is the maximum value of $X_j(t)$ in the time interval $[0,T]$. $X_{\max,T}$ is then approximated by $X_{\max,T} = \max\{Z_1, Z_2, ..., Z_r\}$. This stochastic variable can be used in evaluation of the reliability of the structure considered.

If the partial safety factor method is used the Turkstra rule is often applied together with the Ferry Borges-Castanheta load model.

It is assumed that the load effect (e.g. a cross sectional force) can be written as a function of two (or more loads):

$$S = S(X_1, X_2) \tag{11}$$

According to the Turkstra rule two load combinations are considered:

1. X_1 is dominating and X_2 is the non-dominant load. The resulting load effect is $S = S(X_{1,\max,T}, X_{2C})$
2. X_2 is dominating and X_1 is the non-dominant load. The resulting load effect is $S = S(X_{1C}, X_{2,\max,T})$

In a level 1 code, two combinations for the design load effect corresponding to these two combinations are generally considered:

1. $S_{d,1} = S(x_{d1,1},\ x_{d1,2}) = S(\gamma_1 x_{c1},\ \gamma_2 \psi_2 x_{c2})$
2. $S_{d,2} = S(x_{d2,1},\ x_{d2,2}) = S(\gamma_1 \psi_1 x_{c1},\ \gamma_2 x_{c2})$

where γ_1, γ_2 are partial safety factors, ψ_1, ψ_2 are load combination factors and x_{c1}, x_{c2} are characteristic values (usually 98 % quantiles in the distribution function for the annual maximum load which is $F_{X_j,\max,T=1\text{year}}(x_j), j = 1,2$.

Note that this definition of load combination factors are different from that used in the Danish codes [5] where design values for non-dominating variable loads are calculated as $\psi_j x_{cj}$ i.e. without the partial safety factor γ_j.

The design load effect to be used in design checking is:

$$S_d = \max\{S_{d,1}, S_{d,2}\} \tag{12}$$

In the following it is shown how the above model can be used in a reliability analysis and how it can be used to determine load combination factors.

4.1 Reliability analysis:

Load combination 1: X_1 and X_2 are modeled as stochastic variables with distribution functions $F_{X_1,\max,T}(x_1)$ and $F_{X_{2C}}(x_2)$. A reliability analysis is performed using a given limit state function and the load effect modeled by (11). The result is a reliability index β_1 and design-point values: $x_{1,1}^*$ and $x_{1,2}^*$.

Load combination 2: Similarly, X_1 and X_2 are modeled as stochastic variables with distribution functions $F_{X_{1C}}(x_1)$ and $F_{X_2,\max,T}(x_2)$. A reliability analysis is performed using a given limit state function and the load effect modeled by (11). The result is a reliability index β_2 and design-point values: $x_{2,1}^*$ and $x_{2,2}^*$.

The two cases can be considered as two failure modes and a series system reliability index can be estimated.

Note that if there is a non-zero probability that the load X_j is equal to zero during some of the time intervals τ_j, then the distribution function for X_j has to include this probability, i.e.

$$F_{X_j}(x) = \begin{cases} p_j + (1-p_j)F_{X_j}^*(x) & \text{for } x > 0 \\ p_j & \text{for } x = 0 \end{cases} \tag{13}$$

where $F_{X_j}^*(x)$ is the distribution function for X_j given $x_j > 0$.

4.2 Level 1 approach:

In this subsection the level I approach is considered. First it is described how the load combination factors can be estimated directly on the basis of reliability analyses. Next, the computationally simpler design-value format is used to determine load combination factors.

It is assumed that a target reliability index β^t and characteristic values are given. The partial safety factors and load combination factors can now be estimated.

Load combination factors calculated from reliability analyses: reliability analyses are performed such that for both load combinations the reliability index becomes equal to β^t. From the corre-

sponding design-point values the partial safety factors and load combination factors can be estimated:

$$\gamma_1 = \frac{x_{1,1}^*}{x_{c1}} \qquad \gamma_2 = \frac{x_{2,2}^*}{x_{c2}} \qquad \psi_1 = \frac{x_{2,1}^*}{x_{1,1}^*} = \frac{x_{2,1}^*}{\gamma_1 x_{c1}} \qquad \psi_2 = \frac{x_{1,2}^*}{x_{2,2}^*} = \frac{x_{1,2}^*}{\gamma_2 x_{c2}} \tag{14}$$

Load combination factors calculated from design-value format: In Eurocodes, Basis of Design [5] and ISO [2] it is recommended that design values of dominating variable loads are determined from:

$$F_{X_j,\max,T}(x_{dj,j}) = \Phi(-\alpha_S \beta^t) \tag{15}$$

where α_S=-0.7 is recommended. If the Turkstra rule is applied design values for non-dominating loads can be determined from:

$$F_{X_{1C}}(x_{d2,1}) = F_{X_{1C}}(x_{1C}) = \Phi(-0.4\alpha_S \beta^t) \tag{16}$$

$$F_{X_{2C}}(x_{d1,2}) = F_{X_{2C}}(x_{2C}) = \Phi(-0.4\alpha_S \beta^t) \tag{17}$$

where the factor 0.4 is chosen as a reasonable value. Partial safety factors and load combination factors can then be obtained from (note the similarity with (14):

$$\gamma_1 = \frac{x_{d1,1}}{x_{c1}} \qquad \gamma_2 = \frac{x_{d2,2}}{x_{c2}} \qquad \psi_1 = \frac{x_{d2,1}}{x_{d1,1}} = \frac{x_{d2,1}}{\gamma_1 x_{c1}} \qquad \psi_2 = \frac{x_{d1,2}}{x_{d2,2}} = \frac{x_{d1,2}}{\gamma_2 x_{c2}} \tag{18}$$

Example 3

It is assumed that X_1 and X_2 are modeled as Gumbel distributed stochastic variables with statistical parameters:

Expected values: μ_1 and μ_2
Coefficients of variation: V_1 and V_2 with a reference period $T = 1$ year
Number of repetitions in reference time $T = 1$ year: r_1 and r_2

Characteristic values are assumed to be 98% quantiles in the distribution functions for annual maximum loads. The partial safety factors are determined from:

$$\gamma_j = \frac{x_{dj,j}}{x_{cj}} = \frac{F^{-1}_{X_j,\max,T}\left(\Phi\left(0.7\beta^t\right)\right)}{F^{-1}_{X_j,\max,T}\left(0.98\right)} = \frac{1 - V_j \frac{\sqrt{6}}{\pi}\left\{0.577 + \ln\left(-\ln\left[\Phi\left(0.7\beta^t\right)\right]\right)\right\}}{1 - V_j \frac{\sqrt{6}}{\pi}\left\{0.577 + \ln\left(-\ln\left[0.98\right]\right)\right\}} \tag{19}$$

The load combination factors are determined from:

$$\psi_1 = \frac{x_{1C}}{x_{d1,1}} = \frac{F^{-1}_{X_{1C}}\left(\Phi\left(0.28\beta^t\right)\right)}{F^{-1}_{X_{1,\max,T}}\left(\Phi\left(0.7\beta^t\right)\right)} = \frac{F^{-1}_{X_{1,\max,T}}\left(\Phi\left(0.28\beta^t\right)^{r_1}\right)}{F^{-1}_{X_{1,\max,T}}\left(\Phi\left(0.7\beta^t\right)\right)} = \frac{1 - V_1 \frac{\sqrt{6}}{\pi}\left\{0.577 + \ln\left(-\ln\left[\Phi\left(0.28\beta^t\right)\right]\right) + \ln r_1\right\}}{1 - V_1 \frac{\sqrt{6}}{\pi}\left\{0.577 + \ln\left(-\ln\left[\Phi\left(0.7\beta^t\right)\right]\right)\right\}} \tag{20}$$

$$\psi_2 = \frac{x_{2C}}{x_{d\,2,2}} = \frac{F^{-1}_{X_{2C}}\left(\Phi\left(0.28\beta^t\right)\right)}{F^{-1}_{X_{2,\max,T}}\left(\Phi\left(0.7\beta^t\right)\right)} = \frac{F^{-1}_{X_{2,\max,T}}\left(\Phi\left(0.28\beta^t\right)^{r_1}\right)}{F^{-1}_{X_{2,\max,T}}\left(\Phi\left(0.7\beta^t\right)\right)} = \frac{1-V_2\frac{\sqrt{6}}{\pi}\left\{0.577+\ln\left(-\ln\left[\Phi\left(0.28\beta^t\right)\right]\right)+\ln r_1\right\}}{1-V_2\frac{\sqrt{6}}{\pi}\left\{0.577+\ln\left(-\ln\left[\Phi\left(0.7\beta^t\right)\right]\right)\right\}} \tag{21}$$

* * * * * * * *

Example 4. Load combination factors for imposed loads and wind load

The reference period is assumed to be $T = 1$ year and the corresponding target reliability index β^t =4.3.

	Load	Distribution	Coefficient of Variation	τ_j	r_j
Q_1	Inposed	Gumbel	0.2	τ_1 =0.5 years	r_1 =2
Q_2	Wind	Gumbel	0.4	τ_2 =1 day	r_2 =360

Table 1. Statistical data

If the design-value format is used (18)-(21) give the partial safety factors and load combination factors in table 2.

Load	Characteristic value	Partial safety factor	Load combination factor
Imposed	$q_{c1} = 1.52\mu_1$	γ_1 =1.28	ψ_1 =0.58
Wind	$q_{c2} = 2.04\mu_2$	γ_2 =1.42	ψ_2 =0.44

Table 2. Partial safety factors and load combination factors with design-value method.

Now, consider the following limit state function:

$$g = zR - (0.4G + 0.6Q_1 + 0.3Q_2)$$

where z is a design parameter, R is a Lognormal distributed strength with coefficient of variation = 0.15 and G is a Normal distributed permanent load with expected value 1 and coefficient of variation = 0.1. Q_1 and Q_2 have expected values = 1 and other statistical data as in table 1.

In load combination 1 (imposed load dominating) Q_1 and Q_2 have the distribution functions $F_{Q_{1,\max}}(q_1)$ and $F_{Q_{2C}}(q_2) = \left(F_{Q_{2,\max}}(q_2)\right)^{1/r_1}$. In load combination 2 (wind load dominating) Q_1 and Q_2 have the distribution functions $F_{Q_{1C}}(q_1) = \left(F_{Q_{1,\max}}(q_1)\right)^{1/r_1}$ and $F_{Q_{2,\max}}(q_2)$. $F_{Q_{1,\max}}$ and $F_{Q_{2,\max}}$ refer to the statistical data in table 1. The design values for the two load combinations corresponding to β^t =4.3 and the associated load combination factors obtained by (14) are shown in table 3. The load combination factors are rather large, whereas compared to the results in [6], example 1 and 2 the partial safety factors are small. Therefore, it is of interest to calculate modified load combination factors where the partial safety factors in [6], example 1 are used, see table 4 below. The load combination factors in table 4 are larger but comparable to those obtained by the design-value method, see table 2.

Load	Load combination 1	Load combination 2	Partial safety factor	Load combination factor
Imposed	$q_{1,1}^{*}$=1.62	$q_{2,1}^{*}$=1.51	$\gamma_1 = \frac{q_{1,1}^{*}}{q_{c1}}$=1.07	$\psi_1 = \frac{q_{2,1}^{*}}{q_{1,1}^{*}}$=0.93
Wind	$q_{1,2}^{*}$=2.03	$q_{2,2}^{*}$=2.25	$\gamma_2 = \frac{q_{2,2}^{*}}{q_{c2}}$=1.10	$\psi_2 = \frac{q_{1,2}^{*}}{q_{2,2}^{*}}$=0.90

Table 3. Partial safety factors and load combination factors with reliability method.

Load	Partial safety factor	Load combination factor
Imposed	γ_1=1.43	$\psi_1 = \frac{q_{2,1}^{*}}{\gamma_1 q_{c1}}$=0.69
Wind	γ_2=1.84	$\psi_1 = \frac{q_{1,2}^{*}}{\gamma_2 q_{c2}}$=0.54

Table 4. Modified partial safety factors and load combination factors with reliability method.

* * * * * * * *

Example 5. Load combination factors for snow and wind loads

The reference period is assumed to be $T = 1$ year and the corresponding target reliability index β^t=4.3. The statistical data are shown in table 5. Note that snow load only occurs in the period November – March. The same limit state function as in example 1 is used.

	Load	Distribution	Coefficient of Variation	τ_j	r_j
Q_1	Snow	Gumbel	0.4	τ_1=15 days (nov-mar)	r_1=10
Q_2	Wind	Gumbel	0.4	τ_2=1 day	r_2=360

Table 5. Statistical data.

In load combination 1 (snow load dominating) Q_1 and Q_2 have the distribution functions $F_{Q_{1,\max}}(q_1)$ and $F_{Q_{2C}}(q_2) = \left(F_{Q_{2,\max}}(q_2)\right)^{1/24}$.

In load combination 2 (wind load dominating) Q_1 and Q_2 have the distribution functions $F_{Q_{1C}}(q_1) = \left(F_{Q_{1,\max}}(q_1)\right)^{1/10}$ and $\left(F_{Q_{2,\max}}(q_2)\right)^{150/360}$.

The design values for the two load combinations corresponding to β^t=4.3 and the associated load combination factors obtained by (14) are shown in table 6. Note that the load combination factor for snow is much larger than the load combination factor for wind.

Load	Load combination 1	Load combination 2	Load combination factor
Snow	$q_{1,1}^{*}$=3.70	$q_{2,1}^{*}$=3.06	$\psi_1 = \frac{q_{2,1}^{*}}{q_{1,1}^{*}}$=0.83
Wind	$q_{1,2}^{*}$=0.23	$q_{2,2}^{*}$=0.95	$\psi_2 = \frac{q_{1,2}^{*}}{q_{2,2}^{*}}$=0.24

Table 6. Load combination factors with reliability method.

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6 References

[1] Thoft-Christensen, P. and M.J. Baker: Structural Reliability Theory and Its Applications. Springer Verlag, 1982.

[2] ISO 2394. General principles on reliability for structures. 1998.

[3] Ferry Borges, J. & M. Castanheta: Structural Safety. 2nd edition. Laboratorio Nacional de Engenharia Civil, Lisbon, 1972.

[4] Turkstra, C.J. & H.O. Madsen: Load combinations in codified structural design. J. Struct. Div., ASCE, Vol. 106, No. St. 12, 1980.

[5] DS409:1998 – Code of practice for the safety of structures. Dansk Standard, 1999.

[6] Sørensen, J.D.: Structural reliability: Level 1 approaches. Note, Institute of Building Technology and Structural Engineering, Aalborg University, April 2000.

Note 11: Example: Fatigue / Reliability-Based Inspection Planning

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1 Introduction

This note gives an introduction to the main steps in a probabilistic fatigue analysis and inspection planning for welded joints. As example tubular joints in fixed offshore platforms of the steel jacket type are considered, but the probabilistic modeling is general and can be used for other types of structures. Initially the fatigue loading is described, here as an example wave loading. Next stress analysis is considered. Based on a spectral analysis the stress spectra for critical points (hot spots) in the joint can then be determined using an influence matrix approach. From the stress spectra stress ranges can be determined and the number of stress cycles can be estimated, e.g. by the Rainflow counting method.

Two models for the fatigue strength are described, namely the classical SN approach and a fracture mechanics approach where the size of the crack is compared with a critical crack length, e.g. the thickness of the tubular member. The basic steps in a reliability analysis with respect to fatigue and in reliability-based inspection planning is described and illustrated. Part of this note is based on EFP [1].

2 Fatigue loading

The most important load for fatigue failure of welded offshore structures is wave loading. Current is insignificant because the time variation is very slow compared with wave loading. The fatigue load due to wind excitation can contribute by 10-15 % of the total fatigue load but usually it is of minor importance. In this section we therefore concentrate on wave loading.

The statistical properties of sea waves are most often modeled using so-called short-term sea states. The duration of a sea state is normally taken as 3 hours. Within each sea state the wave elevation is assumed modeled by a stationary, Gaussian stochastic process $\{\eta(t)\}$. The wave elevation $\eta(t)$ is assumed Normal distributed with expected value $\mu_\eta = 0$ and standard deviation σ_η. The auto-spectrum of $\{\eta(t)\}$ can be modeled by a number of different spectra, e.g.

- Pierson-Moskowitz
- JONSWAP

The Pierson-Moskowitz spectrum has the following form

$$S_{\eta\eta}(\omega) = \frac{4\pi^3 H_S^2}{T_Z \omega^5} \exp\left(-16\pi^3 \left(\frac{1}{T_Z \omega} \right)^4 \right) \tag{1}$$

where ω is the cyclical frequency, H_S is the significant wave height and T_Z is the zero up-crossing period. The parameters H_S and T_Z are constant within each sea state. In figure 1 a typical wave spectrum is shown.

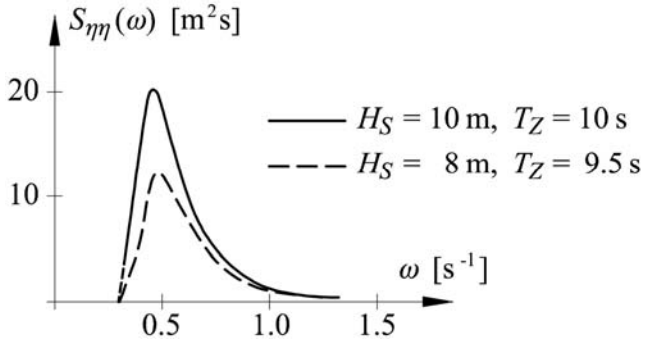


Figure 1. Pierson-Moskowitz spectrum.

T_Z [sec] / H_S [m]	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	10-11	11-12
10.5-11.0											+	
10.0-10.5											+	+
9.5-10.0										1	+	
9.0-9.5										1	+	
8.5-9.0									+	1	+	
8.0-8.5									1	1	1	+
7.5-8.0									1	2	1	+
7.0-7.5									2	2	1	+
6.5-7.0								+	2	3	1	+
6.0-6.5								1	5	4	2	1
5.5-6.0								3	7	5	1	1
5.0-5.5							1	9	11	5	2	1
4.5-5.0							3	18	13	7	2	1
4.0-4.5						1	11	25	10	7	2	1
3.5-4.0						3	22	30	8	4	1	
3.0-3.5					1	20	35	25	5	3	2	
2.5-3.0					3	51	42	18	3	1	1	
2.0-2.5					15	70	30	15	3	1		
1.5-2.0				5	71	58	20	10	2	1		
1.0-1.5				23	91	38	10	3	1			
0.5-1.0				7	32	16	6	3	1			
0.0-0.5				1	1	2	1					

Table 1. Representative scatter diagram for central North Sea. Numbers are probabilities in parts per thousand. +: 0.0<probability<0.0005.

Long-term observations of the sea are usually performed by observing the sea surface for 20 minutes every third hour. For each observation H_S and T_Z are estimated. The relative number of pairs of H_S and T_Z can be represented in so-called scatter diagrams, see table 1. Based on the observations it is also possible to fit the long-term distribution functions for H_S, e.g. by a Weibull distribution

$$F_{H_S}(h) = 1 - \exp\left(-\left(\frac{h - H_0}{H_C - H_0}\right)^{\gamma}\right) \quad , \; h \geq H_0 \tag{2}$$

where γ, H_0 and H_C are parameters.

From table 1 it is seen that H_S and T_Z are dependent. Based on the observations a long-term distribution function for T_Z given H_S can be fitted, for example by a two-parameter Weibull distribution

$$F_{T_Z|H_S}(t_Z \mid h_S) = 1 - \exp\left(-\left(\frac{t_Z}{k_1(h_S)}\right)^{k_2(h_S)}\right) \tag{3}$$

where $k_1(h_S)$ and $k_2(h_S)$ are functions of h_S. In [2] the following models are obtained based on data from the Northern North Sea (h_S in meters).

$$k_1(h_S) = 6.05\exp(0.07h_S) \tag{4}$$
$$k_2(h_S) = 2.35\exp(0.21h_S) \tag{5}$$

Generally the distribution functions for H_S and T_Z are dependent on the wave direction Θ. If eight directions (N, NE, E, SE, S, SW, W, NW) with probabilities of occurrence P_{Θ_i} $,i = 1,2,...,8$ are used, then the distribution function for H_S is written according to (2)

$$F_{H_S}(h,\Theta_i) = 1 - \exp\left(-\left(\frac{h - H_{0,i}}{H_{C,i} - H_{0,i}}\right)^{\gamma_i}\right) \qquad h \geq H_{0,i} \quad , \quad i = 1,2,...,8 \tag{6}$$

The parameters in (3)-(5) can be considered independent of the direction. Together with the parameters in (6) for the 8 directions the probabilities P_{Θ_i} $,i = 1,2,...,8$ for waves in the eight directions constitute the data for the long-term stochastic model.

Measurements of the directional characteristics of the wave elevation show a variation of both the mean direction and a spread with frequency. The spreading of the waves can result in a significant reduction in the wave loading. The directional spectra are assumed modeled by

$$S_{\eta\eta}(\omega,\Theta) = S_{\eta\eta}(\omega)\Psi(\Theta) \tag{7}$$

where the spreading function $\Psi(\Theta)$ e.g. can be modeled by

$$\Psi(\Theta) = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(s+1)}{\Gamma(s+0.5)}\left[\cos\left(0.5\left(\Theta - \overline{\Theta}\right)\right)\right]^{2s} \tag{8}$$

Γ is the Gamma-function, s is a constant and $\overline{\Theta}$ is the mean direction. Usually s=1 is used in practice.

3 Stress Analysis

Above it is described how the wave load can be described by the spectral density $S_{\eta\eta}(\omega)$ and the distribution functions $F_{H_S}(h)$ and $F_{T_Z|H_S}(t_Z \mid h_S)$. Next, it is of interest to calculate the spectral density $S_{\sigma\sigma}(\omega)$ for the stresses in a critical hot spot. One way to calculate $S_{\sigma\sigma}(\omega)$ is to perform a stochastic response analysis to find the cross-spectral density functions $S_{S_kS_l}(\omega)$ for the cross-sectional forces in a given structural element and then calculate $S_{\sigma\sigma}(\omega)$ as described below. Details of such an analysis can be found in e.g. Langen & Sigbjørnson [3].

The cross-spectral density functions $S_{S_kS_l}(\omega)$ can be obtained from

$$S_{S_kS_l}(\omega) = H_{F_i\eta}(\omega)H^*_{F_j\eta}(\omega)S_{\eta\eta}(\omega) \tag{9}$$

where $H_{F_i\eta}(\omega)$ is the transfer function from wave elevation to cross-sectional force no i and * denotes complex conjugate.

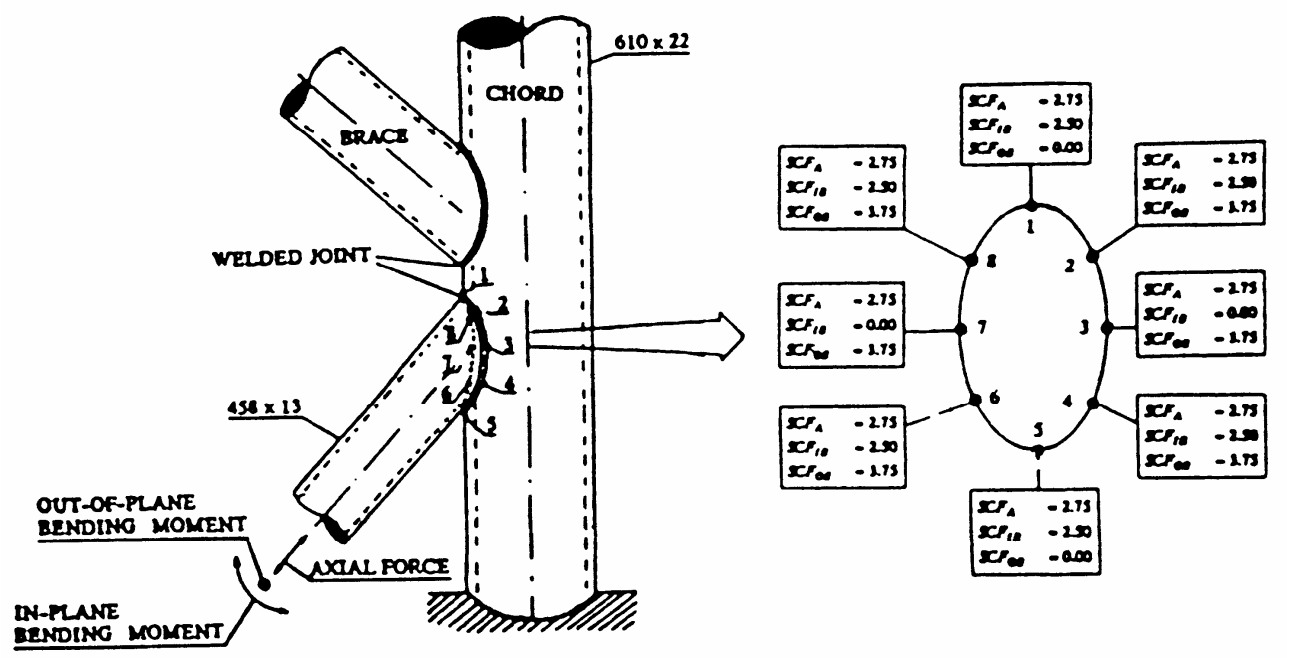


Figure 2. Calculation of influence coefficients (from [4])

In order to illustrate the procedure the K-joint in figure 2 is considered. The cross-sectional forces on the joint can be determined using a beam model of the structure. These forces will be in equilibrium. A local stress analysis of the joint can therefore be performed by fixing one of the cross-sections (see figure 2) and applying the cross-sectional forces from the beam model as external loads on the joint. The cross-sections where the forces are determined should be located in some distance from the joint in order to be able to apply the cross-sectional loads as distributed line loads on a shell element model of the joint, i.e. the stress distribution is unaffected by the joint.

The local fatigue inducing hot spot stress σ in a critical point, namely the principal stress perpendicular to the crack, see figure 3 is estimated by

$$\sigma = \sum_{k=1}^{N} \alpha_k S_k \tag{10}$$

where N is the number of cross-sectional forces applied as loads to the joint (=18 in figure 4 where each cross-section has 6 degrees of freedom). α_k is the coefficient of influence giving the stress in the critical point for a unit load S_k.

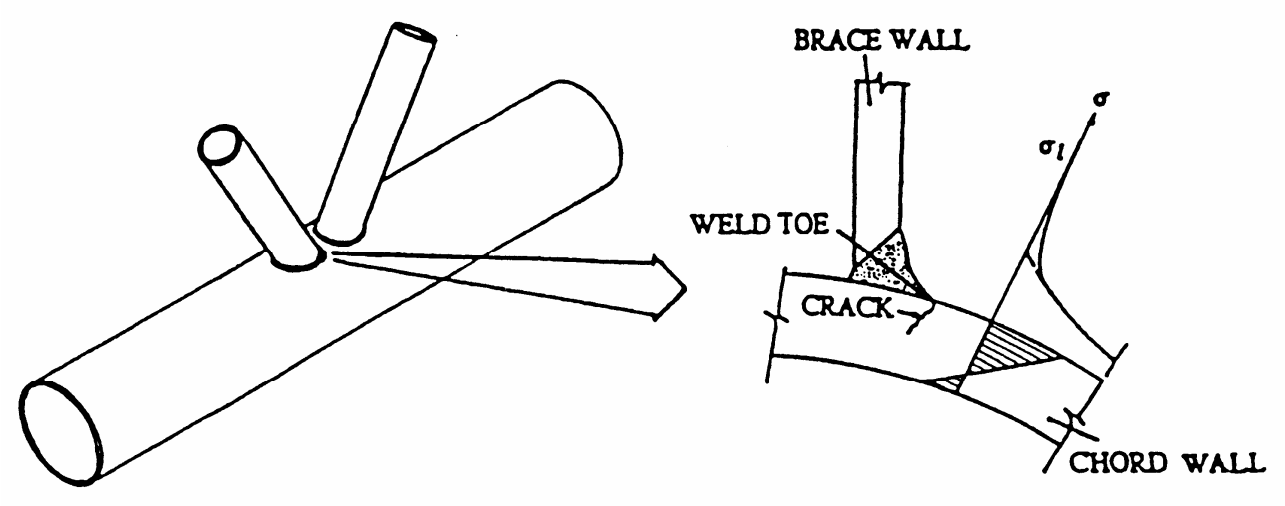


Figure 3. Stress variation through thickness (from [4]).

Based on the cross spectral densities for the cross-sectional forces the auto spectral density of the fatigue hot spot stress σ can be determined from, see also figure 4

$$S_{\sigma\sigma}(\omega) = \sum_{k=1}^{N}\sum_{l=1}^{N} \alpha_k \alpha_l S_{S_k S_l}(\omega) = \sum_{k=1}^{N}\sum_{l=1}^{N} \alpha_k \alpha_l H_{F_k \eta}(\omega) H^*_{F_l \eta}(\omega) S_{\eta\eta}(\omega) \tag{11}$$

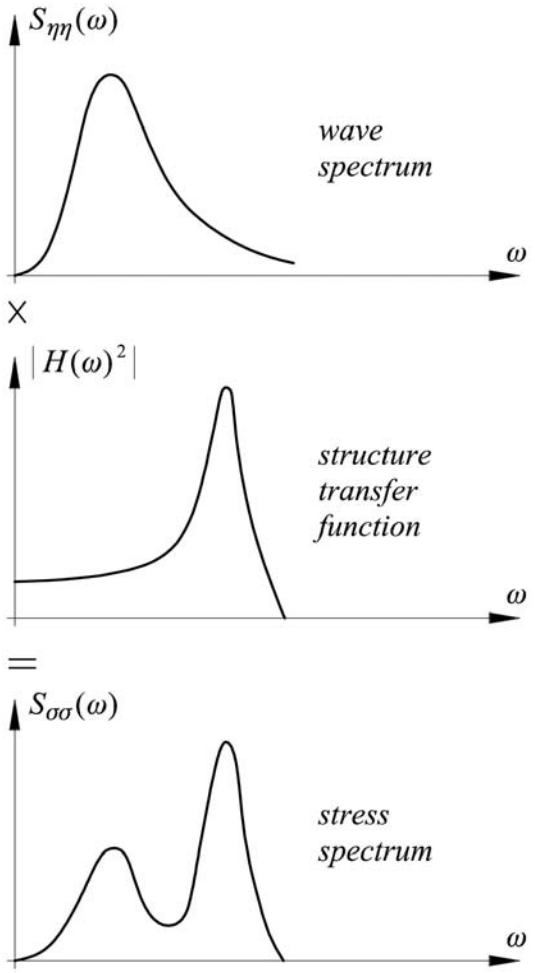


Figure 4. Illustration of calculation of spectral density of hot-spot stresses.

For computational reasons it is more convenient to calculate the cross spectral densities $S_{S_k S_l}(\omega)$ of the load effects first. Next, when the auto-spectral density of a stress is required this can be calculated using (11). If the result of the spectral analysis had been the auto-spectral density of the fatigue stress, a new spectral analysis would be required whenever the fatigue stress in a new location is needed. This would be rather unfortunate, as a full spectral analysis is very time consuming.

The location of the most critical hot spots is usually not known in advance. Therefore 8 (or 12) points located as shown in figure 2 are investigated. The auto-spectral density functions are determined for each location and a fatigue analysis is performed as described in the following sections. This is done for the 8 points in the brace and for the corresponding 8 points on the intersection in the chord.

4 Fatigue strength

4.1 SN approach

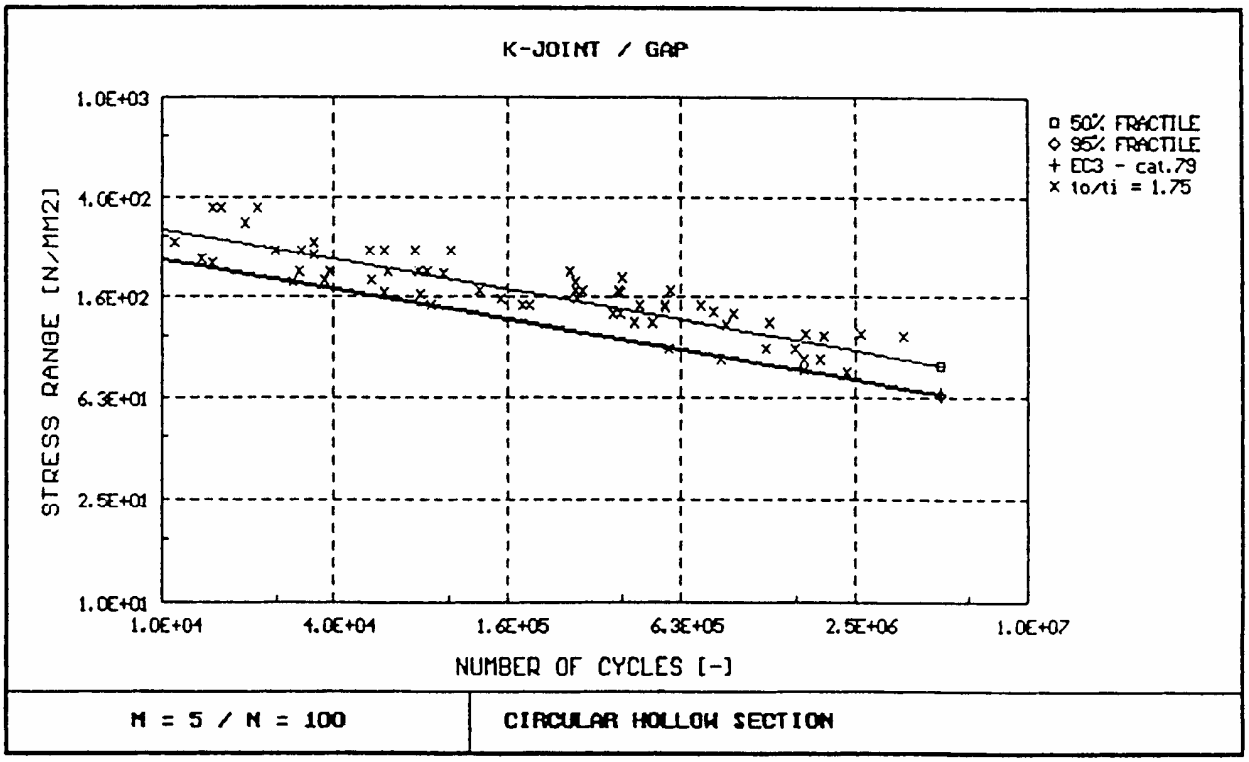


Figure 5. Experimental SN results for circular K-joint. From EC3 background document [7].

Assuming that the fatigue damage is accumulated linearly in an interaction free manner the damage accumulation law attributed to Palmgren [5] and Miner [6] can be applied. Failure occurs when the accumulated damage exceeds 1, i.e. the failure criteria is

$$\sum_i \frac{n_i}{N(\Delta\sigma_i)} \geq 1 \tag{12}$$

where n_i is the number of stress cycles at a particular stress range level $\Delta\sigma_i$ and $N(\Delta\sigma_i)$ is the number of constant amplitude stress cycles at that stress range level which leads to failure. The summation in (12) is over the number of different stress range levels. $N(\Delta\sigma_i)$ is usually determined on the basis of experiments and therefore has a random character, see figure 5.

Most often a relationship of the type

$$N(\Delta\sigma) = K\Delta\sigma^{-m} \quad , \ \Delta\sigma > 0 \tag{13}$$

is assumed and the material parameters m and K are fitted to experimentally obtained data.

It is seen from (12) and (13) that a limit state function can be written as

$$g = \Delta - \frac{1}{K}\sum_i n_i \Delta\sigma_i^m \tag{14}$$

or

$$g = \Delta - \frac{1}{K}\sum_i \Delta\sigma_i^m \tag{15}$$

if the summation is over all individual stress range cycles. Δ is a stochastic variable modeling the uncertainty related to application of the Miner rule for linear accumulation of damage from each stress cycle. Usually, Δ has an expected value equal to 1.

4.2 Fracture mechanics approach

The most simple and generally applicable crack growth equation is due to Paris & Erdogan [8]:

$$\frac{da}{dn} = C(\Delta K)^m \quad , \ \Delta K > 0 \tag{16}$$

where a is the crack size (depth), n is the number of stress cycles, ΔK is the stress intensity factor range in a stress cycle. C and m are material constants.

According to (16) a plot of $\log\frac{da}{dn}$ versus $\log(\Delta K)$ should be linear but a typical plot obtained experimentally would be more like the one shown in figure 6.

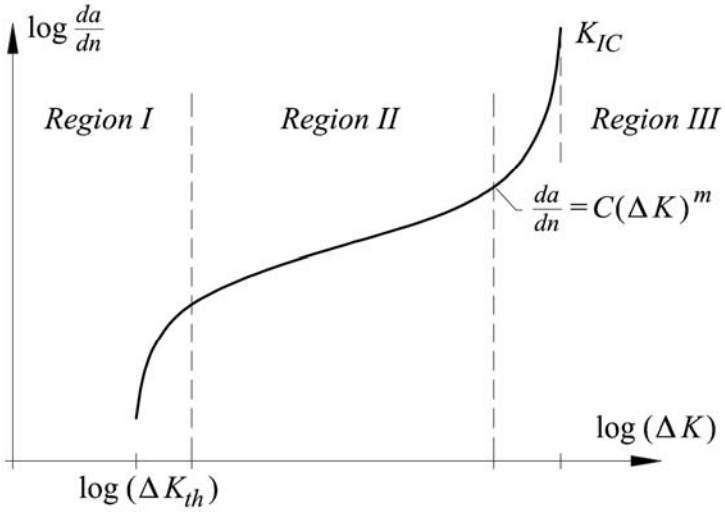


Figure 6. Crack growth rate as function of stress intensity factor.

The agreement between (16) and experiments is seen to be reasonable in region II (almost linear) whereas (16) overestimates the crack growth rate in region I and underestimates the crack growth rate in region III. ΔK_{th} is a threshold stress intensity range below which the crack will not grow. K_{IC} is the value of the stress intensity factor at which the crack becomes unstable and brittle fracture takes place. The stress intensity factor (SIF) can be shown to have the form:

$$\Delta K = Y(a)\Delta\sigma\sqrt{\pi a} \tag{17}$$

where

$Y(a)$ is a geometry function
a is the crack depth/length and
$\Delta\sigma$ is the hot spot fatigue stress range.

ΔK is a factor accounting for a redistribution of the hot spot fatigue stresses. The reason for this redistribution is the influence of the crack itself and other local geometry boundary conditions.

By inserting (17) into (16) we obtain

$$\frac{da}{dn} = CY(a)^m (\Delta\sigma)^m \left(\sqrt{\pi a}\right)^m \tag{18}$$

By integrating (18) we obtain assuming $Y(a)$=1 (infinite plate solution)

$$a(n) = \begin{cases} \left(a_0^{(2-m)/2} + \frac{2-m}{2} C\pi^{m/2} \Delta\sigma^m n \right)^{2/(2-m)} & \text{for } m \neq 2 \\ a_0 \exp\left(C\pi\Delta\sigma^2 n\right) & \text{for } m = 2 \end{cases} \tag{19}$$

where a_0 the initial crack depth/length.

For offshore joints it is generally not sufficient to model cracks as being one-dimensional. This is because both the crack depth a and the crack length c influence the geometry function $Y = Y(a,c)$.

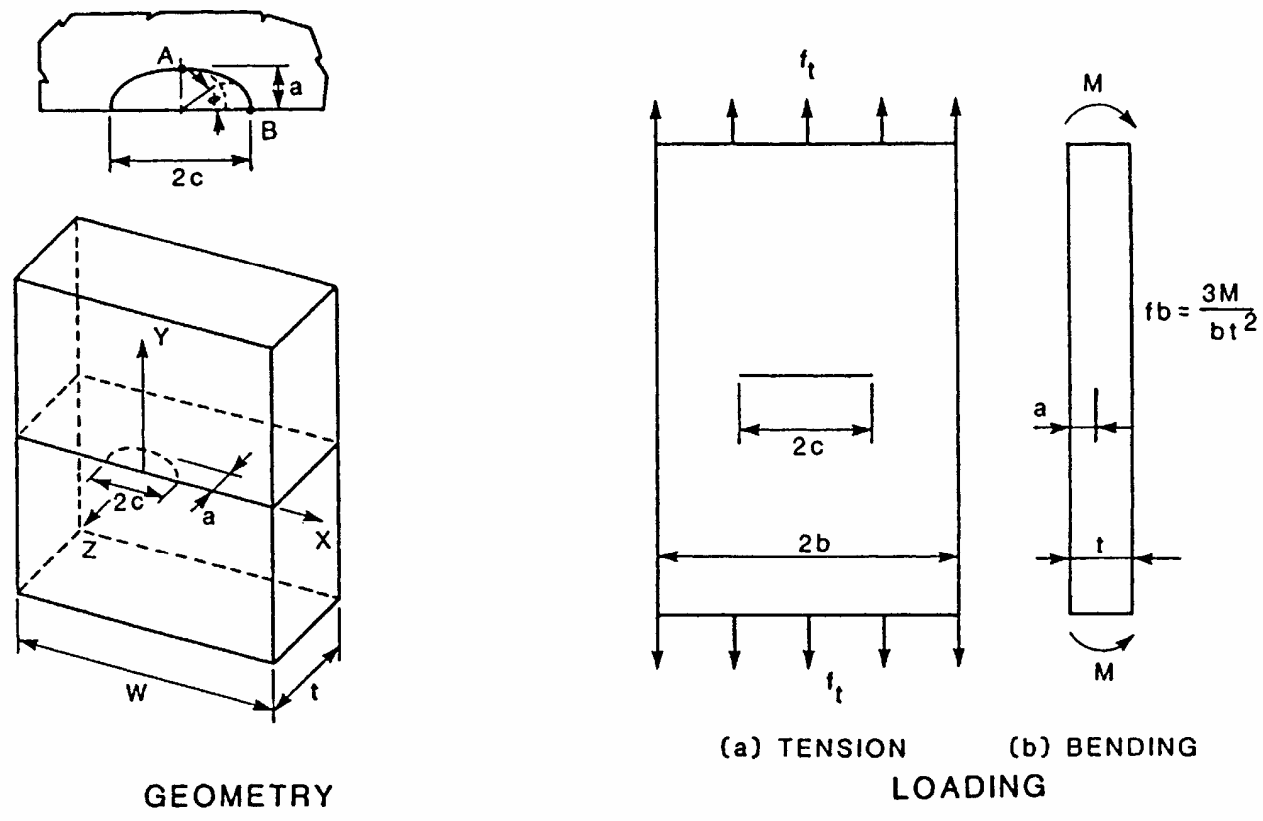


Figure 7. Semi-elliptical surface crack in plate.

Consider a flat plate with a semi-elliptical surface crack under tension or bending fatigue loads, see figure 7. The depth of the crack is a and its length is $2c$, while the thickness of the plate is t. Shang-Xian [9] assumed that the growth rates at the deepest point A and the end point B of the crack follow independently the Paris & Erdogan equations:

$$\frac{da}{dn} = C_a(\Delta K_a)^m \quad \text{with } a(0) = a_0 \tag{20}$$

$$\frac{dc}{dn} = C_c(\Delta K_c)^m \quad \text{with } c(0) = c_0 \tag{21}$$

The variation in the three-dimensional stress field is accounted for by the constants C_a and C_c, while ΔK_a and ΔK_c denote respectively the ranges of the stress intensity factor at the deepest point A and the summit B, see figure 7.

From the two coupled equations, the differential equation of the shape change is derived as

$$\frac{dc}{da} = \frac{C_c}{C_a}\left(\frac{\Delta K_c}{\Delta K_a}\right)^m \quad \text{with } c(a_0) = c_0 \tag{22}$$

together with

$$\frac{dn}{da} = \frac{1}{C_a(\Delta K_a)^m} \quad \text{with } n(a_0) = 0 \tag{23}$$

(22) and (23) are solved numerically.

4.3 Fatigue cycle counting

The statistics of the amplitude or stress-ranges and the corresponding number of stress-ranges in a given time internal must be obtained in order to assess the fatigue damage.

If the fracture mechanics approach (see section 4.2) is used, crack growth is governed by Paris' law. In order to illustrate how fatigue cracks can be counted a one-dimensional crack model is used in the following. Integration of (18) gives for constant stress-range amplitudes $\Delta\sigma$

$$\int_{a_0}^{a_c} \frac{da}{\left(Y(a)\sqrt{\pi a}\right)^m} = C\,\Delta\sigma^m\, n \tag{24}$$

where a_0 and a_c are the initial and the final (critical) crack size, respectively. $Y(a)$ is the geometry function, $\Delta\sigma$ is the constant amplitude stress-range and n is the number of stress cycles during the considered time interval $[0,T]$. A generalization to variable stress-range amplitudes can be obtained by using instead of $\Delta\sigma^m$ the equivalent stress range to power m, $E[\Delta\sigma^m]$

$$E[\Delta\sigma^m] = \frac{1}{n}\sum_{i=1}^{n}\Delta\sigma_i^m \tag{25}$$

neglecting any sequence effects. $\Delta\sigma^m$ is treated as a stochastic variable and $E[.]$ denotes the expectation operation.

If SN curves (see section 4.1) are used to model the fatigue strength it is seen from (15) that also in this case the damage accumulation is governed by (25).

For offshore structures the expectation (25) must be performed for a given sea state because the state term statistics of the stresses are conditional on the sea states. Therefore an expectation over all sea states must be performed:

$$E[\Delta\sigma^m] = \frac{1}{n}\sum_{i=1}^{n}\Delta\sigma_i^m = \sum_{i=1}^{n_S} P(H_{S,i}, T_{Z,i}, \Theta_i)\left(\Delta\sigma(H_{S,i}, T_{Z,i}, \Theta_i)\right)^{\mathrm{m}} \tag{26}$$

where n_S is the number of non-zero boxes in the scatter diagram and $P(H_{S,i}, T_{Z,i}, \Theta_i)$ is the probability having $(H_{S,i}, T_{Z,i}, \Theta_i)$ corresponding to the i th box. $\Delta\sigma(H_{S,i}, T_{Z,i}, \Theta_i)$ is the corresponding stress range.

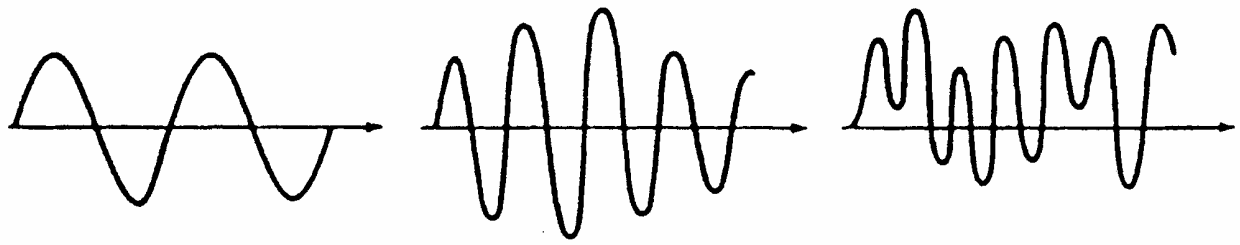


Figure 8. Three examples of stress-variations around the mean stress level.

Figure 8 shows three different sample curves of stress histories. The first case corresponds to constant amplitude loading, where the stress-ranges are the same for all stress cycles. The second case corresponds to a stationary ideal narrow band Gaussian process. Again the stress cycle is easily defined in terms of the stress process between two constitutive up-crossings of the mean value. The third case, which is the more general case with broad banded stress variation, is not quite as obvious. In this case one has to use counting methods.

In section 4.3.1 narrow band stress spectra are considered. Next broad band spectra are considered. In section 4.3.2 and 4.3.3 it is shown how the range counting and the Rainflow counting methods can be used to estimate $E[\Delta\sigma^m]$ and the expected number of stress cycles n.

4.3.1 Narrow band spectra

For a narrow-banded Gaussian process, the stress-ranges are Rayleigh distributed. The mean value in (25) is then

$$E[\Delta\sigma^m] = \left(2\sqrt{2}\right)^m \left(\sqrt{m_0}\right)^m \Gamma\left(1 + m/2\right) \tag{27}$$

where m_0 is the zero'th spectral moment of the stress spectrum, $S_{\sigma\sigma}(\omega)$ ($\sqrt{m_0}$ is the standard deviation of the stress process). Generally, the i th moment is defined by (if $S_{\sigma\sigma}(\omega)$ is a two-sided spectrum):

$$m_i = 2\int_0^\infty \omega^i S_{\sigma\sigma}(\omega) d\omega \tag{28}$$

The number of stress cycles n in the time interval $[0,T]$ is estimated from

$$n = \nu_0 T = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}T \tag{29}$$

where ν_0 is the mean zero-crossing rate and m_2 is given by (28). Note that $\sqrt{m_2}$ is equal to the standard deviation for the derivative process $\{\dot{\sigma}(t)\}$.

4.3.2 Broad band spectra - range counting

In the range counting method a half stress cycle is defined of the difference between successive local extremes, see figure 9. The range counting method uses only local information. Therefore information on larger stress cycles can be lost if small stress reversals are superimposed on the larger stress cycles. The method gives a lower bound on the fatigue damage.

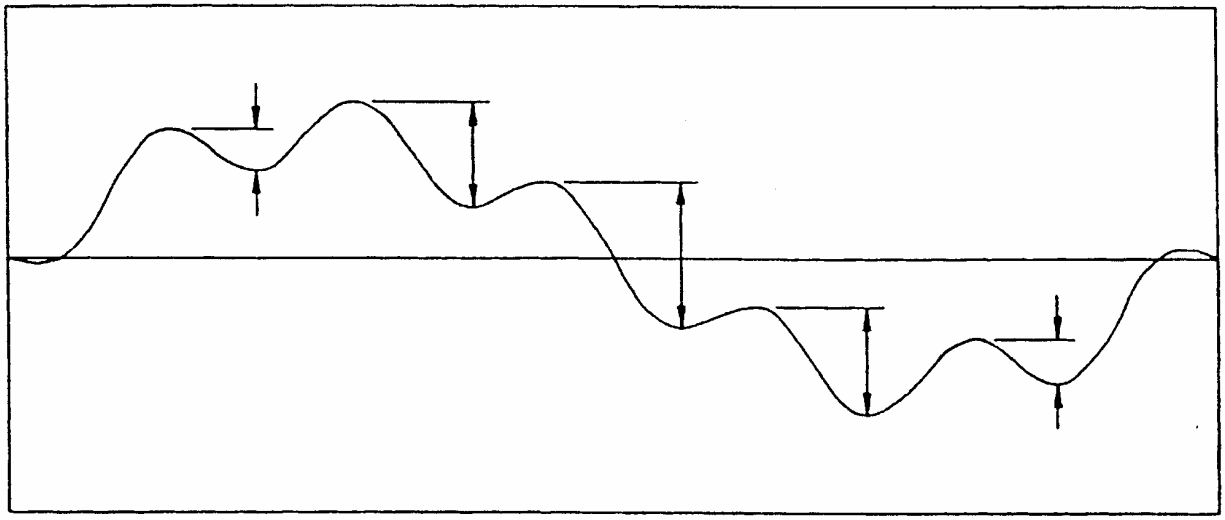


Figure 9. Range counting method.

The mean number of stress cycles in a time interval $[0,T]$ is equal to the mean number of local maxima in the time interval

$$n = \nu_m T = \frac{1}{2\pi}\sqrt{\frac{m_4}{m_2}}T \tag{30}$$

where m_4 is given by (28).

Using a double envelope process to model the stress process it can be shown that, see [10]

$$E[\Delta\sigma^m] = \alpha^m \left(2\sqrt{2}\right)^m \left(\sqrt{m_0}\right)^m \Gamma(1+m/2) \tag{31}$$

where the regularity factor α is defined by

$$\alpha = \frac{m_2}{\sqrt{m_0 m_4}} = \frac{\nu_0}{\nu_m} \tag{32}$$

(31) deviates from (27) by the factor α^m. In the limit where the process is narrow banded (α=1) (31) and (27) are identical.

4.3.3 Broad band spectra - Rainflow counting

Rainflow counting is considered to give the most accurate predictions of the fatigue life when compared to actual fatigue life results. Rainflow counting is widely used. Material hysterises loops are sometimes used to justify its use. Rainflow counting is illustrated in figure 10 where the largest cycles are extracted first and the smaller cycles are considered to be superimposed on the larger cycles, see [11] and [12].

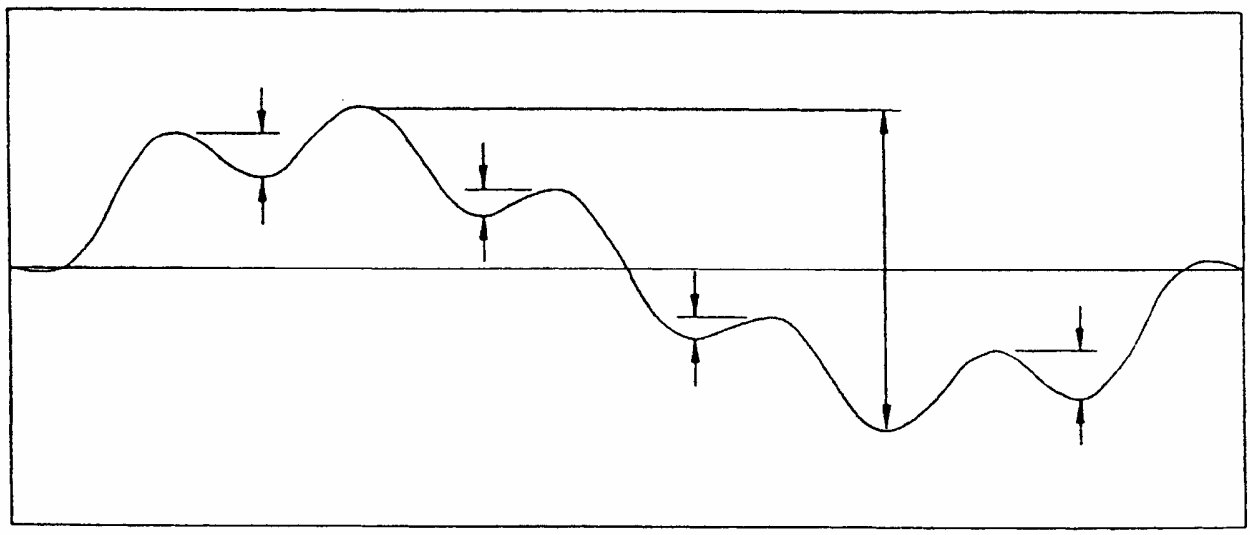


Figure 10. Rainflow counting.

The Rainflow counting method counts the number of stress cycles by converting a realization of the stress process $\{\sigma(t)\}$ to a point process of peaks and troughs as shown in figure 11. The peaks are identified by even numbers and the troughs by odd numbers. The following rules are imposed on "rain dropping on the roofs", so that cycles and half cycles are defined, see Wirshing & Sheheta [13].

1. A rain-flow is started at each peak and trough.
2. When a rain-flow part started at a trough comes to a tip of the roof, the flow stops if the opposite trough is more negative than that at the start of the path under consideration (e.g. in figure 4.6, path [1-8], path [9-10], etc.]). For a path started at a peak, it is stopped by a peak which is

more positive than that at the start of the rain path under consideration (e.g. in figure 11, path [2-3], path [4-5] and path [6-7]).

3. If the rain flowing down a roof intercepts a flow from the previous path, the present path is stopped, (e.g. in figure 4.6, path [3-3a], path [5-5a], etc.)
4. A new path is not started until the path under consideration is stopped.

Half-cycles of trough-originated range magnitudes h_i are projected distances on the x-axis (e.g. in figure 11, [1-8], [3-3a], [5-5a] etc.). If the realization of $\{\sigma(t)\}$ is sufficiently long, any trough-originated half-cycle will be followed by another peak originated half-cycle of the same range.

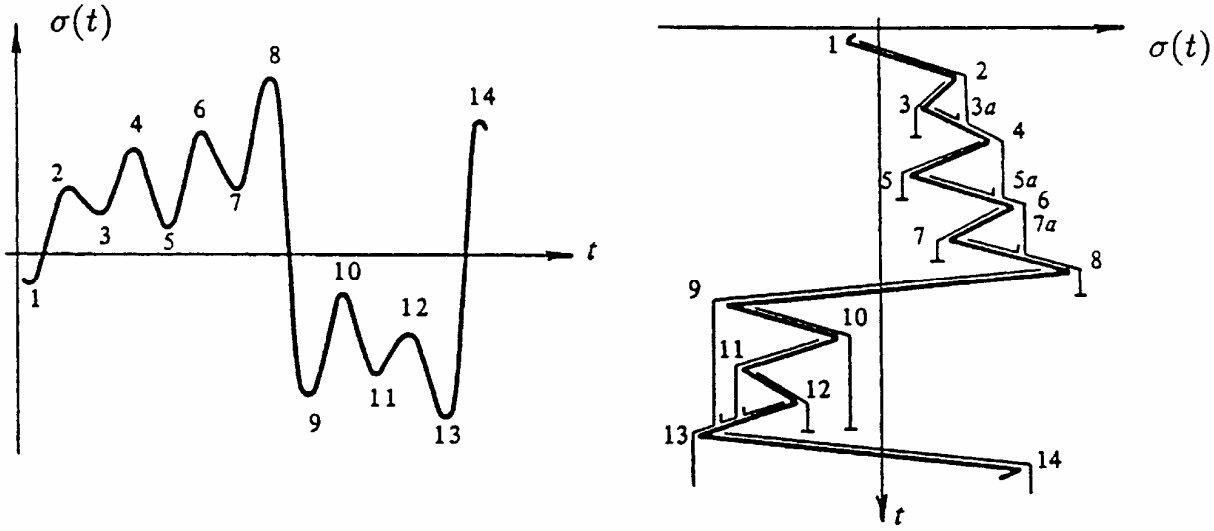


Figure 11. Illustration of Rainflow cycle counting applied to sample of $\{\sigma(t)\}$ (from [13]).

Due to the complexity of the Rainflow algorithm it is very difficult to derive a density function $f_{\Delta\sigma}$ for the stress ranges and to estimate the number of stress cycles.

However, based on extensive computer simulations, Dirlik [14] has derived empirical expressions for $f_{\Delta\sigma}$:

$$f_{\Delta\sigma}(s) = \frac{1}{2\sqrt{m_0}}\left[\frac{D_1}{Q}\exp\left(-\frac{s}{2Q\sqrt{m_0}}\right) + \frac{D_2 s}{2\sqrt{m_0}R^2}\exp\left(-\frac{s^2}{8m_0R^2}\right) + \frac{D_3 s}{2\sqrt{m_0}}\exp\left(-\frac{s^2}{8m_0}\right)\right] \tag{33}$$

where

$$D_1 = 2(\beta - \alpha^2)/(1+\alpha^2) \tag{34}$$

$$R = (\alpha - \beta - D_1^2)/(1 - \alpha - D_1 + D_1^2) \tag{35}$$

$$D_2 = (1 - \alpha - D_1 + D_1^2)/(1 - R) \tag{36}$$

$$D_3 = 1 - D_1 - D_2 \tag{37}$$

$$Q = 1.25(\alpha - D_3 - D_2 R)/D_1 \tag{38}$$

$$\beta = \frac{m_1}{m_0}\sqrt{\frac{m_2}{m_4}} \tag{39}$$

Using (33) $E[\Delta\sigma^m]$ can be estimated numerically.

The expected number of stress cycles n is estimated by

$$n = \nu_m T \tag{40}$$

4.4 Simple example

From (19) and (24) it is seen that if $Y(a)$=1 and $n = \nu T$ then failure can be modeled by the limit state function

$$g = a_0^{(2-m)/2} + \frac{2-m}{2} C\pi^{m/2}\Delta\sigma^m \nu T - a_c^{(2-m)/2}$$

It is assumed that the parameters can be modeled by the values in table 2:

	Variable	Distribution	Expected value	Standard deviation
X_1	a_0	E	0.1	
X_2	a_c	N	40	10
X_3	$\ln C$	N	-33	0.47
X_4	$\Delta\sigma$	W	60	10
	ν	D	10^6 cycles / year	
	m	D	3	

Table 2. N: Normal, E: exponential, W: Weibull, D: deterministic. Dimensions in mm and N.

T [years]	β	α_1	α_2	α_3	α_4
2.5	5.50	0.49	-0.03	0.77	0.41
5.0	4.38	0.52	-0.02	0.74	0.43
7.5	3.78	0.53	-0.02	0.72	0.45
10.0	3.32	0.54	-0.02	0.70	0.46
12.5	2.99	0.55	-0.03	0.69	0.46
15.0	2.72	0.56	-0.02	0.68	0.47
17.5	2.50	0.57	-0.02	0.67	0.48
20.0	2.31	0.58	-0.02	0.66	0.48
22.5	2.15	0.58	-0.01	0.66	0.48
25.0	2.00	0.59	-0.01	0.66	0.49

Table 3. Reliability index and sensitivity coefficients $\alpha_1,\ldots,\alpha_4$ at different times.

The results of a reliability analysis is shown in table 3. It is seen that the reliability index β decreases from 5.50 to 2.00 when T increases from 2.5 year to 25 years. Further it is also seen, that a_0, $\ln C$ and $\Delta\sigma$ are the most important stochastic variables in this example.

5 Example: Reliability analysis of fatigue critical details

The example presented in this section is based on the results in [1]. 4 joints in steel jacket structure is selected, see [15]. The reliability analyses are carried out on the basis of the traditional SN-approach and on the basis of a fracture mechanics model of the crack growth. Subsequently the fracture mechanical model is calibrated to give the same probability distribution function for the time to failure as the SN-approach. Finally, the fracture mechanics model is used to determine an optimal plan for inspection of the selected joints, see section 6. The optimal plan is determined on the basis of simple reliability updating where it is assumed that no crack is detected by the inspections.

On the basis of a deterministic analysis the joints and elements given in table 4 have been identified as being critical with respect to fatigue. The numbers given in table 4 refers to the numbering given in the report "Structure Identification, Platforms" [15].

Joint	Element
154050	111404
152050	111424
253050	52501
253050	52006

Table 4. Critical joints and elements.

5.1 Evaluation of the equivalent stress range

Both the SN-approach and the fracture mechanics approach are based on the evaluation of an equivalent stress range. The equivalent stress range is determined on the basis of the assumption that the stress history in a given sea-state can be modeled as a narrow band Gaussian process. This implies that for a given sea state the stress range, $\Delta\sigma$, follows a Rayleigh distribution

$$F_{\Delta\sigma}\left(\Delta\sigma\middle|H_S,T_Z,\Theta\right)=1-\exp\left(-\frac{\Delta\sigma^2}{8m_0\left(H_S,T_Z,\Theta\right)}\right) \tag{42}$$

where H_S, T_Z and Θ denotes the significant wave height, the wave period and the direction, m_0 is the 0th order spectral moment of the stress history.

The long-term stress distribution can be determined from

$$F_{\Delta\sigma}\left(\Delta\sigma\right)=\sum_{i=1}^{n_S}P\left(H_{S,i},T_{Z,i},\Theta_i\right)F_{\Delta\sigma}\left(\Delta\sigma\middle|H_{S,i},T_{Z,i},\Theta_i\right) \tag{43}$$

where n_S denotes the number of considered sea states and $P\left(H_{S,i},T_{Z,i},\Theta_i\right)$ is the probability of having significant wave height $H_{S,i}$, wave period $T_{Z,i}$ and direction Θ_i in sea state no. i.

The long-term stress distribution may be approximated by a Weibull distribution given by

$$F_{\Delta\sigma}\left(\Delta\sigma\right)=1-\exp\left(-\left(\frac{\Delta\sigma}{k}\right)^{\lambda}\right) \tag{44}$$

The Weibull distribution may be fitted so that it corresponds to the long-term stress distribution in the 95 and 99 % quantiles. Using this approximation it can be found that

$$E\left[\Delta\sigma^m\right] = k^m \Gamma\left(1+\frac{m}{\lambda}\right) \tag{45}$$

The number of cycles in a given period of time, $[0,T]$, is given by

$$N = T\sum_{i=1}^{n_S} P\left(H_{S,i}, T_{Z,i}, \Theta_i\right)\sqrt{\frac{m_{2,i}}{m_{0,i}}} \tag{46}$$

where $m_{0,i} = m_{0,i}\left(H_{S,i}, T_{Z,i}, \Theta_i\right)$ and $m_{2,i} = m_{2,i}\left(H_{S,i}, T_{Z,i}, \Theta_i\right)$ are spectral moments for sea state no. i.

The spectral moments are determined by a spectral analysis, see [1]. The spectral moments determined are deterministic constants. However, due to the uncertainty related to the transfer function and the uncertainty related to the stress concentration factors the spectral moments must be modeled as stochastic variables. The uncertainty related to the spectral moments is introduced by

$$m_0 = U_H U_{SCF}^2 m_0^* \tag{47}$$

where U_H and U_{SCF} are stochastic variables describing the random variation of the transfer function and the stress concentration factors and where m_0^* is the spectral moment determined by the spectral analysis where the mean values of the transfer function and stress concentration factors have been used.

The uncertainty related to the spectral moments is taken into account by determining the joint distribution of the parameters k and λ in the Weibull model.

Joint	Element	Hot spot	k [MPa]	$\frac{1}{\lambda}$
253050	52006	0	LN(10.5;1.93)	1.098
253050	52006	90	LN(8.04;1.48)	1.085
253050	52006	180	LN(9.41;1.73)	1.125
253050	52006	270	LN(12.1;2.21)	1.083
253050	52501	0	LN(9.80;1.80)	1.120
253050	52501	90	LN(7.98;1.47)	1.092
253050	52501	180	LN(9.24;1.70)	1.107
253050	52501	270	LN(11.24;2.06)	1.086
154050	111404	0	LN(1.81;0.331)	1.053
154050	111404	90	LN(1.81;0.332)	0.095
154050	111404	180	LN(8.12;1.49)	1.099
154050	111404	270	LN(11.7;2.14)	1.093
152050	111424	0	LN(7.55;1.39)	1.135
152050	111424	90	LN(1.69;0.331)	1.131
152050	111424	180	LN(1.70;0.313)	1.067
152050	111424	270	LN(10.9;1.99)	1.125

Table 5. Parameters for the equivalent stress range.

The joint distribution of these variables is determined by simulation. For a given outcome of the variables U_H and U_{SCF} the parameters k and λ are determined. On the basis of a large number of simulations (here 1000 simulations), the joint distribution of the variables is determined.

For the joints and elements considered here the distributions of the parameters k and λ are given in table 5.

5.2 SN-Approach

The limit state function for reliability analysis using the SN-approach is written, see section 4.1:

$$g = \Delta - D = \frac{1}{K}\sum_{i=1}^{n}\Delta\sigma_i^m \tag{48}$$

where Δ is the damage limit corresponding to failure and D models the accumulated damage during the considered time interval, see (15).

In table 6 and 7 the input parameters to the limit state function are given.

Parameter	Distribution / Value	Unit
Δ	LN(1,0.3)	-
$\ln K$	N(34.63,0.54)	-
k	See table 5	MPa
$\frac{1}{\lambda}$	See table 5	-
m	4.1	-
n	See table 7	-

Table 6. Distributions and values of the input to the probabilistic model.

Joint	Element	Hot spot	Number of cycles n
253050	52006	270	5589061
253050	52501	270	5632062
154050	111404	270	5757739
152050	111424	270	5760740

Table 7. Number of cycles per year.

In figure 12 the reliability index as a function of time is given for the four different joints and elements for the hot spot 270°. In figure 13 the squared alpha values of the three stochastic variables are shown.

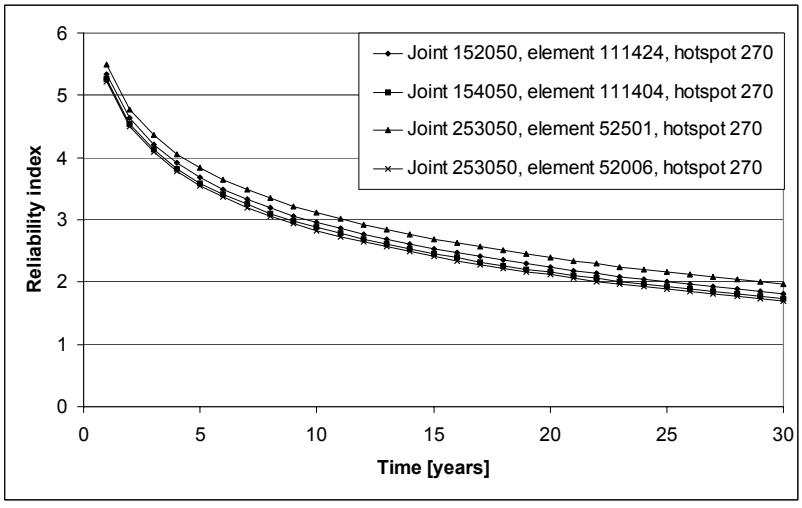


Figure 12. Reliability index as a function of time.

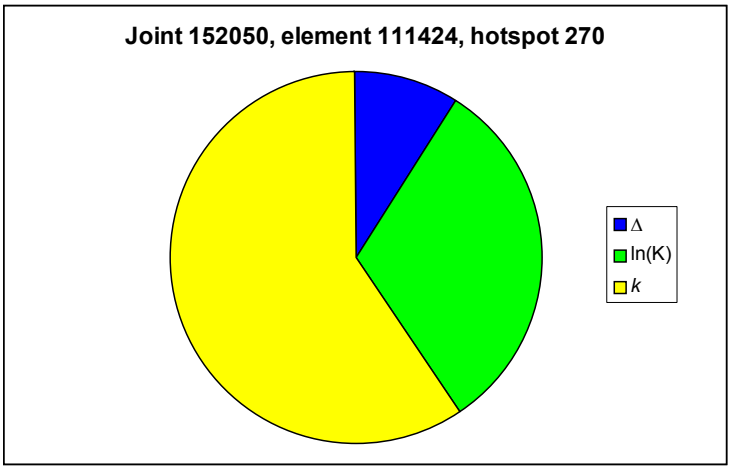


Figure 13. Sensitivity measures of the stochastic variables at time=25 years.

The sensitivity measures are only shown for one of the joints, but they are very similar to the three other joints. The values do not change within time. It is seen that approximately 60% of the uncertainty originate from the mean value of the Weibull distribution modeling the long term stress, k, 30% from the logarithm of the material parameter, ln(K), and 10% from the damage limit corresponding to failure, Δ.

A reliability level corresponding to a safety index of 3.7 will assure the required safety. From figure 12 it is seen that this level is exceeded after approximately 5 years. Therefore it will be necessary to perform an inspection after 5 years in order to check the condition of the joint.

5.3 Fracture Mechanics

The crack growth is assumed to be described by (16) and (17). On the basis of analyses using a two-dimensional crack growth model Kirkemo and Madsen [16] have found the following expression for the geometry function

$$Y(a) = 1.08 - 0.7\frac{a}{t} \tag{49}$$

where t is the thickness of the plate.

This expression is valid for a crack in a plate. For cracks in a tubular joint the stress intensity factors can be determined by multiplying by a correction factor determined by Smith and Hurworth [17] as

$$M_k(a) = 1.0 + 1.24\exp\left(-22.1\frac{a}{t}\right) + 3.17\exp\left(-357\frac{a}{t}\right) \tag{50}$$

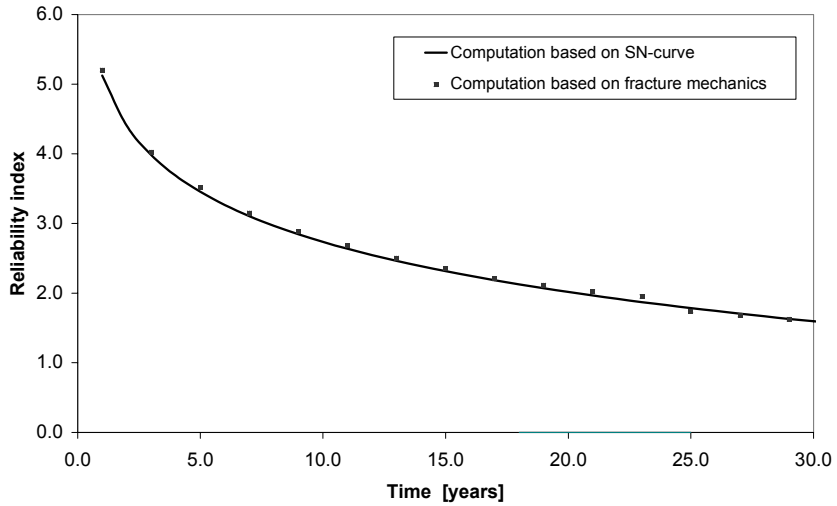
The probabilistic model is given in table 8.

Parameter	Distribution / Value	Unit
$\ln(C_A)$	N(-31.5,0.77)	-
k	See table 5	Mpa
$\frac{1}{\lambda}$	See table 5	-
m	3.2	-
n	See table 7	-

Table 8: Probabilistic model.

The probabilistic models for m and C_A have been chosen such that the difference between the distribution functions for the time to fatigue failure using the fracture mechanical approach and the SN-approach is minimized.

In figure 14 the results of the reliability analyses are shown. It is seen that the results of the analyses using the SN-approach are virtually identical to the results obtained on the basis of the fracture me-



chanics model. The analysis has been carried out for joint 253050, element 52006, hot spot 270^o, i.e. the element with the largest mean stress range.
Figure 14. Results of reliability analysis.

6 Inspection Planning

The inspection planning is performed on the basis of the assumption that no cracks are detected by the inspections. Further, an inspection is carried out when the reliability with respect to the considered event decreases below a given acceptable limit. For a more detailed description of the method see [18].

The observation may be performed by three different methods

- Magnetic Particle Inspection
- Eddy Current
- Close visual inspection

Each of these inspection methods are associated with a given probability of detection depending on the size of the crack and depending on whether the inspection is carried out above or below water. As mentioned it is assumed that no crack is detected by the inspection, i.e. the crack is smaller than the detectable crack size. This event can be modeled by the following limit state function

$$h = a - a_d \tag{51}$$

where a_d is the depth of the detectable crack.

The distribution of the size of the detectable crack or the so-called POD-curve (Probability of Detection) depends on the inspection method. The distribution of the detectable crack can be modelled by an upward bounded Exponential distribution

$$F_{a_d}(a_d) = P_0\left(1 - \exp\left(-a/\lambda\right)\right) \tag{52}$$

where λ and P_0 are model parameters which may be determined by tests.

The inspection planning is carried out using an inspection method with the following parameters.

$\lambda = 2{,}67\ \text{mm}^{-1}$

$P_0 = 1.0$

In figure 15 the results of the inspection planning are shown. On the basis of the probabilistic modeling of the fatigue crack growth described an inspection plan has been determined in accordance with the methodology described in [19]. It is found that 4 inspections performed at year 4, 8, 15 and 23 will ensure as sufficient safety for the considered joint throughout its anticipated service life of 30 years.

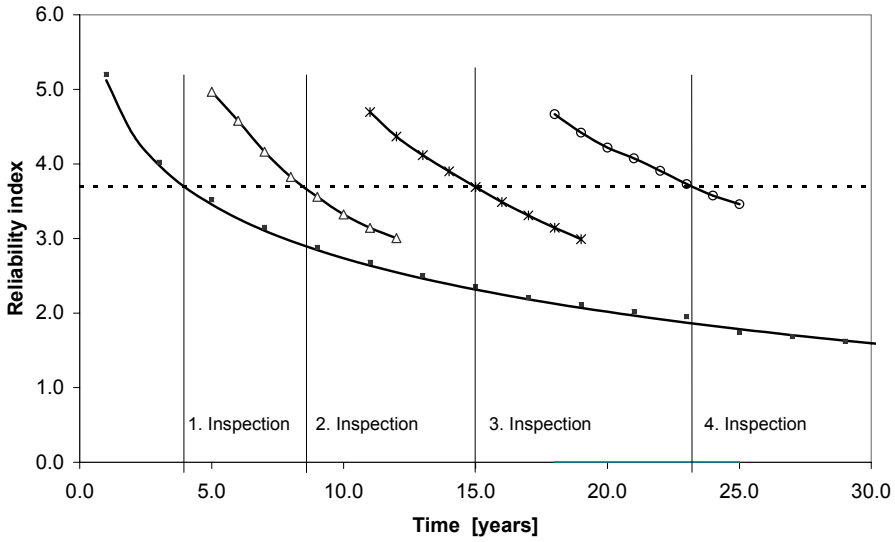


Figure 15. Updated reliability assuming no-find inspection results.

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Note 12: RELIABILITY UPDATING

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1 Introduction

When new information it can be used to update the stochastic models and the estimates of the reliability (probability of failure). In this note it is described how this updating can be performed when the new information consists of

1. Observation of events described by one or more stochastic variables. The observation is modeled by an event margin and the failure event by a safety margin. Updated / conditional probabilities of failure can then be obtained, see section 2.
2. Samples / measurements of a stochastic variable X. Updating can in this case be performed using Bayesian statistics, see section 3.

2 Bayesian updating of stochastic variables

In order to model the observed events an event function

$$H = h(\mathbf{X}) \tag{1}$$

is introduced. The event function h corresponds to the limit state function. The actual observations are considered as realizations (samples) of the stochastic variable H. This type of information can e.g. be

- Inspection events such as measurements of the chloride content in concrete structures or measurements of crack sizes in steel structures exposed to fatigue loading. The event margin can include the uncertainty related to the measurement.
- Proof loading where a well defined load is applied to a structure and the level of damage (usually no damage is observed) is observed.
- Repair events where a certain type of repair or maintenance has been performed.
- No-failure events where the 'simple' observation that the structure / component considered is well-functioning after some time in use.

It is assumed that these observations can be modeled by

a. inequality events $\{H \le 0\}$, i.e. it is observed that the observed quantity is less than or equal to some limit, or
b. equality events $\{H = 0\}$, i.e. it is observed that the observed quantity is equal to some limit.

If inequality events are used the updated probability of failure is estimated by

$$P_F^U = P(g(\mathbf{X}) \le 0 \,|\, h(\mathbf{X}) \le 0) = \frac{P(g(\mathbf{X}) \le 0 \cap h(\mathbf{X}) \le 0)}{P(h(\mathbf{X}) \le 0)} \tag{2}$$

where $M = g(\mathbf{X})$ is the safety margin related to the limit state function $g(\mathbf{x})$ and $\mathbf{X} = (X_1,...,X_n)$ are stochastic variables. In (2) it is used that the probability of an event A given an event B (denoted $P(A|B)$) is equal to $\frac{P(A \cap B)}{P(B)}$. It is seen that $P(g(\mathbf{X}) \le 0 \cap h(\mathbf{X}) \le 0)$ is the probability of a parallel system with two elements. (2) can be evaluated by simulation or FORM/SORM methods, see Madsen et al. [1].

Other observations can be modeled by equality events $\{H = 0\}$, i.e. it is observed that the observed quantity is equal to some limit. In this case the updated probability of failure can be estimated by, see Madsen et al. [1], Madsen [2] and Schall and Rackwitz [3]

$$P_F^U = P(g(\mathbf{X}) \le 0 | h(\mathbf{X}) = 0) = \frac{P(g(\mathbf{X}) \le 0 \cap h(\mathbf{X}) = 0)}{P(h(\mathbf{X}) = 0)} = \frac{\frac{\partial}{\partial z} P(g(\mathbf{X}) \le 0 \cap h(\mathbf{X}) \le z)\Big|_{z=0}}{\frac{\partial}{\partial z} P(h(\mathbf{X}) \le z)\Big|_{z=0}} \tag{3}$$

Equation (3) can also be evaluated by FORM/SORM methods and can easily be generalized if more than one event is observed. In most software packages for reliability analysis efficient algorithms are available for solving this problem.

Example 2.1
Consider the following limit state function modeling crack growth of a crack with initial length a_0 to the critical length a_c. m and C are the parameters in Paris' law. ν is the number of stress cycles per year with stress range $\Delta\sigma$. T is the time in years.

$$g = a_0^{(2-m)/2} + \frac{2-m}{2} C \pi^{m/2} \Delta\sigma^m \nu T - a_c^{(2-m)/2}$$

It is assumed that the parameters can be modeled by the values in table 1.

	Variable	Distribution	Expected value	Standard deviation
X_1	a_0	Exponential	0.1	
X_2	a_c	Normal	50	5
X_3	$\ln C$	Normal	-33	0.47
X_4	$\Delta\sigma$	Weibull	60	10
	ν	Deterministic	6 10^6 cycles / year	
	m	Deterministic	3	

Table 1. Stochastic model. Dimensions in mm and N.
The annual reliability index $\beta(T)$ determined by

$$\beta(T) = -\Phi^{-1}\left(P(g(\mathbf{X},T) \le 0) - P(g(\mathbf{X},T-1) \le 0)\right)$$

is shown in figure 1. It is seen that the annual reliability index becomes smaller than $\beta = 4$ at year 14.

It is now assumed that the fatigue critical detail is inspected at time T_1=14 years with the result that no cracks were detected. The reliability of the inspection is described by the following POD-curve where a_d models the smallest detectable crack size:

$$POD(a) = F_{a_d}(a) = 1 - \exp\left(-\frac{a}{b}\right)$$

b is a parameter modeling the expected value of a_d. It is assumed that b=1 mm. The inspection event (no cracks detected) is modeled by the event margin

$$h = a_d^{(2-m)/2} - a_0^{(2-m)/2} - \frac{2-m}{2} C \pi^{m/2} \Delta\sigma^m \nu T \le 0$$

The corresponding updated annual reliability index shown in figure 1 is determined by

$$\beta^U(T) = -\Phi^{-1}\left(P(g(\mathbf{X},T) \le 0) \middle| h(\mathbf{X}) \le 0) - P(g(\mathbf{X},T-1) \le 0) \middle| h(\mathbf{X}) \le 0)\right) \quad , \; T > T_1$$

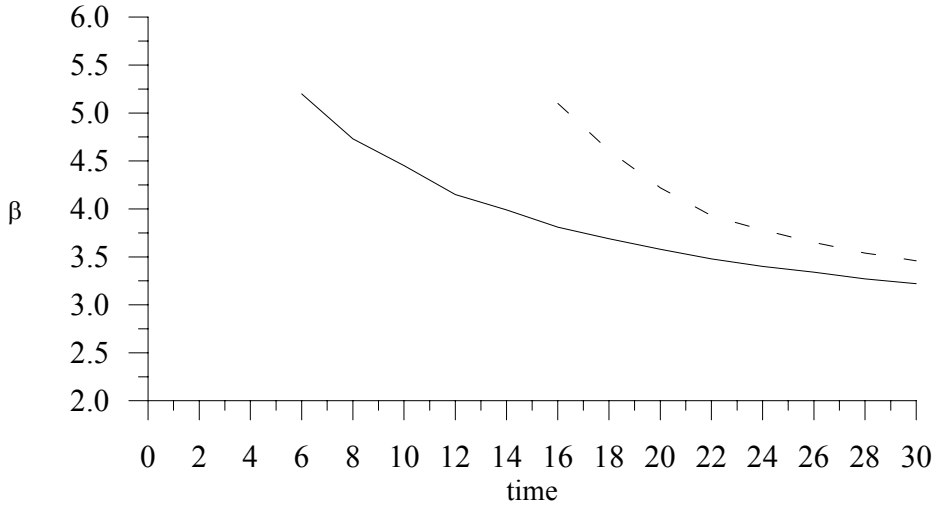

Figure 1. Reliability indices as function of time in years. Full line: $\beta(T)$ and broken line: $\beta^U(T)$.

3 Bayesian updating of stochastic variables

If possible Bayesian techniques should be used for parameter estimation because Bayesian estimation gives an estimate of the statistical uncertainty related to the estimated parameters and because updating of the model when new information becomes available is easy.

If observations of one (or more) of the stochastic variables $\mathbf{X}$ are available, the probabilistic model can be updated and thereby also the probability of failure. Consider a stochastic variable X with density function $f_X(x)$. If $\mathbf{q}$ denotes a vector of parameters defining the distribution for X, the density function of the stochastic variable X can be written

$$f_X(x,\mathbf{q}) \tag{4}$$

If X is normally distributed then $\mathbf{q}$ could contain the mean and the standard deviation of X.

If the parameters $\mathbf{q}$ are uncertain then $f_X(x,\mathbf{q})$ can be considered as a conditional density function : $f_X(x|\mathbf{Q})$ and $\mathbf{q}$ denotes a realization of **Q**. The initial density function for the parameters **Q** is denoted $f'_{\mathbf{Q}}(\mathbf{q})$ and is denoted the **prior density function**.

It is assumed that *n* observations (realizations) of the stochastic variable X are available making up a sample $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$. The realizations are assumed to be independent. The updated density function $f''_{\mathbf{Q}}(\mathbf{q}|\hat{\mathbf{x}})$ of the uncertain parameters **Q** given the realizations is denoted the **posterior density function** and is given by, see textbook on Bayesian statistics, e.g. Box & Tiao [4] and Lindley [5]

$$f''_{\mathbf{Q}}(\mathbf{q}|\hat{\mathbf{x}}) = \frac{f_N(\hat{\mathbf{x}}|\mathbf{q}) f'_{\mathbf{Q}}(\mathbf{q})}{\int f_N(\hat{\mathbf{x}}|\mathbf{q}) f'_{\mathbf{Q}}(\mathbf{q}) d\mathbf{q}} \tag{5}$$

where $f_N(\hat{\mathbf{x}}|\mathbf{q}) = \prod_{i=1}^{N} f_X(\hat{x}_i|\mathbf{q})$ is the probability density at the given observations assuming that the distribution parameters are $\mathbf{q}$. The integration in (5) is over all possible values of $\mathbf{q}$.

The updated density function of the stochastic variable X given the realization $\hat{\mathbf{x}}$ is denoted the **predictive density function** and is defined by,

$$f_X(x|\hat{\mathbf{x}}) = \int f_X(x|\mathbf{q}) f''_{\mathbf{Q}}(\mathbf{q}|\hat{\mathbf{x}}) d\mathbf{q} \tag{6}$$

Given the distribution function for the stochastic variable X, the prior distribution is often chosen such that the posterior distribution will be of the same type as the prior distribution (a so-called conjugated prior). In the literature a number of prior, posterior and predictive distribution functions can be found, see e.g. Raiffa & Schlaifer [6], Aitchison & Dunsmore [7] and Rackwitz & Schrupp [8]. Some of these are shown in appendix A.

By use of the Bayesian method presented here, both the physical uncertainty related to the considered variable as well as the statistical uncertainty related to the model parameters can be quantified. However, as mentioned the probabilistic model must also be formulated such that the measurement uncertainty and the model uncertainty are taken into account.

Due to the ability of Bayesian statistics to incorporate engineering judgement and experience into the statistical modeling, different reassessment engineers may reach different results due to the use of different statistical models. This may obviously be a serious obstacle for the use of such methods. Also in order to avoid this obstacle it is necessary to define and agree on a common basis for such analyses thus ensuring that reliability based reassessment analyses are performed on a consistent and comparable basis

Example 3.1
Consider a Normal distributed stochastic variable X with expected value μ and known standard deviation σ=3. Prior knowledge on μ is modeled by a Normal distributed prior distribution with expected value μ'=10 and standard deviation σ'=4.

Further it is assumed that n=5 observations are available: $\hat{\mathbf{x}}$=(11.3, 12.4, 9.5, 10.7, 11.1) implying that $\bar{x}$=10.9. The posterior distribution the becomes Normal with expected value μ'' and standard deviation σ'', see (A-4)-(A-5):

$$\mu'' = \frac{n\bar{x}\sigma'^2 + \mu'\sigma^2}{n\sigma'^2 + \sigma^2} = 10.9 \quad \text{and} \quad \sigma''^2 = \frac{\sigma'^2\,\sigma^2}{n\sigma'^2 + \sigma^2} = 1.62$$

The predictive distribution also becomes Normal distributed with expected value μ'' and standard deviation σ''', see (A-8)

$$\mu'' = \frac{n\bar{x}\sigma'^2 + \mu'\sigma^2}{n\sigma'^2 + \sigma^2} = 10.9 \quad \text{and} \quad \sigma''' = \sqrt{\sigma''^2 + \sigma^2} = 3.26$$

In figure 2 are shown the prior and posterior distributions for the expected value μ.

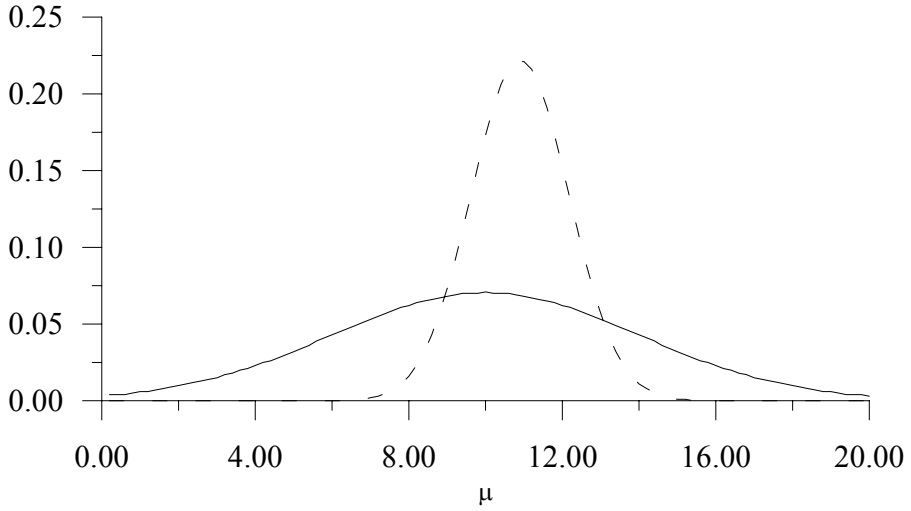

Figure 2. Prior (full line) and posterior (broken line) distributions for the expected value μ.

In figure 3 is shown the predictive distribution for the stochastic variable X .

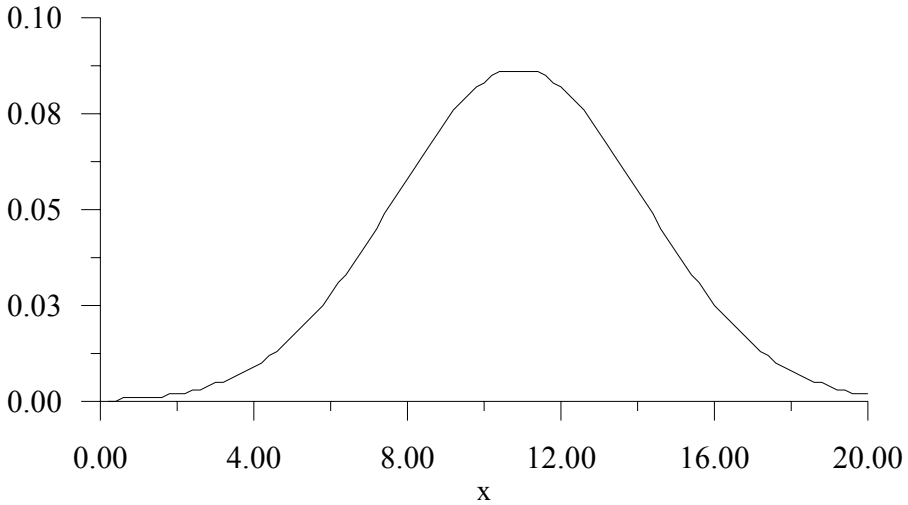

Figure 3. Predictive distribution for the stochastic variable X .

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Appendix A. Bayesian updating of distribution functions

A.1 Normal distribution with unknown mean

The stochastic variable X is normal distributed

$$f_X(x|\mu,\sigma) = f_N(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{A-1}$$

where f_N indicates a Normal distribution. The uncertain parameter is the mean $Q_1 = \mu$.

The prior density function is assumed to be a normal distribution:

$$f'_\mu(\mu) = f_N(\mu|\mu',\sigma') = \frac{1}{\sigma'\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\mu-\mu'}{\sigma'}\right)^2\right) \tag{A-2}$$

The posterior density function of μ then also becomes a normal distribution

$$f''_\mu(\mu|\hat{\mathbf{x}}) = f_N(\mu|\mu'',\sigma'') \tag{A-3}$$

where

$$\mu'' = \frac{n\bar{x}\sigma'^2 + \mu'\sigma^2}{n\sigma'^2 + \sigma^2} \tag{A-4}$$

$$\sigma''^2 = \frac{\sigma'^2\,\sigma^2}{n\sigma'^2 + \sigma^2} \tag{A-5}$$

and the statistic $\bar{x}$ is

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\hat{x}_i \tag{A-6}$$

$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ is n test results.

The predictive density function becomes a normal distribution

$$f_X(x|\hat{\mathbf{x}}) = f_N(x|\mu'',\sigma''') \tag{A-7}$$

where

$$\sigma''' = \sqrt{\sigma''^2 + \sigma^2} \tag{A-8}$$

A.2 Normal distribution with unknown standard deviation

The stochastic variable X is normal distributed

$$f_X(x|\mu,\sigma) = f_N(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{A-9}$$

The uncertain parameter is the mean $Q_1 = \sigma$.

The prior density function is chosen as an Invers-Gamma-2 distribution

$$f'_\sigma(\sigma) = f_{i\gamma 2}(\sigma|\omega', \nu') = \frac{2}{\Gamma\left((\nu'+2)/2\right)} \frac{\left(\frac{1}{2}\right)\left(\nu'\omega'/\sigma^2\right)^{(\nu'+2)/2}}{\left(\frac{1}{2}\right)\left(\nu'\omega'\right)^{1/2}} \exp\left(-\frac{1}{2}\frac{\nu'\omega'}{\sigma^2}\right) \tag{A-10}$$

where $f_{i\gamma 2}$ indicates an Invers-Gamma-2 distribution.

The posterior density function becomes an Invers-Gamma-2 distribution

$$f''_\sigma(\sigma) = f_{i\gamma 2}(\sigma|\omega'', \nu'') \tag{A-11}$$

where

$$\nu'' = \nu' + \nu \tag{A-12}$$

$$\omega'' = (\omega'\nu' + \omega\nu)/\nu'' \tag{A-13}$$

and the statistic ω is

$$\omega = \frac{1}{\nu}\sum_{i=1}^{\nu}(\hat{x}_i - \mu)^2 \tag{A-14}$$

$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\nu)$ are ν test results.

The posterior distribution function corresponding to (A-11) is the Invers-Gamma-2 distribution function

$$F''_\sigma(\sigma) = 1 - \frac{\Gamma\left(\frac{\nu''}{2}\frac{\omega''}{\sigma^2}, \frac{\nu''}{2}\right)}{\Gamma\left(\frac{\nu''}{2}\right)} \tag{A-15}$$

where

$$\Gamma(a) = \int_0^\infty \exp(-t)t^{a-1}dt \tag{A-16}$$

$$\Gamma(a,b) = \int_0^\infty \exp(-t)t^{b-1}dt \tag{A-17}$$

The predictive density function becomes a Student distribution

$$\begin{aligned} f_X(x|\hat{\mathbf{x}}) &= f_S(x|\mu,\omega'',\nu'') = \int f_N(x|\mu,\sigma) f_{i\gamma 2}(\sigma|\omega'',\nu'')d\sigma \\ &= \frac{\nu''^{(\nu''/2)}}{\sqrt{\omega''}B(½,½\nu'')}\left[\nu''+\frac{(x-\mu)^2}{\omega''}\right]^{-(\nu''+1)/2} \end{aligned} \tag{A-18}$$

where f_S indicates the Student distribution and B is the beta function.

A.3 Normal distribution with unknown mean and standard deviation

The normal density function is

$$f_X(x|\mu,\sigma) = f_N(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{A-19}$$

The uncertain statistical parameters are : $(Q_1, Q_2) = (\mu, \sigma)$.

The joint prior density function is chosen as a Normal-Invers-Gamma-2 distribution

$$f'_{\mu,\sigma}(\mu,\sigma) = f_N\left(\mu'\middle|\bar{x}', \frac{\sigma}{\sqrt{n'}}\right) f_{i\gamma 2}\left(\sigma|\upsilon', \nu'\right) \tag{A-20}$$

The posterior density function becomes also a Normal-Invers-Gamma-2 distribution

$$f''_{\mu,\sigma}(\mu,\sigma|\hat{\mathbf{x}}) = f_N\left(\mu'\middle|\bar{x}'', \frac{\sigma}{\sqrt{n''}}\right) f_{i\gamma 2}\left(\sigma|\upsilon'', \nu''\right) \tag{A-21}$$

where

$$n'' = n'+n \tag{A-22}$$

$$\bar{x}'' = \left(n'\bar{x}'+n\bar{x}\right)/n'' \tag{A-23}$$

$$\upsilon'' = \left(\nu'\upsilon'+n'\bar{x}'^2+\nu\upsilon+n\bar{x}^2-n''\bar{x}''^2\right)/\nu'' \tag{A-24}$$

$$\nu'' = \nu'+\delta(n')+\nu+\delta(n)-\delta(n'') \tag{A-25}$$

and the statistics $\bar{x}$ and υ are

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\hat{x}_i \tag{A-26}$$

$$\upsilon = \frac{1}{n-1}\sum_{i=1}^{n}(\hat{x}_i - \bar{x}'')^2 \tag{A-27}$$

$\nu = n-1$ and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ are n test results.

The posterior distribution function for σ becomes

$$F''_{\sigma}(\sigma|\hat{\mathbf{x}}) = 1 - \frac{\Gamma\left(\frac{\nu''}{2}\frac{\upsilon''}{\sigma^2}, \frac{\nu''}{2}\right)}{\Gamma\left(\frac{\nu''}{2}\right)} \tag{A-28}$$

The posterior distribution function for μ given σ becomes

$$F''_{\mu|\sigma}(\mu|\sigma, \hat{\mathbf{x}}) = \Phi\left(\frac{\mu - \bar{x}''}{\sigma / \sqrt{n''}}\right) \tag{A-29}$$

The predictive density function becomes a Student distribution

$$f_X(x|\hat{\mathbf{x}}) = f_S\left(x\middle|\bar{x}'', \upsilon''\frac{n''+1}{n''}, \nu''\right) \tag{A-30}$$

A.4 Lognormal distribution

The density function is

$$f_X(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) \tag{A-31}$$

The Lognormal stochastic variable X can be treated in the same way as a normal distributed variable because $Y = \ln X$ is normal distributed with standard deviation

$$\sigma_Y = \sqrt{\ln\left(\frac{\sigma^2}{\mu^2} + 1\right)} \tag{A-32}$$

and expected value

$$\mu_Y = \ln\mu - \frac{1}{2}\sigma_Y^2 \tag{A-33}$$

A.5 Gumbel distribution

The distribution function is

$$F_X(x|\alpha,u) = \exp(-\exp(-\alpha(x-u))) \tag{A-34}$$

and the density function is

$$f_X(x|\alpha,u) = \alpha \exp\big(-\alpha(x-u) - \exp(-\alpha(x-u))\big) \tag{A-35}$$

The uncertain statistical parameter is : u

The prior density is chosen as

$$f_u^{'}(u) = \exp\big(n'\alpha u - t'\exp(\alpha u)\big)\alpha t'^{(n')}/\Gamma(n') \tag{A-36}$$

The posterior density function becomes

$$f_u^{''}(u|\hat{\mathbf{x}}) = \exp\big(n''\alpha u - t''\exp(\alpha u)\big)\alpha t''^{(n'')}/\Gamma(n'') \tag{A-37}$$

where

$$n'' = n + n' \tag{A-38}$$

$$t'' = t + t' \tag{A-39}$$

$$t = \sum_{i=1}^{n} \exp(-\alpha x_i) \tag{A-40}$$

$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ are n test results.

The posterior distribution function becomes

$$F_u^{''}(u|\hat{\mathbf{x}}) = \int_{-\infty}^{u} \exp\big(n''\alpha z - t''\exp(\alpha z)\big)\alpha t''^{(n'')}/\Gamma(n'')dz \tag{A-41}$$

The predictive density function becomes

$$f_X(x|\hat{\mathbf{x}}) = n''\left[1 + \frac{1}{t''}\exp(-\alpha x)\right]^{-(n''+1)} \frac{\alpha}{t''}\exp(-\alpha x) \tag{A-42}$$

The predictive distribution function becomes

$$F_X(x|\hat{\mathbf{x}}) = \left[1 + \frac{1}{t''}\exp(-\alpha x)\right]^{-n''} \tag{A-43}$$

A.6 Weibull distribution

The distribution function is

$$F_X(x|\varepsilon,u,k) = 1-\exp\left(-\left(\frac{x-\varepsilon}{u}\right)^k\right) \tag{A-44}$$

and the density function is

$$f_X(x|\varepsilon,u,k) = \frac{k}{u}\left(\frac{x-\varepsilon}{u}\right)^{k-1}\exp\left(-\left(\frac{x-\varepsilon}{u}\right)^k\right) \tag{A-45}$$

The uncertain statistical parameter is : u.

The prior density function is chosen as

$$f_u'(u) = \frac{1}{\Gamma\left(n'-1/k\right)}u^{-(n'k)}\exp\left(-u^{-k}t'\right)kt'^{(n'-1/k)} \tag{A-46}$$

The posterior density function becomes

$$f_u''(u|\hat{\mathbf{x}}) = \frac{1}{\Gamma\left(n''-1/k\right)}u^{-(n''k)}\exp\left(-u^{-k}t''\right)kt''^{(n''-1/k)} \tag{A-47}$$

where

$$n'' = n+n' \tag{A-48}$$

$$t'' = t+t' \tag{A-49}$$

$$t = \sum_{i=1}^{n}(\hat{x}_i-\varepsilon)^k \tag{A-50}$$

$\hat{\mathbf{x}} = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$ are n test results.

The posterior distribution function becomes

$$F_u''(u|\hat{\mathbf{x}}) = 1-\frac{\Gamma\left(n''-1/k,t''u^{-k}\right)}{\Gamma\left(n''-1/k\right)} \tag{A-51}$$

The predictive density function becomes

$$f_X(x|\hat{\mathbf{x}}) = \left(n''-1/k\right)\left[1+\frac{\left(x-\varepsilon\right)^k}{t''}\right]^{-(n''-1/k+1)}\frac{k}{t''}\left(x-\varepsilon\right)^{(k-\varepsilon)} \tag{A-52}$$

The predictive distribution becomes

$$F_X(x|\hat{\mathbf{x}}) = 1.\left[1+\frac{(x-\varepsilon)^k}{t''}\right]^{-(n''-1/k)} \tag{A-53}$$

A.7 Exponential distribution

The density function is

$$f_X(x|\lambda) = \lambda \exp(-\lambda) \tag{A-54}$$

The uncertain statistical parameter is : λ .

The prior density function is chosen as a Gamma distribution

$$f'_\lambda(\lambda) = \frac{\nu'^{h'}\ \lambda^{(h'-1)}\ \exp(-\nu'\lambda)}{\Gamma(h')} \tag{A-55}$$

The posterior density function becomes a Gamma distribution

$$f''_\lambda(\lambda|\hat{\mathbf{x}}) = \frac{\nu''^{h''}\ \lambda^{(h''-1)}\ \exp(-\nu''\lambda)}{\Gamma(h'')} \tag{A-56}$$

and the posterior distribution is

$$F''_\lambda(\lambda|\hat{\mathbf{x}}) = \frac{\Gamma(\nu''\lambda, h'')}{\Gamma(h'')} \tag{A-57}$$

where

$$\nu'' = \nu' + \sum_{i=1}^{n} \hat{x}_i \tag{A-58}$$

$$h'' = n + h' \tag{A-59}$$

$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ are n test results.

The predictive density function becomes an Invers-beta distribution

$$f_X(x|\hat{\mathbf{x}}) = \frac{h''\ \nu''^{h''}}{(x+\nu'')^{(h''+1)}} \tag{A-60}$$

and the predictive distribution function becomes

$$F_X(x|\hat{\mathbf{x}}) = \frac{B(x/(x+\nu''),\nu'',h'')}{B(1,h'')} \tag{A-61}$$

where

$$B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \tag{A-62}$$

$$B(x,a,b) = \int_0^x t^{(a-1)}(1-t)^{(b-1)}\,dt \tag{A-63}$$